Communications in Partial Differential Equations

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

Nonuniqueness and nonlinear instability of Gaussons under repulsive harmonic potential

Rémi Carles & Chunmei Su

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
https://doi.org/10.1080/03605302.2022.2050257

Nonuniqueness and nonlinear instability of Gaussons under repulsive harmonic potential

Rémi Carles[a] and Chunmei Su[b]

[a]Univ Rennes, CNRS, IRMAR - UMR 6625, Rennes, France; [b]Yau Mathematical Sciences Center, Tsinghua University, Beijing, China

ABSTRACT
We consider the Schrödinger equation with a nondispersive logarithmic nonlinearity and a repulsive harmonic potential. For a suitable range of the coefficients, there exist two positive stationary solutions, each one generating a continuous family of solitary waves. These solutions are Gaussian, and turn out to be orbitally unstable. We also discuss the notion of ground state in this setting: for any natural definition, the set of ground states is empty.

ARTICLE HISTORY
Received 7 October 2021
Accepted 24 February 2022

KEYWORDS
Nonlinear Schrödinger equation; logarithmic nonlinearity; ground states; instability

2020 MATHEMATICS SUBJECT CLASSIFICATION
Primary: 35Q55; Secondary: 35B35; 35C08; 37K40

1. Introduction

We consider the equation

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2 \frac{|x|^2}{2} u + \lambda u \ln\left(|u|^2\right), \quad x \in \mathbb{R}^d, \tag{1.1}$$

in the case $\omega > 0$ (repulsive harmonic potential) and $\lambda < 0$. The logarithmic Schrödinger equation ((1.1) with $\omega = 0$) was introduced in [1], and has been considered in various fields of physics since; see for example [2–9] and references therein. A special feature of the logarithmic nonlinearity is that it leads to very special solitary waves, called *Gaussons* in [1, 10]: if $\lambda < 0$, for any $\nu \in \mathbb{R}$,

$$e^{i\nu t} e^{\frac{\lambda d - \nu}{2\lambda}} e^{\lambda |x|^2}$$

is a solution to (1.1) (with $\omega = 0$). These solitary waves are orbitally stable, as proved in [11] (radial case) and [12] (general case). In addition, still in the case $\omega = 0$, it is known that for $\lambda < 0$, no solution is dispersive ([11, Proposition 4.3]), while for $\lambda > 0$, every solution is dispersive, with an enhanced rate compared to the usual rate of the free Schrödinger equation ([13]).

The logarithmic Schrödinger equation in the presence of a confining harmonic potential was considered in physics in [14],

CONTACT Rémi Carles Remi.Carles@math.cnrs.fr CNRS, IRMAR - UMR 6625, Univ Rennes, Rennes, France

$$i\partial_t u + \frac{1}{2}\Delta u = \omega^2 \frac{|x|^2}{2} u + \lambda u \ln\left(|u|^2\right), \quad x \in \mathbb{R}^d. \tag{1.2}$$

In the case $\lambda < 0$ ([15]) as well as in the case $\lambda > 0$ ([16]), generalized Gaussons exist, and are orbitally stable, in the sense introduced in [17] (see Definition 1.1 below for the definition in the case of (1.1), the notion being the same for (1.2)).

The case of an inverted, or repulsive harmonic potential as in (1.1), does not seem to correspond to a realistic model, but constitutes an interesting mathematical toy. The potential $V(x) = -\omega^2 \frac{|x|^2}{2}$ is unbounded from below, and goes to $-\infty$ as fast as possible in order to guarantee that the Hamiltonian $-\frac{1}{2}\Delta + V(x)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$; see [18, 19]. In the linear case $\lambda = 0$, classical trajectories go to infinity exponentially fast in time, the solution disperses exponentially in time, and the Sobolev norms grow exponentially in time (see e.g. [20]). Because of that, there are no long range effects (scattering theory) when a power-like nonlinearity is added ([20]), and at least in the case of an L^2-critical focusing nonlinearity,

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2 \frac{|x|^2}{2} u - |u|^{4/d} u, \quad x \in \mathbb{R}^d,$$

there exists no nontrivial solitary wave $u(t,x) = e^{i\nu t}\phi(x)$ with $\phi \in L^2(\mathbb{R}^d)$ [21, 22].

In the case of (1.1), the mass and the energy are formally independent of time: they are given by

$$\begin{aligned} M(u) &= ||u||^2_{L^2(\mathbb{R}^d)}, \\ E(u) &= \frac{1}{2}||\nabla u||^2_{L^2(\mathbb{R}^d)} - \frac{\omega^2}{2}||xu||^2_{L^2} + \lambda \int_{\mathbb{R}^d} |u|^2\left(\ln|u|^2 - 1\right)dx. \end{aligned} \tag{1.3}$$

The energy has no definite sign, for two reasons: the repulsive harmonic potential has a negative contribution in E, and the logarithmic nonlinearity induces a potential energy with indefinite sign (entropy). Introduce the space Σ defined by

$$\Sigma = H^1 \cap \mathcal{F}(H^1) = \left\{f \in H^1(\mathbb{R}^d), \quad x \mapsto |x| f(x) \in L^2(\mathbb{R}^d)\right\},$$

and equipped with the norm

$$\begin{aligned} ||f||^2_\Sigma &= ||f||^2_{L^2(\mathbb{R}^d)} + ||\nabla f||^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx \\ &= ||f||^2_{L^2(\mathbb{R}^d)} + \langle\left(-\Delta + |x|^2\right) f, f\rangle. \end{aligned}$$

It is proved in [16, Proposition 1.3] that for $\lambda \in \mathbb{R}$ and any $u_0 \in \Sigma$, there exists a unique solution $u \in L^\infty_{\text{loc}}(\mathbb{R};\Sigma) \cap C(\mathbb{R}; L^2(\mathbb{R}^d))$ to (1.1), such that $u_{|t=0} = u_0$. In addition, the mass M and the energy E are independent of time. In [16], it is proved in addition that in the case $\lambda > 0$, every solution to (1.1) disperses exponentially fast: in particular, there is no solitary wave in this case.

The situation is different in the case $\lambda < 0$, and leads to features which appear to be quite unique, in the context of the logarithmic Schrödinger equation (with potential), and more generally of nonlinear Schrödinger equations. In [23], it was proven that (1.1) admits at least one positive bound state, under some conditions on the coefficients, recalled below. Under suitable assumptions regarding the parameters λ and ω, we exhibit two positive stationary solutions.

Due to the presence of the potential, (1.1) is not invariant by translation in space, hence the definition below (as in [15]):

Definition 1.1. A standing wave $u(t,x)=\phi(x)e^{i\nu t}$ solution to (1.1) is orbitally stable in the energy space if for any $\varepsilon>0$, there exists $\eta>0$ such that if $u_0\in\Sigma$ satisfies $||u_0-\phi||_\Sigma<\eta$, then the solution u to (1.1) exists for all $t\in\mathbb{R}$, and

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}||u(t)-e^{i\theta}\phi||_\Sigma<\varepsilon.$$

Otherwise, the standing wave is said to be unstable.

The main result of this article is the following:

Theorem 1.2 (i) *Let $-\lambda>\omega>0$. Then (1.1) possesses two positive stationary solutions, which are Gaussons,*

$$\phi_{k_\pm}(x)=e^{-\frac{dk_\pm}{4\lambda}}e^{-k_\pm|x|^2/2},\quad \text{where}\quad k_\pm=-\lambda\pm\sqrt{\lambda^2-\omega^2}.$$

Each stationary solution generates a continuous family of solitary waves,

$$u_{\pm,\nu}(t,x)=\phi_{k_\pm,\nu}(x)e^{i\nu t},\quad \phi_{k_\pm,\nu}(x)=e^{-\frac{\nu}{2\lambda}}\phi_{k_\pm}(x),\quad \nu\in\mathbb{R}.$$

Every such solitary wave is unstable in the sense of Definition 1.1.

(ii). In the limiting case $-\lambda=\omega>0$, $\phi_{k_-}=\phi_{k_+}=\phi_\omega=e^{d/4}e^{-\omega|x|^2/2}$ also generates a continuous family of solitary waves,

$$u_{\omega,\nu}(t,x)=\phi_{\omega,\nu}(x)e^{i\nu t},\quad \phi_{\omega,\nu}(x)=e^{\frac{\nu}{2\omega}}\phi_\omega(x),\quad \nu\in\mathbb{R},$$

and every such solitary wave is unstable in the sense of Definition 1.1.

We note that ϕ_{k_-} and ϕ_{k_+} are two positive solutions to the stationary equation

$$-\frac{1}{2}\Delta\phi-\omega^2\frac{|x|^2}{2}\phi+\lambda\phi\ln\left(|\phi|^2\right)=0. \tag{1.4}$$

As evoked above, it is shown in [23] that (1.1) has at least one positive solution, under suitable assumptions on the coefficients of the equation. More precisely, in [23], a semiclassical parameter ε is present,

$$-\varepsilon^2\Delta u-|x|^2u=u\ln|u|^2.$$

A stationary, positive solution exists for sufficiently small values of the semiclassical parameter ε. Actually a rescaling argument shows that this corresponds to (1.4) in the case $\lambda=-2$, with $\omega=\varepsilon$: for ε small, we indeed have $-\lambda>\omega>0$. In [24], it is shown that for the logarithmic Schrödinger equation with a potential admitting a global minimum reached in $\ell\geq 2$ points sufficiently far one from another, there exist at least ℓ positive stationary solutions, providing a situation where nonuniqueness holds, which is quite different from ours.

Linearizing (1.1) around ϕ_k, for $k = k_-$ or k_+, leads to:

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2 \frac{|x|^2}{2} u - \frac{dk}{2} u - \lambda k |x|^2 u = k^2 \frac{|x|^2}{2} u - \frac{dk}{2} u. \tag{1.5}$$

The underlying Hamiltonian is the (shifted) harmonic oscillator,

$$H_k = -\frac{1}{2}\Delta + k^2 \frac{|x|^2}{2} - \frac{dk}{2}, \tag{1.6}$$

whose point spectrum is $k\mathbb{N}$. This implies *linear and spectral stability* of the stationary states $\phi_{k_\pm}$, like, for example, for the Gausson in the case of the logarithmic KdV equation [25–27]. From this perspective, the nonlinear instability stated in Theorem 1.2 can appear surprising. We actually show several possible mechanisms leading to instability.

Ground states are often characterized as the unique positive solution to an elliptic equation (typically when the nonlinearity is homogeneous, but not only, see e.g. [28, 29]): we discuss more into details the notion of ground state in Section 4, and show that neither ϕ_{k_-} nor ϕ_{k_+} can be considered as a ground state according to standard definitions. Note that the underlying operator $-\Delta - \omega^2|x|^2$ is not elliptic, since its symbol is $|\xi|^2 - \omega^2|x|^2$. In particular, we do not obtain a variational characterization of the Gaussons in the present case, unlike in the case without potential [12], or with a confining harmonic potential [15, 16]. This is consistent with the fact that these solutions are unstable. Note however that in view of the global existence result [16, Proposition 1.3], the instability mechanism is not related to finite time blow-up.

The rest of this article is organized as follows. In Section 2, we show some special invariances and discuss more into details special Gaussian solutions to (1.1). In Section 3, we complete the proof of Theorem 1.2, by showing the instability of ϕ_{k_-} and ϕ_{k_+}; several causes of instability are exhibited. Finally in Section 4, we discuss the notion of ground state associated to (1.1), and show that it should be considered that (1.1) admits no ground state.

2. Special solutions and invariances

2.1. Some invariances

(1.1) is invariant with respect to translation in time, but not with respect to translation in space, due to the potential. It is gauge invariant: if u is a solution, then so is $e^{i\theta}u$ for any constant $\theta \in \mathbb{R}$.

2.1.1. Size effect

The following invariance is a feature of the logarithmic nonlinearity: If u solves (1.1), then for all $c \in \mathbb{C}$, so does

$$u_c(t,x) := c\; u(t,x) e^{-it\lambda \ln |c|^2}. \tag{2.1}$$

Typically, if we find a stationary solution, then the above transform generates a continuum of solitary waves, indexed by $c \in (0,\infty)$, or equivalently by

$$\nu = -\lambda \ln\left(c^2\right) \in \mathbb{R}.$$

Note that the size of these solitary waves is arbitrary, as c ranges $(0,\infty)$.

2.1.2. Galilean invariance

Due to the repulsive harmonic potential, the Galilean invariance reads are follows. If $u(t,x)$ solves (1.1), then for any $v \in \mathbb{R}^d$, so does

$$u\left(t, x - v\frac{\sinh(\omega t)}{\omega}\right) \exp\left(i\cosh(\omega t) v\cdot x - \frac{i|v|^2}{4\omega}\sinh(2\omega t)\right). \tag{2.2}$$

At $t=0$, the above transform is just a multiplication by $e^{iv\cdot x}$.

2.1.3. Space translation

The absence of invariance with respect to translation in space can be specified as follows. If u solves (1.1), then for any $x_0 \in \mathbb{R}^d$, so does

$$u(t, x - x_0\cosh(\omega t)) \exp\left(i\omega \sinh(\omega t) x_0\cdot x - \frac{i\omega|x_0|^2}{4}\sinh(2\omega t)\right). \tag{2.3}$$

At $t=0$, the above transform corresponds to a shift in space.

2.1.4. Tensorization

The logarithmic nonlinearity was introduced in [1] to satisfy the following tensorization property: as the external potential decouples space variables,

$$-\omega^2\frac{|x|^2}{2} = -\frac{\omega^2}{2}\sum_{j=1}^d x_j^2,$$

if the initial datum is a tensor product,

$$u_0(x) = \prod_{j=1}^d u_{0j}(x_j),$$

then the solution to (1.1) is given by

$$u(t,x) = \prod_{j=1}^d u_j(t,x_j),$$

where each u_j solves a one-dimensional equation,

$$i\partial_t u_j + \frac{1}{2}\partial_{x_j}^2 u_j = -\omega^2\frac{x_j^2}{2}u_j + \lambda\ln\left(|u_j|^2\right)u_j, \quad u_{j|t=0} = u_{0j}.$$

2.2. Gaussons

As announced in the introduction, for $-\lambda > \omega > 0$, the stationary Gaussons are given by

$$\phi_k(x) = e^{-\frac{dk}{4\lambda}} e^{-k|x|^2/2},$$

where k is either of the solutions to

$$k^2 + 2\lambda k + \omega^2 = 0, \text{ i.e. } \mathrm{k}_\pm = -\lambda \pm \sqrt{\lambda^2 - \omega^2}. \tag{2.4}$$

If $-\lambda = \omega > 0$, then $k_- = k_+ = \omega$, and we will see in the next subsection that when $\omega > -\lambda > 0$, there exists no Gausson. We compute

$$||\phi_k||^2_{L^2(\mathbb{R}^d)} = e^{-\frac{dk}{2\lambda}} \left(\frac{\pi}{k}\right)^{d/2}.$$

We note that as $\omega \to 0$ with $\lambda < 0$ fixed, $k_- \to 0$, $k_+ \to -2\lambda$, hence

$$||\phi_{k_-}||^2_{L^2(\mathbb{R}^d)} \to \infty, \quad \text{whereas} \quad ||\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} \to e^d \left(\frac{\pi}{-2\lambda}\right)^{d/2}.$$

We have more generally

Lemma 2.1. *Let $-\lambda > \omega > 0$. We have*

$$||\phi_{k_-}||_{L^2(\mathbb{R}^d)} > ||\phi_{k_+}||_{L^2(\mathbb{R}^d)}.$$

Proof. It suffices to prove that

$$\frac{e^{-k_-/\lambda}}{k_-} > \frac{e^{-k_+/\lambda}}{k_+} \iff e^{(k_+ - k_-)/\lambda} > \frac{k_-}{k_+}$$
$$\iff e^{2\sqrt{\lambda^2 - \omega^2}/\lambda} > \frac{-\lambda - \sqrt{\lambda^2 - \omega^2}}{-\lambda + \sqrt{\lambda^2 - \omega^2}}.$$

We view the above inequality as depending on the unknown $\omega \in (0, -\lambda)$, and change the unknown as $\theta = \sqrt{\lambda^2 - \omega^2}/|\lambda| \in (0,1)$, so the above inequality becomes

$$e^{-2\theta} > \frac{1-\theta}{1+\theta} \iff 1 + \theta > (1-\theta)e^{2\theta}.$$

The map $f(\theta) = 1 + \theta - (1-\theta)e^{2\theta}$, defined for $\theta \in (0,1)$, satisfies

$$f''(\theta) = 4e^{2\theta} - 4(1-\theta)e^{2\theta} > 0, \text{ hence } f'(\theta) = 1 + e^{2\theta} - 2(1-\theta)e^{2\theta} > 0,$$

and $f(\theta) > 0$ for all $0 < \theta < 1$. □

In view of (2.1), with $\nu = -\lambda \ln(c^2)$, $c > 0$, we have a continuum of standing waves:

$$u_{\pm,\nu}(t,x) = \phi_{k_\pm,\nu}(x) e^{i\nu t}, \quad \phi_{k_\pm,\nu}(x) = e^{-\frac{\nu}{2\lambda}} \phi_{k_\pm}(x), \quad \nu \in \mathbb{R}.$$

Therefore, to understand the dynamical properties of $u_{\pm,\nu}$ (orbital stability or instability), it is enough to consider the stationary solutions $\phi_{k_\pm}$.

2.3. Gaussian solutions

By Gaussian solutions, we mean solutions which are Gaussian in the space variable, with time-dependent coefficients. We adapt the computations presented in [13] in the case $\omega = 0$. Suppose $d = 1$ (for $d \geq 2$, we may invoke the above tensorization property).

We seek $u(t,x) = b(t)e^{-a(t)x^2/2}$ (in particular u_0 is Gaussian). We find:

$$i\dot{b} = \frac{1}{2}ab + \lambda b \ln|b|^2; \quad i\dot{a} = a^2 + 2\lambda \mathrm{Re}\, a + \omega^2.$$

The function b is given explicitly in terms of a and its initial value b_0,

$$b(t) = b_0 \exp\left(-i\lambda t \ln\left(|b_0|^2\right) - \frac{i}{2}A(t) - i\lambda \mathrm{Im} \int_0^t A(s)ds\right),$$

where we have denoted $A(t) = \int_0^t a(s)ds$. We may write a under the form

$$a = \frac{1}{\tau^2} - i\frac{\dot{\tau}}{\tau}, \quad \tau \in \mathbb{R}, \tag{2.5}$$

and the equation for a leads to

$$\ddot{\tau} = \frac{2\lambda}{\tau} + \frac{1}{\tau^3} + \omega^2 \tau. \tag{2.6}$$

We note that the form (2.5) implies that $b(t)$ can be written as

$$b(t) = b_0 e^{i\theta(t)} \sqrt{\frac{\tau(0)}{\tau(t)}}, \quad \theta(t) \in \mathbb{R}. \tag{2.7}$$

Multiplying (2.6) by $\dot{\tau}$ and integrating, we get

$$(\dot{\tau})^2 + V(\tau) = C_0, \quad V(\tau) = -4\lambda \ln|\tau| + \frac{1}{\tau^2} - \omega^2\tau^2, \tag{2.8}$$

where $C_0 = \dot{\tau}(0)^2 - 4\lambda \ln|\tau(0)| + \frac{1}{\tau(0)^2} - \omega^2 \tau(0)^2$ is related to the initial data. Noticing that $V(q) \to +\infty$ when $q \to 0$, this readily shows that τ remains bounded away from zero, and thus may be supposed positive in view of (2.5):

$$\exists \delta > 0, \quad \tau(t) \geq \delta, \quad \forall t \geq 0.$$

Proposition 2.2. Let $d = 1$, $\lambda < 0 < \omega$.

1. *If $-\lambda > \omega > 0$, then (2.6) has exactly two stationary solutions, $\tau_\mp = 1/\sqrt{k_\pm}$. The other solutions are either periodic, or unbounded, corresponding to time-periodic and dispersive Gaussian solutions to (1.1), respectively.*
2. *If $-\lambda = \omega > 0$, then (2.6) has exactly one stationary solution, $\tau_0 = 1/\sqrt{\omega}$. All the other solutions are unbounded. In other words, any Gaussian solution to (1.1) which is not of the form*

$$e^{(2\nu+\omega)/(4\omega)} e^{i\nu t} e^{-\omega x^2/2}, \quad \nu \in \mathbb{R},$$

is dispersive.

3. *If $\omega > -\lambda > 0$, then every solution to (2.6) is unbounded. More precisely,*

$$e^{\omega t} \lesssim \tau(t) \lesssim e^{\omega t}, \quad t \geqslant 0,$$

and every Gaussian solution to (1.1) disperses exponentially fast.

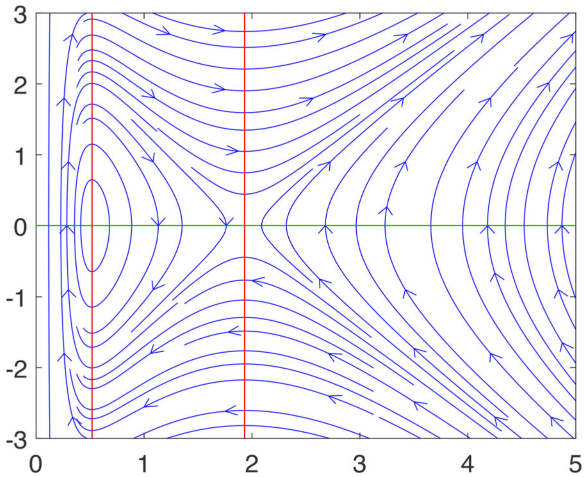

Figure 1. Phase portraits for the ODE (2.6) with $\omega = 1$ and $\lambda = -2$.

Proof. We remark that the righthand side of (2.6) can be rewritten as

$$\ddot{\tau} = P\left(\frac{1}{\tau^2}\right)\tau, \quad P(X) = X^2 + 2\lambda X + \omega^2.$$

When $-\lambda > \omega > 0$, P has exactly two roots, k_- and k_+, so

$$\ddot{\tau} = \left(\frac{1}{\tau^2} - k_-\right)\left(\frac{1}{\tau^2} - k_+\right)\tau.$$

According to the initial data for τ, the value of the constant C_0 in (2.8) varies, leading to bounded trajectories, in which case τ is periodic, or to unbounded trajectories, in which case $\tau(t) \to \infty$ as t goes to infinity. This is illustrated by Figure 1, displaying the phase portrait for the Equation (2.6) with $\omega = 1$ and $\lambda = -2$, where we find

$$\tau_- = \frac{1}{\sqrt{2+\sqrt{3}}} \approx 0.518, \quad \tau_+ = \frac{1}{\sqrt{2-\sqrt{3}}} \approx 1.932.$$

When $-\lambda = \omega > 0$, P has exactly one double root ω, and

$$\ddot{\tau} = \left(\frac{1}{\tau^2} - \omega\right)^2 \tau.$$

If τ is not constant (equal to $1/\sqrt{\omega}$), then τ is strictly convex. If $\tau(t_0) = 1/\sqrt{\omega}$ for some $t_0 \geqslant 0$, then $\dot{\tau}(t_0) \neq 0$, for otherwise τ would be constant, by uniqueness for (2.6): τ can't remain close to $1/\sqrt{\omega}$, and assuming that τ is bounded leads to a contradiction. As τ is positive and convex, $\tau(t)$ goes to infinity as $t \to \infty$. This is illustrated in Figure 2.

When $\omega > -\lambda > 0$, P is uniformly bounded from below on $\mathbb{R}$, $P(X) \geqslant \delta > 0$. If τ is bounded, (2.6) would yield $\ddot{\tau} \gtrsim 1$, since τ is bounded away from zero, hence a contradiction. As τ is convex, $\tau(t)$ goes to infinity as $t \to \infty$, see Figure 3.

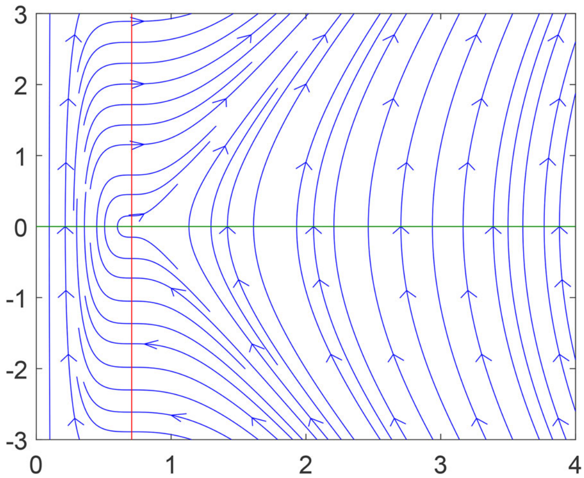

Figure 2. Phase portraits for the ODE (2.6) with $\omega = 2$ and $\lambda = -2$.

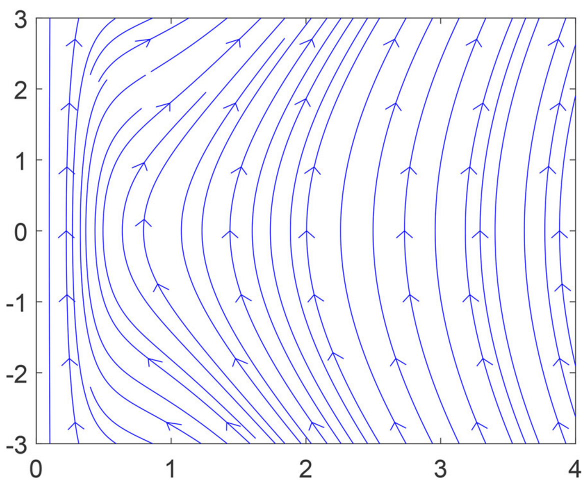

Figure 3. Phase portraits for the ODE (2.6) with $\omega = 2$ and $\lambda = -1$.

As a consequence, for any $\varepsilon > 0$, picking T sufficiently large,

$$\ddot{\tau}(t) \geqslant \omega^2 \tau(t) - \varepsilon, \quad \forall t \geqslant T.$$

The solution to

$$\ddot{\theta}(t) = \omega^2 \theta(t) - \varepsilon, \quad \theta(T) = \tau(T), \quad \dot{\theta}(T) = \dot{\tau}(T),$$

is given by

$$\theta(t) = \tau(T)\cosh(\omega(t-T)) + \dot{\tau}(T)\frac{\sinh(\omega(t-T))}{\omega} - \frac{2\varepsilon}{\omega^2}\sinh^2\left(\frac{\omega}{2}(t-T)\right).$$

As $\tau(T)$ and $\dot{\tau}(T)$ go to infinity as $T \to \infty$, we infer that $\tau(t) \gtrsim e^{\omega t}$. The converse estimate is a direct consequence of (2.8), again because for t sufficiently large, $\ln \tau(t) > 0$, and $\lambda < 0$. □

Remark 2.3. In the linear case $\lambda = 0 < \omega$, there is no solitary wave, as every solution is dispersive. This can be seen for instance *via* the vector field $J(t) = \omega x \sinh(\omega t) + i\cosh(\omega t)\nabla$: as observed in [20, Lemma 2.3], if u solves

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2 \frac{|x|^2}{2} u,$$

then so does Ju, and since J can be factorized as

$$J(t) = i\cosh(\omega t) e^{i\omega\frac{|x|^2}{2}\tanh(\omega t)} \nabla\left(e^{-i\omega\frac{|x|^2}{2}\tanh(\omega t)} \cdot\right), \tag{2.9}$$

Gagliardo–Nirenberg inequality yields, for $2 \leqslant p < \frac{2d}{(d-2)_+}$,

$$\begin{aligned} \|u(t)\|_{L^p(\mathbb{R}^d)} &\leqslant \frac{C(p,d)}{(\cosh(\omega t))^{\delta(p)}} \|u(t)\|_{L^2}^{1-\delta(p)} \|J(t)u\|_{L^2}^{\delta(p)} \\ &= \frac{C(p,d)}{(\cosh(\omega t))^{\delta(p)}} \|u_0\|_{L^2}^{1-\delta(p)} \|\nabla u_0\|_{L^2}^{\delta(p)}, \quad \delta(p) = d\left(\frac{1}{2} - \frac{1}{p}\right), \end{aligned}$$

since the L^2-norm is preserved by the flow. Therefore, if $u_0 \in \Sigma$, the L^p-norm of u decreases exponentially in time, and no solitary wave exists. The existence of solitary waves when $-\lambda \geqslant \omega > 0$ is thus due to the presence of the logarithmic nonlinearity, which is sufficiently strong (due to the singularity of the logarithm at the origin) to counterbalance the exponential linear dispersion.

3. Orbital instability

The instability result that we prove is slightly stronger than instability in the sense of Definition 1.1:

Lemma 3.1. *Let $\nu \in \mathbb{R}$.*

1. *Suppose $-\lambda > \omega > 0$. The solitary waves $\phi_{k_-,\nu}(x)e^{i\nu t}$ and $\phi_{k_+,\nu}(x)e^{i\nu t}$ are unstable. More precisely, for any $\eta > 0$, there exists $u_0 \in \Sigma$ such that*

$$\|u_0 - \phi_{k_+,\nu}\|_\Sigma < \eta,$$

and the solution to (1.1) such that $u_{|t=0} = u_0$ satisfies

$$\sup_{t \geqslant 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\phi_{k_+,\nu}\|_{L^2(\mathbb{R}^d)} \geqslant \frac{1}{2}\|\phi_{k_+,\nu}\|_{L^2(\mathbb{R}^d)}.$$

The same holds when k_+ is replaced by k_-.

2. *Suppose $-\lambda = \omega > 0$. The solitary wave $\phi_{\omega,\nu}(x)e^{i\nu t}$ is unstable in the same sense as above.*

Proof. We present the argument for ϕ_{k_+}, to shorten notations: considering $\phi_{k_\pm,\nu}$ for $\nu \in \mathbb{R}$ goes along the same lines, and the argument includes the limiting case $-\lambda = \omega > 0$. For all $\eta > 0$, then exists $\delta > 0$ such that for $|x_0| < \delta$,

$$||u_0 - \phi_{k_+}||_\Sigma < \eta, \quad u_0(x) = \phi_{k_+}(x - x_0).$$

In view of (2.3), the solution to (1.1) with initial datum u_0 is given by

$$u(t,x) = \phi_{k_+}(x - x_0\cosh(\omega t))e^{i\omega\sinh(\omega t)x_0\cdot x - \frac{i\omega|x_0|^2}{4}\sinh(2\omega t)}$$

Therefore, for any $t > 0$,

$$\inf_{\theta\in\mathbb{R}} ||u(t) - e^{i\theta}\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left|\phi_{k_+}(x - x_0\cosh(\omega t)) - \phi_{k_+}(x)\right|^2 dx.$$

Indeed, denote $u(t,x) = \phi_{k_+}(x - x_0\cosh(\omega t))e^{i\alpha(x_0,x,t)}$ with $\alpha(x_0,x,t) \in \mathbb{R}$ given by the above formula. Then

$$\begin{aligned}||u(t) - e^{i\theta}\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} &= ||\phi_{k_+}(x - x_0\cosh(\omega t)) - e^{i(\theta-\alpha(x_0,x,t))}\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} \\ &= 2||\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} - 2\int_{\mathbb{R}^d} \cos(\theta - \alpha(x_0,x,t))\phi_{k_+}(x - x_0\cosh(\omega t))\phi_{k_+}(x)dx,\end{aligned}$$

which implies

$$\begin{aligned}&\inf_{\theta\in\mathbb{R}} ||u(t) - e^{i\theta}\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} \\ &= 2||\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} - 2\sup_{\theta\in\mathbb{R}}\int_{\mathbb{R}^d} \cos(\theta - \alpha(x_0,x,t))\phi_{k_+}(x - x_0\cosh(\omega t))\phi_{k_+}(x)dx \\ &\geqslant 2||\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} - 2\int_{\mathbb{R}^d} \phi_{k_+}(x - x_0\cosh(\omega t))\phi_{k_+}(x)dx \\ &= ||\phi_{k_+}(x - x_0\cosh(\omega t)) - \phi_{k_+}(x)||^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

It becomes obvious that picking t sufficiently large (in terms of η) leads to

$$\inf_{\theta\in\mathbb{R}} ||u(t) - e^{i\theta}\phi_{k_+}||^2_{L^2(\mathbb{R}^d)} \geqslant \frac{1}{2}||\phi_{k_+}||^2_{L^2(\mathbb{R}^d)}.$$

This rules out orbital stability, even in the L^2-norm, for initial data close to ϕ_{k_+} in the Σ-topology. □

Remark 3.2. We can adapt the above proof by using the Galilean invariance (2.2), and consider instead

$$u_0(x) = \phi_{k_+}(x)e^{iv\cdot x}, \quad |v| \ll 1.$$

Remark 3.3. It is clear from the argument that u_0 is close to ϕ_{k_+} in Σ, but also in stronger norms, while orbital stability is ruled out by measuring only the L^2-norm.

Remark 3.4. (Linearization). The argument of the proof can be compared to the discussion around linearization after the statement of Theorem 1.2. Differentiating (2.3) with respect to x_0, the trace at $x_0 = 0$ yields the infinitesimal generator $iJ(t)$, where the vector-field J is defined in Remark 2.3. Using (2.9) to shorten computations, and the explicit expression of ϕ_{k_+}, we find that $v(t,x) := iJ(t)\phi_{k_+}(x)$ solves

$$i\partial_t v + \frac{1}{2}\Delta v = k_+^2 \frac{|x|^2}{2} v - \frac{dk_+}{2} v + 2\lambda k_+ \frac{\cosh(\omega t)}{k_+ \cosh(\omega t) + i\omega \sinh(\omega t)} v.$$

So we see that the approach of the proof of Lemma 3.1 is different from the direct linearization of (1.1) about ϕ_{k_+}, leading to (1.5).

The above arguments do not rule out orbital stability when the initial datum are restricted to be radially symmetric. In [11], this restriction was considered essentially to obtain compactness properties (the embedding of $H^1_{\mathrm{rad}}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for $2 \leqslant p < \frac{2d}{(d-2)_+}$ is compact). Note that Σ is compactly embedded into $L^p(\mathbb{R}^d)$ for $2 \leqslant p < \frac{2d}{(d-2)_+}$. The lemma below shows instability for ϕ_{k_-} even at the radial level.

Lemma 3.5 . *Let $\nu \in \mathbb{R}$.*

1. *Suppose $-\lambda > \omega > 0$. The solitary wave $\phi_{k_-,\nu}(x)e^{i\nu t}$ is unstable even if we restrict Definition 1.1 to radial solutions.*
2. *The same holds for $\phi_{\omega,\nu}(x)e^{i\nu t}$ in the case $-\lambda = \omega > 0$.*

Proof. Assume $-\lambda > \omega > 0$. We show that $u_{k_-,\nu}$ is unstable even as a Gaussian solution centered at the origin, by linearizing (2.6) about $\tau_- = 1/\sqrt{k_-}$: we compute the linearization as

$$\ddot{h} = \omega^2 h - 2\lambda k_- h - 3k_-^2 h = \Omega_{\mathrm{eff}} h,$$

where

$$\Omega_{\mathrm{eff}} = \omega^2 - 2\lambda k_- - 3k_-^2 = -4k_-^2 - 4\lambda k_- = -4k_-(k_- + \lambda).$$

Since $k_- + \lambda < 0$, the linearized operator is such that $\Omega_{\mathrm{eff}} > 0$, so h grows exponentially. Of course linearizing makes sense only for sufficiently small h, but this is enough to contradict the definition of orbital stability. Indeed, there exists $\delta > 0$ such that as long as $|h(t)| \leqslant \delta$, we can write the solution τ to (2.6) with $\tau(0) = \tau_- + h(0)$ and $\dot{\tau}(0) = 0$ as

$$\tau(t) = \tau_- + h(t) + r(t), \quad \text{with} \quad |r(t)| \leqslant \frac{|h(t)|}{2}.$$

For $0 < \varepsilon < \delta$, let h solve

$$\ddot{h} = \Omega_{\mathrm{eff}} h, \quad h(0) = \varepsilon, \quad \dot{h}(0) = 0.$$

As $h(t) = \varepsilon \cosh\left(t\sqrt{\Omega_{\mathrm{eff}}}\right)$ grows exponentially, there exists $t_0 > 0$ such that $h(t_0) = \delta$, and the triangle inequality yields

$$|\tau(t_0) - \tau_-| \geqslant \frac{\delta}{2}.$$

Now if u denotes the Gaussian solution associated with τ, we see that for all $\eta > 0$, picking $\varepsilon > 0$ sufficiently small ensures

$$\|u(0) - \phi_{k_-}\|_\Sigma < \eta,$$

while, in view of (2.7), setting $k(t) = 1/\tau(t)^2$,

$$\begin{aligned}\sup_{t\geqslant 0}\inf_{\theta\in\mathbb{R}}\|u(t)-e^{i\theta}\phi_{k_-}\|_{L^2(\mathbb{R}^d)} &\geqslant \inf_{\theta\in\mathbb{R}}\|u(t_0)-e^{i\theta}\phi_{k_-}\|_{L^2(\mathbb{R}^d)}\\ &\geqslant e^{-dk_-/(4\lambda)}\left\|\left(\frac{\tau_-}{\tau(t_0)}\right)^{d/2}e^{-k(t_0)|x|^2/2}-e^{-k_-|x|^2/2}\right\|_{L^2(\mathbb{R}^d)}\\ &\geqslant C(\delta)>0,\end{aligned}$$

where $C(\delta)$ is independent of ε, hence independent of η. Thus, we have the same instability results as in Lemma 3.1, at the level of radial Gaussian solutions.

In the case $-\lambda=\omega>0$, we find $\Omega_{\text{eff}}=0$, hence $h(t)=\dot{h}(0)t+h(0)$. We now pick $\dot{h}(0)=\varepsilon$, $h(0)=0$, so h is still unbounded as time grows. We thus consider the solution τ to (2.6) with $\tau(0)=\tau_-\left(=1/\sqrt{\omega}\right)$ and $\dot{\tau}(0)=\varepsilon$, and the above argument can be repeated. □

Remark 3.6. For $-\lambda>\omega>0$, the same argument is not conclusive in the case of k_+, since we then have

$$\Omega_{\text{eff}}=-4k_+(k_++\lambda)<0.$$

The trajectories of the linearized operator are bounded (periodic). This is consistent with the phase portrait corresponding to the Gaussian case, see Figure 1 (recalling that k_+ corresponds to the smaller value τ_-). It is an open question to decide whether $\phi_{k_+,\nu}(x)e^{i\nu t}$ is stable or not under radial perturbations.

4. On the notion of ground state

The most standard notions of ground state seem to be the following:

- Minimizer of the action $E+\nu M$.
- Minimizer of the energy E for a given mass M.
- Positive solution of $dE+\nu dM=0$.

In the case of an homogeneous nonlinearity, the three notions coincide, and the ground state is unique, up to the invariants of the equation; see, for example, [30, Chapter 8]. In the absence of potential ($\omega=0$), the Gausson is the only positive stationary solution to (1.1) [31]. In the present case, we have seen already that for $\lambda>\omega>0$, there are two distinct solutions to the stationary equation $dE=0$, namely ϕ_{k_-} and ϕ_{k_+} : the last notion cannot be relevant. On the other hand, because the potential is unbounded from below, the first two notions are not relevant either: given $u\in\Sigma$,

$$E(u_{x_0})\underset{|x_0|\to\infty}{\longrightarrow}-\infty,\quad u_{x_0}(x):=u(x-x_0).$$

In [32], the second notion is adapted, by requiring in addition that the ground state is a critical point of the energy on the set of function with a given mass M, which is meaningful even when the energy is unbounded from below on this set. The case of the logarithmic nonlinearity turns out to be rather specific: a solitary wave $e^{i\nu t}\phi(x)$ solves (1.1) if and only if ϕ solves

$$-\frac{1}{2}\Delta\phi + \nu\phi - \omega^2\frac{|x|^2}{2}\phi + \lambda\phi\ln|\phi|^2 = 0.$$

Multiplying this equation by $\bar{\phi}$ and integrating shows that ϕ must solve

$$||\nabla\phi||_{L^2}^2 - \omega^2||x\phi||_{L^2}^2 + 2\lambda\int_{\mathbb{R}^d}|\phi|^2\ln|\phi|^2dx + 2\nu||\phi||_{L^2}^2 = 0.$$

This Pohozaev identity defines the *Nehari manifold*. But we see that the above left hand side differs from twice the energy

$$E(u) = \frac{1}{2}||\nabla u||_{L^2}^2 - \frac{\omega^2}{2}||xu||_{L^2}^2 + \lambda\int_{\mathbb{R}^d}|u|^2\left(\ln|u|^2 - 1\right)dx$$

only by the term $2(\lambda+\nu)M$. Following [12, 15] (see also [33, 34]), we thus introduce the action and the Nehari functional,

$$S_\nu(u) := E(u) + \nu||u||_{L^2}^2,$$

$$I_\nu(u) := ||\nabla u||_{L^2}^2 - \omega^2||xu||_{L^2}^2 + 2\lambda\int_{\mathbb{R}^d}|u|^2\ln|u|^2dx + 2\nu||u||_{L^2}^2 = 2S_\nu(u) + 2\lambda||u||_{L^2}^2,$$

and consider the minimization problem

$$\begin{aligned}\delta(\nu) &:= \inf\{S_\nu(u)\ |\ u\in\Sigma\setminus\{0\},\quad I_\nu(u)=0\}\\ &= -\lambda\inf\{||u||_{L^2}^2\ |\ u\in\Sigma\setminus\{0\},\quad I_\nu(u)=0\}.\end{aligned}$$

The set of ground states is defined by

$$\mathcal{G}_\nu := \{\phi\in\Sigma\setminus\{0\}\ |\ I_\nu(u)=0,\quad S_\nu(\phi)=\delta(\nu)\}.$$

We check that

$$I_0(\phi_{k_\pm}) = 0 \quad (\text{hence } I_\nu(\phi_{k_\pm,\nu}) = 0).$$

In view of Lemma 2.1, ϕ_{k_-} does not belong to $\mathcal{G}_0$, and should thus not be considered as a ground state, even though it is a positive solution to (1.4).

It turns out that ϕ_{k_+} is not a ground state either:

Proposition 4.1. *Let $\lambda < 0 < \omega$. For any $\nu\in\mathbb{R}$, $\delta(\nu) = 0$, and $\mathcal{G}_\nu = \emptyset$.*

Proof. Consider the two-parameter family of Gaussians

$$\gamma_{\varepsilon,x_0}(x) = \varepsilon\ e^{-|x-x_0|^2/2}.$$

Naturally, the parameter $\varepsilon > 0$ is aimed at being arbitrarily small, and we use the center x_0 to adjust the size of the momentum so that γ_{ε,x_0} belongs to the Nehari manifold. The choice of a variance equal to one is arbitrary, for the following computation would lead to the same conclusion for any fixed variance. We compute:

$$||\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}^2 = \varepsilon^2\pi^{d/2}, \quad ||\nabla\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}^2 = \frac{d\varepsilon^2}{2}\pi^{d/2},$$

$$||x\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}^2 = \varepsilon^2\int_{\mathbb{R}^d}|y+x_0|^2e^{-|y|^2}dy = \varepsilon^2\frac{d}{2}\pi^{d/2} + \varepsilon^2|x_0|^2\pi^{d/2},$$

$$\int_{\mathbb{R}^d}\gamma_{\varepsilon,x_0}^2\ln\left(\gamma_{\varepsilon,x_0}^2\right) = \ln(\varepsilon^2)||\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}^2 - ||\nabla\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}^2 = \varepsilon^2\pi^{d/2}\left(\ln(\varepsilon^2) - \frac{d}{2}\right),$$

hence:

$$I_\nu(\gamma_{\varepsilon,x_0}) = \varepsilon^2\pi^{d/2}\left((1-2\lambda)\frac{d}{2} - \omega^2\frac{d}{2} - \omega^2|x_0|^2 + 2\lambda\ln(\varepsilon^2) + 2\nu\right).$$

For $\varepsilon > 0$ sufficiently small, $2\lambda\ln(\varepsilon^2) + (1-2\lambda)\frac{d}{2} - \omega^2\frac{d}{2} + 2\nu > 0$ (recall that $\lambda < 0$), and we can find $x_0 \in \mathbb{R}^d$ (with $|x_0|$ of order $\sqrt{-\ln\varepsilon}/\omega$) such that $I_\nu(\gamma_{\varepsilon,x_0}) = 0$. But of course $||\gamma_{\varepsilon,x_0}||_{L^2(\mathbb{R}^d)}$ is arbitrarily small, hence $\delta(\nu) = 0$. The second line in the definition of $\delta(\nu)$ obviously implies that $\mathcal{G}_\nu = \emptyset$. □

The next natural question to complete the picture is then to ask whether ϕ_{k_+} is a local minimum or a saddle point of $S_0(u)$ at $I_0(u) = 0$. Recalling (2.4), we compute, for $w = u + iv$ (u and v real-valued), $\langle S_0''(\phi_{k_+})w, w\rangle = \langle L_1u, u\rangle + \langle L_2v, v\rangle$, where

$$L_1u = -\Delta u + k_+^2|x|^2u - dk_+u + 4\lambda u = 2H_{k_+}u + 4\lambda u, \quad L_2v = 2H_{k_+}v,$$

where the shifted harmonic operator H_k is defined in (1.6). Its spectrum is $k\mathbb{N}$, and its eigenfunctions are Hermite functions. Since $\lambda < 0$, we infer that ϕ_{k_+} is a saddle point.

Acknowledgments

The authors thank the referees for their careful reading of the article and their suggestions.

Funding

Rémi Carles is supported by Rennes Métropole through its AIS program.

ORCID

Rémi Carles http://orcid.org/0000-0002-8866-587X
Chunmei Su http://orcid.org/0000-0002-7934-592X

References

[1] Bialynicki-Birula, I., Mycielski, J. (1976). Nonlinear wave mechanics. *Ann. Physics*. 100(1-2):62–93. DOI: 10.1016/0003-4916(76)90057-9.

[2] Avdeenkov, A. V., Zloshchastiev, K. G. (2011). Quantum Bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent. *J. Phys. B: At. Mol. Opt. Phys*. 44(19):195303. DOI: 10.1088/0953-4075/44/19/195303.

[3] Buljan, H., Šiber, A., Soljačić, M., Schwartz, T., Segev, M., Christodoulides, D. (2003). Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media. *Phys. Rev. E*. 68(3):036607. DOI: 10.1103/PhysRevE.68.036607.

[4] Hansson, T., Anderson, D., Lisak, M. (2009). Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis. *Phys. Rev. A*. 80(3):033819. DOI: 10.1103/PhysRevA.80.033819.

[5] Hefter, E. F. (1985). Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics. *Phys. Rev. A*. 32(2):1201–1204. DOI: 10.1103/physreva.32.1201.

[6] Krolikowski, W., Edmundson, D., Bang, O. (2000). Unified model for partially coherent solitons in logarithmically nonlinear media. *Phys. Rev. E*. 61(3):3122–3126. DOI: 10.1103/PhysRevE.61.3122.
[7] Lauro, G. (2008). A note on a Korteweg fluid and the hydrodynamic form of the logarithmic Schrödinger equation. *Geophys. Astrophys. Fluid Dynn.* 102(4):373–380. DOI: 10.1080/03091920801956957.
[8] Martino, S. D., Falanga, M., Godano, C., Lauro, G. (2003). Logarithmic Schrödinger-like equation as a model for magma transport. *Europhys. Lett*. 63(3):472–475. DOI: 10.1209/epl/i2003-00547-6.
[9] Zloshchastiev, K. G. (2010). Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences. *Gravit. Cosmol.* 16(4):288–297. DOI: 10.1134/S0202289310040067.
[10] Białynicki-Birula, I., Mycielski, J. (1979). Gaussons: Solitons of the logarithmic Schrödinger equation. *Special Issue Solitons Phys Phys. Scripta*. 20:539–544.
[11] Cazenave, T. (1983). Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Anal.* 7(10):1127–1140. DOI: 10.1016/0362-546X(83)90022-6.
[12] Ardila, A. H. (2016). Orbital stability of Gausson solutions to logarithmic Schrödinger equations. *Electron. J. Diff. Eq*. 9.
[13] Carles, R., Gallagher, I. (2018). Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.* 167(9):1761–1801. DOI: 10.1215/00127094-2018-0006.
[14] Bouharia, B. (2015). Stability of logarithmic Bose-Einstein condensate in harmonic trap. *Mod. Phys. Lett. B*. 29(01):1450260. DOI: 10.1142/S0217984914502601.
[15] Ardila, A. H., Cely, L., Squassina, M. (2019). Logarithmic Bose-Einstein condensates with harmonic potential. *Asymptot. Anal.* 116(1):27–40. DOI: 10.3233/ASY-191538.
[16] Carles, R., Ferriere, G. (2021). Logarithmic Schrödinger equation with quadratic potential. *Nonlinearity*. 34(12):8283–8310. DOI: 10.1088/1361-6544/ac3144.
[17] Cazenave, T., Lions, P.-L. (1982). Orbital stability of standing waves for some nonlinear Schrödinger equations. *Commun. Math. Phys.* 85(4):549–561. DOI: 10.1007/BF01403504.
[18] Dunford, N., Schwartz, J. T. (1963). *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*. With the assistance of William G. Bade and Robert G. Bartle. Interscience Publishers John Wiley & Sons New York-London.
[19] Reed, M., Simon, B. (1975). *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York.
[20] Carles, R. (2003). Nonlinear Schrödinger equations with repulsive harmonic potential and applications. *SIAM J. Math. Anal.* 35(4):823–843. DOI: 10.1137/S0036141002416936.
[21] Johnson, R., Pan, X. B. (1993). On an elliptic equation related to the blow-up phenomenon inthe nonlinear Schrödinger equation. *Proc. Roy. Soc. Edinburgh Sect. A*. 123(4):763–782. DOI: 10.1017/S030821050003095X.
[22] Kavian, O., Weissler, F. (1994). Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation. *Michigan Math. J.* 41(1):151–173. DOI: 10.1307/mmj/1029004922.
[23] Zhang, C., Zhang, X. (2020). Bound states for logarithmic Schrödinger equations with potentials unbounded below. *Calc. Var. Partial Differ. Equ.* 59(1):31. DOI: 10.1007/s00526-019-1677-y.
[24] Alves, C. O., Ji, C. (2020). Multiple positive solutions for a Schrödinger logarithmic equation. *Discrete Contin. Dyn. Syst*. 40(5):2671–2685. DOI: 10.3934/dcds.2020145.
[25] Carles, R., Pelinovsky, D. (2014). On the orbital stability of Gaussian solitary waves in the log-KdV equation. *Nonlinearity*. 27(12):3185–3202. DOI: 10.1088/0951-7715/27/12/3185.
[26] James, G., Pelinovsky, D. (2014). Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials. *Proc. Math. Phys. Eng. Sci.* 470(2165):20130420–20130462. DOI: 10.1098/rspa.2013.0462.
[27] Pelinovsky, D. E. (2017). On the linearized log-KdV equation. *Commun. Math. Sci*. 15(3):863–880. DOI: 10.4310/CMS.2017.v15.n3.a13.

[28] Jang, J. (2010). Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$, $N \geq 2$. *Nonlinear Anal.* 73(7):2189–2198.

[29] Lewin, M., Rota Nodari, S. (2020). The double-power nonlinear Schrödinger equations and its generalizations: uniqueness, non-degeneracy and applications. *Calc. Var. Partial Differ. Equ.* 59(6). DOI: 10.1007/s00526-020-01863-w.

[30] Cazenave, T. (2003). *Semilinear Schrödinger Equations, volume 10 of* Courant Lecture Notes in Mathematics. New York University Courant Institute of Mathematical Sciences, New York.

[31] Troy, W. C. (2016). Uniqueness of positive ground state solutions of the logarithmic Schrödinger equation. *Arch. Rational Mech. Anal.* 222(3):1581–1600. DOI: 10.1007/s00205-016-1028-5.

[32] Bellazzini, J., Jeanjean, L. (2016). On dipolar quantum gases in the unstable regime. *SIAM J. Math. Anal.* 48(3):2028–2058. DOI: 10.1137/15M1015959.

[33] Shuai, W. (2019). Multiple solutions for logarithmic Schrödinger equations. *Nonlinearity*. 32(6):2201–2225. DOI: 10.1088/1361-6544/ab08f4.

[34] Squassina, M., Szulkin, A. (2015). Multiple solutions to logarithmic Schrödinger equations with periodic potential. *Calc. Var.* 54(1):585–597. DOI: 10.1007/s00526-014-0796-8.