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Two exponential-type integrators for the "good" Boussinesq equation

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Abstract
We introduce two exponential-type integrators for the "good" Boussinesq equation. They are of orders one and two, respectively, and they require lower spatial regularity of the solution compared to classical exponential integrators. For the first integrator, we prove first-order convergence in H^r for solutions in H^{r+1} with $r > 1/2$. This new integrator even converges (with lower order) in H^r for solutions in H^r, i.e., without any additional smoothness assumptions. For the second integrator, we prove second-order convergence in H^r for solutions in H^{r+3} with $r > 1/2$ and convergence in L^2 for solutions in H^3. Numerical results are reported to illustrate the established error estimates. The experiments clearly demonstrate that the new exponential-type integrators are favorable over classical exponential integrators for initial data with low regularity.

Mathematics Subject Classification 35Q40 · 35Q55 · 65M12 · 65M15 · 81Q05

1 Introduction

We consider the "good" Boussinesq (GB) equation [5]

$$z_{tt} + z_{xxxx} - z_{xx} - (z^2)_{xx} = 0, \tag{1.1}$$

which was originally introduced as a model for one-dimensional weakly nonlinear dispersive waves in shallow water. Similar to the well-known Korteweg–de Vries (KdV) equation and the cubic nonlinear Schrödinger equation, the GB equation

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is one of the important models for describing the interaction between nonlinearity and dispersion. The GB equation has been widely applied in many areas, e.g., plasma physics, coastal engineering, hydraulics studies, elastic crystals, and so on.

The GB equation and its various extensions have been extensively analyzed in the literature. For the well-posedness, we refer to [11–13,18,23,26] and references therein. For the interaction of solitary waves, we refer to [21,22]. Many numerical methods have been developed for solving the GB equation, such as finite difference methods (FDM) [6,24], Petrov–Galerkin methods [21], mesh free methods [9], Fourier spectral methods [7,8,27] and operator splitting methods [28,29]. Regarding the numerical analysis of the GB equation, nonlinear stability and convergence of some finite difference methods were studied in [24]. Specifically, an explicit finite difference scheme was proved to converge quadratically in both space and time under the regularity assumption that $\frac{\partial^6 z}{\partial x^6}$, $\frac{\partial^4 z}{\partial t^4}$ are bounded and the severe time step restriction $\Delta t = \mathcal{O}(\Delta x^2)$, where Δt and Δx represent the discretization parameters in time and space, respectively. For a pseudospectral discretization with periodic boundary conditions, a second order temporal discretization was proposed and analyzed in [8], and full order of convergence was proved in a weak energy norm: the L^2 norm in z combined with the H^{-2} norm in z_t under a similar time step constraint $\Delta t = \mathcal{O}(\Delta x^2)$. Due to the absence of a dissipation mechanism in the GB equation (1.1), it is more challenging to analyze the nonlinear error terms here than for parabolic equations. The presence of a second-order spatial derivative in the nonlinear term brings an essential difficulty for numerical error estimates in a higher order Sobolev norm [7]. The norm was strengthened in [7], where a second order temporal scheme was proposed and convergence was proved in a stronger energy norm: the H^2 norm in z combined with the L^2 norm in z_t. Moreover, such a convergence is unconditional so that the time step restriction $\Delta t = \mathcal{O}(\Delta x^2)$ is avoided. However, it requires the solution to be smooth enough such as $z \in H^{m+4}$, $z_{tt} \in H^4$ and $\partial_t^4 z \in L^2$ to get an error of size $\mathcal{O}(\Delta t^2 + \Delta x^m)$. Such constraints become very restrictive for computation, especially for rough solutions.

In this paper, we present two exponential-type Fourier integrators for the GB equation which enable us to weaken the classical regularity assumptions and to obtain first- and second-order convergence by requiring one and three additional derivatives, respectively. The exponential-type integrators are constructed based on the following strategy:

1. In the first step, we formulate the GB Eq. (1.1) as a first-order equation in the complex domain via the transformation

$$u = z - i\langle \partial_x^2 \rangle_c^{-1} z_t,$$

where for some $c > 0$

$$\langle \partial_x^2 \rangle_c = \sqrt{\partial_x^4 - \partial_x^2 + c}. \tag{1.2}$$

2. In the second step, we rescale the first-order equation in time by considering the so-called *twisted variable*

$$w(t) = \mathrm{e}^{it\partial_x^2} u(t).$$

This essential step will later enable us to treat the dominant term triggered by the nonlinearity in an exact way.
3. Finally, we iterate Duhamel's formula in $w(t)$ and integrate the dominant interactions exactly.

The idea of twisting the variable is widely applied in the analysis of PDEs in low regularity spaces [4]. It has also been applied in the context of numerical analysis for the Schrödinger equation [19,25], the KdV equation [16] and Klein–Gordon type equations [2,3].

For implementation issues, we impose periodic boundary conditions and refer to [13,18,23] for the corresponding well-posedness results. For $m \in \mathbb{R}$, we define the Sobolev norm on $\Omega = (-\pi, \pi)$ by

$$\|f\|_m^2 = \sum_{k=-\infty}^{\infty} (1+k^2)^m |\widehat{f}_k|^2, \quad \text{where} \quad \widehat{f}_k = \frac{1}{2\pi} \int_\Omega f(x)\mathrm{e}^{-ikx}\mathrm{d}x.$$

Moreover, we denote by H^m all the functions defined on Ω with finite norm $\|\cdot\|_m$. For $m = 0$, the space is exactly L^2 and the corresponding norm is denoted as $\|\cdot\|$.

The rest of this paper is organized as follows. In Sect. 2, we introduce a scaling of the GB equation and give some preliminary notations and lemmas. The first- and second-order exponential-type integrators are constructed and analyzed in Sects. 3 and 5, respectively. Low-order convergence without any additional smoothness assumption is addressed in Sect. 4. Numerical results that illustrate the proved convergence results are reported in Sect. 6. Finally, some concluding remarks are given in Sect. 7.

2 Scaling for low regularity exponential-type integrators

First note that the GB equation

$$z_{tt} + z_{xxxx} - z_{xx} - (z^2)_{xx} = 0, \quad x \in (-\pi, \pi), \quad t > 0 \tag{2.1}$$

can be reformulated as a first-order coupled system

$$\begin{aligned}
\partial_t u &= i\langle\partial_x^2\rangle_c u - \frac{i}{4}\langle\partial_x^2\rangle_c^{-1}\left[2c(u+\overline{v}) + \partial_{xx}(u+\overline{v})^2\right],\\
\partial_t v &= i\langle\partial_x^2\rangle_c v - \frac{i}{4}\langle\partial_x^2\rangle_c^{-1}\left[2c(\overline{u}+v) + \partial_{xx}(\overline{u}+v)^2\right],
\end{aligned}$$

where

$$\langle\partial_x^2\rangle_c = \sqrt{\partial_x^4 - \partial_x^2 + c}, \quad u = z - i\langle\partial_x^2\rangle_c^{-1} z_t, \quad v = \overline{z} - i\langle\partial_x^2\rangle_c^{-1}\overline{z_t}, \tag{2.2}$$

and c is a positive number. Since $z \in \mathbb{R}$ we immediately get $u = v$ and

$$z = \frac{1}{2}(u + \overline{u}), \quad z_t = \frac{i}{2}\langle \partial_x^2 \rangle_c (u - \overline{u}). \tag{2.3}$$

Thus the GB Eq. (2.1) is equivalent to the following first-order equation in the complex domain

$$\partial_t u = i\langle \partial_x^2 \rangle_c u - \frac{i}{4}\langle \partial_x^2 \rangle_c^{-1}\Big[2c(u + \overline{u}) + \partial_{xx}(u + \overline{u})^2\Big], \quad x \in (-\pi, \pi), \quad t > 0 \tag{2.4}$$

with initial data

$$u(0) = z(0) - i\langle \partial_x^2 \rangle_c^{-1} z_t(0).$$

Remark 1 The linear term "cz" was added and subtracted in Eq. (2.1) so that u and v in (2.2) are well-defined for functions with $\widehat{(z_t)}_0 \neq 0$.

We note that the leading term in $\langle \partial_x^2 \rangle_c$ is $-\partial_x^2$. This motivates us to introduce the so-called *twisted variable*

$$w(t) = \mathrm{e}^{it\partial_x^2} u(t).$$

In this new variable (2.4) becomes

$$\begin{aligned}
\partial_t w &= iAw - \frac{i}{4}\mathrm{e}^{it\partial_x^2}\langle \partial_x^2 \rangle_c^{-1}\Big[2c(\mathrm{e}^{-it\partial_x^2} w + \mathrm{e}^{it\partial_x^2}\overline{w}) + \partial_x^2(\mathrm{e}^{-it\partial_x^2} w + \mathrm{e}^{it\partial_x^2}\overline{w})^2\Big] \\
&= iAw - \frac{ic}{2}\langle \partial_x^2 \rangle_c^{-1} w - \frac{ic}{2}\mathrm{e}^{2it\partial_x^2}\langle \partial_x^2 \rangle_c^{-1}\overline{w} \\
&\quad - \frac{i}{4}\mathrm{e}^{it\partial_x^2}\langle \partial_x^2 \rangle_c^{-1}\partial_x^2(\mathrm{e}^{-it\partial_x^2} w + \mathrm{e}^{it\partial_x^2}\overline{w})^2,
\end{aligned} \tag{2.5}$$

where $A = \partial_x^2 + \langle \partial_x^2 \rangle_c$. Thus, by Duhamel's formula, we have

$$\begin{aligned}
w(t_n + \tau) &= \mathrm{e}^{i\tau A} w(t_n) \\
&\quad - \frac{ic}{2}\langle \partial_x^2 \rangle_c^{-1}\int_0^\tau \mathrm{e}^{i(\tau - s)A}\Big[w(t_n + s) + \mathrm{e}^{2i(t_n+s)\partial_x^2}\overline{w}(t_n + s)\Big]\,\mathrm{d}s \\
&\quad - \frac{i\partial_x^2}{4}\langle \partial_x^2 \rangle_c^{-1}\int_0^\tau \mathrm{e}^{i(\tau - s)A}\mathrm{e}^{i(t_n+s)\partial_x^2}\Big[\mathrm{e}^{-i(t_n+s)\partial_x^2} w(t_n + s) \\
&\quad + \mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w}(t_n + s)\Big]^2\,\mathrm{d}s.
\end{aligned} \tag{2.6}$$

Unless otherwise stated we assume $r > 1/2$ so that the well-known bilinear estimate holds:

$$\|fg\|_r \leq D_r \|f\|_r \|g\|_r, \tag{2.7}$$

where D_r represents a positive constant depending on r. For simplicity of notation, we will employ the following definitions.

Notations: Throughout the paper we will use the following notations

$$\psi_1(z) = \int_0^1 \mathrm{e}^{zs}\mathrm{d}s, \quad \psi_2(z) = \int_0^1 s\mathrm{e}^{zs}\mathrm{d}s, \quad \text{for} \quad z \in \mathbb{C}.$$

For $f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \mathrm{e}^{ikx}$, the regularization of ∂_x^{-1} is defined through its action in Fourier space by

$$(\partial_x^{-1})_k := \begin{cases} (ik)^{-1} & \text{if} \quad k \neq 0, \\ 0 & \text{if} \quad k = 0, \end{cases} \quad \text{i.e.,} \quad \partial_x^{-1} f(x) = \sum_{k\neq 0} \frac{\widehat{f}_k}{ik} \mathrm{e}^{ikx}.$$

Let $R = R(v, t, s)$ be a term that depends on the function values $v(t+\xi)$ for $0 \leq \xi \leq s$. We say that $R \in \mathcal{R}_\beta(s^\alpha)$ if and only if

$$\|R(v, t, s)\|_r \leq Cs^\alpha,$$

where C depends on $\sup_{0\leq\xi\leq s} \|v(t+\xi)\|_{r+\beta}$. For simplicity, we write $f = g+\mathcal{R}_\beta(s^\alpha)$ whenever $f = g + R$ with $R \in \mathcal{R}_\beta(s^\alpha)$.

Next we present some lemmas which will be used frequently.

Lemma 1 *For all $x, y \in \mathbb{R}$ and $0 \leq \alpha \leq 1$, it holds that*

$$|e^{ix} - 1| \leq 2^{1-\alpha}|x|^\alpha, \quad |e^{ix} - 1 - ix| \leq 2^{1-2\alpha}|x|^{1+\alpha},$$
$$|e^{i(x+y)} + 1 - e^{ix} - e^{iy}| \leq 2^{2-2\alpha}|x|^\alpha|y|^\alpha.$$

Proof The first assertion follows by combining the following two inequalities

$$|\mathrm{e}^{ix} - 1| \leq 2, \qquad |\mathrm{e}^{ix} - 1| \leq |x|.$$

For the second expansion, we note that

$$|\mathrm{e}^{ix} - 1 - ix| \leq x^2/2, \qquad |\mathrm{e}^{ix} - 1 - ix| \leq 2|x|,$$

which directly gives the result $|\mathrm{e}^{ix} - 1 - ix| \leq (x^2/2)^\alpha(2|x|)^{1-\alpha} = 2^{1-2\alpha}|x|^{1+\alpha}$. The last assertion follows from the first bound by noting that

$$|\mathrm{e}^{i(x+y)} + 1 - \mathrm{e}^{ix} - \mathrm{e}^{iy}| = |\mathrm{e}^{ix} - 1||\mathrm{e}^{iy} - 1| \leq 2^{2-2\alpha}|x|^\alpha|y|^\alpha,$$

which completes the proof. □

Lemma 2 *For all $t \in \mathbb{R}$, $\gamma \geq 0$ and $f \in H^\gamma$, we have*

$$\left\| \psi_1(it\partial_x^2) f \right\|_\gamma \leq \|f\|_\gamma, \qquad \left\| \psi_2(it\partial_x^2) f \right\|_\gamma \leq \|f\|_\gamma / 2.$$

Proof For $z \in \mathbb{R}$, it can be easily obtained that

$$|\psi_1(iz)| \leq 1, \quad |\psi_2(iz)| \leq \int_0^1 s \mathrm{d}s \leq \frac{1}{2}.$$

This directly gives

$$\left\| \psi_1(it\partial_x^2) f \right\|_\gamma^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma |\psi_1(-itk^2)|^2 |\widehat{f}_k|^2 \leq \|f\|_\gamma^2,$$
$$\left\| \psi_2(it\partial_x^2) f \right\|_\gamma^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma |\psi_2(-itk^2)|^2 |\widehat{f}_k|^2 \leq \|f\|_\gamma^2 / 4,$$

which completes the proof. □

Lemma 3 *For all $t \in \mathbb{R}$, $\gamma \geq 0$ and $f \in H^\gamma$, we have*

$$\|\langle \partial_x^2 \rangle_c^{-1} f\|_\gamma \leq \|f\|_\gamma / \sqrt{c}, \quad \|\partial_x^2 \langle \partial_x^2 \rangle_c^{-1} f\|_\gamma \leq \|f\|_\gamma,$$
$$\|e^{itA} f\|_\gamma = \|f\|_\gamma, \quad \|Af\|_\gamma \leq C_1 \|f\|_\gamma, \quad \left\| (e^{itA} - 1) f \right\|_\gamma \leq C_1 t \|f\|_\gamma,$$

where $C_1 = \max\{1, c\}$.

Proof The first two assertions are obvious by noting that the Fourier factors satisfy

$$\frac{1}{\sqrt{c+k^2+k^4}} \leq \frac{1}{\sqrt{c}}, \qquad \left| \frac{-k^2}{\sqrt{c+k^2+k^4}} \right| \leq 1.$$

In view of the fact that A acts as the Fourier multiplier $A_k = \sqrt{c+k^2+k^4} - k^2$, it is bounded for $k \neq 0$ as

$$A_k = \sqrt{c+k^2+k^4} - k^2 = k^2\left(\sqrt{1 + 1/k^2 + c/k^4} - 1\right) \leq \frac{1}{2} + \frac{c}{2k^2} \leq C_1, \tag{2.8}$$

where $C_1 = \max\{1, c\}$. Here we have used the bound $\sqrt{1+x} \leq 1 + x/2$ for $x \geq 0$. Hence we get

$$\|Af\|_\gamma^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma A_k^2 |\widehat{f}_k|^2 \leq c|\widehat{f}_0|^2 + C_1^2 \sum_{k\neq 0} (1+k^2)^\gamma |\widehat{f}_k|^2$$
$$\leq C_1^2 \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma |\widehat{f}_k|^2 = C_1^2 \|f\|_\gamma^2.$$

This together with the property that $|\mathrm{e}^{ix}-1| \le |x|$ gives $\left\|(\mathrm{e}^{itA}-1)f\right\|_\gamma \le C_1 t\|f\|_\gamma$, which completes the proof. □

3 A first-order exponential-type integrator

In this section we derive a first-order exponential-type integration scheme for the solution of Eq. (2.4). The construction is based on Duhamel's formula (2.6) and an appropriate first-order approximation.

First, by applying Lemma 3, (2.6) and the bilinear estimate (2.7), we verify that $w(t_n)$ is a first-order approximation to $w(t_n+s)$ for $|s| \le \tau$.

Lemma 4 *For $r > 1/2$, we have*

$$\|w(t_n+s)-w(t_n)\|_r \le 2\,C_1 s \sup_{0\le y\le s} \|w(t_n+y)\|_r + D_r s \sup_{0\le y\le s}\|w(t_n+y)\|_r^2. \tag{3.1}$$

Proof Thanks to the fact that $\|\langle\partial_x^2\rangle_c^{-1} f\|_r \le \frac{1}{\sqrt{c}}\|f\|_r$, it follows from (2.6) and Lemma 3 that

$$\begin{aligned}
&\|w(t_n+s)-w(t_n)\|_r\\
&\quad\le \|(\mathrm{e}^{isA}-1)w(t_n)\|_r + \sqrt{c}\,s \sup_{0\le y\le s}\|w(t_n+y)\|_r\\
&\qquad+\frac{s}{4}\sup_{0\le y\le s}\left\|\left(\mathrm{e}^{-i(t_n+y)\partial_x^2}w(t_n+y)+\mathrm{e}^{i(t_n+y)\partial_x^2}\overline{w}(t_n+y)\right)^2\right\|_r\\
&\quad\le C_1 s\|w(t_n)\|_r + \sqrt{c}\,s \sup_{0\le y\le s}\|w(t_n+y)\|_r + D_r s \sup_{0\le y\le s}\|w(t_n+y)\|_r^2\\
&\quad\le 2\,C_1 s \sup_{0\le y\le s}\|w(t_n+y)\|_r + D_r s \sup_{0\le y\le s}\|w(t_n+y)\|_r^2,
\end{aligned}$$

which completes the proof. □

With this approximation and Lemma 3, we can rewrite (2.6) as

$$\begin{aligned}
w(t_n+\tau) = \mathrm{e}^{i\tau A}w(t_n) &- \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}w(t_n)\\
&-\frac{ic}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\int_0^\tau \mathrm{e}^{2i(t_n+s)\partial_x^2}\overline{w(t_n)}\mathrm{d}s\\
&-\frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\int_0^\tau \mathrm{e}^{i(t_n+s)\partial_x^2}\Big[\mathrm{e}^{-i(t_n+s)\partial_x^2}w(t_n)\\
&+\mathrm{e}^{i(t_n+s)\partial_x^2}\overline{w(t_n)}\Big]^2\mathrm{d}s + \mathcal{R}_0(\tau^2).
\end{aligned}$$

Twisting the variable back, we get

$$\begin{aligned}
u(t_n+\tau) &= \mathrm{e}^{-i(t_n+\tau)\partial_x^2} w(t_n+\tau) \\
&= \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} u(t_n) - \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}u(t_n) - \frac{ic}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} \\
&\quad \int_0^\tau \mathrm{e}^{2is\partial_x^2}\overline{u(t_n)}\mathrm{d}s - \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} \\
&\quad \int_0^\tau \mathrm{e}^{is\partial_x^2}\left[\mathrm{e}^{-is\partial_x^2}u(t_n) + \mathrm{e}^{is\partial_x^2}\overline{u(t_n)}\right]^2\mathrm{d}s + \mathcal{R}_0(\tau^2) \\
&= \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} u(t_n) - \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}u(t_n) \\
&\quad - \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\psi_1(2i\tau\partial_x^2)\overline{u(t_n)} \\
&\quad - \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\left[I_1^\tau(u(t_n)) + 2I_2^\tau(u(t_n)) + I_0^\tau(u(t_n))\right] + \mathcal{R}_0(\tau^2),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
&I_1^\tau(f) = \int_0^\tau \mathrm{e}^{is\partial_x^2}(\mathrm{e}^{-is\partial_x^2}f)^2\mathrm{d}s, \quad I_2^\tau(f) = \int_0^\tau \mathrm{e}^{is\partial_x^2}|\mathrm{e}^{-is\partial_x^2}f|^2\mathrm{d}s, \\
&I_0^\tau(f) = \int_0^\tau \mathrm{e}^{is\partial_x^2}(\mathrm{e}^{is\partial_x^2}\overline{f})^2\mathrm{d}s.
\end{aligned} \tag{3.3}$$

The integral in I_1 can be expressed in terms of the Fourier coefficients as follows

$$\begin{aligned}
I_1^\tau(f) &= \sum_{k_1,k_2\in\mathbb{Z}} \int_0^\tau \mathrm{e}^{is(k_1^2+k_2^2-(k_1+k_2)^2)}\widehat{f}_{k_1}\widehat{f}_{k_2}\mathrm{d}s\, \mathrm{e}^{i(k_1+k_2)x} \\
&= \sum_{k_1\neq 0,k_2\neq 0} \frac{1-\mathrm{e}^{-2i\tau k_1k_2}}{2ik_1k_2}\widehat{f}_{k_1}\widehat{f}_{k_2}\mathrm{e}^{i(k_1+k_2)x} + 2\tau\widehat{f}_0\sum_{k\in\mathbb{Z}}\widehat{f}_k\mathrm{e}^{ikx} - \tau\widehat{f}_0^2 \\
&= \frac{i}{2}\left[(\partial_x^{-1}f)^2 - \mathrm{e}^{i\tau\partial_x^2}(\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}f)^2\right] + 2\tau\widehat{f}_0 f - \tau\widehat{f}_0^2.
\end{aligned} \tag{3.4}$$

A similar calculation yields that

$$\begin{aligned}
I_2^\tau(f) &= \sum_{k_1,k_2\in\mathbb{Z}} \int_0^\tau \mathrm{e}^{is(k_1^2-k_2^2-(k_1-k_2)^2)}\widehat{f}_{k_1}\overline{\widehat{f}_{k_2}}\mathrm{d}s\, \mathrm{e}^{i(k_1-k_2)x} \\
&= \sum_{k_1\neq k_2,k_2\neq 0} \frac{\mathrm{e}^{i\tau(k_1^2-k_2^2-(k_1-k_2)^2)}-1}{2ik_2(k_1-k_2)}\widehat{f}_{k_1}\overline{\widehat{f}_{k_2}}\mathrm{e}^{i(k_1-k_2)x} + \tau\overline{\widehat{f}_0}\sum_{k\in\mathbb{Z}}\widehat{f}_k\mathrm{e}^{ikx} \\
&\quad + \tau\|f\|^2 - \tau|\widehat{f}_0|^2
\end{aligned}$$

$$= -\frac{i}{2}\partial_x^{-1} e^{i\tau\partial_x^2}\left[(e^{-i\tau\partial_x^2} f)(e^{i\tau\partial_x^2}\partial_x^{-1}\overline{f})\right] + \frac{i}{2}\partial_x^{-1}\left[f(\partial_x^{-1}\overline{f})\right]$$
$$+ \tau\overline{\widehat{f}_0} f + \tau\|f\|^2 - \tau|\widehat{f}_0|^2. \tag{3.5}$$

For $I_0^\tau(f)$, a direct computation gives that

$$\begin{aligned} I_0^\tau(f) &= \sum_{k_1,k_2\in\mathbb{Z}} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} e^{-i(k_1+k_2)x} \int_0^\tau e^{-is\left[2(k_1+k_2)^2 - 2k_1k_2\right]} ds \\ &= \sum_{k_1,k_2\in\mathbb{Z}} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} e^{-i(k_1+k_2)x} \int_0^\tau e^{-2is(k_1+k_2)^2}(1 + e^{2isk_1k_2} - 1) ds \\ &= I_3^\tau(f) + P_1^\tau(f), \end{aligned}$$

where

$$I_3^\tau(f) = \tau \sum_{k_1,k_2\in\mathbb{Z}} \psi_1(-2i\tau(k_1+k_2)^2)\overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} e^{-i(k_1+k_2)x} = \tau\psi_1(2i\tau\partial_x^2)(\overline{f})^2, \tag{3.6}$$

and by Lemma 1,

$$\begin{aligned} \|P_1^\tau(f)\|_r^2 &= \sum_{l\in\mathbb{Z}} (1+l^2)^r \left| \sum_{k_1+k_2=l} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} \int_0^\tau e^{-2isl^2}(e^{2isk_1k_2} - 1) ds \right|^2 \\ &\le 4\tau^{2+2\gamma} \sum_{l\in\mathbb{Z}} (1+l^2)^r \Big(\sum_{k_1+k_2=l} |k_1|^\gamma |k_2|^\gamma |\widehat{f}_{k_1}||\widehat{f}_{k_2}| \Big)^2 \\ &\le 4\tau^{2+2\gamma} \sum_{l\in\mathbb{Z}} (1+l^2)^r \Big(\sum_{k_1+k_2=l} (1+k_1^2)^{\gamma/2}(1+k_2^2)^{\gamma/2} |\widehat{f}_{k_1}||\widehat{f}_{k_2}| \Big)^2. \end{aligned} \tag{3.7}$$

Let g denote the function

$$g(x) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\gamma/2} |\widehat{f}_k| e^{ikx}.$$

This implies $\|g\|_r = \|f\|_{r+\gamma}$ and

$$\|P_1^\tau(f)\|_r \le 2\tau^{1+\gamma}\|g^2\|_r \le 2D_r\tau^{1+\gamma}\|f\|_{r+\gamma}^2. \tag{3.8}$$

Further, let

$$\Phi^\tau(f) = \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} f - \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} f - \frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\psi_1(2i\tau\partial_x^2)\overline{f}$$
$$- \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\big[I_1^\tau(f) + 2I_2^\tau(f) + I_3^\tau(f)\big], \tag{3.9}$$

where the terms I_1^τ, I_2^τ and I_3^τ are defined in (3.4), (3.5) and (3.6), respectively. Combining the above approximation and noting that I_1, I_2 are exact, we obtain that

$$u(t_n + \tau) = \Phi^\tau(u(t_n)) + \mathcal{R}_0(\tau^2) + \mathcal{R}_\gamma(\tau^{1+\gamma}). \tag{3.10}$$

The above expansion motivates us to define the scheme as

$$u^{n+1} = \Phi^\tau(u^n), \quad n \geq 0, \quad u^0 = u_0. \tag{3.11}$$

Regarding stability, we have the following estimate.

Lemma 5 *Let $r > 1/2$ and $f, g \in H^r$. Then, for all $\tau \geq 0$ we have*

$$\|\Phi^\tau(f) - \Phi^\tau(g)\|_r \leq (1 + M\tau)\|f - g\|_r,$$

where $M = \sqrt{c} + D_r(\|f\|_r + \|g\|_r)$.

Proof For simplicity of notation, let $Q = \|f\|_r + \|g\|_r$. Using (3.3) and (2.7), we have

$$\begin{aligned}\|I_1^\tau(f) - I_1^\tau(g)\|_r &\leq \tau \sup_{0\leq s\leq\tau} \|(\mathrm{e}^{-is\partial_x^2} f)^2 - (\mathrm{e}^{-is\partial_x^2} g)^2\|_r \\ &\leq D_r\tau \sup_{0\leq s\leq\tau} \|\mathrm{e}^{-is\partial_x^2}(f+g)\|_r\|\mathrm{e}^{-is\partial_x^2}(f-g)\|_r \\ &\leq QD_r\tau\|f-g\|_r. \end{aligned}\tag{3.12}$$

A similar calculation gives that

$$\|I_2^\tau(f) - I_2^\tau(g)\|_r \leq QD_r\tau\|f - g\|_r. \tag{3.13}$$

With the aid of Lemma 2, we obtain

$$\|I_3^\tau(f) - I_3^\tau(g)\|_r \leq \tau\|f^2 - g^2\|_r \leq QD_r\tau\|f - g\|_r. \tag{3.14}$$

Combining (3.12)–(3.14) and applying Lemmas 2 and 3, we finally derive

$$\begin{aligned}\|\Phi^\tau(f) - \Phi^\tau(g)\|_r &\leq (1 + \sqrt{c}\tau)\|f - g\|_r + QD_r\tau\|f - g\|_r \\ &\leq \big[1 + \tau(\sqrt{c} + QD_r)\big]\,\|f - g\|_r,\end{aligned}$$

which completes the proof. □

The local error expansion (3.10) and the stability property allow us to prove the following error bound.

Theorem 1 *Let $r > 1/2$ and $0 < \gamma \leq 1$. Assume that the exact solution of (2.4) satisfies $u \in H^{r+\gamma}$ for $0 \leq t \leq T$. Then there exists a constant $\tau_0 > 0$ such that for all step sizes $0 < \tau \leq \tau_0$ and $t_n = n\tau \leq T$ we have that the global error of (3.11) with (3.9) is bounded by*

$$\|u(t_n) - u^n\|_r \leq C\tau^\gamma,$$

where C depends on T, c, r and $\|u\|_{L^\infty(0,T;H^{r+\gamma})}$.

Proof It follows from (3.2) that

$$\|u(t_{k+1}) - \Phi^\tau(u(t_k))\|_r \leq M_1\tau^{1+\gamma},$$

where M_1 depends on $\|u\|_{L^\infty(0,T;H^{r+\gamma})}$. The triangle inequality yields

$$\begin{aligned}\|u(t_{k+1}) - u^{k+1}\|_r &= \|u(t_{k+1}) - \Phi^\tau(u^k)\|_r \\ &\leq \|u(t_{k+1}) - \Phi^\tau(u(t_k))\|_r + \|\Phi^\tau(u(t_k)) - \Phi^\tau(u^k)\|_r \\ &\leq M_1\tau^{1+\gamma} + \|\Phi^\tau(u(t_k)) - \Phi^\tau(u^k)\|_r.\end{aligned}$$

By applying Lemma 5 for $u^k \in H^r$, $0 \leq k \leq n$, we get

$$\begin{aligned}\|u(t_{n+1}) - u^{n+1}\|_r &\leq M_1\tau^{1+\gamma} + \mathrm{e}^{\tau L}\|u(t_n) - u^n\|_r \\ &\leq M_1\tau^{1+\gamma} + \mathrm{e}^{\tau L}\left(M_1\tau^{1+\gamma} + \mathrm{e}^{\tau L}\|u(t_{n-1}) - u^{n-1}\|_r\right) \\ &\leq M_1\tau^{1+\gamma}\sum_{k=0}^{n}\mathrm{e}^{t_k L} \leq \frac{M_1}{L}\mathrm{e}^{TL}\tau^\gamma,\end{aligned}$$

where L depends on $\sup_{0\leq k\leq n}\|u(t_k)\|_r$ and $\sup_{0\leq k\leq n}\|u^k\|_r$. Then the assertion follows by induction, respectively, a *Lady Windermere's fan* argument (cf. [17,20]). □

For the scheme u^{n+1} defined in (3.11), we set

$$z^{n+1} = \frac{1}{2}(u^{n+1} + \overline{u^{n+1}}), \quad z_t^{n+1} = \frac{i}{2}\langle\partial_x^2\rangle_c(u^{n+1} - \overline{u^{n+1}}). \tag{3.15}$$

In view of the relationship between u and z, Theorem 1 allows us to get the convergence of z^n.

Corollary 1 *Let $r > 1/2$ and $0 < \gamma \leq 1$. Assume that the exact solution of (2.1) satisfies*

$$z \in H^{r+\gamma}, \quad z_t \in H^{r+\gamma-2},$$

which implies that the solution of (2.4) *satisfies* $u \in H^{r+\gamma}$ *for* $0 \le t \le T$*. Then there exists a constant* $\tau_0 > 0$ *such that for all step sizes* $0 < \tau \le \tau_0$ *and* $t_n = n\tau \le T$ *we have that the global error of* (3.15) *is bounded by*

$$\|z(t_n) - z^n\|_r + \|z_t(t_n) - z_t^n\|_{r-2} \le C\tau^\gamma,$$

where C *depends on* T*,* c*,* r *and* $\|z\|_{L^\infty(0,T;H^{r+\gamma})} + \|z_t\|_{L^\infty(0,T;H^{r+\gamma-2})}$.

Remark 2 Due to the well-posedness result from [18], the GB equation is locally well-posed for initial data $z(0) \in H^s$, $z_t(0) \in H^{s-2}$ with $s \ge -1/2$. Thus the regularity required in Corollary 1 can be obtained whenever the initial data has the same smoothness.

4 Convergence without additional smoothness assumptions

Theorem 1 provides convergence of order γ in H^r for solutions in $H^{r+\gamma}$. The additional spatial smoothness of the solution was needed because one term of $I_0^\tau(f)$ was only roughly estimated. In this section, we will establish a modified estimate which shows low-order convergence of our scheme without additional smoothness assumptions, i.e., convergence in H^r for solutions in H^r. We are grateful to the anonymous reviewer for pointing out this approach to us.

The analysis in this section relies on the classical tame estimate

$$\|uv\|_r \le C_{r,\rho}(\|u\|_r\|v\|_\rho + \|u\|_\rho\|v\|_r), \tag{4.1}$$

which holds on the one-dimensional torus for all $r \ge \rho > \frac{1}{2}$ (see, for example, [1, Lemma B.1]). Using this tool, we will prove the following convergence result.

Theorem 2 *Let* $r > 1/2$ *and* $0 < \gamma < \frac{r-1/2}{3r+1/2}$*. Assume that the exact solution of* (2.1) *satisfies*

$$z \in H^r, \quad z_t \in H^{r-2},$$

which implies that the solution of (2.4) *satisfies* $u \in H^r$ *for* $0 \le t \le T$*. Then there exists a constant* $\tau_0 > 0$ *such that for all step sizes* $0 < \tau \le \tau_0$ *and* $t_n = n\tau \le T$*, the global errors of* (3.11) *and* (3.15) *satisfy*

$$\|u(t_n) - u^n\|_r \le C\tau^\gamma, \quad \|z(t_n) - z^n\|_r + \|z_t(t_n) - z_t^n\|_{r-2} \le C\tau^\gamma,$$

where C *depends on* T*,* c*,* r *and* $\|z\|_{L^\infty(0,T;H^r)} + \|z_t\|_{L^\infty(0,T;H^{r-2})}$.

Proof We repeate the proof of Theorem 1 by giving a sharper estimate of the term $P_1^\tau(f)$. For this purpose, we recall

$$\begin{aligned} P_1^\tau(f) &= \sum_{k_1,k_2\in\mathbb{Z}} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} \mathrm{e}^{-i(k_1+k_2)x} \int_0^\tau \mathrm{e}^{-2is(k_1+k_2)^2} (\mathrm{e}^{2isk_1k_2} - 1)\mathrm{d}s \\ &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{-ikx} \sum_{l\in\mathbb{Z}} \overline{\widehat{f}_l}\,\overline{\widehat{f}_{k-l}} \int_0^\tau \mathrm{e}^{-2isk^2} (\mathrm{e}^{2isl(k-l)} - 1)\mathrm{d}s \\ &= \frac{i}{2} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-ikx} \sum_{l\in\mathbb{Z}} \overline{\widehat{f}_l}\,\overline{\widehat{f}_{k-l}} \left[\frac{\mathrm{e}^{-2i\tau(k^2+l^2-kl)} - 1}{k^2+l^2-kl} - \frac{\mathrm{e}^{-2i\tau k^2} - 1}{k^2} \right]. \end{aligned} \tag{4.2}$$

For $w \in H^r$ with $\widehat{w}_k = 0$ for $|k| < N$ and $\rho \le r$, we have

$$\|w\|_\rho^2 = \sum_{|k|\ge N} (1+k^2)^\rho |\widehat{w}_k|^2 \le N^{2(\rho-r)} \sum_{|k|\ge N} (1+k^2)^r |\widehat{w}_k|^2 = N^{2(\rho-r)} \|w\|_r^2,$$

which yields that $\|w\|_\rho \le N^{\rho-r}\|w\|_r$. With the help of (4.1) we are thus able to improve the bilinear estimate in the following way:

$$\|uv\|_r \le 2C_{r,\rho} N^{\rho-r} \|u\|_r \|v\|_r \tag{4.3}$$

for $u, v \in H^r$ with $\widehat{u}_k = \widehat{v}_k = 0$ for $|k| < N$.

For $\beta > \alpha > 0$ still to be chosen, we set $N = [\tau^{-\alpha}] + 1$, where $[\,\cdot\,]$ denotes the integer part, and $M = [\tau^{-\beta}] + 1$. Next, we decompose $P_1^\tau(f)$ into three (not necessarily disjoint) sums, depending on the indices in (4.2)

$$\begin{aligned} &P_{1,1}^\tau(f){:}\quad |l| \ge N \quad \text{and} \quad |k-l| \ge N, \\ &P_{1,2}^\tau(f){:}\quad |l| < M \quad \text{and} \quad |k-l| < M, \\ &P_{1,3}^\tau(f){:}\quad |l| < N \le M \le |k-l| \quad \text{or} \quad |k-l| < N \le M \le |l|. \end{aligned}$$

Let $\widehat{g}_k = \widehat{f}_k$ for $|k| \ge N$ and $\widehat{g}_k = 0$ else. With the help of (4.3), the first term is readily estimated as

$$\begin{aligned} \|P_{1,1}^\tau(f)\|_r &\le \Big(\sum_{k\in\mathbb{Z}} (1+k^2)^r \Big(\sum_{l\in\mathbb{Z}} 2\tau |\widehat{g}_l| |\widehat{g}_{k-l}| \Big)^2 \Big)^{1/2} \\ &\le 2\tau \|g^2\|_r \le 4C_{r,\rho}\tau N^{\rho-r} \|g\|_r^2 \\ &\le 4C_{r,\rho}\tau^{1+\alpha(r-\rho)} \|f\|_r^2. \end{aligned}$$

On the other hand, for the second term, we get by a straightforward estimate

$$\begin{aligned}\|P^{\tau}_{1,2}(f)\|_r^2 &= \Big\| \sum_{k\in\mathbb{Z}} \mathrm{e}^{-ikx} \sum_{|l|,|k-l|<M} \overline{\widehat{f}_l}\, \overline{\widehat{f}_{k-l}} \int_0^{\tau} \mathrm{e}^{-2isk^2}(\mathrm{e}^{2isl(k-l)}-1)\mathrm{d}s \Big\|_r^2 \\ &\le \tau^4 \sum_{k\in\mathbb{Z}} (1+k^2)^r \Big(\sum_{|l|,|k-l|<M} |l||k-l||\widehat{f}_l||\widehat{f}_{k-l}| \Big)^2 \\ &\le \tau^4 (M-1)^4 \sum_{k\in\mathbb{Z}} (1+k^2)^r \Big(\sum_{|l|,|k-l|<M} |\widehat{f}_l||\widehat{f}_{k-l}| \Big)^2 \le \tau^{4-4\beta} D_r^2 \|f\|_r^4,\end{aligned}$$

which directly gives

$$\|P^{\tau}_{1,2}(f)\|_r \le D_r \tau^{2-2\beta} \|f\|_r^2.$$

The remaining term $P^{\tau}_{1,3}(f)$ only includes indices for which $|l| < N \le M \le |k-l|$ or $|k-l| < N \le M \le |l|$. We take the former case as an example, the latter one can be treated similarly. Note that in this case we have

$$|k| \ge |k-l| - |l| > \tfrac{1}{2}\tau^{-\beta}$$

for τ sufficiently small, which implies that

$$|l|/|k| < 2\tau^{\beta-\alpha}.$$

Using

$$\frac{1}{k^2+l^2-kl} = \frac{1}{k^2}\left(1 + \mathcal{O}\left(\frac{l}{k}\right)\right),$$

we expand the term in the bracket in (4.2) to obtain

$$\begin{aligned}\left| \frac{\mathrm{e}^{-2i\tau(k^2+l^2-kl)}-1}{k^2+l^2-kl} - \frac{\mathrm{e}^{-2i\tau k^2}-1}{k^2} \right| &\le \frac{1}{k^2}\left| \mathrm{e}^{-2i\tau(k^2+l^2-kl)} - \mathrm{e}^{-2i\tau k^2} \right| + \mathcal{O}\left(\frac{\tau l}{k}\right) \\ &\le \frac{1}{k^2}\left| \mathrm{e}^{-2i\tau(l^2-kl)} - 1 \right| + \mathcal{O}\left(\frac{\tau l}{k}\right) \\ &= \mathcal{O}(\tau^{1+\beta-\alpha}).\end{aligned}$$

This implies that $P^{\tau}_{1,3}(f)$ can be bounded as

$$\|P^{\tau}_{1,3}(f)\|_r \le C\tau^{1+\beta-\alpha}\|f\|_r^2.$$

We still have to determine appropriate values of α, β, ρ, and γ. Requiring that $P^{\tau}_{1,1}(f)$, $P^{\tau}_{1,2}(f)$, and $P^{\tau}_{1,3}(f)$ have the same order $1+\gamma$ yields the condition

$$1+\gamma = 1+\alpha(r-\rho) = 2-2\beta = 1+\beta-\alpha$$

from where we deduce

$$\beta = \frac{1}{2}(1-\gamma), \quad \alpha = \frac{1-3\gamma}{2}, \quad \rho = r - \frac{2\gamma}{1-3\gamma}.$$

Finally, we have to guarantee that $\frac{1}{2} < \rho \leq r$. Obviously, this condition is equivalent to

$$0 < \gamma < \frac{2r-1}{6r+1}.$$

Altogether we have shown that

$$I_0^\tau(f) = I_3^\tau(f) + \mathcal{R}_0(\tau^{1+\gamma}),$$

which together with (3.2) gives the desired convergence result. □

5 A second-order exponential-type integrator

To construct the second-order scheme, we need the following expansions of the operators e^{isA} and $\mathrm{e}^{is\partial_x^2}$ based on the bounds given in Lemma 1.

Lemma 6 *For all $s \in \mathbb{R}$ and $v \in H^r$ $(r \geq 0)$, we have*

$$e^{\pm isA} v = v \pm isAv + \mathcal{R}_0(s^2). \tag{5.1}$$

If $0 \leq \gamma \leq 1$ and $v \in H^{r+2\gamma}$, then

$$e^{\pm is\partial_x^2} v = v + \mathcal{R}_{2\gamma}(s^\gamma). \tag{5.2}$$

Proof The first assertion (5.1) follows from the second-order expansion in Lemma 1 and the boundedness of the operator A (cf. Lemma 3). Similarly, the first-order expansion in Lemma 1 yields the second identity. □

Using (2.6) and (3.1), we can approximate $w(t_n+s)$ as follows:

$$\begin{aligned} w(t_n+s) = \mathrm{e}^{isA} w(t_n) &- \frac{ic}{2}\langle\partial_x^2\rangle_c^{-1} \int_0^s \mathrm{e}^{i(s-y)A}\left[w(t_n) + \mathrm{e}^{2i(t_n+y)\partial_x^2}\overline{w(t_n)}\right]\mathrm{d}y \\ &- \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1} \int_0^s \mathrm{e}^{i(s-y)A}\mathrm{e}^{i(t_n+y)\partial_x^2}\left[\mathrm{e}^{-iy\partial_x^2}u(t_n) + \mathrm{e}^{iy\partial_x^2}\overline{u(t_n)}\right]^2 \mathrm{d}y \\ &+ \mathcal{R}_0(s^2), \end{aligned}$$

where we used the identity $w(t_n) = \mathrm{e}^{it_n\partial_x^2}u(t_n)$. Let $0 < \gamma \leq 1$. Using (5.1) and (5.2), we get

$$\begin{aligned} w(t_n+s) &= \mathrm{e}^{isA}w(t_n) - \frac{ics}{2}\langle\partial_x^2\rangle_c^{-1}\left[w(t_n)+\mathrm{e}^{2it_n\partial_x^2}\overline{w(t_n)}\right] \\ &\quad - \frac{is\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{it_n\partial_x^2}\left[u(t_n)+\overline{u(t_n)}\right]^2 + \mathcal{R}_0(s^2) + \mathcal{R}_{2\gamma}(s^{1+\gamma}) \\ &= w(t_n) + isv(t_n) + \mathcal{R}_{2\gamma}(s^{1+\gamma}), \end{aligned} \tag{5.3}$$

where

$$v(t_n) = Aw(t_n) - \frac{1}{4}\langle\partial_x^2\rangle_c^{-1}\left[2c\big(w(t_n)+\mathrm{e}^{2it_n\partial_x^2}\overline{w(t_n)}\big) + \partial_x^2\mathrm{e}^{it_n\partial_x^2}(u(t_n)+\overline{u(t_n)})^2\right]. \tag{5.4}$$

Plugging (5.3) into (2.6) and applying Lemma 6, we get

$$\begin{aligned} &w(t_n+\tau) \\ &\quad = \mathrm{e}^{i\tau A}w(t_n) + \mathcal{R}_{2\gamma}(\tau^{2+\gamma}) \\ &\quad - \frac{ic}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\int_0^\tau \mathrm{e}^{-isA}\left[w(t_n)+isv(t_n)+\mathrm{e}^{2i(t_n+s)\partial_x^2}(\overline{w(t_n)}-is\overline{v(t_n)})\right]\mathrm{d}s \\ &\quad - \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\mathrm{e}^{it_n\partial_x^2}\int_0^\tau \mathrm{e}^{is(\partial_x^2-A)}\left[2\mathrm{Re}\left(\mathrm{e}^{-is\partial_x^2}(u(t_n)+is\mu(t_n))\right)\right]^2\mathrm{d}s \\ &\quad = \mathrm{e}^{i\tau A}w(t_n) - \frac{ic}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\Big[\int_0^\tau [(1-isA)w(t_n)+isv(t_n)]\,\mathrm{d}s \\ &\quad + \mathrm{e}^{2it_n\partial_x^2}\int_0^\tau \mathrm{e}^{2is\partial_x^2}\left[(1-isA)\overline{w(t_n)}-is\overline{v(t_n)}\right]\mathrm{d}s\Big] \\ &\quad - \frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau A}\mathrm{e}^{it_n\partial_x^2}\int_0^\tau \mathrm{e}^{is\partial_x^2}\Big[(1-isA)\big(\mathrm{e}^{-is\partial_x^2}u(t_n)+\mathrm{e}^{is\partial_x^2}\overline{u(t_n)}\big)^2 \\ &\quad +2is\big(\mathrm{e}^{-is\partial_x^2}u(t_n)+\mathrm{e}^{is\partial_x^2}\overline{u(t_n)}\big)\big(\mathrm{e}^{-is\partial_x^2}\mu(t_n)-\mathrm{e}^{is\partial_x^2}\overline{\mu(t_n)}\big)\Big]\mathrm{d}s + \mathcal{R}_{2\gamma}(\tau^{2+\gamma}), \end{aligned} \tag{5.5}$$

where

$$\mu(t_n) = \mathrm{e}^{-it_n\partial_x^2}v(t_n) = \mathcal{P}(u(t_n)),$$

with

$$\mathcal{P}(f) = Af - \frac{1}{4}\langle\partial_x^2\rangle_c^{-1}\left[2c\big(f+\overline{f}\big)+\partial_x^2(f+\overline{f})^2\right]. \tag{5.6}$$

Twisting the variable back, we get the approximation for $u(t_n+\tau)$ as

$$\begin{aligned} &u(t_n+\tau) \\ &\quad = \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}u(t_n) - \frac{ic}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\Big[\int_0^\tau [(1-isA)u(t_n)+is\mu(t_n)]\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\tau \mathrm{e}^{2is\partial_x^2}\big[(1-isA)\overline{u(t_n)}-is\overline{\mu(t_n)}\big]\mathrm{d}s\Big] \\
&-\frac{i\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\int_0^\tau \mathrm{e}^{is\partial_x^2}\Big[(1-isA)\big(\mathrm{e}^{-is\partial_x^2}u(t_n)+\mathrm{e}^{is\partial_x^2}\overline{u(t_n)}\big)^2 \\
&+2is\big(\mathrm{e}^{-is\partial_x^2}u(t_n)+\mathrm{e}^{is\partial_x^2}\overline{u(t_n)}\big)\big(\mathrm{e}^{-is\partial_x^2}\mu(t_n)-\mathrm{e}^{is\partial_x^2}\overline{\mu(t_n)}\big)\Big]\mathrm{d}s+\mathcal{R}_{2\gamma}(\tau^{2+\gamma}) \\
=&\ \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}u(t_n)+\mathcal{L}(u(t_n))+\frac{\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\Big(-iI_1^\tau(u(t_n))-2iI_2^\tau(u(t_n)) \\
&-iI_0^\tau(u(t_n))-A\big[J_1^\tau(u(t_n),u(t_n))+2J_2^\tau(u(t_n),u(t_n))+J_0^\tau(u(t_n),u(t_n))\big] \\
&+2J_1^\tau(u(t_n),\mu(t_n)) \\
&+2\big[J_2^\tau(\mu(t_n),u(t_n))-J_0^\tau(u(t_n),\mu(t_n))-J_2^\tau(u(t_n),\mu(t_n))\big]\Big) \\
&+\mathcal{R}_{2\gamma}(\tau^{2+\gamma}),
\end{aligned}
\tag{5.7}
$$

where I_1^τ, I_2^τ, I_0^τ are defined as (3.3),

$$
\begin{aligned}
\mathcal{L}(f)=-\frac{ic\tau}{2}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\Big[&f-\frac{i\tau}{2}(Af-\mathcal{P}(f))+\psi_1(2i\tau\partial_x^2)\overline{f} \\
&-i\tau\psi_2(2i\tau\partial_x^2)\big(A\overline{f}+\overline{\mathcal{P}(f)}\big)\Big],
\end{aligned}
\tag{5.8}
$$

and

$$
\begin{aligned}
J_0^\tau(f,g)&=\int_0^\tau s\mathrm{e}^{is\partial_x^2}\big(\mathrm{e}^{is\partial_x^2}\overline{f}\big)\big(\mathrm{e}^{is\partial_x^2}\overline{g}\big)\mathrm{d}s, \\
J_1^\tau(f,g)&=\int_0^\tau s\mathrm{e}^{is\partial_x^2}\big(\mathrm{e}^{-is\partial_x^2}f\big)\big(\mathrm{e}^{-is\partial_x^2}g\big)\mathrm{d}s, \\
J_2^\tau(f,g)&=\int_0^\tau s\mathrm{e}^{is\partial_x^2}\big(\mathrm{e}^{-is\partial_x^2}f\big)\big(\mathrm{e}^{is\partial_x^2}\overline{g}\big)\mathrm{d}s.
\end{aligned}
$$

The integral in $J_1^\tau(f,g)$ can be expressed in terms of the Fourier coefficients as follows

$$
\begin{aligned}
J_1^\tau(f,g)&=\sum_{k_1,k_2\in\mathbb{Z}}\int_0^\tau s\mathrm{e}^{is(k_1^2+k_2^2-(k_1+k_2)^2)}\widehat{f}_{k_1}\widehat{g}_{k_2}\mathrm{d}s\,\mathrm{e}^{i(k_1+k_2)x} \\
&=\sum_{k_1\neq0,k_2\neq0}\left[\frac{i\tau\mathrm{e}^{-2i\tau k_1k_2}}{2k_1k_2}+\frac{\mathrm{e}^{-2i\tau k_1k_2}-1}{4k_1^2k_2^2}\right]\widehat{f}_{k_1}\widehat{g}_{k_2}\mathrm{e}^{i(k_1+k_2)x} \\
&\quad+\frac{\tau^2}{2}\widehat{f}_0\sum_{k\in\mathbb{Z}}\widehat{g}_k\mathrm{e}^{ikx}+\frac{\tau^2}{2}\widehat{g}_0\sum_{k\in\mathbb{Z}}\widehat{f}_k\mathrm{e}^{ikx}-\frac{\tau^2}{2}\widehat{f}_0\widehat{g}_0
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{i\tau}{2}\mathrm{e}^{i\tau\partial_x^2}\left[\left(\partial_x^{-1}\mathrm{e}^{-i\tau\partial_x^2}f\right)\left(\partial_x^{-1}\mathrm{e}^{-i\tau\partial_x^2}g\right)\right] - \frac{1}{4}\left(\partial_x^{-2}f\right)\left(\partial_x^{-2}g\right)\\
&\quad + \frac{1}{4}\mathrm{e}^{i\tau\partial_x^2}\left[\left(\partial_x^{-2}\mathrm{e}^{-i\tau\partial_x^2}f\right)\left(\partial_x^{-2}\mathrm{e}^{-i\tau\partial_x^2}g\right)\right] + \frac{\tau^2}{2}\widehat{f}_0 g + \frac{\tau^2}{2}\widehat{g}_0 f - \frac{\tau^2}{2}\widehat{f}_0\widehat{g}_0.
\end{aligned} \tag{5.9}
$$

A similar calculation yields that

$$
\begin{aligned}
J_2^\tau(f,g) &= \sum_{k_1,k_2\in\mathbb{Z}} \int_0^\tau s\mathrm{e}^{is(k_1^2-k_2^2-(k_1-k_2)^2)}\widehat{f}_{k_1}\overline{\widehat{g}_{k_2}}\mathrm{d}s\,\mathrm{e}^{i(k_1-k_2)x}\\
&= \sum_{k_1\neq k_2, k_2\neq 0}\left[\frac{\tau\mathrm{e}^{2i\tau k_2(k_1-k_2)}}{2ik_2(k_1-k_2)} + \frac{\mathrm{e}^{2i\tau k_2(k_1-k_2)}-1}{4k_2^2(k_1-k_2)^2}\right]\widehat{f}_{k_1}\overline{\widehat{g}_{k_2}}\mathrm{e}^{i(k_1-k_2)x}\\
&\quad + \frac{\tau^2}{2}\overline{\widehat{g}_0}\sum_k \widehat{f}_k\mathrm{e}^{ikx} + \frac{\tau^2}{2}\sum_k \widehat{f}_k\overline{\widehat{g}_k} - \frac{\tau^2}{2}\widehat{f}_0\overline{\widehat{g}_0}\\
&= \frac{i\tau}{2}\partial_x^{-1}\mathrm{e}^{i\tau\partial_x^2}\left[\left(\mathrm{e}^{-i\tau\partial_x^2}f\right)\left(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\overline{g}\right)\right]\\
&\quad + \frac{1}{4}\partial_x^{-2}\mathrm{e}^{i\tau\partial_x^2}\left[\left(\mathrm{e}^{-i\tau\partial_x^2}f\right)\left(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-2}\overline{g}\right)\right]\\
&\quad - \frac{1}{4}\partial_x^{-2}\left[f\left(\partial_x^{-2}\overline{g}\right)\right] + \frac{\tau^2}{2}\overline{\widehat{g}_0}f + \frac{\tau^2}{2}(f,g) - \frac{\tau^2}{2}\widehat{f}_0\overline{\widehat{g}_0},
\end{aligned} \tag{5.10}
$$

where $(\cdot,\cdot)$ represents the inner product in L^2 defined as $(f,g) = \frac{1}{2\pi}\int_\Omega f(x)\overline{g(x)}\mathrm{d}x$. Similarly, one obtains

$$
\begin{aligned}
J_0^\tau(f,g) &= \sum_{k_1,k_2\in\mathbb{Z}}\int_0^\tau s\mathrm{e}^{-is[2(k_1+k_2)^2-2k_1k_2]}\mathrm{d}s\,\overline{\widehat{f}_{k_1}}\overline{\widehat{g}_{k_2}}\mathrm{e}^{-i(k_1+k_2)x}\\
&= \sum_{k_1,k_2\in\mathbb{Z}}\int_0^\tau s\mathrm{e}^{-2is(k_1+k_2)^2}\mathrm{d}s\,\overline{\widehat{f}_{k_1}}\overline{\widehat{g}_{k_2}}\mathrm{e}^{-i(k_1+k_2)x} + P_0^\tau(f,g)\\
&= \sum_{k_1+k_2\neq 0}\left[\frac{i\tau\mathrm{e}^{-2i\tau(k_1+k_2)^2}}{2(k_1+k_2)^2} + \frac{\mathrm{e}^{-2i\tau(k_1+k_2)^2}-1}{4(k_1+k_2)^4}\right]\overline{\widehat{f}_{k_1}}\overline{\widehat{g}_{k_2}}\mathrm{e}^{-i(k_1+k_2)x}\\
&\quad + \frac{\tau^2}{2}\sum_{k\in\mathbb{Z}}\overline{\widehat{f}_k}\overline{\widehat{g}_{-k}} + P_0^\tau(f,g) = J_3^\tau(f,g) + P_0^\tau(f,g),
\end{aligned} \tag{5.11}
$$

where

$$
J_3^\tau(f,g) = -\frac{i\tau}{2}\mathrm{e}^{2i\tau\partial_x^2}\partial_x^{-2}\left(\overline{fg}\right) + \frac{i\tau}{2}\psi_1(2i\tau\partial_x^2)\partial_x^{-2}\left(\overline{fg}\right) + \frac{\tau^2}{2}(\overline{f},g). \tag{5.12}
$$

By Lemma 1, we have

$$\begin{aligned}\|P_0^\tau(f,g)\|_r^2 &= \sum_{l\in\mathbb{Z}}(1+l^2)^r\Big|\sum_{k_1+k_2=l}\overline{\widehat{f}_{k_1}}\overline{\widehat{g}_{k_2}}\int_0^\tau s\mathrm{e}^{-2isl^2}(\mathrm{e}^{2isk_1k_2}-1)\mathrm{d}s\Big|^2\\ &\le \tau^{4+2\gamma}\sum_{l\in\mathbb{Z}}(1+l^2)^r\Big(\sum_{k_1+k_2=l}|k_1|^\gamma|k_2|^\gamma|\widehat{f}_{k_1}||\widehat{g}_{k_2}|\Big)^2\\ &\le \tau^{4+2\gamma}\sum_{l\in\mathbb{Z}}(1+l^2)^r\Big(\sum_{k_1+k_2=l}(1+k_1^2)^{\gamma/2}(1+k_2^2)^{\gamma/2}|\widehat{f}_{k_1}||\widehat{g}_{k_2}|\Big)^2,\end{aligned}$$

which implies that

$$\|P_0^\tau(f,g)\|_r \le D_r\tau^{2+\gamma}\|f\|_{r+\gamma}\|g\|_{r+\gamma}. \tag{5.13}$$

It follows from Lemma 3 that

$$\|\mu(t_n)\|_{r+\gamma} \le 2C_1\|u(t_n)\|_{r+\gamma} + D_{r+\gamma}\|u(t_n)\|_{r+\gamma}^2,$$

which together with (5.11) yields that

$$\begin{aligned}J_0^\tau(u(t_n),\mu(t_n)) &= J_3^\tau(u(t_n),\mu(t_n)) + \mathcal{R}_\gamma(\tau^{2+\gamma}),\\ J_0^\tau(u(t_n),u(t_n)) &= J_3^\tau(u(t_n),u(t_n)) + \mathcal{R}_\gamma(\tau^{2+\gamma}).\end{aligned} \tag{5.14}$$

Instead of approximating I_0^τ by I_3^τ with the remainder $\mathcal{R}_\gamma(\tau^{1+\gamma})$, it remains to find a refined approximation to the integral I_0^τ. For this aim, we will employ the following decomposition

$$\begin{aligned}\mathrm{e}^{-is[2(k_1+k_2)^2-2k_1k_2]} &= \mathrm{e}^{-2is(k_1+k_2)^2}\mathrm{e}^{2isk_1k_2}\\ &= \mathrm{e}^{-2is(k_1+k_2)^2} + \mathrm{e}^{2isk_1k_2} - 1 + P_2^s(k_1,k_2),\end{aligned}$$

where, by Lemma 1,

$$|P_2^s(k_1,k_2)| = |\mathrm{e}^{-2is(k_1+k_2)^2}-1||\mathrm{e}^{2isk_1k_2}-1| \le 4s^{1+\gamma}|k_1+k_2|^{2\gamma}|k_1||k_2|.$$

Hence we have

$$I_0^\tau(f) = \sum_{k_1,k_2\in\mathbb{Z}}\overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}}\mathrm{e}^{-i(k_1+k_2)x}\int_0^\tau\Big[\mathrm{e}^{-2is(k_1+k_2)^2}+\mathrm{e}^{2isk_1k_2}-1\Big]\mathrm{d}s + P_2^\tau(f),$$

where

$$\|P_2^\tau(f)\|_r^2 = \sum_{l\in\mathbb{Z}}(1+l^2)^r \Big| \sum_{k_1+k_2=l} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}} \int_0^\tau P_2^s(k_1,k_2)\mathrm{d}s\Big|^2$$
$$\le 16\tau^{4+2\gamma}\sum_{l\in\mathbb{Z}}(1+l^2)^{r+2\gamma}\Big(\sum_{k_1+k_2=l}|k_1||k_2||\widehat{f}_{k_1}||\widehat{f}_{k_2}|\Big)^2,$$

which implies that

$$\|P_2^\tau(f)\|_r \le 4D_{r+2\gamma}\tau^{2+\gamma}\|f\|_{r+1+2\gamma}^2.$$

Hence

$$I_0^\tau(f) = \sum_{k_1,k_2\in\mathbb{Z}} \overline{\widehat{f}_{k_1}}\,\overline{\widehat{f}_{k_2}}\mathrm{e}^{-i(k_1+k_2)x}\int_0^\tau \Big[\mathrm{e}^{-2is(k_1+k_2)^2}+\mathrm{e}^{2isk_1k_2}-1\Big]\mathrm{d}s$$
$$+\mathcal{R}_{1+2\gamma}(\tau^{2+\gamma})$$
$$= I_3^\tau(f)+\overline{I_1(f)}-\tau\overline{f}^2+\mathcal{R}_{1+2\gamma}(\tau^{2+\gamma}), \tag{5.15}$$

where I_1^τ and I_3^τ are defined as (3.4) and (3.6), respectively.

Note that I_1, I_2, J_1 and J_2 are exact. Combining (5.7) with the approximations (5.14) and (5.15), we get

$$u(t_{n+1}) = \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}u(t_n)+\mathcal{L}(u(t_n))+\frac{\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\Big(-i\big[I_1^\tau(u(t_n))+\overline{I_1^\tau(u(t_n))}$$
$$+2I_2^\tau(u(t_n))+I_3^\tau(u(t_n))-\tau\overline{u(t_n)}^2\big]$$
$$-A\big[J_1^\tau(u(t_n),u(t_n))+2J_2^\tau(u(t_n),u(t_n))\big]$$
$$-AJ_3^\tau(u(t_n),u(t_n))+2\big[J_1^\tau(u(t_n),\mu(t_n))+J_2^\tau(\mu(t_n),u(t_n))\big]$$
$$-2\big[J_3^\tau(u(t_n),\mu(t_n))+J_2^\tau(u(t_n),\mu(t_n))\big]\Big)+\mathcal{R}_{1+2\gamma}(\tau^{2+\gamma}). \tag{5.16}$$

This motivates us to define the numerical scheme as

$$u^{n+1} = \Psi^\tau(u^n), \tag{5.17}$$

where

$$\Psi^\tau(f) = \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}f+\mathcal{L}(f)$$
$$+\frac{\partial_x^2}{4}\langle\partial_x^2\rangle_c^{-1}\mathrm{e}^{i\tau\langle\partial_x^2\rangle_c}\Big(-2\big[J_3^\tau(f,\mathcal{P}(f))+J_2^\tau(f,\mathcal{P}(f))\big]$$
$$-A\big[J_1^\tau(f,f)+2J_2^\tau(f,f)+J_3^\tau(f,f)\big]$$

$$+ 2\left[J_1^\tau(f, \mathcal{P}(f)) + J_2^\tau(\mathcal{P}(f), f)\right]$$
$$- i\left[I_1^\tau(f) + \overline{I_1^\tau(f)} + 2I_2^\tau(f) + I_3^\tau(f) - \tau \overline{f}^2\right]\Big), \tag{5.18}$$

with $\mathcal{L}$ and $\mathcal{P}$ defined in (5.8) and (5.6), respectively, and I_1^τ, I_2^τ, I_3^τ, J_1^τ, J_2^τ, J_3^τ defined in (3.4), (3.5), (3.6), (5.9), (5.10) and (5.12), respectively.

Regarding stability, we have the following estimate.

Lemma 7 *Let $r > 1/2$ and $f, g \in H^r$. Then, for all $\tau \le 1$, we have*

$$\|\Psi^\tau(f) - \Psi^\tau(g)\|_r \le (1 + M\tau)\|f - g\|_r, \tag{5.19}$$

where M depends on c, r and $\|f\|_r + \|g\|_r$.

Proof We still use the notation $Q = \|f\|_r + \|g\|_r$. By the definitions of $\mathcal{L}$ and $\mathcal{P}$, applying Lemma 3 and (2.7), we get

$$\begin{aligned}
\|\mathcal{P}(f)\|_r &\le (C_1 + \sqrt{c})\|f\|_r + D_r\|f\|_r^2 \le (2C_1 + D_r\|f\|_r)\|f\|_r,\\
\|\mathcal{P}(f) - \mathcal{P}(g)\|_r &\le C_1\|f - g\|_r + \sqrt{c}\|f - g\|_r\\
&\quad + \frac{1}{2}\left(\|f^2 - g^2\|_r + \||f|^2 - |g|^2\|_r\right)\\
&\le M_1\|f - g\|_r,
\end{aligned}$$

where $M_1 = 2C_1 + D_r Q$. This together with Lemma 2 yields that

$$\begin{aligned}
\|\mathcal{L}(f) - \mathcal{L}(g)\|_r &\le \frac{\tau\sqrt{c}}{2}\left[(2 + C_1\tau)\|f - g\|_r + \tau\|\mathcal{P}(f) - \mathcal{P}(g)\|_r\right]\\
&\le \tau M_2\|f - g\|_r,
\end{aligned}$$

where

$$M_2 = \frac{\sqrt{c}}{2}(2 + C_1\tau + M_1\tau) = \frac{\sqrt{c}}{2}(2 + 3C_1\tau + D_r Q\tau).$$

By the definition of J_1 and J_2, we have

$$\begin{aligned}
\|J_1^\tau(f_1, g_1) - J_1^\tau(f_2, g_2)\|_r &\le \left\|\int_0^\tau s\mathrm{e}^{is\partial_x^2}(\mathrm{e}^{-is\partial_x^2} f_1)(\mathrm{e}^{-is\partial_x^2}(g_1 - g_2))\mathrm{d}s\right\|_r\\
&\quad + \left\|\int_0^\tau s\mathrm{e}^{is\partial_x^2}(\mathrm{e}^{-is\partial_x^2}(f_1 - f_2))(\mathrm{e}^{-is\partial_x^2} g_2)\mathrm{d}s\right\|_r\\
&\le D_r\tau^2\left(\|f_1\|_r\|g_1 - g_2\|_r + \|g_2\|_r\|f_1 - f_2\|_r\right),
\end{aligned}$$

which immediately gives that

$$\begin{aligned}
\|J_1^\tau(f, f) - J_1^\tau(g, g)\|_r &\le QD_r\tau^2\|f - g\|_r,\\
\|J_1^\tau(f, \mathcal{P}(f)) - J_1^\tau(g, \mathcal{P}(g))\|_r &\le D_r\tau^2(\|f\|_r\|\mathcal{P}(f) - \mathcal{P}(g)\|_r
\end{aligned}$$

$$\begin{aligned}&+ \|\mathcal{P}(g)\|_r \|f - g\|_r)\\ &\le M_3 \tau^2 \|f - g\|_r,\end{aligned}$$

where

$$M_3 = QD_r(2C_1 + QD_r).$$

Similar calculations show that

$$\begin{aligned}\|J_2^\tau(f, f) - J_2^\tau(g, g)\|_r &\le QD_r\tau^2\|f - g\|_r,\\ \|J_2^\tau(f, \mathcal{P}(f)) - J_2^\tau(g, \mathcal{P}(g))\|_r &\le M_3\tau^2\|f - g\|_r,\\ \|J_2^\tau(\mathcal{P}(f), f) - J_2^\tau(\mathcal{P}(g), g)\|_r &\le M_3\tau^2\|f - g\|_r.\end{aligned}$$

For the approximation term J_3, we estimate the Lipschitz continuity together with the operator in front of it, which allows us to ignore the constant term in (5.12):

$$\begin{aligned}&\left\| \frac{\partial_x^2}{4} \langle \partial_x^2 \rangle_c^{-1} \mathrm{e}^{i\tau\langle\partial_x^2\rangle_c} \left[J_3^\tau(f_1, g_1) - J_3^\tau(f_2, g_2) \right] \right\|_r\\ &= \left\| \frac{\partial_x^2}{4} \langle \partial_x^2 \rangle_c^{-1} \left[J_4^\tau(f_1, g_1) - J_4^\tau(f_2, g_2) \right] \right\|_r\\ &\le \frac{1}{4} \|J_4^\tau(f_1, g_1) - J_4^\tau(f_2, g_2)\|_r, \end{aligned} \tag{5.20}$$

where J_4 is obtained from J_3 by removing the constant

$$J_4^\tau(f, g) = -\frac{i\tau}{2} \mathrm{e}^{2i\tau\partial_x^2} \partial_x^{-2} \left(\overline{fg}\right) + \frac{i\tau}{2} \psi_1(2i\tau\partial_x^2) \partial_x^{-2} \left(\overline{fg}\right). \tag{5.21}$$

Applying Lemma 2, one can easily find that

$$\|J_4^\tau(f_1, g_1) - J_4^\tau(f_2, g_2)\|_r \le D_r\tau \left(\|f_1\|_r\|g_1 - g_2\|_r + \|g_2\|_r\|f_1 - f_2\|_r\right),$$

which implies that

$$\begin{aligned}\|J_4^\tau(f, f) - J_4^\tau(g, g)\|_r &\le QD_r\tau\|f - g\|_r,\\ \|J_4^\tau(f, \mathcal{P}(f)) - J_4^\tau(g, \mathcal{P}(g))\|_r &\le D_r\tau(\|f\|_r\|\mathcal{P}(f) - \mathcal{P}(g)\|_r\\ &\quad + \|\mathcal{P}(g)\|_r\|f - g\|_r)\\ &\le M_3\tau\|f - g\|_r.\end{aligned}$$

Combining the estimates above and noticing (5.20), we derive that

$$\begin{aligned}&\|\Psi^\tau(f) - \Psi^\tau(g)\|_r\\ &\quad \le \|f - g\|_r + \|\mathcal{L}(f) - \mathcal{L}(g)\|_r\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\Big[2\sum_{j=1}^{2}\|I_j^\tau(f)-I_j^\tau(g)\|_r+\|I_3^\tau(f)-I_3^\tau(g)\|_r\\
&+\tau\|f^2-g^2\|_r+2\sum_{j=1}^{2}\|J_j^\tau(f,\mathcal{P}(f))-J_j^\tau(g,\mathcal{P}(g))\|_r\\
&+2\left(\|J_2^\tau(\mathcal{P}(f),f)-J_2^\tau(\mathcal{P}(g),g)\|_r+\|J_4^\tau(f,\mathcal{P}(f))-J_4^\tau(g,\mathcal{P}(g))\|_r\right)\\
&+C_1\left(\|J_1(f,f)-J_1(g,g)\|_r+2\|J_2(f,f)-J_2(g,g)\|_r\right.\\
&\left.+\|J_4(f,f)-J_4(g,g)\|_r\right)\Big]\\
\le\ &(1+M\tau)\|f-g\|_r,
\end{aligned}
$$

where

$$M=M_2+2M_3+(2+C_1)QD_r.$$

This completes the proof. □

Combining the local error expansion (5.16) with the stability estimate (5.19), and applying a similar argument as in the proof of Theorem 1, we get the error bound of the second-order scheme (3.15) with u^n given by (5.17), (5.18) as follows.

Theorem 3 *Let $r > 1/2$ and $0 < \gamma \le 1$. Assume that the exact solution of* (2.4) *satisfies $u \in H^{1+r+2\gamma}$ for $0 \le t \le T$. Then there exists a constant $\tau_0 > 0$ such that for all step sizes $0 < \tau \le \tau_0$ and $t_n = n\tau \le T$ we have that the global error of* (5.18) *is bounded by*

$$\|u(t_n)-u^n\|_r \le C\tau^{1+\gamma},$$

where C depends on T, c, r and $\|u\|_{L^\infty(0,T;H^{1+r+2\gamma})}$.

Corollary 2 *Let $r > 1/2$ and $0 < \gamma \le 1$. Assume that the exact solution of* (2.1) *satisfies $z \in H^{1+r+2\gamma}$ and $z_t \in H^{r+2\gamma-1}$ for $0 \le t \le T$. Then there exists a constant $\tau_0 > 0$ such that for all step sizes $0 < \tau \le \tau_0$ and $t_n = n\tau \le T$ we have that the global error of* (3.15) *combined with* (5.18) *is bounded by*

$$\|z(t_n)-z^n\|_r+\|z_t(t_n)-z_t^n\|_{r-2} \le C\tau^{1+\gamma},$$

where C depends on T, c, r and $\|z\|_{L^\infty(0,T;H^{1+r+2\gamma})}+\|z_t\|_{L^\infty(0,T;H^{r+2\gamma-1})}$.

Note that the error estimate was established under the constraint $r > 1/2$, which enables us to use the the bilinear estimate (2.7) that is crucial for stability (cf. (5.19)). However, we can derive the error bound in L^2 following the approach in [19,20] by using a so-called refined bilinear estimate. First note that Theorem 3 implies that for solutions in H^3, the second-order scheme (5.18) converges with order $\tau^{3/2}$ in H^1.

This gives an a priori bound on the numerical solution u^n in H^1. This together with the refined bilinear estimate

$$\|fg\| \le \|f\|\|g\|_{L^\infty} \le c\|f\|\|g\|_1, \tag{5.22}$$

which is a combination of Hölder's inequality and the Sobolev embedding theorem, yields second-order convergence in L^2.

Corollary 3 *Assume that the exact solution of* (2.1) *satisfies* $z \in H^3$ *and* $z_t \in H^1$ *for* $0 \le t \le T$. *Then there exists a constant* $\tau_0 > 0$ *such that for all step sizes* $0 < \tau \le \tau_0$ *and* $t_n = n\tau \le T$ *we have that the global error of* (3.15) *with* u^n *given by* (5.18) *is bounded by*

$$\|z(t_n) - z^n\| + \|z_t(t_n) - z_t^n\|_{-2} \le C\tau^2,$$

where C *depends on* T, c *and* $\|z\|_{L^\infty(0,T;H^3)} + \|z_t\|_{L^\infty(0,T;H^1)}$.

6 Numerical experiments

In this section, we present some numerical experiments to illustrate our analytic convergence rates given in Corollaries 1, 2 and 3. In the numerical experiments, we use a standard Fourier pseudospectral method for space discretization. We chose the spatial mesh size $\Delta x = 1/2^{10}$ in *Example 1* and $\Delta x = \pi/2^{10}$ in *Example 2*, respectively.

Example 1 In the first experiment, we study the temporal error of the newly developed exponential-type integrators (3.9) and (5.18) for the solitary-wave solution. The well-known solitary-wave solution of the GB Eq. (1.1) is given by

$$z(x,t) = -\frac{3}{2}\lambda^2 \operatorname{sech}^2(\lambda/2(x - vt - x_0)), \quad v = \pm(1-\lambda^2)^{1/2}, \tag{6.1}$$

where $0 < \lambda \le 1$ and x_0 are real parameters. Since the solitary wave decays exponentially in the far field, it is possible to use periodic boundary conditions in case that the domain is chosen large enough. Here we choose $\lambda = 1/2$, $x_0 = 0$ and the torus $\Omega = (-40, 40)$.

Figure 1 displays the H^1-error of the first- and second-order exponential-type schemes (3.9) and (5.18), respectively, at $T = 2$ for various choices of c and step sizes τ. It can be clearly observed that the schemes (3.9) and (5.18) converge linearly and quadratically in time, respectively. Moreover, the error decreases as c gets smaller.

Example 2 In the second experiment, we apply the newly developed exponential-type integrators (3.9) and (5.18) to the GB equation with rough initial data. We compare the results with the classical first- and second-order Gautschi-type [14,15], and the second-order Deuflhard-type [10,30] exponential integrators (see the appendix for more details), applied directly to the original GB Eq. (1.1).

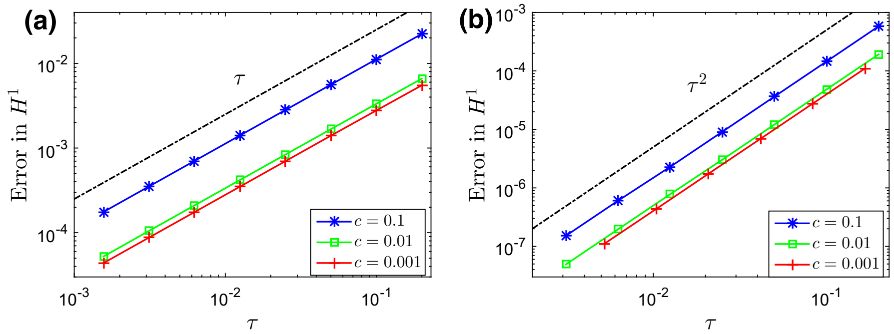

Fig. 1 Numerical simulation for the solitary-wave solution (6.1) at $T = 2$ of the first- and second-order exponential-type integrator schemes with spatial mesh size $\Delta x = 1/2^{10}$. **a** Linear convergence of the first-order scheme (3.9) for various choices of c. The broken line has slope one. **b** Quadratic convergence of the second-order scheme (5.18) for various choices of c. The broken line has slope 2

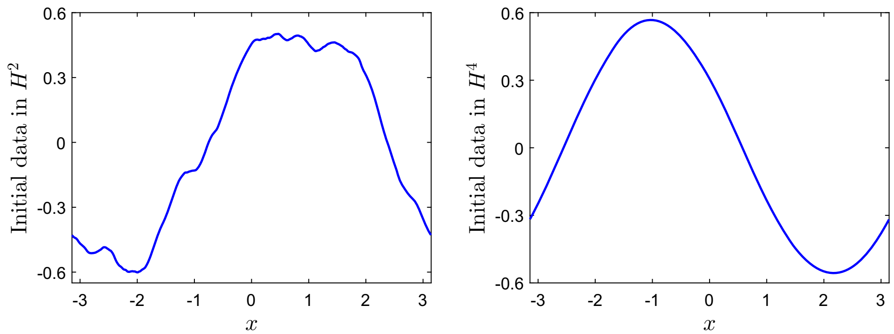

Fig. 2 Initial data z_0 normalized in L^2 for two different values of θ. Left: H^2 initial value for $\theta = 2$. Right: H^4 initial value for $\theta = 4$

For the grid points $x_j = -\pi + j\Delta x, 0 \leq j < M$ with $M = 2^{11}$ and $\Delta x = 2\pi/M$, we denote by Z_j the approximation of $z(x_j)$. For a vector

$$Z^M := [Z_0, \ldots, Z_{M-1}] = \text{rand}(1, M) \in \mathbb{R}^M,$$

we set

$$z_\theta^M = |\partial_{x,M}|^{-\theta} Z^M, \quad (|\partial_{x,M}|^{-\theta})_k := \begin{cases} |k|^{-\theta} & \text{if} \quad k \neq 0, \\ 0 & \text{if} \quad k = 0, \end{cases}$$

for different values of θ normalized in L^2 to represent the initial condition in H^θ. For initial conditions with different regularity, we refer to Fig. 2.

In the following, we study the temporal error at time $T = 2$ measured in the L^2 norm or H^1 norm for non-smooth solutions. The parameter c for the new exponential-type integrators is chosen as $c = 0.01$. The reference solution is obtained by the second-order Deuflhard-type exponential integrator with $\tau = 10^{-6}$ and $\Delta x = \pi/2^{10}$.

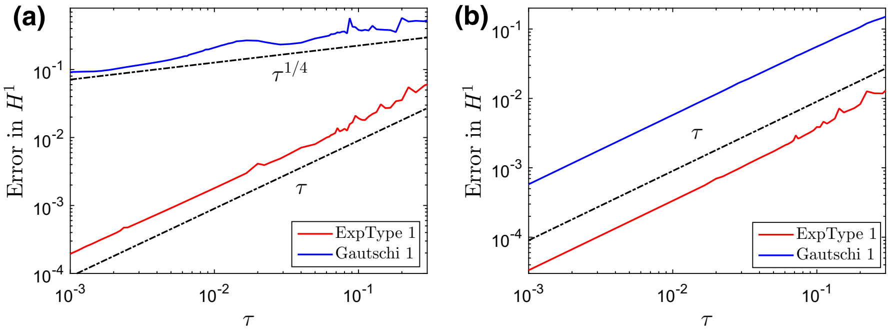

Fig. 3 Temporal error of the first-order exponential-type integrator (3.9) (red) and classical first-order exponential integrator (blue) at $T = 2$. The error is measured in H^1 with initial data in H^2 (**a**) and H^3 (**b**), respectively. The broken lines are of slope a quarter and one, respectively (color figure online)

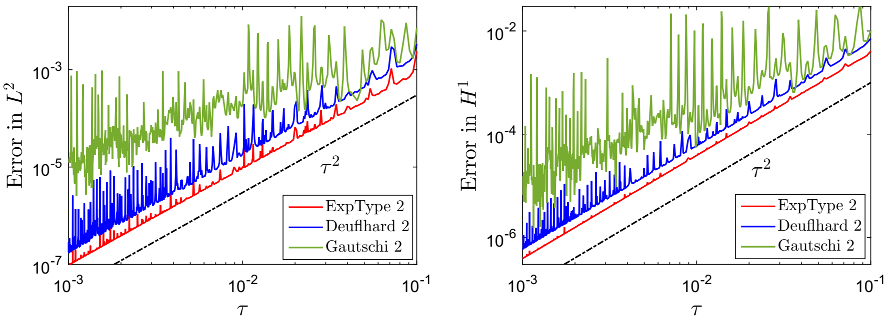

Fig. 4 Temporal error of the second-order exponential-type integrator (5.18) (red), Deuflhard exponential integrator (blue) and Gautschi exponential integrator (green) at $T = 2$. Left: the error is measured in L^2 with initial data in H^3. Right: the error is measured in H^1 with initial data in H^4. The broken line is of slope two (color figure online)

Figure 3 displays the convergence behavior of the first-order schemes. The errors are measured in H^1 for initial data in H^2 and H^3, respectively. For initial data in H^2, the order of the classical first-order Gautschi-type exponential integrator drops to one quarter whereas the new exponential-type integrator is still first-order convergent. The convergence agrees with the error estimate given in Corollary 1. The first-order convergence of the classical exponential integrator is recovered for smoother initial data in H^3. Furthermore, it can be clearly seen from Fig. 3b that the newly developed exponential-type integrator (3.9) is much more accurate than the classical one.

The temporal errors of the second-order schemes are presented in Fig. 4: errors in L^2 for initial data in H^3 (left) and errors in H^1 for initial data in H^4 (right). For such non-smooth initial conditions, it clearly shows that the convergence behavior of the classical second-order exponential integrators, e.g., Deuflhard-type and Gautschi-type integrators, becomes irregular while the newly developed second-order exponential-type integrator (5.18) remains quadratically convergent. This confirms the error estimates in Corollaries 2 and 3. Figure 5 shows the errors for less smooth solutions, where the convergence for all exponential integrators behaves irregularly. This indicates to a cer-

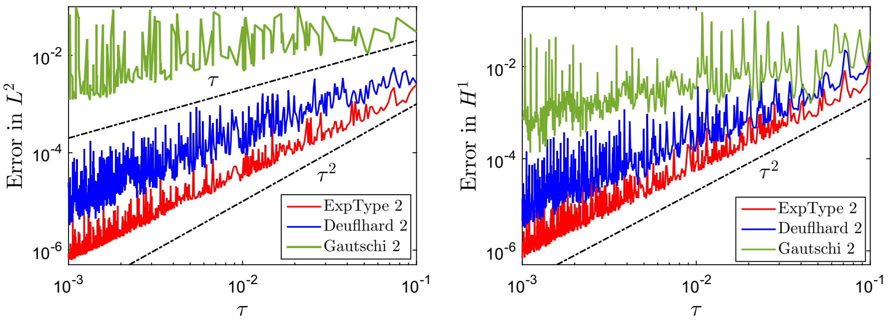

Fig. 5 Temporal error of the second-order exponential-type integrator (5.18) (red), Deuflhard exponential integrator (blue) and Gautschi exponential integrator (green) at $T = 2$. Left: the error is measured in L^2 with initial data in H^2. Right: the error is measured in H^1 with initial data in H^3. The broken line is of slope two (color figure online)

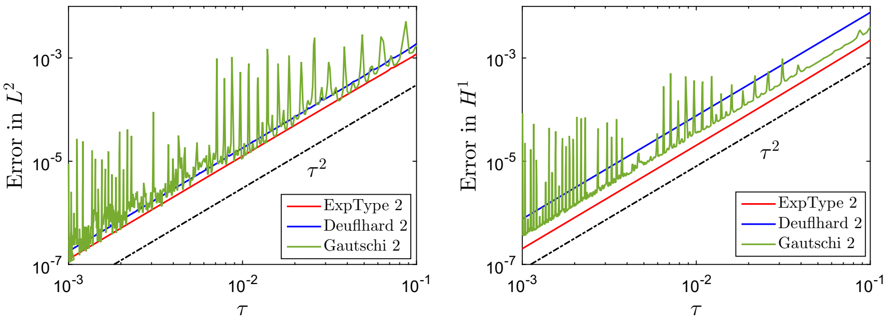

Fig. 6 Temporal error of the second-order exponential-type integrator (5.18) (red), Deuflhard exponential integrator (blue) and Gautschi exponential integrator (green) at $T = 2$. Left: the error is measured in L^2 with initial data in H^4. Right: the error is measured in H^1 with initial data in H^5. The broken line is of slope two (color figure online)

tain degree that our error estimates in Corollaries 2 and 3 are sharp. For smoother initial conditions which enable the classical Deuflhard exponential integrator to recover the quadratic convergence rate, we refer to Fig. 6 for the error in L^2 (left) and H^1 (right), respectively.

7 Conclusions

Two exponential-type integrators were proposed and analyzed for the "good" Boussinesq equation with rough initial data. They both require less spatial regularity of the solution than standard exponential integrators. The newly developed first-order integrator requires solutions with one additional derivative to attain linear convergence. The integrator even converges without any additional regularity assumption on the solution. The newly developed second-order integrator requires solutions with three additional derivatives to attain quadratic convergence. This is in contrast to the high regularity requirements for classical numerical methods, e.g., finite difference methods or pseudospectral methods. Numerical experiments confirm our analytical results. In particular, the numerical experiments with non-smooth solutions show the reliability and superiority of the newly developed exponential-type integrators compared to the classical exponential integrators, which show irregularities and order reductions.

Appendix: Classical exponential integrators

For the purpose of a reference solution and as benchmark for comparisons, we make use of the classical exponential integrators by Gautschi and Deuflhard [10,14,15,30]. In this appendix, we shortly recall these integrators and their application to the GB equation.

Applying Duhamel's formula to (2.1), we get

$$\begin{aligned} z(t_n+s) &= \cos(s\langle\partial_x^2\rangle)z(t_n) + \frac{\sin(s\langle\partial_x^2\rangle)}{\langle\partial_x^2\rangle} z_t(t_n) \\ &\quad + \frac{1}{\langle\partial_x^2\rangle}\int_0^s \sin((s-y)\langle\partial_x^2\rangle)(z^2)_{xx}(t_n+y)\mathrm{d}y, \\ z_t(t_n+s) &= -\langle\partial_x^2\rangle\sin(s\langle\partial_x^2\rangle)z(t_n) + \cos(s\langle\partial_x^2\rangle)z_t(t_n) \\ &\quad + \int_0^s \cos((s-y)\langle\partial_x^2\rangle)(z^2)_{xx}(t_n+y)\mathrm{d}y, \end{aligned}$$

where $\langle\partial_x^2\rangle = \sqrt{\partial_x^4 - \partial_x^2}$. Approximating the integral in different ways, we obtain the following three exponential integrators.

Gautschi 1 EI Setting $s=\tau$ and replacing $(z^2)_{xx}(t_n+y)$ by $(z^2)_{xx}(t_n)$ in the integrals leads to the following one-step scheme of first order (e.g., [15, Example 3.11]):

$$\begin{aligned} z^{n+1} &= \cos(\tau\langle\partial_x^2\rangle)z^n + \frac{\sin(\tau\langle\partial_x^2\rangle)}{\langle\partial_x^2\rangle} z_t^n + \frac{\tau^2}{2}\mathrm{sinc}^2\left(\frac{\tau}{2}\langle\partial_x^2\rangle\right)((z^n)^2)_{xx}, \\ z_t^{n+1} &= -\langle\partial_x^2\rangle\sin(\tau\langle\partial_x^2\rangle)z^n + \cos(\tau\langle\partial_x^2\rangle)z_t^n + \tau\,\mathrm{sinc}(\tau\langle\partial_x^2\rangle)((z^n)^2)_{xx}, \quad n\ge 0. \end{aligned}$$

Gautschi 2 EI Setting $s=\pm\tau$, replacing $(z^2)_{xx}(t_n\pm y)$ by $(z^2)_{xx}(t_n)$ and adding up yields the following symmetric two-step scheme of second order (e.g., [15, Example 3.11]):

$$\begin{aligned} z^{n+1} &= -\,z^{n-1} + 2\cos(\tau\langle\partial_x^2\rangle)z^n + \tau^2\mathrm{sinc}^2\left(\frac{\tau}{2}\langle\partial_x^2\rangle\right)((z^n)^2)_{xx}, \quad n\ge 1, \\ z^1 &= \cos(\tau\langle\partial_x^2\rangle)z^0 + \frac{\sin(\tau\langle\partial_x^2\rangle)}{\langle\partial_x^2\rangle} z_t^0 + \frac{\tau^2}{2}\mathrm{sinc}^2\left(\frac{\tau}{2}\langle\partial_x^2\rangle\right)((z^0)^2)_{xx}. \end{aligned}$$

Deuflhard 2 EI Setting $s=\tau$ and approximating the integrals by using the trapezoidal rule, one gets the one-step scheme of second-order for $n\ge 0$ as follows (e.g., [15, Example 3.12]):

$$z^{n+1} = \cos(\tau\langle\partial_x^2\rangle)z^n + \frac{\sin(\tau\langle\partial_x^2\rangle)}{\langle\partial_x^2\rangle}z_t^n + \frac{\tau^2}{2}\operatorname{sinc}\left(\tau\langle\partial_x^2\rangle\right)((z^n)^2)_{xx},$$

$$z_t^{n+1} = -\langle\partial_x^2\rangle\sin(\tau\langle\partial_x^2\rangle)z^n + \cos(\tau\langle\partial_x^2\rangle)z_t^n + \frac{\tau}{2}\left[\cos(\tau\langle\partial_x^2\rangle)((z^n)^2)_{xx} + ((z^{n+1})^2)_{xx}\right].$$

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