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On deformations of Fano manifolds

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Abstract
In this paper, we provide new necessary and sufficient conditions for the existence of Kähler–Einstein metrics on small deformations of a Fano Kähler–Einstein manifold. We also show that the Weil–Petersson metric can be approximated by the Ricci curvatures of the canonical L^2 metrics on the direct image bundles. In addition, we describe the plurisubharmonicity of the energy functional of harmonic maps on the Kuranishi space of the deformation of compact Kähler–Einstein manifolds of general type.

1 Introduction

The existence of canonical metrics on compact complex manifolds is an important component in understanding the structure of the moduli spaces and metrics on them. Well-known examples include the Weil–Petersson metric on the moduli spaces of hyperbolic Riemann surfaces, and polarized Calabi–Yau manifolds. The classical approach to the Weil–Petersson metric is via the Kodaira–Spencer–Kuranishi theory. In this case, the Weil–Petersson metric is the natural L^2 metric induced by the Kähler–

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Einstein metrics and the harmonic representatives of Kodaira–Spencer classes. The advantage of this classical approach is that we can define the Weil–Petersson metric pointwisely on the Kuranishi space. This is indeed the case when we study the moduli spaces of Kähler–Einstein manifolds of general type. Although the moduli spaces are singular in general, the complex manifold corresponding to a point in the moduli space does admit a unique Kähler–Einstein metric, following the work of Yau [36].

On the other hand, when we study the deformation of a Fano manifold X_0, although the deformation of the complex structure on X_0 is unobstructed, there may not be any Kähler–Einstein metric on such a manifold. By the recent work of Chen–Donaldson–Sun [6–8] on the solution of the Yau's conjecture [38], we know that the existence of Kähler–Einstein metrics on such manifolds is equivalent to the K-stability.

For a Fano Kähler–Einstein manifold (X_0, ω_0) with discrete holomorphic automorphism group Aut (X_0), Koiso [15] showed in 1983 that each small deformation of X_0 admits a Kähler–Einstein metric by using the implicit function theorem. It is more subtle when Aut (X_0) is non-discrete. In the latter case, the existence of canonical metrics such as cscK or extremal metrics were studied by Székelyhidi [30], Brönnle [2], and Rollin–Simanca–Tipler [25] in terms of the Futaki invariant or the linear stability of the action of $\mathrm{Aut}_0\,(X_0)$ on the Kuranishi space of X_0.

In this paper, we study small deformations of Fano Kähler–Einstein manifolds and investigate the Weil–Petersson metric on their moduli spaces. Our first main result is the following new necessary and sufficient conditions for the existence of Kähler–Einstein metrics on small deformations of a Fano Kähler–Einstein manifold.

Theorem 1.1 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$, with $X_t = \pi^{-1}(t)$, be the Kuranishi family of X_0 with respect to ω_0. Then the following statements are equivalent:*

(1) *X_t admits a Kähler–Einstein metric for each $t \in B$.*
(2) *The dimension $h^0\left(X_t, T^{1,0}X_t\right)$ of the space of holomorphic vector fields on X_t is independent of t for all $t \in B$.*
(3) *The automorphism group $\mathrm{Aut}_0\,(X_t)$ is isomorphic to $\mathrm{Aut}_0\,(X_0)$ for each $t \in B$.*

Remark 1 Since $h^0\left(X_t, T^{1,0}X_t\right)$ is upper semi-continuous in t according to [14], Theorem 1.1 includes Koiso's result in [15] as a special case.

Remark 2 In [24], Phong and Sturm introduced a stability condition preventing possible jump of the dimension of the spaces of holomorphic vector fields in the limit metric to study the convergence of the Kähler–Ricci flow on Fano manifolds. This stability condition, namely Condition (B), was further explored in Phong–Song–Sturm–Weinkove [22]; see also [23].

Remark 3 It is well known that semisimple Lie algebras are rigid. Thus the third statement of Theorem 1.1 would follow from the second one directly if the Lie algebra $H^0\left(X_0, T^{1,0}X_0\right) \cong Lie\,(\mathrm{Aut}\,(X_0))$ is semisimple. However, $H^0\left(X_0, T^{1,0}X_0\right)$ is only reductive in general.

Returning to the study of the Weil–Petersson metric, in [10] Fujiki and Schumacher defined generalized Weil–Petersson metrics on the deformation space of a family of extremal manifolds by pushing down the curvature of relative line bundles over the total space. In particular, they showed that the generalized Weil–Petersson metric for a family of Kähler–Einstein manifolds coincides with the classical one. Essentially, assuming the family of Kähler–Einstein metrics is smooth, they showed that the curvature form of the Deligne pairing of the relative canonical bundle (or relative anti-canonical bundle) is precisely the Weil–Petersson curvature form; see also Schumacher [27]. In our case, when (X_0, ω_0) is a Fano Kähler–Einstein manifold with non-discrete automorphism group, the existence of such smooth family of Kähler–Einstein metrics is guaranteed by Theorem 1.1 above, provided each fiber X_t admits a Kähler–Einstein metric. In this case, the Weil–Petersson metric is well-defined. Namely, it is independent of the choices of fiberwise Kähler–Einstein metrics. On the other hand, the L^2 metrics on $R^0\pi_* K^{-k}_{\mathfrak{X}/B}$ and their curvatures do depend on such choices in general. Again, Theorem 1.1 assures that the L^2 metrics are well-defined.

In this paper, we show that the Weil–Petersson metric ω_{WP} can be approximated by the (normalized) Ricci curvatures of the L^2 metrics on the direct image bundles $R^0\pi_* K^{-k}_{\mathfrak{X}/B}$. More precisely, we have

Theorem 1.2 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family of X_0. We assume that each fiber X_t admits a Kähler–Einstein metric. Let $\Omega = \{\omega_t\}$ be any smooth family of Kähler–Einstein metrics. For each positive integer k, let $Ric_k = Ric\,(E_k, H_k)$ be the Ricci form of the L^2 metric $H_k\,(\Omega)$ on $E_k = R^0\pi_* K^{-k}_{\mathfrak{X}/B}$. Then*

$$\lim_{k\to\infty} \frac{\pi^n}{k^{n+1}} Ric_k = -\omega_{WP}.$$

Remark 4 We note that the above approximation is natural due to the work of Fujiki–Schumacher on the curvature of the Deligne pairing, and the Knudsen–Mumford expansion of the determinant bundle of the direct image sheaf $R^0\pi_* K^{-k}_{\mathfrak{X}/B}$ (see Knudsen–Mumford [11], Zhang [40], and Phong–Ross–Sturm [21]). In this paper, see Sect. 4, we take a more direct approach by establishing canonical local holomorphic sections of the direct image sheaves and the deformation of Kähler–Einstein metrics. This leads to a systematic way to derive integral formulas for the full curvature tensors of L^2 metrics. While the Kähler–Einstein metric on each X_t is analytical in nature, the advantage of our approach is that we can approximate the Weil–Petersson metric by using algebraic metrics on each fiber.

The paper is organized as follows. In Sect. 2, in order to give a simple criterion to check the existence of Kähler–Einstein metrics on small deformations of a Fano Kähler–Einstein manifold, we first show that, given a Fano Kähler–Einstein manifold (X_0, ω_0) and its Kuranishi family $(\mathfrak{X}, B, \pi)$ with respect to ω_0, the complex structure on $X_t = \pi^{-1}(t) \subset \mathfrak{X}$ is compatible with the symplectic form ω_0. In this case, the construction of the Kuranishi family is compatible with Donaldson's infinite dimensional GIT picture. One technical key ingredient is the equivalence of the Kuranishi gauge and the divergence gauge; see Theorem 2.2. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we investigate the Weil–Petersson metric and prove Theorem 1.2.

An integral formula of the full curvature tensor of the L^2 metric $H_k(\Omega)$ on E_k is also derived. In addition, we obtain the deformation formulas for the Kähler form ω_t and the volume form V_t on X_t, respectively, for each $t \in B$.

Finally, in the last section, we describe the plurisubharmonicity of the energy functionals of harmonic maps on the Kuranishi spaces of Kähler–Einstein manifolds of general type. It is known that this energy functional plays a crucial role in understanding the Weil–Petersson geometry of such manifolds. When $\mathcal{T}_g$ is the Teichmüller space of Riemann surfaces of genus $g \geq 2$ and (N, h) a Riemannian manifold with Hermitian nonpositive curvature, it was shown by Toledo [34] that the energy function is plurisubharmonic. Here, we consider the higher dimensional analogue. Assume $(\mathfrak{X}, B, \pi)$ is the Kuranishi family of a compact Kähler–Einstein manifold of general type, and let (N, h) be a Riemannian manifold with Hermitian nonpositive curvature. Let $E(t)$ be the energy of a harmonic map from X_t to N in a given homotopy class. By using the deformation theory established in [28] and the Siu–Sampson vanishing theorem in [26], we derive the first and second variation formulas of E and prove its plurisubharmonicity (Theorem 5.2).

2 The Kuranishi Gauge

In this section we derive some special properties of the Kuranishi gauge on a family of compact complex manifolds when the central fiber is a Fano Kähler–Einstein manifold. This leads to an explicit description of the action of the automorphism group of the central fiber on the Kuranishi space via differential geometric data.

For any Kähler metric g on a complex manifold M with local holomorphic coordinates $z_1, \ldots, z_n$, the corresponding Kähler form is

$$\omega_g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$, and the Ricci form is

$$\mathrm{Ric}\left(\omega_g\right) = -\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log\det\left(g_{i\bar{j}}\right).$$

We will often use g and ω_g interchangeably. Throughout this section we assume that (X, ω_0, J_0) is a Fano manifold with complex dimension $\dim_{\mathbb{C}} X = n \geq 2$. Here X is a fixed smooth manifold and we denote by $X_0 = (X, J_0)$ the corresponding complex manifold. Since the canonical line bundle K_{X_0} is negative, by the Serre duality and the Kodaira vanishing theorem, we have

$$H^{0,k}\left(X_0, T^{1,0} X_0\right) = 0 \tag{2.1}$$

for all $2 \leq k \leq n$. In particular, deformations of X_0 are unobstructed.

By the work of Kodaira–Spencer, we know that any almost complex structure J on X close to J_0 can be described by a unique Beltrami differential $\varphi \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$, and J is integrable if and only if

$$\overline{\partial}_0 \varphi = \frac{1}{2}[\varphi, \varphi] \tag{2.2}$$

where $\overline{\partial}_0$ is the $\overline{\partial}$-operator on X with respect to the complex structure J_0. In order to construct a complete family of small deformations of X_0, Kuranishi introduced another equation. Let $\Delta_\varepsilon^m \subset \mathbb{C}^m$ be the open ball with center 0 and radius ε. For any Beltrami differential $\varphi_1 \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$ with $\overline{\partial}_0\varphi_1 = 0$, there exists $\varepsilon > 0$ such that the equation

$$\varphi(t) = t\varphi_1 + \frac{1}{2}\overline{\partial}_0^* G_0 [\varphi(t), \varphi(t)] \tag{2.3}$$

has a unique power series solution $\varphi(t) = \sum_{i \geq 1} t^i \varphi_i \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$ which converges (in some appropriate Hölder norm) for all $t \in \Delta_\varepsilon^1$. Here, the Green's function G_0 and $\overline{\partial}_0^*$ are operators on X_0 with respect to the Kähler metric ω_0. It follows from the standard elliptic estimate and (2.1) that each $\varphi(t)$ satisfies the integrability equation (2.2) and defines a complex structure on X. We also note that

$$\overline{\partial}_0^* (\varphi(t) - t\varphi_1) = 0.$$

By using this construction and the Kodaira–Spencer theory, one can construct a Kuranishi family in the following way. We pick a basis $\varphi_1, \ldots, \varphi_m \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)$, where we use $\mathbb{H}$ to denote the harmonic space or harmonic projection with respect to the metric ω_0. Let $B = \Delta_\varepsilon^m \subset \mathbb{C}^m$ be a ball with coordinates $t = (t_1, \ldots, t_m)$ and denote by

$$\varphi(t) = \sum_{i=1}^{m} t_i \varphi_i + \sum_{|I| \geq 2} t^I \varphi_I \tag{2.4}$$

the unique solution of

$$\begin{cases} \overline{\partial}_0 \varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)], \\ \overline{\partial}_0^* \varphi(t) = 0, \\ \mathbb{H}_0(\varphi(t)) = \sum_{i=1}^m t_i \varphi_i \end{cases} \tag{2.5}$$

where $\mathbb{H}_0$ is the harmonic projection with respect to the metric ω_0.

We note that the second equation of (2.5) is the Kuranishi gauge condition, and the third equation characterizes the flat coordinate system around $0 \in B$ up to affine transformations.

Now we consider the smooth manifold

$$\mathfrak{X} = X \times B \tag{2.6}$$

and define an almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ in the following way: for each point $(p, t) \in \mathfrak{X}$, where $p \in X$ and $t \in B$, we let

$$\Omega^{1,0}_{(p,t)}\mathfrak{X} = (I + \varphi(t)) \left(\Omega^{1,0}_p X_0\right) \oplus \pi^* \Omega^{1,0}_t B, \tag{2.7}$$

where $\varphi(t)$ is given by (2.5). Kodaira and Spencer showed that this almost complex structure $\mathfrak{J}$ on $\mathfrak{X}$ is integrable and $\pi : \mathfrak{X} \to B$ is a Kuranishi family of X_0. For each $t \in B$, we let $X_t = \pi^{-1}(t)$ and denote the corresponding complex structure by J_t.

Thanks to the works of Kuranishi [16,17] and Wavrik [35], we have the following properties of the family $\pi : \mathfrak{X} \to B$; see also [5].

Theorem 2.1 *Let $\pi : \mathfrak{X} \to B$ be the Kuranishi family of X_0 constructed above. Then*

(1) *The Kuranishi family of X_0 parameterizes all small deformations of X_0 and is unique up to isomorphisms;*
(2) *$\pi : \mathfrak{X} \to B$ is semiuniversal at $0 \in B$;*
(3) *$\pi : \mathfrak{X} \to B$ is complete at each point $t \in B$;*
(4) *If $h^0\left(X_t, T^{1,0}X_t\right)$ is constant in $t \in B$, then the Kuranishi family is universal at each $t \in B$.*

In general, the complex structure J_t is not compatible with ω_0, which is viewed as a symplectic form on X. The compatibility property requires $\varphi(t) \lrcorner \omega_0 = 0$. Since $\overline{\partial}_0^* \varphi(t) = 0$, a direct computation shows that $\varphi(t) \lrcorner \omega_0 = 0$ if and only if $\mathrm{div}_0 \varphi(t) = 0$. This divergence gauge was introduced in [28,29], where it was shown that the Kuranishi gauge $\overline{\partial}_0^* \varphi(t) = 0$ is equivalent to the divergence gauge $\mathrm{div}_0 \varphi(t) = 0$ when the fibers X_t are Kähler–Einstein manifolds with negative or zero scalar curvature. In this section, we generalize this equivalence to the Fano case.

Theorem 2.2 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold.*

(1) *If $\varphi(t)$ is the unique solution of equations (2.5), then $div_0 \varphi(t) = 0$ and $\varphi(t) \lrcorner \omega_0 = 0$.*
(2) *If $\varphi \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$ is a Beltrami differential with $\overline{\partial}_0 \varphi = \frac{1}{2}[\varphi, \varphi]$ and $div_0 \varphi = 0$, then $\overline{\partial}_0^* \varphi = 0$ and $\varphi(t) \lrcorner \omega_0 = 0$.*

To prove this theorem, we need the following technical results.

Lemma 2.1 *Let $\left(X, \omega_g\right)$ be a Kähler manifold.*

(a) *If $\varphi \in A^{0,1}\left(X, T^{1,0}X\right)$ with $\overline{\partial}\left(\varphi \lrcorner \omega_g\right) = 0$ and $\overline{\partial}^* \varphi = 0$, then*

$$\Delta_{\overline{\partial}}\left(\varphi \lrcorner \omega_g\right) = \frac{\sqrt{-1}}{2} div\left(\overline{\partial}\varphi\right) + \varphi \lrcorner Ric\left(\omega_g\right).$$

(b) *If $\left(X, \omega_g\right)$ is Fano Kähler–Einstein, and $\eta \in A^{0,2}(X)$ such that $\overline{\partial}\eta = 0$ and $\Delta_{\overline{\partial}}\eta = \eta$, then $\eta = 0$.*

Proof The first claim (a) follows from direct computations; we refer the reader to [28, 29] for details. To prove the second claim, by the assumptions, we have the following Bochner formula,

$$\Delta|\eta|^2 = |\partial\eta|^2 + \left|\overline{\nabla}\eta\right|^2 .$$

This implies $\partial\eta = 0$. Since $\partial^*\eta = 0$, we conclude that $\Delta_\partial\eta = 0$. Thus

$$\eta = \Delta_{\overline{\partial}}\eta = \Delta_\partial\eta = 0.$$

□

Now we can prove Theorem 2.2.

Proof The proof of claim (2) is similar to that in [28]. Indeed, since $\mathrm{div}_0\varphi = 0$, we have

$$\begin{aligned} 0 = \overline{\partial}_0 (\mathrm{div}_0\varphi) =& \mathrm{div}_0 \left(\overline{\partial}_0\varphi\right) - 2\sqrt{-1}\varphi \lrcorner \mathrm{Ric}(\omega_0) \\ =& \frac{1}{2}\mathrm{div}_0 [\varphi, \varphi] - 2\sqrt{-1}\varphi \lrcorner \omega_0 \\ =& \varphi \lrcorner \partial_0 (\mathrm{div}_0\varphi) - 2\sqrt{-1}\varphi \lrcorner \omega_0 \\ =& - 2\sqrt{-1}\varphi \lrcorner \omega_0. \end{aligned}$$

Together with $\mathrm{div}_0\varphi = 0$, a direct computation shows that $\overline{\partial}_0^*\varphi = 0$.

Now we prove claim (1). Consider the power series (2.4) which satisfies equations (2.5). We will use induction on $|I|$ to show that $\mathrm{div}_0\varphi_I = 0$. If $|I| = 1$, then $\varphi_I = \varphi_i$ for some $1 \leq i \leq m$ which is harmonic. Thus

$$\overline{\partial}_0 (\varphi_i \lrcorner \omega_0) = 0 \quad \text{and} \quad \overline{\partial}_0^*\varphi_i = 0. \tag{2.8}$$

Then Lemma 2.1 implies that

$$\Delta_{\overline{\partial}_0} (\varphi_i \lrcorner \omega_0) = \frac{\sqrt{-1}}{2}\mathrm{div}_0 \left(\overline{\partial}_0\varphi_i\right) + \varphi_i \lrcorner \mathrm{Ric}(\omega_0) . \tag{2.9}$$

Since $\overline{\partial}_0\varphi_i = 0$ and $\mathrm{Ric}(\omega_0) = \omega_0$, we have

$$\Delta_{\overline{\partial}_0} (\varphi_i \lrcorner \omega_0) = \varphi_i \lrcorner \omega_0. \tag{2.10}$$

Again by Lemma 2.1, we know that $\varphi_i \lrcorner \omega_0 = 0$. Combining with $\overline{\partial}_0^*\varphi_i = 0$ we get $\mathrm{div}_0\varphi_i = 0$.

Now we assume $\operatorname{div}_0 \varphi_I = 0$ for all $|I| \le k-1$. For any multi-index I with $|I| = k$, we have

$$\begin{aligned}\overline{\partial}_0\left(\varphi_I \lrcorner \omega_0\right) &= \overline{\partial}_0 \varphi_I \lrcorner \omega_0 \\ &= \frac{1}{2}\sum_{J+K=I}\left[\varphi_J, \varphi_K\right] \lrcorner \omega_0 \\ &= \frac{1}{2}\sum_{J+K=I}\left(\varphi_J \lrcorner \partial_0\left(\varphi_K \lrcorner \omega_0\right) + \varphi_K \lrcorner \partial_0\left(\varphi_J \lrcorner \omega_0\right)\right) = 0.\end{aligned}$$

Since $\overline{\partial}_0^* \varphi_I = 0$, we conclude from Lemma 2.1 that

$$\begin{aligned}\Delta_{\overline{\partial}}\left(\varphi_I \lrcorner \omega_0\right) &= \frac{\sqrt{-1}}{2}\operatorname{div}_0\left(\overline{\partial}_0 \varphi_I\right) + \varphi_I \lrcorner \operatorname{Ric}\left(\omega_0\right) \\ &= \frac{\sqrt{-1}}{4}\left(\sum_{J+K=I}\left[\varphi_J, \varphi_K\right]\right) + \varphi_I \lrcorner \omega_0 \\ &= \frac{\sqrt{-1}}{4}\left(\sum_{J+K=I}\varphi_J \lrcorner \partial_0\left(\operatorname{div}_0 \varphi_K\right) + \varphi_K \lrcorner \partial_0\left(\operatorname{div}_0 \varphi_J\right)\right) + \varphi_I \lrcorner \omega_0 \\ &= \varphi_I \lrcorner \omega_0,\end{aligned}$$

where we have used the fact that $\operatorname{div}_0 \varphi_J = \operatorname{div}_0 \varphi_K = 0$ for all $|J|, |K| < |I|$. It then follows from Lemma 2.1 that $\varphi_I \lrcorner \omega_0 = 0$. Together with the assumption $\overline{\partial}_0^* \varphi_I = 0$, we conclude that $\operatorname{div}_0 \varphi_I = 0$. □

Remark 5 Let (X_0, ω_0) be a Fano manifold with $[\omega_0] = \pi c_1(X_0)$ and let $\pi : \mathfrak{X} \to B$ be a Kuranishi family of X_0 defined by (2.6) and (2.7) where $\varphi(t)$ is the unique solution of equations (2.5).

(1) For any Beltrami differentials $\varphi, \psi \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$ with $\varphi \lrcorner \omega_0 = 0$ or $\psi \lrcorner \omega_0 = 0$, the pointwise Hermitian inner product is given by

$$\varphi \cdot \overline{\psi} = \langle \varphi, \psi \rangle_g = \varphi^i_{\bar{j}} \overline{\psi^l_{\bar{k}}} g_{i\bar{l}} g^{k\bar{j}} = \varphi^i_{\bar{j}} \overline{\psi^j_{\bar{i}}}, \tag{2.11}$$

where g is the corresponding Kähler metric.

(2) If ω_0 is a Kähler–Einstein metric, then each J_t is compatible with ω_0. This implies that, in the Kähler–Einstein case, the Kuranishi gauge is compatible with Donaldson's infinite dimensional GIT picture. In fact, let $\omega = \omega_0$ be the symplectic form on X and let $\mathcal{J}^{int}$ be the space of integrable almost complex structures on X which are compatible with ω. Theorem 2.2 shows that B can be viewed naturally as a slice in $\mathcal{J}^{int}$ containing J_0 via Kuranishi's construction described above.

(3) Theorem 2.2 holds in more general situation if we allow appropriate twist. Let f be the normalized Ricci potential satisfying

$$\begin{cases} \mathrm{Ric}\,(\omega_0) = \omega_0 + \frac{\sqrt{-1}}{2}\partial_0\overline{\partial}_0 f \\ \int_{X_0} f\omega_0^n = 0. \end{cases}$$

If we define the twisted operators $\overline{\partial}_f^*$ and div_f with respect to the weighted volume form $e^f\frac{\omega_0^n}{n!}$, then the twisted Kuranishi gauge $\overline{\partial}_f^*\varphi(t) = 0$ is equivalent to the twisted divergence gauge $\mathrm{div}_f\varphi(t) = 0$. In particular, we still have $\varphi(t)\lrcorner\omega_0 = 0$. The proof is essentially the same as that of Theorem 2.2.

An immediate corollary of Theorem 2.2 is the explicit expression of a Ricci potential of the Kähler manifold (X_t, ω_0). This turns out to play an important role in the proof of Theorem 1.1 (Theorem 3.1). As above, let (X_0, ω_0) be a Fano Kähler–Einstein manifold, let $\varphi(t)$ be the solution of equation (2.5) and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family of (X_0, ω_0) constructed above. Then Theorem 2.2 implies that the symplectic form ω_0 is indeed a Kähler form on X_t.

Corollary 2.1 *A Ricci potential of the Kähler manifold* (X_t, ω_0) *is given by*

$$h_t = \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right). \tag{2.12}$$

Namely,

$$Ric\,(X_t, \omega_0) = \omega_0 + \frac{\sqrt{-1}}{2}\partial_t\overline{\partial}_t \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right) \tag{2.13}$$

where $\overline{\partial}_t$ *is the* $\overline{\partial}$*-operator on* X_t.

Proof We want to show that $-\partial_t\overline{\partial}_t \log\left(e^{h_t}\frac{\omega_0^n}{n!}\right) = \omega_0$. Fixing a point $t \in B$ and we let $\varphi = \varphi(t)$, $z = (z_1, \ldots, z_n)$ be local holomorphic coordinates on X_0, $w = (w_1, \ldots, w_n)$ be local holomorphic coordinates on X_t, $\omega_0 = \frac{\sqrt{-1}}{2}g_{i\overline{j}}dz_i \wedge d\overline{z}_j$, $g = \det\left[g_{i\overline{j}}\right]$, $A = [a_{\alpha i}] = \left[\frac{\partial w_\alpha}{\partial z_i}\right]$ and $B = \left[b^{i\alpha}\right] = A^{-1}$. Then

$$e^{h_t}\frac{\omega_0^n}{n!} = c_n\,|\det A|^{-2}\,g dw_1 \wedge \cdots \wedge dw_n \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n$$

where $c_n = (-1)^{\frac{n(n-1)}{2}}\left(\frac{\sqrt{-1}}{2}\right)^n$. On the other hand, we have

$$\frac{\partial w_\alpha}{\partial \overline{z}_j} = \varphi_{\overline{j}}^i a_{\alpha i}, \quad \frac{\partial z_i}{\partial w_\alpha} = (I - \varphi\overline{\varphi})^{ik}\, b^{k\alpha}, \quad \frac{\partial z_i}{\partial \overline{w}_\beta} = -\varphi_{\overline{j}}^i\overline{(I - \varphi\overline{\varphi})^{jl}}\,\overline{b^{l\beta}}, \tag{2.14}$$

where $(I-\varphi\overline{\varphi})^{ik}$ is the (i,k)-entry of the matrix $(I-\varphi\overline{\varphi})^{-1}$. By a direct computation, we have

$$-\frac{\partial^2}{\partial w_\alpha \partial \overline{w}_\beta}\log\left(c_n\left|\det A\right|^{-2}g\right)=\frac{\partial \overline{z}_l}{\partial \overline{w}_\beta}b^{k\alpha}\left[\frac{\partial}{\partial z_k}\varphi^p_{\bar{l}}\mu^p+R_{k\bar{l}}+\frac{\partial}{\partial z_k}\left((\mathrm{div}_0\varphi)_{\bar{l}}\right)\right]$$
$$-\frac{\partial \overline{z}_l}{\partial \overline{w}_\beta}b^{k\alpha}\left(\frac{\partial}{\partial \overline{z}_l}-\varphi^i_{\bar{l}}\frac{\partial}{\partial z_i}\right)\left(\mu^k\right), \tag{2.15}$$

where

$$\mu^k=(I-\varphi\overline{\varphi})^{ik}\left[\overline{\varphi^{j}_{\bar{i}}}\,(\mathrm{div}_0\varphi)_{\bar{j}}-\left(\overline{\mathrm{div}_0\varphi}\right)_i\right],$$

and $\frac{\sqrt{-1}}{2}R_{i\bar{j}}dz_i\wedge d\overline{z}_j$ is the Ricci form of (X_0,ω_0). Since ω_0 is a Kähler–Einstein metric on X_0, we have $R_{i\bar{j}}=g_{i\bar{j}}$. By Theorem 2.2, we know $\mathrm{div}_0\varphi=0$ which implies $\mu^k=0$. Hence the above formula reduces to

$$-\frac{\partial^2}{\partial w_\alpha \partial \overline{w}_\beta}\log\left(c_n\left|\det A\right|^{-2}g\right)=\frac{\partial \overline{z}_l}{\partial \overline{w}_\beta}b^{k\alpha}g_{k\bar{l}}. \tag{2.16}$$

It remains to show that

$$\frac{\sqrt{-1}}{2}\frac{\partial \overline{z}_l}{\partial \overline{w}_\beta}b^{k\alpha}g_{k\bar{l}}dw_\alpha\wedge d\overline{w}_\beta=\omega_0.$$

Again, by Theorem 2.2, we know $\varphi^i_{\bar{j}}g_{i\bar{l}}=\varphi^i_{\bar{l}}g_{i\bar{j}}$ and the above identity follows immediately from formula (2.14). □

Now we look at the action of the automorphism group of X_0 on the Kuranishi space B. For the rest of this section, we assume ω_0 is a Kähler–Einstein metric on X_0.

Let $G=\mathrm{Isom}_0(X_0,\omega_0)$ be the isometry group with Lie algebra $\mathfrak{g}$. By the work of Matsushima [20] and Calabi [3], we know that the complexification $G^{\mathbb{C}}$ of G is isomorphic to the holomorphic automorphism group $\mathrm{Aut}_0(X_0)$ and we have $\mathfrak{g}^{\mathbb{C}}\cong H^0\left(X_0,T^{1,0}X_0\right)$. Furthermore, if we let

$$\Lambda_1^{\mathbb{R}}=\left\{f\in C^\infty(X_0,\mathbb{R})\mid(\Delta_0+1)f=0\right\}$$

be the first eigenspace of the Laplacian on X_0 and let $\Lambda_1^{\mathbb{C}}=\Lambda_1^{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$, then we have

$$\mathfrak{g}\cong\left\{\mathrm{Im}\left(\nabla_0^{1,0}f\right)\middle|f\in\Lambda_1^{\mathbb{R}}\right\} \tag{2.17}$$

and

$$\mathfrak{g}^{\mathbb{C}}\cong\left\{\nabla_0^{1,0}f\middle|f\in\Lambda_1^{\mathbb{C}}\right\}. \tag{2.18}$$

The diffeomorphism group of X acts on the space of complex structures on X via pullback and thus acts locally on the set of Beltrami differentials on X_0 which satisfy the obstruction equation (2.2). Let $D \subset \mathrm{Diff}_0(X)$ be a neighborhood of the identity map and let $Y = (X, J)$ be a complex manifold obtained by deforming the complex structure J_0 via $\varphi \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$. We assume $\|\varphi\|$ is small and $\sigma \in D$. In [17] Kuranishi showed that the Beltrami differential $\psi = \varphi \circ \sigma$ corresponding to the complex structure $\sigma^* J$ is characterized by

$$\frac{\partial \sigma_k}{\partial \overline{z}_j} + \varphi_{\overline{l}}^k(\sigma(z)) \frac{\partial \overline{\sigma}_l}{\partial \overline{z}_j} = \psi_{\overline{j}}^i \left(\frac{\partial \sigma_k}{\partial z_i} + \varphi_{\overline{l}}^k(\sigma(z)) \frac{\partial \overline{\sigma}_l}{\partial z_i} \right), \tag{2.19}$$

where $z_1, \ldots, z_n$ are local holomorphic coordinates on X_0. It follows that

Corollary 2.2 *If $\sigma \in Aut_0(X_0)$ is a biholomorphism of X_0 then $\varphi \circ \sigma = \sigma^* \varphi$. If $\sigma \in Aut_0(Y)$ is a biholomorphism of Y then $\varphi \circ \sigma = \varphi$.*

Now we assume that $\sigma \in G \cap D$ is an isometry of (X_0, ω_0) and $\varphi(t)$ is a solution of equation (2.5). Then $\varphi(t) \circ \sigma = \sigma^* \varphi(t)$ satisfies the first two equations of (2.5) since σ preserves ω_0 and J_0. Thus, for each t with $|t|$ small, $\sigma^* \varphi(t) = \varphi\left(t'\right)$ where t' is characterized by $\sum_i t_i' \varphi_i = \mathbb{H}(\sigma^* \varphi(t))$. Let $V = T_0^{1,0} B \cong H^{0,1}\left(X_0, T^{1,0}X_0\right)$. If we linearize the above action with respect to φ, then we see that the linear action of G on $T_0^{1,0} B$, denoted by $\rho : G \to GL(V)$, is given by

$$\rho(\sigma)([\varphi]) = \left[\sigma^* \varphi\right]. \tag{2.20}$$

This is also true at the form level: $\sigma^* \varphi$ is harmonic when σ is an isometry and φ is harmonic. The representation ρ naturally extends to the representation $\rho^{\mathbb{C}} : G^{\mathbb{C}} \to GL(V)$ which is also given by (2.20). Now we linearize the representation ρ and we have the representation of Lie algebra $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ given by

$$\rho_*(v)([\varphi]) = [L_v \varphi]. \tag{2.21}$$

Again, this holds at the form level: $L_v \varphi$ is harmonic when $v \in \mathfrak{g}$ is a Killing field and φ is harmonic. This representation also extends to a representation $\rho_*^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}(V)$.

Remark 6 We note that, by the construction of Kuranishi family (2.6) and (2.7), both G and $G^{\mathbb{C}}$ act on $\mathfrak{X}$ holomorphically.

Note that if $v \in H^0\left(X_0, T^{1,0}X_0\right)$ is a holomorphic vector field and $\varphi, \psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)$ are harmonic Beltrami differentials, then by direct computations we have

$$\begin{aligned} &L_v \varphi = [v, \varphi], \\ &L_{\overline{v}} \varphi = \overline{\partial}(\overline{v} \lrcorner \varphi), \\ &[v, \varphi] \cdot \overline{\psi} = v\left(\varphi \cdot \overline{\psi}\right) - \mathrm{div}\left(\left(v \lrcorner \overline{\psi}\right) \lrcorner \varphi\right) + \left(v \lrcorner \overline{\psi}\right) \lrcorner (\mathrm{div} \varphi). \end{aligned} \tag{2.22}$$

Now we look at the representation $\rho_* : \mathfrak{g} \to \operatorname{End}(V)$. Let $\xi \in \mathfrak{g}$ be a Killing field. By the identification (2.18), there exists a unique eigenfunction $f \in \Lambda_1^{\mathbb{R}}$ such that $\xi = \operatorname{Im}(v)$, where

$$v = \nabla^{1,0} f \in H^0\left(X_0, T^{1,0}X_0\right).$$

For any harmonic Beltrami differentials $\varphi, \psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)$, we have

$$\langle L_\xi \varphi, \psi\rangle_{L^2} = \frac{1}{2\sqrt{-1}}\left(\langle L_v\varphi, \psi\rangle_{L^2} - \langle L_{\overline{v}}\varphi, \psi\rangle_{L^2}\right).$$

By (2.22), we know that

$$\langle L_{\overline{v}}\varphi, \psi\rangle_{L^2} = \int_{X_0}\left(\overline{\partial}\left(\overline{v}\lrcorner\varphi\right)\right)\cdot\overline{\psi}\, dV_g = 0$$

since ψ is harmonic. By integration by parts and Theorem 2.2, we have

$$\begin{aligned}
\langle L_v\varphi, \psi\rangle_{L^2} &= \int_{X_0}\left(v\left(\varphi\cdot\overline{\psi}\right) - \operatorname{div}\left(\left(v\lrcorner\overline{\psi}\right)\lrcorner\varphi\right) + \left(v\lrcorner\overline{\psi}\right)\lrcorner(\operatorname{div}\varphi)\right)\, dV_g\\
&= \int_{X_0} v\left(\varphi\cdot\overline{\psi}\right)\, dV_g = \int_{X_0}(\operatorname{div} v)\left(\varphi\cdot\overline{\psi}\right)\, dV_g\\
&= -\int_{X_0} f\left(\varphi\cdot\overline{\psi}\right)\, dV_g.
\end{aligned}$$

This implies

$$\langle L_\xi\varphi, \psi\rangle_{L^2} = \frac{\sqrt{-1}}{2}\int_{X_0} f\left(\varphi\cdot\overline{\psi}\right)\, dV_g.$$

Let

$$Q = \left\{\varphi\cdot\overline{\psi} \mid \varphi, \psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)\right\} \subset C^\infty(X_0). \tag{2.23}$$

Since $L_\xi\varphi$ is harmonic, we know that $L_\xi\varphi = 0$ if and only if $f \perp_{L^2} Q$.

In conclusion, we have proved the following

Corollary 2.3 *The representation ρ_* is trivial (and thus $\rho_*^{\mathbb{C}}$, ρ and $\rho^{\mathbb{C}}$ are trivial) if and only if $\Lambda_1^{\mathbb{R}} \perp_{L^2} Q$ (and thus $\Lambda_1^{\mathbb{C}} \perp_{L^2} Q$).*

3 Small deformation of Fano Kähler–Einstein manifolds

Throughout this section, we assume (X_0, ω_0) is a Fano Kähler–Einstein manifold and denote by $(\mathfrak{X}, B, \pi)$ the Kuranishi family with respect to ω_0 as constructed in

Sect. 2. An important question concerning the geometry of the moduli space of X_0 is the existence of Kähler–Einstein metrics on small deformations of (X_0, ω_0). By using the implicit function theorem, Koiso [15] showed that any small deformation of X_0 admits a Kähler–Einstein metric, provided the automorphism group of X_0 is discrete. The case that X_0 has non-trivial holomorphic vector fields is much more delicate. In [30] Szekelyhidi showed that a small deformation of a cscK manifold admits a cscK metric if and only if it is K-polystable. A similar result was established by Brönnle [2] in terms of the polystability of the action of the automorphism group on the Kuranishi space. Later, it was proved by Chen et al. [6–8] that the existence of a Kähler–Einstein metric on a Fano manifold X is equivalent to the K-stability of X. However, it is highly nontrivial to check the K-stability of a Fano manifold in general.

In this section, we provide new and simple necessary and sufficient conditions on the existence of Kähler–Einstein metrics on small deformations of X_0 as stated in Theorem 1.1.

Theorem 3.1 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family with respect to ω_0. By shrinking B if necessary, the following statements are equivalent:*

(1) *X_t admits a Kähler–Einstein metric for each $t \in B$;*
(2) *The dimension $h^0\left(X_t, T^{1,0}X_t\right)$ of the space of holomorphic vector fields on X_t is independent of t for all $t \in B$;*
(3) *The automorphism group $Aut_0\left(X_t\right)$ is isomorphic to $Aut_0\left(X_0\right)$ for all $t \in B$.*

Proof Firstly, we assume that X_t admits Kähler–Einstein metrics for each $t \in B$. By Remark 5, we know that ω_0 defines a Kähler metric on X_t. We shall show that

$$\mathrm{Isom}_0\left(X_0, \omega_0\right) = \mathrm{Isom}_0\left(X_t, \omega_0\right) \tag{3.1}$$

for each $t \in B$. Once we have this, then statements (2) and (3) follow from the upper semi-continuity of $h^0\left(X_t, T^{1,0}X_t\right)$ as a function of t [14]. Indeed, after shrinking B we can assume that $h^0\left(X_t, T^{1,0}X_t\right) \leq h^0\left(X_0, T^{1,0}X_0\right)$ for all $t \in B$. We know that $\left(\mathrm{Isom}_0\left(X_t, \omega_0\right)\right)^{\mathbb{C}}$ is a subgroup of $\mathrm{Aut}_0\left(X_t\right)$ for each t and $\left(\mathrm{Isom}_0\left(X_0, \omega_0\right)\right)^{\mathbb{C}} \cong \mathrm{Aut}_0\left(X_0\right)$. Since

$$\begin{aligned}
\dim_{\mathbb{R}} \mathrm{Isom}_0\left(X_t, \omega_0\right) \leq & \dim_{\mathbb{C}} \mathrm{Aut}_0\left(X_t\right) = h^0\left(X_t, T^{1,0}X_t\right) \\
\leq & h^0\left(X_0, T^{1,0}X_0\right) = \dim_{\mathbb{C}} \mathrm{Aut}_0\left(X_0\right) \\
= & \dim_{\mathbb{R}} \mathrm{Isom}_0\left(X_0, \omega_0\right),
\end{aligned}$$

identity (3.1) would imply that $h^0\left(X_t, T^{1,0}X_t\right) = h^0\left(X_0, T^{1,0}X_0\right)$ and $\left(\mathrm{Isom}_0\left(X_t, \omega_0\right)\right)^{\mathbb{C}} \cong \mathrm{Aut}_0\left(X_t\right)$ for all $t \in B$. It then follows that $\mathrm{Aut}_0\left(X_t\right) \cong \mathrm{Aut}_0\left(X_0\right)$ for all $t \in B$ since they are complexifications of the same compact Lie group.

To prove (3.1), it suffices to show that each isometry $\sigma \in G = \mathrm{Isom}_0\left(X_0, \omega_0\right)$, viewed as a diffeomorphism of X, is also an isometry of (X_t, ω_0). This will give us a natural embedding

$$G \hookrightarrow \mathrm{Isom}_0\,(X_t, \omega_0) \tag{3.2}$$

and (3.1) follows from the dimensional reason as above.

By the discussion in Sect. 2, since each isometry $\sigma \in G$ preserves both the Kähler–Einstein metric ω_0 and the complex structure J_0, it preserves all operators which are canonically associated to ω_0 and J_0. Hence, such σ maps each solution of equation (2.5) to another solution since these solutions are given by the Kuranishi equation (2.3). On the other hand, since a solution $\varphi(t)$ of equation (2.3) is determined by its harmonic part $\mathbb{H}\,(\varphi(t)) = \sum_{i=1}^m t_i \varphi_i$, it is enough to show that the action of G (or $G^{\mathbb{C}}$) on B, as described in Sect. 2, is trivial. Indeed, this would imply that, for each $t \in B$ and $\sigma \in G$, we have $\sigma^* \left(\sum_{i=1}^m t_i \varphi_i\right) = \sum_{i=1}^m t_i \varphi_i$ and thus, by the uniqueness of solution of the Kuranishi equation (2.3), we have $\sigma^* \varphi(t) = \varphi(t)$ which implies σ preserves the complex structure J_t. Since σ preserves ω_0, we conclude that $\sigma \in \mathrm{Isom}_0\,(X_t, \omega_0)$.

It remains to show that the action of $G^{\mathbb{C}}$ on B is trivial. Let us denote by

$$V = T_0^{1,0} B \cong H^{0,1}\left(X_0, T^{1,0} X_0\right)$$

as before. If the action of $G^{\mathbb{C}}$ on B is nontrivial, then there exists a subgroup $\lambda : \mathbb{C}^* \to G^{\mathbb{C}}$ whose action on V is nontrivial. We can then pick a basis $e_1, \ldots, e_m$ of V such that

$$\lambda(s)\,(e_i) = s^{\kappa_i} e_i, \quad s \in \mathbb{C}^*$$

with $\kappa_i \in \mathbb{Z}$ for each i. It follows that at least one of the κ_i's is nonzero. Replacing λ by λ^{-1} if necessary, we can assume $\kappa_i > 0$ for some i. Let

$$\Delta_\varepsilon = \left\{(0, \ldots, 0, t_i, 0, \ldots, 0) \,\middle|\, |t_i| < \varepsilon\right\} \subset B$$

be the one-dimensional disk, in the i-th coordinate line of the Kuranishi space, with center 0 and radius ε. We pick some $t' \in \Delta_\varepsilon^* = \Delta_\varepsilon \setminus \{0\}$.

Let $\mathfrak{X}' = \mathfrak{X}\,|_{\Delta_\varepsilon}$ and consider the subfamily $\left(\mathfrak{X}', \Delta_\varepsilon, \pi\right)$ with an action of $H = \{s \in C^* \mid |s| < 1\}$ on Δ_ε given by $\lambda(s)(t) = s^{\kappa_i} t$. We note that X_t is biholomorphic to $X_{t'}$ if $t \neq 0$ because of the action of H. Furthermore, X_0 is not biholomorphic to $X_{t'}$. To see this, we note that, by Theorem 2.1, the Kodaira–Spencer map $KS_t : T_t^{1,0} B \to H^{0,1}\left(X_t, T^{1,0} X_t\right)$ is an isomorphism if $t = 0$, and is surjective if $t \neq 0$. The above argument shows that the deformation of $X_{t'}$ is trivial along at least one direction due to the action of $\mathbb{C}^*$. Thus

$$h^{0,1}\left(X_{t'}, T^{1,0} X_{t'}\right) < h^{0,1}\left(X_0, T^{1,0} X_0\right).$$

This shows that X_0 is not biholomorphic to $X_{t'}$. By Remark 6, we also get the action of H on $\mathfrak{X}'$. The family $\left(\mathfrak{X}', \Delta_\varepsilon, \pi\right)$ naturally extends to a family $\left(\mathfrak{X}'', \mathbb{C}, \pi\right)$ with a $\mathbb{C}^*$ action on the base $\mathbb{C}$ with weight κ_i and the corresponding action on $\mathfrak{X}''$. By the standard argument of base change, we can assume $\kappa_i = 1$ and we get a nontrivial test configuration of $\left(X_{t'}, K_{X_{t'}}^{-k}\right)$, where the $\mathbb{C}^*$ action on the line bundle $K_{\mathfrak{X}''/\mathbb{C}}^{-k}$ is the induced one. Since $X_{t'}$ admits a Kähler–Einstein metric, it is K-polystable [1,32]. Now the central fiber of the nontrivial test configuration $\left(\mathfrak{X}'', \mathbb{C}, \pi\right)$ is X_0, which also admits a Kähler–Einstein metric and thus the Futaki invariant is 0. This is a contradiction, thus statements (2) and (3) hold.

Conversely, it is obvious that statement (3) implies (2), so it remains to show that (2) implies (1), namely if the dimension $h^0\left(X_t, T^{1,0}X_t\right)$ of the space of holomorphic vector fields on X_t is independent of t:

$$h^0\left(X_t, T^{1,0}X_t\right) = h^0\left(X_0, T^{1,0}X_0\right) = l \quad \text{for all } t \in B,$$

then each X_t admits a Kähler–Einstein metric. Pick a basis $\{v_1, \ldots, v_l\}$ of $H^0\left(X_0, T^{1,0}X_0\right)$. By the above assumption and the work of Kodaira [13], we can extend each v_i to $v_i(t) \in A^0\left(X_0, T^{\mathbb{C}}X_0\right)$ such that $v_i(t) \in H^0\left(X_t, T^{1,0}X_t\right)$ and $v_i(t)$ depends on t holomorphically. By continuity, and by shrinking B if necessary, we know that $\{v_1(t), \ldots, v_l(t)\}$ span $H^0\left(X_t, T^{1,0}X_t\right)$ for each $t \in B$.

Now we define a map

$$\tau_t : A^0\left(X_0, T^{1,0}X_0\right) \to A^0\left(X_t, T^{1,0}X_t\right)$$

by

$$\tau_t(v) = \left(I - \varphi(t)\overline{\varphi(t)}\right)^{-1}(v) - \overline{\varphi(t)}\left(\left(I - \varphi(t)\overline{\varphi(t)}\right)^{-1}(v)\right).$$

Then τ_t is a linear isomorphism for each $t \in B$. Let $\widetilde{v}_i(t) = \tau_t^{-1}(v_i(t))$. Since $\overline{\partial}_t v_i(t) = 0$, a direct computation shows that

$$\overline{\partial}_0 \widetilde{v}_i(t) = -\left[\widetilde{v}_i(t), \varphi(t)\right]. \tag{3.3}$$

Since $\varphi(0) = 0$, we have

$$\overline{\partial}_0\left(\left.\frac{\partial}{\partial t_k}\right|_{t=0} \widetilde{v}_i(t)\right) = \left.\frac{\partial}{\partial t_k}\right|_{t=0} \overline{\partial}_0 \widetilde{v}_i(t) = -\left.\frac{\partial}{\partial t_k}\right|_{t=0} \left[\widetilde{v}_i(t), \varphi(t)\right] = -\left[v_i, \varphi_k\right].$$

This implies that the cohomology class $[[v_i, \varphi_k]] = 0$ for all $1 \le i \le l$ and $1 \le k \le m$. Thus, by (2.22), the action of the Lie algebra $\mathfrak{g}$ on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$ given by (2.21) is trivial which implies that the G-action (2.20) on $T_0^{1,0}B \cong H^{0,1}\left(X_0, T^{1,0}X_0\right)$ is trivial. By the previous arguments, we have the identification $\mathrm{Isom}_0(X_0, \omega_0) = \mathrm{Isom}_0(X_t, \omega_0)$ for each $t \in B$.

We can now restrict our attention to G-invariant Kähler potentials and apply the implicit function theorem as in [2,18,25,30] (which can be further traced back, e.g., to the work of Donaldson–Kronheimer [9]). More specifically, by the work in [18] (see also Corollary 1 in [25]), the above identification $\mathrm{Isom}_0(X_0, \omega_0) = \mathrm{Isom}_0(X_t, \omega_0)$ leads to the existence of an extremal metric on each X_t. On the other hand, by Corollary 2.1, we know that $h_t = \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)$ is a Ricci potential of (X_t, ω_0).

It follows from Corollary 2.2 that each $\sigma \in \mathrm{Aut}_0(X_t)$ preserves $\varphi(t)$, hence the Ricci potential h_t is a σ-invariant function. Thus, for each $\xi \in H^0\left(X_t, T^{1,0}X_t\right)$, we have $\xi(h_t) = 0$ and the vanishing Futaki invariant [12]:

$$f_{X_t}(\omega_0, \xi) = \int_{X_t} \xi(h_t) \frac{\omega_0^n}{n!} = 0.$$

Therefore, the extremal metric on X_t must be a Kähler–Einstein metric. This proves statement (1) and concludes the proof of Theorem 3.1. □

Remark 7 As discussed in [4], under any of the equivalent conditions in Theorem 3.1, any Kähler–Einstein metric ω_0 on X_0 can be extended to a smooth family $\{\omega_t\}_{t\in B}$ such that ω_t is a Kähler–Einstein metric on X_t for each $t \in B$.

Remark 8 Szekelyhidi [30] showed that if X' is a sufficiently small deformation of a Fano Kähler–Einstein manifold X, then either X' admits a Kähler–Einstein metric or there is a test configuration for X' with smooth central fibre X''. Moreover, X'' admits a Kähler–Einstein metric and it is itself a small deformation of X. Combining this result of Szekelyhidi and the assumption that $h^0\left(X_t, T^{1,0}X_t\right)$ is independent of t, one can give an alternative proof of "(2) $\Longrightarrow$ (1)$''$ in Theorem 3.1.

Theorems 3.1 and 2.1 immediately imply the following universal property of the Kuranishi family.

Corollary 3.1 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family with respect to ω_0. If X_t admits a Kähler–Einstein metric for each $t \in B$, then the family $(\mathfrak{X}, B, \pi)$ is universal at each t.*

4 Curvature of the L^2 metrics on direct image sheaves

The Weil–Petersson metric is a L^2 metric on the parameter space of a family of complex manifolds which admit certain canonical metrics. It was first introduced by Weil to study the moduli spaces of hyperbolic Riemann surfaces based on the Petersson pairing. See, e.g., [4] for a brief survey on certain aspects of the Weil–Petersson metric.

In general, we consider a complex analytic family $(\mathcal{Y}, D, p)$ of compact complex manifolds, where $D \subset \mathbb{C}^m$ is the parameter space, and we let $Y_s = p^{-1}(s)$ for each point $s \in D$. If we assume that each fiber Y_s admits a Kähler–Einstein metric ω_s, then we can define the Weil–Petersson metric in the following way. For any $s \in D$ and $u, v \in T_s^{1,0}D$, we let $\varphi, \psi \in \mathbb{H}^{0,1}\left(Y_s, T^{1,0}Y_s\right)$ be the harmonic representatives of the Kodaira–Spencer classes $KS_s(u)$ and $KS_s(v)$ respectively, where we use the chosen Kähler–Einstein metric ω_s on Y_s to determine φ and ψ. Then the Weil–Petersson metric ω_{WP} is given by

$$h_s(u, v) = \int_{Y_s} \langle \varphi, \psi \rangle_{\omega_s} \frac{\omega_s^n}{n!}. \tag{4.1}$$

When Y_s is a Kähler–Einstein manifold of general type or a polarized Calabi–Yau manifold, there is a unique Kähler–Einstein metric on Y_s. Therefore, in this case, φ and ψ are uniquely determined and the Weil–Petersson metric is well-defined. Furthermore, for any submanifold $D' \subset D$, when we consider the restricted family $\left(\mathcal{Y}\mid_{D'}, D', p\right)$, the Weil–Petersson metric on D' defined by (4.1) is just the restriction of the Weil–Petersson metric on D to D'. In fact, one can define the canonical L^2 metric on $H^{0,1}\left(Y_s, T^{1,0}Y_s\right)$ in this case by using the unique Kähler–Einstein metric on Y_s, even when there are obstructions on deforming the complex structure on Y_s. This generalization of the classical Weil–Petersson metric plays an important role in studying the moduli space of Y_s.

Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family constructed in Sect. 2. We assume that each X_t admits a Kähler–Einstein metric. Then, by Theorem 3.1 and Remark 7, we know that each Kähler–Einstein metric on X_0 can be extended to a smooth family of Kähler–Einstein metrics. In this case it is not hard to show that the Weil–Petersson metric is well-defined, namely it is independent of the choice of Kähler–Einstein metrics on each X_t. In fact, following the classical approach, if $\{\omega(s)\}$ is any family of Kähler–Einstein metrics on X_0 and let φ_s and ψ_s be harmonic representatives of any two given Kodaira–Spencer classes with respect to $\omega(s)$. Then a simple computation shows that

$$\frac{d}{ds}\bigg|_{s=0} \langle \varphi_s, \psi_s \rangle_{L^2(\omega(s))} = 0$$

for all choices of $\{\omega(s)\}$ if and only if $\left\{\varphi_i \overline{\varphi}_j\right\} \perp_{L^2(\omega(0))} \Lambda_1^{\mathbb{R}}(\omega(0))$ and the latter condition is guaranteed by Theorem 3.1 and Corollary 2.3.

Now we turn our attention to the approximation of the Weil–Petersson metric. Given a smooth family $(\mathfrak{X}, B, \pi)$ of Kähler–Einstein manifolds of general type, it was shown in [28] that the Ricci curvatures of the L^2 metrics, induced by the fiberwise Kähler–Einstein metrics on the direct image bundle $R^0\pi_* K^k_{\mathfrak{X}/B}$, converge to the Weil–Petersson metric after an appropriate normalization. There are two steps involved in establishing the curvature formula of the L^2 metrics. The first step is to extend sections in $H^0\left(X_0, K^k_{X_0}\right)$ to $H^0\left(X_t, K^k_{X_t}\right)$ in a canonical way in order to obtain local holomorphic sections of $R^0\pi_* K^k_{\mathfrak{X}/B}$. We note that the background smooth pair of $\left(X_t, K_{X_t}\right)$ is independent of t. In [28], a slight different notion inspired by the work of Todorov [33] was used. This technique can be directly applied to more general situations, e.g., a family $(\mathcal{Y}, D, p)$ of compact complex manifolds and a relative ample holomorphic line bundle $\mathcal{L}$ over $\mathcal{Y}$. In [29], this idea was used to construct local holomorphic sections of the bundle over D whose fiber at $s \in D$ is $H^0\left(Y_s, \left(\mathcal{L}^k\mid_{Y_s}\right)\right)$. The second step is to find deformation of the Kähler–Einstein metrics with respect to the Kuranishi-divergence gauge.

It turns out that similar results hold in our situation if we replace the relative canonical bundle used in [28] by the relative anti-canonical bundle. Let (X_0, ω_0) be a Fano Kähler–Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family with respect to ω_0. Note that for any positive integer k, by the Serre duality and Kodaira vanishing theorems, we have

$$h^i\left(X_t, K_{X_t}^{-k}\right) = h^{n-i}\left(X_t, K_{X_t}^{k+1}\right) = 0 \tag{4.2}$$

for all $1 \leq i \leq n$ since $K_{X_t}^{k+1}$ is negative. Thus, by the Riemann–Roch theorem, we know that $h^0\left(X_t, K_{X_t}^{-k}\right)$ remain constant for all $t \in B$. This implies that the direct image sheaf $R^0\pi_* K_{\mathfrak{X}/B}^{-k}$ is a holomorphic vector bundle over B . We denote this bundle by E_k and its rank by N_k.

Similar to the work in [33], we define the linear map $\sigma_t : A^0\left(X_0, K_{X_0}^{-k}\right) \to A^0\left(X_t, K_{X_t}^{-k}\right)$ by

$$\sigma_t(s) = \left(\det\left(I - \varphi(t)\overline{\varphi(t)}\right)\right)^{-k}\left(s^{\frac{1}{k}} \lrcorner e^{-\overline{\varphi(t)}}\right)^k. \tag{4.3}$$

It is easy to see that σ_t is well-defined and is an isomorphism if $|t|$ is small. Furthermore, a direct computation shows that $\sigma_t(s)$ is a holomorphic section of $K_{X_t}^{-k}$ if and only if

$$\overline{\partial}_0 s = \varphi(t) \lrcorner \nabla_0 s, \tag{4.4}$$

where ∇_0 is the metric connection on $K_{X_0}^{-k}$ induced by the Kähler–Einstein metric on X_0. By using (4.2), equation (4.4) can be solved inductively. Indeed, given any holomorphic section $s \in H^0\left(X_0, K_{X_0}^{-k}\right)$, we look for a power series solution

$$s(t) = s + \sum_{|I|\geq 1} s_I t^I \in A^0\left(X_0, K_{X_0}^{-k}\right) \tag{4.5}$$

to equation (4.4) with normalization $\mathbb{H}_0(s(t)) = s$. By induction, it is not hard to see that

$$s_I = \overline{\partial}_0^* G_0\left(\sum_{J+K=I} \varphi_J \lrcorner \nabla_0 s_K\right). \tag{4.6}$$

Furthermore, standard elliptic estimates imply that the power series (4.5) converges in any $C^{p,\alpha}$ norm when t is sufficiently small. Similar to the work in [28], we have

Theorem 4.1 *For any holomorphic section $s \in H^0\left(X_0, K_{X_0}^{-k}\right)$, the power series solution (4.5) satisfies $\mathbb{H}_0(s(t)) = s$ and $\sigma_t(s(t)) \in H^0\left(X_t, K_{X_t}^{-k}\right)$ for each $t \in B$. Furthermore, by shrinking B if necessary, if $\{s_i\}_{1\leq i\leq N_k} \subset H^0\left(X_0, K_{X_0}^{-k}\right)$ is a basis then $\{\sigma_t(s_i(t))\}_{1\leq i\leq N_k} \subset H^0\left(X_t, K_{X_t}^{-k}\right)$ is also a basis for all $t \in B$.*

Remark 9 A direct computation shows that

$$\begin{aligned}\varphi_J \lrcorner \nabla_0 s_K =& \operatorname{div}_0\left(\varphi_J \otimes s_K\right) - \left(\operatorname{div}_0 \varphi_J\right) \otimes s_K \\ =& \operatorname{div}_0\left(\varphi_J \otimes s_K\right),\end{aligned}$$

where the last equality follows from Theorem 2.2. Thus formula (4.6) is equivalent to

$$s_I = \overline{\partial}_0^* G_0 \left(\sum_{J+K=I} \mathrm{div}_0 \left(\varphi_J \otimes s_K \right) \right). \tag{4.7}$$

If we assume each X_t admits a Kähler–Einstein metric g_t with volume form V_t, then the L^2 metric $H_k(V)$ on $E_k = R^0\pi_* K_{\mathfrak{X}/B}^{-k}$ is given by

$$\langle s_1, s_2 \rangle_{H_k(V)} = \int_{X_t} \langle s_1, s_2 \rangle_{g_t^k} dV_t \tag{4.8}$$

for each $t \in B$ and $s_1, s_2 \in H^0\left(X_t, K_{X_t}^{-k}\right)$, where $V = \{V_t\}_{t\in B}$ and g_t^k is the metric on $K_{X_t}^{-k}$ induced by the Kähler–Einstein metric g_t on X_t. It is clear that the L^2 metric $H_k(V)$ on E_k depends on the choice of the smooth family of fiberwise Kähler–Einstein metrics.

In order to compute the curvature of the L^2 metric, we need the deformation formulas of V_t. By using the Kuranishi-divergence gauge, we view each X_t as the background smooth manifold X equipped with the complex structure J_t obtained by deforming the complex structure on X_0 via $\varphi(t)$. Thus we can view $\{V_t\}_{t\in B}$ as families of differential forms on X. Similar to the work in [28], by deforming the corresponding Monge–Ampére equation, we have

Theorem 4.2 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold with Kuranishi family $(\mathfrak{X}, B, \pi)$. We assume that each X_t admits a Kähler–Einstein metric. Let $V = \{V_t\}$ be a smooth family of Kähler–Einstein volume forms and we write $V_t = e^\rho \det\left(I - \varphi(t)\overline{\varphi(t)}\right) V_0$ for some $\rho \in C^\infty(X_0, \mathbb{R})$. Then ρ has an expansion of the form*

$$\rho = \sum_i t_i \rho_i + \sum_j \overline{t}_j \overline{\rho}_j + \sum_{i,j} t_i \overline{t}_j \rho_{i\overline{j}} + O(t_i t_k) + O\left(\overline{t}_j \overline{t}_l\right) + O\left(|t|^3\right), \tag{4.9}$$

where

(1) $(\Delta_0 + 1)\rho_i = 0$;

(2) $(\Delta_0 + 1)\rho_{i\overline{j}} = \varphi_i \cdot \overline{\varphi}_j - g_0^{\alpha\overline{\beta}} g_0^{\gamma\overline{\delta}} \partial_\alpha \partial_{\overline{\delta}} \rho_i \partial_\gamma \partial_{\overline{\beta}} \overline{\rho}_j$.

Remark 10 Since the Kähler forms $\{\omega_t\}$ of the Kähler–Einstein metrics $\{g_t\}$ are given by $\omega_t = -\frac{\sqrt{-1}}{2}\partial_t\overline{\partial}_t \log V_t$, formula (4.9) also leads to the expansion of the Kähler–Einstein Kähler forms $\{\omega_t\}$. Furthermore, we can also eliminate both the $t_i t_k$ and $\overline{t}_j \overline{t}_l$ terms in the expansion (4.9) by modifying V via a biholomorphism of $\mathfrak{X}$. However, we do not need these facts in the following discussion.

To effectively compute the curvature of the L^2 metric $\langle\cdot,\cdot\rangle_{H_k(V)}$ for any chosen family of Kähler–Einstein volume forms V, we need to adjust the total space $\mathfrak{X}$ without altering the Kuranishi gauge. We shall consider a certain special biholomorphism

$F : \mathfrak{X} \to \mathfrak{X}$ which covers the identity map of B and we let $F_t = F\,|_{X_t} \in \operatorname{Aut}(X_t)$. By Corollary 2.2, we know that F_t preserves $\varphi(t)$. Such a map F would induce a biholomorphic bundle map $\widetilde{F} : E_k \to E_k$, which is indeed a Hermitian isometry $\widetilde{F} : \left(E_k, \langle\cdot,\cdot\rangle_{H_k(V)}\right) \to \left(E_k, \langle\cdot,\cdot\rangle_{H_k((F^{-1})^*V)}\right)$.

For any smooth family V of Kähler–Einstein volume forms, we let $\rho = \rho^V$ be the function as in Theorem 4.2. The family V is said to be normalized if $\rho_i^V = 0$ for each i. We now construct the special biholomorphism F of $\mathfrak{X}$, covering the identity map of B, such that F^*V is normalized. For a given family V, by Theorem 4.2 we know that $\rho_i^V \in \Lambda_1^{\mathbb{C}}$ is an eigenfunction of $\Delta_0 + 1$, hence $\mu_i = \nabla_0^{1,0}\rho_i^V \in H^0\left(X_0, T^{1,0}X_0\right)$ is a holomorphic vector field on X_0. Since we have assumed that each X_t admits a Kähler–Einstein metric, by Theorem 3.1 we know that $h^0\left(X_0, T^{1,0}X_0\right) = h^0\left(X_t, T^{1,0}X_t\right)$ for each $t \in B$. It follows from Kodaira's stability theorem that each μ_i can be extended to a family $\mu_i(t)$ of vector fields such that

(i) $\mu_i(t) \in H^0\left(X_t, T^{1,0}X_t\right)$ for each $t \in B$, and
(ii) $\mu_i(t)$ depends on t holomorphically.

We let $\mu(t) = \sum_i t_i \mu_i(t) \in H^0\left(\mathfrak{X}, T^{1,0}_{\mathfrak{X}/B}\right)$ and let F be the time-one flow of $\mu(t)$. Since $\left.\frac{\partial}{\partial t_i}\right|_{t=0} F = \mu_i$ and $\operatorname{div}_0\mu_i = -\Delta_0\rho_i^V = \rho_i^V$, it follows from direct computations that F^*V is normalized. Thus, to compute the curvature of $\left(E_k, \langle\cdot,\cdot\rangle_{H_k(V)}\right)$, we can always assume that V is normalized. In this case, it follows from Theorem 4.2 and $\rho_i^V = 0$ that $(\Delta_0 + 1)\,\rho_{i\bar{j}}^V = \varphi_i \cdot \overline{\varphi}_j$. We denote by $(\Delta_0 + 1)^{-1}\left(\varphi_i \cdot \overline{\varphi}_j\right)$ the unique solution of this equation which is perpendicular to $\Lambda_1^{\mathbb{C}}$. It then follows that

$$\rho_{i\bar{j}}^V = (\Delta_0 + 1)^{-1}\left(\varphi_i \cdot \overline{\varphi}_j\right) + \nu_{i\bar{j}}^V \tag{4.10}$$

for some $\nu_{i\bar{j}}^V \in \Lambda_1^{\mathbb{C}}$.

The above discussion leads to the approximation of the Weil–Petersson metric ω_{WP} on the parameter sapce B by the Ricci curvatures of the L^2 metrics. Such approximations can be seen via the Knudsen–Mumford expansion [11,21,40] and the work of Schumacher [27]. Here we give a simple and direct proof. Moreover, our method gives the curvature tensor of the L^2 metrics on the direct image sheaves rather than their determinant bundles.

In the following, we will use $\Box_0$ to denote the Hodge Laplacian on bundles over X_0 with respect to metrics induced by the Kähler–Einstein metric ω_0.

Theorem 4.3 *Let (X_0, ω_0) be a Fano Kähler–Einstein manifold with Kuranishi family $(\mathfrak{X}, B, \pi)$. We assume that each X_t admits a Kähler–Einstein metric. Let $\left\{s_\alpha\right\} \subset H^0\left(X_0, K_{X_0}^{-k}\right)$ be a basis, V be a smooth family of Kähler–Einstein volume forms, and $Ric_k = Ric\left(E_k, H_k(V)\right)$. Then*

$$\lim_{k\to\infty} \frac{\pi^n}{k^{n+1}} Ric_k = -\omega_{WP}. \tag{4.11}$$

Proof By the above discussion, we can assume that V is normalized. Let $\nu_{i\overline{j}} = \nu^V_{i\overline{j}}$ be the function given by equation (4.10). We first show that the curvature tensor of the L^2 metric $H_k(V)$ on E_k is given by

$$\begin{aligned} R_{\alpha\overline{\beta}i\overline{j}}(0) = (k+1)\int_{X_0}\langle(\Box_0+k+1)^{-1}\left(\varphi_i\otimes s_\alpha\right),\varphi_j\otimes s_\beta\rangle_{g_0}dV_0 \\ -(k+1)\int_{X_0}\langle s_\alpha, s_\beta\rangle_{g_0^k}\left((\Delta_0+1)^{-1}\left(\varphi_i\cdot\overline{\varphi}_j\right)+\nu_{i\overline{j}}\right)dV_0. \end{aligned} \tag{4.12}$$

To prove this formula, since the curvature of H_k is tensorial, we can use the local sections of E_k constructed in Theorem 4.1 to compute it. For each s_α, let $s_\alpha(t) \subset A^0\left(X_0, K_{X_0}^{-k}\right)$ be the sections constructed by formulas (4.5) and (4.6) and let $h_{\alpha\overline{\beta}}(t) = \langle\sigma_t\left(s_\alpha(t)\right),\sigma_t\left(s_\beta(t)\right)\rangle_{H_k(V)}$. By Theorems 4.1 and 4.2 , we have

$$h_{\alpha\overline{\beta}}(t) = \int_{X_0}\langle s_\alpha(t), s_\beta(t)\rangle_{g_0^k}e^{(k+1)\rho}\det\left(I-\varphi(t)\overline{\varphi(t)}\right)dV_0,$$

where ρ is the function defined by

$$V_t = e^\rho\det\left(I-\varphi(t)\overline{\varphi(t)}\right)V_0.$$

Since V is normalized, by formula (4.7), we have

$$\begin{aligned} \left.\frac{\partial h_{\alpha\overline{\beta}}}{\partial t_i}\right|_{t=0} &= \int_{X_0}\langle\overline{\partial}_0^*G_0\mathrm{div}_0\left(\varphi_i\otimes s_\alpha\right), s_\beta\rangle_{g_0^k}dV_0 \\ &= \int_{X_0}\langle G_0\mathrm{div}_0\left(\varphi_i\otimes s_\alpha\right), \overline{\partial}_0 s_\beta\rangle_{g_0^k}dV_0 = 0, \end{aligned} \tag{4.13}$$

because s_β is holomorphic. Similarly, we have $\left.\frac{\partial h_{\alpha\overline{\beta}}}{\partial\overline{t}_j}\right|_{t=0} = 0$ and

$$\begin{aligned} \left.\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial t_i\partial\overline{t}_j}\right|_{t=0} =&\langle\overline{\partial}_0^*G_0\mathrm{div}_0\left(\varphi_i\otimes s_\alpha\right),\overline{\partial}_0^*G_0\mathrm{div}_0\left(\varphi_j\otimes s_\beta\right)\rangle_{L^2} \\ &+\int_{X_0}\langle s_\alpha, s_\beta\rangle_{g_0^k}\left((k{+}1)\left((\Delta_0{+}1)^{-1}\left(\varphi_i\cdot\overline{\varphi}_j\right)+\nu_{i\overline{j}}\right)-\left(\varphi_i\cdot\overline{\varphi}_j\right)\right)dV_0. \end{aligned} \tag{4.14}$$

Now we analyze the first term on the right hand side of the above formula. Note that, by the proof of Theorem 4.1, we have

$$\begin{aligned} \overline{\partial}_0\mathrm{div}_0\left(\varphi_i\otimes s_\alpha\right) =&\overline{\partial}_0\left(\varphi_i\lrcorner\nabla_0 s_\alpha\right) \\ =&\overline{\partial}_0\varphi_i\lrcorner\nabla_0 s_\alpha - 2k\sqrt{-1}\left(\varphi_i\lrcorner\omega_0\right)\otimes s_\alpha = 0. \end{aligned}$$

It follows that

$$\overline{\partial}_0 G_0 \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right) = 0. \tag{4.15}$$

Integrating by parts, we get

$$\langle \overline{\partial}_0^* G_0 \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right), \overline{\partial}_0^* G_0 \mathrm{div}_0 \left(\varphi_j \otimes s_\beta\right)\rangle_{L^2} = \langle \mathrm{div}_0^* G_0 \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right), \varphi_j \otimes s_\beta\rangle_{L^2}.$$

By using Eq. (4.15) and the fact that $\overline{\partial}_0 \left(\varphi_i \otimes s_\alpha\right) = 0$, a simple computation shows that

$$\begin{aligned} \mathrm{div}_0^* G_0 \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right) &= \left(\Box_0 + k + 1\right)^{-1} \left(\mathrm{div}_0^* \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right)\right) \\ &= \left(\Box_0 + k + 1\right)^{-1} \Box_0 \left(\varphi_i \otimes s_\alpha\right) \\ &= \varphi_i \otimes s_\alpha - (k+1)\left(\Box_0 + k + 1\right)^{-1} \left(\varphi_i \otimes s_\alpha\right). \end{aligned}$$

Thus

$$\begin{aligned} &\langle \overline{\partial}_0^* G_0 \mathrm{div}_0 \left(\varphi_i \otimes s_\alpha\right), \overline{\partial}_0^* G_0 \mathrm{div}_0 \left(\varphi_j \otimes s_\beta\right)\rangle_{L^2} \\ &= \langle \varphi_i \otimes s_\alpha, \varphi_j \otimes s_\beta \rangle_{L^2} - (k+1) \langle \left(\Box_0 + k + 1\right)^{-1} \left(\varphi_i \otimes s_\alpha\right), \varphi_j \otimes s_\beta\rangle_{L^2}. \end{aligned}$$

Inserting this into equation (4.14), we get

$$\begin{aligned} \left.\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial t_i \partial \overline{t}_j}\right|_{t=0} &= (k+1) \int_{X_0} \langle s_\alpha, s_\beta \rangle_{g_0^k} \left(\left(\Delta_0 + 1\right)^{-1} \left(\varphi_i \cdot \overline{\varphi}_j\right) + \nu_{i\overline{j}} \right) dV_0 \\ &\quad - (k+1) \int_{X_0} \langle \left(\Box_0 + k + 1\right)^{-1} \left(\varphi_i \otimes s_\alpha\right), \varphi_j \otimes s_\beta \rangle_{g_0} dV_0. \end{aligned} \tag{4.16}$$

The curvature formula (4.12) of the metric $H_k(V)$ now follows easily from the above formula and Eq. (4.13).

To estimate the limit of Ricci curvatures, we take any vector $v \in T_0^{1,0} B$. By rotation and scaling, we can assume $v = \frac{\partial}{\partial t_1}$. Let $\left\{s_\alpha\right\} \subset H^0\left(X_0, K_{X_0}^{-k}\right)$ be an orthonormal basis with respect to the L^2 metric. By formula (4.12), we have

$$\begin{aligned} \frac{1}{k+1} \mathrm{Ric}_k(v, v) = \sum_\alpha \int_{X_0} &\langle \left(\Box_0 + k + 1\right)^{-1} \left(\varphi_1 \otimes s_\alpha\right), \varphi_1 \otimes s_\alpha \rangle_{g_0} dV_0 \\ &- \int_{X_0} \tau_k \left(\left(\Delta_0 + 1\right)^{-1} \left(|\varphi_1|^2 \right) + \nu_{1\overline{1}} \right) dV_0, \end{aligned} \tag{4.17}$$

where $\tau_k = \sum_\alpha \|s_\alpha\|^2_{g_0^k}$ is the Bergman kernel function. Since the operator $\Box_0 + k + 1$ is self-adjoint and its first eigenvalue is at least $k+1$, we have

$$\begin{aligned} 0 &\leq \sum_\alpha \int_{X_0} \langle (\Box_0 + k + 1)^{-1} \left(\varphi_1 \otimes s_\alpha\right), \varphi_1 \otimes s_\alpha \rangle_{g_0} dV_0 \\ &\leq \sum_\alpha \frac{1}{k+1} \int_{X_0} \langle \varphi_1 \otimes s_\alpha, \varphi_1 \otimes s_\alpha \rangle_{g_0} dV_0 \\ &= \sum_\alpha \frac{1}{k+1} \int_{X_0} |\varphi_1|^2 \|s_\alpha\|^2_{g_0^k} dV_0 = \frac{1}{k+1} \int_{X_0} \tau_k |\varphi_1|^2 dV_0. \end{aligned}$$

Combining the above inequality with equation (4.17), we have

$$\begin{aligned} 0 \leq & \frac{1}{k+1} \mathrm{Ric}_k (v, v) + \int_{X_0} \tau_k (\Delta_0 + 1)^{-1} \left(|\varphi_1|^2\right) dV_0 \\ \leq & \frac{1}{k+1} \int_{X_0} \tau_k |\varphi_1|^2 dV_0. \end{aligned} \tag{4.18}$$

By the Bergman kernel expansion (see [19,31,37,39])

$$\tau_k = \frac{k^n}{\pi^n} + \frac{n k^{n-1}}{2\pi^n} + O\left(k^{n-2}\right)$$

and the fact that

$$\omega_{WP}(v, v) = \int_{X_0} (\Delta_0 + 1)^{-1} \left(|\varphi_1|^2\right) dV_0,$$

we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{\pi^n}{k^n} \int_{X_0} \tau_k \left((\Delta_0 + 1)^{-1} \left(|\varphi_1|^2\right) + \nu_{1\bar{1}}\right) dV_0 \\ &= \int_{X_0} \left(1 + \frac{n}{2k} + O\left(k^{-2}\right)\right) (\Delta_0 + 1)^{-1} \left(|\varphi_1|^2\right) dV_0 + \lim_{k\to\infty} \frac{\pi^n}{k^n} \int_{X_0} \tau_k \nu_{1\bar{1}} dV_0 \\ &= \omega_{WP}(v, v) - \lim_{k\to\infty} \int_{X_0} \left(1 + \frac{n}{2k} + O\left(k^{-2}\right)\right) \Delta_0 \nu_{1\bar{1}} dV_0 \\ &= \omega_{WP}(v, v) \end{aligned}$$

and

$$\lim_{k\to\infty} \frac{\pi^n}{k^n} \left(\frac{1}{k+1} \int_{X_0} \tau_k |\varphi_1|^2 dV_0\right) = 0.$$

Thus, (4.11) follows from inequality (4.18) and the above limits directly. $\square$

5 Plurisubharmonicity of energy of harmonic maps

Another application of the deformation of Kähler–Einstein metrics, such as Theorem 4.2, is the variation of energy of harmonic maps. In [34], Toledo studied the harmonic maps from hyperbolic Riemann surfaces to a fixed Riemannian manifold (N, h). For a Riemann surface Σ, fixing a homotopy class A of continuous maps from Σ to N and assuming that the sectional curvature of N is nonpositive, there exist smooth harmonic maps from Σ to N in the homotopy class A. Although such harmonic maps may not be unique, the energy depends only on the conformal structure of Σ, thus one obtains an energy function E on the Teichmüller space $\mathcal{T}$ of Σ. Toledo showed that if one further assumes that the curvature of N is Hermitian nonpositive, then E is a plurisubharmonic function on $\mathcal{T}$. Shortly after Toledo's work, Yau pointed out that such construction can be used to study the Teichmüller spaces of higher dimensional Kähler–Einstein manifolds and the plurisubharmonicity of the energy functions should hold in these cases. This was carried out in [41] back in 2014.

Let (X, ω) be a Kähler manifold with metric g and let (N, h) be a Riemannian manifold. To ensure the existence of harmonic maps, we assume that N has nonpositive sectional curvature. A $W^{1,2}$-map $f : X \to N$ is harmonic if it minimizes the energy

$$E(f) = \int_X |\partial f|^2 \frac{\omega^n}{n!}$$

in its homotopy class. In this case, f is indeed smooth and satisfies the Euler–Lagrange equation

$$\Delta f^\alpha + \Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial z_i} \frac{\partial f^\gamma}{\partial \overline{z}_j} g^{i\overline{j}} = 0, \tag{5.1}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffell symbol of h. Furthermore, the Hopf differential of f is the section

$$H(f) = \frac{\partial f^\alpha}{\partial z_i} \frac{\partial f^\beta}{\partial z_k} h_{\alpha\beta} dz_i \otimes dz_k$$

of $S^2\Omega^{1,0}X$. The curvature of (N, h) is Hermitian nonpositive if $R^N(u, v, \overline{u}, \overline{v}) \leq 0$ for each point $p \in N$ and all complex tangent vectors $u, v \in T_p^{\mathbb{C}}N$. If $f : X \to N$ is harmonic, then by using Eq. (5.1) we have the Siu-Sampson identity

$$\operatorname{div}(\operatorname{div}(H(f))) = -R^N_{\alpha\beta\gamma\delta} \frac{\partial f^\alpha}{\partial z_i} \frac{\partial f^\gamma}{\partial \overline{z}_j} \frac{\partial f^\beta}{\partial z_k} \frac{\partial f^\delta}{\partial \overline{z}_l} g^{i\overline{j}} g^{k\overline{l}} + \|\nabla^{1,0}\overline{\partial} f\|^2. \tag{5.2}$$

The following result was shown in Sampson [26].

Theorem 5.1 *If the curvature of* (N, h) *is Hermitian nonpositive and* $f : X \to N$ *is a harmonic map, then* $\nabla^{1,0}\overline{\partial} f = 0$ *and*

$$R^N_{\alpha\beta\gamma\delta}\frac{\partial f^\alpha}{\partial z_i}\frac{\partial f^\gamma}{\partial \overline{z}_j}\frac{\partial f^\beta}{\partial z_k}\frac{\partial f^\delta}{\partial \overline{z}_l}g^{i\overline{j}}g^{k\overline{l}} = 0.$$

In view of constructing nontrivial plurisubharmonic functions on the Teichmüller spaces of Kähler–Einstein manifolds by using energy of harmonic maps, the Bochner formula implies that the only interesting case is that when each Kähler–Einstein manifold is of general type.

Let (X_0, ω_0) be a Kähler–Einstein manifold of general type, and $\varphi_1, \ldots, \varphi_m \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)$ be a basis of harmonic Beltrami differentials. We consider the power series $\varphi(t)$ as in Eq. (2.4) which is the solution of the Kuranishi equation (2.3). In this section, we give a formal discussion of the plurisubharmonicity of the energy of harmonic maps. The study of nonsmoothness of the Kuranishi space of X_0, the existence of smooth family of harmonic maps and the asymptotic behavior of the energy function will be discussed elsewhere since they are of independent interests. Thus we assume the deformation of the complex structure on X_0 is unobstructed. Let $(\mathfrak{X}, B, \pi)$ be the Kuranishi family of X_0 as constructed in Sect. 2. It was shown in [28] that, in this case, the Kuranishi gauge is equivalent to the divergence gauge. In particular, we have

$$\varphi(t)\lrcorner\omega_0 = 0. \tag{5.3}$$

To simplify the notation, we assume $m = 1$. The general case follows from the same type of computations. The deformation of Kähler–Einstein metrics in this case was established in [28]. We let V_t and ω_t be the volume form and the Kähler form of the Kähler–Einstein metric on X_t, respectively. Then

$$\begin{aligned} dV_t &= \left(1 + |t|^2\Delta_0(1-\Delta_0)^{-1}\left(|\varphi_1|^2\right) + O\left(|t|^3\right)\right)dV_0,\\ \omega_t &= \omega_0 + |t|^2\left(\frac{\sqrt{-1}}{2}\partial_0\overline{\partial}_0\left((1-\Delta_0)^{-1}|\varphi_1|^2\right)\right) + O\left(|t|^3\right). \end{aligned} \tag{5.4}$$

Now we let (N, h) be a Riemannian manifold of nonpositive sectional curvature, A be a homotopy class of maps from X_0 to N, and $F : \mathfrak{X} \to N$ be a smooth map such that each $f_t = F\mid_{X_t}: X_t \to N$ is a harmonic map in the class A. We note that the energy function $E\left(t, \overline{t}\right) = E\left(f_t\right)$ is independent of the choice of F and is a function on B.

Theorem 5.2 *The first variation of* E *is given by*

$$\left.\frac{\partial E}{\partial t}\right|_{t=0} = -\int_{X_0}\Lambda\left(\varphi_1\lrcorner H(f_0)\right)dV_0 \tag{5.5}$$

and the second variation of E is given by

$$\left.\frac{\partial^2 E}{\partial t \partial \overline{t}}\right|_{t=0} = -\int_{X_0} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\gamma \partial_p f_0^\beta \partial_{\overline{q}} f_0^\delta g^{i\overline{j}} g^{p\overline{q}} K \, dV_0 + \int_{X_0} \|\nabla^{1,0}\overline{\partial} f_0\|^2 K \, dV_0 - 2\int_{X_0} g^{i\overline{j}} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\gamma u^\beta \overline{u}^\delta dV_0 + 2\int_{X_0} \|\nabla^{1,0}\overline{u} - \overline{\varphi}_1 \lrcorner \overline{\partial} f_0\|^2 dV_0, \tag{5.6}$$

where $u = \left.\frac{\partial f_t}{\partial t}\right|_{t=0} \in \Gamma\left(f_0^* T^{\mathbb{C}} N\right)$ *and* $K = (1-\Delta_0)^{-1}\left(|\varphi_1|^2\right)$.

Furthermore, if we assume that the curvature of (N, h) *is Hermitian nonpositive then the second variation of E can be expressed as*

$$\left.\frac{\partial^2 E}{\partial t \partial \overline{t}}\right|_{t=0} = -2\int_{X_0} g^{i\overline{j}} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\gamma u^\beta \overline{u}^\delta dV_0 + 2\int_{X_0} \|\nabla^{1,0}\overline{u} - \overline{\varphi}_1 \lrcorner \overline{\partial} f_0\|^2 dV_0. \tag{5.7}$$

In particular, in this case, the energy function E is plurisubharmonic on B.

Proof Formulas (5.4) and (2.7) give us complete information about the operators ∂_t and $\overline{\partial}_t$, as well as the Kähler–Einstein metric on X_t. Thus, by using formula (5.3) and the harmonic map equation (5.1), the first variation formula (5.5) follows from integration by parts. This also leads to the following expression of the second variation of E:

$$\left.\frac{\partial^2 E}{\partial t \partial \overline{t}}\right|_{t=0} = \int_{X_0} h_{\alpha\beta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\beta g^{i\overline{j}} \Delta_0 K \, dV_0 - \int_{X_0} h_{\alpha\beta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\beta g^{i\overline{q}} g^{p\overline{j}} \partial_p \partial_{\overline{q}} K \, dV_0 - 2\int_{X_0} g^{i\overline{j}} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\overline{j}} f_0^\gamma u^\beta \overline{u}^\delta dV_0 + 2\int_{X_0} \|\nabla^{1,0}\overline{u} - \overline{\psi} \lrcorner \overline{\partial} f_0\|^2 dV_0. \tag{5.8}$$

Formula (5.6) now follows from 5.8 by integration by parts. Furthermore, if we assume the curvature of N is Hermitian nonpositive then, by the Siu–Sampson vanishing Theorem 5.1, the first two terms on the right hand side of the second variation formula (5.6) vanish, thus we have formula (5.7). The plurisubharmonicity of E now follows immediately from the Hermitian nonpositivity of the curvature of N. □

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