

## ON THE STRUCTURE OF GRADIENT YAMABE SOLITONS

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ABSTRACT. We show that every complete nontrivial gradient Yamabe soliton admits a special global warped product structure with a one-dimensional base. Based on this, we obtain a general classification theorem for complete nontrivial locally conformally flat gradient Yamabe solitons.

## 1. The results

Self-similar solutions and translating solutions, often called soliton solutions, have emerged in recent years as important objects in geometric flows since they appear as possible singularity models. Much progress has been made recently in the study of soliton solutions of the Ricci flow (i.e., Ricci solitons) and the mean curvature flow. In this paper, we are interested in geometric structures of Yamabe solitons, which are soliton solutions to the Yamabe flow. Note that the Yamabe flow has been studied extensively in recent years, see, e.g., the very recent survey by Brendle [2] and the references therein.

A complete Riemannian metric  $g = g_{ij} dx^i dx^j$  on a smooth manifold  $M^n$  is called a *gradient Yamabe soliton* if there exists a smooth function  $f$  such that its Hessian satisfies the equation

$$(1.1) \quad \nabla_i \nabla_j f = (R - \rho)g_{ij},$$

where  $R$  is the scalar curvature of  $g$  and  $\rho$  is a constant. For  $\rho = 0$  the Yamabe soliton is *steady*, for  $\rho > 0$  it is *shrinking* and for  $\rho < 0$  *expanding*. The function  $f$  is called a *potential function* of the gradient Yamabe soliton. When  $f$  is constant, we call it a *trivial* Yamabe soliton. It has been known (see [8, 11]) that every compact Yamabe soliton is of constant scalar curvature; hence trivial since  $f$  is harmonic and thus is constant.

Recently, inspired by the classification of locally conformally flat Ricci solitons and especially [3], Daskalopoulos and Sesum [10] initiated the investigation of conformally flat Yamabe solitons and proved the following classification result:

**Theorem 1.1 (Daskalopoulos–Sesum [10]).** *All complete locally conformally flat gradient Yamabe solitons with positive sectional curvature  $K > 0$  are rotationally symmetric.*

Moreover, they constructed examples of rotationally symmetric gradient Yamabe solitons on  $\mathbb{R}^n$  with positive sectional curvature  $K > 0$ .

In this paper, motivated by the above Theorem 1.1 and the recent works of the first author and his collaborators [4, 5] on Bach-flat Ricci solitons, we investigate geometric

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structures of gradient Yamabe solitons not necessarily locally conformally flat. It turns out that, by exploring the special nature of the Yamabe soliton equation (1.1), every complete nontrivial gradient Yamabe soliton  $(M^n, g, f)$  admits a special global warped product structure with a 1-dimensional base and the warping function provided by  $|\nabla f|$  (see Theorem 1.2). Based on this special warped product structure, we are able to prove a classification theorem for locally conformally flat gradient Yamabe solitons without any further assumption on the curvature (see Corollary 1.5). In particular, Theorem 1.1 above is a special case of Corollary 1.3 or Corollary 1.6. Our main result is:

**Theorem 1.2.** *Let  $(M^n, g, f)$  be a nontrivial complete gradient Yamabe soliton satisfying equation (1.1). Then  $|\nabla f|^2$  is constant on regular level surfaces of  $f$ , and either*

- (i)  *$f$  has a unique critical point at some point  $x_0 \in M^n$ , and  $(M^n, g, f)$  is rotationally symmetric and equal to the warped product*

$$([0, \infty), dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{\text{can}}),$$

*where  $\bar{g}_{\text{can}}$  is the round metric on  $\mathbb{S}^{n-1}$ , or*

- (ii)  *$f$  has no critical point and  $(M^n, g, f)$  is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}),$$

*where  $(N^{n-1}, \bar{g})$  is a Riemannian manifold of constant scalar curvature, say  $\bar{R}$ . Moreover, if  $(M^n, g, f)$  has nonnegative Ricci curvature  $Rc \geq 0$  then  $(M^n, g)$  is isometric to the Riemannian product  $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$ ; if the scalar curvature  $R \geq 0$  on  $M^n$ , then either  $\bar{R} > 0$ , or  $R = \bar{R} = 0$  and  $(M^n, g)$  is isometric to the Riemannian product  $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$ .*

As an immediate consequence of Theorem 1.2, we have

**Corollary 1.3.** *Let  $(M^n, g, f)$  be a nontrivial complete gradient Yamabe soliton with positive Ricci curvature  $Rc > 0$ , then  $f$  has exactly one critical point and  $(M^n, g, f)$  is rotationally symmetric.*

**Remark 1.1.** Shortly after the first version of our paper appeared in the arXiv, G. Catino, C. Mantegazza and L. Mazziere [6] posted a paper on the global structure of conformal gradient solitons with nonnegative Ricci tensor in which they proved Theorem 1.2 with the additional assumption of nonnegative Ricci curvature (see Theorem 3.2 in [6]). We also remark that, as pointed out in [10], steady and shrinking Yamabe solitons have nonnegative scalar curvatures.

In the special case when  $(M^n, g, f)$  is locally conformally flat, we can say more about the manifold  $(N^{n-1}, \bar{g})$  in case (ii) of Theorem 1.2.

**Theorem 1.4.** *Suppose  $(M^n, g, f)$  is a nontrivial complete locally conformally flat gradient Yamabe soliton satisfying equation (1.1), and suppose  $f$  has no critical point. Then  $(M^n, g, f)$  is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}_N),$$

*where  $(N^{n-1}, \bar{g}_N)$  is a space form (i.e., of constant sectional curvature).*

It is clear that Theorems 1.2 and 1.4 together imply the following classification of locally conformally flat Yamabe solitons:

**Corollary 1.5.** *Let  $(M^n, g, f)$  be a nontrivial complete locally conformally flat gradient Yamabe soliton. Then,  $(M^n, g, f)$  is either*

(a) *defined on  $\mathbb{R}^n$ , rotationally symmetric, and equal to the warped product*

$$([0, \infty), dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{\text{can}}), \quad \text{or}$$

(b) *the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}_N),$$

*where  $(N^{n-1}, \bar{g}_N)$  is a space form.*

By Corollary 1.5 and Theorem 1.2 (ii), we have

**Corollary 1.6.** *Suppose  $(M^n, g, f)$  is a nontrivial, non-flat, complete, and locally conformally flat gradient Yamabe soliton with nonnegative scalar curvature  $R \geq 0$  (as in the steady and shrinking cases). Then  $(M^n, g, f)$  is rotationally symmetric: it is either defined on  $\mathbb{R}^n$  and equal to the warped product*

$$([0, \infty), dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{\text{can}}), \quad \text{or}$$

*is the warped product cylinder*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{\text{can}})/\Gamma$$

*for some finite group  $\Gamma \subset SO(n)$ .*

**Remark 1.2.** If we assume  $Rc \geq 0$  in Corollary 1.6 then, by Theorem 1.2 (ii), the warped product cylinder is in fact the round cylinder

$$(\mathbb{R}, dr^2) \times (\mathbb{S}^{n-1}, \bar{g}_{\text{can}})/\Gamma.$$

**Remark 1.3.** As we mentioned before, examples of rotationally symmetric gradient Yamabe solitons on  $\mathbb{R}^n$  with positive sectional curvature  $K > 0$  have been constructed by Daskalopoulos and Sesum [10]. On the other hand, C. He [12] has shown that any complete gradient steady Yamabe soliton on  $M^n = \mathbb{R} \times_{\varphi} N^{n-1}$  is necessarily isometric to the Riemannian product with constant  $\varphi$  and  $N$  being of zero scalar curvature. Moreover, he showed the existence of complete gradient Yamabe shrinking soliton metrics on  $M^n = \mathbb{R} \times_{\varphi} N^{n-1}$  with  $\rho = 1$ ,  $\bar{R} > 0$  and positive non-constant warping function  $\varphi$ .

## 2. Warped product structures of complete gradient Yamabe solitons

We shall follow the notations in [3, 4, 10]. Let  $(M^n, g_{ij}, f)$  be a complete nontrivial Yamabe soliton, satisfying the Yamabe soliton equation:

$$(R - \rho)g_{ij} = \nabla_i \nabla_j f.$$

For any regular value  $c_0$  of the potential function  $f$ , consider the level surface  $\Sigma_{c_0} = f^{-1}(c_0)$ . Suppose  $I$  is an open interval containing  $c_0$  such that  $f$  has no critical points in the open neighborhood  $U_I = f^{-1}(I)$  of  $\Sigma_{c_0}$ . Then we can express the soliton metric  $g$  on  $U_I$  as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + \bar{g}_{\Sigma_{c_0}},$$

where  $\bar{g}_{\Sigma_{c_0}} = g_{ab}(f, \theta)d\theta^a d\theta^b$  is the induced metric and  $\theta = (\theta^2, \dots, \theta^n)$  is any local coordinates system on  $\Sigma_{c_0}$ .

On the other hand, as shown in [10], we have

$$(2.1) \quad \nabla(|\nabla f|^2) = 2\nabla^2 f(\nabla f, \cdot) = 2(R - \rho)\nabla f.$$

Hence,  $|\nabla f|^2$  is constant on any regular level surface  $\Sigma_c = f^{-1}(c) \subset U_I$ , which are all diffeomorphic to  $\Sigma_{c_0}$ . This allows us to make the change of variable by setting, up to an additive constant,

$$(2.2) \quad r(x) = \int \frac{df}{|\nabla f|},$$

so that we can further express the metric  $g$  on  $U_I$  as

$$(2.3) \quad ds^2 = dr^2 + g_{ab}(r, \theta)d\theta^a d\theta^b.$$

Let  $\nabla r = \frac{\partial}{\partial r}$ , then  $|\nabla r| = 1$  and  $\nabla f = f'(r)\frac{\partial}{\partial r}$  on  $U_I$ . Note that  $f'(r)$  does not change sign on  $U_I$  because  $f$  has no critical points there. Thus, we may assume  $I = (\alpha, \beta)$  with  $f'(r) > 0$  for  $r \in (\alpha, \beta)$ . It is also easy to check that

$$(2.4) \quad \nabla \frac{\partial}{\partial r} = 0,$$

so integral curves to  $\nabla r$  are normal geodesics.

Next, by (2.4) and equation (1.1), it follows that

$$(2.5) \quad R - \rho = \nabla^2 f \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = f''(r).$$

Therefore, we conclude immediately that the scalar curvature  $R$  is also constant on  $\Sigma_c \subset U_I$ . Moreover, the second fundamental form of  $\Sigma_c$  is given by

$$(2.6) \quad h_{ab} = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{f''(r)}{f'(r)} g_{ab}.$$

In particular,  $\Sigma_c$  is umbilical and its mean curvature is given by

$$(2.7) \quad H = (n-1) \frac{f''(r)}{f'(r)},$$

which is again constant along  $\Sigma_c$ .

Now, we fix a local coordinates system

$$(2.8) \quad (x^1, x^2, \dots, x^n) = (r, \theta^2, \dots, \theta^n)$$

in  $U_I$ , where  $(\theta^2, \dots, \theta^n)$  is any local coordinates system on the level surface  $\Sigma_{c_0}$ , and indices  $a, b, c, \dots$  range from 2 to  $n$ . Then, computing in this local coordinates system we obtain that

$$h_{ab} = -\langle \partial_r, \nabla_a \partial_b \rangle = -\langle \partial_r, \Gamma_{ab}^1 \partial_r \rangle = -\Gamma_{ab}^1.$$

But the Christoffel symbol  $\Gamma_{ab}^1$  is given by

$$\Gamma_{ab}^1 = \frac{1}{2} g^{11} \left( -\frac{\partial g_{ab}}{\partial r} \right) = -\frac{1}{2} \frac{\partial g_{ab}}{\partial r}$$

Hence, we get

$$(2.9) \quad \frac{\partial g_{ab}}{\partial r} = 2 \frac{f''(r)}{f'(r)} g_{ab}.$$

Then, it follows easily from (2.9) that

$$(2.10) \quad g_{ab}(r, \theta) = \left( \frac{f'(r)}{f'(r_0)} \right)^2 g_{ab}(r_0, \theta).$$

Here the level surface  $\{r = r_0\}$  corresponds to  $\Sigma_{c_0}$ .

Therefore we have arrived at the following:

**Proposition 2.1.** *Let  $(M^n, g_{ij}, f)$  be a complete gradient Yamabe soliton, satisfying the soliton equation (1.1), and let  $\Sigma_c = f^{-1}(c)$  be a regular level surface. Then*

- (a)  $|\nabla f|^2$  is constant on  $\Sigma_c$ ;
- (b) the scalar curvature  $R$  is constant on  $\Sigma_c$ ;
- (c) the second fundamental form of  $\Sigma_c$  is given by  $h_{ab} = \frac{R-\rho}{|\nabla f|} g_{ab}$ ;
- (d) the mean curvature  $H = (n - 1) \frac{R-\rho}{|\nabla f|}$  is constant on  $\Sigma_c$ ;
- (e) in any open neighborhood  $U_\alpha^\beta = f^{-1}((\alpha, \beta))$  of  $\Sigma_c$  in which  $f$  has no critical points, the soliton metric  $g$  can be expressed as

$$(2.11) \quad ds^2 = dr^2 + \frac{(f'(r))^2}{(f'(r_0))^2} \bar{g}_{r_0}.$$

where  $(\theta^2, \dots, \theta^n)$  is any local coordinates system on  $\Sigma_c$  and  $\bar{g}_{r_0} = g_{ab}(r_0, \theta) d\theta^a d\theta^b$  is the induced metric on  $\Sigma_c = r^{-1}(r_0)$ .

**Remark 2.1.** Proposition 2.1(a) was observed first by Daskalopoulos and Sesum [10]; also Proposition 2.1(b)–(d) were proved in [10] under the additional assumption that  $(M^n, g, f)$  is locally conformally flat.

**Remark 2.2.** Our proof of Proposition 2.1 was motivated by arguments in [3–5] for Ricci solitons. After the first version of our paper appeared in the arXiv, we learned that equations similar to equation (1.1) had been studied long time ago by various people, see, e.g., [15] and the references therein. There are also more recent works, e.g., [14] and [7]. In particular, Cheeger and Colding [7] presented beautifully a characterization of warped product structures on a Riemannian manifold  $M$  in terms of solutions to the more general equation

$$\nabla_i \nabla_j f = h g_{ij},$$

where  $h$  is some smooth function on  $M$ .

Next let us investigate the geometry of the regular level surfaces  $\Sigma_c$ . To do so, we first need the curvature tensor formula of a warped product manifold

$$(2.12) \quad (M^n, g) = (I, dr^2) \times_\varphi (N^{n-1}, \bar{g}),$$

where  $g = dr^2 + \varphi^2(r)\bar{g}$ . Fix any local coordinates system  $\theta = (\theta^2, \dots, \theta^n)$  on  $N^{n-1}$ , and choose  $(x^1, x^2, \dots, x^n) = (r, \theta^2, \dots, \theta^n)$ , as in (2.8) for the local coordinates system on  $M$ . From now on indices  $a, b, c, d$  range from 2 to  $n$ . Also curvature tensors

with bar are the curvature tensors of  $(N, \bar{g})$ . Now from either direct computations or [1, 13], the Riemann curvature tensor of  $(M^n, g)$  is given by

$$(2.13) \quad R_{1a1b} = -\varphi^2 \varphi'' \bar{g}_{ab}, \quad R_{1abc} = 0,$$

and

$$(2.14) \quad R_{abcd} = \varphi^2 \bar{R}_{abcd} - (\varphi \varphi')^2 (\bar{g}_{ac} \bar{g}_{bd} - \bar{g}_{ad} \bar{g}_{bc}).$$

The Ricci tensor of  $(M^n, g)$  is

$$(2.15) \quad R_{11} = -(n-1) \frac{\varphi''}{\varphi}, \quad R_{1a} = 0 \quad (2 \leq a \leq n),$$

and

$$(2.16) \quad R_{ab} = \bar{R}_{ab} - [(n-2)(\varphi')^2 + \varphi \varphi''] \bar{g}_{ab} \quad (2 \leq a, b \leq n).$$

Finally, the scalar curvatures of  $(M^n, g)$  and  $(N^{n-1}, \bar{g})$  are related by

$$(2.17) \quad R = \varphi^{-2} \bar{R} - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2 - 2(n-1) \frac{\varphi''}{\varphi}.$$

From (2.15) and (2.16), we easily see the following basic facts:

**Lemma 2.1.** (a) The “radial” Ricci curvature  $R_{11}$  depends only on  $r$ , hence is constant on level surfaces  $\{r\} \times N$ ;

(b)  $(N, \bar{g})$  is Einstein if and only if the eigenvalues of the Ricci tensor, when restricted to  $\{r\} \times N$ , are the same and depend only on  $r$ :

$$(2.18) \quad R_{22}(r, \theta) = \dots = R_{nn}(r, \theta) = \mu(r).$$

**Remark 2.3.** Note that  $M^n$  is Einstein, with  $Rc = \lambda g$ , if and only if  $N^{n-1}$  is Einstein with  $\bar{R}c = \bar{\lambda} \bar{g}$  and the warping function  $\varphi$  solves the first-order ODE

$$(2.19) \quad \varphi'^2 + \frac{\lambda}{n-1} \varphi^2 = \frac{\bar{\lambda}}{n-2}.$$

More details can be found in [1, 9.110].

Now we are ready to finish the proof Theorem 1.2.

*Proof.* Let  $(M^n, g, f)$  be a complete nontrivial gradient Yamabe soliton. From Proposition 2.1, we know that  $|\nabla f|^2$  is constant on regular level surfaces of  $f$ .

Set  $N^{n-1} = f^{-1}(c_0)$  and  $\bar{g} = (f'(r_0))^{-2} \bar{g}_{r_0}$  as in Proposition 2.1 for some regular value  $c_0$  of  $f$ . Then, since the warping function is  $|\nabla f| = f'(r)$ , the warped product formula (2.11) in Proposition 2.1 implies that the potential function  $f$  has at most two critical values. Thus, formula (2.11) extends to some maximal interval  $I_{\max}$ , which is either a finite closed interval  $[\alpha_0, \beta_0]$  with  $f'(\alpha_0) = f'(\beta_0) = 0$ , or a half-line  $[\alpha_0, \infty)$  with  $f'(\alpha_0) = 0$ , or  $(-\infty, \infty)$ . However, the first case cannot happen, for otherwise  $M^n$  would be compact, but compact Yamabe solitons are trivial as we mentioned in Section 1. Thus,  $f$  has at most one critical point and, after a shift in  $r$  if necessary, either  $I_{\max} = [0, \infty)$ , or  $I_{\max} = (-\infty, \infty)$ .

To see that  $(N^{n-1}, \bar{g})$  has constant scalar curvature, note that for our Yamabe soliton  $(M^n, g, f)$ , we have  $R = f''(r) + \rho$  and the warping function is  $\varphi(r) = f'(r)$ . Thus, from (2.17) we get

$$(2.20) \quad \bar{R} = (f')^2 R + (n - 1)[(n - 2)(f'')^2 + 2f' f'''],$$

which does not depend on  $\theta$ . Therefore,  $\bar{R}$  is a constant.

Also, when  $f$  has a unique critical point  $x_0$ ,  $r(x)$  is simply the distance function  $d(x_0, x)$  from  $x_0$ . So level surfaces of  $f$  are geodesic spheres centered at  $x_0$  which are diffeomorphic to  $(n - 1)$ -sphere  $S^{n-1}$ . In addition, by the smoothness of the metric  $g$  at  $x_0$  we can conclude that the induced metric  $\bar{g}$  on  $N$  is round (see, e.g., Lemma 9.114 in [1]).

Finally, assume we are in the case (ii). Then,  $\varphi = f' > 0$  on  $(-\infty, \infty)$  and  $\varphi' = f'' = R - \rho$ . If  $(M^n, g, f)$  has nonnegative Ricci curvature  $Rc \geq 0$ , then by (2.15) we know  $\varphi'' \leq 0$ , so  $\varphi$  is a positive and weakly concave function on  $\mathbb{R}$ . Thus  $\varphi$  must be a constant and hence  $(M^n, g)$  is isomorphic to the Riemannian product  $(\mathbb{R}, dr^2) \times (N, \bar{g})$ . Now assume  $R \geq 0$ . Again we prove  $\bar{R} > 0$  unless  $\bar{R} = 0$  and  $(M^n, g)$  is the Riemannian product  $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$ . If  $\bar{R} \leq 0$ , by (2.20) we know that  $\varphi'' \leq 0$ . So once again  $\varphi$  is a positive weakly concave function on  $\mathbb{R}$  hence a constant function. Therefore, again by (2.20), we know that  $R = \bar{R} = 0$  and  $(M^n, g)$  is the Riemannian product  $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$ .  $\square$

### 3. Classification of locally conformally flat Yamabe solitons

Now let us discuss the classification of locally conformally flat gradient Yamabe solitons and prove Theorem 1.4.

*Proof.* Let  $(M^n, g, f)$  be a nontrivial complete locally conformally flat gradient Yamabe soliton such that  $f$  has no critical point. By Theorem 1.2,  $(M^n, g)$  is a warped product

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}).$$

Clearly, it remains to prove the following:

**Claim 1:**  $(N^{n-1}, \bar{g})$  is a space form.

Indeed, Claim 1 was first proved by Daskalopoulos and Sesum [10] (see Proposition 2.2 and Lemma 2.4(iii) in [10]) where they used B. Chow’s Li–Yau type differential Harnack for locally conformally flat Yamabe flow [9] to show that property (2.18) holds and then deduced that  $(N, \bar{g})$  is a space form. Here we present a simple and direct proof based on the warped product structure of  $(M^n, g, f)$ .

All we need is the explicit formula of the Weyl tensor  $W$  for an arbitrary warped product manifold (2.12) which can be easily deduced from (2.13) to (2.17):

$$(3.1) \quad W_{1a1b} = \frac{\bar{R}}{(n - 1)(n - 2)} \bar{g}_{ab} - \frac{1}{n - 2} \bar{R}_{ab},$$

$$(3.2) \quad W_{1abc} = 0,$$

and

$$(3.3) \quad W_{abcd} = \varphi^2 \overline{W}_{abcd}.$$

Here  $\overline{W}$  denotes the Weyl tensor of  $(N, \overline{g})$ .

Now  $(M^n, g, f)$  is a warped product, with  $\varphi = f'$ , and is locally conformally flat, thus  $W = 0$ . From (3.1) and (3.3), we see that  $(N, \overline{g})$  is Einstein and  $\overline{W} = 0$ . Thus  $N$  is a space form.

This proves Claim 1 and completes the proof of Theorem 1.4.  $\square$

**Remark 3.1.** By (3.1)–(3.3), it is clear that  $(N, \overline{g})$  is a space form if and only if  $(M, g)$  is locally conformally flat.

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