# Regular circle actions on 2–connected 7–manifolds

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**Abstract**

A circle action $S^1 \times M \to M$ on a manifold $M$ is called *regular* if this action is free and the orbit space is a manifold. In this paper, we determine the homeomorphism (resp. diffeomorphism) types of those 2-connected 7-manifolds (resp. smooth 2-connected 7-manifolds) that admit regular circle actions (resp. smooth regular circle actions).

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## 1 Introduction

In this paper all manifolds under consideration are closed, oriented and topological, unless otherwise stated. Moreover, all homeomorphisms and diffeomorphisms are to be orientation preserving. Given a positive integer $n$ let $S^n$ (resp. $D^{n+1}$) be the diffeomorphism type of the unit $n$–sphere (resp. $(n+1)$–disk) in $(n+1)$–dimensional Euclidean space $\mathbb{R}^{n+1}$.

**Definition 1.1.** A circle action $S^1 \times M \to M$ on a manifold $M$ is called *regular* if this action is free and the orbit space $N := M/S^1$ (with quotient topology) is a manifold.

Similarly, a smooth circle action $S^1 \times M \to M$ on a smooth manifold $M$ is called *regular* if this action is free (see [26, p.38, Proposition 5.2]).

For a given manifold $M$ one can ask

**Problem 1.2.** Does $M$ admit a regular circle action?

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Solutions to Problem 1.2 can have direct implications in contact topology. For example, the Boothby–Wang theorem implies that the existence of a smooth regular circle action on a smooth manifold $M$ is a necessary condition to the existence of a regular contact form on $M$ (see [7, p.341]).

Problem 1.2 has been solved for all 1–connected 5–manifolds by Duan and Liang [5]. In particular, it was shown that all 1–connected 4–manifolds with second Betti number $r$ can be realized as the orbit spaces of some regular circle actions on the single 5–manifold $\#_{r-1}S^2 \times S^3$, the connected sums of $r-1$ copies of the product $S^2 \times S^3$. In this paper we study Problem 1.2 for the 2–connected 7–manifolds.

Our main result is stated in terms of a family $M^c_{l,k}, c \in \{0,1\}, l, k \in \mathbb{Z}$ of 2–connected 7–manifolds. Let $M^0_{l,k} \overset{\pi_M}{\to} S^4$ be the $S^3$–bundle with characteristic map $[f_{l,k}] \in \pi_3(SO(4))$ defined by

$$f_{l,k}(u)v = u^{l+k}vu^{-l},\ v \in \mathbb{R}^4,\ u \in S^3,$$

where the space $\mathbb{R}^4$ and the sphere $S^3$ are identified with the algebra of quaternions and the space of unit quaternions, respectively, and where quaternion multiplication is understood on the right hand side of the formula. The diffeomorphism types of manifolds $M^0_{l,k}$ for $k = \pm 1$ were first studied in [19]. Their complete classification for all values of $(l, k)$ was achieved in [4]. The manifold $M^1_{l,k}$ is a certain non–smoothable topological manifold homotopy equivalent to $M^0_{l,k}$ when $k \equiv 0 \bmod 2$ (see Section 3.2 for more details).

The group $\Gamma_7$ of exotic 7–spheres is cyclic of order 28 with generator $M^0_{1,1}$ [6, Section 6]. Let $\Sigma_r := rM^0_{1,1} \in \Gamma_7$, $r \in \mathbb{Z}$. Our main result is stated below, where $\mathbb{N}$ is the set of all nonnegative integers.

**Theorem 1.3.** All homeomorphism classes of the 2–connected 7–manifolds that admit regular circle actions are represented by

$$M = \#_{2r}S^3 \times S^4 \# M^c_{6m,(1+c)k},\ c \in \{0,1\}, r \in \mathbb{N} \text{ and } m, k \in \mathbb{Z},$$

where $M$ is smoothable if and only if $c = 0$.

All diffeomorphism classes of the smooth 2–connected 7–manifolds that admit smooth regular circle actions are represented by

$$\#_{2r}S^3 \times S^4 \# M^0_{6(a+1)m,(a+1)k} \# \Sigma_{(1-a)m},\ a \in \{0,1\},\ r \in \mathbb{N},\ m, k \in \mathbb{Z}.$$

**Remark 1.4.** Following [3, Theorem 4.8], [4], [29] and [30, Theorem 1], one can easily obtain the diffeomorphism and homeomorphism classification of the smooth manifolds and smoothable topological manifolds appearing in Theorem 1.3, respectively, by comparing the invariants. However, at present the author has no idea how to classify the non–smoothable topological manifolds (the case $c = 1$) appearing in Theorem 1.3, as it is unknown whether the known homeomorphism invariants (see Section 3.1) for such manifolds are complete.

In the course to establish Theorem 1.3 we obtain also a classification of the 6–manifolds that can appear as the orbit spaces of some regular circle actions on 2–connected 7–manifolds, see Lemmas 2.1 and 2.2 in Section 2. In addition, Theorem 1.3 has some direct consequences which are discussed in Section 5.

## 2 The homeomorphism types of the orbit spaces

In this section we determine the homeomorphism types of those 6–manifolds which can appear as the orbit spaces of some regular circle actions on 2–connected 7–manifolds.

A 2–connected 7–manifold $M$ with a regular circle action defines a principal $S^1$–bundle (circle bundle) $M \to N$ with base space $N = M/S^1$ [8]. Fix an orientation on $S^1$ once and for all, and let $N$ be oriented so that $M \to N$ is an oriented principal $S^1$–bundle. From the homotopy exact sequence

$$0 \to \pi_2(M) \to \pi_2(N) \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(N) \to 0$$

of the fibration one finds that

$$\pi_1(N) = 0;\ \pi_2(N) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Consequently $N$ is a 1–connected 6–manifold with $H_2(N) \cong \mathbb{Z}$.

Conversely, for a 1–connected 6–manifold $N$ with $H_2(N) \cong \mathbb{Z}$ let $t \in H^2(N) \cong \mathbb{Z}$ be a generator and let $N_t \to N$ be the oriented circle bundle over $N$ with Euler class $t$. From the homotopy exact sequence of this fibration we find that $N_t$ is 2–connected with the canonical orientation as the total space of the circle bundle. Summarizing we get

**Lemma 2.1.** Let $S^1 \times M \to M$ be a regular circle action on a 2–connected 7–manifold $M$ with orbit space $N$. Then $N$ is a 1–connected 6–manifold with $H_2(N) \cong \mathbb{Z}$.

Conversely, every 1–connected 6–manifold $N$ with $H_2(N) \cong \mathbb{Z}$ can be realized as the orbit space of some regular circle action on a 2–connected 7–manifold.$\square$

In view of Lemma 2.1 the classification of those 1–connected 6–manifolds $N$ with $H_2(N) \cong \mathbb{Z}$ amounts to a crucial step toward a solution to Problem 1.2. In terms of the known invariants for 1–connected 6–manifolds due to Jupp [14] and Wall [27] we can enumerate all these manifolds in the next result.

Denote by $\Theta$ the set of equivalence classes $[N, t]$ of the pairs $(N, t)$ with $N$ a 1–connected 6–manifold whose integral cohomology satisfies

$$H^r(N) = \begin{cases} \mathbb{Z} \text{ if } r = 0, 2, 4, 6, \\ 0 \text{ otherwise,} \end{cases}$$

and with $t \in H^2(N)$ a fixed generator. Two such pairs $(N_1, t_1), (N_2, t_2)$ are called *equivalent* if there is an orientation-preserving homeomorphism $f : N_1 \to N_2$ such that $f^*t_2 = t_1$. For each $(N, t)$ fix a generator $x \in H^4(N)$ such that the value $\langle t \cup x, [N] \rangle$ of the cup product $t \cup x$ on the fundamental class $[N]$ is equal to 1. Consider the functions

$$k : \Theta \to \mathbb{Z};\ \ p : \Theta \to \mathbb{Z};\ \ \varepsilon : \Theta \to \{0, 1\};\ \ \delta : \Theta \to \{0, 1\}$$

determined by the following properties

i) $t^2 = k([N, t])x$;

ii) the second Stiefel–Whitney class $w_2(N)$ and the first Pontrjagin class $p_1(N)$ of $N$ are given by $\varepsilon([N, t])t \bmod 2$ and $p([N, t])x$, respectively;

iii) the class $\Delta(N) \equiv \delta([N, t])x \bmod 2 \in H^4(N; \mathbb{Z}_2)$ is the Kirby–Siebenmann invariant of $N$,

where the Kirby–Siebenmann invariant $\Delta(V)$ of a manifold $V$ is the obstruction to lift the classifying map $V \to BTOP$ for the stable tangent bundle of $V$ to $BPL$, and where $BTOP$ and $BPL$ are the classifying spaces for the stable $TOP$ bundles and $PL$ bundles, respectively (see [15]).

The following Lemma 2.2 follows very easily from [14].

**Lemma 2.2.** For each 1–connected 6–manifold $M$ with $H_2(M) \cong \mathbb{Z}$ there exist an $r \in \mathbb{N}$ and an element $[N, t] \in \Theta$ such that $M \cong \#_r S^3 \times S^3 \# N$.

Moreover, the system $\{k, p, \varepsilon, \delta\}$ is a set of complete invariants for elements $[N, t] \in \Theta$ that is subject to the following constraints:

i)If $k([N, t]) \equiv 1 \bmod 2$, then $\varepsilon([N, t]) = 0$ and

$$p([N, t]) = 24m + 4k([N, t]) + 24\delta([N, t]) \text{ for some } m \in \mathbb{Z};$$

ii)If $k([N, t]) \equiv 0 \bmod 2$, then for some $m \in \mathbb{Z}$

$$p([N, t]) = \begin{cases} 24m + 4k([N, t]) + 24\delta([N, t]) & \text{if } \varepsilon([N, t]) = 0, \\ 48m + k([N, t]) + 24\delta([N, t]) & \text{if } \varepsilon([N, t]) = 1. \end{cases}$$

In addition, the manifold $N$ is smoothable if and only if $\delta([N, t]) = 0$.

**Proof.** This is a direct consequence of [14, Theorem 0; Theorem 1]. In particular, the expression of the function $p$ is deduced from the following relation on $H^6(N)$ which holds for all $d \in \mathbb{Z}$:

$$(2dt + \varepsilon([N, t])t)^3 \equiv (p([N, t])x + 24\delta([N, t])x)(2dt + \varepsilon([N, t])t) \bmod 48. \square$$

## 3 Circle bundles over $[N, t] \in \Theta$

Lemma 2.2 singles out the family $\Theta$ of 1–connected 6–manifolds which plays a key role in presenting the orbit spaces of regular circle actions on 2–connected 7–manifolds. In this section we determine the homeomorphism and diffeomorphism types of the total space $N_t$ of the circle bundle over $[N, t] \in \Theta$, i.e. the oriented circle bundle over $N$ with Euler class $t$. For this purpose we shall recall in Section 3.1 that the definition of the known invariant system for 2–connected 7–manifolds; in Section 3.2 we give an explicit construction of the manifolds $M^1_{l,k}$ appearing in Theorem 1.3. The main results in this section are Lemmas 3.4 and 3.6, which identify the homeomorphism and diffeomorphism types of the manifolds $N_t$ with certain $M^c_{l,k}$ or $M^0_{l,k}\#\Sigma$ for some homotopy spheres $\Sigma$.

### 3.1 Invariants for 2–connected 7–manifolds

Recall from Eells, Kuiper [6], Kreck, Stolz [16] and Wilkens [29] that associated to each 2–connected 7–manifold $M$ there is a system $\{H, \frac{p_1}{2}, b, \Delta, \mu, s_1\}$ of invariants characterized by the following properties:

i) $H$ is the fourth integral cohomology group $H^4(M)$ [29];

ii) $\frac{p_1}{2}(M) \in H$ is the first spin characteristic class [25] introduced by Wilkens [29] in smooth category and extended for topological manifolds $M$ by Kreck and Stolz [16, Lemma 6.5];

iii) $b : \tau(H) \otimes \tau(H) \to \mathbb{Q}/\mathbb{Z}$ is the linking form on the torsion part $\tau(H)$ of the group $H$ [29];

iv) $\Delta(M) \in H^4(M; \mathbb{Z}_2)$ is the Kirby–Siebenmann invariant [16].

Furthermore, if the manifold $M$ is smooth and bounds a smooth 8–manifold $W$ with the induced map $j^* : H^4(W, M; \mathbb{Q}) \to H^4(W; \mathbb{Q})$ an isomorphism, then

v) the invariant $\mu \in \mathbb{Q}/\mathbb{Z}$ is firstly defined in [6] for a spin $W$ and extended in [16] for a general $W$, whose value is given by the formula (see also Remark 3.1)

$$\mu(M) \equiv -\frac{1}{2^5 \cdot 7}\sigma(W) + \frac{1}{2^7 \cdot 7}p_1^2(W) - \frac{1}{2^6 \cdot 3}z^2 \cdot p_1(W) + \frac{1}{2^7 \cdot 3}z^4 \bmod \mathbb{Z},$$

where $z \in H^2(W)$ satisfies $w_2(W) \equiv z \bmod 2$, $\sigma(W)$ is the signature of the intersection form on $H^4(W, M; \mathbb{Q})$, and where $p_1^2(W)$, $z^2 \cdot p_1(W)$ and $z^4$ are the characteristic numbers

$$\langle p_1(W) \cup j^{*-1}p_1(W), [W, M]\rangle, \ \langle z^2 \cup j^{*-1}p_1(W), [W, M]\rangle,$$
$$\langle z^2 \cup j^{*-1}z^2, [W, M]\rangle,$$

respectively. Finally, if $M$ is topological and bounds a topological 8–manifold $W$ with the induced map $j^*: H^4(W,M;\mathbb{Q}) \to H^4(W;\mathbb{Q})$ an isomorphism, then

vi) the topological invariant $s_1 \in \mathbb{Q}/\mathbb{Z}$ is defined in [16] whose value is given by

$$s_1(M) \equiv -\tfrac{1}{2^3}\sigma(W) + \tfrac{1}{2^5}p_1^2(W) - \tfrac{7}{2^4\cdot 3}z^2 \cdot p_1(W) + \tfrac{7}{2^5\cdot 3}z^4 \bmod \mathbb{Z}.$$

**Remark 3.1.** The expression of $\mu(M)$ in v) above can be justified with [16, (2.7)] as follows: Since $M$ is 2–connected, it follows, in the notation of [16, (2.7)], that one may choose $z, c \in H^2(W)$ such that $z = 0$ and $w_2(W) \equiv c \bmod 2$. In this case, the formula of $S_1(W,z,c)$ in [16, (2.7)] can be identified with that in the nonspin case of [16, (2.4)]. In particular, the class $z$ in v) is the class $c$ of [16, (2.7)].

**Example 3.2.** Let $N_t$ be the total space of the circle bundle over $[N,t] \in \Theta$. Then the system $\{H, \frac{p_1}{2}, b, \Delta, \mu, s_1\}$ of invariants for the manifold $N_t$ can be expressed in terms of the invariants for $\Theta$ introduced in Lemma 2.2 as follows. For simplicity we write $p, k, \varepsilon$ and $\delta$ in place of $p([N,t])$, $k([N,t])$, $\varepsilon([N,t])$ and $\delta([N,t])$, respectively.

i) $H^4(N_t) \cong \mathbb{Z}_k$ with generator $\pi^*(x)$, where $\pi: N_t \to N$ is the bundle projection and where

$$\mathbb{Z}_k = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/k\mathbb{Z} & \text{if } k \neq 0; \end{cases}$$

ii) $\Delta(N_t) \equiv \frac{1+(-1)^k}{2} \cdot \delta\pi^*(x) \bmod 2$;

iii) $\frac{p_1}{2}(N_t) \equiv \frac{p+\varepsilon k}{2}\pi^*(x) \bmod k$;

iv) $b(\pi^*(x), \pi^*(x)) \equiv \frac{1}{k} \bmod \mathbb{Z}$;

v) $\mu(N_t) \equiv -\frac{|k|}{2^5\cdot 7k} + \frac{(p+k)^2}{2^7\cdot 7k} + \frac{(\varepsilon-1)(2p+k)}{2^7\cdot 3} \bmod \mathbb{Z}$;

vi) $s_1(N_t) \equiv -\frac{|k|}{2^3k} + \frac{(p+k)^2}{2^5k} + \frac{7(\varepsilon-1)(2p+k)}{2^5\cdot 3} \bmod \mathbb{Z}$.

Firstly, from the section $H^2(N) \xrightarrow{\cup t} H^4(N) \xrightarrow{\pi^*} H^4(N_t) \to 0$ in the Gysin sequence of the fibration $N_t \xrightarrow{\pi} N$ and from the relation $t^2 = kx$ on $H^4(N)$ we find that $H^4(N_t) \cong \mathbb{Z}_k$ with generator $\pi^*(x)$. This shows i).

Next, let $f: N \to BTOP$ be the classifying map for the stable tangent bundle of $N$. In view of the decomposition $TN_t \cong \pi^*TN \oplus \varepsilon^1$($\varepsilon^n$ denotes the trivial real vector bundle of rank $n$) for the tangent bundle of $N_t$, the classifying map for the stable tangent bundle of $N_t$ is given by the composition $f \circ \pi: N_t \to N \to BTOP$. It follows that the Kirby–Siebenmann invariant $\Delta(N_t)$ is given by $\pi^*\Delta(N) \equiv \delta\pi^*(x) \bmod 2$. This shows ii).

To calculate the remaining invariants $\left\{\frac{p_1}{2}, b, \mu, s_1\right\}$ of the manifold $N_t$ we make use of the associated disk bundle $W_t \stackrel{\pi_0}{\to} N$ of the oriented 2–plane bundle $\xi_t$ over $N$ with Euler class $t$. If $\varepsilon = 1$, It follows from the decomposition $TW_t \cong \pi_0^*TN \oplus \pi_0^*\xi_t$ that $w_2(W_t) = 0$ and $\frac{p_1}{2}(W_t) = \frac{p+k}{2}\pi_0^* x$. From the relation $\partial W_t = N_t$ we get

$$\tfrac{p_1}{2}(N_t) = \tfrac{p+k}{2}\pi^* x.$$

If $\varepsilon = 0$, then $\frac{p_1}{2}(\cdot)$ is not defined for the nonspin coboundary $W_t$ but defined for the spin manifold $N$. From the value $\frac{p_1}{2}(N) = \frac{p}{2}x$ and the decomposition $TN_t \cong \pi^*TN \oplus \varepsilon^1$, we obtain that

$$\tfrac{p_1}{2}(N_t) = \pi^* \tfrac{p_1}{2}(N) = \tfrac{p}{2}\pi^*(x).$$

This shows iii).

To compute the linking form $b$ of $N_t$ we can assume that $k \neq 0$. Consider the commutative ladder of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^4(W_t, N_t) & \stackrel{j^*}{\to} & H^4(W_t) & \stackrel{i^*}{\to} & H^4(N_t) & \to & 0 \\ & & \cong\uparrow \phi & & \uparrow \pi_0^* & & \uparrow id & & \\ 0 & \to & H^2(N) & \stackrel{\cup t}{\to} & H^4(N) & \stackrel{\pi^*}{\to} & H^4(N_t) & \to & 0 \end{array}$$

with $\phi$ the Thom isomorphism. Since $\pi^*(x) = i^*\pi_0^*(x)$ and $y := \phi(t)$ is a generator of $H^4(W_t, N_t)$ with $j^*(y) = \pi_0^*(t^2) = k\pi_0^*(x)$ we get

$$b(\pi^*(x), \pi^*(x)) \equiv \tfrac{1}{k} < y \cup \pi_0^* x, [W_t, N_t] > \equiv \tfrac{1}{k} \bmod \mathbb{Z}.$$

This shows iv).

Since the induced map $j^* : H^4(W_t, N_t; \mathbb{Q}) \to H^4(W_t; \mathbb{Q})$ is clearly an isomorphism when $k \neq 0$, the invariants $\mu$ and $s_1$ are defined for $N_t$. Moreover, It follows from the Lefschetz duality and the relation $j^*(y) = k\pi_0^*(x)$ that $\sigma(W_t) = \frac{|k|}{k}$. From the decomposition $TW_t \cong \pi_0^*TN \oplus \pi_0^*\xi_t$ we get

$$p_1(W_t) = \pi_0^*(p_1(N) + t^2) = (p+k)\pi_0^*(x);\ w_2(W_t) \equiv (\varepsilon + 1)\pi_0^*(t) \bmod 2.$$

Thus we can take $z = (1-\varepsilon)\pi_0^*(t)$ in the formulae for $\mu$ and $s_1$, and then

$$z^2 = (1-\varepsilon)^2\pi_0^*(t^2) = k(1-\varepsilon)^2\pi_0^*(x).$$

As the group $H^4(W_t, N_t) \cong \mathbb{Z}$ is generated by $y = \phi(t)$ with the relation $j^*(y) = k\pi_0^*(x)$, the isomorphism $H^4(W_t, N_t) \otimes H^4(W_t) \stackrel{\cup}{\to} H^8(W_t, N_t)$ by the Lefschetz duality, together with the formulae for $p_1(W_t)$ and $z^2$ above, implies the relations below

$$z^2 p_1(W_t) = (1-\varepsilon)^2(p+k);\ p_1^2(W_t) = \tfrac{1}{k}(p+k)^2;\ z^4 = k(1-\varepsilon)^4.$$

Substituting these values in the formulae for $\mu$ and $s_1$ yields v) and vi), respectively. This completes the computation of the invariant system for the manifolds $N_t$.

### 3.2 The construction of the 7−manifolds $M^1_{l,k}$

In Section 1 we introduce the smooth 2−connected 7−manifold $M^0_{l,k}$. In this section we present the topological manifold $M^1_{l,k}$ explicitly.

For an $m$–manifold $W$ with boundary write $\mathcal{S}^{TOP}(W)$ for the set of all equivalence classes $[W', h']$ of the pairs $(W', h')$ with $h' : (W', \partial W') \to (W, \partial W)$ a homotopy equivalence of pairs. Two such pairs $(W', h')$ and $(W'', h'')$ are called *equivalent* if there is a homeomorphism $f : W' \to W''$ such that $h'$ is homotopic to $h'' \circ f$ (see [18, Chapter 2]). Let $W^0_{l,k} \overset{\pi_W}{\to} S^4$ be the $D^4$−bundle with characteristic map $[f_{l,k}]$. Then $M^0_{l,k} = \partial W^0_{l,k}$. Adapting the arguments of [4, Section 5] from the PL case to the TOP case we have the following commutative diagram analogous to the one [4, (7)]

$$\begin{array}{ccccc} \mathcal{S}^{TOP}(W^0_{l,k}) & \underset{\cong}{\overset{\eta}{\to}} & [W^0_{l,k}, G/TOP] & \underset{\cong}{\overset{d}{\to}} & H^4(W^0_{l,k}) \cong \mathbb{Z} \\ i^* \downarrow & & i^* \downarrow & & i^* \downarrow \\ \mathcal{S}^{TOP}(M^0_{l,k}) & \underset{\cong}{\overset{\eta}{\to}} & [M^0_{l,k}, G/TOP] & \underset{\cong}{\overset{d}{\to}} & H^4(M^0_{l,k}) \cong \mathbb{Z}_k \end{array}$$

where the space $G$ (resp. $TOP$) is the direct limit of the set of self homotopy equivalences of $S^{n-1}$ (resp. the topological monoid of origin–preserving homeomorphisms of $\mathbb{R}^n$), $\eta$ is the one to one correspondence in the surgery exact sequence (see [18, p.40-44]), the isomorphism $d$ is induced by the primary obstruction to null–homotopy and where $i^* : \mathcal{S}^{TOP}(W^0_{l,k}) \to \mathcal{S}^{TOP}(M^0_{l,k})$ sends each $[W, h]$ to the restriction $[\partial W, h|_{\partial W}]$. Fix a generator $\iota$ of $H^4(S^4)$ as in [4]. Write $\left[W^1_{l,k}, h_W\right]$ for the generator $(d \circ \eta)^{-1}(\pi^*_W(\iota))$ of the cyclic group $\mathcal{S}^{TOP}(W^0_{l,k}) = \mathbb{Z}$ and set

$$\left(M^1_{l,k}, h_M\right) := \left(\partial W^1_{l,k}, h_W|_{\partial W^1_{l,k}}\right).$$

Clearly, the 7−manifold $M^1_{l,k}$ is 2−connected and unique up to homeomorphism.

**Example 3.3**. The invariant system $\{H, \Delta, \frac{p_1}{2}, b, s_1, \mu\}$ of the manifolds $M^c_{l,k}$ has been computed by Crowley and Escher [4] for the case of $c = 0$. We extend their calculation as to include the exceptional case of $c = 1$.

i) $H^4(M^c_{l,k}) \cong \mathbb{Z}_k$ with generator $\kappa = \begin{cases} \pi^*_M(\iota) & \text{if } c = 0, \\ (\pi_M \circ h_M)^*(\iota) & \text{if } c = 1; \end{cases}$

ii) $b(\kappa, \kappa) \equiv \frac{1}{k} \bmod \mathbb{Z}$;

iii) $\Delta(M^c_{l,k}) \equiv \frac{1+(-1)^k}{2} \cdot c\kappa \bmod 2$;

iv) $\frac{p_1}{2}(M^c_{l,k}) \equiv (2l + 12c)\kappa \bmod k$;

v) $s_1(M^c_{l,k}) \equiv \frac{(2l+k+12c)^2-|k|}{8k} \bmod \mathbb{Z}$;

vi) $\mu(M^0_{l,k}) \equiv \frac{(k+2l)^2-|k|}{28 \cdot 8k} \bmod \mathbb{Z}$.

Firstly, since $h_M : M^1_{l,k} \to M^0_{l,k}$ is a homotopy equivalence we get i) and ii) from the relations $H^4(M^0_{l,k}) \cong \mathbb{Z}_k$ (with generator $\pi^*_M(\iota)$) and $b(\pi^*_M(\iota), \pi^*_M(\iota)) \equiv \frac{1}{k} \bmod \mathbb{Z}$ when $c = 0$.

Next, since the map $\mathcal{S}^{TOP}(M^0_{l,k}) \xrightarrow{\Delta} H^4(M^0_{l,k}; \mathbb{Z}_2)$ of taking Kirby– Siebenmann class is a surjective homomorphism [24, Theorem 15.1], and since $\left[M^1_{l,k}, h_M\right]$ is a generator of the cyclic group $\mathcal{S}^{TOP}(M^0_{l,k}) \cong \mathbb{Z}_k$ we have

$$\Delta(M^1_{l,k}) = \Delta(\left[M^1_{l,k}, h_M\right]) \equiv \tfrac{1+(-1)^k}{2}\kappa \bmod 2.$$

This shows iii).

We use the coboundary $W^c_{l,k}$ constructed above to calculate the remaining invariants $\left\{\frac{p_1}{2}, \mu, s_1\right\}$ of the manifold $M^c_{l,k} = \partial W^c_{l,k}$.

To find the formula of $\frac{p_1}{2}(M^c_{l,k})$ we compute the first Pontrjagin class $p_1(W^c_{l,k})$ of $W^c_{l,k}$. Let $\alpha$ denote the generator of $H^4(W^c_{l,k}) \cong \mathbb{Z}$ satisfying

$$\alpha = \begin{cases} \pi^*_W(\iota) & \text{if } c = 0, \\ (\pi_W \circ h_W)^*(\iota) & \text{if } c = 1, \end{cases}$$

and associate an integer $p(W^c_{l,k})$ to $W^c_{l,k}$ such that $p_1(W^c_{l,k}) = p(W^c_{l,k})\alpha$. Let $\bar{i} : G/TOP \to BTOP$ be the natural inclusion and let $f_c : W^c_{l,k} \to BTOP$ be the classifying map for the stable tangent bundle of $W^c_{l,k}$. It follows from the isomorphism $\mathcal{S}^{TOP}(W^0_{l,k}) \xrightarrow{\eta} [W^0_{l,k}, G/TOP]$ and the proof of [18, Theorem 2.23] that

$$\bar{i}_*\eta([W^1_{l,k}, h_W]) = h^{*-1}_W[f_1] - [f_0]$$

and hence

$$\text{(3.1)}\ p(W^1_{l,k})\pi^*_W(\iota) = h^{*-1}_W p_1(W^1_{l,k}) = p_1(W^0_{l,k}) + f^*\bar{i}^* p_1,$$

where $[f] = d^{-1}(\pi^*_W(\iota)) = \eta([W^1_{l,k}, h_W])$ is the generator of $[W^0_{l,k}, G/TOP]$ and $p_1 \in H^4(BTOP)$ is the first Pontrjagin class [14]. It is shown in [24, Lemma 13.3, Proposition 13.4] that a generator $[g]$ of $[S^4, G/TOP]$ corresponds to a topological bundle $\xi$ with classifying map $\bar{i} \circ g$ and Pontrjagin class $p_1(\xi) = g^*\bar{i}^* p_1 = \pm 24\iota$. With an appropriate choice of $\pm d : [W^0_{l,k}, G/TOP] \to H^4(W^0_{l,k})$ applying $\pi^*_W$ to this equation we get

$$f^*\bar{i}^* p_1 = 24\pi^*_W(\iota).$$

This, together with the fact $p(W^0_{l,k}) = 2(k + 2l)$ [19] and the formula (3.1) above, implies that $p(W^c_{l,k}) = 2k+4l+24c$. Consequently from $M^c_{l,k} = \partial W^c_{l,k}$ we get iv).

Finally, we compute $s_1(M^c_{l,k})$. The exact sequence

$$H^4(W^c_{l,k}, M^c_{l,k}) \xrightarrow{j^*} H^4(W^c_{l,k}) \to H^4(M^c_{l,k}) \to 0,$$

together with the isomorphisms $H^4(M^c_{l,k}) \cong \mathbb{Z}_k$ and $H^4(W^c_{l,k}, M^c_{l,k}) \cong \mathbb{Z}$ by the Lefschetz duality, implies that we can take a generator $\beta$ of $H^4(W^c_{l,k}, M^c_{l,k})$ such that $j^*(\beta) = k\alpha$. Since $j^* : H^4(W^c_{l,k}, M^c_{l,k}; \mathbb{Q}) \to H^4(W^c_{l,k}; \mathbb{Q})$ is an isomorphism for $k \neq 0$ the invariant $s_1$ is defined for $M^c_{l,k}$. It follows from the Lefschetz duality and the relation $j^*(\beta) = k\alpha$ that $\sigma(W^c_{l,k}) = \frac{|k|}{k}$. On the other hand, the formula for $p(W^c_{l,k})$, together with the relation $j^*(\beta) = k\alpha$ and the isomorphism $H^4(W^c_{l,k}, M^c_{l,k}) \otimes H^4(W^c_{l,k}) \xrightarrow{\cup} H^8(W^c_{l,k}, M^c_{l,k})$ by the Lefschetz duality, implies that

$$p_1^2(W^c_{l,k}) = \tfrac{4}{k}(k + 2l + 12c)^2.$$

In addition, the relation $w_2(W^1_{l,k}) = w_2(W^0_{l,k}) = 0$ indicates that we can take $z = 0$ in the formula of $s_1$. Substituting the values of $\sigma(W^c_{l,k}), z, p_1^2(W^c_{l,k})$ in the formula for $s_1$ shows v).

Similarly, we refer vi) to Crowley and Escher [4]. This completes the computation of the invariant system for the manifolds $M^c_{l,k}$.

### 3.3 Circle bundles over $[N, t] \in \Theta$

In this section we will prove Lemmas 3.4 and 3.6 which identify the homeomorphism and diffeomorphism types of the manifolds $N_t$ with certain $M^c_{l,k}$ or $M^0_{l,k}\#\Sigma$ for some homotopy spheres $\Sigma$.

**Lemma 3.4.** Let $N_t$ be the total space of the circle bundle over $[N, t] \in \Theta$. Then there is a homeomorphism $N_t \cong M^c_{l,k}$ where

$$\begin{aligned} (k, c) &= (k([N,t]), \tfrac{1+(-1)^{k([N,t])}}{2} \cdot \delta([N,t])); \\ l &= \tfrac{p([N,t])+(3\varepsilon([N,t])-4)\cdot k([N,t])-(1+(-1)^{k([N,t])})\cdot 12\delta([N,t])}{4}. \end{aligned}$$

**Proof.** We divide the proof into two cases depending on the value of $\Delta(N_t)$.

Case 1. $\Delta(N_t) \equiv 0 \bmod 2$ (i.e. the manifold $N_t$ is smoothable, see [18, p.33], [12] and [24, Theorem 5.4]): Up to a $\mathbb{Z}_2$ ambiguity Wilkens [29] showed that the system $\{H, \frac{p_1}{2}, b\}$ of invariants classifies $N_t$ and $M^0_{l,k}$ up to homeomorphism. Moreover, Crowley and Escher [4] proved that this ambiguity can be realized by some $M^0_{l,k}$ whose homeomorphism types can be distinguished by the invariant $s_1$ and hence the system $\{H, \frac{p_1}{2}, b, s_1\}$ classifies the manifolds $N_t$ and $M^0_{l,k}$. Therefore the proof is completed by comparing these invariants for $N_t$ and $M^0_{l,k}$ obtained in Examples 3.2 and 3.3, respectively.

Case 2. $\Delta(N_t) \equiv 1 \bmod 2$: We only need to show that $N_t$ is homeomorphic to $M^1_{l,k}$ where $[N, t] \in \Theta$ and

$$(k,l) = (k([N,t]), \tfrac{p([N,t])+(3\varepsilon([N,t])-4)\cdot k([N,t])-24}{4}).$$

It suffices to construct a homotopy equivalence $q : N_t \to M^0_{l,k}$ with

$$[N_t, q] = \left[M^1_{l,k}, h_M\right] \in \mathcal{S}^{TOP}(M^0_{l,k}).$$

According to Lemma 2.2 there exists a manifold $N'$ with $[N', t'] \in \Theta$ whose invariant system $(k([N',t']), p([N',t']), \varepsilon([N',t']), \delta([N',t'])$ is

$$(k([N,t]), p([N,t]) - 24, \varepsilon([N,t]), 0).$$

Consider the map $\eta : \mathcal{S}^{TOP}(N') \to [N', G/TOP]$ in the surgery exact sequence of $N'$. By the argument at the end of the proof of [14, Theorem 1] we find a homotopy equivalence $h_N : N \to N'$ such that

i) the homotopy class $\eta([N, h_N])$ is trivial on the 2 skeleton of $N'$;

ii) the primary obstruction to finding a null–homotopy of $\eta([N, h_N])$ is the generator $x' \in H^4(N'; \pi_4(G/TOP))$ with $\langle x' \cup t', [N']\rangle$=1.

Pulling $h_N$ back by the bundle projection $\pi' : N'_t \to N'$ induces a homotopy equivalence $h_t : N_t \to N'_t$. On the other hand, by the result of Case 1 we get a homeomorphism $u : M^0_{l,k} \to N'_t$ such that $u^*(\pi'^*(x')) = \pi^*_M(\iota)$. So it remains to show that $\left[N_t, u^{-1} \circ h_t\right] = \left[M^1_{l,k}, h_M\right]$.

Let $[N', G/TOP]_2$ denote the subset of $[N', G/TOP]$ whose elements are trivial on the 2 skeleton of $N'$ and consider the following two commutative diagrams:

$$\begin{array}{ccc} \mathcal{S}^{TOP}(N') & \stackrel{\pi'^*}{\to} & \mathcal{S}^{TOP}(N'_t) \\ \downarrow \eta & & \cong\downarrow \eta \\ [N', G/TOP] & \stackrel{\pi'^*}{\to} & [N'_t, G/TOP] \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{S}^{TOP}(N'_t) & \underset{\cong}{\stackrel{u^*}{\to}} & \mathcal{S}^{TOP}(M^0_{l,k}) \\ & & \cong\downarrow \eta & & \cong\downarrow \eta \\ [N', G/TOP]_2 & \stackrel{\pi'^*}{\to} & [N'_t, G/TOP] & \underset{\cong}{\stackrel{u^*}{\to}} & \left[M^0_{l,k}, G/TOP\right] \\ \downarrow d & & \cong\downarrow d & & \cong\downarrow d \\ H^4(N') & \stackrel{\pi'^*}{\to} & H^4(N'_t) & \underset{\cong}{\stackrel{u^*}{\to}} & H^4(M^0_{l,k}) \end{array}$$

where

i) $\mathcal{S}^{TOP}(N') \xrightarrow{\pi'^*} \mathcal{S}^{TOP}(N'_t)$ maps $[N'', h'']$ to $[N''_t, h''_t]$ with $h''_t$ a pull–back of $h''$ by the bundle projection $\pi' : N'_t \to N'$;

ii) $\mathcal{S}^{TOP}(N'_t) \xrightarrow{u^*} \mathcal{S}^{TOP}(M^0_{l,k})$ maps $[M', g']$ to $\left[M', u^{-1} \circ g'\right]$;

iii) the maps $d$ are to take the primary obstruction to null–homotopy.

The diagrams above, together with the relations

$$\pi'^* [N, h_N] = [N_t, h_t],\ u^*(\pi'^*(x')) = \pi^*_M(\iota) \text{ and } d(\eta [N, h_N]) = x',$$

imply that $u^* [N_t, h_t] = \left[M^1_{l,k}, h_M\right]$, i.e. $\left[N_t, u^{-1} \circ h_t\right] = \left[M^1_{l,k}, h_M\right]$. This completes the proof of Case 2.□

The following Lemma 3.5 plays a key role in the proof of Lemma 3.6 and its proof will be postponed to the end of this section.

**Lemma 3.5**. Let $N_t$ be the total space of the circle bundle over $[N, t] \in \Theta$ with $\delta([N, t]) = 0$ and $k([N, t]) = 0$. Then there exists an 8–manifold $W$ homotopy equivalent to $S^4$ whose boundary satisfies

$$\partial W \cong \begin{cases} N_t & \text{if } \varepsilon([N, t]) = 1, \\ N_t \# \Sigma_{\frac{p([N,t])}{24}} & \text{if } \varepsilon([N, t]) = 0. \end{cases}$$

**Lemma 3.6**. Let $N_t$ be the total space of the circle bundle over $[N, t] \in \Theta$ with $\delta([N, t]) = 0$. Then one has a diffeomorphism $N_t \cong M^0_{l,k} \# \Sigma_r$ where $N_t$ has the smooth structure as the total space of the circle bundle and where

$$(k, l, r) = (k([N, t]), \tfrac{p([N,t]) + (3\varepsilon([N,t]) - 4) \cdot k([N,t])}{4}, \tfrac{(1-\varepsilon([N,t])) \cdot (p([N,t]) - 4k([N,t]))}{24}).$$

**Proof.** In the case of $k([N, t]) \neq 0$ it is shown in [4] that the system $\{H, \frac{p_1}{2}, b, \mu\}$ classifies $N_t$ and $M^0_{l,k}$ up to diffeomorphism. Hence the proof is done by comparing those invariants for $N_t$ and $M^0_{l,k}$ obtained in Examples 3.2 and 3.3, respectively.

Assume next that $k([N, t]) = 0$ and let $W$ be the 8–manifold given in Lemma 3.5. Represent the homotopy equivalence in Lemma 3.5 by an embedding $h : S^4 \hookrightarrow$ Interior $W$ [20, Lemma 6] and take a closed tubular neighborhood $E$ of $h$. As $H_i(W \backslash \text{Interior} E, \partial E) \cong H_i(W, E) = 0$ for all $i$ by the excision theorem, $W \backslash \text{Interior} E$ is an h–cobordism between $\partial W$ and $\partial E$. Hence we get the diffeomorphisms $\partial W \cong \partial E \cong M^0_{l,k}$ for some $l, k \in \mathbb{Z}$ by the h–cobordism theorem [21, Theorem 9.1] and by the fact that $E$ is the total space of the normal disk bundle of $h$. Comparing the invariants $\{H, \frac{p_1}{2}\}$ of $\partial W$ and $M^0_{l,k}$ given in Examples 3.2 and 3.3, respectively, we find that $l = \frac{p([N,t])}{4}$ and $k = 0$ [4]. Thus the proof is completed by

$$N_t \cong \begin{cases} M^0_{\frac{p([N,t])}{4},0} \# \Sigma_{\frac{p([N,t])}{24}} & \text{if } \varepsilon([N,t]) = 0, \\ M^0_{\frac{p([N,t])}{4},0} & \text{if } \varepsilon([N,t]) = 1. \end{cases} \qquad \square$$

**Proof of Lemma 3.5**. The construction of $W$ and the corresponding calculations will be divided into two cases depending on the value of $\varepsilon([N,t])$. Let $W_t \xrightarrow{\pi_0} N$ be the associated disk bundle of the circle bundle $N_t \xrightarrow{\pi} N$.

Case 1. $\varepsilon([N,t]) = 1$: Take an embedding $f : S^2 \hookrightarrow$ Interior $W_t$ that represents a generator of $H_2(W_t) \cong \mathbb{Z}$. Since $w_2(W_t) = 0$ (see Example 3.2), the map $f$ extends to an embedding $\overline{f} : S^2 \times D^6 \hookrightarrow$ Interior $W_t$. Then $W$ is obtained from $W_t$ by surgery along $\overline{f}$. On one hand, it is clear that $\partial W \cong \partial W_t \cong N_t$. On the other hand, from the homotopy equivalences

$$X \simeq W_t \cup_f D^3 \simeq W \cup D^6$$

with $X$ the trace of the surgery [1, P.83-84] we find that $W$ is 3–connected with $H_4(W) \cong \mathbb{Z}$. This, together with the 2–connectedness of $N_t$ and the Lefschetz duality, implies that $H_i(W) \cong H^{8-i}(W, N_t) = 0$ for $i \geq 5$. Hence from the Whitehead theorem we get $W \simeq S^4$.

Case 2. $\varepsilon([N,t]) = 0$: The desired manifold $W$ is constructed as follows. Since $w_2(N) = 0$ we can take an embedding $\overline{g} : S^2 \times D^4 \hookrightarrow N$ whose restriction $g : S^2 \times 0 \hookrightarrow N$ represents the generator $x \cap [N] \in H_2(N)$. Let $\widetilde{W} := N_t \times [0,1] \cup_{(\overline{h},1)} D^4 \times D^4$ with $\overline{h}$ the pull–back of $\overline{g}$ by $\pi$ as in the diagram

$$\begin{array}{ccc} S^3 \times D^4 & \overset{\overline{h}}{\hookrightarrow} & N_t \\ \downarrow & & \pi \downarrow \\ S^2 \times D^4 & \overset{\overline{g}}{\hookrightarrow} & N \end{array}$$

where the map $S^3 \to S^2$ is the Hopf fibration. Since the map $\overline{h}$ induces an isomorphism $\pi_3(S^3 \times D^4) \to \pi_3(N_t)$, then

(3.2) $\partial\widetilde{W} \cong N_t \sqcup (-\Sigma_r)$ for some $r \in \mathbb{Z}$ [28, Lemma 1].

The manifold $W$ is obtained from $\widetilde{W}$ by removing a tubular neighborhood of a smooth arc $\alpha : [0,1] \to \widetilde{W}$ with $\alpha(0) \in N_t$, $\alpha(1) \in \Sigma_r$ and $\alpha(0,1) \subset$Interior $\widetilde{W}$.

It remains to show that

i) $W \simeq S^4$; ii) $\mu(\Sigma_r) \equiv \frac{p([N,t])}{24 \cdot 28} \bmod \mathbb{Z}$ for $\Sigma_r$ in (3.2).

The property i) follows from the facts that the trace $\widetilde{W}$ of the surgery along $\overline{h}$ has the homotopy type $\Sigma_r \cup D^4$ and the homeomorphism type of $W$ is obtained from $\widetilde{W}$ by collapsing the component $\Sigma_r$ of $\partial\widetilde{W}$ to a point.

The property ii) can be computed from the manifold $W' := W_t \cup_{\overline{h}} D^4 \times D^4$ with $\partial W' \cong \Sigma_r$ (by the collar neighborhood theorem). For the convenience of calculation, we use an alternative decomposition $W' = W_t \cup_{\overline{i_0}} \mathbb{C}P^2 \times D^4$ with $\overline{i_0}$ the pull–back of $\overline{g}$ by $\pi_0$ and where $V$ is the total space of the Hopf disk bundle over $S^2$, considered as a subspace of $\mathbb{C}P^2$ as in the diagram

$$\begin{array}{ccc} V \times D^4 & \stackrel{\overline{i_0}}{\hookrightarrow} & W_t \\ \downarrow & & \pi_0 \downarrow \\ S^2 \times D^4 & \stackrel{\overline{g}}{\hookrightarrow} & N \end{array} .$$

From the isomorphism (by the Mayer–Vietoris sequence)

$$i_{1*} \oplus i_{2*} : H_4(\mathbb{C}P^2) \oplus H_4(W_t) \stackrel{\cong}{\to} H_4(W')$$

with $i_1 : \mathbb{C}P^2 \to W', i_2 : W_t \to W'$ the inclusions, we can see below that the intersection matrix of $W'$ with respect to a basis $x_1, x_2 \in H^4(W', \partial W')$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $i_{1*}[\mathbb{C}P^2] = x_1 \cap [W', \partial W'], i_{2*}\alpha = x_2 \cap [W', \partial W']$ with $\alpha \in H_4(W_t)$ the generator satisfying $\langle \pi_0^* x, \alpha \rangle = 1$ (see Section 2). Firstly as the normal bundle of $i_1$ is trivial we get $\Big\langle x_1 \cup x_1, [W', \partial W'] \Big\rangle = 0$. Next the calculation

$$\Big\langle x_2 \cup x_2, [W', \partial W'] \Big\rangle = \langle j^* D_{W_t}\alpha \cup D_{W_t}\alpha, [W_t, N_t] \rangle = 0 \text{ [1, p.115]}$$

($D_{W_t}\alpha$ is the Lefschetz duality of $\alpha$) follows from the facts that the self–intersection number of $i_{2*}\alpha$ is the same as that of $\alpha$ and the homomorphism $j^* : H^4(W_t, N_t) \to H^4(W_t)$ is trivial (see Example 3.2). Finally we have

$$\Big\langle x_1 \cup x_2, [W', \partial W'] \Big\rangle = \Big\langle j'^* x_1, i_{2*}\alpha \Big\rangle = \Big\langle i_2^* j'^* x_1, \alpha \Big\rangle = \langle \pi_0^* x, \alpha \rangle = 1$$

with $j' : W' \to (W', \partial W')$ the inclusion and where the relation $i_2^* j'^* x_1 = \pi_0^* x$ is obtained from its geometric interpretation $i_2^{-1} i_1 [\mathbb{C}P^2] = \pi_0^{-1} g[S^2]$ under the Poincaré–Lefschetz duality.

We can take $z \in H^2(W') \cong \mathbb{Z}$ to be a generator since $w_2(W') \neq 0$ by $i_1^* TW' \cong T\mathbb{C}P^2 \oplus \varepsilon^4$ and $w_2(\mathbb{C}P^2) \neq 0$. To get the values of $z^2, p_1(W')$, it is necessary to compute their images under the isomorphism

$$i_1^* \oplus i_2^* : H^4(W') \stackrel{\cong}{\to} H^4(\mathbb{C}P^2) \oplus H^4(W_t),$$

whose matrix with respect to the basis $\{j'^* x_1, j'^* x_2\}$ and $\{[\mathbb{C}P^2]^*, \pi_0^* x\}$ is the same as the intersection matrix of $W'$ with respect to the basis $\{x_1, x_2\}$, where $[\mathbb{C}P^2]^* \in H^4(\mathbb{C}P^2)$ satisfies $\big\langle [\mathbb{C}P^2]^*, [\mathbb{C}P^2] \big\rangle = 1$. In fact, the images

$$i_1^* \oplus i_2^*(z^2) = ([\mathbb{C}P^2]^*, 0);\ i_1^* \oplus i_2^* p_1(W') = (3[\mathbb{C}P^2]^*, p([N,t])\pi_0^* x)$$

can be obtained from the isomorphisms $H^2(W') \xrightarrow{i_1^*} H^2(\mathbb{C}P^2), H^2(W') \xrightarrow{i_2^*} H^2(W_t)$, $i_1^* TW' \cong T\mathbb{C}P^2 \oplus \varepsilon^4$ and $i_2^* TW' \cong TW_t$, together with the values $p_1(\mathbb{C}P^2) = 3[\mathbb{C}P^2]^*$ and $p_1(W_t) = p([N,t])\pi_0^* x$. Therefore we can see that

$$z^2 = j'^* x_2;\ p_1(W') = p([N,t]) j'^* x_1 + 3 j'^* x_2.$$

From these relations and the intersection form of $W'$ we get

$$p_1^2(W') = 6p([N,t]);\ z^2 p_1(W') = p([N,t]);\ z^4 = 0;\ \sigma(W') = 0.$$

Substituting these values in the formula of $\mu$ in v) before Remark 3.1, this shows the property ii).□

**Remark 3.7**. In a communication concerning this work, Diarmuid Crowley pointed out that according to a result of Wilkens [30, Theorem 1 (ii)] the decomposition $N_t \cong M_{l,k}^0 \# \Sigma_r$ in Lemma 3.6 can be simplified as $N_t \cong M_{l,k}^0$ when $k([N,t]) = 0$, which will play a role in the proof of Corollary 5.6 in the coming section.

# 4 The proof of Theorem 1.3

We establish Theorem 1.3.

**Proof of Theorem 1.3.** Let $M$ be a 2–connected 7–manifold with a regular circle action. By Lemmas 2.1 and 2.2 $M$ is the total space of the oriented circle bundle over $N\#_r S^3 \times S^3$ with Euler class $\bar{t} \in H^2(N\#_r S^3 \times S^3) \cong \mathbb{Z}$ a generator, where $[N,t] \in \Theta$, $r \in \mathbb{N}$. Identify $\bar{t}$ with the generator $t \in H^2(N) \cong \mathbb{Z}$ under the isomorphism $H^2(N) \to H^2(N\#_r S^3 \times S^3)$ induced by the map $N\#_r S^3 \times S^3 \to N$ collapsing $\#_r S^3 \times S^3$ to a point. By Lemmas 3.4 and 3.6 it suffices to show that $M \cong N_t \#_{2r} S^3 \times S^4$.

Consider the decomposition

$$N\#_r S^3 \times S^3 \cong (N \backslash \overset{\circ}{D}_1) \cup_f (\#_r S^3 \times S^3 \backslash \overset{\circ}{D}_2)$$

with $D_i \cong D^6$, $\overset{\circ}{D}_i$ :=Interior$D_i$ and $f : \partial D_2 \to \partial D_1$ a diffeomorphism. Since the restriction of the bundle $N_t \to N$ on $D_1$ is trivial and $N_t \cong N_t \# S^7$ one has the corresponding decomposition

$$M \cong (N_t \backslash \overset{\circ}{D}_1 \times S^1) \cup_{f \times id} ((\#_r S^3 \times S^3 \backslash \overset{\circ}{D}_2) \times S^1) \cong N_t \# M_0$$

where $id$ is the identity on $S^1$, and where

$$M_0 = (S^7 \backslash \mathring{D}_1 \times S^1) \cup_{f \times id} ((\#_r S^3 \times S^3 \backslash \mathring{D}_2) \times S^1).$$

Since $M_0$ can be easily identified with the total space of the oriented circle bundle over $\mathbb{C}P^3 \#_r S^3 \times S^3$ with Euler class a proper generator of $H^2(\mathbb{C}P^3 \#_r S^3 \times S^3) \cong \mathbb{Z}$, a calculation similar to that in Example 3.2 shows that the invariant system $\{H, \frac{p_1}{2}, b, \mu\}$ for $M_0$ and $\#_{2r} S^3 \times S^4$ coincides. Consequently $M_0$ is diffeomorphic to $\#_{2r} S^3 \times S^4$. This shows that $M \cong N_t \#_{2r} S^3 \times S^4$ which completes the proof.□

# 5 Applications

We present some applications in the final section .

A classical topic is to decide which homotopy spheres admit smooth regular circle actions ( [13] [17] [22] [23]). Combining Theorem 1.3 with Example 3.3 we recover the classical computation of Montgomery and Yang [22] .

**Corollary 5.1.** Among the 28 homotopy 7–spheres $\Sigma_r, 0 \leq r \leq 27$ the following ones admit smooth regular circle actions

$$\Sigma_r,\ r = 0, 4, 6, 8, 10, 14, 18, 20, 22, 24. \square$$

In terms of our notation the unit tangent bundle of the sphere $S^4$ is $M^0_{-1,2}$. The additive property of the Eells–Kuiper invariant $\mu$ implies that $M^0_{-1,2} \# \Sigma_r$ with $0 \leq r \leq 27$ represent all the diffeomorphism types of the smooth manifolds homeomorphic to $M^0_{-1,2}$. One can deduce from Theorem 1.3 and Example 3.3 that

**Corollary 5.2**. All the smooth manifolds homeomorphic to the unit tangent bundle of the sphere $S^4$ and admitting smooth regular circle actions are

$$M^0_{-1,2} \# \Sigma_r,\ r = 0, 2, 6, 7, 8, 12, 14, 15, 16, 19, 20, 23, 26. \square$$

In [11] Grove, Verdiani and Ziller constructed on the manifold $M^0_{-1,2} \# \Sigma_{27}$ a metric with positive sectional curvature (see Goette [9, p.34-35]). According to Corollary 5.2 this manifold does not admit any smooth regular circle action.

Another interesting smooth manifold with a metric of positive sectional curvature is the Berger space $M = SO(5)/SO(3)$ where the embedding of $SO(3)$ in $SO(5)$ is given by the conjugation action of $SO(3)$ on the real $3 \times 3$ symmetric matrices of trace zero [2] [10]. Combine the diffeomorphism $M \cong M^0_{\pm 1, \mp 10}$ [10, Corollary 2] with Theorem 1.3 and Example 3.3 we get

**Corollary 5.3.** The Berger space does not admit any smooth regular circle action.

**Definition 5.4.** Two regular (resp. smooth regular) circle actions

$$S^1 \times M_i \to M_i, i = 1, 2,$$

on two manifolds (resp. smooth manifolds) $M_i$ are called *equivalent* if there is an equivariant homeomorphism (diffeomorphism) $f : M_1 \to M_2$. Let $\rho_T(M)$ (resp.$\rho_S(M)$) be the number of all equivalence classes of regular (resp. smooth regular) circle actions on a given manifold (resp. smooth manifold) $M$.

Since $\rho_T(M)$ can be seen as the number of those elements $[N, t] \in \Theta$ satisfying $N_t \cong M$, we get from Lemmas 2.2 and 3.4 that

**Corollary 5.5.** For the family

$$M = M^c_{6m,(1+c)k} \#_{2r} S^3 \times S^4, c \in \{0, 1\}, r \in \mathbb{N}, m, k \in \mathbb{Z}$$

of manifolds that represent all homeomorphism classes of the 2–connected 7–manifolds with regular circle actions (see Theorem 1.3) we have

$$\rho_T(M) = \begin{cases} 1 & \text{if } k = 0 \text{ and } m \equiv 1 \bmod 2, \\ 2 & \text{if } k = 0 \text{ and } m \equiv 0 \bmod 2, \\ \infty & \text{if } k \neq 0. \end{cases} \quad \square$$

Similarly, in the smooth category we get from Lemmas 2.2 and 3.6, together with Remark 3.7, that

**Corollary 5.6**. For the family

$$M = M^0_{6(1+a)m,(1+a)k} \# \Sigma_{(1-a)m} \#_{2r} S^3 \times S^4, a \in \{0, 1\}, r \in \mathbb{N}, m, k \in \mathbb{Z}$$

of manifolds that represent all diffeomorphism classes of the smooth 2–connected 7–manifolds with smooth regular circle actions (see Theorem 1.3) we have

$$\rho_S(M) = \begin{cases} 1 & \text{if } k = 0, a = 0 \text{ and } m \equiv 1 \bmod 2, \\ 2 & \text{if } k = 0 \text{ and } (1 + a)m \equiv 0 \bmod 2, \\ \infty & \text{if } k \neq 0. \end{cases} \quad \square$$

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## References


[1] W. Browder, *Surgery on Simply–Connected Manifolds*, Springer–Verlag, NewYork, 1972.

[2] M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 179-246.

[3] D. Crowley, *The classification of highly connected manifolds in dimensions 7 and 15*, PhD, Indiana University, 2002. Available at arXiv:0203253.

[4] D. Crowley and C. Escher, *A classification of $S^3$–bundles over $S^4$*, Differential Geometry and its Applications **18** (2003), 363–380.

[5] H. Duan and C. Liang, *Circle bundles over 4–manifolds*, Arch. Math. **85** (2005), 278–282.

[6] J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Annali di Math. **60** (1962), 93–110.

[7] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics **109**, Cambridge, 2008.

[8] A. M. Gleason, *Spaces with a compact Lie group of transformations*, Proc. Amer. Math. Soc. 1, (1950), 35–43.

[9] S. Goette, *Adiabatic limits of Seifert fibrations, Dedekind sums, and the diffeomorphism type of certain 7–manifolds*, preprint(2011), arXiv:1108.5614.

[10] S. Goette, N. Kitchloo and K. Shankar, Diffeomorphism type of the Berger space $SO(5)/SO(3)$, Amer. J. Math. **126** (2004), 395–416.

[11] K. Grove, L. Verdiani and W. Ziller, *An exotic $T_1S^4$ with positive curvature,* Geom. Funct. Anal. 21(2011), 499–524.

[12] M. W. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Ann. of Math. Stud. No. 80, Princeton University Press; University of Tokyo Press 1974.

[13] W. C. Hsiang, *A note on free differentiable actions of $S^1$ and $S^3$ on homotopy spheres.* Ann. of Math. (2) **83** (1966), 266–272.

[14] P. Jupp, *Classification of certain 6–manifolds*, Proc. Camb. Phil. Soc. **73** (1973), 293–300.

[15] R. C. Kirby, L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung.* Bull. Amer. Math. Soc. 75 (1969), 742–749.

[16] M. Kreck, S. Stolz, *Some nondiffeomorphic homeomorphic homogeneous 7–manifolds with positive sectional curvature*, J. Differential Geom. **33** (1991), 465–486.

[17] R. Lee, *Non–existence of free differentiable actions of $S^1$ and $\mathbb{Z}_2$ on homotopy spheres.* Proc. Conference on Transformation Groups (New Orleans, 1967), Springer–Verlag, New York, 1968, 208–209.

[18] I. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Math. Studies **92**, Princeton, 1979.

[19] J. Milnor, *On manifolds homeomorphic to the 7–sphere*, Ann. of Math. **64** (1956), 399–405.

[20] J. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds.* Symposia in Pure Math., Amer. Math. Soc. 3, 39–55 (1961).

[21] J. Milnor, *Lectures on the h–cobordism theorem*, Princeton university press, 1965.

[22] D. Montgomery and C. T. Yang, *Differentiable actions on homotopy seven spheres. II*, Proc. Conference on Transformation Groups (New Orleans, La., 1967), Springer, New York, 1968, 125–134.

[23] R. Schultz, *The nonexistence of free $S^1$ actions on some homotopy spheres*, Proc. Amer. Math. Soc, **27** (1971), 595–597.

[24] L. C. Siebenmann, *Topological manifolds.* Proceedings I.C.M. Nice (1970).

[25] E. Thomas, *On the cohomology groups of the classifying space for the stable spinor group*, Bol. Soc. Mat. Mexicana (2) **7** (1962), 57–69.

[26] T. tom Dieck, *Transformation groups*, de Gruyter Studies in Mathematics **8**, Berlin, 1987.

[27] C. T. C. Wall, *Classification problems in differential topology. V. On certain 6–manifolds*, Invent. Math. **1** (1966), 335–374.

[28] C. T. C., Wall, *Killing the middle homotopy groups of odd dimensional manifolds,* Trans. Amer. Math. Soc., 103 (1962), 421–433.

[29] D. Wilkens, *Closed $(s{-}1)$–connected $(2s+1)$–manifolds, $s = 3, 7$,* Bull. London Math. Soc. **4** (1972), 27–31.

[30] D. L. Wilkens, *On the inertia group of certain manifolds*, J. London Math. Soc. (2) 9 (1975), 537–548.