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# Residue formula for an obstruction to coupled Kähler–Einstein metrics

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**Abstract.** We obtain a residue formula for an obstruction to the existence of coupled Kähler–Einstein metrics described by Futaki–Zhang. We apply it to an example studied separately by Futaki and Hultgren which is a toric Fano manifold with reductive automorphism, does not admit a Kähler–Einstein metric but still admits coupled Kähler–Einstein metrics.

## 1. Introduction.

A $k$-tuple of Kähler metrics $\omega_1, \dots, \omega_k$ on a compact Kähler manifold $M$ is called coupled Kähler metrics if it satisfies

$$\mathrm{Ric}(\omega_1) = \cdots = \mathrm{Ric}(\omega_k) = \lambda \sum_{\alpha=1}^{k} \omega_\alpha \tag{1}$$

for $\lambda = -1,\ 0$ or $1$ where $\mathrm{Ric}(\omega_\alpha)$ is the Ricci form of $\omega_\alpha$ (we do not distinguish Kähler metrics $g_\alpha$ and their Kähler forms $\omega_\alpha$). Such metrics were introduced by Hultgren and Witt Nyström [**16**]. If $\lambda = 0$ this is just a $k$-tuple of Ricci-flat metrics and the existence is well-known for compact Kähler manifolds with $c_1(M) = 0$ by the celebrated solution by Yau [**23**] of the Calabi conjecture. For $\lambda = -1$ or $\lambda = 1$ the existence problem is an extension for the problem for negative or positive Kähler–Einstein metrics, and an obvious condition is $c_1(M) < 0$ or $c_1(M) > 0$. Hultgren and Witt Nyström [**16**] proved the existence of the solution for $\lambda = -1$ under the condition $c_1(M) < 0$ extending [**23**] and [**1**], and there are many interesting results for $\lambda = 1$ under the condition $c_1(M) > 0$ including attempts to extend [**3**] and [**22**]. Further studies of coupled Kähler–Einstein metrics have been done in [**4**], [**5**], [**13**], [**15**], [**18**], [**19**], [**20**], [**21**].

In this paper we derive a residue formula for an obstruction to the existence of positive coupled Kähler–Einstein metrics described in our previous paper [**13**] and apply to a computation of an example which appeared in Hultgren [**15**].

The obstruction is described as follows. Let $M$ be a Fano manifold of complex dimension $m$. Assume the anticanonical line bundle has a splitting $K_M^{-1} = L_1 \otimes \cdots \otimes L_k$ into the tensor product of ample line bundles $L_\alpha \to M$. Then we have $c_1(L_\alpha) = (1/2\pi)[\omega_\alpha]$ for a Kähler form $\omega_\alpha = \sqrt{-1} g_{\alpha\, i\bar{j}} dz^i \wedge d\bar{z}^j$, and thus

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$$c_1(M) = \frac{1}{2\pi} \sum_{\alpha=1}^{k} [\omega_\alpha].$$

For each $\omega_\alpha$ we have $f_\alpha \in C^\infty(M)$ such that

$$\mathrm{Ric}(\omega_\alpha) = \sum_{\beta=1}^{k} \omega_\beta + \sqrt{-1}\,\partial\overline{\partial} f_\alpha,$$

where $f_\alpha$ are normalized by

$$e^{f_1}\omega_1^m = \cdots = e^{f_k}\omega_k^m. \tag{2}$$

Note that this normalization still leaves an ambiguity up to a constant. However, we ignore this ambiguity since it does not cause any problem in later arguments. Of course $\omega_1, \ldots, \omega_k$ are coupled Kähler–Einstein metrics if and only if $f_\alpha$ are all constant.

Let $X$ be a holomorphic vector field. Since a Fano manifold is simply connected there exist complex-valued smooth functions defined up to constant $u_\alpha$ such that

$$i_X\omega_\alpha = \overline{\partial}(\sqrt{-1}u_\alpha). \tag{3}$$

By the abuse of terminology we call $u_\alpha$ the Hamiltonian function of $X$ with respect to $\omega_\alpha$ though $u_\alpha$ is a Hamiltonian function for the imaginary part of $X$ in the usual sense of symplectic geometry only when $u_\alpha$ is real valued. In Theorem 3.3 of [**13**], it is shown for some choices of $u_\alpha$ we have

$$\Delta_\alpha u_\alpha + (\mathrm{grad}_\alpha u_\alpha) f_\alpha = -\sum_{\beta=1}^{k} u_\beta, \tag{4}$$

where $\Delta_\alpha = -\overline{\partial}_\alpha^*\overline{\partial}$ is the Laplacian with respect to $\omega_\alpha$ and $\mathrm{grad}_\alpha u_\alpha$ is the type $(1,0)$-part of the gradient of $u_\alpha$ expressed as $\mathrm{grad}_\alpha u_\alpha = g_\alpha^{i\overline{j}}(\partial u_\alpha/\partial\overline{z}^j)(\partial/\partial z^i)$ in terms of local holomorphic coordinates $(z^1, \ldots, z^m)$. The case of $k = 1$ of this result has been obtained in [**8**]. If we replace $u_\alpha$ by $u_\alpha^{c_\alpha} = u_\alpha + c_\alpha$ the equations (4) are satisfied for $u_\alpha^{c_\alpha}$ if and only if

$$\sum_{\alpha=1}^{k} c_\alpha = 0. \tag{5}$$

Definition 1.1 ([**13**]). With the choice of $u_\alpha$ satisfying (4) the Lie algebra character is defined as

$$\begin{aligned} \mathrm{Fut} : \mathfrak{h}(M) &\to \mathbb{C} \\ X \quad &\mapsto \mathrm{Fut}(X) = \sum_{\alpha=1}^{k} \frac{\int_M u_\alpha\, \omega_\alpha^m}{\int_M \omega_\alpha^m}. \end{aligned} \tag{6}$$

Notice that this definition of Fut is not affected by the ambiguity of the choice of

$u_\alpha$ because of (5). Note also Fut is the coupled infinitesimal form of the group character obtained in [**7**].

To formulate the localization formula let $Z = \bigcup_{\lambda\in\Lambda} Z_\lambda$ be zero set of $X$ where $Z_\lambda$'s are connected components. Let $N_\alpha(Z_\lambda) = (T_M|_{Z_\lambda})/T_{Z_\lambda}$ be the normal bundle of $Z_\lambda$ with respect to $\omega_\alpha$. Then the Levi-Civita connection $\nabla^\alpha$ of $\omega_\alpha$ naturally induces an endomorphism $L^{N_\alpha}(X)$ of $N_\alpha(Z_\lambda)$ by

$$L^{N_\alpha}(X)(Y) = (\nabla^\alpha_Y X)^\perp \in N_\alpha(Z_\lambda), \qquad \text{for any } Y \in N_\alpha(Z_\lambda).$$

We also assume $Z$ is nondegenerate in the sense that $L^{N_\alpha}$ is nondegenerate. Let $K_\alpha$ be the curvature of $N_\alpha(Z_\lambda)$. The localization formula of $\mathrm{Fut}(X)$ we obtain is the following.

Theorem 1.2. *Let $M$ be a Fano manifold with $K_M^{-1} = L_1 \otimes \cdots \otimes L_k$. Let $X$ be a holomorphic vector field with nondegenerate zero set $Z = \bigcup_{\lambda\in\Lambda} Z_\lambda$, then*

$$\begin{aligned}&\mathrm{Fut}(X)\\ &= \frac{1}{m+1}\sum_{\alpha=1}^{k}\left(\frac{\sum\limits_{\lambda\in\Lambda}\int_{Z_\lambda}\big((E_\alpha + c_1(L_\alpha))|_{Z_\lambda}\big)^{m+1}/\det\big((2\pi)^{-1}(L^{N_\alpha}(X)+\sqrt{-1}K_\alpha)\big)}{\sum\limits_{\lambda\in\Lambda}\int_{Z_\lambda}\big((E_\alpha + c_1(L_\alpha))|_{Z_\lambda}\big)^{m}/\det\big((2\pi)^{-1}(L^{N_\alpha}(X)+\sqrt{-1}K_\alpha)\big)}\right), \end{aligned} \tag{7}$$

*where $E_\alpha \in \Gamma(\mathrm{End}(L_\alpha))$ is given by $E_\alpha s = u_\alpha s$ with $L^{N_\alpha}$ and $K_\alpha$ being as above.*

Corollary 1.3. *If $Z$ contains only discrete points, then*

$$\begin{aligned}\mathrm{Fut}(X) &= \frac{1}{m+1}\sum_{\alpha=1}^{k}\left(\frac{\sum\limits_{p\in Z}(u_\alpha(p))^{m+1}/\det(\nabla X)(p)}{\sum\limits_{p\in Z}(u_\alpha(p))^{m}/\det(\nabla X)(p)}\right)\\ &= \frac{1}{m+1}\left(\sum_{\alpha=1}^{k}\frac{\sum\limits_{p\in Z}(u_\alpha(p))^{m+1}}{\sum\limits_{p\in Z}(u_\alpha(p))^{m}}\right).\end{aligned}$$

We can apply the obtained localization formula for the invariant Fut in the coupled situation to verify the example considered in Hultgren's paper [**15**]. This example was first considered by the first author in [**6**], where he showed that the invariant Fut is non-vanishing, hence there does not exist a Kähler–Einstein metric on this example though the automorphism group is reductive and thus Matsushima's condition [**17**] is satisfied. Later, in [**11**], the localization formula in [**12**] was used to show a much simpler computation of the invariant Fut can be done. Hultgren [**15**] considered decompositions of the anticanonical line bundle, and proved in a special case of the decomposition there do exist coupled Kähler–Einstein metrics on this manifold.

The rest of the paper proceeds as follows. In Section 2 we prove Theorem 1.2. In Section 3 we verify the existence result of Hultgren in [**15**] by checking the vanishing of Fut as an application of Theorem 1.2.

## 2. Localization formula.

We first consider an ample line bundle $L \to M$ with $c_1(L) = (1/2\pi)[\omega]$ where $[\omega]$ is a Kähler class of $M$. Let $e_U$ be a non-vanishing local holomorphic section of $L|_U$ where $U$ is an open set of $M$. Then $e_U$ determines a local trivialization of the line bundle $L|_U \cong U \times \mathbb{C}$, given by $ze_U \mapsto (p, z)$, where $z$ is the fiber coordinate. Let $h$ be the Hermitian metric of $L$, and $h_U = h(e_U, e_U)$. The local connection form is given by $\theta_U = \partial \log h_U$. Let

$$\theta = \theta_U + \frac{dz}{z}, \tag{8}$$

then $\theta$ is a globally defined connection form on the associated principle $\mathbb{C}^*$-bundle. To see this, we first remark that $dz/z$ is the Maurer–Cartan form of $\mathbb{C}^*$. If $U \cap V \neq \emptyset$, and we take another trivialization on $L|_V \cong V \times \mathbb{C}$, given by $we_V \mapsto (p, w)$, where $e_V$ is a non-vanishing local holomorphic section and $w$ is the fiber coordinate. Let $f$ be the non-vanishing holomorphic function such that $e_V = fe_U$, then $h_V = |f|^2 h_U$ and $z = fw$. Then,

$$\theta_V + \frac{dw}{w} = \partial \log |f|^2 h_U + \frac{f}{z} d\left(\frac{z}{f}\right) = \frac{df}{f} + \partial \log h_U + \frac{dz}{z} - \frac{df}{f} = \theta_U + \frac{dz}{z}.$$

Hence $\theta = \theta_U + dz/z$ is independent of the trivialization. Obviously $\sqrt{-1}\,\overline{\partial}\theta = \omega$. Let $u$ be a complex-valued smooth function such that

$$i_X \omega = \overline{\partial}(\sqrt{-1}u). \tag{9}$$

It is well-known (c.f. [**10**] for example) that a Hamiltonian vector field $X$ written in this way lifts to $L$ uniquely up to $cz\partial/\partial z$ for a constant $c$. Let $\tilde{X}$ be a lift of $X$ to $L$. Then obviously $u_X := -\theta(\tilde{X})$ is a Hamiltonian function for $X$ and $-\theta(\tilde{X} - cz\partial/\partial z) = u_X + c$. Thus, the ambiguity of $c_\alpha$ for $L_\alpha$ above appears in this way. The connection form $\theta$ determines a horizontal lift $X^h$ of $X$, given by

$$X^h = \tilde{X} - \theta(\tilde{X}) z \frac{\partial}{\partial z}.$$

Apparently, this expression is independent of the lift $\tilde{X}$ and $\theta(X^h) = 0$.

Now, for each ample line bundle $L_\alpha \to M$, $\alpha = 1, \ldots, k$, choose Hermitian metric $h_\alpha$, let $\theta_\alpha$ be corresponding connection form on the associated principal $\mathbb{C}^*$-bundle, and $\Theta_\alpha$ is the curvature form such that $\Theta_\alpha = \overline{\partial}\partial \log h_\alpha = -\sqrt{-1}\omega_\alpha$.

Hence, with a choice of a Hamiltonian function $u_\alpha$, the lifted holomorphic vector field $X_\alpha$ (omitting the tilde) of $X$ on $L_\alpha$ is

$$X_\alpha = X_\alpha^h - u_\alpha z \frac{\partial}{\partial z}$$

where $X_\alpha^h$ is the horizontal lift of $X$. Then of course

$$u_\alpha = -\theta_\alpha(X_\alpha).$$

The infinitesimal action on the space $\Gamma(L_\alpha)$ of holomorphic sections of $L_\alpha$ is given by

$$\begin{aligned}\Lambda_\alpha : \Gamma(L_\alpha) &\to \Gamma(L_\alpha)\\ s &\mapsto \Lambda_\alpha(s) = \nabla^\alpha_X s + u_\alpha s\end{aligned}$$

where $\nabla^\alpha$ is the covariant derivative determined by $\theta_\alpha$.

Then we can check that for $f \in C^\infty(M)$, $s \in \Gamma(L_\alpha)$,

(1) $\Lambda_\alpha$ satisfies the Leibniz rule.

$$\begin{aligned}\Lambda_\alpha(fs) &= \nabla^\alpha_X(fs) + u_\alpha fs\\ &= X(f)s + f\nabla^\alpha_X s + fu_\alpha s\\ &= X(f)s + f\Lambda_\alpha s.\end{aligned}$$

(2) $\overline{\partial}\Lambda_\alpha = \Lambda_\alpha\overline{\partial}$. This follows from

$$\begin{aligned}\overline{\partial}\Lambda_\alpha s &= \overline{\partial}(i_X\nabla^\alpha s + u_\alpha s) = -i_X\overline{\partial}\nabla^\alpha s + \overline{\partial}u_\alpha s\\ &= \big(-i_X\Theta_\alpha + \overline{\partial}u_\alpha\big)s = \sqrt{-1}\big(i_X\omega_\alpha - \overline{\partial}(\sqrt{-1}u_\alpha)\big)s = 0.\end{aligned}$$

(3) It is obvious that $\Lambda_\alpha|_{\mathrm{Zero}(X)} = u_\alpha|_{\mathrm{Zero}(X)}$ is a linear map on $\Gamma(L_\alpha|_{\mathrm{Zero}(X)})$.

This implies $\Lambda_\alpha|_{\mathrm{Zero}(X)} \in \mathrm{End}(L_\alpha|_{\mathrm{Zero}(X)})$. This endomorphism along the zero set of $X$ can be extended to a global endomorphism of $L_\alpha$ by letting for $s \in \Gamma(L_\alpha)$

$$E_\alpha s = \Lambda_\alpha s - \nabla^\alpha_X s = u_\alpha s = -\theta_\alpha(X_\alpha)s.$$

Then $E_\alpha \in \mathrm{End}(L_\alpha)$ and

$$\overline{\partial}E_\alpha = \overline{\partial}u_\alpha = i_X(-\sqrt{-1}\omega_\alpha) = i_X\Theta_\alpha. \tag{10}$$

The above discussion enables us to write the Lie algebra character (6) as

$$\begin{aligned}\mathrm{Fut}(X) &= \sum_{\alpha=1}^k \frac{\int_M u_\alpha\,\omega_\alpha^m}{\int_M \omega_\alpha^m}\\ &= \frac{1}{m+1}\sum_{\alpha=1}^k \frac{\int_M (u_\alpha+\omega_\alpha)^{m+1}}{\int_M (u_\alpha+\omega_\alpha)^m}\\ &= \frac{1}{m+1}\sum_{\alpha=1}^k \frac{\int_M (-\theta_\alpha(X_\alpha)+\sqrt{-1}\Theta_\alpha)^{m+1}}{\int_M (-\theta_\alpha(X_\alpha)+\sqrt{-1}\Theta_\alpha)^m}\\ &= \frac{1}{m+1}\sum_{\alpha=1}^k \frac{\int_M (E_\alpha+\sqrt{-1}\Theta_\alpha)^{m+1}}{\int_M (E_\alpha+\sqrt{-1}\Theta_\alpha)^m}.\end{aligned} \tag{11}$$

Here we remark that the both expressions $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^{m+1}$ and $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^m$ are independent of the choice of Hermitian metric $h_\alpha$. This could either follow from [**12**, Proposition 2.1] or argue as follows. We choose a family of

Hermitian metrics $h_\alpha(t)$, let $h_\alpha(t) = e^{-t\varphi_\alpha} h_\alpha$, for $\varphi_\alpha \in C^\infty(M)$. Then

$$\theta_\alpha(t) = \partial \log h_\alpha(t) + \frac{dz}{z} = \theta_\alpha - t\partial\varphi_\alpha$$

is the corresponding family of connections on associated principle $\mathbb{C}^*$-bundle, and the curvature forms are

$$\Theta_\alpha(t) = \Theta_\alpha + t\partial\overline{\partial}\varphi_\alpha,$$

and we compute that

$$i_X\Theta_\alpha(t) = i_X\Theta_\alpha + i_X(t\partial\overline{\partial}\varphi_\alpha) = \overline{\partial}\big(u_\alpha + tX(\varphi_\alpha)\big),$$

we let $u_\alpha(t) = u_\alpha + tX(\varphi_\alpha)$. This $u_\alpha(t)$ is a Hamiltonian function of $X$ for the Kähler form $\omega_\alpha(t)$ corresponding to $h_\alpha(t)$. As we saw above the lifted vector field on $L_\alpha$ is given by

$$X_\alpha(t) = X_\alpha^h(t) - u_\alpha(t) z\frac{\partial}{\partial z}.$$

Then

$$-\theta_\alpha(t)(X_\alpha(t)) = u_\alpha(t) = -\theta_\alpha(X_\alpha) + tX(\varphi_\alpha).$$

We will check the metric independence of $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^{m+1}$, and similar argument works for $\int_M(-\theta_\alpha(X_\alpha) + \sqrt{-1}\Theta_\alpha)^m$. We compute that

$$\begin{aligned}
&\frac{d}{dt}\int_M \big(-\theta_\alpha(t)(X_\alpha(t)) + \sqrt{-1}\,\Theta_\alpha(t)\big)^{m+1}\\
&= (m+1)\int_M \big(-\theta_\alpha(t)(X_\alpha(t)) + \sqrt{-1}\,\Theta_\alpha(t)\big)^m \wedge \big(X(\varphi_\alpha) + \sqrt{-1}\,\partial\overline{\partial}\varphi_\alpha\big)\\
&= (m+1)\bigg(\int_M X(\varphi_\alpha)(\sqrt{-1}\,\Theta_\alpha(t))^m - m\theta_\alpha(t)(X_\alpha(t))(\sqrt{-1}\,\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\,\partial\overline{\partial}\varphi_\alpha\bigg)\\
&= (m+1)\bigg(\int_M X(\varphi_\alpha)(\sqrt{-1}\,\Theta_\alpha(t))^m - m\int_M \overline{\partial}\big(\theta_\alpha(t)(X_\alpha(t))\big)(\sqrt{-1}\,\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\,\partial\varphi_\alpha\bigg)\\
&= (m+1)\bigg(\int_M X(\varphi_\alpha)(\sqrt{-1}\,\Theta_\alpha(t))^m + m\int_M i_X\Theta_\alpha(t) \wedge (\sqrt{-1}\,\Theta_\alpha(t))^{m-1} \wedge \sqrt{-1}\,\partial\varphi_\alpha\bigg)\\
&= 0.
\end{aligned}$$

PROOF OF THEOREM 1.2. Now, we follow an argument in the book [**9**] (see Theorem 5.2.8), originally due to Bott [**2**] to give the localization formula.

Consider an invariant polynomial $P$ of degree $(m+l)$ for $l = 0, 1$, let

$$P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha) = \sum_{r=0}^{m+l} P_{\alpha,r}(E_\alpha, \sqrt{-1}\,\Theta_\alpha),$$

where

$$P_{\alpha,r}(E_\alpha, \sqrt{-1}\,\Theta_\alpha) = \binom{m+l}{r} P(E_\alpha, \dots, E_\alpha; \underbrace{\sqrt{-1}\,\Theta_\alpha, \dots, \sqrt{-1}\,\Theta_\alpha}_{r}).$$

Since $\overline{\partial} E_\alpha = i_X \Theta_\alpha$, we have

$$\sqrt{-1}\,\overline{\partial} P_\alpha = i_X P_\alpha.$$

Define a $(1,0)$ form $\pi_\alpha$ as follows: for a holomorphic vector field $Y$,

$$i_Y \pi_\alpha = \frac{\omega_\alpha(Y, X)}{\omega_\alpha(X, X)},$$

then

$$i_X \pi_\alpha = 1, \quad \text{and} \quad i_X \overline{\partial} \pi_\alpha = 0.$$

We further define

$$\eta_\alpha = \pi_\alpha \wedge \sum_{i=0}^{m-1} (\sqrt{-1}\,\overline{\partial} \pi_\alpha)^i \wedge P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha),$$

then $\eta_\alpha$ is defined outside zero set of $X$. The computation shows

$$P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha) = -\sqrt{-1}\,\overline{\partial} \eta_\alpha + i_X \eta_\alpha.$$

Let $B_\epsilon(Z)$ be an $\epsilon$-neighbourhood of $Z$. Then, denoting the type $(2m-1)$-part of $\eta_\alpha$ by $\eta_\alpha^{(2m-1)}$ we have

$$\begin{aligned}
&\int_M P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha) \\
&\quad = \lim_{\epsilon \to 0} \int_{M - B_\epsilon(Z)} P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha) \\
&\quad = \sqrt{-1} \lim_{\epsilon \to 0} \int_{M - B_\epsilon(Z)} -\overline{\partial} \eta_\alpha^{(2m-1)} = \sqrt{-1} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(Z)} \eta_\alpha^{(2m-1)} \\
&\quad = \sqrt{-1} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(Z)} \pi_\alpha \wedge \big(1 + (\sqrt{-1}\,\overline{\partial} \pi_\alpha) + (\sqrt{-1}\,\overline{\partial} \pi_\alpha)^2 + \dots + (\sqrt{-1}\,\overline{\partial} \pi_\alpha)^{m-1}\big) \\
&\qquad \wedge \sum_{r=0}^{m-1} P_{\alpha,r}(E_\alpha, \sqrt{-1}\,\Theta_\alpha).
\end{aligned}$$

As computed in Theorem 5.2.8 in [**9**] or [**2**],

$$(2\pi)^{-m} \int_M P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha) = \sum_{\lambda \in \Lambda} \int_{Z_\lambda} \frac{P_\alpha(E_\alpha + \sqrt{-1}\,\Theta_\alpha)|_{Z_\lambda}}{\det\big((2\pi)^{-1}(L^{N_\alpha}(X) + \sqrt{-1}\,K_\alpha)\big)},$$

where $K_\alpha$ is the curvature of the normal bundle $N_\alpha$ with respect to the induced metric. Taking $P = tr^{m+1}$ and $P = tr^m$, and apply above to (11), we obtain the localization formula of $\mathrm{Fut}(X)$ in the coupled case (7). □

## 3. Application of localization formula.

Before computing the example, we remark that by Theorem 3.2 in [**13**], (4) is equivalent to

$$\int_M (u_1 + \cdots + u_k) dV = 0$$

where $dV = e^{f_\alpha}\omega_\alpha^m$ which is independent of $\alpha$ by the normalization (2). By Theorem 5.2 in [**13**] this condition is equivalent to

$$\sum_{\alpha=1}^{k} \mathcal{P}_\alpha = \mathcal{P}_{-K_M} \tag{12}$$

where $\mathcal{P}_\alpha$ is the moment map image of $\omega_\alpha$.

We consider the tautological line bundles $\mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$ and $\mathcal{O}_{\mathbb{CP}^2}(-1) \to \mathbb{CP}^2$, and the bundle $E = \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^2}(-1)$ over $\mathbb{CP}^1 \times \mathbb{CP}^2$. Let $M$ be the total space of the projective line bundle $\mathbb{P}(E)$ over $\mathbb{CP}^1 \times \mathbb{CP}^2$. In local coordinates, we let

$$\mathbb{CP}^1 = \{(b_0 : b_1)\}, \qquad \mathbb{CP}^2 = \{(a_0 : a_1 : a_2)\},$$

$$\mathcal{O}_{\mathbb{CP}^1}(-1) = \{[(w_0, w_1), (b_0 : b_1)] |\ (w_0, w_1) = \lambda(b_0, b_1) \text{ for some } \lambda \in \mathbb{C}\},$$
$$\mathcal{O}_{\mathbb{CP}^2}(-1) = \{[(z_0, z_1, z_2), (a_0 : a_1 : a_2)] |\ (z_0, z_1, z_2) = \mu(a_0, a_1, a_2) \text{ for some } \mu \in \mathbb{C}\},$$

$$\begin{aligned} M = \{&[(z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)] | \\ &(w_0, w_1) = \lambda(b_0, b_1), (z_0, z_1, z_2) = \mu(a_0, a_1, a_2) \text{ for some } (\lambda, \mu) \neq (0,0) \text{ in } \mathbb{C} \times \mathbb{C}\}. \end{aligned}$$

The $(\mathbb{C}^*)^4$-action on $M$ is defined by extending the $\mathbb{C}^*$-action on $\mathbb{CP}^1$ and $(\mathbb{C}^*)^2$-action on $\mathbb{CP}^2$. We let $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$, then

$$\begin{aligned} &(t_1, t_2, t_3, t_4) \cdot [(z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)] \\ &\quad = [(z_0 : t_1 z_1 : t_2 z_2 : t_4 w_0 : t_4 t_3 w_1), (a_0 : t_1 a_1 : t_2 a_2), (b_0 : t_3 b_1)]. \end{aligned}$$

There are totally seven $(\mathbb{C}^*)^4$-invariant divisors;

$$D_1 = \{z_0 = a_0 = 0\}, \quad D_2 = \{z_1 = a_1 = 0\}, \quad D_3 = \{z_2 = a_2 = 0\},$$

which are identified with $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$;

$$D_4 = \{b_0 = w_0 = 0\}, \quad D_5 = \{b_1 = w_1 = 0\},$$

which are identified with $\mathbb{CP}^1$-bundle over $\mathbb{CP}^2$;

$$D_6 = \{z_0 = z_1 = z_2 = 0\}, \quad D_7 = \{w_0 = w_1 = 0\},$$

which are identified with $\mathbb{CP}^1 \times \mathbb{CP}^2$. It is known that

$$K_M^{-1} = \sum_{i=1}^{7} D_i.$$

As in [**15**], we consider the following decomposition for $c \in (1/4, 3/4)$ which is ampleness condition for the line bundles associated with $D(c)$ and $D(1-c)$ below. Define

$$D(c) = \frac{1}{2}K_M^{-1} + \left(c - \frac{1}{2}\right)(D_4 + D_5),$$
$$D(1-c) = \frac{1}{2}K_M^{-1} + \left(\frac{1}{2} - c\right)(D_4 + D_5),$$

then

$$K_M^{-1} = D(c) + D(1-c), \tag{13}$$

in particular, putting $c = 1/2$ corresponds to the canonical decomposition $K_M^{-1} = (1/2)K_M^{-1} + (1/2)K_M^{-1}$. In this case, the coupled setting is completely reduced to the ordinary Kähler–Einstein setting, and one can no longer expect the existence of coupled Kähler–Einstein metric due to [**6**]. So we would like to consider the deformation from this, and try to find $c$ such that the invariant Fut vanishes.

We remark that the torus action preserves the above decomposition (13). Note also that the invariant Fut is invariant under any automorphism of $M$ preserving the decomposition (13). Using the automorphism $(b_0, b_1) \mapsto (b_1, b_0)$ one can see $\mathrm{Fut}(X_3) = \mathrm{Fut}(-X_3)$ and thus $\mathrm{Fut}(X_3) = 0$ for the infinitesimal generator $X_3$ for the $t_3$-action, and similarly $\mathrm{Fut}(X_1) = \mathrm{Fut}(X_2) = 0$ for the infinitesimal generators $X_1$ and $X_2$ of $t_1$ and $t_2$-actions using the automorphisms induced by the odd permutations of the coordinates $(a_0 : a_1 : a_2)$. Hence, to compute the coupled Fut invariant, it is sufficient to consider the action of one parameter subgroup $(1, 1, 1, t_4)$ on $M$ by

$$\begin{aligned}(1,1,1,t_4) \cdot [(z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)]\\ = [(z_0 : z_1 : z_2 : t_4 w_0 : t_4 w_1), (a_0 : a_1 : a_2), (b_0 : b_1)].\end{aligned}$$

For this action, let $\xi = \lambda/\mu$, $\eta = 1/\xi$, then the associated holomorphic vector field is

$$X = \xi\frac{\partial}{\partial \xi} = -\eta\frac{\partial}{\partial \eta}.$$

Zero sets are

$$Z_\infty = \{\mu = 0\} = D_6, \quad \text{and} \quad Z_0 = \{\lambda = 0\} = D_7.$$

Since

$$\begin{aligned}\mathbb{P}(\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^2}(-1)) &= \mathbb{P}\big((\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^2}(-1)) \otimes \mathcal{O}_{\mathbb{CP}^2}(1)\big)\\ &= \mathbb{P}\big((\mathcal{O}_{\mathbb{CP}^1}(-1) \otimes \mathcal{O}_{\mathbb{CP}^2}(1)) \oplus \mathcal{O}_{\mathbb{CP}^2}\big),\end{aligned}$$

the normal bundle of $Z_\infty$ is

$$\nu(Z_\infty) = \mathcal{O}_{\mathbb{CP}^1}(-1) \otimes \mathcal{O}_{\mathbb{CP}^2}(1),$$

similarly, the normal bundle of $Z_0$ is

$$\nu(Z_0) = \mathcal{O}_{\mathbb{CP}^1}(1) \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) = \nu(Z_\infty)^{-1}.$$

Let $a, b$ be the positive generators of $H^2(\mathbb{CP}^1, \mathbb{Z})$ and $H^2(\mathbb{CP}^2, \mathbb{Z})$. Then

$$c_1(\mathbb{CP}^1) = 2a, \quad c_1(\mathbb{CP}^2) = 3b,$$

and

$$c_1(K_M^{-1})|_{Z_\infty} = c_1(Z_\infty) + c_1(\nu(Z_\infty)) = 2a + 3b - a + b = a + 4b.$$

Similarly we have

$$c_1(K_M^{-1})|_{Z_0} = 3a + 2b.$$

Since the line bundle $[D_4]$ restricted to $Z_\infty = D_6$ is isomorphic to the line bundle corresponding to the divisor $\{b_0 = 0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^2$ we have $c_1([D_4])|_{Z_\infty} = a$. Similarly we have

$$c_1([D_4])|_{Z_0} = c_1([D_5])|_{Z_\infty} = c_1([D_5])|_{Z_0} = a.$$

Then

$$c_1(D(c))|_{Z_\infty} = \frac{1}{2}(a + 4b) + \left(c - \frac{1}{2}\right) 2a = \left(2c - \frac{1}{2}\right) a + 2b,$$
$$c_1(D(c))|_{Z_0} = \frac{1}{2}(3a + 2b) + \left(c - \frac{1}{2}\right) 2a = \left(2c + \frac{1}{2}\right) a + b.$$

To see the value of $u$ along the zero set of $X$ we may use the description of the moment polytope $P(c)$ in [**15**]

$$P(c) = \left\{ y \in \mathbb{R}^4 : \langle y, d_i \rangle \le \frac{1}{2}, i \ne 4, 5,\ \langle y, d_i \rangle \le c, i = 4, 5 \right\}$$

where $d_i$ are as described in [**15**]. Since $P(c) + P(1-c) = P_{-K_M}$ the moment polytopes are those obtained by the Hamiltonian functions satisfying (4) as follows from the arguments of the beginning of this section. From this description for $d_6 = (0, 0, 0, -1)$ and $d_7 = (0, 0, 0, 1)$ we see

$$u\big|_{Z_\infty} = -\frac{1}{2}, \qquad u\big|_{Z_0} = \frac{1}{2}.$$

By using the fact

$$a^2 = b^3 = 0,$$

we first compute

$$
\begin{aligned}
\mathrm{Vol}(D(c)) &= \left[\frac{(u|_{Z_\infty}+c_1(D(c))|_{Z_\infty})^4}{u|_{Z_\infty}+c_1(\nu(Z_\infty))}+\frac{(u|_{Z_0}+c_1(D(c))|_{Z_0})^4}{u|_{Z_0}+c_1(\nu(Z_0))}\right][\mathbb{CP}^1\times\mathbb{CP}^2] \\
&= \left[\frac{\big(-1/2+(2c-1/2)a+2b\big)^4}{-1/2-a+b}+\frac{\big(1/2+(2c+1/2)a+b\big)^4}{1/2+a-b}\right][\mathbb{CP}^1\times\mathbb{CP}^2] \\
&= 112c-6, \qquad (14)
\end{aligned}
$$

replacing $c$ by $1-c$, we get

$$
\mathrm{Vol}(D(1-c)) = 106-112c. \qquad (15)
$$

We also need to compute the numerators in the localization formula. For the divisor $D(c)$,

$$
\begin{aligned}
&\left[\frac{(u|_{Z_\infty}+c_1(D(c))|_{Z_\infty})^5}{u|_{Z_\infty}+c_1(\nu(Z_\infty))}+\frac{(u|_{Z_0}+c_1(D(c))|_{Z_0})^5}{u|_{Z_0}+c_1(\nu(Z_0))}\right][\mathbb{CP}^1\times\mathbb{CP}^2] \\
&= \left[\frac{\big(-1/2+(2c-1/2)a+2b\big)^5}{-1/2-a+b}+\frac{\big(1/2+(2c+1/2)a+b\big)^5}{1/2+a-b}\right][\mathbb{CP}^1\times\mathbb{CP}^2] \\
&= -30c+12, \qquad (16)
\end{aligned}
$$

replacing $c$ by $1-c$, we get for divisor $D(1-c)$,

$$
\begin{aligned}
&\left[\frac{(u|_{Z_\infty}+c_1(D(1-c))|_{Z_\infty})^5}{u|_{Z_\infty}+c_1(\nu(Z_\infty))}+\frac{(u|_{Z_0}+c_1(D(1-c))|_{Z_0})^5}{u|_{Z_0}+c_1(\nu(Z_0))}\right][\mathbb{CP}^1\times\mathbb{CP}^2] \\
&= 30c-18. \qquad (17)
\end{aligned}
$$

Plugging above (14), (15), (16), (17) into the localization formula (Theorem 1.2), we obtain

$$
\begin{aligned}
\mathrm{Fut}(X) &= \frac{\left[\frac{(u|_{Z_\infty}+c_1(D(c))|_{Z_\infty})^5}{u|_{Z_\infty}+c_1(\nu(Z_\infty))}+\frac{(u|_{Z_0}+c_1(D(c))|_{Z_0})^5}{u|_{Z_0}+c_1(\nu(Z_0))}\right][\mathbb{CP}^1\times\mathbb{CP}^2]}{\mathrm{Vol}(D(c))} \\
&\quad+\frac{\left[\frac{(u|_{Z_\infty}+c_1(D(1-c))|_{Z_\infty})^5}{u|_{Z_\infty}+c_1(\nu(Z_\infty))}+\frac{(u|_{Z_0}+c_1(D(1-c))|_{Z_0})^5}{u|_{Z_0}+c_1(\nu(Z_0))}\right][\mathbb{CP}^1\times\mathbb{CP}^2]}{\mathrm{Vol}(D(1-c))} \\
&= \frac{-30c+12}{112c-6}+\frac{30c-18}{106-112c} \\
&= \frac{-15(112c^2-112c+23)}{(56c-3)(56c-53)}, \qquad (18)
\end{aligned}
$$

therefore, the invariant Fut character vanishes when

$$
c = \frac{1}{2}\pm\frac{1}{4}\sqrt{\frac{5}{7}}.
$$

This is the same as in [**15**].

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