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# UNIFORM ERROR BOUNDS OF A FINITE DIFFERENCE METHOD FOR THE ZAKHAROV SYSTEM IN THE SUBSONIC LIMIT REGIME VIA AN ASYMPTOTIC CONSISTENT FORMULATION*

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**Abstract.** We present a uniformly accurate finite difference method and establish rigorously its uniform error bounds for the Zakharov system (ZS) with a dimensionless parameter $0 < \varepsilon \leq 1$, which is inversely proportional to the speed of sound. In the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$, the solution propagates highly oscillatory waves and/or rapid outgoing initial layers due to the perturbation of the wave operator in the ZS and/or the incompatibility of the initial data, which is characterized by two parameters $\alpha \geq 0$ and $\beta \geq -1$. Specifically, the solution propagates waves with wavelength of $O(\varepsilon)$ and $O(1)$ in time and space, respectively, and amplitude at $O(\varepsilon^{\min\{2,\alpha,1+\beta\}})$. This high oscillation of the solution in time brings significant difficulties in designing numerical methods and establishing their error bounds, especially in the subsonic limit regime. A uniformly accurate finite difference method is proposed by reformulating the ZS into an asymptotic consistent formulation and adopting an integral approximation of the oscillatory term. By applying the energy method and using the limiting equation via a nonlinear Schrödinger equation with an oscillatory potential, we rigorously establish two independent error bounds at $O(h^2 + \tau^2/\varepsilon)$ and $O(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})$, respectively, with $h$ the mesh size, $\tau$ the time step and $\alpha^* = \min\{1, \alpha, 1+\beta\}$. Thus we obtain error bounds at $O(h^2 + \tau^{4/3})$ and $O(h^2 + \tau^{1+\alpha^*/(2+\alpha^*)})$ for well-prepared ($\alpha^* = 1$) and ill-prepared ($0 \leq \alpha^* < 1$) initial data, respectively, which are uniform in both space and time for $0 < \varepsilon \leq 1$ and optimal at the second order in space. Other techniques in the analysis include the cut-off technique for treating the nonlinearity and inverse estimates to bound the numerical solution. Numerical results are reported to demonstrate our error bounds.

**Key words.** Zakharov system, nonlinear Schrödinger equation, subsonic limit, highly oscillatory, finite difference method, error bound, uniformly accurate

**AMS subject classifications.** 35Q55, 65M06, 65M12, 65M15

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## 1. Introduction.

Consider the dimensionless Zakharov system (ZS) for describing the propagation of Langmuir waves in plasma [32, 35]:

$$\begin{aligned} & i\partial_t E^\varepsilon(\mathbf{x},t) + \Delta E^\varepsilon(\mathbf{x},t) - N^\varepsilon(\mathbf{x},t)E^\varepsilon(\mathbf{x},t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ & \varepsilon^2 \partial_{tt} N^\varepsilon(\mathbf{x},t) - \Delta N^\varepsilon(\mathbf{x},t) - \Delta|E^\varepsilon(\mathbf{x},t)|^2 = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ & E^\varepsilon(\mathbf{x},0) = E_0(\mathbf{x}), \quad N^\varepsilon(\mathbf{x},0) = N_0^\varepsilon(\mathbf{x}), \quad \partial_t N^\varepsilon(\mathbf{x},0) = N_1^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{1.1}$$

Here $t$ is time, $\mathbf{x}$ is the spatial coordinates, the complex function $E^\varepsilon := E^\varepsilon(\mathbf{x},t)$ is the slowly varying envelope of the highly oscillatory electric field, the real function $N^\varepsilon := N^\varepsilon(\mathbf{x},t)$ represents the deviation of the ion density from its equilibrium value,

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$0 < \varepsilon \le 1$ is a dimensionless parameter which is inversely proportional to the acoustic speed, and $E_0(\mathbf{x})$, $N_0^\varepsilon(\mathbf{x})$, and $N_1^\varepsilon(\mathbf{x})$ are given functions satisfying $\int_{\mathbb{R}^d} N_1^\varepsilon(\mathbf{x})d\mathbf{x} = 0$.

There exist extensive analytical and numerical studies in the literature for the standard ZS, i.e., $\varepsilon = 1$ in (1.1). Along the analytical part, for the derivation of the ZS from the Euler–Maxwell equations, we refer to [22, 35], and for the well-posedness, we refer to [14, 18, 22, 35] and references therein. Based on these results, we know that the ZS (1.1) conserves the *wave energy*

$$\mathcal{M}(t) = \|E^\varepsilon(\cdot,t)\|^2 := \int_{\mathbb{R}^d} |E^\varepsilon(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_{\mathbb{R}^d} |E_0(\mathbf{x})|^2 d\mathbf{x} = \mathcal{M}(0), \quad t \ge 0, \tag{1.2}$$

and the *Hamiltonian*

$$\mathcal{L}^\varepsilon(t) := \int_{\mathbb{R}^d} \left[ |\nabla E^\varepsilon|^2 + N^\varepsilon |E^\varepsilon|^2 + \frac{1}{2}\left(|\nabla U^\varepsilon|^2 + |N^\varepsilon|^2\right) \right] d\mathbf{x} \equiv \mathcal{L}^\varepsilon(0), \quad t \ge 0, \tag{1.3}$$

where $U^\varepsilon := U^\varepsilon(\mathbf{x},t)$ is defined as

$$-\Delta U^\varepsilon(\mathbf{x},t) = \varepsilon \partial_t N^\varepsilon(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^d, \qquad \lim_{|\mathbf{x}|\to\infty} U^\varepsilon(\mathbf{x},t) = 0, \qquad t \ge 0. \tag{1.4}$$

Along the numerical part, different numerical methods have been proposed and analyzed in the last two decades. Glassey [24] presented an energy-preserving implicit finite difference scheme and established an error bound at first order in both spatial and temporal discretizations. Later, Chang and Jiang [17] improved it to the optimal second order convergence by considering implicit or semiexplicit conservative finite difference schemes [16]. Other approaches include the exponential-wave-integrator spectral method [10, 33], the Jacobi-type method [13], the Legendre–Galerkin method [27], the discontinuous Galerkin method [38], and the time-splitting spectral method [9, 29]. The analytical and numerical results for the ZS have been extended to the generalized Zakharov system [25, 26], the vector Zakharov system [36], and the vector Zakharov system for multicomponents [25].

When $\varepsilon \to 0^+$, i.e., in the subsonic limit regime, formally we get $E^\varepsilon(\mathbf{x},t) \to E(\mathbf{x},t)$, $\rho^\varepsilon := \rho^\varepsilon(\mathbf{x},t) = |E^\varepsilon|^2 \to |E|^2 = \rho$, and $N^\varepsilon(\mathbf{x},t) \to N(\mathbf{x},t) = -|E(\mathbf{x},t)|^2$, where $E := E(\mathbf{x},t)$ satisfies the cubic nonlinear Schrödinger equation (NLSE) [31, 32]:

$$\begin{aligned} &i\partial_t E(\mathbf{x},t) + \Delta E(\mathbf{x},t) + |E(\mathbf{x},t)|^2 E(\mathbf{x},t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ &E(\mathbf{x},0) = E_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{1.5}$$

The NLSE (1.5) conserves the wave energy (1.2) with $E^\varepsilon$ replaced by $E$ and the Hamiltonian

$$\mathcal{L}(t) := \int_{\mathbb{R}^d} \left[ |\nabla E(\mathbf{x},t)|^2 - \frac{1}{2}|E(\mathbf{x},t)|^4 \right] d\mathbf{x} \equiv \mathcal{L}(0), \quad t \ge 0. \tag{1.6}$$

Convergence rates of the subsonic limit from the ZS (1.1) to the NLSE (1.5) and initial layers, as well as the propagation of oscillatory waves, have been rigorously studied in the literature [31, 32, 34]. Based on the results, when $0 < \varepsilon \ll 1$, the solution of the ZS (1.1) propagates highly oscillatory waves at wavelength of $O(\varepsilon)$ and $O(1)$ in time and space, respectively, and/or rapid outgoing initial layers at speed $O(1/\varepsilon)$ in space. In addition, the initial data $(E_0, N_0^\varepsilon, N_1^\varepsilon)$ in (1.1) can be decomposed as

$$\begin{aligned}&N_0^\varepsilon(\mathbf{x}) = N(\mathbf{x},0) + \varepsilon^\alpha \omega_0(\mathbf{x}), \quad N_1^\varepsilon(\mathbf{x}) = \partial_t N(\mathbf{x},0) + \varepsilon^\beta \omega_1(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d,\\ &N(\mathbf{x},0) = -|E_0(\mathbf{x})|^2, \quad \partial_t N(\mathbf{x},0) = 2\mathrm{Im}\left(\Delta E_0(\mathbf{x})\overline{E_0(\mathbf{x})}\right) := \phi_1(\mathbf{x}),\end{aligned} \tag{1.7}$$

where $\alpha \geq 0$ and $\beta \geq -1$ are parameters describing the incompatibility of the initial data of the ZS (1.1) with respect to that of the NLSE (1.5) in the subsonic limit regime such that the Hamiltonian (1.3) is bounded, $\omega_0(\mathbf{x})$ and $\omega_1(\mathbf{x})$ are two given real functions independent of $\varepsilon$ satisfying $\int_{\mathbb{R}^d} \omega_1(\mathbf{x})d\mathbf{x} = 0$, and $\mathrm{Im}(c)$ and $\overline{c}$ denote the imaginary and conjugate parts of $c$, respectively. In fact, when $\alpha \geq 2$ and $\beta \geq 1$, the leading order oscillation is due to the term $\varepsilon^2\partial_{tt}N$ in the ZS, and when either $0 \leq \alpha < 2$ or $-1 \leq \beta < 1$, the leading order oscillation is due to the initial data.

To illustrate the oscillatory and/or rapid outgoing wave phenomena, Figure 1 shows the solutions $N^\varepsilon(x,1)$, $N^\varepsilon(1,t)$, $\mathrm{Re}(E^\varepsilon(x,1))$, and $\mathrm{Re}(E^\varepsilon(1,t))$ of the ZS (1.1) for $d = 1$, $\alpha = 0$, $\beta = 0$, and

$$\begin{aligned}&E_0(x) = e^{-x^2/2}, \quad \omega_1(x) = e^{-x^2/3}\sin(2x), \quad f(x) = e^{-1/x}\chi_{(0,\infty)},\\ &\omega_0(x) = g\left(\frac{x+25}{10}\right) g\left(\frac{25-x}{10}\right)\sin(2x), \quad g(x) = \frac{f(x)}{f(x)+f(1-x)},\end{aligned} \tag{1.8}$$

with $\chi$ the characteristic function in (1.7) for different $\varepsilon$, which was obtained numerically on a bounded computational interval $[-200, 200]$ with the homogeneous Dirichlet boundary condition [9]. For comparison, here we also plot $F^\varepsilon(x,1)$ and $F^\varepsilon(1,t)$ defined in (2.1).

The highly oscillatory nature of the solution of the ZS (1.1) in time brings significant numerical burdens, especially in the subsonic limit regime. Some numerical results for the ZS with different $0 < \varepsilon \leq 1$ have been reported in the literature [9, 29]. To the best of our knowledge, there are few results concerning error estimates of different numerical methods for the ZS with respect to the mesh size $h$, the time step $\tau$, and the parameter $0 < \varepsilon \leq 1$, except for an error bound of the finite difference Legendre pseudospectral method derived for the ZS in one dimension (1D) when $\alpha \geq 2$ and $\beta \geq 1$ [27]. Very recently, for the conservative finite difference method, Cai and Yuan [15] established uniform error bounds at $O(h^2 + \tau^{4/3})$ for $0 < \varepsilon \leq 1$ when $\alpha \geq 2$ and $\beta \geq 1$, and at $O(h^2 + \tau^{\frac{2}{3}\min\{\alpha,1+\beta\}})$ when $1 \leq \alpha < 2$ and/or $0 \leq \beta < 1$. However, when $\alpha = 0$ or $\beta = -1$, their error bound $O(h^2/\varepsilon + \tau^2/\varepsilon^3)$ requests the meshing strategy (or $\varepsilon$-scalability) $h = O(\varepsilon^{1/2})$ and $\tau = O(\varepsilon^{3/2})$, which is not uniform in either space or time when $0 < \varepsilon \ll 1$. The reason for this is due to the fact that $N^\varepsilon(\mathbf{x},t)$ does not converge to $N(\mathbf{x},t) = -|E(\mathbf{x},t)|^2$ when $\alpha = 0$ or $\beta = -1$ and $\varepsilon \to 0^+$ [31, 34, 36] (cf. Figure 1, top row).

The aim of this work is to design a finite difference method for the ZS, which is uniformly accurate in space and time for $0 < \varepsilon \leq 1$, and to carry out a rigorous error analysis for the finite difference method by paying particular attention to how the error bounds depend explicitly on $h$ and $\tau$ as well as the parameter $\varepsilon$. The key ingredients in designing the uniformly accurate finite difference method are based on (i) reformulating the ZS into an asymptotic consistent formulation and (ii) adapting an integral approximation of the oscillatory term. In establishing error bounds, we adapt the energy method, the cut-off technique for treating the nonlinearity, the inverse estimates to bound the numerical solution, and the limiting equation via an NLSE with an oscillatory potential. The error bounds of our new numerical method significantly improve the results of the standard finite difference method for the ZS in the subsonic limit regime [15], especially for the ill-prepared initial data, i.e., $0 \leq \alpha < 1$ or $-1 \leq \beta < 0$.

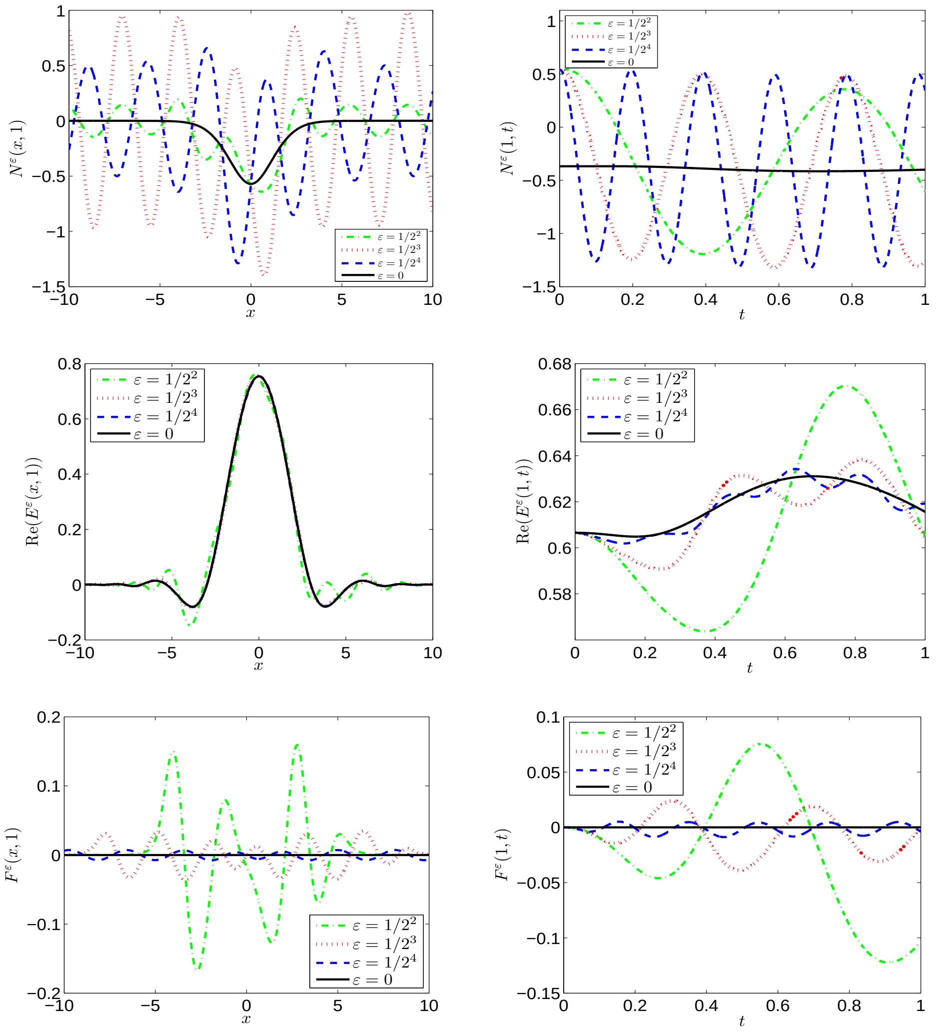

FIG. 1. *The solutions of the ZS* (1.1) *for different* $\varepsilon > 0$ *and the NLSE* ($\varepsilon = 0$) *as well as* $F^\varepsilon$ *defined in* (2.1) *with* $d = 1$. *Here* $\mathrm{Re}(c)$ *denotes the real part of* $c$.

The rest of the paper is organized as follows. In section 2, we introduce an asymptotic consistent formulation of the ZS, present a finite difference method, and state our main results. Section 3 is devoted to the details of the error analysis. Numerical results are reported in section 4 to confirm our error bounds. Finally, some conclusions are drawn in section 5. Throughout the paper, we adopt the standard Sobolev spaces and the corresponding norms and adopt $A \lesssim B$ to mean that there exists a generic constant $C > 0$ independent of $\varepsilon$, $\tau$, and $h$ such that $|A| \le C\,B$.

## 2. A finite difference method and its error bounds.

In this section, we will introduce an asymptotic consistent formulation of the ZS, present a uniformly accurate finite difference method, and state its error bounds.

### 2.1. An asymptotic consistent formulation.

We introduce

$$F^\varepsilon(\mathbf{x},t) = N^\varepsilon(\mathbf{x},t) + |E^\varepsilon(\mathbf{x},t)|^2 - G^\varepsilon(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0, \tag{2.1}$$

where $G^\varepsilon(\mathbf{x}, s)$ is the solution of the linear wave equation

$$\begin{aligned} &\partial_{ss}G^\varepsilon(\mathbf{x}, s) - \frac{1}{\varepsilon^2}\Delta G^\varepsilon(\mathbf{x}, s) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad s > 0,\\ &G^\varepsilon(\mathbf{x}, 0) = \varepsilon^\alpha\omega_0(\mathbf{x}), \quad \partial_s G^\varepsilon(\mathbf{x}, 0) = \varepsilon^\beta\omega_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{2.2}$$

Plugging (2.1) into the ZS (1.1), we can reformulate it into an asymptotic consistent formulation:

$$\begin{aligned} &i\partial_t E^\varepsilon(\mathbf{x}, t) + \Delta E^\varepsilon(\mathbf{x}, t) + \left[|E^\varepsilon(\mathbf{x}, t)|^2 - F^\varepsilon(\mathbf{x}, t) - G^\varepsilon(\mathbf{x}, t)\right] E^\varepsilon(\mathbf{x}, t) = 0,\\ &\varepsilon^2\partial_{tt}F^\varepsilon(\mathbf{x}, t) - \Delta F^\varepsilon(\mathbf{x}, t) - \varepsilon^2\partial_{tt}|E^\varepsilon(\mathbf{x}, t)|^2 = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,\\ &E^\varepsilon(\mathbf{x}, 0) = E_0(\mathbf{x}), \quad F^\varepsilon(\mathbf{x}, 0) \equiv 0, \quad \partial_t F^\varepsilon(\mathbf{x}, 0) \equiv 0, \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{2.3}$$

Now the initial conditions in (2.3) are always well-prepared for any $\alpha \geq 0$ and $\beta \geq -1$. In addition, the above system conserves the wave energy (1.2) and the “modified” Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{L}}^\varepsilon(t) := \int_{\mathbb{R}^d}\Bigg[&|\nabla E^\varepsilon|^2 - \frac{1}{2}|E^\varepsilon|^4 + \frac{1}{2}|F^\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_0^t\int_0^s \nabla F^\varepsilon(\mathbf{x}, s)\cdot\nabla F^\varepsilon(\mathbf{x}, s')ds'ds\\ &+ \int_0^t \left[G^\varepsilon(\mathbf{x}, s)\partial_s|E^\varepsilon(\mathbf{x}, s)|^2 - \phi_1(\mathbf{x})F^\varepsilon(\mathbf{x}, s)\right] ds\Bigg]d\mathbf{x} \equiv \widetilde{\mathcal{L}}^\varepsilon(0), \quad t \geq 0. \end{aligned} \tag{2.4}$$

When $\varepsilon \to 0^+$, i.e., in the subsonic limit regime, formally we get $E^\varepsilon(\mathbf{x}, t) \to E(\mathbf{x}, t)$ and $F^\varepsilon(\mathbf{x}, t) \to 0$, where $E := E(\mathbf{x}, t)$ satisfies the NLSE (1.5). In addition, when $\varepsilon \to 0^+$, formally we can also get $E^\varepsilon(\mathbf{x}, t) \to \widetilde{E}^\varepsilon(\mathbf{x}, t)$, where $\widetilde{E}^\varepsilon := \widetilde{E}^\varepsilon(\mathbf{x}, t)$ satisfies the following nonlinear Schrödinger equation with an oscillatory potential $G^\varepsilon(\mathbf{x}, t)$ (NLSE-OP):

$$\begin{aligned} &i\partial_t\widetilde{E}^\varepsilon(\mathbf{x}, t) + \Delta\widetilde{E}^\varepsilon(\mathbf{x}, t) + \left[\left|\widetilde{E}^\varepsilon(\mathbf{x}, t)\right|^2 - G^\varepsilon(\mathbf{x}, t)\right]\widetilde{E}^\varepsilon(\mathbf{x}, t) = 0, \qquad t > 0,\\ &\widetilde{E}^\varepsilon(\mathbf{x}, 0) = E_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{2.5}$$

It conserves the wave energy (1.2) with $E^\varepsilon = \widetilde{E}^\varepsilon$ and the “modified” Hamiltonian

$$\widetilde{\mathcal{L}}(t) := \int_{\mathbb{R}^d}\left[\left|\nabla\widetilde{E}^\varepsilon\right|^2 - \frac{1}{2}\left|\widetilde{E}^\varepsilon\right|^4 + \int_0^t G^\varepsilon(\mathbf{x}, s)\partial_s\left|\widetilde{E}^\varepsilon(\mathbf{x}, s)\right|^2 ds\right]d\mathbf{x} \equiv \widetilde{\mathcal{L}}(0),\ t \geq 0. \tag{2.6}$$

Following the proof of the subsonic limit of the ZS (1.1) to the NLSE (1.5), one can easily obtain the following quadratic convergence rate from the ZS (2.3) to the NLSE-OP (2.5):

$$\|F^\varepsilon\|_{L^\infty} + \left\|E^\varepsilon(\cdot, t) - \widetilde{E}^\varepsilon(\cdot, t)\right\|_{H^1} \leq C_T\varepsilon^2, \qquad 0 \leq t \leq T, \tag{2.7}$$

where $0 < T < T^*$, with $T^* > 0$ being the maximum common existence time for the solutions of the ZS (2.3) and the NLSE-OP (2.5), and where $C_T$ is a constant which depends on $T$, but is independent of $0 < \varepsilon \leq 1$. To illustrate the above convergence rate, Figure 2 depicts $\|F^\varepsilon(\cdot, t)\|_{L^\infty}$ and $\eta^\varepsilon(t) := \|E^\varepsilon(\cdot, t) - \widetilde{E}^\varepsilon(\cdot, t)\|_{H^1}$ for different $\varepsilon$ with $d = 1$, $\alpha = 0$, and $\beta = 0$ and the initial data chosen as (1.8).

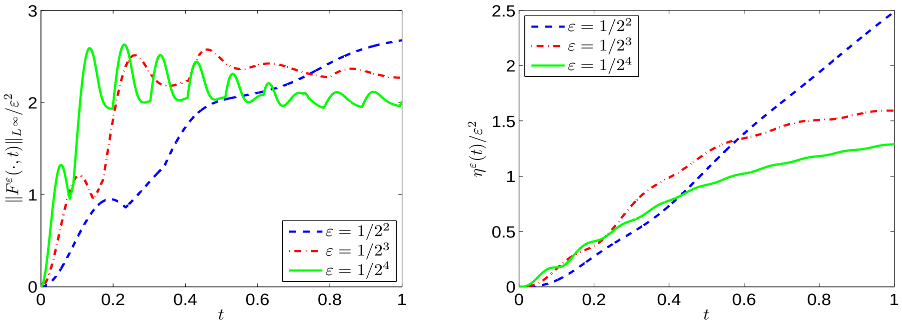

FIG. 2. *Time evolution of* $\|F^\varepsilon\|_{L^\infty}$ *and* $\eta^\varepsilon(t) = \|E^\varepsilon(\cdot,t) - \widetilde{E}^\varepsilon(\cdot,t)\|_{H^1}$ *for different* $\varepsilon$.

### 2.2. A uniformly accurate finite difference method.

For simplicity of notation, we will only present the numerical method for the ZS (2.3) in 1D; extensions to higher dimensions are straightforward. When $d = 1$, we truncate the ZS on a bounded computational interval $\Omega = (a, b)$ with the homogeneous Dirichlet boundary condition (here $|a|$ and $b$ are chosen large enough that the truncation error is negligible):

$$\begin{aligned} &i\partial_t E^\varepsilon(x,t) + \partial_{xx}E^\varepsilon(x,t) + [|E^\varepsilon(x,t)|^2 - F^\varepsilon(x,t) - G^\varepsilon(x,t)]E^\varepsilon(x,t) = 0, \\ &\varepsilon^2\partial_{tt}F^\varepsilon(x,t) - \partial_{xx}F^\varepsilon(x,t) - \varepsilon^2\partial_{tt}|E^\varepsilon(x,t)|^2 = 0, \quad x \in \Omega, \quad t > 0, \\ &E^\varepsilon(x,0) = E_0(x), \quad F^\varepsilon(x,0) \equiv 0, \quad \partial_t F^\varepsilon(x,0) \equiv 0, \quad x \in \overline{\Omega}, \\ &E^\varepsilon(a,t) = E^\varepsilon(b,t) = 0, \qquad F^\varepsilon(a,t) = F^\varepsilon(b,t) = 0, \qquad t \geq 0, \end{aligned} \tag{2.8}$$

where $G^\varepsilon(x,s)$ is defined as (2.2) for $d = 1$ with the homogeneous boundary condition

$$\begin{aligned} &\partial_{ss}G^\varepsilon(x,s) - \frac{1}{\varepsilon^2}\partial_{xx}G^\varepsilon(x,s) = 0, \quad x \in \Omega, \quad s > 0, \\ &G^\varepsilon(x,0) = \varepsilon^\alpha\omega_0(x),\ \partial_s G^\varepsilon(x,0) = \varepsilon^\beta\omega_1(x); \quad G^\varepsilon(a,s) = G^\varepsilon(b,s) = 0,\ s \geq 0. \end{aligned} \tag{2.9}$$

When $\varepsilon \to 0^+$, formally we get $E^\varepsilon(x,t) \to \widetilde{E}^\varepsilon(x,t)$ and $F^\varepsilon(x,t) \to 0$, where $\widetilde{E}^\varepsilon := \widetilde{E}^\varepsilon(x,t)$ satisfies the truncated NLSE-OP

$$\begin{aligned} &i\partial_t\widetilde{E}^\varepsilon(x,t) + \partial_{xx}\widetilde{E}^\varepsilon(x,t) + \left[\left|\widetilde{E}^\varepsilon(x,t)\right|^2 - G^\varepsilon(x,t)\right]\widetilde{E}^\varepsilon(x,t) = 0, \quad t > 0, \\ &\widetilde{E}^\varepsilon(x,0) = E_0(x), \quad x \in \overline{\Omega}; \quad \widetilde{E}^\varepsilon(a,t) = \widetilde{E}^\varepsilon(b,t) = 0, \quad t \geq 0. \end{aligned} \tag{2.10}$$

Choose a mesh size $h := \Delta x = (b-a)/M$, with $M$ being a positive integer and a time step $\tau := \Delta t > 0$, and denote the grid points and time steps as

$$x_j := a + jh, \quad j = 0, 1, \ldots, M; \quad t_k := k\tau, \quad k = 0, 1, 2, \ldots.$$

Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1, 2, \ldots, M-1\}, \quad \mathcal{T}_M^0 = \{j \mid j = 0, 1, \ldots, M\}.$$

Let $E_j^{\varepsilon,k}$ and $F_j^{\varepsilon,k}$ be the approximations of $E^\varepsilon(x_j, t_k)$ and $F^\varepsilon(x_j, t_k)$, respectively, and denote $E^{\varepsilon,k} = (E_0^{\varepsilon,k}, E_1^{\varepsilon,k}, \ldots, E_M^{\varepsilon,k})^T \in \mathbb{C}^{(M+1)}$, $F^{\varepsilon,k} = (F_0^{\varepsilon,k}, F_1^{\varepsilon,k}, \ldots, F_M^{\varepsilon,k})^T \in$

$\mathbb{R}^{(M+1)}$ as the numerical solution vectors at $t = t_k$. Define the standard finite difference operators

$$\delta_t^+ E_j^k = \frac{E_j^{k+1} - E_j^k}{\tau}, \quad \delta_t^c E_j^k = \frac{E_j^{k+1} - E_j^{k-1}}{2\tau}, \quad \delta_t^2 E_j^k = \frac{E_j^{k+1} - 2E_j^k + E_j^{k-1}}{\tau^2},$$
$$\delta_x^+ E_j^k = \frac{E_{j+1}^k - E_j^k}{h}, \quad \delta_x^2 E_j^k = \frac{E_{j+1}^k - 2E_j^k + E_{j-1}^k}{h^2}.$$

We present a finite difference discretization of (2.8) as follows:

$$\begin{aligned} i\delta_t^c E_j^{\varepsilon,k} &= \left(-\delta_x^2 - \left|E_j^{\varepsilon,k}\right|^2 + H_j^{\varepsilon,k} + \frac{F_j^{\varepsilon,k+1} + F_j^{\varepsilon,k-1}}{2}\right) \frac{E_j^{\varepsilon,k+1} + E_j^{\varepsilon,k-1}}{2}, \\ \varepsilon^2 \delta_t^2 F_j^{\varepsilon,k} &= \frac{1}{2}\delta_x^2 \left(F_j^{\varepsilon,k+1} + F_j^{\varepsilon,k-1}\right) + \varepsilon^2 \delta_t^2 \left|E_j^{\varepsilon,k}\right|^2, \quad j \in \mathcal{T}_M, \quad k \geq 1, \end{aligned} \tag{2.11}$$

where an average of the oscillatory potential $G^\varepsilon$ over the interval $[t_{k-1}, t_{k+1}]$ is used:

$$H_j^{\varepsilon,k} = \frac{1}{2\tau} \int_{t_{k-1}}^{t_{k+1}} G^\varepsilon(x_j, s) ds, \quad j \in \mathcal{T}_M, \quad k \geq 1. \tag{2.12}$$

The boundary and initial conditions are discretized as

$$E_0^{\varepsilon,k} = E_M^{\varepsilon,k} = F_0^{\varepsilon,k} = F_M^{\varepsilon,k} = 0, \ k \geq 0; \quad E_j^{\varepsilon,0} = E_0(x_j), \ F_j^{\varepsilon,0} = 0, \quad j \in \mathcal{T}_M^0. \tag{2.13}$$

In addition, the values for the first step $E_j^{\varepsilon,1}$ and $F_j^{\varepsilon,1}$ can be obtained via (2.8) and the Taylor expansion as

$$E_j^{\varepsilon,1} = E_0(x_j) + \tau\phi_2(x_j) + \frac{\tau^2}{2}\phi_3(x_j), \quad F_j^{\varepsilon,1} = \frac{\tau^2}{2}\phi_4(x_j), \quad j \in \mathcal{T}_M, \tag{2.14}$$

where

$$\begin{aligned} \phi_2(x) &:= \partial_t E^\varepsilon(x,0) = i\left[E_0''(x) - N_0^\varepsilon(x)E_0(x)\right], \\ \phi_3(x) &:= \partial_{tt} E^\varepsilon(x,0) = i\left[\phi_2''(x) - N_1^\varepsilon(x)E_0(x) - N_0^\varepsilon(x)\phi_2(x)\right], \quad x \in \Omega, \\ \phi_4(x) &:= \partial_{tt} F^\varepsilon(x,0) = \partial_{tt}\rho^\varepsilon(x,0) = 2\mathrm{Im}\left[\phi_2(x)\overline{E_0''(x)} + E_0(x)\overline{\phi_2''(x)}\right]. \end{aligned} \tag{2.15}$$

Noticing (1.7), the above approximation for $E_j^{\varepsilon,1}$ implies $\max_{0\leq j\leq N} |E_j^{\varepsilon,1}| = O(\tau^2\varepsilon^\beta)$ when $-1 \leq \beta < 0$. In such a case, in order to make sure that $E^{\varepsilon,1}$ is uniformly bounded for $\varepsilon \in (0,1]$, $\tau$ has to be taken as $\tau \leq O(\varepsilon^{-\beta/2})$, which is too restrictive. To remedy this, we replace $\phi_3(x)$ in (2.14) by a modified version [8]

$$\phi_3(x) = i\left[\phi_2''(x) - \left(\phi_1(x) + \frac{\varepsilon^{1+\beta}}{\tau}\sin\left(\frac{\tau}{\varepsilon}\right)\omega_1(x)\right)E_0(x) - N_0^\varepsilon(x)\phi_2(x)\right], \tag{2.16}$$

which yields the first step value with second order accuracy as

$$\begin{aligned} E_j^{\varepsilon,1} &= E_0(x_j) + \tau\phi_2(x_j) + \frac{i\tau^2}{2}\left[\phi_2''(x_j) - \phi_1(x_j)E_0(x_j) - N_0^\varepsilon(x_j)\phi_2(x_j)\right] \\ &\quad - \frac{i\tau}{2}\varepsilon^{1+\beta}\sin\left(\frac{\tau}{\varepsilon}\right)E_0(x_j)\omega_1(x_j). \end{aligned} \tag{2.17}$$

In addition, $H_j^{\varepsilon,k}$ in (2.12) can be approximated by solving the wave equations (2.9) via the sine pseudospectral method in space and then integrating in time in phase space *exactly* as

$$\begin{aligned} H_j^{\varepsilon,k} &\approx \frac{1}{2\tau}\sum_{l=1}^{M-1}\sin\left(\frac{lj\pi}{M}\right)\int_{t_{k-1}}^{t_{k+1}}\left[\varepsilon^{\alpha}\widetilde{(\omega_0)}_l\cos\left(\frac{\mu_l}{\varepsilon}u\right)+\frac{\varepsilon^{1+\beta}}{\mu_l}\widetilde{(\omega_1)}_l\sin\left(\frac{\mu_l}{\varepsilon}u\right)\right]du\\ &=\sum_{l=1}^{M-1}\frac{\varepsilon}{\tau\mu_l}\sin\left(\frac{lj\pi}{M}\right)\sin\left(\frac{\mu_l\tau}{\varepsilon}\right)\left[\varepsilon^{\alpha}\widetilde{(\omega_0)}_l\cos\left(\frac{\mu_l t_k}{\varepsilon}\right)+\frac{\varepsilon^{1+\beta}}{\mu_l}\widetilde{(\omega_1)}_l\sin\left(\frac{\mu_l t_k}{\varepsilon}\right)\right], \end{aligned}$$

where for $l=1,2,\ldots,M-1$,

$$\mu_l=\frac{l\pi}{b-a},\quad \widetilde{(\omega_0)}_l=\frac{2}{M}\sum_{j=1}^{M-1}\omega_0(x_j)\sin\left(\frac{lj\pi}{M}\right),\quad \widetilde{(\omega_1)}_l=\frac{2}{M}\sum_{j=1}^{M-1}\omega_1(x_j)\sin\left(\frac{lj\pi}{M}\right).$$

### 2.3. Main results.

For convenience of notation, denote

$$0\le\alpha^*=\min\{1,\alpha,1+\beta\}\le 1.$$

Let $T^*>0$ be the maximum common existence time for the solutions of the ZS (2.8) and the NLSE-OP (2.10). Then, for any fixed $0<T<T^*$, according to the known results in [1, 31, 32, 34], it is natural to assume that the solution $(E^\varepsilon,F^\varepsilon)$ of the ZS (2.8) and the solution $\widetilde{E}^\varepsilon$ of the NLSE-OP (2.10) are smooth enough over $\Omega_T:=\Omega\times[0,T]$ and satisfy

$$\text{(A)}\quad \begin{aligned} &\|E^\varepsilon\|_{W^{5,\infty}}+\|\partial_t E^\varepsilon\|_{W^{1,\infty}}+\|\partial_{tt}F^\varepsilon\|_{W^{2,\infty}}+\left\|\widetilde{E}^\varepsilon\right\|_{W^{5,\infty}}+\left\|\partial_t\widetilde{E}^\varepsilon\right\|_{W^{1,\infty}}\lesssim 1,\\ &\|F^\varepsilon\|_{W^{4,\infty}}\lesssim\varepsilon^2,\qquad \|\partial_t F^\varepsilon\|_{W^{4,\infty}}\lesssim\varepsilon,\qquad \left\|\partial_{tt}\widetilde{E}^\varepsilon\right\|_{W^{4,\infty}}\lesssim\frac{1}{\varepsilon^{1-\alpha^*}},\\ &\|\partial_{tt}E^\varepsilon\|_{W^{4,\infty}}+\left\|\partial_t^3F^\varepsilon\right\|_{W^{2,\infty}}\lesssim\frac{1}{\varepsilon},\qquad \left\|\partial_t^3E^\varepsilon\right\|_{W^{4,\infty}}+\left\|\partial_t^4F^\varepsilon\right\|_{W^{2,\infty}}\lesssim\frac{1}{\varepsilon^2}. \end{aligned}$$

In addition, it is natural to assume that the initial data satisfies

$$\text{(B)}\qquad \|E_0\|_{W^{5,\infty}(\Omega)}+\|\omega_0\|_{W^{3,\infty}(\Omega)}+\|\omega_1\|_{W^{3,\infty}(\Omega)}\lesssim 1.$$

Then one can obtain

$$\|\partial_s^m G^\varepsilon(\cdot,s)\|_{W^{3,\infty}(\Omega)}\lesssim\varepsilon^{\alpha^*-m},\quad m=0,1,2,3. \tag{2.18}$$

Denote

$$X_M=\left\{v=(v_0,v_1,\ldots,v_M)^T\ \mid\ v_0=v_M=0\right\}\subseteq\mathbb{C}^{M+1},$$

equipped with norms and inner products defined as

$$\begin{aligned} &\|u\|^2=h\sum_{j=1}^{M-1}|u_j|^2,\quad \|\delta_x^+u\|^2=h\sum_{j=0}^{M-1}|\delta_x^+u_j|^2,\quad \|u\|_\infty=\sup_{j\in\mathcal{T}_M^0}|u_j|,\\ &(u,v)=h\sum_{j=1}^{M-1}u_j\overline{v_j},\quad \langle\delta_x^+u,\delta_x^+v\rangle=h\sum_{j=0}^{M-1}(\delta_x^+u_j)\,(\delta_x^+\overline{v_j}),\quad u,v\in X_M. \end{aligned}$$

Then we have

$$\left(-\delta_x^2 u, v\right) = \langle \delta_x^+ u, \delta_x^+ v\rangle, \quad \left(\left(-\delta_x^2\right)^{-1} u, v\right) = \left(u, \left(-\delta_x^2\right)^{-1} v\right), \quad u, v \in X_M. \tag{2.19}$$

Define the error functions $e^{\varepsilon,k} \in X_M$ and $f^{\varepsilon,k} \in X_M$ as

$$e_j^{\varepsilon,k} = E^\varepsilon(x_j, t_k) - E_j^{\varepsilon,k}, \quad f_j^{\varepsilon,k} = F^\varepsilon(x_j, t_k) - F_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}. \tag{2.20}$$

Then we have the following error estimates for (2.11) with (2.12)–(2.14).

Theorem 2.1. *Under assumptions* (A)–(B)*, there exist* $h_0 > 0$ *and* $\tau_0 > 0$ *sufficiently small and independent of* $0 < \varepsilon \le 1$ *such that, when* $0 < h \le h_0$ *and* $0 < \tau \le \tau_0$*, the following two error estimates of the scheme* (2.11) *with* (2.12)–(2.14) *hold:*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \qquad 0 \le k \le \frac{T}{\tau}, \quad 0 < \varepsilon \le 1, \tag{2.21}$$

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}. \tag{2.22}$$

*Thus, by taking the minimum among the two error bounds for* $\varepsilon \in (0, 1]$*, we obtain a uniform error estimate for well-prepared initial data, i.e.,* $\alpha \ge 1$ *and* $\beta \ge 0$*,*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \min_{0<\varepsilon\le 1}\left\{\tau^2 + \tau\varepsilon + \varepsilon^2, \frac{\tau^2}{\varepsilon}\right\} \lesssim h^2 + \tau^{4/3}, \tag{2.23}$$

*and respectively, for ill-prepared initial data, i.e.,* $0 \le \alpha < 1$ *or* $-1 \le \beta < 0$*,*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \min_{0<\varepsilon\le 1}\left\{\tau^2 + \varepsilon^{\alpha^*}(\tau + \varepsilon), \frac{\tau^2}{\varepsilon}\right\} \lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}. \tag{2.24}$$

## 3. Error analysis.

In order to prove Theorem 2.1, we will use the energy method to obtain one error bound (2.21) and use the limiting equation NLSE-OP (2.10) to get the other one (2.22), which is shown in the following diagram [3, 4, 6, 19, 28].

$$\begin{tikzcd}
(E^{\varepsilon,k}, F^{\varepsilon,k}) \arrow[r, "O(h^2+\tau^2+\tau\varepsilon^{\alpha^*}+\varepsilon^{1+\alpha^*})"] \arrow[dr, "O(h^2+\tau^2/\varepsilon)"'] & \left(\widetilde{E}^\varepsilon, 0\right) \arrow[d, "O(\varepsilon^2)"] \\
 & (E^\varepsilon, F^\varepsilon)
\end{tikzcd}$$

To simplify notation, for a function $V := V(x,t)$ and a grid function $V^k \in X_M$ with $k \ge 0$, we denote, for $k \ge 1$,

$$\llparenthesis V \rrparenthesis(x, t_k) = \frac{V(x, t_{k+1}) + V(x, t_{k-1})}{2}, \quad x \in \overline{\Omega}; \quad \llbracket V \rrbracket_j^k = \frac{V_j^{k+1} + V_j^{k-1}}{2}, \quad j \in \mathcal{T}_M^0.$$

In order to deal with the nonlinearity and to bound the numerical solution, we adapt the cut-off technique which has been widely used in the literature [2, 4, 7, 37], i.e., the nonlinearity is first truncated to a global Lipschitz function with compact support and then the error bound can be achieved if the exact solution is bounded and the numerical solution is close to the exact solution under some conditions on the mesh size and time step. Choose a smooth function $\gamma(s) \in C^\infty(\mathbb{R})$ such that

$$\gamma(s) = \begin{cases} 1, & |s| \le 1, \\ \in [0,1], & |s| \le 2, \\ 0, & |s| \ge 2, \end{cases}$$

and by assumption (A) we can choose $M_0 > 0$ as

$$M_0 = \max\left\{ \sup_{\varepsilon\in(0,1]} \|E^\varepsilon\|_{L^\infty(\Omega_T)}, \sup_{\varepsilon\in(0,1]} \left\|\widetilde{E}^\varepsilon\right\|_{L^\infty(\Omega_T)} \right\}.$$

For $s \ge 0$, $y_1, y_2 \in \mathbb{C}$, define

$$\gamma_B(s) = s\,\gamma\left(\frac{s}{B}\right) \quad \text{with} \quad B = (M_0+1)^2,$$

and

$$g(y_1,y_2) = \frac{y_1+y_2}{2}\int_0^1 \gamma_B'(s|y_1|^2 + (1-s)|y_2|^2)ds = \frac{\gamma_B(|y_1|^2) - \gamma_B(|y_2|^2)}{|y_1|^2 - |y_2|^2}\cdot\frac{y_1+y_2}{2}.$$

Then $\gamma_B(s)$ is global Lipschitz and there exists $C_B > 0$ such that

$$|\gamma_B(s_1) - \gamma_B(s_2)| \le \sqrt{C_B}\,|\sqrt{s_1} - \sqrt{s_2}| \quad \forall s_1, s_2 \ge 0. \tag{3.1}$$

Let $\hat{E}^{\varepsilon,k}$, $\hat{F}^{\varepsilon,k} \in X_M$ $(k \ge 0)$ be the solution of the following:

(3.2)

$$\begin{aligned} i\delta_t^c \hat{E}_j^{\varepsilon,k} &= \left(-\delta_x^2 + H_j^{\varepsilon,k}\right)[\![\hat{E}^\varepsilon]\!]_j^k + \left(-\gamma_B\left(\left|\hat{E}_j^{\varepsilon,k}\right|^2\right) + [\![\hat{F}^\varepsilon]\!]_j^k\right) g\left(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\right), \\ \varepsilon^2\delta_t^2 \hat{F}_j^{\varepsilon,k} &= \frac{1}{2}\delta_x^2\left(\hat{F}_j^{\varepsilon,k+1} + \hat{F}_j^{\varepsilon,k-1}\right) + \varepsilon^2\delta_t^2\gamma_B\left(\left|\hat{E}_j^{\varepsilon,k}\right|^2\right), \qquad j \in \mathcal{T}_M, \quad k \ge 1, \\ \hat{E}_j^{\varepsilon,0} &= E_j^{\varepsilon,0}, \quad \hat{F}_j^{\varepsilon,0} = F_j^{\varepsilon,0} = 0, \quad \hat{E}_j^{\varepsilon,1} = E_j^{\varepsilon,1}, \quad \hat{F}_j^{\varepsilon,1} = F_j^{\varepsilon,1}, \quad j \in \mathcal{T}_M^0. \end{aligned}$$

Here $(\hat{E}^{\varepsilon,k}, \hat{F}^{\varepsilon,k})$ can be viewed as another approximation of the solution $(E^\varepsilon, F^\varepsilon)$ of the ZS with a cut-off Lipschitz nonlinearity. Define error functions $\hat{e}^{\varepsilon,k}$, $\hat{f}^{\varepsilon,k} \in X_M$ as

$$\hat{e}_j^{\varepsilon,k} = E^\varepsilon(x_j,t_k) - \hat{E}_j^{\varepsilon,k}, \quad \hat{f}_j^{\varepsilon,k} = F^\varepsilon(x_j,t_k) - \hat{F}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad k \ge 0. \tag{3.3}$$

For $(\hat{e}^{\varepsilon,k}, \hat{f}^{\varepsilon,k})$, we have the following estimates.

Theorem 3.1. *Under assumption* (A), *there exists $\tau_1 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that, when $0 < \tau \le \tau_1$ and $0 < h \le \frac{1}{2}$, we have the following error estimate for the scheme* (3.2)*:*

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+\hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad 0 \le k \le \frac{T}{\tau}, \quad 0 < \varepsilon \le 1. \tag{3.4}$$

Introduce local truncation errors $\hat{\xi}_j^{\varepsilon,k}$, $\hat{\eta}_j^{\varepsilon,k} \in X_M$ as

$$\begin{aligned} \hat{\xi}_j^{\varepsilon,k} &= i\delta_t^c E^\varepsilon(x_j,t_k) + \left(\delta_x^2 - H_j^{\varepsilon,k}\right)(\!|E^\varepsilon|\!)(x_j,t_k) \\ &\quad + \left(\gamma_B(|E^\varepsilon(x_j,t_k)|^2) - (\!|F^\varepsilon|\!)(x_j,t_k)\right) g\left(E^\varepsilon(x_j,t_{k+1}), E^\varepsilon(x_j,t_{k-1})\right) \\ &= i\delta_t^c E^\varepsilon(x_j,t_k) + \left(\delta_x^2 + |E^\varepsilon(x_j,t_k)|^2 - H_j^{\varepsilon,k} - (\!|F^\varepsilon|\!)(x_j,t_k)\right)(\!|E^\varepsilon|\!)(x_j,t_k), \\ \hat{\eta}_j^{\varepsilon,k} &= \varepsilon^2\delta_t^2 F^\varepsilon(x_j,t_k) - \delta_x^2(\!|F^\varepsilon|\!)(x_j,t_k) - \varepsilon^2\delta_t^2\gamma_B(|E^\varepsilon(x_j,t_k)|^2) \\ &= \varepsilon^2\delta_t^2 F^\varepsilon(x_j,t_k) - \delta_x^2(\!|F^\varepsilon|\!)(x_j,t_k) - \varepsilon^2\delta_t^2|E^\varepsilon(x_j,t_k)|^2, \quad j \in \mathcal{T}_M, \quad k \ge 1. \end{aligned} \tag{3.5}$$

Then we have the local errors as follows.

LEMMA 3.2. *Under assumption* (A), *when* $0 < h \le \frac{1}{2}$ *and* $0 < \tau \le \frac{1}{2}$, *we have*

$$\big|\hat{\xi}_j^{\varepsilon,k}\big| + \big|\delta_x^+ \hat{\xi}_j^{\varepsilon,k}\big| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad \big|\hat{\eta}_j^{\varepsilon,k}\big| \lesssim \varepsilon^2 h^2 + \tau^2, \quad \big|\delta_t^c \hat{\eta}_j^{\varepsilon,k}\big| \lesssim \varepsilon h^2 + \frac{\tau^2}{\varepsilon}, \quad j \in \mathcal{T}_M. \tag{3.6}$$

*Proof.* By (2.8), and using the Taylor expansion, we get

$$\begin{aligned}
i\delta_t^c E^\varepsilon(x_j, t_k) &= \frac{i}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \partial_t E^\varepsilon(x_j, s)ds \\
&= \frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \big[(-\partial_{xx}E^\varepsilon - |E^\varepsilon|^2 E^\varepsilon + E^\varepsilon F^\varepsilon)(x_j, s) + E^\varepsilon(x_j, s)G^\varepsilon(x_j, s)\big]\, ds \\
&= -E_{xx}^\varepsilon(x_j, t_k) - |E^\varepsilon(x_j, t_k)|^2 E^\varepsilon(x_j, t_k) + E^\varepsilon(x_j, t_k)F^\varepsilon(x_j, t_k) \\
&\quad - \frac{\tau^2}{4}\int_{-1}^{1}(1-|s|)^2 \partial_{tt}(E_{xx}^\varepsilon + |E^\varepsilon|^2 E^\varepsilon - E^\varepsilon F^\varepsilon)(x_j, t_k + s\tau)ds \\
&\quad + \frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} E^\varepsilon(x_j, s)G^\varepsilon(x_j, s)ds, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned}$$

Similarly, by the Taylor expansion, we have

$$\begin{aligned}
&\Big(\delta_x^2 + |E^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} - \llbracket F^\varepsilon \rrbracket(x_j, t_k)\Big) \llbracket E^\varepsilon \rrbracket(x_j, t_k) \\
&\quad = E_{xx}^\varepsilon(x_j, t_k) + \big(|E^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} - F^\varepsilon(x_j, t_k)\big)E^\varepsilon(x_j, t_k) \\
&\qquad + \frac{h^2}{12}\int_{-1}^{1}(1-|s|)^3 \big(\partial_x^4 E^\varepsilon(x_j + sh, t_k + \tau) + \partial_x^4 E^\varepsilon(x_j + sh, t_k - \tau)\big)\, ds \\
&\qquad + \frac{\tau^2}{2}\int_{-1}^{1}(1-|s|)\big(E_{xxtt}^\varepsilon(x_j, t_k + s\tau) - E^\varepsilon(x_j, t_k)F_{tt}^\varepsilon(x_j, t_k + s\tau)\big)ds \\
&\qquad + \frac{\tau^2}{2}\big(|E^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} - \llbracket F^\varepsilon \rrbracket(x_j, t_k)\big)\int_{-1}^{1}(1-|s|)E_{tt}^\varepsilon(x_j, t_k + s\tau)ds.
\end{aligned}$$

Note that by (2.12), we have

$$\begin{aligned}
&\frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} E^\varepsilon(x_j, s)G^\varepsilon(x_j, s)ds - E^\varepsilon(x_j, t_k)H_j^{\varepsilon,k} \\
&\quad = A_j^k + \frac{1}{2\tau}E_t^\varepsilon(x_j, t_k)\int_{-\tau}^{\tau} s\, G^\varepsilon(x_j, t_k + s)ds \\
&\quad = A_j^k + \frac{\tau^2}{2}E_t^\varepsilon(x_j, t_k)\int_0^1 s\int_{-s}^{s} G_t^\varepsilon(x_j, t_k + \theta\tau)d\theta ds,
\end{aligned}$$

where

$$A_j^k = \frac{\tau^2}{2}\int_{-1}^{1}\int_0^s (s-\theta)\, G^\varepsilon(x_j, t_k + s\tau)E_{tt}^\varepsilon(x_j, t_k + \theta\tau)d\theta ds.$$

Accordingly, by assumption (A) and (2.18), we conclude that

$$\begin{aligned}
\big|\hat{\xi}_j^{\varepsilon,k}\big| &\lesssim h^2\|\partial_x^4 E^\varepsilon\|_{L^\infty} + \tau^2\,\big[\|E_{xxtt}^\varepsilon\|_{L^\infty} + \|\partial_{tt}(|E^\varepsilon|^2 E^\varepsilon)\|_{L^\infty} + \|E^\varepsilon\|_{L^\infty}\|F_{tt}^\varepsilon\|_{L^\infty} \\
&\quad + \|E_t^\varepsilon\|_{L^\infty}\,(\|G_t^\varepsilon\|_{L^\infty} + \|F_t^\varepsilon\|_{L^\infty}) + |E_{tt}^\varepsilon\|_{L^\infty}\,\big(\|G^\varepsilon\|_{L^\infty} + \|F^\varepsilon\|_{L^\infty} + \|E^\varepsilon\|_{L^\infty}^2\big)\big] \\
&\lesssim h^2 + \frac{\tau^2}{\varepsilon}, \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned}$$

Applying $\delta_x^+$ to $\hat{\xi}^{\varepsilon,k}$ and using the same approach, we get

$$\left|\delta_x^+ \hat{\xi}_j^{\varepsilon,k}\right| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Similarly, we obtain

$$\begin{aligned}\hat{\eta}_j^{\varepsilon,k} = {}& \frac{\varepsilon^2\tau^2}{6}\int_{-1}^{1}(1-|s|)^3\left(\partial_t^4 F^\varepsilon(x_j, t_k+s\tau) - \partial_t^4(|E^\varepsilon|^2)(x_j,t_k+s\tau)\right)ds\\ &- \frac{\tau^2}{2}\int_{-1}^{1}(1-|s|)F^\varepsilon_{xxtt}(x_j,t_k+s\tau)ds\\ &- \frac{h^2}{12}\int_{-1}^{1}(1-|s|)^3\left(\partial_x^4F^\varepsilon(x_j+sh,t_k+\tau)+\partial_x^4F^\varepsilon(x_j+sh,t_k-\tau)\right)ds,\end{aligned}$$

which implies

$$\begin{aligned}\left|\hat{\eta}_j^{\varepsilon,k}\right| &\lesssim h^2\|\partial_x^4F^\varepsilon\|_{L^\infty} + \tau^2\left(\|F^\varepsilon_{xxtt}\|_{L^\infty} + \varepsilon^2\|\partial_t^4F^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_t^4|E^\varepsilon|^2\|_{L^\infty}\right)\\ &\lesssim \varepsilon^2h^2 + \tau^2, \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.\end{aligned}$$

Applying $\delta_t^c$ to $\hat{\eta}_j^{\varepsilon,k}$, we have

$$\begin{aligned}\left|\delta_t^c\hat{\eta}_j^{\varepsilon,k}\right| &\lesssim h^2\|\partial_x^4F_t^\varepsilon\|_{L^\infty} + \tau^2(\|\partial_t^3F^\varepsilon_{xx}\|_{L^\infty} + \varepsilon^2\|\partial_t^5F^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_t^5|E^\varepsilon|^2\|_{L^\infty})\\ &\lesssim \varepsilon h^2 + \frac{\tau^2}{\varepsilon}, \qquad j \in \mathcal{T}_M, \quad 2 \le k \le \frac{T}{\tau} - 2.\end{aligned}$$

Thus the proof is completed. ∎

For the initial step, we have the following estimates.

Lemma 3.3. *Under assumption* (A), *when* $0 < \tau \le \frac{1}{2}$, *the first step errors of the discretization* (3.2) *with* (2.13) *and* (2.17) *satisfy*

$$\begin{aligned}&\hat{e}_j^{\varepsilon,0} = 0, \quad \left|\hat{e}_j^{\varepsilon,1}\right| + \left|\delta_x^+\hat{e}_j^{\varepsilon,1}\right| \lesssim \frac{\tau^2}{\varepsilon}, \quad \left|\delta_t^+\hat{e}_j^{\varepsilon,0}\right| \lesssim \frac{\tau^2}{\varepsilon^2},\\ &\hat{f}_j^{\varepsilon,0} = 0, \quad \left|\hat{f}_j^{\varepsilon,1}\right| \lesssim \frac{\tau^3}{\varepsilon}, \quad \left|\delta_t^+\hat{f}_j^{\varepsilon,0}\right| \lesssim \frac{\tau^2}{\varepsilon}.\end{aligned} \tag{3.7}$$

*Proof.* By the definition of $\hat{E}_j^{\varepsilon,1}$ (2.17), and noticing that $\beta \ge -1$, we obtain

$$\begin{aligned}\left|\hat{e}_j^{\varepsilon,1}\right| &\le \tau^2\left|\int_0^1(1-s)E^\varepsilon_{tt}(x_j,s\tau)ds - \frac{1}{2}E^\varepsilon_{tt}(x_j,0)\right|\\ &\quad + \frac{\varepsilon^\beta\tau^2}{2}\left|E_0(x_j)\omega_1(x_j)\right|\left|1-\frac{\sin(\tau/\varepsilon)}{\tau/\varepsilon}\right|\\ &\lesssim \tau^2\left(\|E^\varepsilon_{tt}\|_{L^\infty} + \varepsilon^\beta\|E_0\|_{L^\infty}\|\omega_1\|_{L^\infty}\right) \lesssim \frac{\tau^2}{\varepsilon}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\left|\hat{e}_j^{\varepsilon,1}\right| &= \frac{\tau^3}{2}\left|\int_0^1(1-s)^2E^\varepsilon_{ttt}(x_j,s\tau)ds\right|\\ &\quad + \frac{\tau^3}{2\varepsilon^{1-\beta}}\left|E_0(x_j)\omega_1(x_j)\right|\left|\int_0^1(1-s)\sin\left(\frac{\tau s}{\varepsilon}\right)ds\right|\\ &\lesssim \tau^3\left(\|E^\varepsilon_{ttt}\|_{L^\infty} + \varepsilon^{-2}\|E_0\|_{L^\infty}\|\omega_1\|_{L^\infty}\right) \lesssim \frac{\tau^3}{\varepsilon^2},\end{aligned}$$

which implies that $|\delta_t^+ \hat{e}_j^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon^2}$. Similarly, $|\delta_x^+ \hat{e}_j^{\varepsilon,1}| \lesssim \frac{\tau^2}{\varepsilon}$. It follows from (2.14) and assumption (A) that

$$\left|\hat{f}_j^{\varepsilon,1}\right| = \frac{\tau^3}{2}\left|\int_0^1 (1-s)^2 F_{ttt}^\varepsilon(x_j, s\tau)ds\right| \lesssim \tau^3 \|F_{ttt}^\varepsilon\|_{L^\infty} \lesssim \frac{\tau^3}{\varepsilon}.$$

Recalling that $\hat{f}_j^{\varepsilon,0} = 0$, we can get that $|\delta_t^+ \hat{f}_j^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon}$, which completes the proof. $\square$

Subtracting (3.2) from (3.5), we have the error equations

$$\begin{aligned} i\delta_t^c \hat{e}_j^{\varepsilon,k} &= \left(-\delta_x^2 + H_j^{\varepsilon,k}\right) [\![\hat{e}^\varepsilon]\!]_j^k + r_j^k + \hat{\xi}_j^{\varepsilon,k}, \\ \varepsilon^2 \delta_t^2 \hat{f}_j^{\varepsilon,k} &= \delta_x^2 [\![\hat{f}^\varepsilon]\!]_j^k + \varepsilon^2 \delta_t^2 p_j^k + \hat{\eta}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M,\ 1 \le k \le \frac{T}{\tau} - 1, \end{aligned} \tag{3.8}$$

where $r^k \in X_M$ and $p^k \in X_M$ are defined as

$$\begin{aligned} r_j^k &= \left[-|E^\varepsilon|^2 + (\![F^\varepsilon]\!)\right] (\![E^\varepsilon]\!)\big|_{(x_j,t_k)} + \left[\gamma_{_B}\left(|\hat{E}_j^{\varepsilon,k}|^2\right) - [\![\hat{F}^\varepsilon]\!]_j^k\right] g\big(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\big), \\ p_j^k &= |E^\varepsilon(x_j,t_k)|^2 - \gamma_{_B}\big(|\hat{E}_j^{\varepsilon,k}|^2\big), \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1. \end{aligned} \tag{3.9}$$

By the property of $\gamma_{_B}$ in (3.1), we get, for $0 \le k \le \frac{T}{\tau}$,

$$|p_j^k| = \left|\gamma_{_B}\left(|E^\varepsilon(x_j,t_k)|^2\right) - \gamma_{_B}\left(|\hat{E}_j^{\varepsilon,k}|^2\right)\right| \le \sqrt{C_{_B}}|\hat{e}_j^{\varepsilon,k}|, \quad j \in \mathcal{T}_M. \tag{3.10}$$

Recalling the definition of $g(\cdot,\cdot)$ and noting that $(\![E^\varepsilon]\!)(x_j,t_k) = g(E^\varepsilon(x_j,t_{k+1}), E^\varepsilon(x_j,t_{k-1}))$, similar to the proof in [3, 15] with the details omitted here for brevity, we have, for $j \in \mathcal{T}_M$ and $1 \le k \le \frac{T}{\tau} - 1$,

$$\begin{aligned} &\left|g\left(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\right)\right| \lesssim 1, \quad \left|(\![E^\varepsilon]\!)(x_j,t_k) - g\left(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\right)\right| \lesssim \sum_{l=k\pm1} |\hat{e}_j^{\varepsilon,l}|, \\ &\left|\delta_x^+\left((\![E^\varepsilon]\!)(x_j,t_k) - g\left(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\right)\right)\right| \lesssim \sum_{l=k\pm1}\left(|\hat{e}_j^{\varepsilon,l}| + |\hat{e}_{j+1}^{\varepsilon,l}| + |\delta_x^+ \hat{e}_j^{\varepsilon,l}|\right). \end{aligned} \tag{3.11}$$

*Proof of Theorem* 3.1. Multiplying both sides of the first equation in (3.8) by $4\tau\overline{[\![\hat{e^\varepsilon}]\!]_j^k}$, summing together for $j \in \mathcal{T}_M$ and taking the imaginary parts, we obtain

$$\|\hat{e}^{\varepsilon,k+1}\|^2 - \|\hat{e}^{\varepsilon,k-1}\|^2 \quad = 4\tau\,\mathrm{Im}\left(r^k + \hat{\xi}^{\varepsilon,k}, [\![\hat{e}^\varepsilon]\!]^k\right), \qquad 1 \le k \le \frac{T}{\tau} - 1. \tag{3.12}$$

Using the same approach by multiplying $4\tau\delta_t^c\overline{\hat{e}_j^{\varepsilon,k}}$ and taking the real parts, we get

$$\|\delta_x^+ \hat{e}^{\varepsilon,k+1}\|^2 - \|\delta_x^+ \hat{e}^{\varepsilon,k-1}\|^2 = -4\,\mathrm{Re}\left(H^{\varepsilon,k}[\![\hat{e}^\varepsilon]\!]^k + r^k + \hat{\xi}^{\varepsilon,k}, \tau\delta_t^c \hat{e}^{\varepsilon,k}\right). \tag{3.13}$$

Introduce $\hat{u}^{\varepsilon,k+1/2} \in X_M$ satisfying

$$-\delta_x^2 \hat{u}_j^{\varepsilon,k+1/2} = \delta_t^+\big(\hat{f}_j^{\varepsilon,k} - p_j^k\big), \qquad j \in \mathcal{T}_M.$$

Multiplying both sides of the second equation in (3.8) by $\tau(\hat{u}_j^{\varepsilon,k+1/2}+\hat{u}_j^{\varepsilon,k-1/2})$ and summing them together for $j\in\mathcal{T}_M$, we obtain

$$
\begin{aligned}
&\varepsilon^2\left(\|\delta_x^+\hat{u}^{\varepsilon,k+1/2}\|^2-\|\delta_x^+\hat{u}^{\varepsilon,k-1/2}\|^2\right)+\frac{1}{2}\left(\left\|\hat{f}^{\varepsilon,k+1}\right\|^2-\left\|\hat{f}^{\varepsilon,k-1}\right\|^2\right)\\
&=\left(\llbracket\hat{f}^\varepsilon\rrbracket^k,2\tau\delta_t^c p^k\right)+\tau\left(\hat{\eta}^{\varepsilon,k},\hat{u}^{\varepsilon,k+1/2}+\hat{u}^{\varepsilon,k-1/2}\right),\qquad 1\le k\le\frac{T}{\tau}-1.
\end{aligned}
\tag{3.14}
$$

Define a discrete "energy"

$$
\begin{aligned}
\mathcal{A}^k&=C_{B}\left(\|\hat{e}^{\varepsilon,k}\|^2+\hat{e}^{\varepsilon,k+1}\|^2\right)+\|\delta_x^+\hat{e}^{\varepsilon,k}\|^2+\|\delta_x^+\hat{e}^{\varepsilon,k+1}\|^2\\
&\quad+\varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,k+1/2}\|^2+\frac{1}{2}\left(\|\hat{f}^{\varepsilon,k}\|^2+\|\hat{f}^{\varepsilon,k+1}\|^2\right),\qquad 0\le k\le\frac{T}{\tau}-1.
\end{aligned}
\tag{3.15}
$$

Multiplying (3.12) by $C_{B}>0$ and then summing with (3.13) and (3.14), we get

$$
\begin{aligned}
\mathcal{A}^k-\mathcal{A}^{k-1}&=4\tau C_{B}\,\mathrm{Im}\left(r^k+\hat{\xi}^{\varepsilon,k},\llbracket\hat{e}^\varepsilon\rrbracket^k\right)-4\,\mathrm{Re}\left(H^{\varepsilon,k}\llbracket\hat{e}^\varepsilon\rrbracket^k+r^k+\hat{\xi}^{\varepsilon,k},\tau\delta_t^c\hat{e}^{\varepsilon,k}\right)\\
&\quad+\left(\llbracket\hat{f}^\varepsilon\rrbracket^k,2\tau\delta_t^c p^k\right)+\tau\left(\hat{\eta}^{\varepsilon,k},\hat{u}^{\varepsilon,k+1/2}+\hat{u}^{\varepsilon,k-1/2}\right),\quad 1\le k\le\frac{T}{\tau}-1.
\end{aligned}
\tag{3.16}
$$

Now we estimate different terms in the right-hand side of (3.16). Let $q_1^k\in X_M$ and $q_2^k\in X_M$, defined as

$$
\begin{aligned}
q_{1j}^k&=\big(-|E^\varepsilon(x_j,t_k)|^2+\llparenthesis F^\varepsilon\rrparenthesis(x_j,t_k)\big)\left(\llparenthesis E^\varepsilon\rrparenthesis(x_j,t_k)-g\big(\hat{E}_j^{\varepsilon,k+1},\hat{E}_j^{\varepsilon,k-1}\big)\right),\\
q_{2j}^k&=-g\big(\hat{E}_j^{\varepsilon,k+1},\hat{E}_j^{\varepsilon,k-1}\big)\big(p_j^k-\llbracket\hat{f}^\varepsilon\rrbracket_j^k\big),\qquad j\in\mathcal{T}_M.
\end{aligned}
\tag{3.17}
$$

Then we have

$$
r^k=q_1^k+q_2^k,\qquad 1\le k\le\frac{T}{\tau}-1.
\tag{3.18}
$$

In view of assumption (A), noting (3.10) and (3.11), we get

$$
\big|r_j^k\big|\lesssim\big|\hat{e}_j^{\varepsilon,k+1}\big|+\big|\hat{e}_j^{\varepsilon,k}\big|+\big|\hat{e}_j^{\varepsilon,k-1}\big|+\big|\hat{f}_j^{\varepsilon,k+1}\big|+\big|\hat{f}_j^{\varepsilon,k-1}\big|,\qquad j\in\mathcal{T}_M.
\tag{3.19}
$$

This implies that

$$
\big|\big(r^k,\llbracket\hat{e}^\varepsilon\rrbracket^k\big)\big|\lesssim\mathcal{A}^k+\mathcal{A}^{k-1},\qquad 1\le k\le\frac{T}{\tau}-1.
\tag{3.20}
$$

By the Cauchy inequality, we have

$$
\left|\mathrm{Im}\big(\hat{\xi}^{\varepsilon,k},\llbracket\hat{e}^\varepsilon\rrbracket^k\big)\right|\lesssim\|\hat{\xi}^{\varepsilon,k}\|^2+\|\hat{e}^{\varepsilon,k+1}\|^2+\|\hat{e}^{\varepsilon,k-1}\|^2\lesssim\|\hat{\xi}^{\varepsilon,k}\|^2+\mathcal{A}^k+\mathcal{A}^{k-1}.
\tag{3.21}
$$

In view of (3.8), (3.19), and (2.18), and using the Cauchy inequality, we find

$$
\begin{aligned}
&\big|\mathrm{Re}\big(H^{\varepsilon,k}\llbracket\hat{e}^\varepsilon\rrbracket^k+\hat{\xi}^{\varepsilon,k},\tau\delta_t^c\hat{e}^{\varepsilon,k}\big)\big|\\
&\quad=\tau\left|\mathrm{Im}\big(H^{\varepsilon,k}\llbracket\hat{e}^\varepsilon\rrbracket^k+\hat{\xi}^{\varepsilon,k},(-\delta_x^2+H^{\varepsilon,k})\llbracket\hat{e}^\varepsilon\rrbracket^k+r^k+\hat{\xi}^{\varepsilon,k}\big)\right|\\
&\quad\lesssim\tau\left(1+\|H^{\varepsilon,k}\|_\infty+\|\delta_x^+H^{\varepsilon,k}\|_\infty\right)\left(\left\|\hat{\xi}^{\varepsilon,k}\right\|^2+\left\|\delta_x^+\hat{\xi}^{\varepsilon,k}\right\|^2+\mathcal{A}^k+\mathcal{A}^{k-1}\right)\\
&\quad\lesssim\tau\left(\left\|\hat{\xi}^{\varepsilon,k}\right\|^2+\left\|\delta_x^+\hat{\xi}^{\varepsilon,k}\right\|^2+\mathcal{A}^k+\mathcal{A}^{k-1}\right),\qquad 1\le k\le\frac{T}{\tau}-1.
\end{aligned}
\tag{3.22}
$$

Combining (3.17) and (3.11), we have

$$
\begin{aligned}
\left|\mathrm{Re}(q_1^k, 4\tau\delta_t^c \hat{e}^{\varepsilon,k})\right| &= 4\tau \left|\mathrm{Im}\left(q_1^k, (-\delta_x^2 + H^{\varepsilon,k})\llbracket \hat{e}^\varepsilon \rrbracket^k + r^k + \hat{\xi}^{\varepsilon,k}\right)\right| \\
&\lesssim \tau(1 + \|H^{\varepsilon,k}\|_\infty)(\|\delta_x^+ \hat{e}^{\varepsilon,k+1}\|^2 + \|\delta_x^+ \hat{e}^{\varepsilon,k-1}\|^2 + \|\delta_x^+ q_1^k\|^2 + \|q_1^k\|^2 \\
&\quad + \|r^k\|^2 + \|\hat{\xi}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k+1}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2) \\
&\lesssim \tau(\|\hat{\xi}^{\varepsilon,k}\|^2 + \mathcal{A}^k + \mathcal{A}^{k-1}), \qquad 1 \le k \le \frac{T}{\tau} - 1. \qquad (3.23)
\end{aligned}
$$

In view of (3.17), we get

$$
\begin{aligned}
\mathrm{Re}(q_2^k, 4\tau\delta_t^c \hat{e}^{\varepsilon,k}) &= 2\,\mathrm{Re}\big(g(\hat{E}^{\varepsilon,k+1}, \hat{E}^{\varepsilon,k-1})(\llbracket \hat{f}^\varepsilon \rrbracket^k - p^k), E^\varepsilon(\cdot, t_{k+1}) - E^\varepsilon(\cdot, t_{k-1})\big) \\
&\quad - \big(\llbracket \hat{f}^\varepsilon \rrbracket^k - p^k, 2\tau\delta_t^c(\gamma_{\scriptscriptstyle B}(|\hat{E}^{\varepsilon,k}|^2))\big) = q^k + \big(\llbracket \hat{f}^\varepsilon \rrbracket^k - p^k, 2\tau\delta_t^c p^k\big), \qquad (3.24)
\end{aligned}
$$

where

$$
q^k = 2\,\mathrm{Re}\big((g(\hat{E}^{\varepsilon,k+1}, \hat{E}^{\varepsilon,k-1}) - \llparenthesis E^\varepsilon \rrparenthesis(\cdot, t_k))(\llbracket \hat{f}^\varepsilon \rrbracket^k - p^k), E^\varepsilon(\cdot, t_{k+1}) - E^\varepsilon(\cdot, t_{k-1})\big).
$$

By assumption (A) and (3.11), we have

$$
|q^k| \lesssim \tau\|\partial_t E^\varepsilon\|_{L^\infty}(\mathcal{A}^k + \mathcal{A}^{k-1}) \lesssim \tau(\mathcal{A}^k + \mathcal{A}^{k-1}), \qquad 1 \le k \le \frac{T}{\tau} - 1.
$$

Combining the above inequalities, we obtain

$$
\left|4\,\mathrm{Re}\left(r^k, \tau\delta_t^c \hat{e}^{\varepsilon,k}\right) - \left(\llbracket \hat{f}^\varepsilon \rrbracket^k - p^k, 2\tau\delta_t^c p^k\right)\right| \lesssim \tau(\|\hat{\xi}^{\varepsilon,k}\|^2 + \mathcal{A}^k + \mathcal{A}^{k-1}). \qquad (3.25)
$$

Hence it can be concluded from (3.16), (3.20), (3.21), (3.22), and (3.25) that

$$
\begin{aligned}
&\mathcal{A}^k - \mathcal{A}^{k-1} - (p^k, p^{k+1} - p^{k-1}) - \tau(\hat{\eta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+1/2} + \hat{u}^{\varepsilon,k-1/2}) \\
&\quad \lesssim \tau(\|\hat{\xi}^{\varepsilon,k}\|^2 + \|\delta_x^+ \hat{\xi}^{\varepsilon,k}\|^2 + \mathcal{A}^k + \mathcal{A}^{k-1}), \qquad 1 \le k \le \frac{T}{\tau} - 1. \qquad (3.26)
\end{aligned}
$$

Summing the above equation for $k = 1, 2, \ldots, m \le \frac{T}{\tau} - 1$ and noting $p^0 = \mathbf{0}$ in (3.9), we have

$$
\begin{aligned}
&\mathcal{A}^m - \mathcal{A}^0 - (p^m, p^{m+1}) - \tau\sum_{l=1}^{m}(\hat{\eta}^{\varepsilon,l}, \hat{u}^{\varepsilon,l+1/2} + \hat{u}^{\varepsilon,l-1/2}) \\
&\quad \lesssim \tau\mathcal{A}^0 + \tau\sum_{l=1}^{m}(\|\hat{\xi}^{\varepsilon,l}\|^2 + \|\delta_x^+ \hat{\xi}^{\varepsilon,l}\|^2 + \mathcal{A}^l), \qquad 1 \le m \le \frac{T}{\tau} - 1. \qquad (3.27)
\end{aligned}
$$

Noting (2.19) and using the Sobolev and Cauchy inequalities, we obtain

$$
\begin{aligned}
&-\frac{\mathcal{A}^m}{4} + \tau\sum_{l=1}^{m}\left(\hat{\eta}^{\varepsilon,l}, \hat{u}^{\varepsilon,l+1/2} + \hat{u}^{\varepsilon,l-1/2}\right) \\
&= -\frac{\mathcal{A}^m}{4} + \sum_{l=1}^{m}\left((-\delta_x^2)^{-1}\hat{\eta}^{\varepsilon,l}, \hat{f}^{\varepsilon,l+1} - p^{l+1} - (\hat{f}^{\varepsilon,l-1} - p^{l-1})\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\mathcal{A}^m}{4} - 2\tau \sum_{l=2}^{m-1} \left( \delta_t^c (-\delta_x^2)^{-1} \hat{\eta}^{\varepsilon,l}, \hat{f}^{\varepsilon,l} - p^l \right) \\
&\quad + \sum_{l=m}^{m+1} \left( (-\delta_x^2)^{-1} \hat{\eta}^{\varepsilon,l-1}, \hat{f}^{\varepsilon,l} - p^l \right) - \sum_{l=0}^{1} \left( (-\delta_x^2)^{-1} \hat{\eta}^{\varepsilon,l+1}, \hat{f}^{\varepsilon,l} - p^l \right) \\
&\lesssim \mathcal{A}^0 + \tau \sum_{l=2}^{m-1} (\|\delta_t^c \hat{\eta}^{\varepsilon,l}\|^2 + \mathcal{A}^l) + \sum_{l=1}^{2} \|\hat{\eta}^{\varepsilon,l}\|^2 + \sum_{l=m-1}^{m} \|\hat{\eta}^{\varepsilon,l}\|^2. \qquad (3.28)
\end{aligned}
$$

Recalling that

$$
(3.29) \qquad \left(p^m, p^{m+1}\right) \le \frac{C_{\scriptscriptstyle B}}{2} (\|\hat{e}^{\varepsilon,m}\|^2 + \|\hat{e}^{\varepsilon,m+1}\|^2) \le \frac{1}{2} \mathcal{A}^m, \qquad 1 \le m \le \frac{T}{\tau} - 1.
$$

Combining (3.27), (3.28), and (3.29), there exists $0 < \tau_1 \le \frac{1}{16}$ such that, when $0 < \tau \le \tau_1$, we have

$$
\begin{aligned}
\mathcal{A}^m &\lesssim \mathcal{A}^0 + \tau \sum_{l=1}^{m-1} \mathcal{A}^l + \sum_{l=1}^{2} \|\hat{\eta}^{\varepsilon,l}\|^2 + \sum_{l=m-1}^{m} \|\hat{\eta}^{\varepsilon,l}\|^2 \\
&\quad + \tau \sum_{l=1}^{m} (\|\hat{\xi}^{\varepsilon,l}\|^2 + \|\delta_x^+ \hat{\xi}^{\varepsilon,l}\|^2) + \tau \sum_{l=2}^{m-1} \|\delta_t^c \hat{\eta}^{\varepsilon,l}\|^2, \qquad 1 \le m \le \frac{T}{\tau} - 1. \qquad (3.30)
\end{aligned}
$$

By Lemma 3.3 and using the discrete Sobolev inequality, we have

$$
(3.31) \qquad \varepsilon \|\delta_x^+ \hat{u}^{\varepsilon,1/2}\| \lesssim \varepsilon \|\delta_t^+ (\hat{f}^{\varepsilon,0} - p^0)\| \lesssim \varepsilon \|\delta_t^+ \hat{f}^{\varepsilon,0}\| + \varepsilon \|\delta_t^+ \hat{e}^{\varepsilon,0}\| \lesssim \frac{\tau^2}{\varepsilon},
$$

which, together with Lemma 3.3, yields

$$
(3.32) \qquad \mathcal{A}^0 \lesssim \left( h^2 + \frac{\tau^2}{\varepsilon} \right)^2.
$$

Plugging (3.32) into (3.30) and noting Lemma 3.2, we get

$$
(3.33) \qquad \mathcal{A}^m \lesssim \left( h^2 + \frac{\tau^2}{\varepsilon} \right)^2 + \tau \sum_{l=1}^{m-1} \mathcal{A}^l, \qquad 1 \le m \le \frac{T}{\tau} - 1.
$$

Applying the discrete Gronwall inequality, when $0 < \tau \le \tau_1$, we obtain

$$
\mathcal{A}^m \lesssim \left( h^2 + \frac{\tau^2}{\varepsilon} \right)^2, \qquad 0 \le m \le \frac{T}{\tau} - 1,
$$

which completes the proof of Theorem 3.1 by noting (3.15). ◻

THEOREM 3.4. *Under assumptions* (A)–(B)*, there exists $\tau_2 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that, when $0 < \tau \le \tau_2$ and $0 < h \le \frac{1}{2}$, we have the following error estimate of the scheme* (3.2)*:*

$$
(3.34) \qquad \|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau \varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}, \quad 0 \le k \le \frac{T}{\tau}.
$$

Define another set of error functions $\widetilde{e}^{\varepsilon,k} \in X_M$ and $\widetilde{f}^{\varepsilon,k} \in X_M$ as

$$\widetilde{e}^{\varepsilon,k}_j = \widetilde{E}^\varepsilon(x_j, t_k) - \hat{E}^{\varepsilon,k}_j, \quad \widetilde{f}^{\varepsilon,k}_j = -\hat{F}^{\varepsilon,k}_j, \quad j \in \mathcal{T}^0_M, \quad 0 \le k \le \frac{T}{\tau}, \tag{3.35}$$

where $\widetilde{E}^\varepsilon$ is the solution of the NLSE-OP (2.10), with their corresponding local truncation errors $\widetilde{\xi}^{\varepsilon,k} \in X_M$ and $\widetilde{\eta}^{\varepsilon,k} \in X_M$ as

$$\begin{aligned}
\widetilde{\xi}^{\varepsilon,k}_j &= i\delta^c_t \widetilde{E}^\varepsilon(x_j, t_k) + (\delta^2_x - H^{\varepsilon,k}_j)[\![\widetilde{E}^\varepsilon]\!](x_j, t_k) \\
&\quad + \gamma_{\scriptscriptstyle B}(|\widetilde{E}^\varepsilon(x_j, t_k)|^2) g\big(\widetilde{E}^\varepsilon(x_j, t_{k+1}), \widetilde{E}^\varepsilon(x_j, t_{k-1})\big) \\
&= i\delta^c_t \widetilde{E}^\varepsilon(x_j, t_k) + \big(\delta^2_x + |\widetilde{E}^\varepsilon(x_j, t_k)|^2 - H^{\varepsilon,k}_j\big)[\![\widetilde{E}^\varepsilon]\!](x_j, t_k), \\
\widetilde{\eta}^{\varepsilon,k}_j &= -\varepsilon^2 \delta^2_t \gamma_{\scriptscriptstyle B}(|\widetilde{E}^\varepsilon(x_j, t_k)|^2) = -\varepsilon^2 \delta^2_t(|\widetilde{E}^\varepsilon(x_j, t_k)|^2), \quad j \in \mathcal{T}_M.
\end{aligned} \tag{3.36}$$

LEMMA 3.5. *Under assumption* (A)*, when* $0 < h \le \frac{1}{2}$ *and* $0 < \tau \le \frac{1}{2}$*, we have*

$$\|\widetilde{\xi}^{\varepsilon,k}\| + \|\delta^+_x \widetilde{\xi}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*}, \quad \|\widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^2, \quad \|\delta^c_t \widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^{1+\alpha^*}. \tag{3.37}$$

*Proof.* Similar to the proof of Lemma 3.2, we can get that

$$\begin{aligned}
\widetilde{\xi}^{\varepsilon,k}_j &= B^k_j + \frac{h^2}{12}\int_{-1}^{1}(1-|s|)^3 \sum_{m=\pm 1} \widetilde{E}^\varepsilon_{xxxx}(x_j + sh, t_k + m\tau)ds \\
&\quad - \frac{\tau^2}{4}\int_{-1}^{1}(1-|s|)^2 \partial_{tt}(\widetilde{E}^\varepsilon_{xx} + |\widetilde{E}^\varepsilon|^2\widetilde{E}^\varepsilon)(x_j, t_k + s\tau)ds \\
&\quad + \frac{\tau^2}{2}\int_{-1}^{1}(1-|s|)\widetilde{E}^\varepsilon_{xxtt}(x_j, t_k + s\tau)ds \\
&\quad + \frac{\tau^2}{2}\big(|\widetilde{E}^\varepsilon(x_j, t_k)|^2 - H^{\varepsilon,k}_j\big)\int_{-1}^{1}(1-|s|)\widetilde{E}^\varepsilon_{tt}(x_j, t_k + s\tau)ds,
\end{aligned}$$

where

$$\begin{aligned}
|B^k_j| &= \left|\frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \widetilde{E}^\varepsilon(x_j, s)G^\varepsilon(x_j, s)ds - \widetilde{E}^\varepsilon(x_j, t_k)H^{\varepsilon,k}_j\right| \\
&= \left|\frac{\tau}{2}\int_{-1}^{1} G^\varepsilon(x_j, t_k + s\tau)\int_0^s \widetilde{E}^\varepsilon_t(x_j, t_k + \theta\tau)d\theta ds\right| \\
&\lesssim \tau\|G^\varepsilon\|_{L^\infty}\|\widetilde{E}^\varepsilon_t\|_{L^\infty} \lesssim \tau\varepsilon^{\alpha^*}, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned}$$

Recalling (2.5), (2.18), and assumption (A), and using integration by parts, we have

$$\begin{aligned}
&\tau^2\left|\int_{-1}^{1}(1-|s|)\widetilde{E}^\varepsilon_{tt}(x_j, t_k + s\tau)ds\right| \\
&\quad = \left|\tau^2\int_{-1}^{1}(1-|s|)\left(\widetilde{E}^\varepsilon_{xxt} + \big(|\widetilde{E}^\varepsilon|^2\widetilde{E}^\varepsilon\big)_t - \big(\widetilde{E}^\varepsilon G^\varepsilon\big)_t\right)(x_j, t_k + s\tau)ds\right| \\
&\quad \le \tau^2\left|\int_{-1}^{1}(1-|s|)\big(\widetilde{E}^\varepsilon_{xxt} + \big(|\widetilde{E}^\varepsilon|^2\widetilde{E}^\varepsilon\big)_t\big)(x_j, t_k + s\tau)ds\right| \\
&\qquad + \tau\left|\int_0^1\Big[\big(\widetilde{E}^\varepsilon G^\varepsilon\big)(x_j, t_k + s\tau) - \big(\widetilde{E}^\varepsilon G^\varepsilon\big)(x_j, t_k - s\tau)\Big]ds\right| \\
&\quad \lesssim \tau^2 + \tau\varepsilon^{\alpha^*}, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned} \tag{3.38}$$

Similarly, we can get that

$$\tau^2\left|\int_{-1}^{1}(1-|s|)^2\partial_{tt}\big(\widetilde{E}^\varepsilon_{xx}+|\widetilde{E}^\varepsilon|^2\widetilde{E}^\varepsilon\big)(x_j,t_k+s\tau)ds\right|\lesssim\tau^2+\tau\varepsilon^{\alpha^*},$$
$$\tau^2\left|\int_{-1}^{1}(1-|s|)\widetilde{E}^\varepsilon_{xxtt}(x_j,t_k+s\tau)ds\right|\lesssim\tau^2+\tau\varepsilon^{\alpha^*}.$$

Hence we can conclude that

$$\|\widetilde{\xi}^{\varepsilon,k}\|\lesssim h^2+\tau^2+\tau\varepsilon^{\alpha^*},\quad 1\le k\le\frac{T}{\tau}-1.$$

Similarly, we can get

$$\|\delta_x^+\widetilde{\xi}^{\varepsilon,k}\|\lesssim h^2+\tau^2+\tau\varepsilon^{\alpha^*},\quad 1\le k\le\frac{T}{\tau}-1.$$

By assumption (A), it is easy to get that

$$\left|\partial_{tt}|\widetilde{E}^\varepsilon(x,t)|^2\right|=\left|-2\,\mathrm{Im}\left(\overline{\widetilde{E}^\varepsilon_t}\widetilde{E}^\varepsilon_{xx}+\overline{\widetilde{E}^\varepsilon}\widetilde{E}^\varepsilon_{xxt}\right)\right|\lesssim 1,\quad x\in\Omega,\quad 0\le t\le T,$$
$$\left|\partial_{ttt}|\widetilde{E}^\varepsilon(x,t)|^2\right|=\left|-2\,\mathrm{Im}\left(\overline{\widetilde{E}^\varepsilon_{tt}}\widetilde{E}^\varepsilon_{xx}+2\overline{\widetilde{E}^\varepsilon_t}\widetilde{E}^\varepsilon_{xxt}+\overline{\widetilde{E}^\varepsilon}\widetilde{E}^\varepsilon_{xxtt}\right)\right|\lesssim\varepsilon^{\alpha^*-1},$$

which indicate that

$$\|\widetilde{\eta}^{\varepsilon,k}\|\lesssim\varepsilon^2,\quad 1\le k\le\frac{T}{\tau}-1;\quad \|\delta_t^c\widetilde{\eta}^{\varepsilon,k}\|\lesssim\varepsilon^{1+\alpha^*},\quad 2\le k\le\frac{T}{\tau}-2.$$

Thus the proof is completed. ☐

Analogous to Lemma 3.3, we have error bounds of $\widetilde{e}^{\varepsilon,k}$, $\widetilde{f}^{\varepsilon,k}$ at the first step.

LEMMA 3.6. *Under assumptions* (A) *and* (B), *when* $0<h\le\frac{1}{2}$ *and* $0<\tau\le\frac{1}{2}$, *we have*

$$\widetilde{e}^{\varepsilon,0}_j=\widetilde{f}^{\varepsilon,0}_j=0,\quad \left|\widetilde{e}^{\varepsilon,1}_j\right|+\left|\delta_x^+\widetilde{e}^{\varepsilon,1}_j\right|\lesssim\tau^2+\tau\varepsilon^{\alpha^*},$$
$$\left|\delta_t^+\widetilde{e}^{\varepsilon,0}_j\right|\lesssim\tau+\varepsilon^{\alpha^*},\quad \left|\widetilde{f}^{\varepsilon,1}_j\right|\lesssim\tau^2,\quad \left|\delta_t^+\widetilde{f}^{\varepsilon,0}_j\right|\lesssim\tau,\quad j\in\mathcal{T}_M.$$

*Proof.* It follows from (2.8) and (2.5) that $\partial_t E^\varepsilon(x_j,0)=\partial_t\widetilde{E}^\varepsilon(x_j,0)=\phi_2(x_j)$ for $j\in\mathcal{T}_M^0$. By (2.17), (3.38), and assumption (B), we get

$$\left|\widetilde{e}^{\varepsilon,1}_j\right|=\left|\tau^2\int_0^1(1-s)\widetilde{E}^\varepsilon_{tt}(x_j,s\tau)ds-\frac{i\tau^2}{2}\left[\phi_2''(x_j)-\phi_1(x_j)E_0(x_j)-N_0^\varepsilon(x_j)\phi_2(x_j)\right]\right.$$
$$\left.-\frac{i\tau\varepsilon^{1+\beta}}{2}\sin\left(\frac{\tau}{\varepsilon}\right)E_0(x_j)\omega_1(x_j)\right|\lesssim\tau^2+\tau\varepsilon^{\alpha^*},\quad j\in\mathcal{T}_M.$$

Similarly, we have

$$\left|\delta_x^+\widetilde{e}^{\varepsilon,1}_j\right|\lesssim\tau^2+\tau\varepsilon^{\alpha^*},\quad j\in\mathcal{T}_M.$$

Moreover, it is easy to get that

$$|\widetilde{f}^{\varepsilon,1}_j|=|F^{\varepsilon,1}_j|\lesssim\tau^2|F^\varepsilon_{tt}(x_j,0)|\lesssim\tau^2,\quad j\in\mathcal{T}_M.$$

The rest can be obtained similarly; details are omitted here for brevity. ☐

*Proof of Theorem* 3.4. Subtracting (3.2) from (3.36), we obtain the error equations

$$i\delta_t^c \widetilde{e}_j^{\,\varepsilon,k} = \big(-\delta_x^2 + H_j^{\varepsilon,k}\big)[\![\widetilde{e}^{\varepsilon}]\!]_j^k + \widetilde{r}_j^k + \widetilde{\xi}_j^{\,\varepsilon,k}, \tag{3.39}$$

$$\varepsilon^2 \delta_t^2 \widetilde{f}_j^{\,\varepsilon,k} = \delta_x^2[\![\widetilde{f}^{\varepsilon}]\!]_j^k + \varepsilon^2\delta_t^2 \widetilde{p}_j^k + \widetilde{\eta}_j^{\,\varepsilon,k}, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1, \tag{3.40}$$

where $\widetilde{r}^k \in X_M$ and $\widetilde{p}^k \in X_M$, defined as

$$\begin{aligned}\widetilde{r}_j^k &= -\big|\widetilde{E}^{\varepsilon}(x_j,t_k)\big|^2 \llparenthesis\widetilde{E}^{\varepsilon}\rrparenthesis(x_j,t_k) + \Big(\gamma_{B}\big(\big|\hat{E}_j^{\varepsilon,k}\big|^2\big) - [\![\hat{F}^{\varepsilon}]\!]_j^k\Big)\, g\big(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}\big),\\ \widetilde{p}_j^k &= \big|\widetilde{E}^{\varepsilon}(x_j,t_k)\big|^2 - \gamma_{B}\big(\big|\hat{E}_j^{\varepsilon,k}\big|^2\big), \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.\end{aligned}$$

Let $\widetilde{u}^{\varepsilon,k+\frac{1}{2}} \in X_M$ be the solution of the equation

$$-\delta_x^2 \widetilde{u}_j^{\,\varepsilon,k+\frac{1}{2}} = \delta_t^+\big(\widetilde{f}_j^{\,\varepsilon,k} - \widetilde{p}_j^k\big), \quad j \in \mathcal{T}_M, \quad 0 \le k \le \frac{T}{\tau} - 1.$$

Define another discrete "energy"

$$\begin{aligned}\widetilde{\mathcal{A}}^k = {} & C_{B}(\|\widetilde{e}^{\,\varepsilon,k}\|^2 + \widetilde{e}^{\,\varepsilon,k+1}\|^2) + \|\delta_x^+\widetilde{e}^{\,\varepsilon,k}\|^2 + \|\delta_x^+\widetilde{e}^{\,\varepsilon,k+1}\|^2\\ & + \varepsilon^2\|\delta_x^+\widetilde{u}^{\varepsilon,k+1/2}\|^2 + \frac{1}{2}\big(\|\widetilde{f}^{\varepsilon,k}\|^2 + \|\widetilde{f}^{\varepsilon,k+1}\|^2\big), \quad 0 \le k \le \frac{T}{\tau} - 1.\end{aligned} \tag{3.41}$$

Applying the same approach as in the proof of Theorem 3.1 and noting that $(\widetilde{p}^k, \widetilde{p}^{k+1}) \le \frac{1}{2}\widetilde{\mathcal{A}}^k$, there exists $0 < \tau_2 \le \frac{1}{16}$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that, when $0 < \tau \le \tau_2$,

$$\begin{aligned}\widetilde{\mathcal{A}}^k \lesssim {} & \widetilde{\mathcal{A}}^0 + \tau\sum_{l=1}^{k-1}\widetilde{\mathcal{A}}^l + \sum_{l=1}^{2}\|\widetilde{\eta}^{\,\varepsilon,l}\|^2 + \sum_{l=k-1}^{k}\|\widetilde{\eta}^{\,\varepsilon,l}\|^2\\ & + \tau\sum_{l=1}^{k}(\|\widetilde{\xi}^{\,\varepsilon,l}\|^2 + \|\delta_x^+\widetilde{\xi}^{\,\varepsilon,l}\|^2) + \tau\sum_{l=2}^{k-1}\|\delta_t^c\widetilde{\eta}^{\,\varepsilon,l}\|^2.\end{aligned}$$

By Lemma 3.6 and the discrete Sobolev inequality, we deduce that

$$\varepsilon\|\delta_x^+\widetilde{u}^{\varepsilon,1/2}\| \lesssim \varepsilon\|\delta_t^+\widetilde{f}^{\varepsilon,0}\| + \varepsilon\|\delta_t^+\widetilde{e}^{\,\varepsilon,0}\| \lesssim \varepsilon\tau + \varepsilon^{1+\alpha^*},$$

which, together with Lemma 3.6, yields

$$\widetilde{\mathcal{A}}^0 \lesssim (\tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})^2.$$

By Lemma 3.5, when $0 < \tau \le \tau_2$ and $0 < h \le \frac{1}{2}$, we have

$$\widetilde{\mathcal{A}}^k \lesssim \Big(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}\Big)^2 + \tau\sum_{l=1}^{k-1}\widetilde{\mathcal{A}}^l, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Using the discrete Gronwall inequality, when $0 < \tau \le \tau_2$, we have

$$\widetilde{\mathcal{A}}^k \lesssim \Big(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}\Big)^2, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Noting (3.41), we get

$$\|\widetilde{e}^{\varepsilon,k}\| + \|\delta_x^+\widetilde{e}^{\varepsilon,k}\| + \left\|\widetilde{f}^{\varepsilon,k}\right\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}, \quad 0 \le k \le \frac{T}{\tau}.$$

Combining the above inequality and (2.18), using the triangle inequality, and noting (2.7), we obtain

$$\begin{aligned}\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+\hat{e}^{\varepsilon,k}\| &\lesssim \|\widetilde{e}^{\varepsilon,k}\| + \|\delta_x^+\widetilde{e}^{\varepsilon,k}\| + \|E^\varepsilon(\cdot,t_k) - \widetilde{E}^\varepsilon(\cdot,t_k)\|_{H^1}\\ &\lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}, \quad 0 \le k \le \frac{T}{\tau},\\ \|\hat{f}^{\varepsilon,k}\| &\lesssim \|\widetilde{f}^{\varepsilon,k}\| + \|F^\varepsilon(\cdot,t_k)\|_{L^2} \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*},\end{aligned}$$

which completes the proof of Theorem 3.4. ◻

*Proof of Theorem* 2.1. When $0 < \tau \le \min\left\{\frac{1}{16}, \tau_1, \tau_2\right\}$ and $0 < h \le \frac{1}{2}$, combining (3.4) and (3.34), we have, for $0 \le k \le \frac{T}{\tau}$,

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+\hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \min_{0<\varepsilon\le 1}\left\{\tau^2 + \varepsilon^{\alpha^*}(\tau+\varepsilon), \frac{\tau^2}{\varepsilon}\right\} \lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}. \tag{3.42}$$

This, together with the inverse inequality [37], implies

$$\|\hat{E}^{\varepsilon,k}\|_\infty - \|E^\varepsilon(\cdot,t_k)\|_\infty \le \|\hat{e}^{\varepsilon,k}\|_\infty \lesssim \|\delta_x^+\hat{e}^{\varepsilon,k}\| \lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}, \quad 0 \le k \le \frac{T}{\tau}.$$

Thus, there exist $h_1 > 0$ and $\tau_3 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that, when $0 < h \le h_1$ and $0 < \tau \le \tau_3$,

$$\|\hat{E}^{\varepsilon,k}\|_\infty \le 1 + \|E^\varepsilon(\cdot,t_k)\|_\infty \le 1 + M_0, \quad 0 \le k \le \frac{T}{\tau}.$$

Taking $h_0 = \min\left\{\frac{1}{2}, h_1\right\}$ and $\tau_0 = \min\left\{\frac{1}{16}, \tau_1, \tau_2, \tau_3\right\}$, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, the numerical method (3.2) collapses to (2.11), i.e.,

$$\hat{E}_j^{\varepsilon,k} = E_j^{\varepsilon,k}, \quad \hat{F}_j^{\varepsilon,k} = F_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}.$$

Thus the proof is completed. ◻

*Remark* 3.7. The error bounds in Theorem 2.1 are still valid in high dimensions, e.g., $d = 2, 3$, provided that an additional condition on the time step $\tau$ is added:

$$\tau = o\left(C_d(h)^{1-\frac{\alpha^*}{2+2\alpha^*}}\right),$$

with

$$C_d(h) \sim \begin{cases}\dfrac{1}{|\ln h|}, & d = 2,\\ h^{1/2}, & d = 3.\end{cases}$$

The reason for this is due to the discrete Sobolev inequality [3, 4, 5]

$$\|\psi_h\|_\infty \le \frac{1}{C_d(h)}\|\psi_h\|_{H^1},$$

where $\psi_h$ is a mesh function over $\Omega$ with the homogeneous Dirichlet boundary condition.

TABLE 1
*Spatial error analysis at time $t = 1$ for Case* II, *i.e.,* $\alpha = \beta = 0$.

| $e^\varepsilon(t=1)$ | $h_0 = 0.2$ | $h_0/2$ | $h_0/2^2$ | $h_0/2^3$ | $h_0/2^4$ | $h_0/2^5$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 1.27E-2 | 3.27E-3 | 8.19E-4 | 2.05E-4 | 5.13E-5 | 1.28E-5 |
| rate | - | 1.96 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 1.22E-2 | 3.12E-3 | 7.84E-4 | 1.96E-4 | 4.92E-5 | 1.23E-5 |
| rate | - | 1.96 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^2$ | 1.21E-2 | 3.10E-3 | 7.74E-4 | 1.94E-4 | 4.86E-5 | 1.22E-5 |
| rate | - | 1.97 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^3$ | 1.26E-2 | 3.22E-3 | 8.08E-4 | 2.02E-4 | 5.07E-5 | 1.27E-5 |
| rate | - | 1.97 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^4$ | 1.25E-2 | 3.17E-3 | 7.93E-4 | 1.98E-4 | 4.99E-5 | 1.25E-5 |
| rate | - | 1.98 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^6$ | 1.27E-2 | 3.24E-3 | 8.10E-4 | 2.03E-4 | 5.07E-5 | 1.27E-5 |
| rate | - | 1.98 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^8$ | 1.27E-2 | 3.24E-3 | 8.10E-4 | 2.03E-4 | 5.07E-5 | 1.27E-5 |
| rate | - | 1.98 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^{10}$ | 1.27E-2 | 3.24E-3 | 8.10E-4 | 2.03E-4 | 5.07E-5 | 1.27E-5 |
| rate | - | 1.98 | 2.00 | 2.00 | 2.00 | 2.00 |

| $n^\varepsilon(t=1)$ | $h_0 = 0.2$ | $h_0/2$ | $h_0/2^2$ | $h_0/2^3$ | $h_0/2^4$ | $h_0/2^5$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 7.06E-3 | 1.75E-3 | 4.39E-4 | 1.09E-4 | 2.74E-5 | 6.85E-6 |
| rate | - | 2.01 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 9.60E-3 | 2.38E-3 | 5.95E-4 | 1.48E-4 | 3.72E-5 | 9.31E-6 |
| rate | - | 2.01 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^2$ | 5.35E-3 | 1.34E-3 | 3.34E-4 | 8.33E-5 | 2.09E-5 | 5.23E-6 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^3$ | 3.36E-3 | 8.33E-4 | 2.08E-4 | 5.19E-5 | 1.30E-5 | 3.25E-6 |
| rate | - | 2.01 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^4$ | 3.25E-3 | 8.06E-4 | 2.00E-4 | 5.00E-5 | 1.25E-5 | 3.12E-6 |
| rate | - | 2.02 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^6$ | 3.23E-3 | 7.95E-4 | 1.99E-4 | 4.97E-5 | 1.25E-5 | 3.12E-6 |
| rate | - | 2.02 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^8$ | 3.23E-3 | 7.95E-4 | 1.99E-4 | 4.97E-5 | 1.25E-5 | 3.12E-6 |
| rate | - | 2.02 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^{10}$ | 3.23E-3 | 7.95E-4 | 1.99E-4 | 4.97E-5 | 1.25E-5 | 3.12E-6 |
| rate | - | 2.02 | 2.00 | 2.00 | 2.00 | 2.00 |

## 4. Numerical results.

In this section, we present numerical results for the ZS (1.1) by our proposed finite difference method. In order to do so, we take $d = 1$ in (1.1), and the initial condition is taken as

$$E_0(x) = e^{-x^2/2}, \quad \omega_0(x) = e^{-x^2/4}, \quad \omega_1(x) = e^{-x^2/3}\sin(x), \quad x \in \mathbb{R}.$$

We mainly consider two types of initial data:

Case I. Well-prepared initial data, i.e., $\alpha = 1$ and $\beta = 0$.

Case II. Ill-prepared initial data, i.e., $\alpha = 0$ and $\beta = 0$.

In practical computation, the problem is truncated on a bounded interval $\Omega_\varepsilon = [-30 - \frac{1}{\varepsilon}, 30 + \frac{1}{\varepsilon}]$, which is large enough that the truncation error of (2.8) to the original whole space problem (2.3) can be negligible due to the homogeneous Dirichlet boundary condition. We remark here that the bounded computational domain $\Omega_\varepsilon$ has to be chosen as $\varepsilon$-dependent due to the fact that (i) the rapid outgoing waves are at wave speed $O\left(\frac{1}{\varepsilon}\right)$ and (ii) the simple homogeneous Dirichlet boundary condition was adopted at $\partial\Omega_\varepsilon$ for simplicity of notation. Of course, if one adapts the accu-

TABLE 2
*Temporal error analysis at time $t = 1$ for Case* I, *i.e.,* $\alpha = 1$ *and* $\beta = 0$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ | $\tau_0/2^6$ | $\tau_0/2^7$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 5.35E-2 | 2.01E-2 | 7.42E-3 | 2.17E-3 | 5.62E-4 | 1.42E-4 | 3.55E-5 | 8.95E-6 |
| rate | - | 1.42 | 1.44 | 1.77 | 1.95 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2$ | 3.67E-2 | 1.72E-2 | 6.87E-3 | 2.04E-3 | 5.27E-4 | 1.33E-4 | 3.34E-5 | 8.42E-6 |
| rate | - | 1.09 | 1.33 | 1.76 | 1.94 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 3.42E-2 | 1.52E-2 | 6.05E-3 | 1.84E-3 | 4.81E-4 | 1.21E-4 | 3.04E-5 | 7.67E-6 |
| rate | - | 1.17 | 1.32 | 1.72 | 1.94 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 3.38E-2 | 1.74E-2 | 6.40E-3 | 1.80E-3 | 4.64E-4 | 1.17E-4 | 2.94E-5 | 7.40E-6 |
| rate | - | 0.96 | 1.45 | 1.83 | 1.96 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^4$ | 3.31E-2 | 1.72E-2 | 6.40E-3 | 1.82E-3 | 4.69E-4 | 1.18E-4 | 2.96E-5 | 7.44E-6 |
| rate | - | 0.94 | 1.43 | 1.81 | 1.96 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^6$ | 3.30E-2 | 1.72E-2 | 6.43E-3 | 1.84E-3 | 4.73E-4 | 1.19E-4 | 2.98E-5 | 7.45E-6 |
| rate | - | 0.94 | 1.42 | 1.81 | 1.96 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2^8$ | 3.31E-2 | 1.72E-2 | 6.43E-3 | 1.84E-3 | 4.73E-4 | 1.19E-4 | 2.98E-5 | 7.45E-6 |
| rate | - | 0.94 | 1.42 | 1.81 | 1.96 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2^{10}$ | 3.31E-2 | 1.72E-2 | 6.43E-3 | 1.84E-3 | 4.73E-4 | 1.19E-4 | 2.98E-5 | 7.45E-6 |
| rate | - | 0.94 | 1.42 | 1.81 | 1.96 | 1.99 | 2.00 | 2.00 |

| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ | $\tau_0/2^6$ | $\tau_0/2^7$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 1.06E-2 | 2.82E-3 | 7.28E-4 | 1.85E-4 | 4.67E-5 | 1.18E-5 | 2.95E-6 | 7.44E-7 |
| rate | - | 1.91 | 1.95 | 1.98 | 1.99 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2$ | 2.55E-2 | 7.23E-3 | 1.89E-3 | 4.81E-4 | 1.21E-4 | 3.04E-5 | 7.67E-6 | 1.93E-6 |
| rate | - | 1.82 | 1.94 | 1.97 | 1.99 | 1.99 | 1.99 | 1.99 |
| $\varepsilon = 1/2^2$ | 3.37E-2 | 1.46E-2 | 4.63E-3 | 1.23E-3 | 3.13E-4 | 7.86E-5 | 1.97E-5 | 4.96E-6 |
| rate | - | 1.22 | 1.65 | 1.91 | 1.98 | 1.99 | 1.99 | 1.99 |
| $\varepsilon = 1/2^3$ | 1.87E-2 | 7.82E-3 | 3.15E-3 | 1.42E-3 | 4.13E-4 | 1.06E-4 | 2.66E-5 | 6.69E-6 |
| rate | - | 1.26 | 1.31 | 1.14 | 1.79 | 1.96 | 1.99 | 2.00 |
| $\varepsilon = 1/2^4$ | 6.98E-3 | 4.70E-3 | 2.83E-3 | 1.12E-3 | 4.92E-4 | 1.66E-4 | 4.32E-5 | 1.09E-5 |
| rate | - | 0.57 | 0.73 | 1.33 | 1.19 | 1.57 | 1.94 | 1.99 |
| $\varepsilon = 1/2^5$ | 8.84E-3 | 2.36E-3 | 1.29E-3 | 9.12E-4 | 4.80E-4 | 1.73E-4 | 7.42E-5 | 2.07E-5 |
| rate | - | 1.90 | 0.87 | 0.51 | 0.93 | 1.47 | 1.22 | 1.84 |
| $\varepsilon = 1/2^6$ | 9.14E-3 | 3.26E-3 | 7.40E-4 | 3.44E-4 | 2.67E-4 | 1.77E-4 | 7.04E-5 | 2.99E-5 |
| rate | - | 1.49 | 2.14 | 1.10 | 0.36 | 0.59 | 1.33 | 1.23 |
| $\varepsilon = 1/2^7$ | 8.79E-3 | 2.07E-3 | 7.42E-4 | 1.70E-4 | 8.99E-5 | 7.41E-5 | 5.66E-5 | 3.01E-5 |
| rate | - | 2.08 | 1.48 | 2.12 | 0.92 | 0.28 | 0.39 | 0.91 |
| $\varepsilon = 1/2^8$ | 8.67E-3 | 1.67E-3 | 6.51E-4 | 1.36E-4 | 4.34E-5 | 2.32E-5 | 1.99E-5 | 1.66E-5 |
| rate | - | 2.38 | 1.36 | 2.26 | 1.65 | 0.90 | 0.22 | 0.26 |
| $\varepsilon = 1/2^9$ | 8.64E-3 | 1.66E-3 | 4.15E-4 | 2.01E-4 | 4.30E-5 | 1.09E-5 | 5.95E-6 | 5.24E-6 |
| rate | - | 2.38 | 2.00 | 1.05 | 2.23 | 1.98 | 0.87 | 0.18 |
| $\varepsilon = 1/2^{10}$ | 8.63E-3 | 1.68E-3 | 3.90E-4 | 1.23E-4 | 4.58E-5 | 9.80E-6 | 2.73E-6 | 1.52E-6 |
| rate | - | 2.36 | 2.10 | 1.67 | 1.42 | 2.22 | 1.84 | 0.85 |

rate and/or high order perfect matched layer [11], transparent boundary condition [21, 23], or absorbing boundary condition [20] for the wave-type equations in (2.9) and (2.8) (or (2.3) and (2.2)) during the truncation, then one can choose the bounded computational domain as $\varepsilon$-independent, which can be significantly smaller compared to $\Omega_\varepsilon$ in the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$.

In order to quantify the numerical errors, we introduce the following error functions:

$$e^\varepsilon(t_k) := \frac{\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\|}{\|E^\varepsilon(\cdot,t_k)\|_{H^1}}, \quad n^\varepsilon(t_k) := \frac{\|N^\varepsilon(\cdot,t_k) - N^{\varepsilon,k}\|}{\|N^\varepsilon(\cdot,t_k)\|_{L^2}}, \qquad k \geq 0,$$

TABLE 3
*Temporal error analysis at time $t = 1$ for Case* II*, i.e.,* $\alpha = \beta = 0$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ | $\tau_0/2^6$ | $\tau_0/2^7$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1/2$ | 5.12E-2 | 1.96E-2 | 7.42E-3 | 2.20E-3 | 5.72E-4 | 1.44E-4 | 3.61E-5 | 9.09E-6 |
| rate | - | 1.38 | 1.40 | 1.76 | 1.94 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 4.98E-2 | 1.99E-2 | 7.23E-3 | 2.17E-3 | 5.68E-4 | 1.43E-4 | 3.58E-5 | 9.04E-6 |
| rate | - | 1.33 | 1.46 | 1.73 | 1.94 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 3.76E-2 | 1.95E-2 | 7.50E-3 | 2.22E-3 | 5.81E-4 | 1.47E-4 | 3.67E-5 | 9.23E-6 |
| rate | - | 0.95 | 1.38 | 1.75 | 1.94 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^4$ | 3.35E-2 | 1.74E-2 | 6.42E-3 | 1.81E-3 | 4.64E-4 | 1.17E-4 | 2.92E-5 | 7.36E-6 |
| rate | - | 0.95 | 1.44 | 1.83 | 1.96 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^5$ | 3.34E-2 | 1.70E-2 | 6.42E-3 | 1.83E-3 | 4.73E-4 | 1.19E-4 | 2.98E-5 | 7.46E-6 |
| rate | - | 0.97 | 1.41 | 1.81 | 1.96 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2^6$ | 3.31E-2 | 1.71E-2 | 6.39E-3 | 1.83E-3 | 4.72E-4 | 1.19E-4 | 2.98E-5 | 7.47E-6 |
| rate | - | 0.95 | 1.42 | 1.81 | 1.95 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2^7$ | 3.29E-2 | 1.71E-2 | 6.50E-3 | 1.85E-3 | 4.70E-4 | 1.18E-4 | 2.97E-5 | 7.44E-6 |
| rate | - | 0.94 | 1.40 | 1.81 | 1.97 | 1.99 | 1.99 | 2.00 |
| $\varepsilon = 1/2^8$ | 3.28E-2 | 1.71E-2 | 6.52E-3 | 1.97E-3 | 5.06E-4 | 1.21E-4 | 2.96E-5 | 7.39E-6 |
| rate | - | 0.94 | 1.39 | 1.72 | 1.96 | 2.07 | 2.03 | 2.00 |
| $\varepsilon = 1/2^9$ | 3.28E-2 | 1.71E-2 | 6.50E-3 | 2.00E-3 | 6.21E-4 | 1.56E-4 | 3.33E-5 | 7.57E-6 |
| rate | - | 0.94 | 1.39 | 1.70 | 1.69 | 2.00 | 2.23 | 2.14 |
| $\varepsilon = 1/2^{10}$ | 3.28E-2 | 1.71E-2 | 6.50E-3 | 1.99E-3 | 6.47E-4 | 2.39E-4 | 5.95E-5 | 1.10E-5 |
| rate | - | 0.94 | 1.39 | 1.70 | 1.62 | 1.44 | 2.00 | 2.44 |

| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ | $\tau_0/2^6$ | $\tau_0/2^7$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1/2$ | 2.64E-2 | 7.43E-3 | 1.94E-3 | 4.94E-4 | 1.25E-4 | 3.13E-5 | 7.85E-6 | 1.98E-6 |
| rate | - | 1.83 | 1.94 | 1.97 | 1.99 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 2.57E-2 | 1.02E-2 | 3.19E-3 | 8.48E-4 | 2.16E-4 | 5.42E-5 | 1.35E-5 | 3.41E-6 |
| rate | - | 1.33 | 1.68 | 1.91 | 1.98 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 1.56E-2 | 7.53E-3 | 2.62E-3 | 9.26E-4 | 2.61E-4 | 6.67E-5 | 1.68E-5 | 4.21E-6 |
| rate | - | 1.05 | 1.52 | 1.50 | 1.83 | 1.97 | 1.99 | 2.00 |
| $\varepsilon = 1/2^4$ | 4.48E-3 | 3.68E-3 | 2.64E-3 | 1.15E-3 | 3.65E-4 | 1.09E-4 | 2.82E-5 | 7.07E-6 |
| rate | - | 0.28 | 0.48 | 1.20 | 1.66 | 1.74 | 1.96 | 1.99 |
| $\varepsilon = 1/2^5$ | 4.10E-3 | 1.67E-3 | 9.58E-4 | 7.92E-4 | 4.83E-4 | 1.61E-4 | 5.07E-5 | 1.36E-5 |
| rate | - | 1.29 | 0.80 | 0.27 | 0.72 | 1.58 | 1.67 | 1.90 |
| $\varepsilon = 1/2^6$ | 5.46E-3 | 3.18E-3 | 5.99E-4 | 2.49E-4 | 2.13E-4 | 1.68E-4 | 7.45E-5 | 2.27E-5 |
| rate | - | 0.78 | 2.41 | 1.26 | 0.22 | 0.35 | 1.17 | 1.71 |
| $\varepsilon = 1/2^7$ | 5.91E-3 | 2.51E-3 | 8.90E-4 | 1.99E-4 | 6.73E-5 | 5.55E-5 | 4.95E-5 | 3.07E-5 |
| rate | - | 1.24 | 1.49 | 2.16 | 1.57 | 0.28 | 0.17 | 0.69 |
| $\varepsilon = 1/2^8$ | 6.04E-3 | 1.90E-3 | 1.01E-3 | 3.19E-4 | 8.42E-5 | 2.01E-5 | 1.44E-5 | 1.33E-5 |
| rate | - | 1.67 | 0.91 | 1.67 | 1.92 | 2.07 | 0.48 | 0.11 |
| $\varepsilon = 1/2^9$ | 6.07E-3 | 1.73E-3 | 7.98E-4 | 4.07E-4 | 1.58E-4 | 3.97E-5 | 7.28E-6 | 3.78E-6 |
| rate | - | 1.81 | 1.12 | 0.97 | 1.36 | 1.99 | 2.45 | 0.95 |
| $\varepsilon = 1/2^{10}$ | 6.08E-3 | 1.69E-3 | 7.04E-4 | 3.69E-4 | 1.68E-4 | 7.75E-5 | 1.95E-5 | 3.18E-6 |
| rate | - | 1.85 | 1.26 | 0.93 | 1.14 | 1.12 | 1.99 | 2.62 |

where $e_j^{\varepsilon,k} = E^\varepsilon(x_j, t_k) - E_j^{\varepsilon,k}$ and $N_j^{\varepsilon,k} = -|E_j^{\varepsilon,k}|^2 + F_j^{\varepsilon,k} + G^\varepsilon(x_j, t_k)$ for $0 \leq j \leq M$. The "exact" solution is obtained by the time-splitting spectral method [9] with very small mesh size $h = 1/64$ and time step $\tau = 10^{-6}$.

Table 1 depicts the spatial errors at $t = 1$ with a fixed time step $\tau = 10^{-5}$ and Case II initial data for different mesh sizes $h$ and $0 < \varepsilon \leq 1$. It clearly demonstrates that our new finite difference method is uniformly second order accurate in space for all $\varepsilon \in (0, 1]$. The results for other initial data are analogous, e.g., different $\alpha \geq 0$ and $\beta \geq -1$, and are thus omitted for brevity.

TABLE 4

*Temporal error analysis at time $t = 1$ for well-prepared and ill-prepared initial data in the resonance regions with different $\tau$ and $\varepsilon$.*

| Case I<br>$\tau = O(\varepsilon^{3/2})$ | $\varepsilon_0 = 1/2$<br>$\tau_0 = 0.1$ | $\varepsilon_0/2^2$<br>$\tau_0/2^3$ | $\varepsilon_0/2^4$<br>$\tau_0/2^6$ | $\varepsilon_0/2^6$<br>$\tau_0/2^9$ |
|---|---|---|---|---|
| $n^\varepsilon(t = 1)$ | 2.55E-2 | 1.42E-3 | 7.42E-5 | 4.34E-6 |
| rate in time | - | 4.17/3 | 4.26/3 | 4.09/3 |

| Case II<br>$\tau = O(\varepsilon)$ | $\varepsilon_0 = 1/2$<br>$\tau_0 = 0.1/2^2$ | $\varepsilon_0/2$<br>$\tau_0/2$ | $\varepsilon_0/2^2$<br>$\tau_0/2^2$ | $\varepsilon_0/2^3$<br>$\tau_0/2^3$ | $\varepsilon_0/2^4$<br>$\tau_0/2^4$ | $\varepsilon_0/2^5$<br>$\tau_0/2^5$ |
|---|---|---|---|---|---|---|
| $n^\varepsilon(t = 1)$ | 1.94E-3 | 8.48E-4 | 2.61E-4 | 1.09E-4 | 5.07E-5 | 2.27E-5 |
| rate in time | - | 1.19 | 1.70 | 1.26 | 1.10 | 1.16 |

Table 2 presents the temporal errors at $t = 1$ with a fixed mesh size $h = 2.5 \times 10^{-4}$ and Case I initial data for different time steps $\tau$ and $0 < \varepsilon \leq 1$, and respectively, Table 3 depicts similar results for the Case II initial data.

From Tables 2 and 3, we can see that our numerical method is "essentially" second order in time for any fixed $0 < \varepsilon \leq 1$ for both well-prepared and ill-prepared initial data. In fact, for each fixed $0 < \varepsilon \leq 1$, second order convergence in time is observed for $0 < \tau \leq \tau_0$ with $\tau_0 > 0$ independent of $\varepsilon$ except for a small resonance region (cf. each row in Tables 2 and 3), e.g., at $\tau = O(\varepsilon^{3/2})$ for the well-prepared Case I initial data and at $\tau = O(\varepsilon)$ for the ill-prepared Case II initial data. In fact, for the well-prepared Case I initial data, in the resonance region $\tau = O(\varepsilon^{3/2})$, the convergence rate is downgraded to 4/3; and respectively, for the ill-prepared Case II initial data, it is downgraded to first order in the resonance region $\tau = O(\varepsilon)$; these results are listed in Table 4.

## 5. Conclusion.

A uniformly accurate finite difference method was presented for the Zakharov system (ZS) with a dimensionless parameter $0 < \varepsilon \leq 1$ which is inversely proportional to the speed of sound. When $0 < \varepsilon \ll 1$, i.e., in the subsonic limit regime, the solution of the ZS propagates highly oscillatory waves in time and/or rapid outgoing waves in space. Our method was designed by reformulating the ZS into an asymptotic consistent formulation and applying an integral approximation of the oscillating term. Two error bounds were established by using the energy method and the limiting equation, respectively, which depend explicitly on the mesh size $h$ and the time step $\tau$ as well as the parameter $0 < \varepsilon \leq 1$. From the two error bounds, uniform error estimates were obtained for $0 < \varepsilon \leq 1$. Numerical results were reported to demonstrate that the error bounds are sharp.

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