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WEIL-PETERSSON METRICS ON DEFORMATION SPACES

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Abstract. In this paper we survey certain aspects of the classical Weil-Petersson metric and its generalizations. Being a natural L^2 metric on the parameter space of a family of complex manifolds or holomorphic vector bundles which admit some canonical metrics, the Weil-Petersson metric is well defined when the automorphism group of each fiber is discrete and the curvature of the Weil-Petersson metric can be computed via certain integrals over each fiber. We will discuss the case when these fibers have continuous automorphism groups. We also discuss the relation between the Weil-Petersson metric and energy of harmonic maps.

1. Introduction

The Weil-Petersson metric is an important tool in studying the geometry of various moduli spaces of geometric objects equipped with certain canonical metrics. It was first introduced by Weil [30] in the late 1950s by using the Petersson pairing on the space of holomorphic quadratic differentials. Since then the Weil-Petersson metric has been generalized to study the geometry of moduli spaces of manifolds and vector bundles which admit certain canonical metrics.

To start with, let us consider the moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus g with $g \geq 2$. For any point p in $\mathcal{M}_g$ represented by a Riemann surface Σ, it is well-known that the holomorphic tangent space of $\mathcal{M}_g$ at p can be identified with $H^{0,1}\left(\Sigma, T^{1,0}\Sigma\right)$, and that the dual space $\Omega_p^{1,0}\mathcal{M}_g$ can

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be identifies with $H^0\left(K_\Sigma^2\right)$, the space of holomorphic quadratic differentials on Σ. The Weil-Petersson co-metric, as defined by Weil, is given by

$$\|\eta\|_{WP}^2=\int_\Sigma |\eta|_\lambda^2 dV_\lambda,$$

where λ is the hyperbolic metric on Σ and $\eta\in H^0\left(K_\Sigma^2\right)$. It follows that the Weil-Petersson metric on $H^{0,1}\left(\Sigma,T^{1,0}\Sigma\right)$ is defined by

$$\|\varphi\|_{WP}^2=\int_\Sigma |\varphi|_\lambda^2 dV_\lambda$$

for any harmonic Beltrami differential φ with respect to λ.

In the 1960s Ahlfors [1,2] showed that the above Weil-Petersson metric is Kähler and has negative curvature. The latter property also follows from the integral formula of the curvature of the Weil-Petersson metric established by Wolpert [33]: for any local holomorphic coordinates $(t_1,\cdots,t_{3g-3})$ around p, the curvature tensor of the Weil-Petersson metric is given by

$$R_{i\bar{j}k\bar{l}}=\int_\Sigma \left((\Delta-1)^{-1}\left(\varphi_i\cdot\overline{\varphi}_j\right)\left(\varphi_k\cdot\overline{\varphi}_l\right)+(\Delta-1)^{-1}\left(\varphi_i\cdot\overline{\varphi}_l\right)\left(\varphi_k\cdot\overline{\varphi}_j\right)\right)dV_\lambda,$$

where φ_i is the harmonic representative of the Kodaira-Spencer class $KS\left(\frac{\partial}{\partial t_i}\right)$, and the Laplacian Δ has nonpositive spectrum. Moreover, Chu [8] and Masur [15] computed the asymptotics of the Weil-Petersson metric near the compactification divisor of Deligne-Mumford moduli spaces by using degeneration of hyperbolic metrics. In particular, they showed that the Weil-Petersson metric is incomplete and has finite volume.

Koiso [14] first studied the Weil-Petersson metric on the moduli space of Kähler-Einstein manifolds and proved that it is Kähler. Royden and Siu [21] developed the canonical lifting method to compute the curvature of the Weil-Petersson metric on the moduli spaces of Kähler-Einstein manifolds of general type. Their method was adopted by Nannicini [17] to establish the integral formula of the curvature of the Weil-Petersson metric on the moduli spaces of polarized Calabi-Yau manifolds. Later, Candelas, Green and Hübsch [6] wrote down a local Kähler potential of the Weil-Petersson metric by using the L^2 metric on the first Hodge bundle. Thus, in the polarized Calabi-Yau case, the Weil-Petersson metric is the curvature of the L^2 metric on the first Hodge bundle. It follows that the curvature formula of the Weil-Petersson metric can be derived from the Taylor expansion of canonical local sections of the first Hodge bundle whose construction was due to Todorov [25]. The lifting method was refined by Schumacher [19] by using Kähler-Einstein metrics which lead to the integral curvature formula of the Weil-Petersson metric on the moduli space of Kähler-Einstein manifolds in general, including the Fano case.

In the above cases the Weil-Petersson metric is defined as the L^2 metric on the space of harmonic Beltrami differentials with respect to fiberwise Kähler-Einstein metrics. Precisely, let $\pi:\mathfrak{X}\to B$ be a family of Kähler-Einstein manifolds with fiber $X_t=\pi^{-1}(t)$ and Kähler-Einstein metric ω_t on X_t. For any tangent vector $u\in T_t^{1,0}B$, we let $KS(u)\in H^{0,1}\left(X_t,T^{1,0}X_t\right)$ be its image under the

Kodaira-Spencer map and let φ be the harmonic representative of $KS(u)$ with respect ω_t. Then,

$$\|u\|_{WP}^2 = \int_{X_t} |\varphi|_{\omega_t}^2 \frac{\omega_t^n}{n!},$$

where n is the complex dimension of X_t. The Weil-Petersson metric in these cases **is** well-defined as long as the Kähler-Einstein metric on each X_t is unique. This is the case when the first Chern class $c_1(X_t) < 0$, or $c_1(X_t) = 0$ with a fixed polarization, or $c_1(X_t) > 0$ with a discrete automorphism group.

However, when X_t is a Fano Kähler-Einstein manifold with non-discrete automorphism group, there is a family of Kähler-Einstein metrics on X_t. In this case the L^2 inner product and harmonic representatives of the Kodaira-Spencer classes vary according to the choices of Kähler-Einstein metrics. It is not clear a priori that the Weil-Petersson metric is well-defined. We will discuss this problem in later sections and present in Section 3 a sufficient condition that guarantees the Weil-Petersson metric **to be** well-defined. There is also a canonical Weil-Petersson metric on the (complexified) Kähler moduli spaces which was defined in a similar fashion [27, 31]. These Weil-Petersson metrics play an important role in mirror symmetry.

We also note that there is a natural Weil-Petersson metric on the moduli space of stable vector bundles over a Kähler manifold X. For a family B of stable bundles over X, if we denote by E the bundle over X corresponding to a point $t \in B$, then the Kodaira-Spencer map is

$$KS : T_t^{1,0}B \to H^{0,1}(X, \mathrm{End}(E)).$$

The Weil-Petersson metric is defined in a similar way by using the Hermitian-Einstein metric on E together with the Kähler metric on the base manifold X. Schumacher and Toma [20] established an integral curvature formula **in a situation as above.** Recently, Fan, Kanazana and Yau [11] defined a Weil-Petersson metric on the space of Bridgeland stability conditions which lead to the study of stringy Kähler moduli spaces. Being a classical L^2 metric on the parameter space of a family of complex manifolds, by pulling back to the total space, the Weil-Petersson metric can be used to construct Kähler metrics on the total space.

We remark that there are also infinite dimensional analogs of the Weil-Petersson metric. For example, let (X, ω) be a symplectic manifold and let $\mathcal{J}$ be the space of almost complex structures on X compatible with ω. One can then identify $\mathcal{J}$ with a subset of $\Gamma\left(T^{1,0}X_{J_0} \otimes T^{*\,0,1}X_{J_0}\right)$ for any $J_0 \in \mathcal{J}$. The Donaldson-Fujiki metric [9, 12] is the Weil-Petersson type L^2 metric on $\Gamma\left(T^{1,0}X_{J_0} \otimes T^{*\,0,1}X_{J_0}\right)$.

2. Kuranishi space of Fano Kähler-Einstein manifolds

It turns out that the behavior of the Weil-Petersson metric on the moduli space of Fano Kähler-Einstein manifolds is closely related to the action of the automorphism group of a Kähler-Einstein manifold on its Kuranishi space. Let (X_0, ω_0) be a Fano manifold of dimension $n \geq 2$. In the following, we will use $\mathbb{H}$ to denote the harmonic spaces or harmonic projections with respect to the metric ω_0.

We first note that, by Serre duality, we have

$$H^{0,k}\left(X_0, T^{1,0}X_0\right) \cong H^{0,n-k}\left(X_0, K_{X_0} \otimes \Omega^{1,0}X_0\right)^\vee \cong H^{1,n-k}\left(X_0, K_{X_0}\right)^\vee.$$

Since $c_1(X_0) > 0$, we know that K_{X_0} is a negative line bundle. Thus, by the Kodaira vanishing theorem, we have $H^{1,n-k}(X_0, K_{X_0}) = 0$ for $1+n-k < n$. In particular, $H^{0,2}\left(X_0, T^{1,0}X_0\right) = 0$, which implies that deformation of the complex structure on X_0 is unobstructed. Now we take a basis $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}X_0\right)$ and identify this space with $\mathbb{C}^N$. The Kodaira-Spencer-Kuranishi theory implies that the **system of equations**

$$\begin{cases} \overline{\partial}_0 \varphi(t) = \frac{1}{2}\left[\varphi(t), \varphi(t)\right] \\ \overline{\partial}_0^* \varphi(t) = 0 \\ \mathbb{H}\left(\varphi(t)\right) = \sum_{i=1}^N t_i \varphi_i \end{cases} \tag{2.1}$$

have a unique power series solution $\varphi(t) = \sum_{i=1}^N t_i\varphi_i + \sum_{|I|\geq 2} t^I \varphi_I$, where $\varphi_I \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$, and the power series converges in a ball $B \subset \mathbb{C}^N$. In this case we can construct the Kuranishi family $(\mathfrak{X}, B, \pi)$ of X_0 by letting $\mathfrak{X} \overset{C^\infty}{\cong} X \times B$ as a smooth manifold with X being the underlying smooth manifold of X_0, $\pi : \mathfrak{X} \to B$ **being** the projection to the second factor, and

$$\Omega^{1,0}_{p,t}\mathfrak{X} = \left(I + \varphi(t)\right)\left(\Omega^{1,0}_p X_0\right) \oplus \pi^* \Omega^{1,0}_t B.$$

It is well-known that the Kuranishi space B parameterizes all small deformations of the complex manifold X_0, and such **coordinate** $(t_1, \ldots, t_N)$ on the Kuranishi space is unique up to affine transformations. This is the flat coordinates when ω_0 is a Kähler-Einstein metric. In particular, if a Beltrami differential $\varphi \in A^{0,1}\left(X_0, T^{1,0}X_0\right)$ defines an integrable complex structure and satisfies the Kuranishi gauge $\overline{\partial}_0^* \varphi = 0$ then $\varphi = \varphi(t)$, where $\mathbb{H}(\varphi) = \sum_{i=1}^N t_i\varphi_i$, provided $\|\varphi\|$ is small. Namely, such φ is determined by its harmonic part.

Now we let $G = \mathrm{Aut}_0(X_0)$ and $K = \mathrm{Isom}_0(X_0)$ be the identity components of the holomorphic automorphism group and isometry group, respectively. Then, the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$ are **respectively** the spaces of holomorphic vector fields and Killing fields. If the Kuranishi family $(\mathfrak{X}, B, \pi)$ is universal at $0 \in B$, then $\mathrm{Aut}(X_0)$ acts naturally on B by universality. However, the Kuranishi family of a Fano manifold may not be universal in general, and thus we define the action analytically. We first look at the action of the isometry group on B. For each $\sigma \in K$, since it is isotopic to the identity map**,** it acts trivially on the cohomology and thus preserves the harmonic form ω_0 **since** it preserves the metric. It follows that σ is a biholomorphism.

The diffeomorphism group of X acts on the space $\mathcal{J}$ of complex structures on X by pulling back and **thereby** induces an action on the space of integrable Beltrami differentials. Let $U \subset \mathrm{Diff}_0(X)$ be a small neighborhood of the identity map. For each $\sigma \in U$ and Beltrami differential φ with $\overline{\partial}_0\varphi = \frac{1}{2}[\varphi, \varphi]$ and $\|\varphi\|$ small, the Beltrami differential $\psi = \varphi \circ \sigma$ defines a complex structure corresponding to the action of σ on $\mathcal{J}$, where ψ is characterized by

$$\frac{\partial \sigma_k}{\partial \overline{z}_j} + \varphi^k_{\overline{l}} \frac{\partial \overline{\sigma}_l}{\partial \overline{z}_j} = \psi^i_{\overline{j}} \left(\frac{\partial \sigma_k}{\partial z_i} + \varphi^k_{\overline{l}} \frac{\partial \overline{\sigma}_l}{\partial z_i} \right). \tag{2.2}$$

It follows from this formula of Kodaira that $\varphi \circ \sigma = \sigma^* \varphi$ if σ is a biholomorphism of X_0, and $\varphi \circ \sigma = \varphi$ if σ is a biholomorphism of X_φ.

For any $\sigma \in K \cap U$ and $t \in B$, where we shrink B if necessary, we know that $\varphi(t) \circ \sigma = \sigma^* \varphi(t)$ which defines a complex structure on X and satisfies the Kuranishi gauge $\overline{\partial}_0^* (\sigma^* \varphi(t)) = 0$ since σ preserves the Kähler structure on X_0. It follows that $\varphi(t) \circ \sigma = \varphi(t')$, where $\sum_{i=1}^N t_i' \varphi_i = \mathbb{H}_0 (\sigma^* \varphi(t))$, and this gives an action of $K \cap U$ on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$. Clearly this action extends to an action of K on $T_0^{1,0} B \cong H^{0,1}\left(X_0, T^{1,0} X_0\right)$.

When ω_0 is a (Fano) Kähler-Einstein metric, by the work of Matsushima [16] and Calabi [5], we know that G is the complexification of K and

$$\mathfrak{g} \cong \left\{ \nabla^{1,0} f \mid f \in \Lambda_1^{\mathbb{C}} \right\}$$

$$\mathfrak{k} \cong \left\{ \operatorname{Im}\left(\nabla^{1,0} f\right) \mid f \in \Lambda_1^{\mathbb{R}} \right\},$$

where $\Lambda_1^{\mathbb{R}}$ is the eigenspace $\{ f \in C^\infty (X, \mathbb{R}) \mid (\Delta_0 + 1) f = 0 \}$, $\Delta_0 = g^{i\bar{j}} \partial_i \partial_{\bar{j}}$ is the Laplace operator with respect to the Kähler-Einstein metric ω_0, and $\Lambda_1^{\mathbb{C}} = \Lambda_1^{\mathbb{R}} \otimes \mathbb{C}$. In this case, the action of K on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$ naturally extends to an action of $G = K^{\mathbb{C}}$. **At the Lie algebra level,** the action of $\mathfrak{g}$ on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$ is given by

$$v([\varphi]) = [L_v \varphi]$$

for any $\varphi \in A^{0,1}\left(X_0, T^{1,0} X_0\right)$ with $\overline{\partial}_0 \varphi = 0$ and any holomorphic vector field $v \in H^0\left(X_0, T^{1,0} X_0\right)$. Here, L_v denotes the Lie derivative in the direction of v. It is clear that the action of G (or K) on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$ is trivial if and only if the action of $\mathfrak{g}$ is trivial.

In the rest of this section and Section 3, we assume that ω_0 is a Fano Kähler-Einstein metric on X_0. To study the relation between the action of G on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$ and the existence of Kähler-Einstein metrics on each fiber X_t, we first observe that the complex structure on X_t is compatible with the symplectic form ω_0. In fact, by using the Kähler-Einstein condition, we have the identity

$$\Delta_{\overline{\partial}_0} (\varphi \lrcorner \omega_0) = \frac{\sqrt{-1}}{2} \operatorname{div}_0 \left(\overline{\partial} \varphi\right) + \varphi \lrcorner \omega_0$$

for any $\varphi \in A^{0,1}\left(X_0, T^{1,0} X_0\right)$ with $\overline{\partial}_0 (\varphi \lrcorner \omega_0) = 0$ and $\overline{\partial}_0^* \varphi = 0$. Here $\Delta_{\overline{\partial}_0}$ is the Hodge Laplacian. By using a family of Bochner identities, we obtained the following result in [7].

Proposition 2.1. *The Kuranishi gauge $\overline{\partial}_0^* \varphi(t) = 0$ is equivalent to the divergence gauge $\operatorname{div}_0 \varphi(t) = 0$. As a consequence, we have*

$$\varphi(t) \lrcorner \omega_0 = 0 \tag{2.3}$$

for each $t \in B$. In particular, ω_0 is a Kähler form on X_t.

If the action of K (thus G and $\mathfrak{g}$) on $H^{0,1}\left(X_0, T^{1,0} X_0\right)$ is trivial, then for each $\sigma \in K$ we have $\varphi(t) \circ \sigma = \sigma^* \varphi(t) = \varphi(t)$, which implies that $\sigma \in \operatorname{Aut}_0(X_t)$. Namely, σ is a biholomorphism of X_t. Since σ preserves ω_0, we conclude that σ is an isometry of the Kähler manifold (X_t, ω_0). Here, we identify X_0 and X_t as smooth manifolds via the Kuranishi gauge. This implies that K can be embedded into the isometry group $\mathrm{Isom}_0(X_t, \omega_0)$, and we have

$$\dim_{\mathbb{C}} \mathrm{Aut}_0(X_t) \geq \dim_{\mathbb{R}} \mathrm{Isom}_0(X_t, \omega_0) \geq \dim_{\mathbb{R}} K = \dim_{\mathbb{C}} G.$$

At the Lie algebra level, we have $h^0\left(X_t, T^{1,0}X_t\right) \geq h^0\left(X_0, T^{1,0}X_0\right)$. By the upper semi-continuity of $h^0\left(X_t, T^{1,0}X_t\right)$ in t, we have

$$h^0\left(X_t, T^{1,0}X_t\right) = h^0\left(X_0, T^{1,0}X_0\right)$$

for each $t \in B$. Namely, the dimension of the space of holomorphic vector fields on X_t remains constant.

On the other hand, if $\dim H^0\left(X_t, T^{1,0}X_t\right)$ is constant in t then G acts trivially on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$. To see this, we note that, by the work of Kodaira, the above assumption implies that every holomorphic vector field $v \in H^0\left(X_0, T^{1,0}X_0\right)$ can be extended to $v(t) \in H^0\left(X_t, T^{1,0}X_t\right)$ for each $t \in B$. By linearizing the equation $\overline{\partial}_t v(t) = 0$ at $t = 0$, one easily obtains the identity

$$[v, \varphi_i] = -\overline{\partial}_0 \left(\left.\frac{\partial}{\partial t_i}\right|_{t=0} \tau_t^{-1} v(t) \right),$$

where

$$\tau_t : A^{0,1}\left(X_0, T^{1,0}X_0\right) \to A^{0,1}\left(X_t, T^{1,0}X_t\right)$$

is a linear isomorphism canonically associated to the deformation. This implies **vanishing of** the cohomology class $[[v, \varphi_i]] = 0$ since the section $[v, \varphi_i]$ is $\overline{\partial}_0$-exact. Since $L_v \varphi_i = [v, \varphi_i]$, we can conclude that the action of the Lie algebra $\mathfrak{g}$ on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$ is trivial and thus the actions of G and K are trivial.

The above claims are equivalent to the existence of Kähler-Einstein metrics on small deformations of X_0. Indeed, in [7] we proved the following

Theorem 2.2. *Let X_0 be a Fano Kähler-Einstein manifold and let $(\mathfrak{X}, B, \pi)$ be a Kuranishi family of X_0. Then the following statements are equivalent:*

1. *There exists a Kähler-Einstein metric on X_t for each $t \in B$;*
2. *The action of G on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$ is trivial;*
3. *The dimension of $H^{0,1}\left(X_t, T^{1,0}X_t\right)$ remains constant for all $t \in B$.*

We remark that it was pointed out by Donaldson [10] that if the action of G on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$ is nontrivial, then the Kähler-Einstein metrics cannot form a smooth family. On the other hand, by direct computations, the equivalence of the Kuranishi gauge and the divergence gauge implies that the action of $\mathfrak{g}$ on $H^{0,1}\left(X_0, T^{1,0}X_0\right)$ is trivial if and only if

$$P = \left\{ \varphi_i \cdot \overline{\varphi}_j \mid \varphi_i, \varphi_j \in \mathcal{H}^{0,1}(X_0, T^{1,0}X_0), 1 \leq i, j \leq N \right\} \perp_{L^2} \Lambda_1^{\mathbb{C}}. \tag{2.4}$$

Schumacher [19] studied the curvature of the relative anti-canonical bundle $K_{\mathfrak{X}/B}^{-1}$ with respect to the fiberwise Kähler-Einstein metric and showed that the above space P lies in the image of the self-adjoint operator $\Delta_0 + 1$. Thus, the first statement in Theorem 2.2 implies the second since $\Lambda_1^{\mathbb{C}}$ is the kernel of $\Delta_0 + 1$. Schumacher's work was based on the existence of smooth family of Kähler-Einstein metrics.

In the current situation this follows from the works of Brönnle [4] and Székelyhidi [24] by using the implicit function theorem.

On the other hand, if we assume **that** the dimensions of the spaces of holomorphic vector fields do not jump, then, by an inverse function theorem type argument, we can show that there exists an extremal metric on X_t in the class $c_1(X_t)$ for each $t \in B$. Thus, the first statement follows if we can show that the Futaki invariant of X_t vanishes. Now by using identity (2.3) in Proposition 2.1, we find that the Ricci potential of the Kähler manifold (X_t, ω_0) is given by $\log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)$. As discussed above, each biholomorphism $\rho \in U \subset \mathrm{Aut}_0(X_t)$ preserves $\varphi(t)$, where U is some neighborhood of the identity. It follows that

$$\int_{X_t} \rho^*\left(\log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)\right)\frac{\omega_0^n}{n!} = \int_{X_t} \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)\frac{\omega_0^n}{n!}.$$

Linearizing this equation at the identity, we obtain

$$\mathrm{Fut}_{X_t}(v) = \int_{X_t} v\left(\log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)\right)\frac{\omega_0^n}{n!} = 0$$

for each holomorphic vector field $v \in H^0\left(X_t, T^{1,0}X_t\right)$, hence the first statement follows. We refer the reader to [7] for more details of the proof.

It turns out, as we shall explain in the next section, that the second statement in Theorem 2.2 also guarantees the Weil-Petersson metric **to be** well-defined.

3. The Weil-Petersson metric

When (X_0, ω_0) is a Fano Kähler-Einstein manifold and $G = \mathrm{Aut}_0(X_0)$ is nontrivial, Bando and Mabuchi [3] showed that any Kähler-Einstein metric ω on X_0 is the pull back $\omega = \rho^*\omega_0$ for some biholomorphism ρ. For any cohomology class $A \in H^{0,1}\left(X_0, T^{1,0}X_0\right)$, the harmonic representative of A depends on the choice of the Kähler-Einstein metric, and so does the L^2 inner product. Thus the Weil-Petersson metric at $0 \in B$ may **a priori** depend on the choice of the Kähler-Einstein metric.

We pick a one-parameter family of biholomorphisms ρ_t such that $\rho_0 = id$ and let $\omega_t = \rho_t^*\omega_0$. For any two cohomology classes $A, B \in H^{0,1}\left(X_0, T^{1,0}X_0\right)$, let φ_t and ψ_t be the harmonic representative of A and B with respect to ω_t, respectively. Then the Weil-Petersson metric on $T_0^{1,0}B$ with respect to ω_t is

$$h_t(A, B) = \int_{X_0} \langle \varphi_t, \psi_t \rangle_{\omega_t} \frac{\omega_t^n}{n!}.$$

Thus the Weil-Petersson metric is well-defined if $h_t(A, B)$ is independent of t. This is equivalent to $\left.\frac{d}{dt}\right|_{t=0} h_t(A, B) = 0$ since ω_0 is an arbitrary Kähler-Einstein metric. Note that on a Kähler manifold (X, ω) with Beltrami differentials $\varphi, \psi \in A^{0,1}\left(X, T^{1,0}X\right)$, if $\varphi \lrcorner \omega = 0$ or $\psi \lrcorner \omega = 0$ then $\langle \varphi, \psi \rangle_\omega = \mathrm{Tr}\left(\varphi\overline{\psi}\right)$. Furthermore, since $[\varphi_t] = A$, we have $\varphi_t = \varphi_0 + \overline{\partial}\eta_t$ for a smooth family $\{\eta_t\} \subset A^0\left(X_0, T^{1,0}X_0\right)$ with $\eta_0 = 0$. Similarly, $\psi_t = \psi_0 + \overline{\partial}\tau_t$. We let $\widetilde{\eta} = \left.\frac{d}{dt}\right|_{t=0} \eta_t$ and $\widetilde{\tau} = \left.\frac{d}{dt}\right|_{t=0} \tau_t$.

It follows from identity (2.3) that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\langle\varphi_t,\psi_t\rangle_{\omega_t} =&\frac{d}{dt}\bigg|_{t=0}\mathrm{Tr}\left(\varphi_t\overline{\psi_t}\right)\\ =&\mathrm{Tr}\left(\left(\overline{\partial}\widetilde{\eta}\right)\overline{\psi_0}\right)+\mathrm{Tr}\left(\varphi_0\left(\overline{\overline{\partial}\widetilde{\tau}}\right)\right)\\ =&\langle\overline{\partial}\widetilde{\eta},\psi_0\rangle_{\omega_0}+\langle\varphi_0,\overline{\partial}\widetilde{\tau}\rangle_{\omega_0}.\end{aligned}$$

Thus, after integration by parts, we have

$$\frac{d}{dt}\bigg|_{t=0}h_t(A,B)=\int_{X_0}\langle\varphi_0,\psi_0\rangle_{\omega_0}\frac{d}{dt}\bigg|_{t=0}\left(\frac{\omega_t^n}{n!}\right)=\int_{X_0}\langle\varphi_0,\psi_0\rangle_{\omega_0}L_v\frac{\omega_t^n}{n!},$$

where $v=\frac{d}{dt}\Big|_{t=0}\rho_t$ is a real holomorphic vector field. Since the space of such vector fields on X_0 is spanned by $\left\{\mathrm{Re}\left(\nabla^{1,0}f\right),\mathrm{Im}\left(\nabla^{1,0}f\right)\mid f\in\Lambda_1^{\mathbb{R}}\right\}$ and each isometry $\sigma\in K$ preserves ω_0, we can take $v=\mathrm{Re}\left(\nabla^{1,0}f\right)$ for some $f\in\Lambda_1^{\mathbb{R}}$. In this case, $L_v\frac{\omega_t^n}{n!}=\Delta_{\omega_0}f\frac{\omega_0^n}{n!}$ and

$$\frac{d}{dt}\bigg|_{t=0}h_t(A,B)=\int_{X_0}\langle\varphi_0,\psi_0\rangle_{\omega_0}\Delta_{\omega_0}f\frac{\omega_0^n}{n!}=-\int_{X_0}\langle\varphi_0,\psi_0\rangle_{\omega_0}f\frac{\omega_0^n}{n!},$$

which vanishes by (2.4) if any one of the three equivalent statements in Theorem 2.2 holds. In [7] we have shown

Theorem 3.1. *Let X_0 be a Fano Kähler-Einstein manifold and let $(\mathfrak{X},B,\pi)$ be its Kuranishi family. If for each $t\in B$ there exists a Kähler-Einstein metric on X_t, then the Weil-Petersson metric is well-defined.*

Another interesting question is to approximate the Weil-Petersson metric by algebraic quantities. In the case when X_0 is a Kähler-Einstein manifold with $c_1(X_0)<0$ and the deformation of the complex structure on X_0 is unobstructed, if $(\mathfrak{X},B,\pi)$ is the Kuranishi family of X_0 then there is a natural L^2 metric on the direct image sheaf $R^0\pi_*K^m_{\mathfrak{X}/B}$. In this case, it was shown that the curvature of the determinant bundle $\Lambda^{\mathrm{top}}R^0\pi_*K^m_{\mathfrak{X}/B}$, under suitable normalization, converges to the Weil-Petersson metric as m goes to infinity [22,23]. Similar results hold in the case $c_1(X_0)>0$ when the automorphism group G is discrete. Indeed, in [7] we obtained the following result.

Theorem 3.2. *Let $\pi:\mathcal{X}\to B$ be an analytic family of Fano Kähler-Einstein manifolds of dimension n with central fiber $\pi^{-1}(0)=(X_0,\omega_0)$, and assume that $Aut_0(X_0)$ is trivial. Let $R^m_{i\bar{j}}$ be the Ricci curvature of the L^2 metric on the direct image sheaf $R^0\pi_*(K^m_{\mathcal{X}/B})$. Then the Weil-Petersson metric can be approximated by the Ricci curvatures with normalization. Precisely, we have*

$$\lim_{m\to\infty}\frac{\pi^n}{m^n}R^m_{i\bar{j}}=-\int_{X_0}\left(\varphi_i\cdot\overline{\varphi}_j\right)dV_0,$$

where φ_i is the harmonic representative of the Kodaira-Spencer class.

In both cases, the Kähler-Einstein metric on each X_t is unique. Thus, if we identify X_t with X_0 as smooth manifolds via the Kuranishi-divergence gauge, the Kähler forms of the Kähler-Einstein metrics can be viewed as a family of symplectic forms on a fixed manifold and have an expansion in term of

t. In particular, the second order term in the expansion is closely tied to the Weil-Petersson metric. This plays a crucial role in the above approximation.

When $c_1(X_0) > 0$ and G is non-discrete, if we assume that each X_t admits a Kähler-Einstein metric, then the Kähler-Einstein metrics on X_t form a smooth fiber bundle over the base B of the Kuranishi family. It would be interesting to see whether one can find a natural section of this bundle such that the above approximation still holds.

4. Energy of harmonic maps

For Riemann surfaces of genus $g \geq 2$, the energy of harmonic maps is tightly connected to the Weil-Petersson geometry of the Teichmüller space $\mathcal{T}_g$ via the works of Tromba [28, 29], Wolf [32], Jost-Yau [13], etc.

For each point $p \in \mathcal{T}_g$, let Σ_p be a corresponding Riemann surface equipped with the hyperbolic metric. For any **given** point $p \in \mathcal{T}_g$, Tromba showed that the function $E' : \mathcal{T}_g \to \mathbb{R}$ provides a Kähler potential of the Weil-Petersson metric on $\mathcal{T}_g$ at point p, where $E'(q)$ is the energy of the harmonic map from Σ_q to Σ_p which is isotopic to the identity map. In another direction, Wolf [32] showed that the energy function E'', where $E''(q)$ is the energy of the harmonic map from Σ_p to Σ_q in the isotopy class of the identity map, is also a Kähler potential of the Weil-Petersson metric at point p.

Toledo [26] generalized Tromba's work in the following way. Let (N, h) be a Riemannian manifold whose curvature is Hermitian nonpositive, and let $E : \mathcal{T}_g \to \mathbb{R}$ be the energy function such that $E(p)$ is the energy of a harmonic map from Σ_p to N in a fixed homotopy class. Toledo showed that the function E is plurisubharmonic on $\mathcal{T}_g$. By the work of Schoen and Yau [18], one can further show that this function is proper and thus **give** a proof of the pseudoconvexity of $\mathcal{T}_g$. Soon after Toledo's work, Yau pointed out that such functions could be used to study the Teichmüller spaces of higher dimensional Kähler-Einstein manifolds of general type **of which** very little is known even in the case that the deformation of complex structures is unobstructed. This is indeed **possibel** as was carried out in [34] back in 2014. One of the key components in computing the complex Hessian of E is the expansion of the Kähler forms of Kähler-Einstein metrics. In [7], we derived the deformation of Kähler-Einstein metrics with respect to the Kuranishi gauge by deforming the Monge-Ampére equation

$$\omega_t = \omega_0 + |t|^2 \left(\sqrt{-1}\partial_0\overline{\partial}_0 \left((1-\Delta_0)^{-1} |\varphi|^2 \right) \right) + O\left(|t|^3\right),$$

where φ is the harmonic Beltrami differential representing the Kodaira-Spencer class of $\frac{\partial}{\partial t}$ at 0.

Theorem 4.1. *Let X_0 be a Kähler-Einstein manifold of general type such that the deformation of its complex structure is unobstructed. Let (N, h) be a Riemannian manifold with Hermitian nonpositive curvature. Let $\mathcal{T}$ be the Teichmüller space of X_0 and let $E : \mathcal{T} \to \mathbb{R}$ be the energy function such that $E(p)$ is the energy of a harmonic map from X_p to N in a fixed homotopy class. Then E is a plurisubharmonic function on $\mathcal{T}$.*

In fact, if we only assume **that** the sectional curvature of N is nonpositive then the complex Hessian of E is given by

$$\frac{\partial^2 E}{\partial t \partial \bar{t}}\bigg|_{t=0} = -\int_{X_0} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\bar{j}} f_0^\gamma \partial_p f_0^\beta \partial_{\bar{q}} f_0^\delta g^{i\bar{j}} g^{p\bar{q}} K \, dV_0 + \int_{X_0} \|\nabla^{1,0}\overline{\partial} f_0\|^2 K \, dV_0$$
$$- 2\int_{X_0} g^{i\bar{j}} R^N_{\alpha\beta\gamma\delta} \partial_i f_0^\alpha \partial_{\bar{j}} f_0^\gamma u^\beta \overline{u}^\delta dV_0 + 2\int_{X_0} \|\nabla^{1,0}\overline{u} - \overline{\varphi}_1 \lrcorner \overline{\partial} f_0\|^2 dV_0,$$

where $u = \frac{\partial f_t}{\partial t}\Big|_{t=0} \in \Gamma\left(f_0^* T^{\mathbb{C}} N\right)$ and $K = (1-\Delta_0)^{-1}\left(|\varphi|^2\right)$. If the curvature of N is Hermitian nonpositive, then the third term on the right side of the above formula is non-negative and the first two terms vanish due to the Siu-Sampson vanishing theorem. The plurisubharmonicity of E then follows immediately. We refer the interested readers to [7] for further details.

Clearly, it would be very interesting to know if the function E in Theorem 4.1 is proper. This will be discussed elsewhere.

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