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ERROR ESTIMATES OF A REGULARIZED FINITE DIFFERENCE METHOD FOR THE LOGARITHMIC SCHRÖDINGER EQUATION*

WEIZHU BAO†, RÉMI CARLES‡, CHUNMEI SU§, AND QINGLIN TANG¶

Abstract. We present a regularized finite difference method for the logarithmic Schrödinger equation (LogSE) and establish its error bound. Due to the blowup of the logarithmic nonlinearity, i.e., $\ln \rho \to -\infty$ when $\rho \to 0^+$ with $\rho = |u|^2$ being the density and u being the complex-valued wave function or order parameter, there are significant difficulties in designing numerical methods and establishing their error bounds for the LogSE. In order to suppress the roundoff error and to avoid blowup, a regularized LogSE (RLogSE) is proposed with a small regularization parameter $0 < \varepsilon \ll 1$ and linear convergence is established between the solutions of RLogSE and LogSE in term of ε. Then a semi-implicit finite difference method is presented for discretizing the RLogSE and error estimates are established in terms of the mesh size h and time step τ as well as the small regularization parameter ε. Finally numerical results are reported to illustrate our error bounds.

Key words. logarithmic Schrödinger equation, logarithmic nonlinearity, regularized logarithmic Schrödinger equation, semi-implicit finite difference method, error estimates, convergence rate

AMS subject classifications. 35Q40, 35Q55, 65M15, 81Q05

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1. Introduction.

We consider the logarithmic Schrödinger equation (LogSE) which arises in a model of nonlinear wave mechanics (cf. [7]),

$$\begin{cases} i\partial_t u(\mathbf{x},t) + \Delta u(\mathbf{x},t) = \lambda u(\mathbf{x},t)\ \ln(|u(\mathbf{x},t)|^2), & \mathbf{x}\in\Omega, \quad t>0,\\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}, \end{cases} \tag{1.1}$$

where t is time, $\mathbf{x} \in \mathbb{R}^d$ $(d = 1,2,3)$ is the spatial coordinate, $\lambda \in \mathbb{R}\backslash\{0\}$ measures the force of the nonlinear interaction, $u := u(\mathbf{x},t) \in \mathbb{C}$ is the dimensionless wave function or order parameter, and $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition[1] fixed on the boundary. It admits applications to quantum mechanics [7, 8], quantum optics [9, 21], nuclear physics [18], transport and diffusion phenomena [16, 23], open quantum systems [19, 27], effective quantum gravity [28], theory of superfluidity, and Bose–Einstein condensation [3]. The LogSE enjoys three conservation laws, *mass*, *momentum*, and *energy* [12, 13], like in the case of the nonlinear Schrödinger equation with a power-like nonlinearity (e.g.,

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†Department of Mathematics, National University of Singapore, Singapore 119076 (matbaowz@nus.edu.sg, http://www.math.nus.edu.sg/~bao/).
‡CNRS, IRMAR, Université de Rennes 1 & ENS Rennes, France (Remi.Carles@math.cnrs.fr, http://carles.perso.math.cnrs.fr).
§Corresponding author. Department of Mathematics, University of Innsbruck, Innsbruck 6020, Austria (sucm13@163.com).
¶Co-corresponding author. School of Mathematics, State Key Laboratory of Hydraulics and Mountain River Engineering, Sichuan University, Chengdu 610064, People's Republic of China (qinglin_tang@scu.edu.cn).
[1]Whenever we consider this case, it is assumed that the boundary is Lipschitz continuous.

cubic):

$$
\begin{aligned}
M(t) &:= \|u(\cdot,t)\|^2_{L^2(\Omega)} = \int_\Omega |u(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x} = M(0),\\
P(t) &:= \operatorname{Im} \int_\Omega \overline{u}(\mathbf{x},t)\nabla u(\mathbf{x},t) d\mathbf{x} \equiv \operatorname{Im} \int_\Omega \overline{u_0}(\mathbf{x})\nabla u_0(\mathbf{x}) d\mathbf{x} = P(0), \quad t \geq 0,\\
E(t) &:= \int_\Omega \left[|\nabla u(\mathbf{x},t)|^2 d\mathbf{x} + \lambda F(|u(\mathbf{x},t)|^2)\right] d\mathbf{x}\\
&\equiv \int_\Omega \left[|\nabla u_0(\mathbf{x})|^2 + \lambda F(|u_0(\mathbf{x})|^2)\right] d\mathbf{x} = E(0),
\end{aligned}
\tag{1.2}
$$

where $\operatorname{Im} f$ and $\overline{f}$ denote the imaginary part and complex conjugate of f, respectively, and

$$
F(\rho) = \int_0^\rho \ln(s) ds = \rho \ln \rho - \rho, \qquad \rho \geq 0. \tag{1.3}
$$

Note that the expression of F is the same as the usual entropy from gas dynamics (up to the sign, according to the community).

On a mathematical level, the logarithmic nonlinearity possesses several features that make it quite different from more standard nonlinear Schrödinger equations. First, the nonlinearity is not locally Lipschitz continuous because of the behavior of the logarithm function at the origin. Note that in view of numerical simulation, this singularity of the "nonlinear potential" $\lambda \ln(|u(\mathbf{x},t)|^2)$ makes the choice of a discretization quite delicate. The second aspect is that whichever the sign of λ, the nonlinear potential energy in E has no definite sign. In fact, whether the nonlinearity is repulsive/attractive (or defocusing/focusing) depends on both λ and the value of the density $\rho := \rho(\mathbf{x},t) = |u(\mathbf{x},t)|^2$. When $\lambda > 0$, then the nonlinearity $\lambda\rho \ln \rho$ is repulsive when $\rho > 1$ and, respectively, it is attractive when $0 < \rho < 1$. On the other hand, when $\lambda < 0$, then the nonlinearity $\lambda\rho \ln \rho$ is attractive when $\rho > 1$ and, respectively, it is repulsive when $0 < \rho < 1$. Therefore, solving the Cauchy problem for (1.1) is not a trivial issue, and constructing solutions which are defined for all time requires some work; see [10, 13, 15]. Essentially, the outcome is that if u_0 belongs to (a subset of) $H^1(\Omega)$, (1.1) has a unique, global solution, regardless of the space dimension d (see also Theorem 2.2 below).

Next, the large time behavior reveals new phenomena. A first remark suggests that nonlinear effects are weak. Indeed, unlike what happens in the case of a homogeneous nonlinearity (classically in the form $\lambda|u|^p u$), replacing u with ku ($k \in \mathbb{C} \setminus \{0\}$) in (1.1) has only little effect, since we have

$$
i\partial_t(ku) + \Delta(ku) = \lambda ku \ln\left(|ku|^2\right) - \lambda(\ln|k|^2)ku\,.
$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$
ku(\mathbf{x},t)e^{-it\lambda \ln|k|^2}
$$

solves (1.1) (with initial datum ku_0). In particular, the size of the initial datum does not influence the dynamics of the solution. In spite of this property which is reminiscent of linear equations, nonlinear effects are stronger in (1.1) than in, say, cubic Schrödinger equations in several respects. For $\Omega = \mathbb{R}^d$, it was established in [11] that in the case $\lambda < 0$, no solution is dispersive (not even for small data, in view of the above remark), while if $\lambda > 0$, the results from [10] show that every solution disperses, at a faster rate than for the linear equation.

In view of the gauge invariance of the nonlinearity, for $\Omega = \mathbb{R}^d$, (1.1) enjoys the standard Galilean invariance: if $u(\mathbf{x}, t)$ solves (1.1), then, for any $\mathbf{v} \in \mathbb{R}^d$, so does

$$u(\mathbf{x} - 2\mathbf{v}t, t)e^{i\mathbf{v}\cdot\mathbf{x} - i|\mathbf{v}|^2 t}.$$

A remarkable feature of (1.1) is that it possesses a large set of explicit solutions. In the case $\Omega = \mathbb{R}^d$: if u_0 is Gaussian, $u(\cdot, t)$ is Gaussian for all time, and solving (1.1) amounts to solving ordinary differential equations [7]. For simplicity of notation, we take the one-dimensional case as an example. If the initial data in (1.1) with $\Omega = \mathbb{R}$ is taken as

$$u_0(x) = b_0 e^{-\frac{a_0}{2}x^2 + ivx}, \qquad x \in \mathbb{R},$$

where $a_0, b_0 \in \mathbb{C}$ and $v \in \mathbb{R}$ are given constants satisfying $\alpha_0 := \operatorname{Re} a_0 > 0$ with $\operatorname{Re} f$ denoting the real part of f, then the solution of (1.1) is given by [2, 10]

$$u(x,t) = \frac{b_0}{\sqrt{r(t)}} e^{i(vx - v^2 t) + Y(x - 2vt, t)}, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.4}$$

with

$$Y(x,t) = -i\phi(t) - \alpha_0 \frac{x^2}{2r(t)^2} + i\frac{\dot{r}(t)}{r(t)}\frac{x^2}{4}, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.5}$$

where $\phi := \phi(t) \in \mathbb{R}$ and $r := r(t) > 0$ solve the ODEs [2, 10]

$$\begin{aligned} \dot{\phi} &= \frac{\alpha_0}{r^2} + \lambda \ln |b_0|^2 - \lambda \ln r, \quad \phi(0) = 0, \\ \ddot{r} &= \frac{4\alpha_0^2}{r^3} + \frac{4\lambda\alpha_0}{r}, \quad r(0) = 1, \; \dot{r}(0) = -2 \operatorname{Im} a_0. \end{aligned} \tag{1.6}$$

In the case $\lambda < 0$, the function r is (time) periodic (in agreement with the absence of dispersive effects). In particular, if $a_0 = -\lambda > 0$, it follows from (1.6) that $r(t) \equiv 1$ and $\phi(t) = \phi_0 t$ with $\phi_0 = \lambda\left[\ln(|b_0|^2) - 1\right]$, which generates the uniformly *moving Gausson* as [2, 10]

$$u(x,t) = b_0 e^{\frac{\lambda}{2}(x-2vt)^2 + i(vx - (\phi_0 + v^2)t)}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.7}$$

As a very special case with $b_0 = e^{1/2}$ such that $\phi_0 = 0$ and moreover $v = 0$, one can get the *static Gausson* as

$$u(x,t) = e^{1/2} e^{\lambda |x|^2/2}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.8}$$

This special solution is orbitally stable [11, 14]. On the other hand, in the case $\lambda > 0$, it is proven in [10] that for general initial data (not necessarily Gaussian), there exists a universal dynamics. For extensions to higher dimensions, we refer to [2, 10] and references therein. Therefore, (1.1) possesses several specific features, which make it quite different from the nonlinear Schrödinger equation.

Different numerical methods have been proposed and analyzed for the nonlinear Schrödinger equation with smooth nonlinearity (e.g., cubic nonlinearity) in the literature, such as the finite difference methods [4, 5], finite element methods [1, 20], and the time-splitting pseudospectral methods [6, 25]. However, they cannot be applied to the LogSE (1.1) directly due to the blowup of the logarithmic nonlinearity, i.e., $\ln\rho \to -\infty$ when $\rho \to 0^+$. The main aim of this paper is to present a regularized finite difference method for the LogSE (1.1) by introducing a proper regularized LogSE (RLogSE) and then discretizing the RLogSE via a semi-implicit finite difference method. Error estimates will be established between the solutions of LogSE and RLogSE as well as their numerical approximations.

The rest of the paper is organized as follows. In section 2, we propose a regularized version of (1.1) with a small regularization parameter $0 < \varepsilon \ll 1$, and analyze its properties, as well as the convergence of its solution to the solution of (1.1). In section 3, we introduce a semi-implicit finite difference method for discretizing the RLogSE, and prove an error estimate, in which the dependence of the constants with respect to the regularization parameter ε is tracked very explicitly. Finally, numerical results are provided in section 4 to confirm our error bounds and to demonstrate the efficiency and accuracy of the proposed numerical method.

Throughout the paper, we use $H^m(\Omega)$ and $\|\cdot\|_{H^m(\Omega)}$ to denote the standard Sobolev spaces and their norms, respectively. In particular, the norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_{L^2(\Omega)}$ and $(\cdot,\cdot)$, respectively. Moreover, we adopt $A \lesssim B$ to mean that there exists a generic constant $C > 0$ independent of the regularization parameter ε, the time step τ, and the mesh size h such that $A \le C\,B$, and $\lesssim_c$ means the constant C depends on c.

2. A regularized logarithmic Schrödinger equation. It turns out that a direct simulation of the solution of (1.1) is very delicate, due to the singularity of the logarithm at the origin, as discussed in the introduction. Instead of working directly with (1.1), we shall consider the following RLogSE with a small regularized parameter $0 < \varepsilon \ll 1$ as

$$
\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) + \Delta u^\varepsilon(\mathbf{x},t) = \lambda u^\varepsilon(\mathbf{x},t)\,\ln\left(\varepsilon + |u^\varepsilon(\mathbf{x},t)|\right)^2, \quad \mathbf{x}\in\Omega, \quad t>0,\\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}. \end{cases} \tag{2.1}
$$

2.1. Conserved quantities. For the RLogSE (2.1), it can be similarly deduced that the mass, momentum, and energy are conserved.

PROPOSITION 2.1. *The mass, momentum, and "regularized" energy are formally conserved for the RLogSE* (2.1)*:*

$$
\begin{aligned}
M^\varepsilon(t) &:= \int_\Omega |u^\varepsilon(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x} = M(0),\\
P^\varepsilon(t) &:= \mathrm{Im}\int_\Omega \overline{u^\varepsilon}(\mathbf{x},t)\nabla u^\varepsilon(\mathbf{x},t)d\mathbf{x} \equiv \mathrm{Im}\int_\Omega \overline{u_0}(\mathbf{x})\nabla u_0(\mathbf{x})d\mathbf{x} = P(0), \quad t\ge 0,\\
E^\varepsilon(t) &:= \int_\Omega \left[|\nabla u^\varepsilon(\mathbf{x},t)|^2 + \lambda F_\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)\right]d\mathbf{x}\\
&\equiv \int_\Omega \left[|\nabla u_0(\mathbf{x})|^2 + \lambda F_\varepsilon(|u_0(\mathbf{x})|^2)\right]d\mathbf{x} = E^\varepsilon(0),
\end{aligned} \tag{2.2}
$$

where

$$
\begin{aligned}
F_\varepsilon(\rho) &= \int_0^\rho \ln(\varepsilon+\sqrt{s})^2 ds\\
&= \rho\ln\left(\varepsilon+\sqrt{\rho}\right)^2 - \rho + 2\varepsilon\sqrt{\rho} - \varepsilon^2\ln\left(1+\sqrt{\rho}/\varepsilon\right)^2, \qquad \rho\ge 0.
\end{aligned} \tag{2.3}
$$

Proof. The conservation for mass and momentum is standard, and relies on the fact that the right-hand side of (2.1) involves u^ε multiplied by a *real* number. For the energy $E^\varepsilon(t)$, we compute

$$\begin{aligned}\frac{d}{dt}E^\varepsilon(t) &= 2\,\mathrm{Re}\int_\Omega \left[\nabla u^\varepsilon\cdot\nabla\partial_t\overline{u^\varepsilon} + \lambda u^\varepsilon\partial_t\overline{u^\varepsilon}\ln(\varepsilon+|u^\varepsilon|)^2 - \lambda u^\varepsilon\partial_t\overline{u^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\ &\quad + 2\lambda\int_\Omega \partial_t|u^\varepsilon|\Big[\varepsilon + \frac{|u^\varepsilon|^2-\varepsilon^2}{\varepsilon+|u^\varepsilon|}\Big](\mathbf{x},t)d\mathbf{x}\\ &= 2\,\mathrm{Re}\int_\Omega\left[\partial_t\overline{u^\varepsilon}\left(-\Delta u^\varepsilon + \lambda u^\varepsilon\ln(\varepsilon+|u^\varepsilon|)^2\right)\right](\mathbf{x},t)d\mathbf{x}\\ &= 2\,\mathrm{Re}\int_\Omega i|\partial_t u^\varepsilon(\mathbf{x},t)|^2d\mathbf{x} = 0,\quad t\ge 0,\end{aligned}$$

which completes the proof. ∎

Note however that since the above regularized energy involves the L^1-norm of u^ε for any $\varepsilon>0$, E^ε is obviously well-defined for $u_0\in H^1(\Omega)$ when Ω has finite measure, but not when $\Omega=\mathbb{R}^d$. This aspect is discussed in more detail in subsections 2.3.3 and 2.4.

2.2. The Cauchy problem. For $\alpha>0$ and $\Omega=\mathbb{R}^d$, denote by L^2_α the weighted L^2 space

$$L^2_\alpha := \{v\in L^2(\mathbb{R}^d),\quad \mathbf{x}\longmapsto\langle\mathbf{x}\rangle^\alpha v(\mathbf{x})\in L^2(\mathbb{R}^d)\},$$

where $\langle\mathbf{x}\rangle := \sqrt{1+|\mathbf{x}|^2}$, with norm

$$\|v\|_{L^2_\alpha} := \|\langle\mathbf{x}\rangle^\alpha v(\mathbf{x})\|_{L^2(\mathbb{R}^d)}.$$

In the case where Ω is bounded, we simply set $L^2_\alpha = L^2(\Omega)$. Regarding the Cauchy problems (1.1) and (2.1), we have the following result.

THEOREM 2.2. *Let $\lambda\in\mathbb{R}$ and $0<\varepsilon\le 1$. Consider* (1.1) *and* (2.1), *with an initial datum $u_0\in H^1_0(\Omega)\cap L^2_\alpha$ for some $0<\alpha\le 1$.*

1. *Suppose that Ω is bounded, and consider* (1.1) *and* (2.1) *with homogeneous Dirichlet or periodic boundary condition:*
   - *There exists a unique, global weak solution $u\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1_0(\Omega))$ to* (1.1) *and a unique, global solution $u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1_0(\Omega))$ to* (2.1).
   - *If in addition $u_0\in H^2(\Omega)$, then $u,u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^2(\Omega))$.*
2. *Suppose that $\Omega=\mathbb{R}^d$:*
   - *There exists a unique, global weak solution $u\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{R}^d)\cap L^2_\alpha)$ to* (1.1) *and a unique, global solution $u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{R}^d)\cap L^2_\alpha)$ to* (2.1).
   - *If in addition $u_0\in H^2(\mathbb{R}^d)$, then $u,u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^2(\mathbb{R}^d))$.*
   - *If $u_0\in H^2(\mathbb{R}^d)\cap L^2_2$, then $u,u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^2(\mathbb{R}^d)\cap L^2_2)$.*

Remark 2.1. We emphasize that the solution to (1.1) thus constructed is a weak solution, because we rely on compactness arguments, from the regularized equation (2.1). It is rather surprising in these conditions that uniqueness is always granted. This was noticed already in [13], as recalled in the course of our proof. Note that in the recent paper [17], solutions to (1.1) are obtained in the case $\lambda<0$ as strong limits of approximating sequences, which are different from ours. The approach consists in replacing the logarithmic (entropy) function F by its Taylor expansion at ε on the interval $(0,\varepsilon)$, following the strategy from [13]. Note, however, that the arguments there rely on energy estimates bound to the case $\lambda<0$, and that this approximating procedure is more delicate to implement numerically than the use of (2.1).

Proof. This result can be proved by using more or less directly the arguments invoked in [10]. First, for fixed $\varepsilon>0$, the nonlinearity in (2.1) is locally Lipschitz, and grows more slowly than any power for large $|u^\varepsilon|$. Therefore, the standard Cauchy theory for nonlinear Schrödinger equations applies (see, in particular, [12, Corollary 3.3.11 and Theorem 3.4.1]), and so if $u_0\in H_0^1(\Omega)$, then (2.1) has a unique solution $u^\varepsilon\in L^\infty_{\rm loc}(\mathbb{R};H_0^1(\Omega))$. Higher Sobolev regularity is propagated, with controls depending on ε in general.

A solution u of (1.1) can be obtained by compactness arguments, by letting $\varepsilon\to 0$ in (2.1), provided that we have suitable bounds independent of $\varepsilon>0$. We have

$$i\partial_t\nabla u^\varepsilon+\Delta\nabla u^\varepsilon=2\lambda\ln\left(\varepsilon+|u^\varepsilon|\right)\nabla u^\varepsilon+2\lambda\frac{u^\varepsilon}{\varepsilon+|u^\varepsilon|}\nabla|u^\varepsilon|.$$

We first present the proof in the case where Ω is bounded, and then explain how to modify the arguments in the case where $\Omega=\mathbb{R}^d$.

The standard energy estimate (multiply the above equation by $\nabla\overline{u^\varepsilon}$, integrate over Ω, and take the imaginary part) yields, in the case of a periodic boundary condition,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}\le 2|\lambda|\int_\Omega\frac{|u^\varepsilon|}{\varepsilon+|u^\varepsilon|}\,|\nabla|u^\varepsilon||\,|\nabla u^\varepsilon|d\mathbf{x}\le 2|\lambda|\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}.$$

Gronwall's lemma yields a bound for u^ε in $L^\infty(0,T;H^1(\Omega))$, uniformly in $\varepsilon>0$, for any given $T>0$. Indeed, the above estimate uses the property

$$\mathrm{Im}\int_\Omega\nabla\overline{u^\varepsilon}\cdot\Delta\nabla u^\varepsilon\,d\mathbf{x}=0, \tag{2.4}$$

which needs not be true when u^ε satisfies homogeneous Dirichlet boundary conditions. In that case, we use the conservation of the energy E^ε (Proposition 2.1), and write

$$\begin{aligned}\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega)}&\le E^\varepsilon(u_0)+2|\lambda|\int_\Omega|u^\varepsilon(\mathbf{x},t)|^2\left|\ln\left(\varepsilon+|u^\varepsilon(\mathbf{x},t)|\right)\right|d\mathbf{x}\\&\quad+2\varepsilon|\lambda|\|u^\varepsilon(t)\|_{L^1(\Omega)}+2|\lambda|\varepsilon^2\int_\Omega\left|\ln\left(1+|u^\varepsilon(\mathbf{x},t)|/\varepsilon\right)\right|d\mathbf{x}\\&\lesssim 1+\varepsilon|\Omega|^{1/2}\|u^\varepsilon(t)\|_{L^2(\Omega)}+\int_\Omega|u^\varepsilon(\mathbf{x},t)|^2\left|\ln\left(\varepsilon+|u^\varepsilon(\mathbf{x},t)|\right)\right|d\mathbf{x}\\&\lesssim 1+\int_\Omega|u^\varepsilon(\mathbf{x},t)|^2\left|\ln\left(\varepsilon+|u^\varepsilon(\mathbf{x},t)|\right)\right|d\mathbf{x},\quad t\ge 0,\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the conservation of the mass $M^\varepsilon(t)$. Noticing that for $0<\eta\le 1$, we have

$$\ln x\le\frac{x^\eta}{\eta e},\quad x\ge 1;\qquad (1+x)^\eta\le 1+x^\eta,\quad x\ge 0,$$

which gives

$$
\begin{aligned}
&\int_\Omega |u^\varepsilon|^2 \left|\ln\left(\varepsilon+|u^\varepsilon|\right)\right| d\mathbf{x}\\
&\le \frac{1}{\eta e}\int_{\varepsilon+|u^\varepsilon|>1} |u^\varepsilon|^2\left(\varepsilon+|u^\varepsilon|\right)^\eta d\mathbf{x} + \frac{1}{\eta e}\int_{\varepsilon+|u^\varepsilon|<1}|u^\varepsilon|^2\left(\varepsilon+|u^\varepsilon|\right)^{-\eta} d\mathbf{x}\\
&\le \frac{1}{\eta e}\int_\Omega |u^\varepsilon|^2\left(1+|u^\varepsilon|\right)^\eta d\mathbf{x} + \frac{1}{\eta e}\int_\Omega |u^\varepsilon|^{2-\eta} d\mathbf{x}\\
&\le \frac{1}{\eta e}\left[\|u^\varepsilon\|^2_{L^2(\Omega)} + \|u^\varepsilon\|^{2+\eta}_{L^{2+\eta}(\Omega)} + |\Omega|^{\eta/2}\|u^\varepsilon\|^{2-\eta}_{L^2(\Omega)}\right] \lesssim 1 + \|\nabla u^\varepsilon\|^{d\eta/2}_{L^2(\Omega)},
\end{aligned}
$$

where for the last inequality we used the conservation of the mass, the fact that Ω is bounded, and the interpolation inequality (see, e.g., [24] and here we take $p = 2+\eta$)

$$
\|u\|_{L^p(\Omega)} \lesssim \|u\|^{1-\alpha}_{L^2(\Omega)}\|\nabla u\|^\alpha_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \quad \text{for} \quad \frac{1}{p} = \frac{1}{2} - \frac{\alpha}{d}, \quad 0 \le \alpha < 1.
$$

Thus we obtain again that u^ε is bounded in $L^\infty(0,T;H^1(\Omega))$, uniformly in $\varepsilon > 0$, for any given $T > 0$.

Compactness arguments show that u^ε converges to a solution u to (1.1); see [12, 13] for details (compactness in time follows from (2.1)). Uniqueness of such a solution for (1.1) follows from the arguments of [13], involving a specific algebraic inequality, generalized in Lemma 2.4 below. Note that at this stage, we know that u^ε converges to u by compactness arguments, so we have no convergence estimate. Such estimates are established in subsection 2.3.

To prove the propagation of the H^2 regularity, we note that differentiating the nonlinearity in (2.1) twice makes it unrealistic to expect direct bounds which are uniform in ε. To overcome this difficulty, the argument proposed in [10] relies on Kato's idea: instead of differentiating the equation twice in space, differentiate it once in time, and use the equation to infer H^2 regularity. This yields the propagation of the H^2 regularity.

When $\Omega = \mathbb{R}^d$, we can resume the above arguments: the identity (2.4) is valid on $\mathbb{R}^d$, so the uniform bound for ∇u^ε follows like in the periodic case. On the other hand, compactness in space is provided by multiplying (2.1) with $\langle \mathbf{x}\rangle^{2\alpha}\overline{u^\varepsilon}$ and integrating in space:

$$
\frac{d}{dt}\|u^\varepsilon\|^2_{L^2_\alpha} = 4\alpha \operatorname{Im}\int_{\mathbb{R}^d} \frac{\mathbf{x}\cdot\nabla u^\varepsilon}{\langle \mathbf{x}\rangle^{2-2\alpha}}\,\overline{u^\varepsilon}(t)\, d\mathbf{x} \lesssim \|\langle \mathbf{x}\rangle^{2\alpha-1} u^\varepsilon\|_{L^2(\mathbb{R}^d)}\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)},
$$

where we have used the Cauchy–Schwarz inequality. Recalling that $0 < \alpha \le 1$,

$$
\|\langle \mathbf{x}\rangle^{2\alpha-1} u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \|\langle \mathbf{x}\rangle^{\alpha} u^\varepsilon\|_{L^2(\mathbb{R}^d)} = \|u^\varepsilon\|_{L^2_\alpha},
$$

and we obtain a bound for u^ε in $L^\infty(0,T;H^1(\mathbb{R}^d)\cap L^2_\alpha)$ which is uniform in ε. The propagation of the H^2 regularity follows by using the same argument as in the case when Ω is bounded. To establish the last part of the theorem, we prove that $u \in L^\infty_{\text{loc}}(\mathbb{R};L^2_2)$ and the same approach applies to u^ε. It follows from (1.1) that

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|^2_{L^2_2} &= -2\operatorname{Im}\int_{\mathbb{R}^d}\langle \mathbf{x}\rangle^4\overline{u(\mathbf{x},t)}\Delta u(\mathbf{x},t)d\mathbf{x}\\
&= 8\operatorname{Im}\int_{\mathbb{R}^d}\langle \mathbf{x}\rangle^2\overline{u(\mathbf{x},t)}\,\mathbf{x}\cdot\nabla u(\mathbf{x},t)d\mathbf{x} \le 8\,\|u(t)\|_{L^2_2}\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{2.5}
$$

By the Cauchy–Schwarz inequality and integration by parts, we have

$$\begin{aligned}
&\|\mathbf{x}\cdot\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \sum_{j=1}^d\sum_{k=1}^d\int_{\mathbb{R}^d} x_j^2\frac{\partial u(\mathbf{x},t)}{\partial x_k}\frac{\partial\overline{u(\mathbf{x},t)}}{\partial x_k}d\mathbf{x}\\
&= -2\int_{\mathbb{R}^d}\overline{u(\mathbf{x},t)}\,\mathbf{x}\cdot\nabla u(\mathbf{x},t)d\mathbf{x} - \int_{\mathbb{R}^d}|\mathbf{x}|^2\overline{u(\mathbf{x},t)}\Delta u(\mathbf{x},t)d\mathbf{x}\\
&\le \frac{1}{2}\|\mathbf{x}\cdot\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\|u(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|u(t)\|^2_{L^2_2} + \frac{1}{2}\|\Delta u(t)\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which yields directly that

$$\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)} \le 2\|u(t)\|_{L^2(\mathbb{R}^d)} + \|u(t)\|_{L^2_2} + \|\Delta u(t)\|_{L^2(\mathbb{R}^d)}.$$

This together with (2.5) gives that

$$\frac{d}{dt}\|u(t)\|_{L^2_2} \le 4\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)} \le 4\|u(t)\|_{L^2_2} + 8\|u(t)\|_{L^2(\mathbb{R}^d)} + 4\|\Delta u(t)\|_{L^2(\mathbb{R}^d)}.$$

Since we already know that $u \in L^\infty_{\text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d))$, Gronwall's lemma completes the proof. ◻

Remark 2.2. We emphasize that if $u_0 \in H^k(\mathbb{R}^d)$, $k \ge 3$, we cannot guarantee in general that this higher regularity is propagated in (1.1), due to the singularities stemming from the logarithm. Still, this property is fulfilled in the case where u_0 is Gaussian, since then u remains Gaussian for all time. However, our numerical tests, in the case where the initial datum is chosen as the dark soliton of the cubic Schrödinger equation multiplied by a Gaussian, suggest that even the H^3 regularity is not propagated in general.

2.3. Convergence of the regularized model. In this subsection, we show the approximation property of the regularized model (2.1) to (1.1).

2.3.1. A general estimate. We prove the following.

Lemma 2.3. *Suppose the equation is set on Ω, where $\Omega = \mathbb{R}^d$, or $\Omega \subset \mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition, then we have the general estimate*

$$\frac{d}{dt}\|u^\varepsilon(t) - u(t)\|^2_{L^2(\Omega)} \le 4|\lambda|\left(\|u^\varepsilon(t) - u(t)\|^2_{L^2(\Omega)} + \varepsilon\|u^\varepsilon(t) - u(t)\|_{L^1(\Omega)}\right). \tag{2.6}$$

Before giving the proof of Lemma 2.3, we introduce the following lemma, which is a variant of [12, Lemma 9.3.5], established initially in [13, Lemma 1.1.1].

Lemma 2.4. *Let $\varepsilon \ge 0$ and denote $f_\varepsilon(z) = z\ln(\varepsilon + |z|)$, then we have*

$$|\text{Im}\left((f_\varepsilon(z_1) - f_\varepsilon(z_2))(\overline{z_1} - \overline{z_2})\right)| \le |z_1 - z_2|^2, \quad z_1, z_2 \in \mathbb{C}.$$

Proof. Notice that

$$\text{Im}\left[(f_\varepsilon(z_1) - f_\varepsilon(z_2))(\overline{z_1} - \overline{z_2})\right] = \frac{1}{2}\left[\ln(\varepsilon + |z_1|) - \ln(\varepsilon + |z_2|)\right]\text{Im}(\overline{z_1}z_2 - z_1\overline{z_2}).$$

Supposing, for example, $0 < |z_2| \le |z_1|$, we can obtain that

$$|\ln(\varepsilon + |z_1|) - \ln(\varepsilon + |z_2|)| = \ln\left(1 + \frac{|z_1| - |z_2|}{\varepsilon + |z_2|}\right) \le \frac{|z_1| - |z_2|}{\varepsilon + |z_2|} \le \frac{|z_1 - z_2|}{|z_2|}$$

and

$$|\mathrm{Im}(\overline{z_1}z_2 - z_1\overline{z_2})| = |z_2(\overline{z_1} - \overline{z_2}) + \overline{z_2}(z_2 - z_1)| \le 2|z_2|\,|z_1 - z_2|.$$

Otherwise the result follows by exchanging z_1 and z_2. ∎

Proof of Lemma 2.3. Subtracting (1.1) from (2.1), we see that the error function $e^\varepsilon := u^\varepsilon - u$ satisfies

$$i\partial_t e^\varepsilon + \Delta e^\varepsilon = \lambda\left[u^\varepsilon \ln(\varepsilon + |u^\varepsilon|)^2 - u\ln(|u|^2)\right].$$

Multiplying the error equation by $\overline{e^\varepsilon(t)}$, integrating in space, and taking the imaginary parts, we can get by using Lemma 2.4 that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e^\varepsilon(t)\|^2_{L^2(\Omega)} &= 2\lambda\,\mathrm{Im}\int_\Omega \left[u^\varepsilon \ln(\varepsilon + |u^\varepsilon|) - u\ln(|u|)\right](\overline{u^\varepsilon} - \overline{u})(\mathbf{x},t)d\mathbf{x}\\ &\le 2|\lambda|\|e^\varepsilon(t)\|^2_{L^2(\Omega)} + 2|\lambda|\left|\int_\Omega \overline{e^\varepsilon}u\left[\ln(\varepsilon + |u|) - \ln(|u|)\right](\mathbf{x},t)d\mathbf{x}\right|\\ &\le 2|\lambda|\|e^\varepsilon(t)\|^2_{L^2(\Omega)} + 2\varepsilon|\lambda|\|e^\varepsilon(t)\|_{L^1(\Omega)},\end{aligned}$$

where we have used the general estimate $0 \le \ln(1 + |x|) \le |x|$. ∎

2.3.2. Convergence for bounded domain. If Ω has finite measure, then we can have the following convergence behavior.

PROPOSITION 2.5. *Assume that Ω has finite measure, and let $u_0 \in H^2(\Omega)$. For any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \le C_1\varepsilon, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\Omega))} \le C_2\varepsilon^{1/2}, \tag{2.7}$$

where C_1 depends on $|\lambda|$, T, $|\Omega|$ and C_2 depends on $|\lambda|$, T, $|\Omega|$, and $\|u_0\|_{H^2(\Omega)}$.

Proof. Note that $\|e^\varepsilon(t)\|_{L^1(\Omega)} \le |\Omega|^{1/2}\|e^\varepsilon(t)\|_{L^2(\Omega)}$, then it follows from (2.6) that

$$\frac{d}{dt}\|e^\varepsilon(t)\|_{L^2(\Omega)} \le 2|\lambda|\|e^\varepsilon(t)\|_{L^2(\Omega)} + 2\varepsilon|\lambda||\Omega|^{1/2}.$$

Applying Gronwall's inequality, we immediately get that

$$\|e^\varepsilon(t)\|_{L^2(\Omega)} \le \left(\|e^\varepsilon(0)\|_{L^2(\Omega)} + \varepsilon|\Omega|^{1/2}\right)e^{2|\lambda|t} = \varepsilon|\Omega|^{1/2}e^{2|\lambda|t}.$$

The convergence rate in H^1 follows from the property $u^\varepsilon, u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2(\Omega))$ and the Gagliardo–Nirenberg inequality [22],

$$\|\nabla v\|_{L^2(\Omega)} \lesssim \|v\|^{1/2}_{L^2(\Omega)}\|\Delta v\|^{1/2}_{L^2(\Omega)},$$

which completes the proof. ∎

Remark 2.3. The weaker rate in the H^1 estimate is due to the fact that Lemma 2.3 is not easily adapted to H^1 estimates, because of the presence of the logarithm. Differentiating (1.1) and (2.1) makes it hard to obtain the analogue in Lemma 2.3. This is why we bypass this difficulty by invoking boundedness in H^2 and interpolating with the error bound at the L^2 level. If we have u^ε, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^k(\Omega))$ for $k > 2$, then the convergence rate in $H^1(\Omega)$ can be improved as

$$\|e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \lesssim \varepsilon^{\frac{k-1}{k}},$$

by using the inequality (see, e.g., [24])

$$\|v\|_{H^1(\Omega)} \lesssim \|v\|^{1-1/k}_{L^2(\Omega)}\|v\|^{1/k}_{H^k(\Omega)}.$$

2.3.3. Convergence for the whole space. In order to prove the convergence rate of the regularized model (2.1) to (1.1) for the whole space, we need the following lemma.

LEMMA 2.6. *For $d = 1, 2, 3$, if $v \in L^2(\mathbb{R}^d) \cap L^2_2$, then we have*

$$\|v\|_{L^1(\mathbb{R}^d)} \leq C_d \|v\|^{1-d/4}_{L^2(\mathbb{R}^d)} \|v\|^{d/4}_{L^2_2}, \tag{2.8}$$

where $C_d > 0$ depends on d.

Proof. Applying the Cauchy–Schwarz inequality, we can get for fixed $r > 0$,

$$\begin{aligned}\|v\|_{L^1(\mathbb{R}^d)} &= \int_{|\mathbf{x}|\leq r} |v(\mathbf{x})| d\mathbf{x} + \int_{|\mathbf{x}|\geq r} \frac{|\mathbf{x}|^2 |v(\mathbf{x})|}{|\mathbf{x}|^2} d\mathbf{x} \\ &\lesssim r^{d/2} \left(\int_{|\mathbf{x}|\leq r} |v(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}} + \left(\int_{|\mathbf{x}|\geq r} |\mathbf{x}|^4 |v(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}} \left(\int_{|\mathbf{x}|\geq r} \frac{1}{|\mathbf{x}|^4} d\mathbf{x} \right)^{\frac{1}{2}} \\ &\lesssim r^{d/2} \|v\|_{L^2(\mathbb{R}^d)} + r^{d/2-2} \|v\|_{L^2_2}.\end{aligned}$$

Then (2.8) can be obtained by setting $r = \left(\|v\|_{L^2_2} / \|v\|_{L^2(\mathbb{R}^d)} \right)^{1/2}$. ◻

PROPOSITION 2.7. *Assume that $\Omega = \mathbb{R}^d$, $1 \leq d \leq 3$, and let $u_0 \in H^2(\mathbb{R}^d) \cap L^2_2$. For any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \leq C_1 \varepsilon^{\frac{4}{4+d}}, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\mathbb{R}^d)))} \leq C_2 \varepsilon^{\frac{2}{4+d}},$$

where C_1 depends on d, $|\lambda|$, T, $\|u_0\|_{L^2_2}$ and C_2 depends on $\|u_0\|_{H^2(\mathbb{R}^d)}$ additionally.

Proof. Applying (2.8) and Young's inequality, we deduce that

$$\varepsilon \|e^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq \varepsilon C_d \|e^\varepsilon(t)\|^{1-d/4}_{L^2(\mathbb{R}^d)} \|e^\varepsilon(t)\|^{d/4}_{L^2_2} \leq C_d \left(\|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{\frac{8}{4+d}} \|e^\varepsilon(t)\|^{\frac{2d}{4+d}}_{L^2_2} \right),$$

which together with (2.6) gives that

$$\frac{d}{dt} \|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} \leq 4|\lambda|(1 + C_d) \|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} + 4C_d |\lambda| \varepsilon^{\frac{8}{4+d}} \|e^\varepsilon(t)\|^{\frac{2d}{4+d}}_{L^2_2}.$$

Gronwall's lemma yields

$$\|e^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon^{\frac{4}{4+d}} \|e^\varepsilon(t)\|^{\frac{d}{4+d}}_{L^2_2} e^{tC_{d,|\lambda|}}.$$

The proposition follows by recalling that u^ε, $u \in L^\infty_{\text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d) \cap L^2_2)$. ◻

Remark 2.4. If we have $u^\varepsilon, u \in L^\infty_{\text{loc}}(\mathbb{R}; L^2_m)$ for $m > 2$, then by applying the inequality

$$\varepsilon \|v\|_{L^1(\mathbb{R}^d)} \lesssim \varepsilon \|v\|^{1-\frac{d}{2m}}_{L^2(\mathbb{R}^d)} \|v\|^{\frac{d}{2m}}_{L^2_m} \lesssim \|v\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{\frac{4m}{2m+d}} \|v\|^{\frac{2d}{2m+d}}_{L^2_m},$$

which can be proved like (2.8), the convergence rate can be improved as

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \lesssim \varepsilon^{\frac{2m}{2m+d}}.$$

Remark 2.5. If, in addition, $u^\varepsilon, u \in L^\infty_{\rm loc}(\mathbb{R}; H^s(\mathbb{R}^d))$ for $s > 2$, then the convergence rate in $H^1(\mathbb{R}^d)$ can be improved as

$$\|e^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^d))} \le C\varepsilon^{\frac{2m}{2m+d}\frac{s-1}{s}}$$

by using the Gagliardo–Nirenberg inequality:

$$\|\nabla v\|_{L^2(\mathbb{R}^d)} \le C\|v\|^{1-1/s}_{L^2(\mathbb{R}^d)}\|\nabla^s v\|^{1/s}_{L^2(\mathbb{R}^d)}.$$

The previous two remarks apply typically in the case of Gaussian initial data.

2.4. Convergence of the energy. In this subsection we will show the convergence of the energy $E^\varepsilon(u_0) \to E(u_0)$.

PROPOSITION 2.8. *For $u_0 \in H^1(\Omega)\cap L^1(\Omega)$, the energy $E^\varepsilon(u_0)$ converges to $E(u_0)$ with*

$$|E^\varepsilon(u_0) - E(u_0)| \le 4\,\varepsilon|\lambda|\|u_0\|_{L^1(\Omega)}.$$

Proof. It can be deduced from the definition that

$$\begin{aligned}|E^\varepsilon(u_0) - E(u_0)| &= 2|\lambda|\left|\varepsilon\|u_0\|_{L^1(\Omega)} + \int_\Omega |u_0(\mathbf{x})|^2\left[\ln(\varepsilon + |u_0(\mathbf{x})| - \ln(|u_0(\mathbf{x})|)\right]d\mathbf{x}\right.\\ &\qquad\left. - \varepsilon^2\int_\Omega \ln\left(1 + |u_0(\mathbf{x})|/\varepsilon\right)d\mathbf{x}\right|\\ &\le 4\,\varepsilon|\lambda|\|u_0\|_{L^1(\Omega)},\end{aligned}$$

which completes the proof. ∎

Remark 2.6. If Ω is bounded, then $H^1(\Omega) \subseteq L^1(\Omega)$. If $\Omega = \mathbb{R}^d$, then Lemma 2.6 (and its natural generalizations) shows that $H^1(\mathbb{R}) \cap L^2_1 \subseteq L^1(\mathbb{R})$, and if $d = 2, 3$, $H^1(\mathbb{R}^d) \cap L^2_2 \subseteq L^1(\mathbb{R}^d)$.

Remark 2.7. This regularization is reminiscent of the one considered in [10] in order to prove (by compactness arguments) that (1.1) has a solution,

$$i\partial_t u^\varepsilon(\mathbf{x}, t) + \Delta u^\varepsilon(\mathbf{x}, t) = \lambda u^\varepsilon(\mathbf{x}, t)\,\ln\left(\varepsilon + |u^\varepsilon(\mathbf{x}, t)|^2\right), \quad \mathbf{x} \in \Omega, \quad t > 0. \tag{2.9}$$

With that regularization, it is easy to adapt the error estimates established above for (2.1). Essentially, ε must be replaced by $\sqrt{\varepsilon}$ (in Lemma 2.3 and, hence, in its corollaries).

3. A regularized semi-implicit finite difference method. In this section, we study the approximation properties of a finite difference method for solving the regularized model (2.1). For simplicity of notation, we set $\lambda = 1$ and only present the numerical method for the RLogSE (2.1) in one dimension, as extensions to higher dimensions are straightforward. When $d = 1$, we truncate the RLogSE on a bounded computational interval $\Omega = (a, b)$ with homogeneous Dirichlet boundary condition (here $|a|$ and b are chosen large enough such that the truncation error is negligible):

$$\begin{cases} i\partial_t u^\varepsilon(x,t) + \partial_{xx} u^\varepsilon(x,t) = u^\varepsilon(x,t)\,\ln(\varepsilon + |u^\varepsilon(x,t)|)^2, & x \in \Omega, \quad t > 0, \\ u^\varepsilon(x,0) = u_0(x), \quad x \in \overline{\Omega}; & u^\varepsilon(a,t) = u^\varepsilon(b,t) = 0, \quad t \geq 0, \end{cases} \tag{3.1}$$

3.1. A finite difference scheme and main results on error bounds. Choose a mesh size $h := \Delta x = (b-a)/M$ with M being a positive integer and a time step $\tau := \Delta t > 0$ and denote the grid points and time steps as

$$x_j := a + jh, \quad j = 0, 1, \ldots, M; \quad t_k := k\tau, \quad k = 0, 1, 2, \ldots.$$

Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1, 2, \ldots, M-1\}, \quad \mathcal{T}_M^0 = \{j \mid j = 0, 1, \ldots, M\}.$$

Let $u_j^{\varepsilon,k}$ be the approximation of $u^\varepsilon(x_j, t_k)$, and denote $u^{\varepsilon,k} = (u_0^{\varepsilon,k}, u_1^{\varepsilon,k}, \ldots, u_M^{\varepsilon,k})^T \in \mathbb{C}^{M+1}$ as the numerical solution vector at $t = t_k$. Define the standard finite difference operators

$$\delta_t^c u_j^k = \frac{u_j^{k+1} - u_j^{k-1}}{2\tau}, \quad \delta_x^+ u_j^k = \frac{u_{j+1}^k - u_j^k}{h}, \quad \delta_x^2 u_j^k = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2}.$$

Denote

$$X_M = \left\{ v = (v_0, v_1, \ldots, v_M)^T \mid v_0 = v_M = 0 \right\} \subseteq \mathbb{C}^{M+1},$$

equipped with inner products and norms defined as (recall that $u_0 = v_0 = u_M = v_M = 0$ by the Dirichlet boundary condition)

$$\begin{aligned} &(u, v) = h \sum_{j=1}^{M-1} u_j \overline{v_j}, \quad \langle u, v \rangle = h \sum_{j=0}^{M-1} u_j \overline{v_j}, \quad \|u\|_\infty = \sup_{j \in \mathcal{T}_M^0} |u_j|, \\ &\|u\|^2 = (u, u), \quad |u|_{H^1}^2 = \left\langle \delta_x^+ u, \delta_x^+ u \right\rangle, \quad \|u\|_{H^1}^2 = \|u\|^2 + |u|_{H^1}^2. \end{aligned} \tag{3.2}$$

Then we have for u, $v \in X_M$,

$$(-\delta_x^2 u, v) = \left\langle \delta_x^+ u, \delta_x^+ v \right\rangle = (u, -\delta_x^2 v). \tag{3.3}$$

Consider a semi-implicit finite difference (SIFD) discretization of (3.1) as following

$$i\delta_t^c u_j^{\varepsilon,k} = -\frac{1}{2}\delta_x^2 (u_j^{\varepsilon,k+1} + u_j^{\varepsilon,k-1}) + u_j^{\varepsilon,k} \ln(\varepsilon + |u_j^{\varepsilon,k}|)^2, \quad j \in \mathcal{T}_M, \quad k \geq 1. \tag{3.4}$$

The boundary and initial conditions are discretized as

$$u_0^{\varepsilon,k} = u_M^{\varepsilon,k} = 0, \quad k \geq 0, \quad u_j^{\varepsilon,0} = u_0(x_j), \quad j \in \mathcal{T}_M^0. \tag{3.5}$$

In addition, the first step $u_j^{\varepsilon,1}$ can be obtained via the Taylor expansion as

$$u_j^{\varepsilon,1} = u_j^{\varepsilon,0} + \tau u_1(x_j), \quad j \in \mathcal{T}_M^0, \tag{3.6}$$

where

$$u_1(x) := \partial_t u^\varepsilon(x, 0) = i\left[u_0''(x) - u_0(x)\ln(\varepsilon + |u_0(x)|)^2\right], \quad a \leq x \leq b.$$

Let $0 < T < T_{\max}$ with $T_{\max}$ the maximum existence time of the solution u^ε to the problem (3.1) for a fixed $0 \le \varepsilon \ll 1$. Similarly to the stability analysis of the SIFD method for the nonlinear Schrödinger equation by using the standard von Neumann analysis via a frozen coefficients technique [4, 5], we can show that the discretization (3.4) is conditionally stable under the stability condition

$$0 < \tau \le \frac{1}{2\max\{|\ln\varepsilon|, \ln(\varepsilon + \max_{j\in\mathcal{T}_M}|u_j^{\varepsilon,k}|)\}}, \qquad 0 \le k \le \frac{T}{\tau}. \tag{3.7}$$

Define the error functions $e^{\varepsilon,k} \in X_M$ as

$$e_j^{\varepsilon,k} = u^\varepsilon(x_j,t_k) - u_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}, \tag{3.8}$$

where u^ε is the solution of (3.1). Then we have the following error estimates for (3.4) with (3.5) and (3.6).

THEOREM 3.1 (main result). *Assume that the solution u^ε is smooth enough over $\Omega_T := \Omega \times [0,T]$, i.e.,*

$$u^\varepsilon \in C\left([0,T]; H^5(\Omega)\right) \cap C^2\left([0,T]; H^4(\Omega)\right) \cap C^3\left([0,T]; H^2(\Omega)\right), \tag{A}$$

and there exist $\varepsilon_0 > 0$ and $C_0 > 0$ independent of ε such that

$$\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} + \|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \le C_0,$$

uniformly in $0 \le \varepsilon \le \varepsilon_0$. Then there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small with $h_0^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$ and $\tau_0^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$ such that, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$ satisfying the stability condition (3.7)*, we have the following error estimates*

$$\begin{aligned} &\|e^{\varepsilon,k}\| \le C_3(\varepsilon,T)(h^2+\tau^2), \qquad 0 \le k \le \frac{T}{\tau}, \\ &\|e^{\varepsilon,k}\|_{H^1} \le C_4(\varepsilon,T)(h^2+\tau^2), \quad \|u^{\varepsilon,k}\|_\infty \le \Lambda + 1, \end{aligned} \tag{3.9}$$

where $\Lambda = \|u^\varepsilon\|_{L^\infty(\Omega_T)}$, $C_3(\varepsilon,T) \sim e^{CT|\ln(\varepsilon)|^2}$, $C_4(\varepsilon,T) \sim \frac{1}{\varepsilon}e^{CT|\ln(\varepsilon)|^2}$, and C depends on C_0.

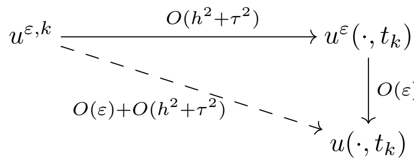
Remark 3.1. Note that the regularity assumptions made in the above statement supercede the regularity that we could establish in a theoretical level as in Theorem 2.2, where the exact solution u is not known to be better than H^2. This implies that in general, to apply the above result, we have to check (numerically) that the above bound for u^ε is indeed uniform with respect to ε. Note that in the Gaussian case, we already know that the above assumption is satisfied.

The error bounds in this theorem show not only the quadratical convergence in terms of the mesh size h and time step τ but also the explicit dependence on the regularization parameter ε. Here we remark that assumption (A) is valid at least in the case of taking the Gaussian as the initial datum.

Define the error functions $\widetilde{e}^{\varepsilon,k} \in X_M$ as

$$\widetilde{e}_j^{\varepsilon,k} = u(x_j,t_k) - u_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}, \tag{3.10}$$

where u^ε is the solution of the LogSE (1.1) with $\Omega = (a,b)$. Combining Proposition 2.5 and Theorem 3.1, we immediately obtain the following corollary (see an illustration in the following diagram)



Corollary 3.2. *Under the assumptions of Proposition* 2.5 *and Theorem* 3.1*, we have the following error estimates:*

$$\begin{aligned} &\|\widetilde{e}^{\varepsilon,k}\| \le C_1\varepsilon + C_3(\varepsilon,T)(h^2+\tau^2),\\ &\|\widetilde{e}^{\varepsilon,k}\|_{H^1} \le C_2\varepsilon^{1/2} + C_4(\varepsilon,T)(h^2+\tau^2), \qquad 0\le k\le \frac{T}{\tau}, \end{aligned} \tag{3.11}$$

where C_1 *and* C_2 *are presented as in Proposition* 2.5*, and* $C_3(\varepsilon,T)$ *and* $C_4(\varepsilon,T)$ *are given in Theorem* 3.1*.*

Remark 3.2. If the goal of a simulation is to study LogSE by using the scheme (3.4) with (3.5) and (3.6), then the error bound in Corollary 3.2 formally suggests the following choice of h and τ versus ε in order to balance the errors introduced by the regularization and spatial/temporal discretization: $h^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$ and $\tau^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$. In fact, our numerical results also suggest the errors will be saturated when $h^2\sim\varepsilon$ and $\tau^2\sim\varepsilon$, i.e., the errors will no longer decrease further as long as $h^2\sim\varepsilon$ and $\tau^2\sim\varepsilon$ for any fixed $0<\varepsilon\ll 1$ (cf. upper triangle in Table 4.1).

3.2. Error estimates. Define the local truncation error $\xi_j^{\varepsilon,k}\in X_M$ for $k\ge 1$ as

$$\begin{aligned} \xi_j^{\varepsilon,k} &= i\delta_t^c u^\varepsilon(x_j,t_k) + \frac{1}{2}\left(\delta_x^2 u^\varepsilon(x_j,t_{k+1}) + \delta_x^2 u^\varepsilon(x_j,t_{k-1})\right)\\ &\quad - u^\varepsilon(x_j,t_k)\ln(\varepsilon+|u^\varepsilon(x_j,t_k)|)^2, \qquad j\in\mathcal{T}_M, \quad 1\le k<\frac{T}{\tau}, \end{aligned} \tag{3.12}$$

then we have the following bounds for the local truncation error.

Lemma 3.3 (local truncation error). *Under assumption (A), we have*

$$\|\xi^{\varepsilon,k}\|_{H^1} \lesssim h^2+\tau^2, \quad 1\le k<\frac{T}{\tau}.$$

Proof. By the Taylor expansion, we have

$$\xi_j^{\varepsilon,k} = \frac{i\tau^2}{4}\alpha_j^{\varepsilon,k} + \frac{\tau^2}{2}\beta_j^{\varepsilon,k} + \frac{h^2}{12}\gamma_j^{\varepsilon,k}, \tag{3.13}$$

where

$$\begin{aligned} &\alpha_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)^2\partial_t^3 u^\varepsilon(x_j,t_k+s\tau)ds, \quad \beta_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)\partial_t^2 u_{xx}^\varepsilon(x_j,t_k+s\tau)ds,\\ &\gamma_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)^3\left(\partial_x^4 u^\varepsilon(x_j+sh,t_{k+1}) + \partial_x^4 u^\varepsilon(x_j+sh,t_{k-1})\right)ds. \end{aligned}$$

By the Cauchy–Schwarz inequality, we can get that

$$
\begin{aligned}
\|\alpha^{\varepsilon,k}\|^2 &= h\sum_{j=1}^{M-1}|\alpha_j^{\varepsilon,k}|^2 \le h\int_{-1}^{1}(1-|s|)^4 ds\sum_{j=1}^{M-1}\int_{-1}^{1}\left|\partial_t^3 u^\varepsilon(x_j,t_k+s\tau)\right|^2 ds\\
&= \frac{2}{5}\Big[\int_{-1}^{1}\|\partial_t^3 u^\varepsilon(\cdot,t_k+s\tau)\|_{L^2(\Omega)}^2 ds\\
&\qquad -\int_{-1}^{1}\sum_{j=0}^{M-1}\int_{x_j}^{x_{j+1}}(|\partial_t^3 u^\varepsilon(x,t_k+s\tau)|^2-|\partial_t^3 u^\varepsilon(x_j,t_k+s\tau)|^2)dxds\Big]\\
&= \frac{2}{5}\Big[\int_{-1}^{1}\|\partial_t^3 u^\varepsilon(\cdot,t_k+s\tau)\|_{L^2(\Omega)}^2 ds\\
&\qquad -\int_{-1}^{1}\sum_{j=0}^{M-1}\int_{x_j}^{x_{j+1}}\int_{x_j}^{\omega}\partial_x|\partial_t^3 u^\varepsilon(x',t_k+s\tau)|^2 dx'd\omega ds\Big]\\
&\le \frac{2}{5}\int_{-1}^{1}\Big[\|\partial_t^3 u^\varepsilon(\cdot,t_k+s\tau)\|_{L^2(\Omega)}^2\\
&\qquad\qquad +2h\|\partial_t^3 u_x^\varepsilon(\cdot,t_k+s\tau)\|_{L^2(\Omega)}\|\partial_t^3 u^\varepsilon(\cdot,t_k+s\tau)\|_{L^2(\Omega)}\Big]ds\\
&\le \max_{0\le t\le T}\left(\|\partial_t^3 u^\varepsilon\|_{L^2(\Omega)}+h\|\partial_t^3 u_x^\varepsilon\|_{L^2(\Omega)}\right)^2,
\end{aligned}
$$

which yields that when $h\le 1$,

$$\|\alpha^{\varepsilon,k}\|\le\|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}.$$

Applying a similar approach, it can be established that

$$\|\beta^{\varepsilon,k}\|\le 2\|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))}.$$

On the other hand, we can obtain that

$$
\begin{aligned}
\|\gamma^{\varepsilon,k}\|^2 &\le h\int_{-1}^{1}(1-|s|)^6 ds\sum_{j=1}^{M-1}\int_{-1}^{1}\left|\partial_x^4 u^\varepsilon(x_j+sh,t_{k+1})+\partial_x^4 u^\varepsilon(x_j+sh,t_{k-1})\right|^2 ds\\
&\le \frac{4h}{7}\sum_{j=1}^{M-1}\int_{-1}^{1}\left(\left|\partial_x^4 u^\varepsilon(x_j+sh,t_{k+1})\right|^2+\left|\partial_x^4 u^\varepsilon(x_j+sh,t_{k-1})\right|^2\right)ds\\
&\le \frac{8}{7}\left(\|\partial_x^4 u^\varepsilon(\cdot,t_{k-1})\|_{L^2(\Omega)}^2+\|\partial_x^4 u^\varepsilon(\cdot,t_{k+1})\|_{L^2(\Omega)}^2\right)\\
&\le 4\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}^2,
\end{aligned}
$$

which implies that $\|\gamma^{\varepsilon,k}\|\le 2\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}$. Hence by assumption (A), we get

$$
\begin{aligned}
\|\xi^{\varepsilon,k}\| &\lesssim \tau^2\left(\|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}+\|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))}\right)+h^2\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}\\
&\lesssim_{C_0}\tau^2+h^2.
\end{aligned}
$$

Applying δ_x^+ to $\xi^{\varepsilon,k}$ and using the same approach, we can get that

$$
\begin{aligned}
|\xi^{\varepsilon,k}|_{H^1} &\lesssim \tau^2\left(\|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))}+\|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}\right)+h^2\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))}\\
&\lesssim_{C_0}\tau^2+h^2,
\end{aligned}
$$

which completes the proof. ∎

For the first step, we have the following estimates.

LEMMA 3.4 (error bounds for $k = 1$). *Under assumption (A), the first step errors of the discretization* (3.6) *satisfy*

$$e^{\varepsilon,0} = 0, \quad \|e^{\varepsilon,1}\|_{H^1} \lesssim \tau^2.$$

Proof. By the definition of $u_j^{\varepsilon,1}$ in (3.6), we have

$$e_j^{\varepsilon,1} = \tau^2 \int_0^1 (1-s) u_{tt}^\varepsilon(x_j, s\tau) ds,$$

which implies that

$$\|e^{\varepsilon,1}\| \lesssim \tau^2 \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \lesssim \tau^2, \quad |e^{\varepsilon,1}|_{H^1} \lesssim \tau^2 \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \lesssim \tau^2,$$

and the proof is completed. ∎

Proof of Theorem 3.1. We prove (3.9) by induction. It follows from Lemma 3.4 that (3.9) is true for $k = 0, 1$.

Assume (3.9) is valid for $k \le n \le \frac{T}{\tau} - 1$. Next we need to show that (3.9) still holds for $k = n + 1$. Subtracting (3.4) from (3.12), we get the error equations

$$(3.14)\quad i\delta_t^c e_j^{\varepsilon,m} = -\frac{1}{2}(\delta_x^2 e_j^{\varepsilon,m+1} + \delta_x^2 e_j^{\varepsilon,m-1}) + r_j^{\varepsilon,m} + \xi_j^{\varepsilon,m}, \quad j \in \mathcal{T}_M, \quad 1 \le m \le \frac{T}{\tau} - 1,$$

where $r^{\varepsilon,m} \in X_M$ represents the difference between the logarithmic nonlinearities

$$(3.15)\quad r_j^{\varepsilon,m} = u^\varepsilon(x_j, t_m)\ln(\varepsilon + |u^\varepsilon(x_j, t_m)|)^2 - u_j^{\varepsilon,m}\ln(\varepsilon + |u_j^{\varepsilon,m}|)^2, \quad 1 \le m \le \frac{T}{\tau} - 1.$$

Multiplying both sides of (3.14) by $2\tau\,\overline{(e_j^{\varepsilon,m+1} + e_j^{\varepsilon,m-1})}$, summing together for $j \in \mathcal{T}_M$, and taking the imaginary parts, we obtain for $1 \le m < T/\tau$,

$$(3.16)\quad \begin{aligned} \|e^{\varepsilon,m+1}\|^2 - \|e^{\varepsilon,m-1}\|^2 &= 2\tau\,\mathrm{Im}(r^{\varepsilon,m} + \xi^{\varepsilon,m}, e^{\varepsilon,m+1} + e^{\varepsilon,m-1}) \\ &\le 2\tau\left(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2 + \|e^{\varepsilon,m-1}\|^2\right). \end{aligned}$$

Summing (3.16) for $m = 1, 2, \ldots, n$ ($n \le \frac{T}{\tau} - 1$), we obtain

$$\begin{aligned} \|e^{\varepsilon,n+1}\|^2 + \|e^{\varepsilon,n}\|^2 \le \|e^{\varepsilon,0}\|^2 + \|e^{\varepsilon,1}\|^2 + 2\tau\|e^{\varepsilon,n+1}\|^2 + 2\tau\sum_{m=0}^{n-1}(\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2) \\ (3.17)\qquad + 2\tau\sum_{m=1}^{n}\left(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2\right). \end{aligned}$$

For $m \le n$, when $|u_j^{\varepsilon,m}| \le |u^\varepsilon(x_j, t_m)|$, we write $r_j^{\varepsilon,m}$ as

$$\begin{aligned} |r_j^{\varepsilon,m}| &= \left| e_j^{\varepsilon,m}\ln(\varepsilon + |u^\varepsilon(x_j, t_m)|)^2 + 2u_j^{\varepsilon,m}\ln\left(\frac{\varepsilon + |u^\varepsilon(x_j, t_m)|}{\varepsilon + |u_j^{\varepsilon,m}|}\right)\right| \\ &\le 2\max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon + \Lambda)|\}|e_j^{\varepsilon,m}| + 2|u_j^{\varepsilon,m}|\ln\left(1 + \frac{|u^\varepsilon(x_j, t_m)| - |u_j^{\varepsilon,m}|}{\varepsilon + |u_j^{\varepsilon,m}|}\right) \\ &\le 2|e_j^{\varepsilon,m}|(1 + \max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon + \Lambda)|\}). \end{aligned}$$

On the other hand, when $|u^\varepsilon(x_j,t_m)| \le |u_j^{\varepsilon,m}|$, we write $r_j^{\varepsilon,m}$ as

$$\begin{aligned}|r_j^{\varepsilon,m}| &= \Big|e_j^{\varepsilon,m}\ln(\varepsilon+|u_j^{\varepsilon,m}|)^2 + 2u^\varepsilon(x_j,t_m)\ln\Big(\frac{\varepsilon+|u^\varepsilon(x_j,t_m)|}{\varepsilon+|u_j^{\varepsilon,m}|}\Big)\Big|\\ &\le 2\max\{\ln(\varepsilon^{-1}),|\ln(\varepsilon+1+\Lambda)|\}|e_j^{\varepsilon,m}|\\ &\quad + 2|u^\varepsilon(x_j,t_m)|\ln\Big(1+\frac{|u_j^{\varepsilon,m}|-|u^\varepsilon(x_j,t_m)|}{\varepsilon+|u^\varepsilon(x_j,t_m)|}\Big)\\ &\le 2|e_j^{\varepsilon,m}|(1+\max\{\ln(\varepsilon^{-1}),|\ln(\varepsilon+1+\Lambda)|\}),\end{aligned}$$

where we use the assumption that $\|u^{\varepsilon,m}\|_\infty \le \Lambda+1$ for $m \le n$. Thus it follows that

$$\|r^{\varepsilon,m}\|^2 \lesssim |\ln(\varepsilon)|^2\|e^{\varepsilon,m}\|^2,$$

when ε is sufficiently small. Thus when $\tau \le \frac{1}{2}$, by using Lemmas 3.3 and 3.4 and (3.17), we have

$$\begin{aligned}\|e^{\varepsilon,n+1}\|^2 + \|e^{\varepsilon,n}\|^2 &\lesssim \|e^{\varepsilon,0}\|^2 + \|e^{\varepsilon,1}\|^2 + \tau\sum_{m=0}^{n-1}(\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2)\\ &\quad + \tau\sum_{m=1}^{n}\left(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2\right)\\ &\lesssim (h^2+\tau^2)^2 + \tau|\ln(\varepsilon)|^2\sum_{m=0}^{n-1}(\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2).\end{aligned}$$

We emphasize here that the implicit multiplicative constant in this inequality depends only on C_0, but not on n. Applying the discrete Gronwall inequality, we can conclude that

$$\|e^{\varepsilon,n+1}\|^2 \lesssim e^{CT|\ln(\varepsilon)|^2}(h^2+\tau^2)^2$$

for some C depending on C_0, which gives the error bound for $\|e^{\varepsilon,k}\|$ with $k=n+1$ in (3.9) immediately.

To estimate $|e^{\varepsilon,n+1}|_{H^1}$, multiplying both sides of (3.14) by $2\overline{(e_j^{\varepsilon,m+1} - e_j^{\varepsilon,m-1})}$ for $m \le n$, summing together for $j \in \mathcal{T}_M$, and taking the real parts, we obtain

$$\begin{aligned}&|e^{\varepsilon,m+1}|_{H^1}^2 - |e^{\varepsilon,m-1}|_{H^1}^2\\ &\quad = -2\,\mathrm{Re}\big(r^{\varepsilon,m}+\xi^{\varepsilon,m}, e^{\varepsilon,m+1}-e^{\varepsilon,m-1}\big)\\ &\quad = 2\tau\,\mathrm{Im}\big(r^{\varepsilon,m}+\xi^{\varepsilon,m}, -\delta_x^2(e^{\varepsilon,m+1}+e^{\varepsilon,m-1})\big)\\ &\quad = 2\tau\,\mathrm{Im}\big\langle\delta_x^+(r^{\varepsilon,m}+\xi^{\varepsilon,m}), \delta_x^+(e^{\varepsilon,m+1}+e^{\varepsilon,m-1})\big\rangle\\ &\quad \le 2\tau\left(|r^{\varepsilon,m}|_{H^1}^2 + |\xi^{\varepsilon,m}|_{H^1}^2 + |e^{\varepsilon,m+1}|_{H^1}^2 + |e^{\varepsilon,m-1}|_{H^1}^2\right).\end{aligned} \tag{3.18}$$

To give the bound for $\delta_x^+ r^{\varepsilon,m}$, for simplicity of notation, denote

$$u_{j,\theta}^{\varepsilon,m} = \theta u^\varepsilon(x_{j+1},t_m) + (1-\theta)u^\varepsilon(x_j,t_m), \quad v_{j,\theta}^{\varepsilon,m} = \theta u_{j+1}^{\varepsilon,m} + (1-\theta)u_j^{\varepsilon,m}$$

for $j \in \mathcal{T}_M$ and $\theta \in [0,1]$. Then we have

$$\begin{aligned}\delta_x^+ r_j^{\varepsilon,m} &= 2\delta_x^+(u^\varepsilon(x_j,t_m)\ln(\varepsilon+|u^\varepsilon(x_j,t_m)|)) - 2\delta_x^+(u_j^{\varepsilon,m}\ln(\varepsilon+|u_j^{\varepsilon,m}|))\\ &= \frac{2}{h}\int_0^1[u_{j,\theta}^{\varepsilon,m}\ln(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)]'(\theta)d\theta - \frac{2}{h}\int_0^1[v_{j,\theta}^{\varepsilon,m}\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)]'(\theta)d\theta\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$
\begin{aligned}
I_1 &:= 2\delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \ln(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)d\theta - 2\delta_x^+ u_j^{\varepsilon,m}\int_0^1 \ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)d\theta,\\
I_2 &:= \delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \frac{|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|}d\theta - \delta_x^+ u_j^{\varepsilon,m}\int_0^1 \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}d\theta,\\
I_3 &:= \delta_x^+ \overline{u^\varepsilon(x_j,t_m)}\int_0^1 \frac{(u_{j,\theta}^{\varepsilon,m})^2}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)}d\theta - \delta_x^+ \overline{u_j^{\varepsilon,m}}\int_0^1 \frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}d\theta.
\end{aligned}
$$

Then we estimate I_1, I_2, and I_3, separately. Similarly as before, we have

$$
\begin{aligned}
|I_1| &\le 2|\delta_x^+ u^\varepsilon(x_j,t_m)| \int_0^1 \Big|\ln\Big(\frac{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}\Big)\Big|d\theta + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right| \int_0^1 \Big|\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)\Big|d\theta\\
&= 2|\delta_x^+ u^\varepsilon(x_j,t_m)| \int_0^1 \ln\Big(1+\frac{\Big||u_{j,\theta}^{\varepsilon,m}|-|v_{j,\theta}^{\varepsilon,m}|\Big|}{\varepsilon+\min\{|u_{j,\theta}^{\varepsilon,m}|,|v_{j,\theta}^{\varepsilon,m}|\}}\Big)d\theta\\
&\quad + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right| \int_0^1 \big|\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)\big|d\theta\\
&\le \frac{2}{\varepsilon}|\delta_x^+ u^\varepsilon(x_j,t_m)|\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right) + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right|\max\{\ln(\varepsilon^{-1}),|\ln(\varepsilon+1+\Lambda)|\}\\
&\lesssim \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right) + \ln(\varepsilon^{-1})\left|\delta_x^+ e_j^{\varepsilon,m}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &= \Big|\delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \Big(\frac{|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|} - \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}\Big)d\theta + \delta_x^+ e_j^{\varepsilon,m}\int_0^1 \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_j^{\varepsilon,m}|}d\theta\Big|\\
&\le |\delta_x^+ e_j^{\varepsilon,m}| + |\delta_x^+ u^\varepsilon(x_j,t_m)|\int_0^1 \frac{\varepsilon|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}d\theta\\
&\le |\delta_x^+ e_j^{\varepsilon,m}| + \frac{|\delta_x^+ u^\varepsilon(x_j,t_m)|}{\varepsilon}\int_0^1 |u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|d\theta\\
&\lesssim |\delta_x^+ e_j^{\varepsilon,m}| + \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right).
\end{aligned}
$$

In view of the inequality that

$$
\begin{aligned}
&\Big|\frac{(u_{j,\theta}^{\varepsilon,m})^2}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)} - \frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\Big|\\
&= \Big|\frac{(u_{j,\theta}^{\varepsilon,m})^2-u_{j,\theta}^{\varepsilon,m}v_{j,\theta}^{\varepsilon,m}}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)} + \frac{u_{j,\theta}^{\varepsilon,m}v_{j,\theta}^{\varepsilon,m}}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)} - \frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\Big|\\
&\le \frac{|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{\varepsilon} + \frac{\Big|u_{j,\theta}^{\varepsilon,m}(v_{j,\theta}^{\varepsilon,m})^2(\overline{u_{j,\theta}^{\varepsilon,m}}-\overline{v_{j,\theta}^{\varepsilon,m}})+\varepsilon v_{j,\theta}^{\varepsilon,m}(u_{j,\theta}^{\varepsilon,m}|v_{j,\theta}^{\varepsilon,m}|-|u_{j,\theta}^{\varepsilon,m}|v_{j,\theta}^{\varepsilon,m})\Big|}{|u_{j,\theta}^{\varepsilon,m}||v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\\
&\le \frac{4|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{\varepsilon},
\end{aligned}
$$

we can obtain that

$$
I_3 \lesssim |\delta_x^+ e_j^{\varepsilon,m}| + \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right).
$$

Thus we can conclude that

$$|\delta_x^+ r_j^{\varepsilon,m}| \lesssim \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}| + |e_{j+1}^{\varepsilon,m}|\right) + \ln(\varepsilon^{-1})\left|\delta_x^+ e_j^{\varepsilon,m}\right|.$$

Summing (3.18) for $m = 1, 2, \ldots, n$ ($n \le \frac{T}{\tau} - 1$), we obtain

$$|e^{\varepsilon,n+1}|_{H^1}^2 + |e^{\varepsilon,n}|_{H^1}^2 \le |e^{\varepsilon,0}|_{H^1}^2 + |e^{\varepsilon,1}|_{H^1}^2 + \tau\sum_{m=1}^{n}\left(|r^{\varepsilon,m}|_{H^1}^2 + |\xi^{\varepsilon,m}|_{H^1}^2\right) + \tau|e^{\varepsilon,n+1}|_{H^1}^2 + \tau\sum_{m=0}^{n-1}(|e^{\varepsilon,m}|_{H^1}^2 + |e^{\varepsilon,m+1}|_{H^1}^2).$$

Thus when $\tau \le 1/2$, by using Lemmas 3.3 and 3.4, we have

$$\begin{aligned}|e^{\varepsilon,n+1}|_{H^1}^2 + |e^{\varepsilon,n}|_{H^1}^2 &\lesssim |e^{\varepsilon,0}|_{H^1}^2 + |e^{\varepsilon,1}|_{H^1}^2 + \tau\sum_{m=1}^{n}\left(\frac{1}{\varepsilon^2}\|e^{\varepsilon,m}\|^2 + |\xi^{\varepsilon,m}|_{H^1}^2\right)\\ &\quad + \tau|\ln(\varepsilon)|^2\sum_{m=0}^{n-1}\left(|e^{\varepsilon,m}|_{H^1}^2 + |e^{\varepsilon,m+1}|_{H^1}^2\right)\\ &\lesssim \frac{e^{CT|\ln(\varepsilon)|^2}}{\varepsilon^2}(h^2+\tau^2)^2 + \tau|\ln(\varepsilon)|^2\sum_{m=0}^{n-1}(|e^{\varepsilon,m}|_{H^1}^2 + |e^{\varepsilon,m+1}|_{H^1}^2).\end{aligned}$$

Applying the discrete Gronwall's inequality, we can get that

$$|e^{\varepsilon,n+1}|_{H^1}^2 \lesssim e^{CT|\ln(\varepsilon)|^2}(h^2+\tau^2)^2/\varepsilon^2,$$

which establishes the error estimate for $\|e^{\varepsilon,k}\|_{H^1}$ for $k = n+1$. Finally the boundedness for the solution $u^{\varepsilon,k}$ can be obtained by the triangle inequality

$$\|u^{\varepsilon,k}\|_\infty \le \|u^\varepsilon(\cdot,t_k)\|_{L^\infty(\Omega)} + \|e^{\varepsilon,k}\|_\infty,$$

and the inverse Sobolev inequality [26]

$$\|e^{\varepsilon,k}\|_\infty \lesssim \|e^{\varepsilon,k}\|_{H^1},$$

which completes the proof of Theorem 3.1. ∎

4. Numerical results.

In this section, we test the convergence rate of the regularized model (2.1) and the SIFD (3.4). To this end, we take $d = 1$, $\Omega = \mathbb{R}$, and $\lambda = -1$ in the LogSE (1.1) and consider two different initial data:

Case I: A Gaussian initial data, i.e., u_0 in (1.1) is chosen as

$$u_0(x) = \sqrt[4]{-\lambda/\pi}\, e^{ivx + \frac{\lambda}{2}x^2}, \qquad x \in \mathbb{R}, \tag{4.1}$$

with $v = 1$. In this case, the LogSE (1.1) admits the moving Gausson solution (1.7) with $v = 1$ and $b_0 = \sqrt[4]{-\lambda/\pi}$ as the exact solution.

Case II: A general initial data, i.e., u_0 in (1.1) is chosen as

$$u_0(x) = \tanh(x)e^{-x^2}, \qquad x \in \mathbb{R}, \tag{4.2}$$

which is the multiplication of a dark soliton of the cubic nonlinear Schrödinger equation and a Gaussian. Notice that in this case, the logarithmic term $\ln|u_0|^2$ is singular at $x = 0$.

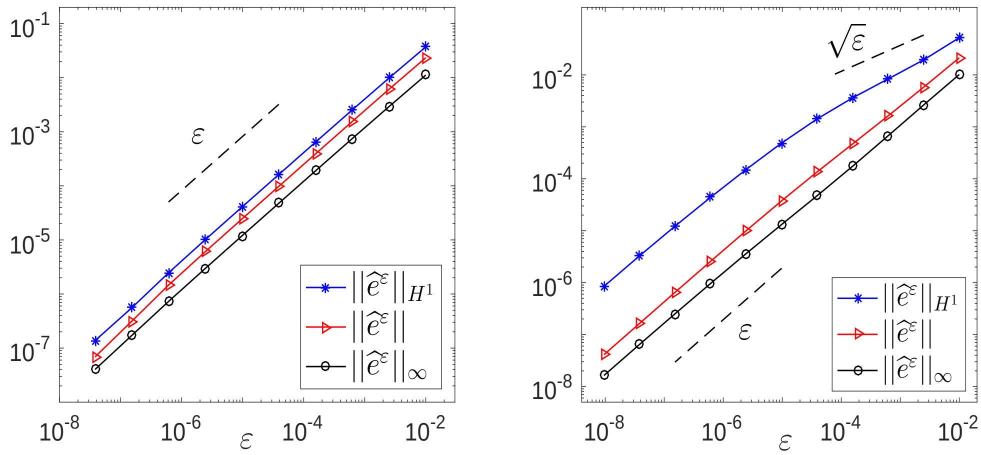


FIG. 4.1. *Convergence of the RLogSE* (2.1) *to the LogSE* (1.1)*, i.e., the error* $\widehat{e}^\varepsilon(0.5)$ *in different norms versus the regularization parameter* ε *for Case* I *(left) and Case* II *(right).*

The RLogSE (2.1) is solved numerically by the SIFD (3.4) on domains $\Omega = [-12, 12]$ and $\Omega = [-16, 16]$ for Cases I and II, respectively. To quantify the numerical errors, we introduce the following error functions:

$$\begin{aligned} &\widehat{e}^\varepsilon(t_k) := u(\cdot, t_k) - u^\varepsilon(\cdot, t_k), \qquad && e^\varepsilon(t_k) := u^\varepsilon(\cdot, t_k) - u^{\varepsilon,k}, \\ &\widetilde{e}^\varepsilon(t_k) := u(\cdot, t_k) - u^{\varepsilon,k}, && e_E^\varepsilon := |E(u) - E^\varepsilon(u^\varepsilon)|. \end{aligned} \tag{4.3}$$

Here u and u^ε are the exact solutions of the LogSE (1.1) and RLogSE (2.1), respectively, while $u^{\varepsilon,k}$ is the numerical solution of the RLogSE (2.1) obtained by the SIFD (3.4). The exact solution u^ε is obtained numerically by the SIFD (3.4) with a very small time step, e.g., $\tau = 0.01/2^9$ and a very fine mesh size, e.g., $h = 1/2^{15}$. Similarly, the exact solution u in Case II is obtained numerically by the SIFD (3.4) with a very small time step and a very fine mesh size as well as a very small regularization parameter ε, e.g., $\varepsilon = 10^{-14}$. The energy is obtained by the trapezoidal rule for approximating the integrals in the energy (1.2) and (2.2).

4.1. Convergence rate of the regularized model. Here we consider the error between the solutions of the RLogSE (2.1) and the LogSE (1.1). Figure 4.1 shows $\|\widehat{e}^\varepsilon\|$, $\|\widehat{e}^\varepsilon\|_{H^1}$, $\|\widehat{e}^\varepsilon\|_\infty$ (the definition of the norms is given in (3.2)) at time $t = 0.5$ for Cases I and II, while Figure 4.2 depicts $e_E^\varepsilon(0.5)$ for Cases I and II and time evolution of $\widehat{e}^\varepsilon(t)$ with different ε for Case I. For comparison, similarly to Figure 4.1, Figure 4.3 displays the convergent results from (2.9) to (1.1).

From Figures 4.1, 4.2, and 4.3 and additional numerical results not shown here for brevity, we can draw the following conclusions: (i) The solution of the RLogSE (2.1) converges linearly to that of the LogSE (1.1) in terms of the regularization parameter ε in both the L^2-norm and L^∞-norm, and, respectively, the convergence rate becomes $O(\sqrt{\varepsilon})$ in the H^1-norm for Case II. (ii) The regularized energy $E^\varepsilon(u^\varepsilon)$ converges linearly to the energy $E(u)$ in terms of ε. (iii) The constant C in (2.7) may grow linearly with time T and it is independent of ε. (iv) The solution of (2.9) converges at $O(\sqrt{\varepsilon})$ to that of (1.1) in both the L^2-norm and L^∞-norm, and, respectively, the convergence rate becomes $O(\varepsilon^{1/4})$ in the H^1-norm for Case II. Thus (2.1) is much

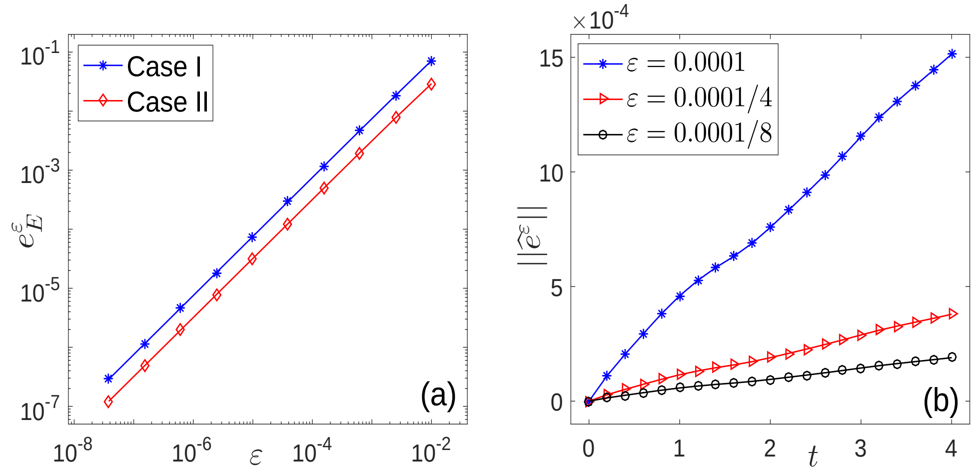


FIG. 4.2. *Convergence of the RLogSE* (2.1) *to the LogSE* (1.1)*:* (a) *error in energy* $e^{\varepsilon}_{\mathrm{E}}(0.5)$ *versus* ε *for Cases* I *and* II*, and* (b) *time evolution of* $\|\widehat{e}^{\varepsilon}(t)\|$ *versus time* t *under different* ε *for Case* I.

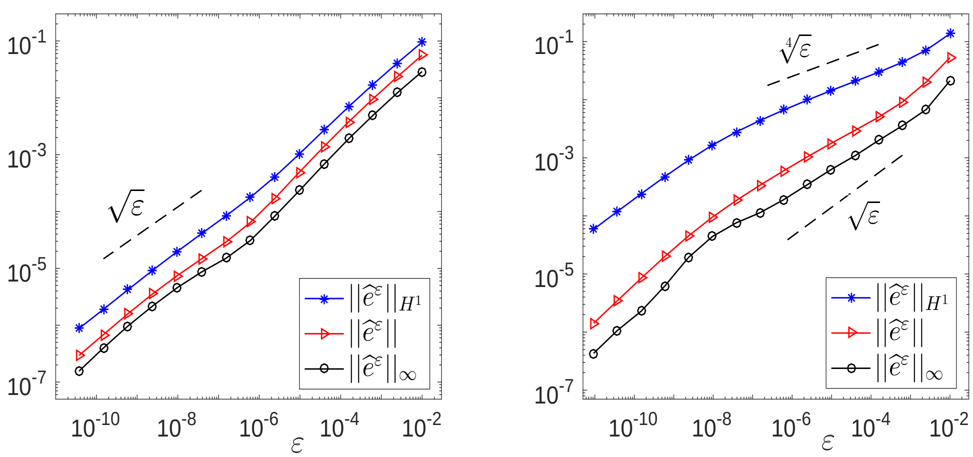


FIG. 4.3. *Convergence of the RLogSE* (2.9) *to the LogSE* (1.1)*, i.e., the error* $\widehat{e}^{\varepsilon}(0.5)$ *in different norms versus the regularization parameter* ε *for Case* I *(left) and Case* II *(right).*

more accurate than (2.9) for the regularization of the LogSE (1.1). (v) The numerical results agree and confirm our analytical results in section 2.

4.2. Convergence rate of the finite difference method. Here we test the convergence rate of the SIFD (3.4) to the RLogSE (2.1) or the LogSE (1.1) in terms of mesh size h and time step τ under any fixed $0 < \varepsilon \ll 1$ for Case I. Figure 4.4 shows the errors $\|e^{\varepsilon}(0.5)\|$ versus time step τ (with a fixed ratio between mesh size h and time step τ at $h = 75\tau/64$) under different ε. In addition, Table 4.1 displays $\|\widetilde{e}^{\varepsilon}(1)\|$ for varying ε, τ, and h.

From Figure 4.4, we can see that the SIFD (3.4) converges quadratically at $O(\tau^2 + h^2)$ to the RLogSE (2.1) for any fixed $\varepsilon > 0$, which confirms our error estimates in Theorem 3.1. From Table 4.1, we can observe that (i) the SIFD (3.4)

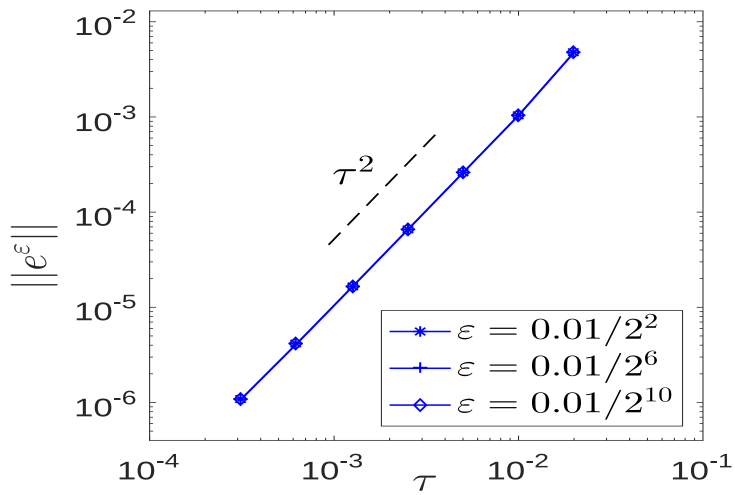


FIG. 4.4. *Convergence of the SIFD* (3.4) *to the RLogSE* (2.1), *i.e., errors* $\|e^{\varepsilon}(0.5)\|$ *versus* τ *(with* $h = 75\tau/64$*) under different* ε *for Case* I *initial data.*

TABLE 4.1
Convergence of the SIFD (3.4) *to the LogSE* (1.1), *i.e.,* $\|\widetilde{e}^{\varepsilon}(1)\|$ *for different* ε, τ, *and* h *for Case* I.

	$h=0.1$ $\tau=0.1$	$h/2$ $\tau/2$	$h/2^2$ $\tau/2^2$	$h/2^3$ $\tau/2^3$	$h/2^4$ $\tau/2^4$	$h/2^5$ $\tau/2^5$	$h/2^6$ $\tau/2^6$	$h/2^7$ $\tau/2^7$	$h/2^8$ $\tau/2^8$	$h/2^9$ $\tau/2^9$
ε=0.001	1.84E-1	**4.84E-2**	1.34E-2	5.96E-3	4.79E-3	4.62E-3	4.58E-3	4.57E-3	4.57E-3	4.57E-3
rate	–	**1.93**	1.85	1.17	0.31	0.05	0.01	0.00	0.00	0.00
$\varepsilon/4$	1.84E-1	4.75E-2	**1.19E-2**	3.36E-3	1.49E-3	1.20E-3	1.16E-3	1.15E-3	1.15E-3	1.15E-3
rate	–	1.96	**1.99**	1.83	1.17	0.31	0.05	0.01	0.00	0.00
$\varepsilon/4^2$	1.84E-1	4.73E-2	1.17E-2	**2.97E-3**	8.39E-4	3.74E-4	3.01E-4	2.90E-4	2.88E-4	2.88E-4
rate	–	1.96	2.01	**1.98**	1.83	1.17	0.31	0.05	0.01	0.00
$\varepsilon/4^3$	1.84E-1	4.72E-2	1.16E-2	2.91E-3	**7.43E-4**	2.10E-4	9.35E-5	7.54E-5	7.27E-5	7.21E-5
rate	–	1.96	2.02	2.00	**1.97**	1.83	1.16	0.31	0.05	0.01
$\varepsilon/4^4$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.27E-4	**1.86E-4**	5.24E-5	2.34E-5	1.89E-5	1.82E-5
rate	–	1.96	2.02	2.00	2.00	**1.97**	1.83	1.16	0.31	0.05
$\varepsilon/4^5$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.24E-4	1.82E-4	**4.64E-5**	1.31E-5	5.85E-6	4.72E-6
rate	–	1.96	2.02	2.01	2.00	1.99	**1.97**	1.83	1.16	0.31
$\varepsilon_0/4^6$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.23E-4	1.81E-4	4.54E-5	**1.16E-5**	3.28E-6	1.47E-6
rate	–	1.96	2.02	2.01	2.00	2.00	1.99	**1.97**	1.83	1.16
$\varepsilon_0/4^7$	1.84E-1	4.72E-2	1.16E-2	2.89E-3	7.23E-4	1.81E-4	4.52E-5	1.14E-5	**2.90E-6**	8.22E-7
rate	–	1.96	2.02	2.01	2.00	2.00	2.00	2.00	**1.97**	1.82

converges quadratically at $O(\tau^2 + h^2)$ to the LogSE (1.1) only when ε is sufficiently small, e.g., $\varepsilon \lesssim h^2$ and $\varepsilon \lesssim \tau^2$ (cf. lower triangle below the diagonal in bold letter in Table 4.1), and (ii) when τ and h are sufficiently small, i.e., $\tau^2 \lesssim \varepsilon$ & $h^2 \lesssim \varepsilon$, the RLogSE (2.1) converge linearly at $O(\varepsilon)$ to the LogSE (1.1) (cf. each column in the right most of Table 4.1), which confirms the error bounds in Corollary 3.2.

5. Conclusion. In order to overcome the singularity of the log-nonlinearity in the LogSE, we proposed an RLogSE with a regularization parameter $0 < \varepsilon \ll 1$ and established linear convergence between RLogSE and LogSE in terms of the small regularization parameter. Then we presented an SIFD method for discretizing RLogSE and proved second-order convergence rates in terms of mesh size h and time step τ. Finally, we established error bounds of the semi-implicit finite difference method to LogSE, which depend explicitly on the mesh size h and time step τ as well as the small regularization parameter ε. Our numerical results confirmed our error bounds.

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