On negatively curved bundles with hyperbolic fibers outside the Igusa stable range

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Abstract

We prove that the Teichmüller space $\mathcal{T}^{<0}(M)$ of negatively curved metrics on a hyperbolic manifold M has nontrivial i-th rational homotopy groups for some $i > \dim M$. Moreover, some elements of infinite order in $\pi_i B\mathrm{Diff}(M)$ can be represented by bundles over S^i with fiberwise negatively curved metrics.

1 Introduction

Let M be a closed smooth manifold. Denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on M, endowed with the smooth topology and let $\mathrm{Diff}_0(M)$ be the group of all smooth self-diffeomorphisms of M which are homotopic to the identity 1_M. The group $\mathcal{D}_0(M) := \mathbb{R}^+ \times \mathrm{Diff}_0(M)$ acts on $\mathcal{MET}(M)$ by scaling and pulling back metrics, i.e.

$$(\lambda, \phi)g = \lambda(\phi^{-1})^* g$$

for $g \in \mathcal{MET}(M)$ and $(\lambda, \phi) \in \mathcal{D}_0(M) = \mathbb{R}^+ \times \mathrm{Diff}_0(M)$. The quotient space

$$\mathcal{T}(M) := \mathcal{MET}(M)/\mathcal{D}_0(M)$$

is called the *Teichmüller space of all Riemannian metrics* on M. Furthermore, if M admits a Riemannian metric with negative sectional curvature, the *Teichmüller space* $\mathcal{T}^{<0}(M)$ *of all negatively curved metrics* on M is defined to be the quotient space $\mathcal{MET}^{<0}(M)/\mathcal{D}_0(M)$ with $\mathcal{MET}^{<0}(M)$ the subspace of $\mathcal{MET}(M)$ consisting of all negatively curved metrics on M. Moreover, in that case the action of $\mathcal{D}_0(M)$ is free and $\mathcal{T}(M)$ is a model for the classifying space $B\mathrm{Diff}_0(M)$ of $\mathrm{Diff}_0(M)$ (see [FO09, Lemma 1.1]).

It is proved in [FO09] that if M is real hyperbolic, the Teichmüller space $\mathcal{T}^{<0}(M)$ is, in general, not contractible. More precisely, they prove that the inclusion map $F : \mathcal{T}^{<0}(M) \to \mathcal{T}(M)$ (which forgets the negatively curved structure) is in general homotopically nontrivial. Similar results are obtained in [Sor14, FS17] when M is a Gromov–Thurston manifold or a complex hyperbolic manifold. These results can be translated in the language of bundle theory (c.f. [FO10, Far16]), which we now recall. A smooth $M-$bundle $p : E \to B$ is said to be *negatively curved* if its concrete fibers $p^{-1}(x), x \in B$ can be endowed with negatively curved Riemannian metrics varying continuously from fiber to fiber. The Teichmüller space $\mathcal{T}(M) = B\mathrm{Diff}_0(M)$ classifies equivalence classes of smooth bundles with abstract fiber M and equipped with a fiber homotopy trivialization, whereas $\mathcal{T}^{<0}(M)$ classifies equivalence classes of negatively curved bundles with abstract fiber M and equipped with a fiber homotopy trivialization. It follows from the results obtained in [FO09, Sor14, FS17] that there are negatively curved bundles which are nontrivial as smooth bundles with abstract fiber a hyperbolic manifold, a Gromov–Thurston manifold or a complex hyperbolic manifold. However they all represent elements of finite order in the homotopy groups of $\mathcal{T}(M)$. We show in this paper that there are negatively curved bundles representing elements of infinite order in the homotopy groups of $\mathcal{T}(M)$. Consequently there are closed real hyperbolic manifolds M such that $\pi_i\mathcal{T}^{<0}(M)\otimes\mathbb{Q}\neq 0$ for some i augmenting the main result of [FO09].

In order to ease the notation we write $\Phi^{\mathbb{Q}}$ for $\Phi\otimes id_{\mathbb{Q}}$ when Φ is a homomorphism between abelian groups. Occasionally we denote $\pi_*(\cdot)\otimes\mathbb{Q}$ by $\pi_*^{\mathbb{Q}}(\cdot)$.

Theorem 1.1. There is a positive constant c such that for every closed real hyperbolic manifold M^n of dimension $n\geq c$ there exists a finite sheeted cover $\tilde{M}$ of M such that for every finite sheeted cover $\hat{M}$ of $\tilde{M}$, the homomorphism

$$F_*^{\mathbb{Q}}:\pi_i\mathcal{T}^{<0}(\hat{M})\otimes\mathbb{Q}\to\pi_i\mathcal{T}(\hat{M})\otimes\mathbb{Q}$$

induced by the forget structure map is nontrivial for some $i\in[n+1,2n]$.

Remark 1.2. Note that the range $i\geq n+1$ in Theorem 1.1 is outside the Igusa stable range which is $i\leq\min\{\frac{n-7}{2},\frac{n-4}{3}\}$(c.f. [Igu88] and [Igu02, p.252]). We show in a companion paper [BFJ17] that inside Igusa stable range, negatively curved bundles over spheres can only represent elements of finite order in $\pi_i\mathcal{T}(M^n)$.

We prove Theorem 1.1 in Section 4. This theorem will follow from Theorem 1.3 below together with the discovery of nontrivial rational Pontryagin classes in $H^{4m+4k}(B\mathrm{Top}(2m);\mathbb{Q})$ for some $k>0$ by Weiss [Wei16].

Let $\mathrm{Diff}(\mathbb{D}^n,\partial)$ be the group of all self-diffeomorphisms of $\mathbb{D}^n$ that restrict to the identity on the boundary $\partial\mathbb{D}^n$, equipped with the $C^\infty-$topology. Let $\Omega\mathrm{Diff}(\mathbb{D}^n,\partial)$ be the space of all continuous loops in $\mathrm{Diff}(\mathbb{D}^n,\partial)$ based at the identity $1_{\mathbb{D}^n}$. When necessary, we will assume the loops $t\mapsto f_t\in\mathrm{Diff}(\mathbb{D}^n,\partial)$ for $t\in[0,1]$ are smooth, i.e. the map $(x,t)\mapsto f_t(x)$ is smooth. Note that the inclusion of the space of all smooth loops into $\Omega\mathrm{Diff}(\mathbb{D}^n,\partial)$ is a homotopy equivalence (c.f. [FO09, p.55]). Define a map

$$\alpha_n:\Omega\mathrm{Diff}(\mathbb{D}^{n-1},\partial)\to\mathrm{Diff}(\mathbb{D}^n,\partial)$$

by $\alpha_n(f)(x,t)=(f_t(x),t)$ for $(x,t)\in\mathbb{D}^{n-1}\times[0,1]=\mathbb{D}^n$, where $f\in\Omega\mathrm{Diff}(\mathbb{D}^{n-1},\partial)$ is a loop based at $1_{\mathbb{D}^n}$.

Theorem 1.3. Let i,n be two positive integers with $i\geq n+1$. Assume that the composition of homomorphisms

$$\pi_{i+1}\mathrm{Diff}(\mathbb{D}^{n-1},\partial)\otimes\mathbb{Q}\xrightarrow{\alpha_{n*}^{\mathbb{Q}}}\pi_i\mathrm{Diff}(\mathbb{D}^n,\partial)\otimes\mathbb{Q}\xrightarrow{\alpha_{n+1*}^{\mathbb{Q}}}\pi_{i-1}\mathrm{Diff}(\mathbb{D}^{n+1},\partial)\otimes\mathbb{Q}$$

is nontrivial. Then for every closed real hyperbolic manifold M of dimension n there exists a finite sheeted cover $\tilde{M}$ of M such that for every finite sheeted cover $\hat{M}$ of $\tilde{M}$, the homomorphism $F_*\otimes id_{\mathbb{Q}}:\pi_{i+1}\mathcal{T}^{<0}(\hat{M})\otimes\mathbb{Q}\to\pi_{i+1}\mathcal{T}(\hat{M})\otimes\mathbb{Q}$ induced by the forget structure map is nontrivial.

Remark 1.4. The map α_n has appeared in Gromoll's fundamental work [Gro66] on positive curvature problems and it has also been used by the second author and Ontaneda in [FO09] to study the Teichmüller space of negatively curved metrics.

The proof of Theorem 1.3 is presented in the next two sections.

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2 Nontrivial elements in $\pi_*\mathcal{T}(M)\otimes\mathbb{Q}$

The goal of this section is to construct nontrivial elements in $\pi^{\mathbb{Q}}_{i+1}\mathcal{T}(M)$ out of elements in $\pi^{\mathbb{Q}}_{i}\mathrm{Diff}(\mathbb{D}^n,\partial)$. Recall that a smooth manifold is *parallelizable* if its tangent bundle is trivial.

Throughout the paper $\mathrm{Top}(n)$ denotes the topological group with the compact open topology of all self-homeomorphisms of $\mathbb{R}^n$ and $O(n)\subset\mathrm{Top}(n)$ is the subgroup of all orthogonal transformations of $\mathbb{R}^n$. Let $[X;A,Y;B]$ denote the set of all homotopy classes of maps $(X,A)\to(Y,B)$ between pairs of topological spaces.

Lemma 2.1. *Let M^n be a closed parallelizable manifold which admits a Riemannian metric of nonpositive sectional curvature and let $\iota:\mathrm{Diff}(\mathbb{D}^n,\partial)\to\mathrm{Diff}_0(M)$ be the map given by extending each $f\in\mathrm{Diff}(\mathbb{D}^n,\partial)$ by the identity outside a fixed embedding $\nu:\mathbb{D}^n\subset M$. If $i\geq n\geq 5$, then the homomorphism $\iota_*:\pi_i\mathrm{Diff}(\mathbb{D}^n,\partial)\to\pi_i\mathrm{Diff}_0(M)$ is injective.*

Proof. Let $\mathrm{Map}(\mathbb{D}^n;\partial,\frac{\mathrm{Top}(n)}{O(n)})$ denote the space of all continuous maps $(\mathbb{D}^n,\partial\mathbb{D}^n)\to\left(\frac{\mathrm{Top}(n)}{O(n)},*\right)$ with the base point $*\in\frac{\mathrm{Top}(n)}{O(n)}$ the coset of the identity element. Define a map $\eta:\mathrm{Map}(\mathbb{D}^n;\partial,\frac{\mathrm{Top}(n)}{O(n)})\to\mathrm{Map}(M,\frac{\mathrm{Top}(n)}{O(n)})$ by extending each $f\in\mathrm{Map}(\mathbb{D}^n;\partial,\frac{\mathrm{Top}(n)}{O(n)})$ via the constant map outside $\mathbb{D}^n\subset M$.

Claim: For every $i\geq 1$, if the homomorphism $\eta_*:\pi_{i+1}\mathrm{Map}(\mathbb{D}^n;\partial,\frac{\mathrm{Top}(n)}{O(n)})\to\pi_{i+1}\mathrm{Map}(M,\frac{\mathrm{Top}(n)}{O(n)})$ is injective, then $\iota_*:\pi_i\mathrm{Diff}(\mathbb{D}^n,\partial)\to\pi_i\mathrm{Diff}_0(M)$ is injective.

Proof of Claim. Firstly, let $\mathbb{TOP}_0(M)$ (resp. $\mathbb{DIFF}_0(M)$) denote the singular complex (resp. singular differentiable complex) of $\mathrm{Top}_0(M)$ (resp. $\mathrm{Diff}_0(M)$) and let

$$\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}:=\left|\frac{\mathbb{TOP}_0(M)}{\mathbb{DIFF}_0(M)}\right|$$

be the geometric realization of the simplicial set $\frac{\mathbb{TOP}_0(M)}{\mathbb{DIFF}_0(M)}$. The space $\frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)}$ is similarly defined. Note that the map $\iota:\mathrm{Diff}(\mathbb{D}^n,\partial)\to\mathrm{Diff}_0(M)$ extends canonically to a map $\mathrm{Top}(\mathbb{D}^n,\partial)\to\mathrm{Top}_0(M)$, which in turn gives rise to a map $\bar\iota:\frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)}\to\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}$. Then consider the commutative diagram:

$$\begin{array}{ccccc}
\pi_{i+1}\frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)} & \xrightarrow[\cong]{\partial_{\mathbb{D}^n}} & \pi_i|\mathbb{DIFF}(\mathbb{D}^n,\partial)| & \xrightarrow[\cong]{} & \pi_i\mathrm{Diff}(\mathbb{D}^n,\partial) \\
\downarrow{\scriptstyle\bar\iota_*} & & \downarrow{\scriptstyle S\iota_*} & & \downarrow{\scriptstyle\iota_*} \\
\pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} & \xrightarrow{\partial_M} & \pi_i|\mathbb{DIFF}_0(M)| & \xrightarrow[\cong]{} & \pi_i\mathrm{Diff}_0(M)
\end{array} \tag{1}$$

where $S\iota$ is induced by ι in the canonical way and $\partial_{\mathbb{D}^n}$ (resp. ∂_M) denotes the connecting homomorphism in the homotopy exact sequence associated to the fibration $|\mathbb{TOP}(\mathbb{D}^n,\partial)| \to \frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)}$ (resp. $|\mathbb{TOP}_0(M)| \to \frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}$). Since M is closed and admits a Riemannian metric with nonpositive sectional curvature and $n \geq 5$, then by [FW91, Corollary 4.3], the canonical map $\mathrm{Diff}(M) \to \mathrm{Top}(M)$ induces surjective homomorphisms on π_i for all $i \geq 2$. Furthermore, since the canonical maps

$$|\mathbb{TOP}_0(M)| \xrightarrow[\cong]{} \mathrm{Top}_0(M) \longrightarrow \mathrm{Top}(M)$$

$$|\mathbb{DIFF}_0(M)| \xrightarrow[\cong]{} \mathrm{Diff}_0(M) \longrightarrow \mathrm{Diff}(M)$$

induce isomorphisms on π_i for $i \geq 1$, then the map $|\mathbb{DIFF}_0(M)| \to |\mathbb{TOP}_0(M)|$ also induces epimorphisms on π_i for $i \geq 2$, and hence $\partial_M : \pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} \to \pi_i|\mathbb{DIFF}_0(M)|$ is injective, for $i \geq 1$, by the homotopy exact sequence associated to the fibration $|\mathbb{TOP}_0(M)| \to \frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}$. Consequently, by the commutative diagram (1), to prove ι_* is injective, it suffices to show that $\bar{\iota}_* : \pi_{i+1}\frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)} \to \pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}$ is injective. Hence the claim follows from the commutativity of the next diagram:

$$\begin{array}{ccc}
\pi_{i+1}\frac{\mathrm{Top}(\mathbb{D}^n,\partial)}{\mathrm{Diff}(\mathbb{D}^n,\partial)} & \xrightarrow[\cong]{\mu(\mathbb{D}^n)_*} & \pi_{i+1}\mathrm{Map}(\mathbb{D}^n;\partial,\frac{\mathrm{Top}(n)}{O(n)}) \\
\bar{\iota}_*\downarrow & & \downarrow\eta_* \\
\pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} & \xrightarrow[\cong]{\mu(M)_*} & \pi_{i+1}\mathrm{Map}(M,\frac{\mathrm{Top}(n)}{O(n)})
\end{array}$$

where $\mu(\mathbb{D}^n)_*$, $\mu(M)_*$ the Morlet isomorphisms[BL74, Theorem 4.2].

Thus it remains to show η_* is injective. Let $\hat{\eta} : M \times \mathbb{D}^{i+1}/\partial \to S^{n+i+1} = \mathbb{D}^n \times \mathbb{D}^{i+1}/\partial$ be the degree 1 map which collapses the union of $M \times \partial\mathbb{D}^{i+1}$ and the complement of $\nu \times id_{\mathbb{D}^{i+1}} : \mathbb{D}^n \times \mathbb{D}^{i+1} \to M \times \mathbb{D}^{i+1}$ to a point. Then η_* can be identified with the map $\hat{\eta}^* : [S^{n+i+1};*,\frac{\mathrm{Top}(n)}{O(n)};*] \to [M \times \mathbb{D}^{i+1};\partial,\frac{\mathrm{Top}(n)}{O(n)};*]$. Since $i \geq n$, then by Whitney embedding theorem, there is an embedding $M \subset S^{n+i+1}$ with a closed tubular neighborhood $N(M)$, which induces a degree 1 map $q : S^{n+i+1} \to N(M)/\partial$ collapsing the complement of $N(M) \subset S^{n+i+1}$ into a point. Since M is parallelizable, then $N(M) = M \times \mathbb{D}^{i+1}$. Now the composition $\hat{\eta} \circ q : S^{n+i+1} \to M \times \mathbb{D}^{i+1}/\partial \to S^{n+i+1}$ is of degree 1 and hence homotopic to the identity. This shows that $q^* \circ \hat{\eta}^*$ is the identity on $[S^{n+i+1};*,\mathrm{Top}(n)/O(n);*]$. Consequently, $\hat{\eta}^*$ and hence η_* are injective. This completes the proof of the lemma. □

Recall that a smooth manifold is *stably parallelizable* if the Whitney sum of its tangent bundle with a trivial line bundle is trivial.

Lemma 2.2. Let M^n be a closed stably parallelizable manifold admitting a Riemannian metric with nonpositive sectional curvature and assume $i \geq n+1$. If the image of a class $x \in \pi_i^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^n,\partial)$ under the Gromoll-map

$$\pi_i^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^n,\partial) \xrightarrow{\alpha^{\mathbb{Q}}_{n+1*}} \pi_{i-1}^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^{n+1},\partial)$$

is nonzero, then $\iota_*^{\mathbb{Q}}x \neq 0 \in \pi_i^{\mathbb{Q}}\mathrm{Diff}_0(M)$ where $\iota : \mathrm{Diff}(\mathbb{D}^n,\partial) \to \mathrm{Diff}_0(M)$ is the map given by extending each $f \in \mathrm{Diff}(\mathbb{D}^n,\partial)$ by the identity outside a fixed embedding $\nu : \mathbb{D}^n \hookrightarrow M$.

Proof. The theorem is vacuous if $n = 1, n = 2$ by Smale[Sma59] and if $n = 3$ by Hatcher[Hat83]. Hence we may assume $n \geq 4$. Analogous to the map α_{n+1}, a map $\sigma : \Omega\mathrm{Diff}(M) \to \mathrm{Diff}(M \times [0,1], \partial)$ can be defined such that the following diagram commutes:

$$\begin{array}{ccc} \pi_i \mathrm{Diff}(\mathbb{D}^n, \partial) & \xrightarrow{\alpha_{n+1*}} & \pi_{i-1}\mathrm{Diff}(\mathbb{D}^{n+1}, \partial) \\ \iota_* \big\downarrow & & \big\downarrow \tau_* \\ \pi_i \mathrm{Diff}_0(M) & \xrightarrow{\sigma_*} & \pi_{i-1}\mathrm{Diff}_0(M \times [0,1], \partial) \end{array} \tag{2}$$

where τ is the map given by extending each diffeomorphism in $\mathrm{Diff}(\mathbb{D}^{n+1}, \partial)$ via the identity outside the embedding

$$\mathbb{D}^{n+1} = \mathbb{D}^n \times [0,1] \xrightarrow{\nu \times id_{[0,1]}} M \times [0,1] \; .$$

By the commutativity of the diagram (2), it suffices to show $\tau_*^{\mathbb{Q}} \alpha_{n+1*}^{\mathbb{Q}} x \neq 0$. This can be seen as follows: Let $q : \mathrm{Diff}_0(M \times [0,1], \partial) \to \mathrm{Diff}_0(M \times S^1)$ be the map given by extending each diffeomorphism in $\mathrm{Diff}_0(M \times [0,1], \partial)$ via the identity outside a fixed embedding $M \times [0,1] \subset M \times S^1$. Since $M \times S^1$ is parallelizable and admits a Riemannian metric with nonpositive sectional curvature, then Lemma 2.1 yields that $(q \circ \tau)_* : \pi_{i-1}\mathrm{Diff}(\mathbb{D}^{n+1}, \partial) \to \pi_{i-1}\mathrm{Diff}_0(M \times S^1)$ is injective, which implies that τ_* is injective and so must be $\tau_* \otimes id_{\mathbb{Q}}$. Hence

$$\tau_*^{\mathbb{Q}} \alpha_{n+1*}^{\mathbb{Q}} x \neq 0.$$

This completes the proof. ☐

Remark 2.3. The condition $\alpha_{n+1*}^{\mathbb{Q}} x \neq 0$ is never satisfied inside Igusa stable range by [FH78].

3 Elements in the image of the forgetful homomorphism $F^{\mathbb{Q}} : \pi_* \mathcal{T}^{<0}(M) \otimes \mathbb{Q} \to \pi_* \mathcal{T}(M) \otimes \mathbb{Q}$

In this section, we give a sufficient condition (in Lemma 3.1 below) for an element in $\pi_*^{\mathbb{Q}} \mathcal{T}(M)$ to belong to the image of $F_*^{\mathbb{Q}} : \pi_*^{\mathbb{Q}} \mathcal{T}^{<0}(M) \to \pi_*^{\mathbb{Q}} \mathcal{T}(M)$ and use it together with Lemma 2.2 to prove Theorem 1.3.

For any embedding $\nu : \mathbb{D}^n \to M$, let

$$\iota(\nu) : \mathrm{Diff}(\mathbb{D}^n, \partial) \to \mathrm{Diff}_0(M)$$

be the map given by extending each $\phi \in \mathrm{Diff}(\mathbb{D}^n, \partial)$ by the identity outside the embedding $\nu : \mathbb{D}^n \to M$. The following lemma is a direct consequence of [FO09, Theorem 2].

Lemma 3.1. Let i be a nonnegative integer and n be a positive integer. Suppose that a class $\overline{x} \in \pi_i \mathrm{Diff}(\mathbb{D}^n, \partial)$ lies in the image of the Gromoll homomorphism

$$\alpha_{n*} : \pi_{i+1}\mathrm{Diff}(\mathbb{D}^{n-1}, \partial) \to \pi_i \mathrm{Diff}(\mathbb{D}^n, \partial).$$

Then there exists a real number $r > 0$ with the following property: for any closed real hyperbolic manifold (M, g) with injectivity radius at least $3r$ at some point, there is an embedding $\nu : \mathbb{D}^n \to M$ such that the class

$$\iota(\nu)_* \overline{x} \in \pi_i \mathrm{Diff}_0(M) = \pi_{i+1} \mathcal{T}(M)$$

is in the image of the forget structure homomorphism $F_* : \pi_{i+1} \mathcal{T}^{<0}(M) \to \pi_{i+1} \mathcal{T}(M)$.

The proof of this lemma is given below, after we make the following considerations. Let

$$\Gamma_g : \mathrm{Diff}_0(M) \to \mathcal{MET}^{<0}(M)$$

denote the orbit map given by $\Gamma_g(\phi) = (\phi^{-1})^* g$. Since the sequence

$$\pi_{i+1}\mathcal{T}^{<0}(M) \xrightarrow{F_*} \pi_{i+1}\mathcal{T}(M) = \pi_i \mathrm{Diff}_0(M) \xrightarrow{\Gamma_{g*}} \pi_i \mathcal{MET}^{<0}(M)$$

is exact (c.f. [FO09, Far16]), then to show Lemma 3.1 it suffices to prove that there is an embedding $\nu : \mathbb{D}^n \to M$ such that the image of $\overline{x}$ under the composition

$$\pi_i \mathrm{Diff}(\mathbb{D}^n, \partial) \xrightarrow{\iota(\nu)_*} \pi_i \mathrm{Diff}_0(M) \xrightarrow{\Gamma_{g*}} \pi_i \mathcal{MET}^{<0}(M)$$

is zero. To see how this follows, we review some notions from [FO09].

Let $\mathrm{Diff}_0(S^{n-1} \times [1,2], \partial)$ denote the group of all self-diffeomorphisms of $S^{n-1} \times [1,2]$ that are the identity near $S^{n-1} \times \{1,2\}$ and are homotopic to the identity by a homotopy that is constant near $S^{n-1} \times \{1,2\}$. Let $\mathcal{G}$ be the subgroup of $\mathrm{Diff}_0(S^{n-1} \times [1,2], \partial)$ whose elements are all smooth isotopies ϕ of S^{n-1}, namely $\phi(\cdot, t) \in \mathrm{Diff}(S^{n-1})$ for all $t \in [1,2]$. Let N be a real hyperbolic manifold of dimension n, with a geodesic ball B of radius $2r$ centered at some point $p \in N$, hence $B \backslash p$ can be identified with $S^{n-1} \times (0, 2r]$. Furthermore, identify $S^{n-1} \times [r, 2r]$ with $S^{n-1} \times [1,2]$ by $(x,t) \mapsto (x, \frac{t}{r})$ for any $(x,t) \in [r, 2r]$. Then under these identifications, a map

$$\Lambda(N, p, r) : \mathrm{Diff}_0(S^{n-1} \times [1,2], \partial) \to \mathrm{Diff}_0(N)$$

can be defined by extending each $\varphi \in \mathrm{Diff}_0(S^{n-1} \times [1,2], \partial)$ via the identity outside $S^{n-1} \times [r, 2r] \subset N$. Now we can restate a special case of [FO09, Theorem 2] as follows:

Theorem 3.2 (Farrell-Ontaneda)**.** Given a compact subset $K \subset \mathcal{G}$, there is a real number $r > 0$ such that the following holds: let (N, g) be a closed real hyperbolic manifold and let $p \in N$ with radius of injectivity at p larger than $3r$. Then the map

$$K \xrightarrow{\Lambda(N,p,r)} \mathrm{Diff}_0(N) \xrightarrow{\Gamma_g} \mathcal{MET}^{<0}(N)$$

is homotopic to a constant map.

Proof of Lemma 3.1. Identify $\mathbb{D}^{n-1}$ with the northern hemisphere of the sphere S^{n-1}. Then this induces an inclusion $\mathbb{D}^n = \mathbb{D}^{n-1} \times [1,2] \to S^{n-1} \times [1,2]$, which gives rise to a map

$$j : \mathrm{Diff}(\mathbb{D}^n, \partial) \to \mathrm{Diff}_0(S^{n-1} \times [1,2], \partial).$$

Let $f : S^i \to \Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ be a continuous map such that $\alpha_n \circ f$ represents the homotopy class $\overline{x}$ and let $K = \{(j \circ \alpha_n \circ f)(u) | u \in S^i\}$. Then K is a compact subset of $\mathcal{G}$ and hence Theorem 3.2 applies. Let r be the real number in Theorem 3.2. By the assumption of Lemma 3.1 we can find a $p \in M$ with radius of injectivity at p larger than $3r$. Theorem 3.2 implies that the homomorphism $(\Gamma_g \circ \Lambda(M,p,r))_* : \pi_i K \to \pi_i \mathcal{MET}^{<0}(M)$ is trivial. In particular, if we define $\psi : S^i \to K$ by $\psi(u) = j(\alpha_n(f(u)))$ then the composition $\Gamma_g \circ \Lambda(M,p,r) \circ \psi$ is null-homotopic. Let $\iota := \Lambda(M,p,r) \circ j$, then ι is obviously equal to $\iota(\nu)$ for some embedding $\nu : \mathbb{D}^n \to M$. Therefore the composition of maps $\Gamma_g \circ \iota \circ \alpha_n \circ f$ is null-homotopic, which implies $\Gamma_{g*}\iota_*\overline{x} = 0$. This completes the proof of the lemma.

□

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. By the assumption of Theorem 1.3, there exists $y \in \pi_{i+1}^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ such that $(\alpha_{n+1} \circ \alpha_n)_*^{\mathbb{Q}} y \neq 0$ and hence there is $N >> 0$ and $\overline{y} \in \pi_{i+1}\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ such that

$$N \cdot y = \overline{y} \otimes 1 \in \pi_{i+1}^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^{n-1}, \partial).$$

Let $\overline{x} = \alpha_{n*}\overline{y} \in \pi_i\mathrm{Diff}(\mathbb{D}^n, \partial)$, then Lemma 3.1 applies. Given a closed real hyperbolic manifold N, there is a finite sheeted cover of M which is stably parallelizable (the proof of this result is sketched in [Sul79, p.553]. Another proof appears in [Oku01]). We can further pass to another finite sheeted cover $\tilde{M}$ of the latter so that $\tilde{M}$ has a geodesic ball with large enough radius as required in Lemma 3.1 (c.f. [FJ89, p.901]). Then for any finite sheeted cover $\hat{M}$ of $\tilde{M}$ there is an embedding $\nu : \mathbb{D}^n \to \hat{M}$ such that $\iota(\nu)_*\overline{x}$ lives in the image of the homomorphism $F_* : \pi_{i+1}\mathcal{T}^{<0}(\hat{M}) \to \pi_{i+1}\mathcal{T}(\hat{M})$. Let now $x = \overline{x} \otimes 1 \in \pi_i^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^n, \partial)$. Then $\iota(\nu)_*^{\mathbb{Q}} x = \iota(\nu)_*\overline{x} \otimes 1$ lies in the image of the homomorphism $F_*^{\mathbb{Q}} : \pi_{i+1}^{\mathbb{Q}}\mathcal{T}^{<0}(\hat{M}) \to \pi_{i+1}^{\mathbb{Q}}\mathcal{T}(\hat{M})$ and

$$\alpha_{n+1*}^{\mathbb{Q}} x = N(\alpha_{n+1} \circ \alpha_n)_*^{\mathbb{Q}} y \neq 0.$$

Also by Lemma 2.2, the class $\iota_*^{\mathbb{Q}} x \in \pi_i^{\mathbb{Q}}\mathrm{Diff}_0(\hat{M})$ is nonzero. This completes the proof of the theorem. □

4 An example coming from Pontryagin classes in $H^*(B\mathbf{Top}(m); \mathbb{Q})$

In this section we apply a recent striking result of M. Weiss [Wei16] to exhibit examples where the conditions of Theorem 1.3 are satisfied. As a byproduct we will obtain a proof of Theorem 1.1.

Let $B\mathrm{Top}(m)$ denote the classifying space of $\mathrm{Top}(m)$. The inclusion $\mathrm{Top}(m) \hookrightarrow \mathrm{Top}(m+1)$ induces the map $B\mathrm{Top}(m) \hookrightarrow B\mathrm{Top}(m+1)$. Let $B\mathrm{Top} := \varinjlim B\mathrm{Top}(m)$ and $j_l : B\mathrm{Top}(l) \to B\mathrm{Top}$ be the canonical map. Denote by $p_i \in H^{4i}(B\mathrm{Top}; \mathbb{Q})$ the $i-$th universal rational Pontryagin class.

Theorem 4.1 (Weiss [Wei16])**.** There exist positive constants c_1 and c_2 such that, for all positive integers m and k where $m \geq c_1$ and $k < 5m/4 - c_2$, the rational Pontryagin class $p_{m+k} \in H^{4m+4k}(B\mathrm{Top}(2m); \mathbb{Q})$ evaluates nontrivially on $\pi_{4m+4k}^{\mathbb{Q}}(B\mathrm{Top}(2m))$.

Let c_1 and c_2 be the positive numbers given by Theorem 4.1, then we have the following lemma.

Lemma 4.2. Let n and i be nonnegative integers satisfying the conditions $n \geq 2c_1 + 2$, $i \equiv -n + 2 \mod 4$ and $n \leq i < \frac{7}{2}n - 11 - 4c_2$, then the composite

$$\pi_{i+1}^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^{n-1}, \partial) \xrightarrow{\alpha_{n*}^{\mathbb{Q}}} \pi_i^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^n, \partial) \xrightarrow{\alpha_{n+1*}^{\mathbb{Q}}} \pi_{i-1}^{\mathbb{Q}}\mathrm{Diff}(\mathbb{D}^{n+1}, \partial) \qquad (3)$$

is nontrivial.

Proof. Let $\mathrm{Top}(l)/O(l)$ be the homotopy fiber of the forget structure map $BO(l) \to$

$B\mathrm{Top}(l)$. Consider the commutative diagram

$$\begin{array}{ccccc}
\pi^{\mathbb{Q}}_{i+1}\mathrm{Diff}(\mathbb{D}^{n-1},\partial) & \xrightarrow{\alpha^{\mathbb{Q}}_{n*}} & \pi^{\mathbb{Q}}_{i}\mathrm{Diff}(\mathbb{D}^{n},\partial) & \xrightarrow{\alpha^{\mathbb{Q}}_{n+1*}} & \pi^{\mathbb{Q}}_{i-1}\mathrm{Diff}(\mathbb{D}^{n+1},\partial) \\
\mu^{\mathbb{Q}}_{n-1*}\downarrow\cong & & \mu^{\mathbb{Q}}_{n*}\downarrow\cong & & \mu^{\mathbb{Q}}_{n+1*}\downarrow\cong \\
\pi^{\mathbb{Q}}_{i+n+1}\mathrm{Top}(n-1)/O(n-1) & \longrightarrow & \pi^{\mathbb{Q}}_{i+n+1}\mathrm{Top}(n)/O(n) & \longrightarrow & \pi^{\mathbb{Q}}_{i+n+1}\mathrm{Top}(n+1)/O(n+1) \\
\partial^{\mathbb{Q}}_{*}\uparrow\cong & & \partial^{\mathbb{Q}}_{*}\uparrow\cong & & \partial^{\mathbb{Q}}_{*}\uparrow\cong \\
\pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(n-1) & \xrightarrow{i^{\mathbb{Q}}_{n*}} & \pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(n) & \xrightarrow{i^{\mathbb{Q}}_{n+1*}} & \pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(n+1)
\end{array}$$

where the vertical maps $\mu^{\mathbb{Q}}_{l*}, n-1 \leq l \leq n+1$ are the Morlet isomorphisms[BL74, Theorem 4.2]; the horizontal maps in the middle and bottom rows are induced by the standard stabilizations $\mathrm{Top}(n-1) \to \mathrm{Top}(n)$ and $\mathrm{Top}(n) \to \mathrm{Top}(n+1)$, and the vertical maps $\partial^{\mathbb{Q}}_{*}$ are the connecting homomorphisms in the homotopy exact sequences associated to the corresponding fibrations; the commutativity of the two top squares is guaranteed by [Bur73, Theorem 1.3]. Since $i \geq n$, then the rational homotopy groups $\pi^{\mathbb{Q}}_{i+n+2}BO(l)$ and $\pi^{\mathbb{Q}}_{i+n+1}BO(l)$ are zero for $n-1 \leq l \leq n+1$ and hence the connecting homomorphisms $\partial^{\mathbb{Q}}_{*}$ are isomorphisms. Hence to prove that $\alpha^{\mathbb{Q}}_{n+1*} \circ \alpha^{\mathbb{Q}}_{n*}$ is nontrivial, it suffices to show $i^{\mathbb{Q}}_{n+1*} \circ i^{\mathbb{Q}}_{n*}$ is nontrivial.

We claim that for any pair of integers n, i such that $n \geq 2c_1 + 2$, $i \equiv -n+2 \mod 4$ and $n \leq i < \frac{7}{2}n - 11 - 4c_2$, there is an integer m such that $n-1 \geq 2m$ and such that the rational Pontryagin class $j^*_{2m} p_{\frac{i+n+2}{4}} \in H^{i+n+2}(B\mathrm{Top}(2m);\mathbb{Q})$ evaluates nontrivially on $\pi_{i+n+2}B\mathrm{Top}(2m)$. In fact, considers a pair of integers

$$(m,k) = \begin{cases} (\frac{n+1}{2}, \frac{i-n}{4}), & \text{if } n \text{ is odd.} \\ (\frac{n-2}{2}, \frac{i-n+6}{4}), & \text{if } n \text{ is even.} \end{cases}$$

Then $j^*_{2m} p_{\frac{i+n+2}{4}} = j^*_{2m} p_{m+k} \in H^{i+n+2}(B\mathrm{Top}(2m);\mathbb{Q}) = H^{4m+4k}(B\mathrm{Top}(2m);\mathbb{Q})$ lives in the family of Pontryagin classes in Theorem 4.1 and hence the claim follows.

Let now $i : B\mathrm{Top}(2m) \to B\mathrm{Top}(n-1)$ be induced by the standard stabilization $\mathrm{Top}(2m) \to \mathrm{Top}(n-1)$. Since

$$j_{n+1} \circ i_{n+1} \circ i_n \circ i = j_{2m} : \mathrm{Top}(2m) \to \mathrm{Top}.$$

We have by the claim above that there is $x \in \pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(2m)$ such that

$$\left\langle j^*_{n+1} p_{\frac{i+n+2}{4}}, [i^{\mathbb{Q}}_{n+1*} i^{\mathbb{Q}}_{n*} i^{\mathbb{Q}}_{*} x] \right\rangle = \left\langle i^* i^*_n i^*_{n+1} j^*_{n+1} p_{\frac{i+n+2}{4}}, [x] \right\rangle = \left\langle j^*_{2m} p_{\frac{i+n+2}{4}}, [x] \right\rangle \neq 0$$

where the bracket denotes the Kronecker product and $[x] \in H_{i+n+2}(B\mathrm{Top}(2m);\mathbb{Q})$ (resp. $[i_{n+1*} i^{\mathbb{Q}}_{n*} i^{\mathbb{Q}}_{*} x] \in H_{i+n+2}(B\mathrm{Top}(n+1);\mathbb{Q}))$ denotes the image of x (resp. $i^{\mathbb{Q}}_{n+1*} i^{\mathbb{Q}}_{n*} i^{\mathbb{Q}}_{*} x$) under the Hurewicz homomorphisms. This implies $i^{\mathbb{Q}}_{n+1*} \circ i^{\mathbb{Q}}_{n*}$ is nontrivial and completes the proof. □

Proof of Theorem 1.1. Let n and i be nonnegative integers satisfy that $n \geq 2c_1 + 2$, $i \equiv -n+2 \mod 4$ and $n+1 \leq i < \frac{7}{2}n - 11 - 4c_2$, Then by Lemma 4.2 and Theorem 1.3 we have that for any large enough finite sheeted cover M of a closed real hyperbolic manifold, the forgetful map $\pi^{\mathbb{Q}}_{i+1}\mathcal{T}^{<0}(M) \to \pi^{\mathbb{Q}}_{i+1}\mathcal{T}(M)$ is nonzero. The result follows if we take $c = \max\{2c_1 + 2, \frac{25+8c_2}{3}\}$. □

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