Rigidity and characteristic classes of smooth bundles with nonpositively curved fibers

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Abstract

We prove vanishing results for the generalized Miller-Morita-Mumford classes of some smooth bundles whose fiber is a closed manifold that supports a nonpositively curved Riemannian metric. We also find, under some extra conditions, that the vertical tangent bundle is topologically rigid.

1 Introduction and statement of results

In this paper we study the topology of smooth fiber bundles $M \to E \to B$ whose fiber M is a closed manifold that supports a nonpositively curved Riemannian metric. We obtain two types of results, namely, topological rigidity for the associated vertical tangent bundle of $E \to B$ and vanishing of the tautological classes in positive degrees (also known as generalized Miller-Morita-Mumford classes).

Rigidity results have appeared in the context of negatively curved bundles already, i.e. smooth bundles whose (concrete) fibers can be given negatively curved Riemannian metrics varying continuously from fiber to fiber. In fact, it is proven in [FG15] that the associated vertical *sphere* bundle of a negatively curved bundle is topologically trivial, provided the base space B is a simply connected simplicial complex. Our results, stated below, concern with the associated vertical *tangent* bundle of a bundle whose (abstract) fiber is a nonpositively curved manifold.

With regard to the rational tautological classes, we show that they vanish in many instances for bundles with nonpositively curved fibers, provided the dimension of the fiber is greater than 2. This contrasts heavily with the recent results of Galatius and Randal-Williams [GRW14], which show that tautological classes *do not* vanish for some smooth bundles with highly connected fibers.

Our results also point in the direction of the following conjecture by Farrell and Ontaneda:

Conjecture 1 (Farrell-Ontaneda)**.** *Every negatively curved fiber bundle over a paracompact, Hausdorff space with the homotopy type of a simply connected finite simplicial complex is topologically trivial.*

We now proceed to state the results of this paper.

A *smooth fiber bundle* is a fiber bundle $M \to E \xrightarrow{q} B$ whose (*abstract*) fiber M is a smooth finite dimensional manifold and whose structure group is the group $\mathrm{Diff}(M)$ of smooth self-diffeomorphisms of M. We denote the (*concrete*) fiber $q^{-1}(x)$ over $x \in B$ by M_x.

Throughout this paper we assume that M is a closed smooth manifold. We say that a smooth fiber bundle $M \to E \xrightarrow{q} B$ is *fiber homotopically trivial* (resp. *topologically trivial*) if there exists a continuous map $\Phi : E \to M$ such that its restriction to each fiber $\Phi|_{M_x} : M_x \to M$ is a homotopy equivalence (resp. homeomorphism). Note that in this case the map $E \to B \times M$ given by $a \mapsto (q(a), \Phi(a))$ is a fiber-preserving homotopy equivalence (resp. homeomorphism).

A smooth fiber bundle $M \to E \to B$ over B is said to have *nonpositively curved fibers* if the abstract fiber M supports a complete Riemannian metric whose sectional curvatures are all nonpositive.

Given a smooth fiber bundle $M \to E \xrightarrow{q} B$ we define its *associated vertical tangent bundle* $\mathfrak{t}E \xrightarrow{\pi} B$ to be the smooth fiber bundle of vectors tangent to the fibers M_x of $q : E \to B$, that is to say

$$\mathfrak{t}E = \bigcup_{x \in B} TM_x,$$

where TM_x is the tangent bundle of M_x. The projection map $\pi : \mathfrak{t}E \to B$ is nothing but the composite $\mathfrak{t}E \xrightarrow{r} E \xrightarrow{q} B$, where the map $r : \mathfrak{t}E \to E$ sends a vector in $\mathfrak{t}E$ to its "foot" in E. Thus $\pi : \mathfrak{t}E \to B$ is a smooth fiber bundle with fiber TM. In fact it is an associated bundle to $q : E \to B$ in the sense of Steenrod [Ste51, p. 43]. Notice that $r : \mathfrak{t}E \to E$ is a vector bundle of rank equal to the dimension of M. Also, if E and B are smooth manifolds and π is a smooth map, $\mathfrak{t}E$ can be identified with the the subbundle of the tangent bundle TE of E, consisting of all the vectors $v \in TE$ in the kernel of the differential of q.

Our first theorem is a rigidity result for the associated vertical tangent bundle of a smooth bundle with nonpositively curved fibers.

Theorem A. *Let $M^n \to E \to B$ be a smooth fiber bundle with nonpositively curved n-dimensional ($n \neq 3$) closed fiber over a finite simplicial complex B. Assume that the bundle is fiber homotopically trivial. Then there exists a continuous map $\Psi : \mathfrak{t}E \to TM$ such that $\Psi|_{TM_x} : TM_x \to TM$ is a homeomorphism for each $x \in B$.*

We invite the reader to compare this result with [FG15, Theorem 1.6], where they find fiber homotopically trivial smooth bundles with real hyperbolic fibers, whose associated *sphere* bundles are *not* topologically trivial.

Using obstruction theory, one can show that a smooth fiber bundle $M \to E \to B$ is fiber homotopically trivial if the fiber M is aspherical, $\pi_1(M)$ is centerless and B is a simply connected finite simplicial complex (see for example [FG15, Proposition 1.4]). Hence Theorem A (cf. Remark 7 below) implies the following:

Corollary A.1. *If $M^n \to E \to B$ is a smooth bundle ($n \neq 3$), with M a closed negatively curved manifold and B a simply connected finite simplicial complex, then there exists a continuous map $\Psi : \mathfrak{t}E \to TM$ such that $\Psi|_{TM_x} : TM_x \to TM$ is a homeomorphism for each $x \in B$.*

We now recall the definition of the tautological classes (also known as generalized Miller-Morita-Mumford classes) of an oriented smooth fiber bundle. Let $M^n \to E \xrightarrow{q}$

B be an *oriented* smooth fiber bundle with closed n-dimensional fiber M, i.e. M is oriented and the structure group of the bundle is the group of orientation preserving self-diffeomorphisms of M. Let $\beta : E \to BSO(n)$ be the classifying map for the vector bundle $\mathbb{R}^n \to \mathfrak{t}E \xrightarrow{r} E$. For every cohomology class $c \in H^i(BSO(n);\mathbb{Z})$, we define a *tautological class* $\tau_c(E)$ for the smooth bundle $M \to E \xrightarrow{q} B$ by the formula

$$\tau_c(E) := q_! \, \beta^*(c) \in H^{i-n}(B;\mathbb{Z}),$$

where $q_! : H^*(E;\mathbb{Z}) \to H^{*-n}(B;\mathbb{Z})$ is the Gysin map ("integration along the fiber"), arising from the Serre spectral sequence for the fiber bundle $E \to B$. (See [Mor01, pag. 148-150].) We say that $\tau_c(E) \in H^{i-n}(B;\mathbb{Z})$ is a class of *positive degree* (resp. *degree zero*) if $i > n$ (resp. $i = n$).

Our next result is another instance of rigidity for the associated vertical tangent bundle. It has the consequence that the rational tautological characteristic classes of a smooth bundle with nonpositively curved fiber depend only on the characteristic classes of the abstract fiber.

Recall that a *topological* $\mathbb{R}^n$*-bundle* is a fiber bundle whose fiber is homeomorphic to $\mathbb{R}^n$. Its structure group is denoted by $\mathrm{TOP}(n)$ and consists of all self-homeomorphisms of $\mathbb{R}^n$. Additionally we can assume that every topological $\mathbb{R}^n$-bundle has a zero section. This is possible because the map $p : \mathrm{TOP}(n) \to \mathbb{R}^n$, defined by $p(f) = f(0)$ is a fibration whose base space is contractible. Then $\mathrm{TOP}(n)$ is homotopy equivalent to the fiber of p, namely the group of self-homeomorphisms that fix the origin. The latter corresponds to the structure group for topological $\mathbb{R}^n$-bundles with a zero-section.

Let $\rho : B \times M \to M$ be the projection map onto the second factor.

Theorem B. *Let $M^n \to E \to B$ be a smooth fiber bundle over a finite simplicial complex B, with fiber a closed nonpositively curved n-dimensional manifold ($n \neq 3$). Assume that the bundle is fiber homotopically trivial and let $\Phi : E \to B \times M$ be a fiber homotopy equivalence. Then $\mathfrak{t}E$ and the pullback of the tangent bundle of M along the composition $E \xrightarrow{\Phi} B \times M \xrightarrow{\rho} M$ are isomorphic as topological $\mathbb{R}^n$-bundles.*

Remark 1. *Although Theorems A and B appear to be essentially the same, they are not. In fact in Theorem A we obtain a homeomorphism which does not necessarily cover the given fiber homotopy equivalence between the smooth bundles. Thus Theorem B is not a consequence of Theorem A, and viceversa.*

As a consequence of Theorem B we have:

Corollary B.1. *Let $M^n \to E \to B$ be a smooth fiber bundle as in the statement of Theorem B (without excluding $n = 3$). Assume that the bundle is oriented. Then for all $i > n$ and for any rational cohomology class $c \in H^i(BSO(n);\mathbb{Q})$,*

$$\tau_c(E) = 0. \tag{1}$$

Remark 2. *If one assumes that the fiber in Corollary B.1 is negatively curved, it is possible to drop the condition of fiber homotopy triviality and still get the vanishing of the rational tautological classes in positive degrees (cf. Theorem F and Example a) below).*

Remark 3. *Tautological classes of degree zero of (not necessarily fiber homotopically trivial) oriented smooth M^n-bundles $q : E \to B$ over a path-connected space B satisfy the following equation:*

$$\langle \tau_c(E), [1_B] \rangle = \langle \alpha^*(c), [M] \rangle\,, \quad \textit{if} \;\; c \in H^n(BSO(n); \mathbb{Q}), \tag{2}$$

where $\alpha : M \to BSO(n)$ denotes the classifying map for the tangent bundle of M and $[1_B] \in H_0(B; \mathbb{Q})$ denotes the image of the generator of $H_0(B; \mathbb{Z})$ and $[M] \in H_n(M; \mathbb{Q})$ the fundamental class of M.

To prove this equation, identify the abstract fiber M with some concrete fiber M_*, $* \in B$. Now, if $\iota : M \hookrightarrow E$ denotes the inclusion map we obtain a morphism of smooth M-bundles

$$\begin{array}{ccc} M & \xrightarrow{\;\iota\;} & E \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ * & \xrightarrow{\;\subset\;} & B \end{array}$$

Thus equation (2) follows straightforwardly from the naturality of the tautological classes.

One can actually realize many non-vanishing classes (in degree zero) appearing in the previous remark by bundles with nonpositively curved fibers, indeed with negatively curved fibers. Here is an example: let $c \in H^{4k}(BSO(4k); \mathbb{Q})$ be a monomial of degree $4k$ in the rational Pontrjagin classes $p_1, \ldots, p_k$ such that the corresponding Pontrjagin number of the complex projective space $\mathbb{C}P^{2k}$ is non-zero [MS74, p.194]. By Ontaneda's work on smooth Riemannian hyperbolization [Ont14], the cobordism class of $\mathbb{C}P^{2k}$ can be represented by a closed negatively curved smooth $4k$-manifold M. Hence the corresponding tautological class $\tau_c(M)$ of the smooth bundle $M \to M \to *$ over a point does not vanish, in fact it equals its corresponding Pontrjagin number by the equation (2).

A modification of the proofs of Theorem B and Corollary B.1 yields the following theorem and its corollary. Recall that a smooth bundle $M \to E \to B$ is a nonpositively curved bundle if each concrete fiber M_x can be endowed with a nonpositively curved metric g_x in a way that the metrics g_x vary continuously from fiber to fiber (c.f. [FO10]).

Caveat. Not every smooth bundle with nonpositively curved fibers is a nonpositively curved bundle. In fact, for every closed negatively curved manifold M of dimension greater than 9, it is possible to construct smooth M-bundles over a circle which cannot be given a fiberwise nonpositively curved metric [FO15].

Theorem C. *Let $E \to B$ and $E' \to B$ be fiber homotopy equivalent smooth M^n-bundles over a finite simplicial complex B, where M^n is a closed smooth manifold of dimension $n \neq 3$. Assume that $M \to E \to B$ is a nonpositively curved bundle. If $\Phi : E \to E'$ is a fiber homotopy equivalence then $\Phi^*(\mathfrak{t}E')$ and $\mathfrak{t}E$ are isomorphic as topological $\mathbb{R}^n$-bundles over E.*

Corollary C.1. *Under the hypothesis of the Theorem C (including n=3), the bundles $E \to B$ and $E' \to B$ have the same rational tautological characteristic classes, provided they are oriented and the fiber homotopy equivalence Φ preserves the orientation.*

This invariance of the characteristic classes under fiber homotopy equivalence can be used to obtain vanishing results for characteristic classes in concrete cases where the fiber is a torus or a closed hyperbolic manifold, without the assumption of fiber homotopy triviality.

Theorem D. *Let $\mathbb{T}^n \to E \to B$ be a smooth fiber bundle over a compact smooth manifold B with fiber an n-dimensional torus $\mathbb{T}^n$. Then the rational Pontrjagin classes of its associated vector bundle $r : \mathfrak{t}E \to E$ over E, vanish.*

Recall that every rational cohomology class $c \in H^*(BSO(n);\mathbb{Q})$ can be expressed as a polynomial in the Pontrjagin classes and the Euler class, such that the Euler class in each monomoial has degree at most 1. Also, equation (2) and the fact that the Euler characteristic of a torus is identically zero, imply that the tautological class $\tau_e(E)$ corresponding to the Euler class $e \in H^n(BSO(n);\mathbb{Q})$ of an oriented torus bundle $\mathbb{T}^n \to E \to B$ vanishes. Hence we obtain the following corollary:

Corollary D.1. *Let $\mathbb{T}^n \to E \to B$ be an oriented torus bundle over a compact smooth manifold B. Then $\tau_c(E) = 0$ for all $c \in H^*(BSO(n);\mathbb{Q})$.*

Theorem E. *Let $M \to E \to B$ be an oriented smooth fiber bundle over a finite simplicial complex B, with fiber an n-dimensional ($n \geq 3$) (real, complex or quaternionic) hyperbolic manifold. Then $\tau_c(E) = 0$ for all $c \in H^i(BSO(n);\mathbb{Q})$ with $i > n$.*

Theorem E also follows from the next more general result, which is proved by different methods in the Appendix.

Theorem F. *Let $M \to E \xrightarrow{q} B$ be an oriented smooth M-bundle over a finite simplicial complex B. Assume that M^n ($n \geq 3$) is a nonpositively curved closed manifold such that $Out(\pi_1 M)$ is finite and $\pi_1 M$ is centerless. Then $\tau_c(E) = 0$ for all $c \in H^i(BSO(n);\mathbb{Q})$ with $i > n$.*

Examples.

a) *Theorem F holds when M is a closed negatively curved Riemannian manifold of dimension at least 3 ([Gro87, Theorem 5.4.A], cf. Remark 7).*

b) *Theorem F also holds if M is a nonpositively curved locally symmetric space of non-compact type, such that it has no finite sheeted cover $\widehat{M}$ such that $\widehat{M}$ is a metric product $A \times B$ of Riemannian manifolds where A is 2-dimensional. Notice that in this case $Out(\pi_1 M)$ is finite by Mostow's strong rigidity theorem [Mos73, Theorem 24.1], and the fundamental group is centerless as shown for example in [Ebe83, p.210].*

c) *The conclusion of Theorem F fails if one allows the outer automorphisms group of the fundamental group of the fiber to be infinite, yet the fundamental group itself can be centerless. For an example, consider the product $E \times E$ of an oriented surface bundle $N \to E \xrightarrow{q} B$ over a closed surface B with itself. Denote by $\tau_{\mathcal{L}}(E \times E)$ the tautological class corresponding to the $\mathcal{L}$-genus of $\mathfrak{t}(E \times E) \to E \times E$, i.e.*

$$\tau_{\mathcal{L}}(E \times E) = (q \times q)_! \left(\frac{7}{45} p_2(\mathfrak{t}(E \times E)) + \frac{1}{45} p_1^2(\mathfrak{t}(E \times E)) \right) \in H^4(B \times B;\mathbb{Q}).$$

It is not difficult to conclude that

$$\langle \tau_{\mathcal{L}}(E \times E), [B \times B] \rangle = sign\,(E \times E) = (sign\,E)^2.$$

And it is known (see [Ati69] or [Mor01, p.155-160]) that there exist surface bundles over surfaces for which $sign\,E \neq 0$.

Note that the example b) does not apply to the bundle $E \times E \to B \times B$ *since* $N \times N$ *obviously has a 2-dimensional metric factor. On the other hand, there are closed irreducible nonpositively curved 4-dimensional Riemannian manifolds* M *(i.e. no finite sheeted cover of* M *is a nontrivial metric product) which are locally isometric to* $N \times N$ *(i.e. the universal cover of* M *is isometric to the product of hyperbolic spaces* $\mathbb{H} \times \mathbb{H}$*) and hence satisfy the conditions of example b). Therefore the fundamental group* $\pi_1 M$ *is centerless and Out(*$\pi_1 M$*) is finite. Consequently Theorem F applies to oriented smooth bundles with fiber* M*. The existence of such manifolds* N *is a consequence of [Joh88, Theorem C] and [Shi63, §6].*

Under the additional assumption that the fiber M of a smooth bundle satisfies the Strong Borel Conjecture, we can show the invariance of the tautological classes under fiber homotopy equivalence of M-bundles. We say that a closed manifold M *satisfies the Strong Borel Conjecture (SBC)* if, for all $k \geq 0$, every self-homotopy equivalence of pairs $(M \times D^k, M \times S^{k-1}) \to (M \times D^k, M \times S^{k-1})$ which is a homeomorphism when restricted to the boundary $M \times S^{k-1}$ is homotopic (relative to the boundary) to a homeomorphism, where D^k is a k-dimensional closed disc. For example, every closed nonpositively curved Riemannian manifold of dimension ≥ 5 satisfies SBC, since it is topologically rigid [FJ90]. (See [FJ98] for more results about topological rigidity).

Theorem G. *Let* $\Phi : E \to E'$ *be a fiber homotopy equivalence between* M*-bundles* $p : E \to B$ *and* $q : E' \to B$ *over a finite simplicial complex* B*. Assume that* M *is a closed smooth manifold which satisfies SBC. Then* $\Phi^*(p_i(\mathfrak{t}E')) = p_i(\mathfrak{t}E)$*, where* $p_i(\mathfrak{t}E) \in H^{4i}(E;\mathbb{Q})$ *denotes the* i*-th rational Pontrjagin class of* $\mathfrak{t}E \to E$*.*

Corollary G.1. *Let* $E \to B$ *and* $E' \to B$ *be oriented smooth* M*-bundles over a finite simplicial complex* B*. Assume that* M *satisfies SBC. If there is an orientation preserving fiber homotopy equivalence* $E \to E'$*, then*

$$\tau_c(E) = \tau_c(E')$$

for all $c \in H^*(BSO(n);\mathbb{Q})$*.*

Remark 4. *With the exception of Theorem D and Corollary D.1, all the other results stated above can be proven under the more general assumption that the base space* B *is a paracompact, Hausdorff space which is homotopy equivalent to a finite simplicial complex* $\mathcal{B}$ *(for example,* B *could be a compact topological manifold [KS69]). In that case the proofs follow from rather direct arguments involving pull-backs of bundles along some homotopy equivalence* $\mathcal{B} \to B$ *and a homotopy inverse. We do not state them in that generality in order not to obscure the main ideas.*

This article is organized as follows: In Section 2 we prove Theorem A. In Section 3 we prove B and Corollary B.1. Theorem C and Corollary C.1 are proven in Section 4. In Section 5 we prove Theorem D, Corollary D.1 and Theorem E. Since the proofs of Theorems F and G and Corollary G.1 follow a different strategy than that used to prove the other results, we do them in the Appendix at the end of this article. In the same appendix we prove a lemma about the invariance of the Euler class under fiber homotopy equivalence (Lemma 4.4, see Section 4).

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2 Topological type of the vertical tangent bundle

In this section we prove Theorem A. We handle the cases $n \geq 4$, $n = 2$ and $n = 1$ separately. We will make use of the following:

Lemma 2.1. *[FW91, Corollary 2.7] Let $n \geq 4$ and k be given. Then there is a $\delta > 0$ such that if X^k is a k-dimensional complex, $p : E \to X^k$ is a vector bundle with fiber $\mathbb{R}^n$, $p' : E' \to X'$ is a topological $\mathbb{R}^n$-bundle, and $\hat{f} : E \to E'$ is a bundle map that covers $f : X \to X'$ and such that the restriction of $\hat{f}$ to each fiber is a δ-map, then $f^*(E')$ is isomorphic to E as a topological $\mathbb{R}^n$-bundle.*

A map $g : Y \to Z$ between two metric spaces Y and Z is a *δ-map* if $\operatorname{diam}(g^{-1}(z)) < \delta$, for each $z \in Z$.

Here and in the next sections we will need the following construction: Let Γ be a group. Recall that if X is a right Γ-space and Y is a left Γ-space, then we can define $X \times_\Gamma Y$ to be the quotient space $X \times Y/\sim$, where the equivalence relation on $X \times Y$ is given by $(x, y) \sim (x', y')$ if there exists $g \in \Gamma$ such that $x' = xg^{-1}$ and $y' = gy$.

Let $\Gamma \to E\Gamma \xrightarrow{\pi} B\Gamma$ be the universal principal Γ-bundle over $B\Gamma$ and F be a left Γ-space. The *associated* fiber bundle over $B\Gamma$ with fiber F is denoted by

$$F \to E\Gamma \times_\Gamma F \to B\Gamma,$$

where the bundle projection sends a class $[e, x] \in E\Gamma \times_\Gamma F$ to $\pi(e)$.

Proof of Theorem A.
Case $n \geq 4$: Our proof in this case is essentially a parametrized version of the proof of a theorem of Ferry and Weinberger [FW91, Theorem 2]. Let $\Phi : E \to B \times M$ be a

homotopy trivialization of the smooth bundle $M \to E \xrightarrow{q} B$. Identify M with the concrete fiber $M_* = q^{-1}(*)$, $* \in B$, and endow it with a nonpositively curved Riemannian metric.

Fix a universal covering $\rho_M : \widetilde{M} \to M$ and denote by Γ its group of deck transformations. Let $\widetilde{E}$ be the pullback of the covering $B \times \widetilde{M} \xrightarrow{id_B \times \rho_M} B \times M$ along $\Phi : E \to B \times M$ and $\widetilde{\Phi} : \widetilde{E} \to B \times \widetilde{M}$ the map that covers Φ.

Consider now the following diagram:

$$\begin{array}{ccccc} B \times TM & \xrightarrow{id_B \times Exp} & B \times \widetilde{M} \times_\Gamma \widetilde{M} & \xrightarrow{\widetilde{\Upsilon} \times_\Gamma id} & \widetilde{E} \times_\Gamma \widetilde{M} \\ & {\scriptstyle id_B \times \tau_M} \searrow & \downarrow {\scriptstyle id_B \times u} & \swarrow {\scriptstyle (q \circ \rho_E) \times_\Gamma \rho_M} & \\ & & B \times M & & \end{array} \tag{3}$$

where the maps are defined as follows:

- $\tau_M : TM \to M$ is the tangent bundle projection.
- To define $Exp : TM \to \widetilde{M} \times_\Gamma \widetilde{M}$, first consider the map $T\widetilde{M} \to \widetilde{M} \times \widetilde{M}$ given by $v \mapsto (\gamma_v(1), \gamma_v(0))$, where γ_v is the unique geodesic in $\widetilde{M}$ such that $\gamma_v(0) = \tau_{\widetilde{M}}(v)$ and $\dot{\gamma}_v(0) = v$. This map is Γ-equivariant respect to the action of Γ on $T\widetilde{M}$ induced by the Γ-action on $\widetilde{M}$. The map $Exp : TM \to \widetilde{M} \times_\Gamma \widetilde{M}$ is the induced map on the orbit space.
- $u : \widetilde{M} \times_\Gamma \widetilde{M} \to M$ is the projection onto the second factor mod Γ.
- The map $\widetilde{\Upsilon} \times_\Gamma id$ is defined as follows: Let $\Upsilon : B \times M \to E$ be a (fiber preserving) homotopy inverse of Φ (cf. [Dol55]). Thus $\Upsilon^* \widetilde{E}$ is isomorphic to $B \times \widetilde{M}$. Consequently we obtain an isomorphism of Γ-coverings which in turn gives rise to a lifting $\widetilde{\Upsilon} : B \times \widetilde{M} \to \widetilde{E}$ of Υ. The map $\widetilde{\Upsilon} \times_\Gamma id$ corresponds to the map induced by $\widetilde{\Upsilon} \times id$ on the Γ-orbit space.
- $\rho_E : \widetilde{E} \to E$ is the natural projection.

Notice that the left diagonal map is a vector bundle and the right diagonal map is a topological $\mathbb{R}^n$-bundle. One easily checks that every triangle in the diagram 3 above commutes.

Hence $(\widetilde{\Upsilon} \times_\Gamma id) \circ (id_B \times Exp) : B \times TM \to \widetilde{E} \times_\Gamma \widetilde{M}$ is a fiber preserving map. Moreover, for each $x \in B$, the restriction $\widetilde{\Upsilon}|_{\{x\} \times \widetilde{M}} : \{x\} \times \widetilde{M} \to \widetilde{M_x}$ is a δ_x-map for some $\delta_x \geq 0$. Thus, since B is compact, the restriction of $\widetilde{\Upsilon}$ to each fiber is a δ-map for any $\delta \geq \max\{\delta_x\}$. Since we are choosing a nonpositively curved metric on M, the exponential map Exp is a weakly expanding map by the Cartan-Hadamard theorem, i.e. for each $p \in M$, and $v_1, v_2 \in T_pM$

$$d(Exp(v_1), Exp(v_2)) \geq d(v_1, v_2).$$

Hence (using the linear structure on the fibers of TM to obtain a suitable δ if necessary) the map $(\widetilde{\Upsilon} \times_\Gamma id) \circ (id_B \times Exp) : B \times TM \to \widetilde{E} \times_\Gamma \widetilde{M}$ satisfies the conditions of Lemma

2.1, and this implies that there exists an $\mathbb{R}^n$-bundle isomorphism

$$h : B \times TM \to \widetilde{E} \times_\Gamma \widetilde{M}.$$

We now consider the $\mathbb{R}^n$-bundle $\tau E \xrightarrow{t} E$, which is the pullback of the natural projection $\rho_E \times_\Gamma (q \circ \rho_E) : \widetilde{E} \times_\Gamma \widetilde{E} \to E \times B$ along the map $(id_E, q) : E \to E \times B$

Another n-vector bundle V over E can be obtained as follows: Denote by $pr_{\widetilde{M}} : B \times \widetilde{M} \to \widetilde{M}$ the projection onto the second factor and let $s : E \to \widetilde{E} \times_\Gamma \widetilde{M}$ be the section induced by $(id_{\widetilde{E}}, pr_{\widetilde{M}} \circ \widetilde{\Phi}) : \widetilde{E} \to \widetilde{E} \times \widetilde{M}$ on orbit spaces . The tangent bundle projection $T\widetilde{M} \to \widetilde{M}$ induces a vector bundle $\widetilde{E} \times_\Gamma T\widetilde{M} \to \widetilde{E} \times_\Gamma \widetilde{M}$. Let $V \to E$ be the pullback of this vector bundle along s. By composing the map $V \to \widetilde{E} \times_\Gamma T\widetilde{M}$ that covers $s : E \to \widetilde{E} \times_\Gamma \widetilde{M}$ with the natural map $\widetilde{E} \times_\Gamma T\widetilde{M} \to \widetilde{E} \times_\Gamma \widetilde{M}$ induced by the exponential map (defined with respect to the complete nonpositively curved metric on $\widetilde{M}$), we obtain a topological $\mathbb{R}^n$-bundle homeomorphism denoted by $\omega : V \to \widetilde{E} \times_\Gamma \widetilde{M}$.

In addition, note that the map

$$\widetilde{E} \times_\Gamma \widetilde{M} \to \widetilde{E} \times_\Gamma \widetilde{E},$$

that sends a class $[e, y] \in \widetilde{E} \times_\Gamma \widetilde{M}$ to $[e, \widetilde{\Upsilon}(q(\rho_E(e)), y)]$, covers the map $(id_E, q) : E \to E \times B$. Hence there is a map $\lambda : \widetilde{E} \times_\Gamma \widetilde{M} \to \tau E$ that makes the following diagram commutative:

$$\begin{array}{ccccc} V & \xrightarrow{\ \omega\ } & \widetilde{E} \times_\Gamma \widetilde{M} & \xrightarrow{\ \lambda\ } & \tau E \\ & \searrow & \downarrow \rho_E & \swarrow t & \\ & & E & & \end{array} \tag{4}$$

Arguing as a above, we find that the composition $\lambda \circ \omega$ in diagram 4 is a bundle map whose restriction to each fiber is a δ-map. We invoke Lemma 2.1 again to conclude that there is a fiber-preserving homeomorphism $h' : V \to \tau E$.

Note that the restriction of the bundle map $\tau E \xrightarrow{(h' \circ \omega^{-1} \circ h)^{-1}} B \times TM \xrightarrow{pr} TM$ to each fiber over $x \in B$ is a homeomorphism. Thus the proof of Theorem A in the case $n \geq 4$ will be completed as soon as we prove the following lemma:

Lemma 2.2. *$\mathfrak{t}E$ and τE are isomorphic topological $\mathbb{R}^n$-bundles over E.*

Proof. We prove this lemma by first showing that $\mathfrak{t}E$ and τE are isomorphic as microbundles. (We refer the reader to [Mil64] for the basic material about microbundles.) To see this, first choose a continuously varying family of Riemannian metrics g_x, $x \in B$, on each fiber M_x. Let $exp^{g_x} : T\widetilde{M}_x \to \widetilde{M}_x$ be the exponential map associated to the Riemannian metric on $\widetilde{M}_x$ induced by g_x. Define $\mathfrak{t}\widetilde{E} \to \widetilde{E} \times \widetilde{E}$ by sending $v \in \mathfrak{t}\widetilde{E}$ to $[\widetilde{r}(v), exp^{g_x}(v)]$, where $\widetilde{r} : \mathfrak{t}\widetilde{E} \to \widetilde{E}$ is the bundle projection and $x = q(\rho_E(\widetilde{r}(v)))$. Since the metrics vary continuously from fiber to fiber, we obtain a Γ-equivariant map $\mathfrak{t}\widetilde{E} \to \widetilde{E} \times \widetilde{E}$, and moding out by Γ we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{t}E & \longrightarrow & \widetilde{E} \times_\Gamma \widetilde{E} \\ \downarrow & & \downarrow \\ E & \xrightarrow{(1,q)} & E \times B \end{array}$$

where the vertical maps are the obvious bundle projections. This diagram, in turn, gives rise to a map

$$\zeta : \mathfrak{t}E \to \tau E = (1,q)^*(\widetilde{E} \times_\Gamma \widetilde{E})$$

As g_x depends continuously on x, the function on M_x that assigns to each point in M_x its injectivity radius, varies continuously over $x \in B$. Since B and M_x are both compact, let $r > 0$ be the minimum among all these injectivity radii. Denote by $N_r(M_x)$ an r-neighborhood of M_x in TM_x. Then the restriction of the map ζ to the neighborhood $N_r(E) = \bigcup_{x \in B} N_r(M_x)$ of the zero section E in $\mathfrak{t}E$, is an embedding into a neighborhood of the zero section in τE. This proves that $\mathfrak{t}E$ and τE are isomorphic microbundles. Hence, by Kister-Mazur's theorem [Kis64, Theorem 2], $\mathfrak{t}E$ and τE must be isomorphic as topological $\mathbb{R}^n$-bundles. This completes the proof of Lemma 2.2 and Theorem A when $n \geq 4$. □

We now prove Theorem A in the remaining cases. Let $B\mathrm{Diff}(M)$ be the classifying space for the group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M and let $BG(M)$ be the classifying space for the associative H-space $G(M)$ of self-homotopy equivalences of M (see [DL59]). Define $\sigma : B\mathrm{Diff}(M) \to BG(M)$ to be the map functorially induced by the inclusion (which is an H-space morphism) $\iota : \mathrm{Diff}(M) \hookrightarrow G(M)$. It follows from [DL59] that two smooth bundles $f_1, f_2 : B \to B\mathrm{Diff}(M)$ are fiber homotopy equivalent if and only if $\sigma \circ f_1$ is homotopic to $\sigma \circ f_2$. Thus, if σ is a weak homotopy equivalence, then every fiber homotopically trivial smooth M-bundle over B is also smoothly trivial (see [Hat02, Proposition 4.22]). This implies that the associated vertical tangent bundle will be topologically (in fact smoothly) trivial.

Our strategy to prove Theorem A in lower dimensions consists of proving that the map σ is a weak homotopy equivalence. For this it suffices to show that $\iota : \mathrm{Diff}(M) \to G(M)$ is a weak homotopy equivalence.

Recall also that the space $G(X)$ of self-homotopy equivalences of any aspherical complex X has the homotopy type of $\mathrm{Out}(\pi_1 X) \times K(\mathrm{Center}(\pi_1 X), 1)$ (see [Got65]). It will also be useful to keep in mind that $\mathrm{Diff}(M) = \pi_0\mathrm{Diff}(M) \times \mathrm{Diff}^0(M)$, where $\mathrm{Diff}^0(M)$ denotes the connected component of the identity.

With all this preparation, we can prove the theorem in dimensions 2 and 1.

Case $n = 2$: The closed surfaces that support a nonpositively curved metric are the orientable surfaces of genus ≥ 1 and non-orientable surfaces of genus $g \geq 2$.

It follows from classical theorems of Dehn, Nielsen, Baer (see [FM12, Theorem 8.1]) and Mangler [Man39], that for all nonpositively curved closed surfaces M, the inclusion $\mathrm{Top}(M) \hookrightarrow G(M)$ induces a bijection on connected components. It is also known that the inclusion $\mathrm{Diff}(M) \hookrightarrow \mathrm{Top}(M)$ induces an isomorphism $\pi_0(\mathrm{Diff}(M)) \simeq \pi_0(\mathrm{Top}(M))$ (see [FM12, Theorem 1.13] and [EE69]). Therefore we have an isomorphism $\iota_* : \pi_0(\mathrm{Diff}(M)) \to \pi_0(G(M))$ induced by the inclusion $\mathrm{Diff}(M) \hookrightarrow G(M)$. Earle and Eells [EE69] showed that $\pi_i(\mathrm{Diff}^0(M))$ is trivial for all $i \geq 2$ and for all nonpositvely curved surfaces M. Likewise $\pi_i(G^0(M)) = 0$ if $i \geq 2$, where $G^0(M)$ is the identity component of $G(M)$.

Hence it remains to show that ι induces an isomorphism on fundamental groups. This has to be divided into several cases:

- (M has genus ≥ 2): Earle and Eells also prove in [EE69] that $\mathrm{Diff}^0(M)$ is contractible. Since the center of $\pi_1(M)$ is trivial, the components of $G(M)$ are also contractible. Therefore $\iota : \mathrm{Diff}(M) \to G(M)$ induces an isomorphism on fundamental groups.

- (M is a 2-torus): We regard M as a Lie group whose identity element we denote by $e \in M$. Let $p : G(M) \to M$ be evaluation at e, i.e. $p(f) = f(e)$. Also, for $g \in M$, denote by $L_g : M \to M$ the map $L_g(g') = gg'$. Note that the identity map $id : M \to M$ factors through

$$M \xrightarrow{L} \mathrm{Diff}(M) \xrightarrow{\iota} G(M) \xrightarrow{p} M,$$

  where $L(g) = L_g$. This induces an isomorphism

$$\pi_1(M, e) \xrightarrow{L_*} \pi_1(\mathrm{Diff}(M), id) \xrightarrow{\iota_*} \pi_1(G(M), id) \xrightarrow{p_*} \pi_1(M, e).$$

  Earle and Eells [EE69] show that $\pi_1(\mathrm{Diff}(M), id)$ is a free abelian group of rank 2. Then observe that each group in the sequence above is free abelian of rank 2, and the composition is the identity. This forces ι_* to be an isomorphism.

- (M is the Klein bottle): There is a smooth action $L : S^1 \to \mathrm{Diff}(M)$ of the circle S^1 on M. In order to describe the smooth homomorphism L we think of $S^1 = [0, 2]/0 \sim 2$ as obtained by identifying the end points of the interval $[0, 2]$. and we identify M with the quotient space $S^1 \times [0, 1]/\sim$, where $(z, 0) \sim (\bar{z}, 1)$ (here S^1 should be thought of as the unit circle in the complex plane, and $\bar{z}$ is complex conjugation). The action L is then given by $L(s)(z, t) = (z, t + s)$, for $s \in [0, 2]/0 \sim 2$, where "+" here is understood to be addition mod $\mathbb{Z}$.

  Let $* = (1, 0) \in M$ be a base point. Then we have a sequence of continuous maps

$$S^1 \xrightarrow{L} \mathrm{Diff}(M) \xrightarrow{\iota} G(M) \xrightarrow{p} M,$$

  where p is evaluation at $* \in M$. Now recall that $\mathrm{Center}(\pi_1(M))$ is an infinity cyclic group generated by the class in $\pi_1(M, *)$ represented by the orbit of $* \in M$ under this circle action. Then it is easy to see that the induced map

$$\pi_1(S^1) \xrightarrow{L_*} \pi_1(\mathrm{Diff}(M), id) \xrightarrow{\iota_*} \pi_1(G(M), id) \xrightarrow{p_*} \mathrm{Center}(\pi_1(M, *))$$

  is an isomorphism.
  It is shown also in [EE69] that $\pi_1(\mathrm{Diff}(M), id)$ is infinite cyclic. Likewise $\pi_1(G(M), id)$ is infinite cyclic since $\mathrm{Center}(\pi_1(M, *))$ is infinite cyclic. Therefore ι_* must be an isomorphism.

This concludes the proof of the theorem when $n = 2$.

Case $n = 1$**:** This case follows exactly as with the torus. We only need to recall that $\mathrm{Diff}(S^1)$ has only two components each of which has the homotopy type of a circle. □

Remark 5. *When M is a closed hyperbolic 3-manifold, Gabai shows in [Gab01] that the identity component of Diff(M) is contractible. It is also shown in [Gab97] and [GMT03] that the connected components of Diff(M) are in canonical bijection with the connected components of the isometry group Isom(M) of M. This, combined with Mostow rigidity theorem, shows that ι : Diff(M) $\hookrightarrow G(M)$ is a homotopy equivalence.*

It would be interesting to know whether Theorem A holds for all smooth bundles with nonpositively curved 3-dimensional fiber.

3 Vanishing of tautological classes

In this section we prove Theorem B and Corollary B.1. Let $\rho : B \times M \to M$ be the projection onto the second factor.

We now prove Theorem B.

Proof of Theorem B.

Case $n \geq 4$: We make use of Ferry and Weinberger's result (Lemma 2.1) again. Put a nonpositively curved metric on M and consider the following diagram (with the same notation as in Section 2)

$$\begin{array}{ccc} B \times TM & \longrightarrow & \tau E \\ {\scriptstyle id_B \times \tau_M} \downarrow & & \downarrow {\scriptstyle t} \\ B \times M & \xrightarrow{\Upsilon} & E \end{array}$$

where the top horizontal map is defined by

$$(x, v) \mapsto \Big(\Upsilon(x, \tau_M(v)), \Big[\widetilde{\Upsilon}(x, \gamma_v(0)), \widetilde{\Upsilon}(x, \gamma_v(1))\Big]\Big).$$

Here $\gamma_v : [0, 1] \to \widetilde{M}$ is the unique geodesic with $\gamma_v(0) = \tau_M(v)$ and $\dot{\gamma}_v(0) = v$, and the square brackets denote an equivalence class in $\widetilde{E} \times_\Gamma \widetilde{E}$. This is a morphism of topological $\mathbb{R}^n$-bundles and one easily checks (by compactness of B) that the horizontal top map is a δ-map when restricted to each fiber, for some $\delta > 0$. The linear structure on the fibers of TM can be used to adjust this δ so that it is the same as in Lemma 2.1. Therefore $\psi^*(\tau E) \simeq B \times TM$. Thus we have the following isomorphisms of topological $\mathbb{R}^n$-bundles:

$$\begin{aligned} \mathfrak{t}E &\simeq \tau E \\ &\simeq \Phi^*(B \times TM) \\ &\simeq \Phi^* \rho^*(TM), \end{aligned}$$

where the first isomorphism comes from Lemma 2.2. This completes the proof of Theorem B in the case $n \geq 4$.

Case $n = 1, 2$. Since every fiber homotopically trivial smooth M-bundle is smoothly trivial (by the proof of Theorem A), then there exists an isomorphism $\Lambda : E \to B \times M$ of smooth M-bundles over B. Since the inclusion ι : Diff(M) $\hookrightarrow G(M)$ is a (weak) homotopy equivalence, it follows that $\Phi \circ \Lambda^{-1} : B \times M \to B \times M$ is fiberwise homotopic to an isomorphism between smooth M-bundles. This yields to an isomorphism between vector bundles over $B \times M$, namely:

$$(\rho \circ \Phi \circ \Lambda^{-1})^*(TM) \simeq B \times TM,$$

and hence an isomorphism between vector bundles over E:

$$(\rho \circ \Phi)^*(TM) \simeq \Lambda^*(B \times TM) \simeq \mathfrak{t}E.$$

This completes the proof of Theorem B in the case $n = 1, 2$. □

Proof of Corollary B.1. Suppose first that $n \neq 3$. Denote by $\mathfrak{F} : BSO(n) \to B\mathrm{TOP}(n)$ the "forget both orientation and linear structure" map. Then Theorem B can be rephrased by saying that

$$\mathfrak{F} \circ \beta \sim \mathfrak{F} \circ \alpha \circ (\rho \circ \Phi),$$

where $\beta : E \to BSO(n)$ and $\alpha : M \to BSO(n)$ are the classifying maps for the associated vertical tangent bundle of $q : E \to B$ and the tangent bundle of M, respectively. Note that the rational Pontrjagin classes can be viewed as characteristic classes for topological $\mathbb{R}^n$-bundles (see [KS77]), i.e. each universal Pontrjagin class $p_i \in H^{4i}(BSO(n); \mathbb{Q})$ has a preimage under the induced homomorphism

$$\mathfrak{F}^* : H^*(\mathrm{BTOP}(n); \mathbb{Q}) \to H^*(BSO(n); \mathbb{Q}).$$

This implies that

$$\beta^* p_i = (\rho \circ \Phi)^* \alpha^* p_i,$$

for all i. On the other hand, observe that the Euler class is well defined for oriented topological $\mathbb{R}^n$-bundles and natural with respect to orientation-preserving bundle maps of topological $\mathbb{R}^n$-bundles, hence

$$\beta^* e = \pm(\rho \circ \Phi)^* \alpha^* e,$$

where $e \in H^n(BSO(n); \mathbb{Q})$ is the Euler class.

Since $H^*(BSO(n); \mathbb{Q})$ is a polynomial ring with rational coefficients generated by the Pontrjagin classes and the Euler class, then the last two equations imply

$$\beta^* c = \pm(\rho \circ \Phi)^* \alpha^* c$$

for any $c \in H^*(BSO(n); \mathbb{Q})$. Hence every rational tautological class $\tau_c(E)$ of positive degree must vanish.

Let now $n = 3$. Fix a point $* \in S^1$. If $M \to E \xrightarrow{q} B$ is a smooth bundle with nonpositively curved fibers, then so is $M \times S^1 \to E \times S^1 \xrightarrow{q'} B$, where $q'(a, z) = q(a)$. Note that we only have to analyze one possible nonzero cohomology class, namely the first Pontrjagin class $p_1 \in H^4(BSO(n); \mathbb{Q})$. It is easy to see that $\sigma^* \mathfrak{t}(E \times S^1) \simeq \mathfrak{t}E \oplus \varepsilon$, where ε is a trivial line bundle over E and $\sigma : E \hookrightarrow E \times S^1$ sends $a \in E$ to $(a, *)$. Since the dimension of $M \times S^1$ is 4, we conclude that:

$$\begin{aligned} p_1(\mathfrak{t}E) &= p_1(\mathfrak{t}E \oplus \varepsilon) \\ &= p_1(\sigma^* \mathfrak{t}(E \times S^1)) \\ &= \sigma^* \circ (\rho \times id_{S^1} \circ \Phi \times id_{S^1})^* p_1(M \times S^1) = 0, \end{aligned}$$

where the last equation follows from the fact that $M \times S^1$ bounds a closed oriented 5-dimensional manifold-with-boundary. $\square$

Remark 6. *The vanishing of the tautological classes for smooth bundles with* 3*-dimensional fibers (without curvature assumptions) is proved by J. Ebert in [Ebe13].*

4 Tautological classes of nonpositively curved bundles

In this section we prove Theorem C. The strategy of proof is the same as in Theorem B. We construct a space $\widehat{\mathfrak{t}E}$ which plays the same role as $\widetilde{E} \times_\Gamma \widetilde{M}$ when $E \to B$ is fiber homotopically trivial (see Section 2). Assume that M is just a closed smooth manifold (without curvature conditions). For any $* \in M$ define $\mathcal{P}_*(M)$ to be the set of homotopy classes (relative to the end points) of paths $\alpha : [0,1] \to M$ whose initial point is $*$. For a path α and a neighborhood $U \subset M$ of $\alpha(1)$, let (α, U) denote the set of all paths $\beta : [0,1] \to M$ such that $\beta(0) = *$, $\beta(1) \in U$ and there exists a path $\gamma : [0,1] \to U$ with the property that $\gamma(0) = \alpha(1)$, $\gamma(1) = \beta(1)$ such that the loop $\alpha * \gamma * \beta^{-1}$ is nullhomotopic.

The sets (α, U) depend on the homotopy class of α and on the open set U. In fact they form a basis for the topology of $\mathcal{P}_*(M)$. With this topology $\mathcal{P}_*(M)$ is a simply connected space and the map $\mathcal{P}_*(M) \to M$ sending a class $[\alpha]$ to the endpoint $\alpha(1)$, is a covering map (see for example [GH81, I.6]).

Let $\mathcal{P}(M)$ be the disjoint union $\displaystyle\bigsqcup_{*\in M} \mathcal{P}_*(M)$ topologized as follows: For a path $\alpha : [0,1] \to M$, and neighborhoods U and V of $\alpha(0)$ and $\alpha(1)$ respectively, let (α, U, V) denote the set of all paths $\beta : [0,1] \to M$ such that $\beta(0) \in U$, $\beta(1) \in V$ and there exist paths $\gamma_U : [0,1] \to U$ and $\gamma_V : [0,1] \to V$ with the property that $\gamma_U(0) = \alpha(0)$, $\gamma_U(1) = \beta(0)$, $\gamma_V(0) = \alpha(1)$, $\gamma_V(1) = \beta(1)$ and the loop $\alpha * \gamma_V * \beta^{-1} * \gamma_U^{-1}$ is nullhomotopic. It is straightforward that the sets (α, U, V) are well-defined and that they form a basis for the topology of $\mathcal{P}(M)$. With this topology the onto map $r_0 : \mathcal{P}(M) \to M$, sending a class $[\alpha]$ to its initial point $\alpha(0)$ is continuous. In fact we have the following lemma.

Lemma 4.1. *If the universal cover of M is homeomorphic to $\mathbb{R}^n$, then*

$$\mathcal{P}(M) \xrightarrow{r_0} M$$

is a topological $\mathbb{R}^n$-bundle over M.

Proof. Let $U \subset M$ be a simply connected neighborhood of a point $* \in M$. Note that $r_0^{-1}(U) = \bigsqcup_{x \in U} \mathcal{P}_x(M)$. Let us define a map

$$\rho_U : r_0^{-1}(U) \to U \times \mathcal{P}_*(M)$$

by $\rho_U([\alpha]) = (\alpha(0), [\gamma * \alpha])$, where $\gamma : [0,1] \to U$ is any path in U starting at $*$ and ending at $\alpha(0)$. It is clear that that this is a fiber preserving homeomorphism, so it defines a local trivialization for $r_0 : \mathcal{P}(M) \to M$. This completes the proof of the lemma as the universal cover $\widetilde{M} \approx \mathcal{P}_*(M)$ of M is homeomorphic to $\mathbb{R}^n$. $\square$

We shall need a parametrized version of this construction. Let $M \to E \to B$ be a smooth fiber bundle. Define $\widehat{\mathfrak{t}E} := \coprod_{x\in B} \mathcal{P}(M_x)$. This set can be topologized as follows: associated to the fiber bundle $E \to B$, one can construct a concrete principal Diff(M)-bundle $Q \to B$ whose (concrete) fiber over $x \in B$ is the space Diff(M, M_x) of all diffeomorphisms between M and M_x. Then a topology on $\widehat{\mathfrak{t}E}$ is given via the bijection from $Q \times_{\mathrm{Diff}(M)} \mathcal{P}(M)$ onto $\widehat{\mathfrak{t}E}$ that sends $(\phi, \alpha) \in \mathrm{Diff}(M, M_x) \times \mathcal{P}(M)$ to the path $\phi \circ \alpha$. Observe that the natural projection $\widehat{\mathfrak{t}E} \to B$ is, in fact, the $\mathcal{P}(M)$-bundle associated to $E \to B$.

Define the projection

$$s_0 : \widehat{\mathfrak{t}E} \to E$$

by $s_0([\alpha]) = \alpha(0)$. Using the fact that $\mathcal{P}(M) \to M$ is Diff(M)-equivariant and that the natural homeomorphism $Q \times_{\mathrm{Diff}(M)} M \to E$ is covered by the homeomorphism $Q \times_{\mathrm{Diff}(M)} \mathcal{P}(M) \to \widehat{\mathfrak{t}E}$, the next lemma follows obviously from Lemma 4.1.

Lemma 4.2. *If the universal cover of M is homeomorphic to $\mathbb{R}^n$, then $s_0 : \widehat{\mathfrak{t}E} \to E$ is a topological $\mathbb{R}^n$-bundle.*

Now assume that the bundle $M \to E \to B$ is given a fiberwise Riemannian metric g_x, $x \in B$.

Let $v \in \mathfrak{t}E$. Then $v \in T_*(M_x)$ for some $x \in B$ and $* \in M_x$. Let γ_v be the unique g_x-geodesic in M_x such that $\gamma_v(0) = *$ and $\dot\gamma_v(0) = v$. Then define

$$\textsc{exp} : \mathfrak{t}E \to \widehat{\mathfrak{t}E}$$

by

$$\textsc{exp}(v) = [\gamma_v].$$

Since the restriction of $\textsc{exp}$ to each fiber is continuous and the Riemannian metrics g_x vary continuously from fiber to fiber, $\textsc{exp}$ is a fiber preserving continuous map. Let $s_1 : \widehat{\mathfrak{t}E} \to E$ be defined by $s_1[\alpha] = \alpha(1)$. For each $x \in B$ and $* \in M_x$, the map

$$s_1|_{s_0^{-1}(*)} : \mathcal{P}_*(M_x) \to M_x$$

is a covering map. Thus we can pull-back both the smooth structure and the Riemannian metric g_x from M_x along $s_1|_{s_0^{-1}(y)}$. In this way $\widehat{\mathfrak{t}E} \xrightarrow{s_0} E$ acquires a fiberwise Riemannian metric (and thus a distance function on each fiber).

Lemma 4.3. *The map* $\textsc{exp} : \mathfrak{t}E \to \widehat{\mathfrak{t}E}$ *is an expanding map when restricted to each fiber.*

Proof. Let $\widetilde{M_x} \xrightarrow{p} M_x$ be the universal cover of M_x and $\tilde{*} \in p^{-1}(*)$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
T_{\tilde{*}}\widetilde{M_x} & \xrightarrow{\textsc{exp}} & \mathcal{P}_{\tilde{*}}(\widetilde{M_x}) & \xrightarrow{r_1} & \widetilde{M_x} \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} \\
T_*M_x & \xrightarrow{\textsc{exp}} & \mathcal{P}_*(M_x) & \xrightarrow{r_1} & M_x
\end{array}$$

where $\hat{p}$ is the lifting of p, i.e., $\hat{p}([\alpha]) = [p \circ \alpha]$ and p_* is the derivative of p at $\tilde{*}$, the map r_1 is defined by $r_1([\alpha]) = \alpha(1)$ and $\text{EXP} : T_{\tilde{*}}\widetilde{M}_x \to \mathcal{P}_{\tilde{*}}(\widetilde{M}_x)$ is defined with respect to the pull-back metric p^*g_x on $\widetilde{M}_x$.

Note that, when $\mathcal{P}_*(\widetilde{M}_x)$ is equipped with the pull-back metric $(p \circ r_1)^*g_x$, the map $r_1 : \mathcal{P}_{\tilde{*}}(\widetilde{M}_x) \to \widetilde{M}_x$ becomes an isometry and $r_1 \circ \text{EXP} : T_{\tilde{*}}\widetilde{M}_x \to \widetilde{M}_x$ is the usual exponential map. This forces $\text{EXP} : T_{\tilde{*}}\widetilde{M}_x \to \mathcal{P}_{\tilde{*}}(\widetilde{M}_x)$ to be an expanding map (by Cartan-Hadamard theorem). Since $\hat{p}$ and p_* are isometries, the map $\text{EXP} : T_*M_x \to \mathcal{P}_*(M_x)$ is an expanding map as well. □

Proof of Theorem C. Suppose first that $n \geq 4$. For each $x \in B$, the following diagram is commutative

$$\begin{array}{ccccc} TM_x & \xrightarrow{\text{EXP}} & \mathcal{P}(M_x) & \xrightarrow{\widehat{\Phi}_x} & \mathcal{P}(M'_x) \\ \downarrow & & & & \downarrow r_0 \\ M_x & & \xrightarrow{\Phi_x} & & M'_x \end{array}$$

where $\widehat{\Phi}_x$ is the lifting of the restriction Φ_x of the fiber homotopy equivalence Φ to the fiber over $x \in B$. The diagram above then induces a map between topological $\mathbb{R}^n$-bundles

$$\begin{array}{ccc} \mathfrak{t}E & \longrightarrow & \widehat{\mathfrak{t}E'} \\ \downarrow & & \downarrow \\ E & \xrightarrow{\Phi} & E' \end{array}$$

Compactness of B and a covering space theory argument (e.g. [FW91, p. 409]) show that, for all $x \in B$, Φ_x is a δ-map for some $\delta \geq 0$. Hence, by Lemma 4.3, the map $\mathfrak{t}E \to \widehat{\mathfrak{t}E'}$ is a fiberwise δ-map. By Lemma 2.1, $\Phi^*(\widehat{\mathfrak{t}E'})$ and $\mathfrak{t}E$ are isomorphic as topological $\mathbb{R}^n$-bundles. Proceeding as in the proof of Lemma 2.2, one can easily verify that $\widehat{\mathfrak{t}E'}$ and $\mathfrak{t}E'$ are isomorphic as microbundles and by Kister-Mazur theorem they must be isomorphic as topological $\mathbb{R}^n$-bundles. Therefore $\Phi^*(\mathfrak{t}E')$ and $\mathfrak{t}E$ are isomorphic as topological $\mathbb{R}^n$-bundles.

When $n = 1, 2$, the result follows from the fact that the fiber homotopy equivalence Φ is homotopic to an M-bundle isomorphism, provided $\text{Diff}(M) \hookrightarrow G(M)$ is a weak homotopy equivalence and B is a finite simplicial complex. That Φ is homotopic to an M-bundle isomorphism can be easily obtained by an inductive argument over the skeleta of B, using the vanishing of the relative homotopy groups of the pair $(G(M), \text{Diff}(M))$ and the homotopy extension property. □

In order to prove Corollary C.1 we use the following lemma from [GGRW15, Theorem 5.6]. We will provide a different proof of this lemma in the Appendix.

Lemma 4.4. *Let $\Phi : E \to E'$ be an orientation-preserving fiber homotopy equivalence between oriented smooth M^n-bundles over a finite simplicial complex B, where M is a closed smooth n-manifold. Then $\Phi^*(e(\mathfrak{t}E')) = e(\mathfrak{t}E)$, where $e(\mathfrak{t}E) \in H^n(E; \mathbb{Q})$ denotes the Euler class of $\mathfrak{t}E \to E$.*

Proof of Corollary C.1. By an obvious modification of the argument given in the proof of Corollary B.1, one can show that $p_i(\mathfrak{t}E) = \Phi^* p_i(\mathfrak{t}E')$ for all i. This, together with Lemma 4.4, implies that $\tau_c(E) = q_! \, \Phi^* \beta'^* c$ for any $c \in H^*(BSO(n); \mathbb{Q})$, where $\beta' : E' \to BSO(n)$ is the classifying map for $\mathfrak{t}E'$. The result follows from the naturality of the Gysin map. □

5 Tautological classes of torus and hyperbolic bundles

Let G denote the group of affine diffeomorphisms of the n-dimensional torus $\mathbb{T}^n$, i.e. we identify $\mathbb{T}^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$ (so that $GL_n(\mathbb{Z})$ acts naturally on $\mathbb{T}^n$) and define

$$G = \{f \in \mathrm{Diff}(\mathbb{T}^n) | f(x) = Ax + v, \text{ where } A \in GL_n(\mathbb{Z}) \text{ and } v \in \mathbb{T}^n\}.$$

The proof of Theorem D follows from the following three lemmas:

Lemma 5.1. *Every smooth torus bundle $\mathbb{T}^n \to E \to B$ over a CW-complex B is fiber homotopy equivalent to the pull-back of the bundle $\mathbb{T}^n \to EG \times_G \mathbb{T}^n \to BG$ along some continuous map $B \to BG$.*

Proof. It suffices to show that the inclusion $\iota : G \hookrightarrow G(\mathbb{T}^n)$ is a weak homotopy equivalence. For in that case the induced map between classifying spaces $BG \to BG(\mathbb{T}^n)$ is also a weak homotopy equivalence and hence we obtain a bijection between the sets $[B, BG]$ and $[B, BG(\mathbb{T}^n)]$ of homotopy classes of continuous maps from B to BG and to $BG(M)$ respectively.

To prove that ι is a weak homotopy equivalence we note first that the group G is isomorphic to a semidirect product $GL_n(\mathbb{Z}) \ltimes \mathbb{T}^n$. Thus $\pi_i G = 0 = \pi_i G(\mathbb{T}^n)$ for all $i \geq 2$. It remains to show that ι induces a bijection between path components and an isomorphism in π_1.

Recall that $GL_n(\mathbb{Z})$ acts naturally on $\mathbb{T}^n$ yielding a decomposition of the identity map on $GL_n(\mathbb{Z})$ as

$$\mathrm{Out}(\pi_1(\mathbb{T}^n)) = GL_n(\mathbb{Z}) \to G \xrightarrow{\iota} G(\mathbb{T}^n) \to GL_n(\mathbb{Z}) = \mathrm{Out}(\pi_1(\mathbb{T}^n)),$$

where the first map is the action of $GL_n(\mathbb{Z})$ on $\mathbb{T}^n$ and the last map is the one sending a self-homotopy equivalence of $\mathbb{T}^n$ to its induced (outer) automorphism in $\pi_1(\mathbb{T}^n)$. These two maps induce bijections at the π_0-level. Hence $\iota : G \to G(\mathbb{T}^n)$ must induce a bijection at the π_0-level as well.

Let now $0 \in \mathbb{T}^n$ denote the identity element of (the abelian group) $\mathbb{T}^n$ and for each $v \in \mathbb{T}^n$ let $L_v : \mathbb{T}^n \to \mathbb{T}^n$ be the map that sends $x \in \mathbb{T}^n$ to $x + v$. The identity map $id : \mathbb{T}^n \to \mathbb{T}^n$ decomposes as

$$\mathbb{T}^n \xrightarrow{L} G \xrightarrow{\iota} G(\mathbb{T}^n) \xrightarrow{p} \mathbb{T}^n,$$

where $L(v) = L_v$ and $p : G(\mathbb{T}^n) \to \mathbb{T}^n$ is the evaluation at $0 \in \mathbb{T}^n$. Thus we have an isomorphism

$$\pi_1(\mathbb{T}^n, 0) \xrightarrow{L_*} \pi_1(G, id) \xrightarrow{\iota_*} \pi_1(G(\mathbb{T}^n), id) \xrightarrow{p_*} \pi_1(\mathbb{T}^n, 0),$$

where each group in the sequence is a free abelian group of rank n. Therefore ι_* must be an isomorphism. This completes the proof of the lemma. □

Lemma 5.2. *Let $\mathbb{T}^n \to E \to B$ be a smooth torus bundle over a compact space B and let $E' \xrightarrow{p} B$ be the pull-back bundle of $\mathbb{T}^n \to EG \times_G \mathbb{T}^n \to BG$ along a continuous map $B \to BG$. Suppose that there is a fiber homotopy equivalence $\Psi : E' \to E$. Then $\Psi^*\mathfrak{t}E$ and $\mathfrak{t}E'$ are isomorphic as topological $\mathbb{R}^n$ bundles over E.*

Proof. First note that the bundle $\mathbb{T}^n \to E' \xrightarrow{p} B$ can be given a fiberwise affine flat connection in the following way: let $V \subset B$ be an open subset such that there is a fiber preserving diffeomorphism $\phi_V : V \times \mathbb{T}^n \to p^{-1}(V)$. Identify $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$ and endow it with a flat Riemannian metric g^F induced by the Euclidean metric on $\mathbb{R}^n$. Denote by ∇^F the Levi-Civita connection of g^F. Then we put a connection ∇_x on $p^{-1}(x) = \mathbb{T}^n_x$ by "pushing forward" ∇^F via ϕ_V, i.e.

$$\nabla_x = \left(\phi_V|_{\{x\}\times\mathbb{T}^n}\right)_* \nabla^F \quad \text{for } x \in V.$$

Now cover B with finitely many open sets $V_1, \dots, V_m$, as above and define affine connections in a similar way by pushing forward ∇^F via diffeomorphisms $\phi_{V_i} : V_i \times \mathbb{T}^n \to p^{-1}(V_i)$. Since the structure group of the bundle $E' \xrightarrow{p} B$ is the group G of affine diffeomorphisms of the torus, these connections can be glued together to give rise to a desired fiberwise affine flat connection ∇_x, $x \in B$.

Having a fiberwise affine connection allows us to define an exponential map. Let $v \in \mathfrak{t}E'$. Then $v \in T_*(\mathbb{T}^n_x)$ for some $x \in B$ and $* \in \mathbb{T}^n_x$. Let β_v be the unique geodesic (with respect to the connection ∇_x), such that $\beta_v(0) = *$ and $\dot\beta_v(0) = v$. We define a map $\exp^\nabla : \mathfrak{t}E' \to \widehat{\mathfrak{t}}E'$ between $\mathbb{R}^n$-bundles by

$$\exp^\nabla(v) = [\beta_v],$$

where $\widehat{\mathfrak{t}}E'$ was defined in the previous section. This map is continuous because the affine connections ∇_x vary continuously with x.

If we "fiberwise lift" the fiber homotopy equivalence $\Psi : E' \to E$ to a map $\widehat{\Psi} : \widehat{\mathfrak{t}}E' \to \widehat{\mathfrak{t}}E$ (cf. proof of Theorem C), we obtain a map between $\mathbb{R}^n$-bundles

$$\mathfrak{t}E' \xrightarrow{\exp^\nabla} \widehat{\mathfrak{t}}E' \xrightarrow{\widehat{\Psi}} \widehat{\mathfrak{t}}E$$

covering $\Psi : E' \to E$. Using an argument similar to that of Lemma 2.2, it is not hard to see that $\widehat{\mathfrak{t}}E$ is isomorphic to $\mathfrak{t}E$ as $\mathbb{R}^n$-bundles. Thus by Lemma 2.1, it only remains to show that $\widehat{\Psi} \circ \exp^\nabla$ is a δ-map (for a suitable $\delta \geq 0$) when restricted to each fiber.

Let g_x be the push-forward metric of the flat Riemannian metric on $\mathbb{T}^n$ along the restriction of ϕ_V to $\{x\} \times \mathbb{T}^n$. By construction, the Levi-Civita connection of g_x is ∇_x. Thus, for all $x \in V$ and $* \in \mathbb{T}^n_x$, we have

$$d_{g_x}(\exp^\nabla(v), \exp^\nabla(w)) \geq d_{\hat{g}_x}(v, w), \text{ for all } v, w \in T_*\mathbb{T}^n_x,$$

where d_{g_x} and $d_{\hat{g}_x}$ are the distance functions on $\mathbb{T}^n_x$ and $T_*\mathbb{T}^n_x$ respectively, induced by the Riemannian metric g_x.

Now let $\{h_x\}_{x\in B}$ be a continuously varying family of Riemannian metrics on the bundle $\mathbb{T}^n \to E' \xrightarrow{q} B$. It is not hard to see that there exists a number $K_V \geq 1$ that depends on the open set V, such that for all $x \in V$,

$$\begin{aligned} K_V d_{h_x}(\exp^\nabla(v), \exp^\nabla(w)) &\geq d_{g_x}(\exp^\nabla(v), \exp^\nabla(w)) \\ &\geq d_{\hat{g}_x}(v, w) \\ &\geq \frac{1}{K_V} d_{\hat{h}_x}(v, w). \end{aligned} \tag{5}$$

Then we have that for all $x \in B$

$$d_{h_x}(\exp^\nabla(v), \exp^\nabla(w)) \geq \frac{1}{K^2} d_{\hat{h}_x}(v, w),$$

where $K = \max\{K_{V_i} | i \in \{1, \dots, m\}\}$.

By using this inequality, the compactness of B, the fact that $\widehat{\Psi}$ is a δ'-map when restricted to each fiber (see [FW91, p. 409]), and the linear structure on the fibers of $\mathfrak{t}E'$, it is straightforward that the composite $\widehat{\Psi} \circ \exp^\nabla$ is a δ-map (for a $\delta \geq 0$ as in the statement of Lemma 2.1) when restricted to each fiber. This completes the proof of Lemma 5.2. □

Lemma 5.3. *Suppose that B is a smooth manifold. Then, for any continuous map $f : B \to BG$, the rational Pontrjagin classes of the associated vertical tangent bundle $\eta : \mathfrak{t}(f^*(EG \times_G \mathbb{T}^n)) \to f^*(EG \times_G \mathbb{T}^n)$ vanish.*

Proof. It suffices to prove that the structure group of $r : \mathfrak{t}(EG \times_G \mathbb{T}^n) \to EG \times_G \mathbb{T}^n$ (and hence of $\eta : \mathfrak{t}(f^*(EG \times_G \mathbb{T}^n)) \to f^*(EG \times_G \mathbb{T}^n)$) can be reduced to the discrete group $GL_n(\mathbb{Z})$. For if that is the case then η is a flat vector bundle over a smooth manifold and so its rational Pontrjagin classes vanish. (However the rational Euler class does not necessarily vanish for flat bundles. See [MS74, Appendix C, p. 308, 312]).

To prove that the structure group can be reduced to a discrete group we first note that there is a left action of G on the tangent bundle $T\mathbb{T}^n$ of the torus given by the derivative of the G-action on $\mathbb{T}^n$. Consider the vector bundle

$$\mathbb{R}^n \to EG \times_G T\mathbb{T}^n \xrightarrow{\hat{r}} EG \times_G \mathbb{T}^n,$$

where $\hat{r}$ is defined by

$$\hat{r}[(e, V)] = [(e, \tau(V))],$$

and $\tau : T\mathbb{T}^n \to \mathbb{T}^n$ is the tangent bundle projection. Now, since $GL_n(\mathbb{Z})$ sits in an exact sequence of groups

$$1 \to \mathbb{T}^n \to G \to GL_n(\mathbb{Z}) \to 1,$$

the action of G on $EG \times \mathbb{T}^n$ induces a free action of $GL_n(\mathbb{Z})$ on $EG \times_{\mathbb{T}^n} \mathbb{T}^n$ whose orbit space is $EG \times_G \mathbb{T}^n$. On the other hand, we have the usual action of $GL_n(\mathbb{Z})$ on $\mathbb{R}^n$. Thus we can form a vector bundle

$$\mathbb{R}^n \to (EG \times_{\mathbb{T}^n} \mathbb{T}^n) \times_{GL_n(\mathbb{Z})} \mathbb{R}^n \to EG \times_G \mathbb{T}^n.$$

We claim that $\mathfrak{t}(EG \times_G \mathbb{T}^n)$ and $(EG \times_{\mathbb{T}^n} \mathbb{T}^n) \times_{GL_n(\mathbb{Z})} \mathbb{R}^n$ are isomorphic vector bundles over $EG \times_G \mathbb{T}^n$. This claim proves the lemma because $GL_n(\mathbb{Z})$ is the structure group of the latter vector bundle.

To prove the claim we observe that $\mathfrak{t}(EG \times_G \mathbb{T}^n)$ is canonically isomorphic to $EG \times_G T\mathbb{T}^n$. On the other hand, the derivative of left translation on $\mathbb{T}^n$ gives rise to an isomorphism $\mathbb{T}^n \times \mathbb{R}^n \simeq T\mathbb{T}^n$ and hence to continuous map $(EG \times_{\mathbb{T}^n} \mathbb{T}^n) \times_{GL_n(\mathbb{Z})} \mathbb{R}^n \to EG \times_G T\mathbb{T}^n$ which covers the identity map on $EG \times_G \mathbb{T}^n$ and is a linear isomorphism on each fiber. Hence $(EG \times_{\mathbb{T}^n} \mathbb{T}^n) \times_{GL_n(\mathbb{Z})} \mathbb{R}^n$ and $EG \times_G T\mathbb{T}^n$ are isomorphic vector bundles. □

Proof of Theorem D. Using Lemma 5.1, we can find a continuous map $f : B \to BG$ and a fiber homotopy equivalence $\Psi : f^*(EG \times_G \mathbb{T}^n) \to E$ between torus bundles over the smooth manifold B.

By Lemma 5.2, $\Psi^*(\mathfrak{t}E)$ and $\mathfrak{t}(f^*(EG \times_G \mathbb{T}^n))$ are isomorphic topological $\mathbb{R}^n$-bundles. Thus these two bundles have the same rational Pontrjagin classes. But they vanish on the latter by Lemma 5.3. □

We conclude this section by proving Theorem E. Let M be a (real, complex or quaternionic) hyperbolic manifold. By Mostow rigidity theorem [Mos73], there is a group isomorphism

$$\mathrm{Out}(\pi_1 M) \to \mathrm{Isom}(M)$$

such that

$$\mathrm{Out}(\pi_1 M) \to \mathrm{Isom}(M) \xrightarrow{\iota} G(M) \to \mathrm{Out}(\pi_1 M)$$

is the identity map.

Lemma 5.4. *Every smooth bundle $M \to E \to B$ as in the statement of Theorem E is fiber homotopy equivalent to the pull-back of $M \to E\,Out(\pi_1 M) \times_{Out(\pi_1 M)} M \to B\,Out(\pi_1 M)$ along some continuous map $B \to B\,Out(\pi_1 M)$.*

Proof. The inclusion map $\iota : \mathrm{Isom}(M) \to G(M)$ must induce a bijection between components. Also, $\pi_i G(M) = 0$ for all $i > 0$ since $\pi_1 M$ is centerless (see Remark 7 below). This implies that the induced map $B\mathrm{Out}(\pi_1 M) \to BG(M)$ is a weak homotopy equivalence and the result follows. □

Remark 7. *That $\pi_1 M^n$ $(n \geq 2)$ is centerless for a closed negatively curved manifold M is implicitly proven in [EO73, Section 9]. For the reader's convenience, here is an outline of the argument. Let $\widetilde{M}$ denote a universal cover of M and identify $\pi_1 M$ with the group of deck transformations of $\widetilde{M} \to M$. Each element $f \in \pi_1 M - id$ leaves invariant a unique geodesic line L_f in $\widetilde{M}$ (cf. [BGS85, 7.1]). Suppose $Center(\pi_1 M) \neq 1$ and pick an element $g \in Center(\pi_1 M)$ different from the identity. Let $f \in \pi_1 M$. Since $fg = gf$, f must leave L_g invariant. Hence the deck transformation action restricts to a free and properly discontinuous action of $\pi_1 M$ on L_g. Therefore the orbit map $p : L_g \to L_g/\pi_1 M$ is the universal covering space for the 1-dimensional manifold $L_g/\pi_1 M$. Hence both M^n and $L_g/\pi_1 M$ are Eilenberg-MacLane spaces $K(\pi_1 M, 1)$. Therefore the closed manifold M is homotopy equivalent to a closed 1-dimensional manifold. Hence M itself must be 1-dimensional, which is a contradiction.*

Proof of Theorem E. Let $f : B \to B\mathrm{Out}(\pi_1 M)$ be a continuous map such that there is a fiber homotopy equivalence $E' \to E$ where $E' = f^*(E\mathrm{Out}(\pi_1 M) \times_{\mathrm{Out}(\pi_1 M)} M)$. Since $\mathrm{Out}(\pi_1 M)$ acts on M by isometries, it is easy to equip the associated bundle

$M \to E\mathrm{Out}(\pi_1 M) \times_{\mathrm{Out}(\pi_1 M)} M \to B\mathrm{Out}(\pi_1 M)$ over $B\mathrm{Out}(\pi_1 M)$ with a nonpositively curved fiberwise Riemannian metric. Thus, by Corollary C.1, $\tau_c(E') = \tau_c(E)$. But since $\mathrm{Out}(\pi_1 M)$ is a finite group [Bor69], the rational cohomology of $B\mathrm{Out}(\pi_1 M)$ vanishes in positive degrees and so the rational tautological classes of the bundle E' must be zero in positive degrees. □

Remark 8. *Theorem E also follows as particular case of Theorem F in the Appendix.*

6 Appendix

We begin with the proof of Theorem F.

Proof of Theorem F. Let $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ be the subgroup of $\mathrm{Diff}(M)$ consisting of all self-diffeomorphisms of M which are *homotopic* to the identity (note that $\mathrm{Diff}_0(M)$ is different from the identity component of $\mathrm{Diff}(M)$. In fact the former contains the latter).

Consider the following pull-back diagram

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B\mathrm{Diff}_0(M) \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & B\mathrm{Diff}(M) \end{array}$$

where $B \to B\mathrm{Diff}(M)$ is the classifying map for the bundle $q : E \to B$, and $B\mathrm{Diff}_0(M) \to B\mathrm{Diff}(M)$ is the map induced by the inclusion $\mathrm{Diff}_0(M) \hookrightarrow \mathrm{Diff}(M)$. We claim that $\widetilde{B} \xrightarrow{\sigma} B$ is a finite sheeted cover of B. To see this, it suffices to prove that the fiber $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ of $B\mathrm{Diff}_0(M) \to B\mathrm{Diff}(M)$ is finite. But this follows because there is a one-to-one map $\mathrm{Diff}(M)/\mathrm{Diff}_0(M) \to \pi_0(G(M)) \simeq \mathrm{Out}(\pi_1 M)$, and $\mathrm{Out}(\pi_1 M)$ is finite by hypothesis.

We now consider the pull-back bundle $M \to \sigma^* E \to \widetilde{B}$ of $M \to E \xrightarrow{q} B$ along σ. This bundle is fiber homotopically trivial since the map $\widetilde{B} \to B\mathrm{Diff}_0(M) \to B\mathrm{Diff}(M) \to BG(M) \approx B\pi_0 G(M)$ is null-homotopic; which follows from the fact that the composition $\mathrm{Diff}_0(M) \hookrightarrow \mathrm{Diff}(M) \hookrightarrow G(M) \to \pi_0 G(M)$ is a constant map, and $BG(M) \to B\pi_0 G(M)$ is a weak homotopy equivalence (because M is aspherical and $\pi_1 M$ is centerless by assumption).

Hence, by Corollary B.1,

$$\tau_c(\sigma^* E) = 0$$

for $c \in H^i(BSO(n); \mathbb{Q})$ and $i > n$.

Note also that by naturality of the tautological classes

$$\tau_c(\sigma^* E) = \sigma^* \tau_c(E).$$

Thus, it suffices to prove that $\sigma^* : H^*(B; \mathbb{Q}) \to H^*(\widetilde{B}; \mathbb{Q})$ is monic. But this follows easily from the fact that $\sigma : \widetilde{B} \to B$ is finite sheeted covering space, hence there is a transfer map $H^*(\widetilde{B}; \mathbb{Q}) \to H^*(B; \mathbb{Q})$ whose composition with σ^* is multiplication by the number of sheets. □

We now prove Theorem G. Recall that a closed manifold M satisfies the Strong Borel Conjecture (SBC) if, for all $k \geq 0$, every self-homotopy equivalence of pairs $(M \times D^k, M \times S^{k-1}) \to (M \times D^k, M \times S^{k-1})$ which is a homeomorphism when restricted to the boundary $M \times S^{k-1}$ is homotopic (relative to the boundary) to a homeomorphism.

Proof of Theorem G. The first step is proving that the fiber homotopy equivalence Φ is homotopic to a homeomorphism. Let $B^0 \subset B^1 \subset \cdots \subset B^k = B$ be the skeletal filtration of B.

First consider a 0-simplex $v \in B$. The restriction $\Phi|_{p^{-1}(v)} : p^{-1}(v) \to q^{-1}(v)$ is a homotopy equivalence. Thus, by SBC, it is homotopic to a homeomorphism, that is, there is a homotopy $h_t^v : p^{-1}(v) \to q^{-1}(v)$ such that $h_0^v = \Phi|_{p^{-1}(v)}$ and h_1^v is a homeomorphism. By taking the corresponding homeomorphism on each 0-simplex of B, we obtain a homeomorphism $h^0 : p^{-1}(B^0) \to q^{-1}(B^0)$, which is compatible with the bundle projections and homotopic to the restriction of Φ to $p^{-1}(B^0)$.

We now want to extend this homotopy over the 1-skeleton. Let $\Delta^1 \subset B$ be a 1-simplex in B with vertices $v, w \in B$. Define subspaces $E^1 \subset E \times [0,1]$ and $E'^1 \subset E' \times [0,1]$ by

$$E^1 = p^{-1}(v) \times [0,1] \cup p^{-1}(\Delta^1) \times \{0\} \cup p^{-1}(w) \times [0,1]$$

and

$$E'^1 = q^{-1}(v) \times [0,1] \cup q^{-1}(\Delta^1) \times \{0\} \cup q^{-1}(w) \times [0,1],$$

and a homotopy equivalence $\psi : E^1 \to E'^1$ by

$$\psi(y,t) = \begin{cases} (h_t^v(y), t) & \text{if } (y,t) \in p^{-1}(v) \times [0,1] \\ (h_t^w(y), t) & \text{if } (y,t) \in p^{-1}(w) \times [0,1] \\ (\Phi(y), 0) & \text{if } y \in p^{-1}(\Delta^1). \end{cases}$$

The figure below illustrates the construction.

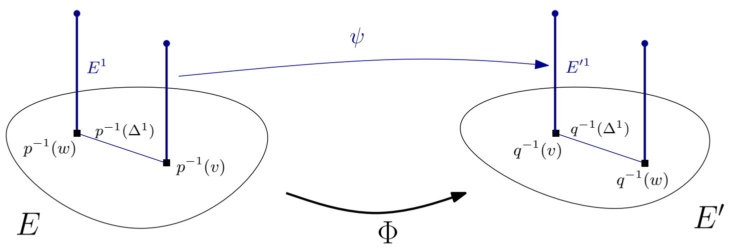

Figure 1: Construction of a homeomorphism homotopic to Φ.

Note that both E^1 and E'^1 are homeomorphic to $M \times D^1$. Thus map ψ can be thought of as a homotopy equivalence of pairs $(M \times D^1, M \times S^0) \to (M \times D^1, M \times S^0)$ which is the homeomorphism h^0 when restricted to the boundary $p^{-1}(v) \times \{1\} \cup p^{-1}(w) \times \{1\} \approx M \times S^0$. Thus, by SBC, we can find a homotopy

$$h_t^{\Delta^1} : p^{-1}(\Delta^1) \to q^{-1}(\Delta^1)$$

which extends h_t^v and h_t^w and such that $h_0^{\Delta^1} = \Phi|_{p^{-1}(\Delta^1)}$ and $h_1^{\Delta^1}$ is a homeomorphism. Repeating this process for each 1-simplex of B we obtain a homeomorphism $h^1 : p^{-1}(B^1) \to q^{-1}(B^1)$ homotopic to the restriction of Φ to $p^{-1}(B^1)$.

Continuing this way on each skeleton we obtain the desired homeomorphism $E \to E'$ homotopic to Φ.

The second and final step to prove the theorem is showing that $\Phi^*\mathfrak{t}E'$ and $\mathfrak{t}E$ are stably isomorphic as topological $\mathbb{R}^n$-bundles. For this, embed B into the Euclidean space $\mathbb{R}^N$ for some N large enough. Take a compact regular neighborhood $\mathcal{B}$ of B in $\mathbb{R}^N$ whose tangent bundle $T\mathcal{B}$ is trivial and such that there is a retraction $r : \mathcal{B} \to B$ of $\mathcal{B}$ onto B. Let $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}$ denote the pull-back bundle of $q : E \to B$ and $q' : E' \to B$ along r respectively. Observe that there are natural fiber homotopy equivalence $\varphi : \mathcal{E} \to \mathcal{E}'$ and bundle maps $\hat{\imath} : E \to \mathcal{E}$, $\hat{\imath}' : E' \to \mathcal{E}'$ covering the inclusion $B \hookrightarrow \mathcal{B}$, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\hat{\imath}} & \mathcal{E} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ E' & \xrightarrow{\hat{\imath}'} & \mathcal{E}' \end{array}$$

Hence, by the first step, $\varphi^*T\mathcal{E}'$ and $T\mathcal{E}$ are isomorphic as topological $\mathbb{R}^n$-bundles. Therefore $\varphi^*\mathfrak{t}\mathcal{E}' \oplus q^*T\mathcal{B}$ and $\mathfrak{t}\mathcal{E} \oplus q^*T\mathcal{B}$ are isomorphic as topological $\mathbb{R}^n$-bundles. But since $T\mathcal{B}$ is a trivial vector bundle of rank, say r, then $\varphi^*\mathfrak{t}\mathcal{E}' \oplus \varepsilon^r_{\mathcal{E}}$ and $\mathfrak{t}\mathcal{E} \oplus \varepsilon^r_{\mathcal{E}}$, where $\varepsilon^r_{\mathcal{E}}$ is the trivial bundle of rank r over $\mathcal{E}$, are isomorphic as topological $\mathbb{R}^n$-bundles.

Finally, the topological $\mathbb{R}^n$-bundle isomorphism above, together with the isomorphisms $\hat{\imath}^*\mathcal{E} \simeq \mathfrak{t}E$ and $\hat{\imath}^*\varphi^*\mathfrak{t}\mathcal{E}' \simeq \Phi^*\hat{\imath}'^*\mathfrak{t}\mathcal{E}' \simeq \Phi^*\mathfrak{t}E'$, imply that $\mathfrak{t}E \oplus \varepsilon^r_E$ and $\Phi^*\mathfrak{t}E' \oplus \varepsilon^r_E$ are isomorphic as topological $\mathbb{R}^n$-bundles. This completes the proof of the theorem. □

Proof of Corollary G.1. Recall that

$$H^*(BSO(n);\mathbb{Q}) \simeq \begin{cases} \mathbb{Q}[p_1,\dots,p_k] & \text{if } n = 2k+1, \\ \mathbb{Q}[p_1,\dots,p_k,e]/(e^2 - p_k) & \text{if } n = 2k \end{cases}$$

Then the corollary follows immediately from Lemma 4.4 and Theorem G. □

We conclude this article proving Lemma 4.4 which appears in Section 4.

Lemma 4.4. *Let $\Phi : E \to E'$ be an orientation-preserving fiber homotopy equivalence between oriented smooth M^n-bundles over a finite simplicial complex B, where M is a closed smooth n-manifold. Then $\Phi^*(e(\mathfrak{t}E')) = e(\mathfrak{t}E)$, where $e(\mathfrak{t}E) \in H^n(E;\mathbb{Q})$ denotes the Euler class of $\mathfrak{t}E \to E$.*

Proof of Lemma 4.4. Assume first that the base space B is a k-dimensional oriented manifold (possibly with boundary). In that case both E and E' are compact $(n+k)$-manifolds (possibly with boundary).

Let $E \times_B E \to B$ be the pull-back of the product bundle $E \times E \to B \times B$ along the diagonal map $B \to B \times B$ and denote by $\delta : E \to E \times_B E$ the fiberwise diagonal map.

Define similarly the bundle $E' \times_B E' \to B$ and the diagonal map $\delta' : E' \to E' \times_B E'$. Observe that the Euler class of $\mathfrak{t}E$ (resp. $\mathfrak{t}E'$) can be computed by the formula

$$e(\mathfrak{t}E) = \delta^*\delta_!(1) \text{ (resp. } e(\mathfrak{t}E') = \delta'^*\delta'_!(1)).$$

Here $\delta_!$ is defined via Lefschetz duality as the map making the following diagram commutative:

$$\begin{array}{ccc} H^0(E) & \xrightarrow{\quad\delta_!\quad} & H^n(E \times_B E) \\ {\scriptstyle \eta_E\cap}\downarrow & & \downarrow{\scriptstyle \eta_{E\times_B E}\cap} \\ H_{n+k}(E, \partial(E)) & \xrightarrow{\delta_*} & H_{n+k}(E \times_B E, \partial(E \times_B E)) \end{array}$$

where $\eta_E \in H_{n+k}(E, \partial E)$ denotes the orientation class of E. We also have the following commutative diagram of pairs:

$$\begin{array}{ccc} (E, \partial E) & \xrightarrow{\delta} & (E \times_B E, \partial(E \times_B E)) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi\times_B\Phi} \\ (E', \partial E') & \xrightarrow{\delta'} & (E' \times_B E', \partial(E' \times_B E')) \end{array}$$

We now make use of the two diagrams above to compute:

$$\begin{aligned} \Phi^* e(\mathfrak{t}E') &= \Phi^*\delta'^*\delta'_!(1) \\ &= \delta^*(\Phi \times_B \Phi)^*\delta'_!(1) \\ &= \delta^*(\Phi \times_B \Phi)^*\overline{\delta'_*\eta_{E'}} \\ &= \delta^*(\Phi \times_B \Phi)^*\overline{\delta'_*\Phi_*\eta_E} \\ &= \delta^*(\Phi \times_B \Phi)^*\overline{(\Phi \times_B \Phi)_*\delta_*\eta_E} \\ &= \delta^*\overline{\delta_*\eta_E} \\ &= \delta^*\delta_!(1) = e(\mathfrak{t}E). \end{aligned}$$

Here, a bar on a homology class denotes its Poincaré - Lefschetz dual and the sixth equation follows from the naturality of the cap product and the fact that $\Phi \times_B \Phi$ is a homotopy equivalence. This completes the proof of the theorem when B is manifold with boundary.

Now suppose that B is a finite dimensional simplicial complex. Embed B in $\mathbb{R}^N$, for some N sufficiently large. Let W be an oriented regular neighborhood of B in $\mathbb{R}^N$ such that W is a manifold-with-boundary. Denote by $r : W \to B$ a retraction of W onto B. We can pull-back the bundles $E \to B$ and $E' \to B$ along the retraction $r : W \to B$ to obtain M-bundles $\mathcal{E} \to W$ and $\mathcal{E}' \to W$ respectively. These two bundles are fiber homotopy equivalent via the restriction of $id_W \times \Phi : W \times E \to W \times E'$ to $\mathcal{E}$. Hence, if we denote this restriction map by Ψ, we have

$$\Psi^*(e(\mathfrak{t}\mathcal{E}')) = e(\mathfrak{t}\mathcal{E}).$$

The theorem then follows by noticing that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\hat{\imath}} & \mathcal{E} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ E' & \xrightarrow{\hat{\imath}'} & \mathcal{E}', \end{array}$$

where $\hat{\imath} : E \to \mathcal{E}$ (resp. $\hat{\imath}' : E' \to \mathcal{E}'$) is the bundle map covering the inclusion $B \hookrightarrow W$, and that $\hat{\imath}^* \mathfrak{t}\mathcal{E} \simeq \mathfrak{t}E$ (resp. $\hat{\imath}'^* \mathfrak{t}\mathcal{E}' \simeq \mathfrak{t}E'$). □

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