

THE EQUIVALENCE OF FRIEDLANDER-MAZUR AND STANDARD CONJECTURES FOR THREEFOLDS

JIN CAO AND WENCHUAN HU

ABSTRACT. We show that the Friedlander-Mazur conjecture holds for a complex smooth projective variety X of dimension three implies the standard conjectures hold for X . This together with a result of Friedlander yields the equivalence of the two conjectures in dimension three. From this we provide some new examples whose standard conjectures hold.

1. THE MAIN RESULT

Let X be a complex projective variety of dimension n . The Lawson homology $L_p H_k(X)$ of p -cycles for a projective variety is defined by

$$L_p H_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad \text{for } k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is the space of algebraic p -cycles on X provided with a natural topology (see [3], [8]).

In [5], Friedlander and Mazur define a cycle class map

$$(1.1) \quad \Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X, \mathbb{Z})$$

for all $k \geq 2p \geq 0$. Suslin conjectures that $\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X, \mathbb{Z})$ is an injection for $k = n + p - 1$ and an isomorphism for $k \geq n + p$.

From now on, all the (co)homology have the rational coefficients. Let X be a complex projective variety of dimension 3. In this case, the second author showed in [6] that the Friedlander-Mazur conjecture [5, §7] can be reduced to (equivalent to):

Conjecture 1.2. $\Phi_{1,4}$ is surjective.

We want to study the relations of this conjecture to Grothendieck standard conjectures that are fundamental conjectures in the theory of algebraic cycles. We refer to [7] for the statements of standard conjectures. Recall that the standard conjectures reduce to the standard conjecture of the Lefschetz type.

We borrow notations from [1, Section 2.3]:

- $L(W)$: the Lefschetz type standard conjecture for W ;
- $l(W)$: for every $i > 0$, one can find a finite correspondence on W that yields an isomorphism $H^{\dim W+i}(W) \cong H^{\dim W-i}(W)$;
- $S(W)$: For any $j > 0$, one can find a finite correspondence $f_j : W \rightarrow Y_j$ with Y_j projective smooth of dimension j such that $f_j^* : H^j(Y_j) \rightarrow H^j(W)$ is surjective.

Date: November 4, 2021.

- Let $L(n)$ be the assertion that $L(W)$ is true for all W of dimension $\leq n$; same for $l(n), S(n)$.

Note that for any smooth projective variety Y with $\dim Y \leq 2$, $L(Y)$ is known. See [9] for example. Therefore $L(2)$ holds.

In [1, Section 2.3], Beilinson has proved that $S(n)$ implies $L(W)$ for any smooth projective variety W of dimension n via two steps:

- $S(n)$ and $L(n-1)$ implies $l(n)$;
- $l(W)$ implies $L(W)$.

However the same proof of the first step can be used to prove that $S(X)$, where X is a smooth projective threefold, implies $l(X)$ since $L(2)$ holds. Combining this with his proof of the second step $l(X) \Rightarrow L(X)$, we conclude that $S(X) \Rightarrow L(X)$ for a smooth projective threefold X .

On the other hand, the Friedlander-Mazur conjecture (1.2) holding for X implies that $S(X)$ via the proof of (i) \Rightarrow (iii) in [1, Proposition 2.2].

In summary, we have the following result.

Theorem 1.1. *The Friedlander-Mazur conjecture for a smooth projective threefold X implies that the Grothendieck standard conjecture holds for X .*

It is already known that Grothendieck standard conjecture holds for a projective threefold X implies that the Friedlander-Mazur conjecture for X (see [4, §4]). Hence the two conjectures are equivalent in dimension three.

We remark that the Friedlander-Mazur conjecture for all smooth projective varieties is equivalent to that the Grothendieck standard conjecture of the Lefschetz type holds. We emphasize here whether the Friedlander-Mazur conjecture for a given smooth projective variety X is equivalent to the Grothendieck standard conjecture of the Lefschetz type is still open in dimension four or higher.

2. EXAMPLES

In most earlier literatures, one verified the Grothendieck standard conjecture first, then use it to check the Friedlander-Mazur conjecture. For some threefolds, it is easier to check the Friedlander-Mazur conjecture than to check the standard conjectures. This is one of the main purpose in this note.

In [6], it has been proved that the Friedlander-Mazur conjecture holds for a smooth projective threefold X with $h^{2,0}(X) = 0$. Hence the standard conjectures also hold for such an X . For example, a smooth projective threefold with representable Chow group $\text{Ch}_0(X)$ of zero cycles satisfying $h^{2,0}(X) = 0$. Fano threefolds are examples in this case. A Calabi-Yau threefold X also satisfies $h^{2,0}(X) = 0$. The last two examples are of threefolds with the Kodaira dimension less than three, the Grothendieck standard conjectures have been proved by Tankeev [10] in a different method.

Surely there exist lots of threefolds X of general type satisfying $h^{2,0}(X) = 0$ but not a complete intersection or a product of a projective curve and surface. Concrete examples of this type can be found in [2] and references therein. They are new examples of threefolds for which the Grothendieck standard conjectures of the Lefschetz type hold.

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JIN CAO, YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA
Email address: caojin@mail.tsinghua.edu.cn

WENCHUAN HU, SCHOOL OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, CHINA
Email address: huwenchuan@gmail.com