Topology of certain symplectic conifold transitions of $\mathbb{C}P^1-$bundles

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Abstract

In this paper, we first extend Smith, Thomas and Yau's examples of certain symplectic conifold transitions on trivial $\mathbb{C}P^1-$bundles over Kähler surfaces to all $\mathbb{C}P^1-$bundles over symplectic 4–manifolds. Then we determine the diffeomorphism types of all these symplectic conifold transitions. In particular, this implies that in the case of trivial $\mathbb{C}P^1-$bundles over projective complex surfaces, Smith, Thomas and Yau's examples of symplectic conifold transitions are diffeomorphic to Kähler 3–folds.

1 Introduction

In this paper, all manifolds under consideration are closed, oriented and differentiable, unless otherwise stated. By a $\mathbb{C}P^1-$bundle, we always mean the projectivization $\mathbb{P}(E)$ of a rank two complex vector bundle E.

Symplectic conifold transitions introduced by Smith, Thomas and Yau [17] are symplectic surgeries on symplectic 6–manifolds which collapse embedded Lagrangian 3–spheres and replace them by symplectic 2–spheres. It is unknown that if symplectic conifold transitions of Kähler manifolds are still Kähler. This problem was raised in [17] and it was also shown in [16, Proposition 4.3] that this problem is related to a question of Donaldson [5, Question 4].

In trivial $\mathbb{C}P^1-$bundles over Kähler surfaces, Smith, Thomas and Yau [17] gave examples of symplectic conifold transitions along *local* embedded Lagrangian 3–spheres (see Definition 2.1). They pointed out that it should be possible for these examples to contain non–Kähler manifolds. In fact, Corti and Smith [4] stated that such symplectic conifold transitions of the trivial $\mathbb{C}P^1-$bundle over certain projective complex surface were not deformation equivalent to any Kähler 3–fold. However, Corti and Smith have now withdrawn their paper from the arXiv since a mistake was found in their proof.

In this paper, we will show that in the case of trivial $\mathbb{C}P^1-$bundles over projective complex surfaces, Smith, Thomas and Yau's examples of symplectic conifold transitions are diffeomorphic to Kähler 3–folds. Actually, we have a similar result for more general $\mathbb{C}P^1-$bundles (see Corollary 1.3). This result can be deduced from our main theorem, which determines the diffeomorphism types of symplectic conifold transitions of all $\mathbb{C}P^1-$bundles over symplectic 4–manifolds along local embedded Lagrangian 3–spheres, where the existence of such Lagrangian 3–spheres is given in Lemma 2.2. The main theorem is stated below.

For simplicity, denote $\overline{\mathbb{C}P^2}$ and S^4 by N_k, $k=1,2$, respectively, where $\overline{\mathbb{C}P^n}$ denotes the complex projective space $\mathbb{C}P^n$ with the opposite orientation. For $k=1,2$, let $\sigma_k^* \in H^2(N_k;\mathbb{Z})$ such that $\sigma_2^*=0$ and σ_1^* is the dual class of the preferred generator $\sigma_1 \in H_2(\overline{\mathbb{C}P^2};\mathbb{Z})$, i.e. $\langle \sigma_1^*, \sigma_1 \rangle = 1$. Denote $[M]$ for the fundamental class of a manifold M. Unless otherwise stated, we always choose symplectic forms on $\mathbb{C}P^1-$bundles over symplectic 4–manifolds to be the ones compactible with the fibrations[12, Theorem 6.3]. As there are exactly two distinct conifold transitions along an embedded Lagrangian 3–sphere up to diffeomorphism[17], we can state our main results as follows.

Theorem 1.1. *Let $\mathbb{P}(E)$ be the projectivization of a rank two complex vector bundle E over a symplectic 4–manifold N. Then the two symplectic conifold transitions of $\mathbb{P}(E)$ along a local embedded Lagrangian 3–sphere are diffeomorphic to $\mathbb{P}(E_1)$ and the one point blowup $\mathbb{P}(E_2)\sharp\overline{\mathbb{C}P^3}$, respectively, where E_k, $k = 1, 2$ are the rank two complex bundles over $N\sharp N_k$ with Chern classes determined by*

$$\begin{aligned} c_1(E_k) &= (c_1(E), -\sigma_k^*) \in H^2(N\sharp N_k; \mathbb{Z}) \cong H^2(N; \mathbb{Z}) \oplus H^2(N_k; \mathbb{Z}); \\ \langle c_2(E_k), [N\sharp N_k]\rangle &= \langle c_2(E), [N]\rangle - 1. \end{aligned}$$

Moreover, the diffeomorphisms above can be chosen to preserve the homotopy classes of almost complex structures.

Remark 1.2. *For a 4–manifold N, every pair in $H^2(N; \mathbb{Z}) \times H^4(N; \mathbb{Z})$ can be realized as the Chern classes of a rank two complex vector bundles E over N and the isomorphism classes of the bundles E_k in Theorem 1.1 can be completely determined by $c_1(E_k), c_2(E_k)$[8, Theorem 1.4.20]. Moreover, it is not hard to prove the manifolds $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)\sharp\overline{\mathbb{C}P^3}$ are in different diffeomorphism classes by comparing their cohomology rings.*

For almost complex 6–manifolds, a diffeomorphism preserves the homotopy classes of almost complex structures if and only if it preserves the first Chern classes c_1[19, Theorem 9]. The existence of diffeomorphisms preserving c_1 has been used to define an equivalence relation between symplectic manifolds[14, 2.1(D)]; .

As it is well–known that the projectivization of a holomorphic vector bundle over a Kähler manifold admits a Kähler structure[18, Proposition 3.18], Theorem 1.1 can imply the corollary below.

Corollary 1.3. *Let $\mathbb{P}(E)$ be the projectivization of a rank two holomorphic vector bundle E over a projective complex surface N, then the symplectic conifold transitions of $\mathbb{P}(E)$ along a local embedded Lagrangian 3–sphere are diffeomorphic to Kähler 3–folds.*

Remark 1.4. *It is easy to extend Theorem 1.1 and Corollary 1.3 to the case of symplectic conifold transitions of $\mathbb{P}(E)$ along arbitrarily many disjoint local embedded Lagrangian 3–spheres.*

We will finish the proof of Theorem 1.1 and Corollary 1.3 in Section 3.3. In the course of establishing Theorem 1.1, we also compute the topological invariants of $\mathbb{C}P^1-$bundles over simply–connected 4–manifolds in Example 3.1. According to [19] and [18], combining this computation with Theorem 1.1 will give diffeomorphism classification of symplectic conifold transitions of simply–connected $\mathbb{C}P^1-$bundles along local embedded Lagrangian 3–spheres.

2 Symplectic conifold transitions on $\mathbb{C}P^1-$bundles

We first recall the definition of conifold transitions [17]. Begin with a Lagrangian embedding $f : S^3 \to X$ in a symplectic 6–manifold X. By the Lagrangian neighborhood theorem [12, Theorem 3.33], the embedding f can extend to a symplectic embedding $f' : T^*_\epsilon S^3 \to X$ with $T^*_\epsilon S^3 \subset T^*S^3$ a neighborhood of the zero section of the cotangent bundle. Define a *conifold transition* along f to be the smooth manifold

$$Y_k := X\backslash f[S^3] \cup_{f'\circ\psi_k} W_k^\epsilon$$

for $k = 1, 2$, where W_k are two small resolutions of the complex singularity $W = \left\{\sum z_j^2 = 0\right\} \subset \mathbb{C}^4$ with exceptional set $\mathbb{C}P^1$ over $\{0\} \in W$ and either of W_k is a complex vector bundle over

$\mathbb{C}P^1$ with first Chern number -2; fixing coordinates on T^*S^3 as

$$T^*S^3 = \left\{(u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 | \, |u| = 1, \langle u, v\rangle = 0\right\},$$

the maps $\psi_k : (W_k \backslash \mathbb{C}P^1 \cong W\backslash\{0\}, \omega_{\mathbb{C}}) \to (T^*S^3\backslash\{v = 0\}, d(vdu))$ are symplectomorphisms defined in [17, (2.1)] with $\omega_{\mathbb{C}}$ the restriction of the symplectic form $\frac{i}{2}\sum_j dz_j \wedge d\overline{z_j}$ on $\mathbb{C}^4$; the submanifolds $W_k^\epsilon \subset W_k$ are neighborhoods of the exceptional set $\mathbb{C}P^1$ such that $W_k^\epsilon \backslash \mathbb{C}P^1 = \psi_k^{-1}[T^*_\epsilon S^3 \backslash \{v = 0\}]$.

There are more choices in conifold transitions along a Lagrangian 3–sphere than along a Lagrangian embedding $S^3 \to X$, as changing the orientation of the Lagrangian 3–sphere $f[S^3]$ would induce a new Lagrangian embedding $S^3 \to X$ different from f. However, this change would just swap the diffeomorphism types of the conifold transitions, so there are exactly two distinct conifold transitions Y_k, $k = 1, 2$ along the Lagrangian 3–sphere $f[S^3]$ up to diffeomorphism[17]. It follows from [17, Theorem 2.9] that the two conifold transitions of a symplectic 6–manifold along a nullhomologous Lagrangian 3–sphere both admit distinguished symplectic structures. Hence to realize such symplectic conifold transitions on $\mathbb{C}P^1-$bundles, it suffices to find nullhomologous Lagrangian 3–spheres.

Inside the product $(\mathbb{C}^2 \times \mathbb{C}P^1, \omega_0 \times \omega_{\mathbb{C}P^1})$ of symplectic manifolds with $\omega_0 = \frac{i}{2}\sum_j dz_j \wedge d\overline{z_j}$ on $\mathbb{C}^2$ and $\omega_{\mathbb{C}P^1}$ the Fubini–Study form on $\mathbb{C}P^1$, a well–known construction[1] of a nullhomologous Lagrangian 3–sphere is given by the composition of maps

$$f : S^3 \overset{(i,h)}{\to} \mathbb{C}^2 \times \mathbb{C}P^1 \overset{\iota\times id_{\mathbb{C}P^1}}{\to} \mathbb{C}^2 \times \mathbb{C}P^1 \tag{1.1}$$

where $i : S^3 \subset \mathbb{C}^2$ is the inclusion of the unit sphere, $h : S^3 \to \mathbb{C}P^1$ is the Hopf map and ι is the complex conjugation on $\mathbb{C}^2$. As the image $f[S^3]$ is entirely contained in $B^4(l) \times \mathbb{C}P^1$ with $B^4(l)$ a ball in $\mathbb{C}^2$ of radius $l > 1$, finding symplectic embeddings of $B^4(l) \times \mathbb{C}P^1$ in $\mathbb{C}P^1-$bundles would induce nullhomologous Lagrangian 3–spheres in these bundles. This may lead to the following definition.

Definition 2.1. *Let $\mathbb{P}(E)$ be a symplectic manifold which is a $\mathbb{C}P^1-$bundle over a 4–manifold N. A Lagrangian 3–sphere in $\mathbb{P}(E)$ is called local if it is the image of the composition of embeddings*

$$S^3 \overset{f}{\to} B^4(l) \times \mathbb{C}P^1 \overset{\eta}{\to} \mathbb{P}(E)$$

for $l > 1$ where the symplectic embedding η can induce a local trivialization of the bundle $\pi : \mathbb{P}(E) \to N$, i.e. there is a differentiable embedding $k : B^4(l) \to N$ such that

$$\pi^{-1}[k[B^4(l)]] = \eta[B^4(l) \times \mathbb{C}P^1] \overset{\eta^{-1}}{\to} B^4(l) \times \mathbb{C}P^1 \overset{k\times id_{\mathbb{C}P^1}}{\to} k[B^4(l)] \times \mathbb{C}P^1$$

is a local trivialization of the $\mathbb{C}P^1-$bundle $\mathbb{P}(E)$.

[17] and [4] have shown the existence of local Lagrangian 3–spheres in trivial $\mathbb{C}P^1-$bundles over Kähler surfaces. We generalize their result in the following Lemma by using Thurston's construction of symplectic form[12, Theorem 6.3] and the construction of Kähler forms on $\mathbb{P}(E)$[18, Proposition 3.18].

Lemma 2.2. *Let $\mathbb{P}(E)$ be the projectivization of a rank two complex vector bundle E over a symplectic 4–manifold N. Then for any positive integer K, the total space $\mathbb{P}(E)$ admits a symplectic form compatible with the fibration such that there are K disjoint local Lagrangian 3–spheres in $\mathbb{P}(E)$. Moreover, if N is Kähler and E admits a holomorphic structure, then for any positive integer K, the total space $\mathbb{P}(E)$ admits a Kähler form compatible with the fibration such that there are K disjoint local Lagrangian 3–spheres in $\mathbb{P}(E)$.*

Proof. By Definition 2.1, it suffices to construct appropriate symplectic forms on $\mathbb{P}(E)$ and find symplectic embeddings $B^4(l) \times \mathbb{C}P^1 \to \mathbb{P}(E)$ for $l > 1$ which can induce local trivializations. The key point is to note that there exists a system of projective–unitary local trivializations $\{(U_j, \phi_j)\}_{j=0}^m$ of the $\mathbb{C}P^1$–bundle $\pi : \mathbb{P}(E) \to N$ and a partition of unity $\rho_j : N \to [0,1]$ subordinating to the open cover $\{U_j\}_{j=0}^m$ of N such that each U_j is contractible and $\rho_0 \equiv 1$ on a nonempty open subset $V \subset U_0$. In fact, such U_j and ρ_0 can be obtained by using [2, Corollary 5.2] and shrinking all open sets U_j except U_0 if necessary.

For the symplectic case, apply Thurston's construction. According to the proof of [12, Theorem 6.3], the local trivializations $\{(U_j, \phi_j)\}_{j=0}^m$ and the partition of unity $\rho_j : N \to [0,1]$, together with the first Chern class of the dual bundle of the tautological line bundle over $\mathbb{P}(E)$, can contribute to define a closed 2–form τ on $\mathbb{P}(E)$ such that the restriction of τ on each fiber $\mathbb{C}P^1$ is just $\omega_{\mathbb{C}P^1}$, and the 2–form $\tau + \lambda\pi^*\omega_N$ on $\mathbb{P}(E)$ is symplectic for $\lambda > 0$ sufficiently large where ω_N denotes the symplectic form on N. Since $\rho_0 \equiv 1$ on V, then the restriction of the form τ to $\pi^{-1}[V]$ is equal to the pullback $\phi_0^*(0 \times \omega_{\mathbb{C}P^1})$ of the form $0 \times \omega_{\mathbb{C}P^1}$ on $V \times \mathbb{C}P^1$. By the Darboux neighborhood theorem, for any positive integer K, there are K disjoint symplectic embeddings $B^4(l) \to (V, \lambda\omega_N)$ with $l > 1$ for λ sufficiently large. So in this case, we have K disjoint compositions of symplectic embeddings

$$B^4(l) \times \mathbb{C}P^1 \to (V \times \mathbb{C}P^1, \lambda\omega_N \times \omega_{\mathbb{C}P^1}) \overset{\phi_0^{-1}}{\to} (\mathbb{P}(E), \tau + \lambda\pi^*\omega_N)$$

which are the desired embeddings.

Now for the Kähler case, assume ω_N is the Kähler form on N and E is holomorphic. Using the previous partition of unity ρ_j and the system of local trivializations $\{(U_j, \varphi_j)\}_{j=0}^m$ of E associated to $\{(U_j, \phi_j)\}_{j=0}^m$, we can obtain a Hermitian metric h on E such that on the restriction $E|_V$ of E to V, the metric h is the pullback of the local Hermitian metric on $\mathbb{C}^2$ via the projection $E|_V \overset{\varphi_0}{\to} V \times \mathbb{C}^2 \to \mathbb{C}^2$. [18, Proposition 3.18] shows that h induces a Hermitian metric on the bundle L^* and the Chern form ω_E associated to this metric can contribute to obtain a Kähler form $\omega_E + \lambda\pi^*\omega_N$ for $\lambda > 0$ sufficiently large. Replacing τ by ω_E in proof of the symplectic case can produce the desired symplectic embeddings. This completes the proof. □

3 Topology of symplectic conifold transitions of $\mathbb{C}P^1$–bundles

The aim of this section is to study the topology of symplectic conifold transitions of $\mathbb{C}P^1$–bundles along local Lagrangian 3–spheres and prove Theorem 1.1 and Corollary 1.3. For this purpose, we first recall in Section 3.1 the invariants of simply–connected 6–manifolds with torsion–free homology, and compute the invariants of $\mathbb{C}P^1$–bundles over simply–connected 4–manifolds; then determine in Section 3.2 the topology of conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along $f[S^3]$, i.e. to establish Lemma 3.3, which is a key ingredient in the proof of Theorem 1.1.

3.1 Invariants of simply–connected 6–manifolds with torsion–free homology

By Wall[19] and Jupp[11], the third Betti number b_3, the integral cohomology ring H^*, the first Pontrjagin class p_1 and the second Whitney–Stiefel class w_2 form a system of invariants, which can distinguish all diffeomorphism classes of simply–connected 6–manifolds with torsion–free homology. As an example, we will compute these invariants for $\mathbb{C}P^1$–bundles over simply–connected 4–manifolds.

Example 3.1. *Let $\pi : \mathbb{P}(E) \to N$ be the projectivization of a rank two complex vector bundle E over a simply–connected 4–manifold. Then $\mathbb{P}(E)$ has a natural orientation which is compatible* *with that of the base and fibers. By the homotopy exact sequence and Gysin sequence, the 6–manifold $\mathbb{P}(E)$ is a simply–connected with $b_3 = 0$. The cohomology ring and the characteristic classes w_2, p_1 can be computed as follows.*

(i)*By the definition of Chern classes[2, Section 20], we have*

$$H^*(\mathbb{P}(E)) \cong H^*(N)[a]/\left\langle a^2 + \pi^* c_1(E)\cdot a + \pi^* c_2(E)\right\rangle$$

where $a = c_1(L^*)$ with L^* the dual bundle of the tautological line bundle $L = \{(l, v) \in \mathbb{P}(E) \times E \mid v \in l\}$ over $\mathbb{P}(E)$. Let $\{y_i\}$ be a basis of the free $\mathbb{Z}-$module $H^2(N)$ and then $\{a, \pi^* y_i\}$ forms a basis of $H^2(\mathbb{P}(E))$. By the relations $a^2 + \pi^* c_1(E)\cdot a + \pi^* c_2(E) = 0$ and $\langle [N]^* \cup a, [\mathbb{P}(E)]\rangle = 1$ with $[N]^* \in H^4(N)$ satisfying $\langle [N]^*, [N]\rangle = 1$, we can obtain

$$\begin{aligned}
\left\langle a^3, [\mathbb{P}(E)]\right\rangle &= \left\langle c_1(E)^2 - c_2(E), [N]\right\rangle;\\
\left\langle a^2 \cup \pi^* y_i, [\mathbb{P}(E)]\right\rangle &= -\left\langle c_1(E) y_i, [N]\right\rangle;\\
\left\langle a \cup \pi^* y_i \cup \pi^* y_j, [\mathbb{P}(E)]\right\rangle &= \left\langle y_i y_j, [N]\right\rangle;\\
\left\langle \pi^* y_i \cup \pi^* y_j \cup \pi^* y_k, [\mathbb{P}(E)]\right\rangle &= 0.
\end{aligned}$$

(ii)*Similar to the proof of [13, Theorem 14.10], as the tautological line bundle L is a subbundle of the pullback $\pi^* E$ and a Hermitian metric on E pulls back to a Hermitian metric on $\pi^* E$, we have a splitting $\pi^* E = L \oplus L^\perp$ where $L^\perp$ is the orthogonal complement bundle of L and hence*

$$\begin{aligned}
T\mathbb{P}(E) &\cong \pi^* TN \oplus Hom_{\mathbb{C}}(L, L^\perp); \qquad (1)\\
Hom_{\mathbb{C}}(L, L^\perp) \oplus \varepsilon^1_{\mathbb{C}} &\cong L^* \otimes \pi^* E
\end{aligned}$$

with $\varepsilon^1_{\mathbb{C}}$ the trivial complex line bundle. These isomorphisms, together with the relations $a^2 + \pi^* c_1(E)\cdot a + \pi^* c_2(E) = 0$, $p_1 = {c_1}^2 - 2c_2$ and

$$c_1(L_1 \otimes L_2) = 2c_1(L_1) + c_1(L_2); c_2(L_1 \otimes L_2) = c_1(L_1)^2 + c_1(L_1)c_1(L_2) + c_2(L_2)$$

with L_i a complex vector bundle of rank $i = 1, 2$[13, Problem 16-B], imply

$$\begin{aligned}
w_2(T\mathbb{P}(E)) &\equiv \pi^*(w_2(TN) + w_2(E));\\
p_1(T\mathbb{P}(E)) &= \pi^*(p_1(TN) + c_1(E)^2 - 4c_2(E)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\left\langle a^2 \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)]\right\rangle &= \left\langle w_2(E) \cup (w_2(E) + w_2(TN)), [N]\right\rangle;\\
\left\langle a \cup \pi^* y_i \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)]\right\rangle &= \left\langle y_i \cup (w_2(E) + w_2(TN)), [N]\right\rangle;\\
\left\langle \pi^* y_i \cup \pi^* y_j \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)]\right\rangle &= 0.\\
\left\langle a \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)]\right\rangle &= 3\sigma(N) + \left\langle c_1(E)^2 - 4c_2(E), [N]\right\rangle\\
\left\langle \pi^* y_i \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)]\right\rangle &= 0
\end{aligned}$$

where $\sigma(N)$ is the signature of the 4–manifold N[13, SIGNATURE THEOREM 19.4].

Remark 3.2. *In Example 3.1, the isomorphisms (1) still hold in the case when the 4–manifold N is not simply–connected, and it follows that*

$$c_1(T\mathbb{P}(E)) = 2a + \pi^*(c_1(TN) + c_1(E))$$

which will be applied to the proof of Theorem 1.1 in Section 3.3.

3.2 Topology of conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along $f[S^3]$

It is easy to see that the definition of conifold transitions can extend to symplectic manifolds with boundaries. In this subsection we will prove Lemma 3.3, determining the topology of Y_k, $k = 1, 2$, which denote the two conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along the Lagrangian embedding $f : S^3 \to B^4(l) \times \mathbb{C}P^1$ in (1.1).

As in Section 1, denote $\overline{\mathbb{C}P^2}$ and S^4 by N_k, $k = 1, 2$, respectively. Let $\sigma_k \in H_2(N_k)$ and $\sigma_k^* \in H^2(N_k)$ such that σ_1^* is the dual class of the preferred generator σ_1 and $\sigma_2 = 0$, $\sigma_2^* = 0$. As $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$, the lemma can be stated as follows.

Lemma 3.3. *Let id_∂ denote the identity map of $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$. Then there are two diffeomorphisms*

$$\begin{aligned} \phi_1 &: B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_1 \to \mathbb{P}(E_1'); \\ \phi_2 &: B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_2 \to \mathbb{P}(E_2') \sharp \overline{\mathbb{C}P^3} \end{aligned}$$

such that the restriction of ϕ_k on $B^4(l) \times \mathbb{C}P^1$ can induce a local trivialization of the bundle $\mathbb{P}(E_k')$ for $k = 1, 2$, where E_k' is the rank two complex bundle over N_k with $c_1(E_k') = -\sigma_k^*$ and $c_2(E_k') = -1$.

To show this lemma, it needs to compute the topological invariants of $M_k := B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$. As Smith and Thomas [16, Proposition 4.2] have computed the intersection forms of the conifold transitions of $\mathbb{C}P^2 \times \mathbb{C}P^1$ along a local Lagrangian 3–sphere, we will extend their computation to the topological invariants of M_k in Lemma 3.4 and Example 3.5.

The following Lemma will be very useful for the computation of invariants of M_k. Referring to the definition of conifold transitions recalled in Section 2, as we have inclusions of the exceptional set $\mathbb{C}P^1 \subset W_k^\epsilon$ and the set $O \times \mathbb{C}P^1 \subset B^4(l) \times \mathbb{C}P^1 \backslash f[S^3]$ with $O \in B^4(l)$ the origin, let C_k and P_k denote the images of the exceptional set $\mathbb{C}P^1$ and the set $O \times \mathbb{C}P^1$ under the natural inclusions $W_k^\epsilon \to Y_k \to M_k$ and $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \to Y_k \to M_k$, respectively.

Lemma 3.4. *Let $\sigma \in H_2(\mathbb{C}P^2)$ be the preferred generator. Then there are two differentiable embeddings $r_k : \mathbb{C}P^2 \sharp N_k \to M_k$, $k = 1, 2$ satisfying the following conditions:*

(i)Under the homomorphism

$$r_{k*} : H_2(\mathbb{C}P^2 \sharp N_k) \cong H_2(\mathbb{C}P^2) \oplus H_2(N_k) \to H_2(M_k),$$

the images of σ and σ_k are the homology classes $[P_k]$ and $\frac{1-(-1)^k}{2} \cdot [C_k]$ in $H_2(M_k)$, respectively.

(ii)The Euler class of the normal bundle of r_k is

$$(-\sigma^*, -\sigma_k^*) \in H^2(\mathbb{C}P^2 \sharp N_k) \cong H^2(\mathbb{C}P^2) \oplus H^2(N_k),$$

where $\sigma^* \in H^2(\mathbb{C}P^2)$ is the dual cohomology class of σ;

(iii)In M_k, the intersection number of the submanifolds $r_k[\mathbb{C}P^2 \sharp N_k]$ and C_k is $(-1)^k$.

To show this lemma, first recall some results in the proof of [17, Theorem 2.9] and [12, Theorem 3.33]. Let

$$\Delta_\epsilon = \left\{(u, v) \in T_\epsilon^* S^3 | (v_1, v_2, v_3, v_4) = \lambda(-u_2, u_1, -u_4, u_3); \lambda \geq 0\right\}$$

and fix W_k, $k = 1, 2$ as $W^\pm$ in [17], respectively. [17, Theorem 2.9] finds 4–dimensional submanifolds $\widehat{S}_k \subset W_k^\epsilon$, $k = 1, 2$ such that

(1)$\widehat{S}_1$ is the complex line bundle over the exceptional set $\mathbb{C}P^1 \subset W_1^\epsilon$ with Euler class -1 and $\psi_1^{-1}[\Delta_\epsilon \backslash \{v=0\}] = \widehat{S}_1 \backslash \mathbb{C}P^1$;
(2)$\widehat{S}_2$ is diffeomorphic to $\mathbb{R}^4$ and $\psi_2^{-1}[\Delta_\epsilon \backslash \{v=0\}]$ is equal to $\widehat{S}_2$ with a point removed.
(3)The intersection number of $\widehat{S}_k$ and the exceptional set $\mathbb{C}P^1$ in W_k^ϵ is $(-1)^k$.

Considering the symplectic form $d(vdu)$ on T^*S^3 and applying [12, Theorem 3.33] to the Lagrangian embedding f, this defines an embedding $\overline{f} : T_\epsilon^* S^3 \to B^4(l) \times \mathbb{C}P^1$ by $\overline{f}(u,v) = \exp_{f(u)}(-J_{f(u)} \circ df_u \circ \Phi_u(v))$, where J is a compatible almost complex structure on $(B^4(l) \times \mathbb{C}P^1, \omega_0 \times \omega_{\mathbb{C}P^1})$ and $\Phi_u : T_u^* S^3 \to T_u S^3$ is an isomorphism determined by the relation $\omega_0 \times \omega_{\mathbb{C}P^1}(df_u \circ \Phi_u(v), J_{f(u)} \circ df_u(v')) = v(v')$ for $v' \in T_q S^3$.

Proof of Lemma 3.4. As [12, Theorem 3.33] shows that $\overline{f}$ is isotopic to a symplectic embedding which represents a Lagrangian neighborhood of f, thus Y_k is diffeomorphic to $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \cup_{\overline{f} \circ \psi_k} W_k^\epsilon$. We claim that the restriction of $\overline{f}$ on $\Delta_\epsilon \backslash \{v=0\}$ is a diffeomorphism onto the relative complement of a closed neighborhood of

$$O \times \mathbb{C}P^1 \subset R_0 = \left\{(\overline{w},[w]) \in B^4(l) \times \mathbb{C}P^1 | \, |w| < 1\right\}.$$

If it is true, then combining this claim with the conditions (1), (2), (3) above and the fact that R_0 is the open disc bundle over $O \times \mathbb{C}P^1$ with Euler class 1, it would imply that $R_0 \cup_{\overline{f} \circ \psi_k | \psi_k^{-1}[\Delta_\epsilon \backslash \{v=0\}]} \widehat{S}_k \cong \mathbb{C}P^2 \sharp N_k$ are well–defined differentiable submanifolds of $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \cup_{\overline{f} \circ \psi_k} W_k^\epsilon \cong Y_k \subset M_k$ for $k=1,2$, respectively, which gives embeddings $r_k : \mathbb{C}P^2 \sharp N_k \hookrightarrow M_k$ satisfying (i)(iii). (ii) would follow from the fact that the restriction of the normal bundle of $R_0 \subset B^4(l) \times \mathbb{C}P^1$ to $O \times \mathbb{C}P^1$ has Euler class -1 and so does the restriction of the normal bundle of $\widehat{S}_1 \subset W_1^\epsilon$ to the exceptional set $\mathbb{C}P^1$.

Now it remains to show our claim. Under the identifications

$$\begin{aligned} TS^3 &= T^*S^3 = \left\{(u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 | \, |u| = 1, \langle u, v \rangle = 0\right\}, \\ \mathbb{R}^4 &= \mathbb{C}^2 : (r_1, r_2, r_3, r_4) \longmapsto (r_1 + ir_2, r_3 + ir_4), \end{aligned}$$

it is easy to see that $v(v') = \omega_0(v, Jv') = \omega_0(\overline{v}, J\overline{v'})$ with $(\overline{v}, \overline{v'})$ the complex conjugate of $(v, v') \in T_u^* S^3 \times T_u S^3$. Thus for any $(u,v) \in \Delta_\epsilon \backslash \{v=0\}$, we have

$$\begin{aligned} v &= \lambda i u = \lambda \sqrt{-1} u, \lambda > 0; \\ df_u(v) &= (\overline{v}, [v]) = (\overline{v}, [0]) \in T_{(\overline{u},[u])} B^4(l) \times \mathbb{C}P^1 = \mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{C}u \end{aligned}$$

and hence $\Phi_u(v) = v$. These relations, together with the fact that the complex structure $J_{f(u)}$ on $T_{f(u)} B^4(l) \times \mathbb{C}P^1$ is induced by the multiplication by $i = \sqrt{-1}$ on $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{C}u$, imply

$$\overline{f}(u,v) = \exp_{(\overline{u},[u])}(-\lambda \overline{u}, [0]) = ((1-\lambda)\overline{u}, [u]) \in R_0 \backslash O \times \mathbb{C}P^1.$$

This completes the proof. □

Using Lemma 3.4, we can compute the topological invariants of $M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$ for $k = 1, 2$.

Example 3.5. *We first claim that M_k is a simply–connected 6–manifold with $b_3 = 0$ and $H^2(M_k)$ has a basis consists of z_k and x_k, where z_k is the Poincaré dual of the submanifold $R_k = r_k[\mathbb{C}P^2 \sharp N_k] \subset M_k$ and the definition of x_k is contained in the following proof of the claim.*

Since M_k is obtained by surgery along an embedding $S^3 \times D^3 \hookrightarrow S^2 \times S^4$ with C_k the resulting 2–sphere[3], then M_k is simply–connected and there is a cobordism W_k between $S^2 \times S^4$ and M_k, assuming $j_k : S^2 \times S^4 \hookrightarrow W_k$ and $j'_k : M_k \hookrightarrow W_k$ are the inclusions. From the cohomology exact sequence of the pairs (W_k, M_k) and $(W_k, S^2 \times S^4)$, it is easy to show that $H_3(M_k) \cong H^3(M_k)$ is trivial. Furthermore, consider the exact sequence

$$0 \to H^2(W_k) \xrightarrow{j'^*_k} H^2(M_k) \xrightarrow{\delta} H^3(W_k, M_k), \tag{3.2}$$

then δz_k is a generator of $H^3(W_k, M_k)$ because the value of δz_k on the generator of $H_3(W_k, M_k)$ is equal to $\langle z_k, [C_k]\rangle = (-1)^k$ by Lemma 3.4 (iii). This, together with the isomorphism $H^2(W_k) \xrightarrow{j^*_k} H^2(S^2 \times S^4)$ and the exact sequence (3.2), implies that $x_k := j'^*_k j^{*-1}_k a$ and z_k form a basis of $H^2(M_k)$, where $a \in H^2(S^2 \times S^4)$ is the dual class of the preferred generator $\left[S^2\right]$ of $H_2(S^2 \times S^4)$.
(i)The cohomology ring of M_k: The relations $j'_{k*}[P_k] = j_{k*}[S^2]$ and $\delta x_k = 0$, together with Lemma 3.4 (i) and the fact that $\langle x_k, [C_k]\rangle$ is equal to the value of $\delta x_k \in H^3(W_k, M_k)$ on the generator of $H_3(W_k, M_k)$, imply

$$\langle r^*_k x_k, \sigma\rangle = \langle x_k, [P_k]\rangle = \left\langle a, [S^2]\right\rangle = 1; \langle r^*_k x_k, \sigma_k\rangle = 0 \tag{3.3}$$

for the basis $\sigma, \sigma_k \in H_2(\mathbb{C}P^2 \sharp N_k) \cong H_2(\mathbb{C}P^2) \oplus H_2(N_k)$. Let $e(\nu r_k)$ denote the Euler class of the normal bundle νr_k of r_k, then it follows from the values(3.3) and Lemma 3.4 (ii) that

$$\begin{aligned}
\left\langle z_k^3, [M_k]\right\rangle &= \left\langle z_k^2, [R_k]\right\rangle = \left\langle e(\nu r_k)^2, [\mathbb{C}P^2 \sharp N_k]\right\rangle = \frac{1 + (-1)^k}{2};\\
\left\langle z_k x_k^2, [M_k]\right\rangle &= \left\langle x_k^2, [R_k]\right\rangle = \left\langle r^*_k x_k^2, [\mathbb{C}P^2 \sharp N_k]\right\rangle = 1;\\
\left\langle x_k z_k^2, [M_k]\right\rangle &= \langle x_k z_k, [R_k]\rangle = \left\langle r^*_k x_k \cup e(\nu r_k), [\mathbb{C}P^2 \sharp N_k]\right\rangle = -1;\\
\left\langle x_k^3, [M_k]\right\rangle &= \left\langle j'^*_k j^{*-1}_k a^3, [M_k]\right\rangle = 0.
\end{aligned}$$

(ii)The first Pontrjagin class of M_k: The exact sequence

$$H_7(W_k, \partial W_k) \xrightarrow{\partial} H_6(S^2 \times S^4 \sqcup M_k) \to H_6(W_k),$$

together with the relations $\partial [W_k] = [M_k] - [S^2 \times S^4]$, $p_1(M_k) = j'^*_k p_1(W_k)$ and $j^*_k p_1(W_k) = p_1(S^2 \times S^4) = 0$, imply

$$\langle p_1(M_k) x_k, [M_k]\rangle = \left\langle p_1(W_k) \cup j^{*-1}_k a, j'_{k*}[M_k] - j_{k*}[S^2 \times S^4]\right\rangle = 0.$$

From the relations $p_1(\nu r_k) = e(\nu r_k)^2$, $\left\langle p_1(\mathbb{C}P^2 \sharp N_k), \left[\mathbb{C}P^2 \sharp N_k\right]\right\rangle = 3 \cdot \frac{1+(-1)^k}{2}$ and $z_k \cap [M_k] = r_{k*}[\mathbb{C}P^2 \sharp N_k]$, together with Lemma 3.4 (ii) and the decomposition $r^*_k TM_k = T(\mathbb{C}P^2 \sharp N_k) \oplus \nu r_k$, we get

$$\langle p_1(M_k) z_k, [M_k]\rangle = \left\langle r^*_k p_1(M_k), [\mathbb{C}P^2 \sharp N_k]\right\rangle = 2 \times (1 + (-1)^k).$$

(iii)The second Whitney class of M_k: As the value $w_2(S^2 \times S^4) = 0$ and the isomorphism $j^*_k : H^2(W_k) \to H^2(S^2 \times S^4)$ imply $w_2(W_k) = 0$, thus

$$w_2(M_k) = j'^*_k w_2(W_k) = 0.$$

Now we can prove the Lemma 3.3.

Proof of Lemma 3.3. Denote S^6 and $\overline{\mathbb{C}P^3}$ by Q_1 and Q_2, respectively. By Wall and Jupp's classification of simply–connected 6–manifolds with torsion–free homology[19][11], comparing the invariants of M_k and $\mathbb{P}(E'_k)$ (see Example 3.5 and Example 3.1), we get two diffeomorphisms

$$\varphi_k : M_k \to \mathbb{P}(E'_k)\sharp Q_k,\ k = 1, 2$$

such that $\varphi_k^* a_k = x_k + \frac{1+(-1)^k}{2} \cdot z_k$ for $k = 1, 2$, $\varphi_1^*\pi_1^*(-\sigma_1^*) = z_1$ and $\varphi_2^* z' = z_2$, where

$$a_k \in H^2(\mathbb{P}(E'_k)\sharp Q_k) \cong H^2(\mathbb{P}(E'_k)) \oplus H^2(Q_k)$$

denote the first Chern classes of the dual bundles of the tautological line bundles over $\mathbb{P}(E'_k)$ for $k = 1, 2$, respectively, $\pi_1 : P(E'_1) \to \overline{\mathbb{C}P^2}$ is the bundle projection, and

$$z' \in H^2(\mathbb{P}(E'_2)\sharp\overline{\mathbb{C}P^3}) \cong H^2(\mathbb{P}(E'_2)) \oplus H^2(\overline{\mathbb{C}P^3})$$

is the Poincaré dual of the submanifold $\mathbb{C}P^2 \subset \overline{\mathbb{C}P^3}$.

We claim that $\varphi_{k*}[O \times \mathbb{C}P^1] = f_{k*}[\mathbb{C}P^1]$ for the submanifold $O \times \mathbb{C}P^1 \subset B^4(l) \times \mathbb{C}P^1 \subset M_k$ and embeddings $f_k : \mathbb{C}P^1 \to \mathbb{P}(E'_k)\sharp Q_k$ representing a fiber of $\mathbb{P}(E'_k)$. As the relations $\langle z_k, [O \times \mathbb{C}P^1]\rangle = 0$ and $j'_{k*}[P_k] = j_{k*}[S^2] = j'_{k*}[O \times \mathbb{C}P^1]$ imply that $[O \times \mathbb{C}P^1]$ is the dual base of $x_k + \frac{1+(-1)^k}{2} \cdot z_k = \varphi_k^* a_k$ in the basis

$$\left\{x_k + \frac{1+(-1)^k}{2} \cdot z_k, z_k\right\} = \left\{ \begin{array}{c} \{\varphi_1^* a_1, \varphi_1^*\pi_1^*(-\sigma_1^*)\} \text{ for } k = 1, \\ \{\varphi_2^* a_2, \varphi_2^* z'\} \text{ for } k = 2, \end{array} \right.$$

comparing this with the fact that $f_{k*}[\mathbb{C}P^1]$ is the dual base of a_k in the basis $\{a_1, \pi_1^*(-\sigma_1^*)\}$ for $k = 1$ and in the basis $\{a_2, z'\}$ for $k = 2$, respectively, shows the claim.

Since the claim above implies that $\varphi_k|_{O\times\mathbb{C}P^1}$ is homotopic to f_k, then by [9, THEOREM 1] and the isotopy extension theorem[10, Chapter 8, 1.3. Theorem], there is an isotopy $F_t^k : \mathbb{P}(E'_k)\sharp Q_k \to \mathbb{P}(E'_k)\sharp Q_k$, $0 \le t \le 1$, such that $F_0^k = id$ and $F_1^k \circ \varphi_k|_{O\times\mathbb{C}P^1} = f_k$. Let $\overline{f_k} : B^4(l) \times \mathbb{C}P^1 \to \mathbb{P}(E'_k)\sharp Q_k$ be an extension of f_k which can induce a local trivialization of the bundle $\mathbb{P}(E'_k)$, then $F_1^k \circ \varphi_k|_{B^4(l)\times\mathbb{C}P^1}$ and $\overline{f_k}$ determine two closed tubular neighborhoods of $f_k[\mathbb{C}P^1]$. By the ambient isotopy theorem for closed tubular neighborhoods[10, Chapter 4, Section 6, Exercises 9], there exists an isotopy $H_t^k : \mathbb{P}(E'_k)\sharp Q_k \to \mathbb{P}(E'_k)\sharp Q_k$, $0 \le t \le 1$, such that $H_0^k = id$, $H_1^k \circ F_1^k \circ \varphi_k[B^4(l) \times \mathbb{C}P^1] = \overline{f_k}[B^4(l) \times \mathbb{C}P^1]$ and

$$g := \overline{f_k}^{-1} \circ H_1^k \circ F_1^k \circ \varphi_k|_{B^4(l)\times\mathbb{C}P^1} : B^4(l) \times \mathbb{C}P^1 \to B^4(l) \times \mathbb{C}P^1$$

is a $B^4(l)-$bundle isomorphism. As the homotopy group $\pi_2(O(4))$ of the real orthogonal group $O(4)$ is trivial, this implies $g|_{\partial B^4(l)\times\mathbb{C}P^1}$ is isotopic to the identity map of $\partial B^4(l) \times \mathbb{C}P^1$ and then similar to the proof of [10, Chapter 8, 2.3], we can extend g to a self–diffeomorphism ϕ of $M_k = B^4(l)\times\mathbb{C}P^1 \cup_{id_\partial} Y_k$ which is identity outside a neighborhood of $B^4(l)\times\mathbb{C}P^1$. Consequently, the restriction of $\phi_k := H_1^k \circ F_1^k \circ \varphi_k \circ \phi^{-1}$ on $B^4(l) \times \mathbb{C}P^1$ is equal to $\overline{f_k}$ and hence $\phi_k, k = 1, 2$, are the desired diffeomorphisms. □

3.3 Topology of symplectic conifold transitions of $\mathbb{C}P^1-$bundles

The establishment of Lemma 3.3 make it possible to prove Theorem 1.1, which determines the diffeomorphism types of symplectic conifold transitions of $\mathbb{C}P^1-$bundles over 4–manifolds along local Lagrangian 3–spheres. In this section, we show this theorem and Corollary 1.3.

Proof of Theorem 1.1. From [17, Theorem 2.9] and the definition of the two symplectic conifold transitions Z_k, $k=1,2$ along a local Lagrangian embedding $S^3 \xrightarrow{f} B^4(l)\times \mathbb{C}P^1 \xrightarrow{\eta} \mathbb{P}(E)$, we get the identification

$$Z_k = \mathbb{P}(E)\cup_\eta M_k \diagdown (\text{Interior } \eta[B^4(l)\times \mathbb{C}P^1])$$

as almost complex manifolds with $B^4(l)\times \mathbb{C}P^1$ seen as a subset of $M_k = B^4(l)\times \mathbb{C}P^1 \cup_{id_\partial} Y_k$. Denote S^6 and $\overline{\mathbb{C}P^3}$ by Q_1 and Q_2, respectively, and let $E\cup_{\mathbb{C}^2} E'_k$ denote the complex vector bundle over the one point union $N\vee N_k$ obtained by identifying one fiber $\mathbb{C}^2$ of the two bundles E and E'_k, respectively. The identity map id of $\mathbb{P}(E)$ and the diffeomorphisms $\phi_k : M_k \to \mathbb{P}(E'_k)\sharp Q_k$ in Lemma 3.3 contribute to define diffeomorphisms

$$\Psi_k : Z_k \overset{\cong}{\to} \mathbb{P}(E_k)\sharp Q_k, k=1,2$$

where E_k is the pullback bundle of the bundle $E\cup_{\mathbb{C}^2} E'_k$ under the natural map $N\sharp N_k \to N\vee N_k$. It is very easy to get the Chern class of E_k from the isomorphism $H^2(N\vee N_k)\cong H^2(N\sharp N_k)$, the homomorphism

$$H^4(N\vee N_k)\cong \mathbb{Z}\oplus\mathbb{Z} \to H^4(N\sharp N_k)\cong \mathbb{Z} : (a,b)\longmapsto a+b$$

and the values

$$c_j(E\cup_{\mathbb{C}^2} E'_k) = (c_j(E), c_j(E_k)) \in H^{2j}(N)\oplus H^{2j}(N_k)\cong H^{2j}(N\vee N_k)$$

for $j=1,2$.

To prove the diffeomorphisms Ψ_k preserve the homotopy classes of almost complex structure, it suffices to show it preserves c_1[19, Theorem 9]. Consider the commutative diagram

$$\begin{array}{ccc} H^2(\mathbb{P}(E_k)\sharp Q_k) & \overset{\Psi_k^*}{\to} & H^2(Z_k) \\ \uparrow\cong & & \uparrow\cong \\ H^2(\mathbb{P}(E)\cup_{\eta\circ\overline{f_k}^{-1}} \mathbb{P}(E'_k)\sharp Q_k) & \overset{(id\cup\phi_k)^*}{\to} & H^2(\mathbb{P}(E)\cup_\eta M_k) \\ \downarrow & & \downarrow \\ H^2(\mathbb{P}(E))\oplus H^2(\mathbb{P}(E'_k)\sharp Q_k) & \overset{id^*\oplus\phi_k^*}{\to} & H^2(\mathbb{P}(E))\oplus H^2(M_k) \end{array} \tag{3.4}$$

where $\overline{f_k} : B^4(l)\times \mathbb{C}P^1 \to \mathbb{P}(E'_k)\sharp Q_k$ is the restriction of ϕ_k as in the proof of Lemma 3.3 and the vertical homomorphisms are induced by the natural inclusions. As the conifold transitions is an almost complex operation preserving the first Chern class [17][4, Lemma 2], the formula of the first Chern class of a one point blowup[6, p.608][7] and Remark 3.2 imply that the images of $c_1(T\mathbb{P}(E_k)\sharp Q_k)$ and $c_1(TZ_k)$ under the vertical composite homomorphisms are

$$\Big(2a+\pi^*(c_1(TN)+c_1(E)), 2a_k-(1+(-1)^k)\cdot z'\Big), \tag{3.5}$$

$$(2a+\pi^*(c_1(TN)+c_1(E)), 2x_k), \tag{3.6}$$

respectively, with a_k, z' and x_k defined in the proof of Lemma 3.3 and Example 3.5. Since $\phi_k^* a_k = x_k + \frac{1+(-1)^k}{2}\cdot z_k$, $\phi_2^* z' = z_2$ by the proof of Lemma 3.3, then the horizontal homomorphism $id^*\oplus\phi_k^*$ maps the class (3.5) to (3.6) and hence $c_1(TZ_k) = \Psi_k^* c_1(T\mathbb{P}(E_k))$ as the vertical homomorphisms in the diagram (3.4) are injective. This completes the proof. □

Now we turn to show Corollary 1.3.

Proof of Corollary 1.3. As the blowup of a Kähler manifold at a point is also Kähler[18, Proposition 3.24], this Corollary follows easily from Theorem 1.1 and the claim that both E_k over the projective complex surfaces $N\sharp N_k$ admit holomorphic structures. To prove the claim, it suffices to note Schwarzenberger [15, Theorem 9] showed that a complex vector bundle over a projective complex surface S admits a holomorphic structure if and only if the first Chern class of the bundle belongs to $H^{1,1}(S)$. As $c_1(E_2) = c_1(E)$ and $c_1(E_1)$ is equal to $c_1(E)$ plus the exceptional divisor $-\sigma_1^*$, so $c_1(E_k) \in H^{1,1}(N\sharp N_k)$ by the Lefschetz theorem on (1,1) classes[18, Theorem 11.30]. This completes the proof. □

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