

DEGENERATION OF CURVES ON SOME POLARIZED TORIC SURFACES

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ABSTRACT. We address the following question: *Given a polarized toric surface (S, \mathcal{L}) , and a general integral curve C of geometric genus g in the linear system $|\mathcal{L}|$, do there exist degenerations of C in $|\mathcal{L}|$ to general integral curves of smaller geometric genera?* We give an affirmative answer to this question for surfaces associated to h -transverse polygons, provided that the characteristic of the ground field is large enough. We give examples of surfaces in small characteristic, for which the answer to the question is negative. In case the answer is affirmative, we deduce that a general curve C as above is nodal. In characteristic 0, we use the result to show the irreducibility of Severi varieties of a large class of polarized toric surfaces with h -transverse polygon.

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1. INTRODUCTION

In algebraic geometry, degenerations are one of the main tools in the study of algebraic curves. They play a central role in enumerative questions [CH98a], Brill-Noether theory [Kle76, GH80], irreducibility problems [Har86], etc. In the current paper, we investigate degenerations of curves on toric surfaces. Our main motivation comes from the Severi problem and Harris' approach to it, which we briefly recall. Roughly speaking, Severi varieties parametrize integral curves of a fixed geometric genus in a given linear system. The problem of the irreducibility of Severi varieties in the planar case, the classical Severi problem, had been a long-standing open problem until settled in characteristic 0 by Harris in 1986 [Har86]. In our preceding paper [CHT20], we extended his result to arbitrary characteristic. The present paper is concerned with the analogous problem for toric surfaces.

Harris' approach goes back to the original ideas of Severi, and consists of two parts: first, one proves that the closure of any irreducible component of the Severi variety contains the Severi

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variety $V_{0,d}^{\text{irr}}$ parametrizing integral rational curves; this amounts to a statement about the possible degenerations of curves in the given linear system. Second, one describes the branches of the original Severi variety along $V_{0,d}^{\text{irr}}$, and uses monodromy arguments to show that all branches belong to the same irreducible component. A similar approach was used by the third author in [Tyo07] to prove the irreducibility of Severi varieties on Hirzebruch surfaces in characteristic 0.

In this paper, we prove the degeneration part of the argument. We restrict our attention to the case of polarized toric surfaces associated to so-called h -transverse polygons. This is a rich class of surfaces that includes the projective plane, Hirzebruch surfaces, and many other examples. In particular, this class includes examples of toric surfaces admitting reducible Severi varieties [Tyo14, LT20]; see Example 7.2. Our main result asserts that any irreducible component of a Severi variety on such a surface contains in its closure the Severi variety parametrizing integral rational curves, provided that the characteristic of the ground field is either 0 or large enough. Moreover, any integral curve of geometric genus g in the linear system associated to an h -transverse polygon can be degenerated to an integral curve of any genus $0 \leq g' \leq g$.

In the planar case, the monodromy part of the argument is relatively easy, and the main difficulty lies in the degeneration step. We shall mention, however, that in general the monodromy argument is subtle, and there are examples of toric surfaces for which it fails. Moreover, some toric surfaces admit reducible Severi varieties, and the structure of the monodromy groups seems to be the main reason for their reducibility. An interested reader can find more about such examples in the recent paper of Lang and the third author [LT20], and about the computation of the monodromy groups in the case of toric surfaces in Lang's [Lan20].

1.1. The main results. We work over an algebraically closed field K . To formulate the main result, we need the notion of the width of an h -transverse polygon. Recall that an h -transverse polygon in \mathbb{R}^2 is a convex lattice polygon Δ , whose horizontal slices at integral heights $y = k \in \mathbb{Z}$ are either empty or intervals with integral boundary points. We define the *width* of an h -transverse polygon Δ to be the maximal length of its horizontal slices and denote it by $w(\Delta)$.

Main Theorem. *Let $\Delta \subset \mathbb{R}^2$ be an h -transverse lattice polygon, and $(S_\Delta, \mathcal{L}_\Delta)$ the associated polarized toric surface. If $\text{char}(K) = 0$ or $\text{char}(K) > w(\Delta)/2$, then any irreducible component of the Severi variety parametrizing integral curves of geometric genus $g \geq 0$ in the linear system $|\mathcal{L}_\Delta|$ contains in its closure the Severi variety parametrizing integral rational curves in $|\mathcal{L}_\Delta|$.*

We refer the reader to Theorem 5.6 for a stronger version of this statement, in which we allow for some fixed tangency conditions with the toric boundary of S_Δ . The bound $\text{char}(K) > w(\Delta)/2$ is sharp for $\text{char}(K) \geq 3$: In Proposition 5.14 we provide for every prime p an explicit example of a polarized toric surface with h -transverse polygon of width $\max\{6, 2p\}$ and a component of the corresponding Severi variety parametrizing genus 1 curves, that violates the assertion of the theorem if $\text{char}(K) = p$.

We give two applications of the Main Theorem. First, we generalize Zariski's Theorem in Theorem 6.1; that is, we show under the assumptions of the Main Theorem, except requiring $\text{char}(K) > w(\Delta)$ instead of $\text{char}(K) > w(\Delta)/2$, that a general element in any irreducible component of a Severi variety is a nodal curve. This was previously known only in characteristic 0 [KS13], and there are examples of polarized toric surfaces, for which the assertion fails in positive characteristic [Tyo13]. In fact, the bound $\text{char}(K) > w(\Delta)$ in Zariski's Theorem is sharp as Example 6.4 shows. Our argument goes as follows: we prove by explicit calculations that a general integral rational curve is nodal. By the Main Theorem, any irreducible component of a Severi variety contains in its closure the general integral rational curves, and the claim follows since being nodal is an open condition.

The second application returns to our original motivation, and we deduce irreducibility of Severi varieties whenever the assumptions of the Main Theorem are satisfied and the results of Lang [Lan20] ensure that the monodromy acts as a full symmetric group on the set of nodes of a general integral rational curve; that is when also the monodromy part in Harris' approach is known. This is the case when $\text{char}(K) = 0$, the primitive outer normals to the sides of an h -transverse polygon Δ generate the whole lattice \mathbb{Z}^2 , and the polarization is ample enough; see Theorem 7.1 for a precise statement. In [Bou16, Corollary 1.11], Bourqui obtains irreducibility for some smooth polarized toric surfaces. Note however the crucial difference that, unlike in *loc. cit.*, the ampleness condition in our Theorem 7.1 is independent of the fixed geometric genus g .

The tangency conditions we allow for in Theorem 5.6 recover for $S_\Delta = \mathbb{P}^2$ some of the generalized Severi varieties of Caporaso and Harris [CH98a]; namely, the ones where the position of the point with tangency to the line is not fixed; see Remark 5.3 for details. Recently, the irreducibility of generalized Severi varieties of irreducible curves has been proven in characteristic 0 by Zahariuc [Zah19]. We expect that our methods can be used to obtain a more detailed understanding of degenerations of curves with fixed tangency profiles on toric surfaces; this however was not the focus of the current paper and the tangencies we allow for in Theorem 5.6 are the ones we get for free in an argument designed for curves that intersect the toric boundary transversally. Note also, that we ensure in the current paper that the curves of smaller geometric genus constructed in each degeneration are irreducible, which we did not require in [CHT20]; this is a second aspect, in which we get a more detailed picture even in the planar case.

The degeneration argument of [CHT20] works for a larger class of toric surfaces, and not just \mathbb{P}^2 . While this class does not contain all surfaces with h -transverse polygons considered in the present paper, it does contain for example Hirzebruch surfaces and toric del Pezzo surfaces. In an upcoming third part [CHT22] of the series, we give a characteristic-free tropical proof that involves no monodromy arguments in these cases, thus establishing the irreducibility of Severi varieties for such surfaces in any characteristic. As a consequence, we obtain the irreducibility of Hurwitz schemes in any characteristic.

1.2. The approach. Our approach is based on tropical geometry and develops further the techniques introduced in [CHT20]. The main idea is to control the degenerations by studying the tropicalizations of one-parameter families of algebraic curves and the induced maps to the moduli spaces of tropical curves. The polygon Δ being h -transverse allows us to work with floor decomposed tropical curves – a particularly convenient class of tropical curves introduced by Brugallé and Mikhalkin [BM09]. Let us explain our approach in some detail.

For the closure V of a given irreducible component of a Severi variety, we consider the locus V^{FD} of algebraic curves, whose tropicalizations are floor decomposed, weightless, 3-valent, and immersed except at contracted legs; see § 2.3 for the definition of weight function and weightless tropical curves. We analyze the tropicalizations of one-parameter families of curves in V and show that in the closure of V^{FD} there are curves, whose tropicalization contains either a contracted loop or a contracted edge that belongs to a cycle of the tropicalization. Proceeding similarly to [CHT20], we show that the length of the contracted edge/loop can be increased indefinitely, and conclude that the closure of V^{FD} contains general irreducible curves of smaller genera. From this, the Main Theorem follows by induction and a dimension count.

Although the strategy of the proof is similar to the proof in [CHT20], it is more complicated. In the planar case discussed in *loc. cit.*, the one-parameter family connecting a general tropical curve with one that contains a contracted edge can be constructed such that all its fibers are weightless, 3-valent tropical curves, except for possibly one 4-valent vertex. In the more general

setting of the current paper, this is no longer the case, and we have to allow for additional types of tropical curves. For each of them, we have to prove a local liftability result to ensure that the curves we construct in the tropical family are tropicalizations of curves in the closure of V^{FD} . Among the new types considered in this paper is the case of tropical curves, which are weightless and 3-valent except for a unique 2-valent vertex of weight 1. To show local liftability in this case, we introduce a local version of Speyer’s well-spacedness condition [Spe14] in Proposition 3.8. Then we use an argument that uses properties of modular curves parametrizing elliptic curves together with a torsion point of a fixed order, that depends on the characteristic; see Lemma 4.6 for details. This is the reason why unlike the result in [CHT20], the Main Theorem is not stated for arbitrary characteristic. As mentioned above, the theorem may fail in low characteristic: a related construction (also involving modular curves) gives explicit examples when this happens; see Proposition 5.14 for details.

1.3. The structure of the paper. In §2 we recall basic definitions and fix some terminology. In §3 we discuss h -transverse polygons, floor decomposed curves, and the tropicalization of parametrized curves. We establish in Proposition 3.8 a local version of Speyer’s well-spacedness condition for realizable tropical curves. In §4 we recall the local liftability results established in [CHT20] and prove in Lemmas 4.5 and 4.6 the additional ones needed in the degeneration argument for h -transverse Δ . In §5 we prove Theorem 5.6, which immediately implies the Main Theorem. In Proposition 5.14 we show that in low characteristic the claim of the theorem may fail. In §6 we prove Zariski’s Theorem, and in §7 we combine the results with those of [Lan20] to establish irreducibility of Severi varieties in some cases.

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2. NOTATION AND TERMINOLOGY

We follow the conventions of [CHT20]. For the convenience of the reader, we recall them here and refer to *loc. cit.* for further details.

2.1. Families of curves. By a *family of curves* we mean a flat, projective morphism of finite presentation and relative dimension one. By a collection of *marked points* on a family of curves, we mean a collection of disjoint sections contained in the smooth locus of the family. A family of curves with marked points is *prestable* if its fibers have at-worst-nodal singularities. It is called *(semi-)stable* if so are its geometric fibers. A prestable curve with marked points defined over a field is called *split* if the irreducible components of its normalization are geometrically irreducible and smooth and the preimages of the nodes in the normalization are defined over the ground field. A family of prestable curves with marked points is called *split* if all of its fibers are so. If $U \subset Z$ is open, and $(\mathcal{X}, \sigma_\bullet)$ is a family of curves with marked points over U , then by a *model* of $(\mathcal{X}, \sigma_\bullet)$ over Z we mean a family of curves with marked points over Z , whose restriction to U is $(\mathcal{X}, \sigma_\bullet)$.

2.2. Toric varieties and parametrized curves. For a toric variety S , we denote the lattice of characters by M , of cocharacters by N , and the monomial functions by x^m for $m \in M$. The polarized toric variety associated to a lattice polytope $\Delta \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ is denoted by $(S_{\Delta}, \mathcal{L}_{\Delta})$, and the facets of Δ are denoted by $\partial\Delta_j$. In the current paper, we mostly work with toric surfaces. For simplicity, we will always fix an identification $M = N = \mathbb{Z}^2$ such that the duality between the lattices is given by the usual dot product pairing.

A *parametrized curve* in a toric variety S is a smooth projective curve with marked points (X, σ_\bullet) together with a map $f: X \rightarrow S$ such that $f(X)$ does not intersect orbits of codimension greater than one, and the image of $X \setminus (\bigcup_i \sigma_i)$ under f is contained in the dense torus $T \subset S$. A *family of parametrized curves* $f: \mathcal{X} \dashrightarrow S_\Delta$ over K consists of the following data:

- (1) a smooth, projective base curve (B, τ_\bullet) with marked points,
- (2) a family of stable marked curves $(\mathcal{X} \rightarrow B, \sigma_\bullet)$, smooth over $B' := B \setminus (\bigcup_i \tau_i)$, and
- (3) a rational map $f: \mathcal{X} \dashrightarrow S_\Delta$, defined over B' , such that for any closed point $b \in B'$ the restriction $\mathcal{X}_b \rightarrow S_\Delta$ is a parametrized curve.

2.3. Tropical curves. The graphs we consider have vertices, edges, and half-edges, called *legs*. We denote the set of vertices of a graph \mathbb{G} by $V(\mathbb{G})$, of edges by $E(\mathbb{G})$, and of legs by $L(\mathbb{G})$. We set $\overline{E}(\mathbb{G}) := E(\mathbb{G}) \cup L(\mathbb{G})$. By a *tropical curve* Γ we mean a finite graph \mathbb{G} with ordered legs equipped with a *length function* $\ell: E(\mathbb{G}) \rightarrow \mathbb{R}_{>0}$ and a *weight (or genus) function* $g: V(\mathbb{G}) \rightarrow \mathbb{Z}_{\geq 0}$. We say that a curve is *weightless* if the weight function is identically zero. For any leg $l \in L(\mathbb{G})$ we set $\ell(l) := \infty$. We usually view tropical curves as polyhedral complexes by identifying the edges of \mathbb{G} with bounded closed intervals of corresponding lengths in \mathbb{R} and identifying the legs of \mathbb{G} with semi-bounded closed intervals in \mathbb{R} .

For $e \in \overline{E}(\mathbb{G})$ we denote by e° the interior of e , and use \vec{e} to indicate a choice of orientation on e . The legs are always oriented away from the adjacent vertex. For $v \in V(\mathbb{G})$, we denote by $\text{Star}(v)$ the *star* of v , i.e., the collection of oriented edges and legs having v as their tail. In particular, $\text{Star}(v)$ contains two oriented edges for every loop based at v . The number of edges and legs in $\text{Star}(v)$ is called the *valence* of v . The *genus* of Γ is defined to be $g(\Gamma) = g(\mathbb{G}) := 1 - \chi(\mathbb{G}) + \sum_{v \in V(\mathbb{G})} g(v)$, where $\chi(\mathbb{G}) := b_0(\mathbb{G}) - b_1(\mathbb{G})$ denotes the Euler characteristic of \mathbb{G} . Finally, a tropical curve Γ is called *stable* if every vertex of weight zero has valence at least three, and every vertex of weight one has valence at least one.

A *parametrized tropical curve* is a balanced piecewise integral affine map $h: \Gamma \rightarrow N_{\mathbb{R}}$ from a tropical curve Γ to $N_{\mathbb{R}}$, i.e., a continuous map, whose restrictions to the edges and legs are integral affine, and for any vertex $v \in V(\Gamma)$ the following balancing condition holds: $\sum_{\vec{e} \in \text{Star}(v)} \frac{\partial h}{\partial \vec{e}} = 0$. Note that the slope $\frac{\partial h}{\partial \vec{e}} \in N$ is not necessarily primitive, and its integral length is the stretching factor of the affine map $h|_e$. We call it the *multiplicity* of h along \vec{e} , and denote it by $\mathfrak{w}(h, e)$, or $\mathfrak{w}(e)$ if the parametrization h is clear from the context. In the literature, the multiplicity $\mathfrak{w}(e)$ is often called the weight of e , but we shall not use this terminology in the current paper.

A parametrized tropical curve (Γ, h) is called *stable* if so is Γ . Its *combinatorial type* Θ is defined to be the weighted underlying graph \mathbb{G} with ordered legs equipped with the collection of slopes $\frac{\partial h}{\partial \vec{e}}$ for $e \in \overline{E}(\mathbb{G})$. We denote the group of automorphisms of a combinatorial type Θ by $\text{Aut}(\Theta)$, and the isomorphism class of Θ by $[\Theta]$.

The *degree* ∇ of a parametrized tropical curve is the sequence $\left(\frac{\partial h}{\partial l_i} \right)$, where l_i 's are the legs not contracted by h . If $\text{rank}(N) = 2$, then we say that a degree ∇ is *dual* to a polygon Δ , if each slope $\frac{\partial h}{\partial l_i}$ is a multiple of an outer normal of Δ and the multiplicities of slopes that correspond to a given side $\partial\Delta_i$ in this way sum to the integral length of $\partial\Delta_i$.

2.4. Moduli and families of parametrized tropical curves. We denote by $M_{g,n,\nabla}^{\text{trop}}$ the moduli space of parametrized tropical curves of genus g , degree ∇ , and with n contracted legs. We always assume that the first n legs l_1, \dots, l_n are the contracted ones. The space $M_{g,n,\nabla}^{\text{trop}}$ is a generalized polyhedral complex. It is stratified by subsets $M_{[\Theta]}$, that are indexed by combinatorial types Θ with the fixed invariants g, ∇ , and n . If \mathbb{G} is the underlying graph of a combinatorial type

Θ , then $M_{[\Theta]} = M_\Theta / \text{Aut}(\Theta)$, where M_Θ is a polyhedron in $\mathbb{R}^{|E(\mathbb{G})|} \times N_{\mathbb{R}}^{|V(\mathbb{G})|}$ parametrizing tropical curves $h: \Gamma \rightarrow N_{\mathbb{R}}$ of type Θ . The e -coordinate for an edge $e \in E(\mathbb{G})$ is the length $\ell(e)$ of e ; the v -coordinates for a vertex $v \in V(\mathbb{G})$ are the coordinates of $h(v) \in N_{\mathbb{R}}$.

Let Λ be a tropical curve without loops. By a *family* of parametrized tropical curves over Λ we mean a continuous family of curves $h_q: \Gamma_q \rightarrow N_{\mathbb{R}}$ for $q \in \Lambda$ such that the following holds:

- (1) The degree and the number of contracted legs are the same for all fibers.
- (2) Along the interior of an edge/leg e of Λ , the combinatorial type of the fibers is constant.
- (3) If a vertex $v \in V(\Lambda)$ is adjacent to an edge/leg e of Λ , there is a fixed contraction of weighted graphs with marked legs $\varphi_{\tilde{e}}: \mathbb{G}_e \rightarrow \mathbb{G}_v$ of the underlying graphs of the fibers.
- (4) The length of an edge $\gamma \in E(\mathbb{G}_e)$ in Γ_q along an edge/leg e of Λ is given by an integral affine function. Here we identify edges over an adjacent vertex of e with those over e° via $\varphi_{\tilde{e}}$, setting their length to zero in case they get contracted.
- (5) Similarly, the coordinates of the images $h(u)$ for a vertex u of \mathbb{G}_e in $h(\Gamma_q)$ along an edge/leg e of Λ is given by an integral affine function, where we again identify vertices over the interior of e with those over vertices adjacent to e via $\varphi_{\tilde{e}}$.

Any family of parametrized tropical curves $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ induces a piecewise integral affine map $\alpha: \Lambda \rightarrow M_{g,n,\nabla}^{\text{trop}}$ by sending $q \in \Lambda$ to the point parametrizing the isomorphism class of the fiber $h_q: \Gamma_q \rightarrow N_{\mathbb{R}}$. Furthermore, α lifts to an integral affine map from the interior of each edge/leg e of Λ to the corresponding stratum M_Θ . We refer the reader to [CHT20, §3.1.3, 3.1.4] for formal definitions.

2.5. Valued fields. For a valued field K , we denote by K^0 the ring of integers, by $K^{00} \subset K^0$ the maximal ideal, and by \tilde{K} the residue field. As a general convention, we will similarly indicate when working over K^0 and \tilde{K} by adding a superscript 0, respectively an overlining tilde; for example, X^0 for a model over K^0 of a curve $X \rightarrow \text{Spec}(K)$ and \tilde{X} for its reduction.

3. REALIZABLE TROPICAL CURVES AND h -TRANSVERSE POLYGONS

In this section and the next, we will work over the algebraic closure K of a complete, discretely valued field, with valuation $v: K \rightarrow \mathbb{R} \cup \{\infty\}$.

3.1. h -transverse polygons. The toric surfaces S_Δ that we consider in this paper are the ones with h -transverse polygon Δ . We restrict ourselves to h -transverse polygons since the theory of floor decomposed tropical curves has been developed only for this case in [BM09].

Definition 3.1. A polygon $\Delta \subset M_{\mathbb{R}} = \mathbb{R}^2$ is called *h -transverse*, if its normal vectors all have integral or infinite slopes with respect to the x -coordinate.

Clearly, whether Δ is h -transverse depends only on its normal fan; thus the notion encodes a property of the surface S_Δ and does not depend on the polarization \mathcal{L}_Δ . On the other hand, it does depend on a choice of coordinates, i.e., on the identification $M \cong \mathbb{Z}^2$. For the results we will discuss, it, of course, suffices that there is one choice of coordinates, for which Δ is h -transverse.

More explicitly, h -transversality of a polygon Δ is characterized by the following. Denote by $y_0 := \min_{q \in \Delta} y_q$ the minimal y -coordinate of lattice points in Δ . Set $y_i := y_0 + i$ with $i \in \mathbb{N}$ up to $y_k := \max_{q \in \Delta} y_q$, and denote by Y_i the horizontal line given by $y = y_i$. Then Y_0 and Y_k intersect Δ in a bounded interval, possibly of length 0. Now, Δ is h -transverse if and only if the lines Y_i for $0 \leq i \leq k$ intersect the boundary of Δ at two lattice points; or in other words, the intervals $Y_i \cap \Delta$ are lattice intervals for all i . See Figure 1 for an illustration. We denote by $w(\Delta)$ the *width* of Δ ,

that is, the maximum length of the intersections $Y_i \cap \Delta$. If they are not of length 0, we call the lattice intervals $Y_0 \cap \Delta$ and $Y_k \cap \Delta$ the *horizontal sides* of Δ .

Example 3.2. Many smooth toric surfaces are induced by h -transverse polygons, for example, \mathbb{P}^2 , Hirzebruch surfaces, toric del Pezzo surfaces, and similarly blow-ups of Hirzebruch surfaces at any subset of the 4 torus invariant points. An interesting class of examples of singular polarized toric surfaces induced by h -transverse polygons are the surfaces associated to the so-called *kites* introduced in [Lan20]; see also [LT20, §1.1]. These are lattice polygons with vertices $(0, 0), (k, \pm 1)$ and $(k + k', 0)$ for non-negative integers k, k' with $k' \geq k$ and $k' > 0$ (the choice of coordinates differs from *loc. cit.*), see Figure 1. Some of these surfaces admit reducible Severi varieties, and we will discuss them in more detail in Proposition 5.14 and Example 7.2 below.

Example 3.3. Let S_Δ be the weighted projective plane with Δ given by vertices $(0, 1), (1, -1)$ and $(6, -1)$; see Figure 1. Then Δ is not h -transverse for any choice of coordinates. Indeed, if there was such a choice of coordinates, then the x -coordinate would give a non-zero linear function on N , that would assign each of the 3 primitive vectors along the rays of the dual fan of Δ values ± 1 or 0. The above conditions on linear functions on N correspond to 3 triples of parallel lines in M , and one checks that for Δ as above no 3 have a common point of intersection other than the origin; hence Δ is not h -transverse for any choice of coordinates. Note also that the same holds for any Δ' whose dual fan is a refinement of the dual fan of Δ . Thus, an appropriate toric blow up $S_{\Delta'}$ of S_Δ gives a smooth toric surface that is not h -transverse for any choice of coordinates.

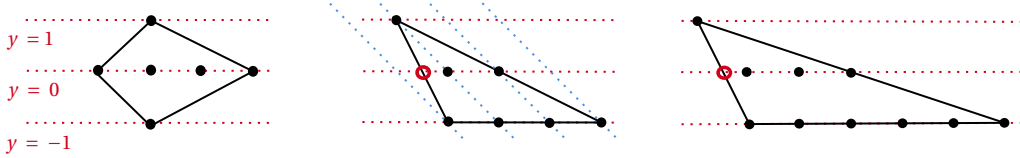


FIGURE 1. The kite $(k, k') = (1, 2)$ and two triangles. The right triangle is not h -transverse in any coordinate system, cf. Example 3.3, while the middle triangle is h -transverse with respect to the x -axis parallel to the blue lines, but not with respect to the usual x -axis.

3.2. Floor decomposed curves. Recall that a *floor decomposed tropical curve* is defined to be a parametrized tropical curve $h: \Gamma \rightarrow N_{\mathbb{R}} = \mathbb{R}^2$, in which the x -coordinate of the slope of each edge/leg is either 0 or ± 1 . In the former case, a non-contracted edge/leg is called an *elevator*. If a connected component of the graph obtained from Γ by removing the interiors of all elevators contains a non-contracted edge/leg, it is called a *floor*.

Lemma 3.4. *Let $h: \Gamma \rightarrow N_{\mathbb{R}}$ be a floor decomposed tropical curve of degree ∇ dual to an h -transverse polygon Δ . Then the multiplicity of any elevator $e \in \overline{E}(\Gamma)$ is bounded above by the width $w(\Delta)$.*

Proof. Recall that the multiplicity $w(h, e)$ is defined to be the integral length of $\frac{\partial h}{\partial e}$. Therefore, for embedded tropical curves, this bound follows from Legendre duality. In general, we argue as follows. For any $m \in M$, consider the parametrized tropical curve $h_m: \Gamma \rightarrow \mathbb{R}$ given by the composition of h with the linear functional $m: N_{\mathbb{R}} \rightarrow \mathbb{R}$. Let d_m be the sum of multiplicities of the legs of Γ having positive h_m -slope. If $p \in \mathbb{R}$ is a point that is not an image of a vertex of Γ , then by balancing, the number of preimages $h_m^{-1}(p)$ counted with edge multiplicities equals d_m . Thus, to show the claim, it suffices to find an $m \in M$ for which $d_m = w(\Delta)$ and h_m does not contract elevators of Γ .

In order to do so, note first that we can inscribe Δ in an h -transverse integral parallelogram $\square \subset M_{\mathbb{R}}$ with two horizontal sides and width $w(\square) = w(\Delta)$. We now claim that the primitive integral vector m along a non-horizontal side of \square and having positive y -component satisfies the assertion. Indeed, h_m does not contract elevators, since the y -coordinate of m cannot be 0. To calculate the degree of h_m , pick a point $p < \min_{v \in V(\Gamma)} \{h_m(v)\}$ in \mathbb{R} . All preimages $h_m^{-1}(p)$ are contained in the set of legs l_i of Γ , whose slopes in $N_{\mathbb{R}}$ have negative inner product with m , i.e., $\left(\frac{\partial h}{\partial l_i}, m\right) < 0$. Let \square_l^- (resp., \square_r^-) be the connected component of $\square \setminus \Delta$ in the bottom left (respectively, bottom right) of the parallelogram \square ; both are (possibly empty, non-convex) polygons. Note that outer normals along \square_l^- and \square_r^- sum to zero, and m has y -coordinate 1 since Δ is h -transverse. Thus, the sum of multiplicities $\sum w(h_m, l_i)$ over the legs l_i , whose slopes have negative inner product with m , equals the sum of the integral lengths of the bottoms of Δ , \square_l^- and \square_r^- . The latter is nothing but the length of the bottom of \square , i.e., $w(\square) = w(\Delta)$. Thus, for the tropical curve $h_m: \Gamma \rightarrow \mathbb{R}$, we have $d_m = w(\Delta)$, as claimed. \square

In our setup, a vertical segment connecting two floors in a floor decomposed tropical curve can contain several elevators. It will be convenient to give a name to such segments, which are special cases of elevators in the sense of [BM09]:

Definition 3.5. Let $h: \Gamma \rightarrow N_{\mathbb{R}}$ be a floor decomposed tropical curve and $E \subset \Gamma$ a subgraph such that $h(E)$ is a bounded interval with constant x -coordinate in $N_{\mathbb{R}}$. We will call E a *basic floor-to-floor elevator* if it satisfies the following properties:

- (1) for any edge/leg e of $\Gamma \setminus E$ adjacent to E , $h(e) \cap h(E)$ is one of the endpoints of $h(E)$,
- (2) for each of the two endpoints of $h(E)$ there is a unique vertex u_i of E that gets mapped to it, and
- (3) the u_i have exactly 2 adjacent edges/legs not contained in E and their slopes have x -coordinates 1 and -1 .

3.3. Tropicalization and realizability. In this section, we give a local version of Speyer's well-spacedness condition for the realizability of elliptic curves [Spe14]; see also [Kat12, Theorem 1.1], [BPR16, Theorem 6.9] and [Ran17, Theorem D] for other generalizations.

If a parametrized curve $f: X \rightarrow S_{\Delta}$ is defined over the valued field K , one can naturally construct its *tropicalization*, that is, a parametrized tropical curve $h: \Gamma \rightarrow N_{\mathbb{R}}$. Parametrized tropical curves arising this way are called *realizable*. The tropicalization construction is well-known and can be found, e.g., in [CHT20, §4.2]. For the convenience of the reader, we recall it in the special case of a non-constant rational function f on a smooth proper marked curve X . Notice that to turn (X, f) into a parametrized curve $f: X \rightarrow \mathbb{P}^1$, one must first add the zeroes and poles of f to the collection of marked points on X . We always do so when talking about tropicalizations of rational functions, without necessarily mentioning it.

So, let $f: X \rightarrow \mathbb{P}^1$ be a parametrized curve, and X^0 the stable model of X over $\text{Spec}(K^0)$. The underlying graph of $\Gamma := \text{trop}(X)$ is the dual graph of the central fiber \tilde{X} , i.e., the vertices correspond to irreducible components of \tilde{X} , the edges – to nodes, the legs – to marked points, and the natural incidence relation holds. The weight of a vertex is given by the geometric genus of the corresponding irreducible component. Finally, the length of an edge corresponding to a node z is given by the valuation $v(\lambda)$, where $\lambda \in K^{00}$ is such that étale locally at z , the total space of X^0 is given by $xy = \lambda$. It remains to define the map $h := \text{trop}(f): \Gamma \rightarrow \mathbb{R}$. Since h is affine along edges and legs, it is sufficient to specify its values at vertices together with its slopes along legs, which are given as follows. For the irreducible component \tilde{X}_v of \tilde{X} corresponding to a vertex $v \in V(\Gamma)$, let $\lambda_v \in K^{\times}$ be such that $\lambda_v f$ is an invertible function at the generic point of \tilde{X}_v . Then

$h(v) := v(\lambda_v)$. The slope of h along a leg $l \in L(\Gamma)$ is defined to be the order of pole of f at the corresponding marked point.

Let λ_v be as above. The *scaled reduction* of f at \tilde{X}_v with respect to λ_v is defined to be the non-zero rational function $(\lambda_v f)|_{\tilde{X}_v}$ on \tilde{X}_v . Notice that although scaled reductions of f at \tilde{X}_v depend on λ_v , their divisors do not. In fact, it is straightforward to check that the divisor of a scaled reduction is given by $-\sum_{\tilde{e} \in \text{Star}(v)} \mathfrak{w}(h, e) p_e$, where $p_e \in \tilde{X}_v$ is either the node of \tilde{X} or the reduction of the marked point corresponding to e . In particular, h has integral slopes, satisfies the balancing condition, and therefore (Γ, h) is a parametrized tropical curve, cf. [Tyo12, Lemma 2.23].

We say that a parametrized tropical curve $h: \Gamma \rightarrow N_{\mathbb{R}}$ is *weakly faithful* at a vertex $v \in V(\Gamma)$ if h does not contract the star of v to a point. If (Γ, h) is the tropicalization of a parametrized curve, then being not weakly faithful means that the scaled reductions of all monomial functions at \tilde{X}_v are constant.

Lemma 3.6. *Let f be a non-constant rational function on a smooth proper marked curve X , and v a vertex of $\text{trop}(X)$. Then there is $\mu \in K$ such that $\text{trop}(f - \mu)$ is weakly faithful at v .*

Proof. Let η be a K -point of X at which f is defined and such that the reduction of η lies on the component \tilde{X}_v of the reduction of X corresponding to the vertex v . Set $\mu := f(\eta) \in K$. Since $f - \mu$ has a zero at η , every scaled reduction of $f - \mu$ at \tilde{X}_v needs to vanish at the reduction of η . However, since f is non-constant, every scaled reduction of $f - \mu$ is non-zero. Therefore, the scaled reductions of $f - \mu$ at \tilde{X}_v are non-constant, and $\text{trop}(f - \mu)$ is weakly faithful at v . \square

A *cycle* of a graph \mathbb{G} is a connected subgraph of \mathbb{G} in which each vertex has valence 2. A cycle O of a parametrized tropical curve $h: \Gamma \rightarrow N_{\mathbb{R}}$ is a subgraph of Γ induced by a cycle of the underlying graph \mathbb{G} , possibly containing additional legs that get contracted by h .

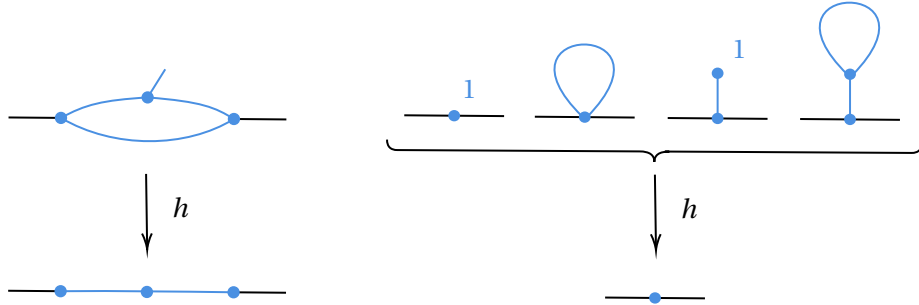


FIGURE 2. From left to right: a flattened cycle, the two versions of an elliptic component, and the two versions of a contracted elliptic tail.

Definition 3.7. Let $h: \Gamma \rightarrow N_{\mathbb{R}}$ be a parametrized tropical curve. We give the following names to certain subgraphs O of Γ (see Figure 2):

- (1) O is called a *flattened cycle* if it is a cycle, whose image in $N_{\mathbb{R}}$ is a bounded interval, and such that for each endpoint of $h(O)$ there is exactly one vertex of O mapped to it;
- (2) O is called an *elliptic component* if it is either a single vertex of weight 1 or a single vertex of weight 0 with an adjacent loop (which necessarily gets contracted by h);
- (3) O is called a *contracted elliptic tail* if $h(O)$ is a point, and O contains 2 vertices v_1 and v_2 connected by a single edge e , such that v_1 has weight zero, and all edges adjacent to v_2 are contained in O . Furthermore, v_2 has either weight 1 and is adjacent only to e , or it has weight 0 and is adjacent to an additional loop.

Proposition 3.8. *Let $h: \Gamma \rightarrow N_{\mathbb{R}}$ be a realizable tropical curve. Assume that $E \subset \Gamma$ is a subgraph that for some choice of coordinates satisfies the three axioms of Definition 3.5 for a basic floor-to-floor elevator. Suppose furthermore, that $O \subset E$ is a subgraph which is either a flattened cycle, an elliptic component, or a contracted elliptic tail. Finally, suppose that $\overline{E \setminus O}$ consists of at most two connected components, each of which is a weightless subgraph of Γ that maps injectively to $h(E)$ away from contracted legs. Then either $h(O) = h(E)$ or the complement of $h(O)$ in $h(E)$ consists of two intervals of the same length.*

Proof. Since $h: \Gamma \rightarrow N_{\mathbb{R}}$ is realizable, there is a parametrized curve $f: X \rightarrow S_{\Delta}$, whose tropicalization is $h: \Gamma \rightarrow N_{\mathbb{R}}$. Let \tilde{X} be the central fiber of the stable model of X , and let \tilde{X}_E be the subcurve of \tilde{X} that corresponds to vertices in E ; similarly, let \tilde{X}_{u_1} and \tilde{X}_{u_2} be the components corresponding to the (unique) vertices $u_1, u_2 \in E$ that get mapped to the endpoints of $h(E)$.

Without loss of generality we may assume that E satisfies the three axioms of Definition 3.5 for the standard coordinates in $N = \mathbb{Z}^2$. In particular, $h(E)$ is vertical. Set $m := (1, 0) \in M = \mathbb{Z}^2$. Since $h(E)$ is vertical and the only adjacent edges/legs to E whose slope has non-zero x -coordinate are by assumption adjacent to u_1 or u_2 , we have that any scaled reduction of $f^*(x^m)$ at components of \tilde{X}_E different from the \tilde{X}_{u_i} has no zeroes or poles – thus, it needs to be constant. In particular, no poles of $f^*(x^m)$ on X specialize to components of \tilde{X}_E different from the \tilde{X}_{u_i} . On the other hand, and again by assumption, for each u_i there are exactly two adjacent edges/legs whose slope has non-zero x -coordinate, and that slope is ± 1 . Accordingly, any scaled reduction of $f^*(x^m)$ at \tilde{X}_{u_i} has a simple pole and a simple zero on \tilde{X}_{u_i} – thus, it necessarily has degree 1.

Case 1: O is an elliptic component or a contracted elliptic tail. Let u be the (unique) vertex in O that either has weight 1, or is adjacent to a loop. Let \tilde{X}_u be the component of \tilde{X} corresponding to u . Note first that we must have $u \neq u_i$ since there can be no rational function of degree 1 on a curve of arithmetic genus 1. By Lemma 3.6, we may choose $\mu \in K$ such that any scaled reduction of $G := f^*(x^m) - \mu$ at \tilde{X}_u is non-constant. The rational functions $f^*(x^m)$ and G may have different zeroes on X , and to consider the tropicalization of G , we add the new zeroes as marked points to X . Note, however, that the poles of $f^*(x^m)$ on X remain unaffected when subtracting μ .

Let $L \subset E$ be the subgraph of E spanned by the edges of E that do not get contracted by h . By the assumptions on E , we can naturally identify L with a bounded line segment. Let γ_i denote the edge in L adjacent to u_i and let $p_i \in \tilde{X}_{u_i}$ denote the point corresponding to γ_i . Let $\lambda_{m,E} \in K$ be such that $\lambda_{m,E} f^*(x^m)$ is regular and invertible at generic points of \tilde{X}_E ; such a $\lambda_{m,E}$ exists since $h(E)$ has constant x -coordinate and $m = (1, 0)$. The corresponding scaled reduction $(\lambda_{m,E} f^*(x^m))|_{\tilde{X}_E}$ has, as explained above, constant value \tilde{c} along components of \tilde{X}_E different from the \tilde{X}_{u_i} , and has degree 1 on the \tilde{X}_{u_i} . Furthermore, the restriction of $\lambda_{m,E} G$ to \tilde{X}_E is $(\lambda_{m,E} f^*(x^m))|_{\tilde{X}_E} - \tilde{c}$. In particular, $\lambda_{m,E} G$ is still a non-zero rational function on the \tilde{X}_{u_i} and we get $\text{trop}(G)(u_1) = \text{trop}(G)(u_2) = v(\lambda_{m,E})$. Furthermore, the scaled reduction $(\lambda_{m,E} G)|_{\tilde{X}_{u_i}}$ has degree 1 with a simple zero at p_i ; thus $\text{trop}(G)$ has outgoing slope -1 at u_i along γ_i .

We can now deduce the claim from the properties of $\text{trop}(G)$, see Figure 3 for an illustration. Let $u' \in L$ be the unique vertex with $h(u') = h(u)$; that is, if O is an elliptic component we set $u' := u$, and if it is a contracted elliptic tail, then u and u' are the two vertices of O . Recall first that $\text{trop}(G)$ is harmonic, no poles of G on X specialize to components of \tilde{X}_E different from the \tilde{X}_{u_i} , and the outgoing slope of $\text{trop}(G)$ at u_i along γ_i is -1 . We conclude that the sum $s_v := \sum_{\tilde{\gamma} \in \text{Star}(v) \cap L} \frac{\partial \text{trop}(G)}{\partial \tilde{\gamma}}$ of outgoing slopes of $\text{trop}(G)$ at any $v \in L \setminus \{u_1, u_2\}$ along edges of L is non-negative. It follows that $\text{trop}(G)$ is convex along L , considered as a line segment. In particular, the outgoing slope of $\text{trop}(G)$ along the two edges of L at u' are both at most 1. On the other hand, since \tilde{X}_u has arithmetic genus 1 and any scaled reduction \tilde{G}_u of G at \tilde{X}_u is, by the construction

of G , not constant, the negative part of $\text{div}(\tilde{G}_u)$ needs to have multiplicity at least 2; this means that $\text{trop}(G)$ needs to have positive outgoing slopes at u that sum to at least 2. Since no poles of G specialize to \tilde{X}_u or $\tilde{X}_{u'}$, it follows that $\text{trop}(G)$ needs to have outgoing slopes at u' along edges contained in L that sum to at least 2. Combining these observations, we get that $\text{trop}(G)$ has outgoing slope 1 at u' along each of the 2 adjacent edges in L . Therefore, by convexity of $\text{trop}(G)$, it is affine on L except at u' , and its slope on both connected components of $L \setminus \{u'\}$ is 1. Since $\text{trop}(G)(u_1) = \text{trop}(G)(u_2)$, this implies that u' is the midpoint of L ; balancing of h then gives that also $h(u) = h(u')$ is the midpoint of $h(E) = h(L)$, as claimed.

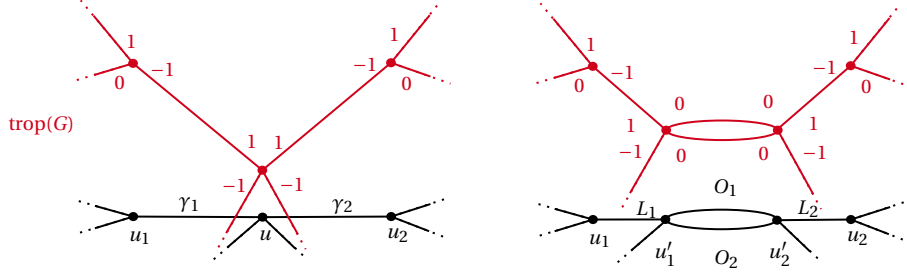


FIGURE 3. In red, the graph of $\text{trop}(G)$, where the numbers indicate outgoing slopes. On the left, Case 1, where O contains a single vertex of weight 1, $u = u'$, and L contains only the two edges γ_1 and γ_2 ; if there are contracted legs contained in E , L may contain more edges. On the right, Case 2, with $\text{trop}(G)$ constant along both O_i ; in general, it is non-constant along at most one of the O_i .

Case 2: O is a flattened cycle. As in Case 1, let u_1, u_2 be the vertices that get mapped to endpoints of $h(E)$, and let u'_1 and u'_2 be the vertices of O that get mapped to the endpoints of $h(O)$. If $u_1 = u'_1$ and $u_2 = u'_2$, there is nothing to show. So suppose $u_1 \neq u'_1$ and we first claim that then also $u_2 \neq u'_2$. Assume to the contrary that $u_2 = u'_2$. Since O is a cycle and $u_1 \neq u'_1$, the closure of $\tilde{X}_E \setminus (\tilde{X}_{u_1} \cup \tilde{X}_{u_2})$ intersects \tilde{X}_{u_2} in two points p_1, p_2 . Since a scaled reduction $(\lambda_{m,E} f^*(x^m))|_{\tilde{X}_E}$ as in Case 1 has constant value \tilde{c} along components of \tilde{X}_E different from the \tilde{X}_{u_i} , the scaled reduction $(\lambda_{m,E} f^*(x^m))|_{\tilde{X}_{u_2}}$ at \tilde{X}_{u_2} needs to achieve the same value \tilde{c} at both intersection points p_1 and p_2 ; since the scaled reduction has degree 1 on \tilde{X}_{u_2} , this is not possible, and hence $u_2 \neq u'_2$.

The remaining argument is similar to Case 1 and we omit some of the details. Denote by L the subgraph of E spanned by edges that do not get contracted by h ; this time, we may identify L with two line segments L_i between u_i and u'_i and two line segments O_1 and O_2 between u'_1 and u'_2 . By Lemma 3.6, we may choose μ such that the scaled reduction of $G := f^*(x^m) - \mu$ at $\tilde{X}_{u'_1}$ is not constant. Arguing as in Case 1, $\text{trop}(G)(u_1) = \text{trop}(G)(u_2)$, $\text{trop}(G)$ has outgoing slope -1 at u_i along L_i , and $\text{trop}(G)$ is convex on both line segments $L_1 \cup O_1 \cup L_2$. It follows that $\text{trop}(G)$ needs to be constant along one of the O_i ; thus $\text{trop}(G)(u'_1) = \text{trop}(G)(u'_2)$. Since $\text{trop}(G)$ is not constant at u'_1 , it furthermore follows that $\text{trop}(G)$ is linear along L_1 . Combining these observations, we get $\ell(L_1) \leq \ell(L_2)$ for the lengths of the L_i . Repeating the argument for G' with scaled reduction at $\tilde{X}_{u'_2}$ non-constant gives $\ell(L_2) \leq \ell(L_1)$, and therefore $\ell(L_1) = \ell(L_2)$ as claimed. \square

Remark 3.9. In the proof of Proposition 3.8 we considered the tropicalization of the function $G = f^*(x^m) - \mu$, and one may view the obtained function as an additional tropical coordinate (cf. Figure 3). This is a version of a *tropical modification*; we refer the interested reader to [Kal18] for a survey and further references.

4. LOCAL LIFTABILITY RESULTS FOR FAMILIES OF TROPICAL CURVES

In [CHT20] we introduced the tropicalization of one-parameter families of parametrized curves and studied the properties of the induced moduli maps $\alpha: \Lambda \rightarrow M_{g,n,\mathbb{V}}^{\text{trop}}$ to the tropical moduli space over certain strata, called nice strata and simple walls. This was then used to locally lift the family of tropical curves, an essential ingredient in the degeneration argument for \mathbb{P}^2 . For the convenience of the reader, we recall this construction in § 4.1 and the properties of α in § 4.2.

For the remainder of this section, we fix a family of parametrized curves $f: \mathcal{X} \dashrightarrow S_\Delta$ such that $\mathcal{X} \rightarrow B$ admits a split stable model $\mathcal{X}^0 \rightarrow B^0$ over $\text{Spec}(K^0)$, where B^0 is a prestable model of the projective curve with marked points (B, τ_\bullet) , cf. § 2. We assume, in addition, that the irreducible components of the reduction \tilde{B} of B^0 are smooth.

4.1. Tropicalization for one-parameter families. The *tropicalization of the family of parametrized curves* $f: \mathcal{X} \dashrightarrow S_\Delta$ with respect to a fixed model $\mathcal{X}^0 \rightarrow B^0$ is a family of parametrized tropical curves $h: \Gamma_\Lambda \rightarrow N_\mathbb{R}$ as in § 2.4. We now sketch its construction, and refer to [CHT20, § 4] for details.

The base Λ of the tropical family is the tropicalization of the base curve (B, τ_\bullet) with respect to the model B^0 . For a vertex $v \in V(\Lambda)$ denote by \tilde{B}_v the irreducible component of the reduction \tilde{B} corresponding to v . Let $\eta \in B(K)$ be a K -point, whose reduction $s := \tilde{\eta}$ is a non-special point of \tilde{B}_v , i.e., s is neither a node of \tilde{B} , nor a marked point. Then the fiber $h_v: \Gamma_v \rightarrow N_\mathbb{R}$ of the family $h: \Gamma_\Lambda \rightarrow N_\mathbb{R}$ over v is defined to be the tropicalization of $f|_{\mathcal{X}_\eta}: \mathcal{X}_\eta \rightarrow S_\Delta$. As long as $s \in \tilde{B}_v$ is non-special, one checks that the parametrized tropical curves obtained this way are isomorphic for different choices of η . Furthermore, the splitness of the model $\mathcal{X}^0 \rightarrow B^0$ ensures that they can be identified in a *canonical* way. More generally, if $e \in \bar{E}(\Lambda)$ and $q \in e$ is a $v(K^\times)$ -rational point, then we can refine the model B^0 of B , such that q becomes a vertex in the tropicalization of the base with respect to the refined model. The fiber $h_q: \Gamma_q \rightarrow N_\mathbb{R}$ over q now is defined as before. One can check that the fibers over the $v(K^\times)$ -rational points of Λ satisfy the 5 axioms of § 2.4, and therefore can be extended uniquely by continuity to a family over the whole of Λ .

To verify the first axiom, recall that the slopes of h_q along the legs of Γ_q are given by the order of pole along the marked points of the pull-backs of monomials $(f|_{\mathcal{X}_\eta})^*(x^m)$. Since marked points of \mathcal{X}_η are restrictions of marked points σ_\bullet of \mathcal{X} , the slopes of h_q along the legs of Γ_q are globally determined by the order of pole of $f^*(x^m)$ along the σ_\bullet , and hence independent of q .

If $s \in \tilde{B}_v$ is a special point corresponding to an edge/leg e of Λ , the underlying graph \mathbb{G}_e of fibers over e° is by construction the dual graph of the fiber $\tilde{\mathcal{X}}_s$ over s . In particular, it is constant along e° , which verifies the second axiom. Furthermore, the family $\mathcal{X}^0 \rightarrow B^0$ prescribes a one-parameter degeneration of stable curves in a neighbourhood of s in \tilde{B}_v . Such a degeneration naturally induces an edge contraction $\varphi_{\tilde{e}}: \mathbb{G}_e \rightarrow \mathbb{G}_v$ between the dual graphs, as required by the third axiom. It remains to check that the edge lengths $\ell(\gamma)$ and the images of vertices $h(u)$ in the fibers are given by integral affine functions along the edges/legs e of Λ .

Let us show how to verify this in the case of $\ell(\gamma)$ over an edge e , as the remaining cases can be verified in a similar way. Consider the toroidal embedding $B \setminus (\cup_i \tau_i) \subset B^0$, whose boundary $\tilde{B} \cup (\cup_i \tau_i)$ contains all points over which fibers of $\mathcal{X}^0 \rightarrow B^0$ fail to be smooth. All models are defined over some discretely valued subfield $F \subset K$ with a uniformizer $\pi \in F^{00}$. Let $s \in \tilde{B}_v \cap \tilde{B}_w$ be the node corresponding to e . Then in an étale neighbourhood U of s we have $ab = \pi^{k_s}$, where $a = 0$ and $b = 0$ define the components \tilde{B}_v and \tilde{B}_w , respectively. Thus, the length of e in Λ is given by $\ell_\Lambda(e) = k_s v(\pi)$.

Let z be the node of $\tilde{\mathcal{X}}_s$ corresponding to γ . Then, possibly after shrinking U , an étale neighbourhood of z in the total space \mathcal{X}^0 is given by $xy = g_z$ for some regular function $g_z \in \mathcal{O}_U(U)$. Let

$\psi: \text{Spec}(F) \rightarrow B$ be an F -point with image η and reduction s . Then the length of γ in $\text{trop}(\mathcal{X}_\eta)$ satisfies $\ell(\gamma) = v(\psi^*(g_z))$ by the tropicalization construction, cf. § 3.3. Since g_z vanishes only along $\tilde{B}_v \cup \tilde{B}_w$ in U , there are $k_a, k_b, k_\pi \in \mathbb{N}$ such that $a^{-k_a} b^{-k_b} \pi^{-k_\pi} g_z$ is regular and invertible on U . Thus, $v(\psi^*(g_z)) = k_a v(\psi^*(a)) + k_b v(\psi^*(b)) + k_\pi v(\pi)$. Since $ab = \pi^{k_s}$, it follows that $v(\psi^*(a)) + v(\psi^*(b)) = k_s v(\pi)$, and therefore

$$v(\psi^*(g_z)) = (k_a - k_b)v(\psi^*(a)) + k_b \ell_\Lambda(e) + k_\pi v(\pi).$$

By the construction, $v(\psi^* a)$ is nothing but the distance from $\text{trop}(\eta) \in e$ from the vertex w , and $k_a \ell_\Lambda(e) + k_\pi v(\pi)$ and $k_b \ell_\Lambda(e) + k_\pi v(\pi)$ are the lengths of γ in \mathbb{G}_v and \mathbb{G}_w , respectively, identified via the contraction maps described above. Thus, the length of γ along e is given by an integral affine function of slope $k_a - k_b$, as claimed.

Remark 4.1. Note that $k_a - k_b$ is the vanishing order at s of the scaled reduction of g_z at \tilde{B}_w since $(ab)^{-k_b} \pi^{-k_\pi} g_z$ is regular and invertible at the generic point of \tilde{B}_w . Thus, the slope $\frac{\partial \ell(\gamma)}{\partial \tilde{e}}$ is given by the vanishing order at s of a scaled reduction of g_z at \tilde{B}_w . Similarly, for a leg l of Λ adjacent to w and corresponding to a marked point $\tau \in B$, the slope $\frac{\partial \ell(\gamma)}{\partial l}$ is the vanishing order of g_z at τ , which coincides with the vanishing order at $\tilde{\tau}$ of a scaled reduction of g_z at \tilde{B}_w .

The slopes of $\frac{\partial h(u)}{\partial \tilde{e}}$ can be described similarly. Let s be a special point of \tilde{B}_w corresponding to an edge/leg e . Then a vertex u of \mathbb{G}_e corresponds to an irreducible component $\tilde{\mathcal{X}}_{s,u}$ of the fiber $\tilde{\mathcal{X}}_s$. There is a unique irreducible component $\tilde{\mathcal{X}}_u$ of the surface $\tilde{\mathcal{X}}|_{\tilde{B}_w}$ that contains $\tilde{\mathcal{X}}_{s,u}$. Then for $m \in M$, the slope $\frac{\partial h(u)(m)}{\partial \tilde{e}}$ is the order of pole along $\tilde{\mathcal{X}}_{s,u}$ of a scaled reduction of $f^*(x^m)$ at $\tilde{\mathcal{X}}_u$, which in the case of a leg e is nothing but the order of pole of $f^*(x^m)$ along the irreducible component of the fiber \mathcal{X}_τ that contains $\tilde{\mathcal{X}}_{s,u}$.

4.2. Simple walls and nice strata. Let $\alpha: \Lambda \rightarrow M_{g,n,\nabla}^{\text{trop}}$ be the map induced by tropicalizing a one-parameter family, v be a vertex of Λ , and Θ the combinatorial type such that $\alpha(v) \in M_{[\Theta]}$. The map α is *harmonic* at v if α lifts to a map to M_Θ locally at v and the slopes $\frac{\partial \alpha}{\partial \tilde{e}}$ along outgoing edges/legs \tilde{e} in $\text{Star}(v)$ sum to zero. The map α is *locally combinatorially surjective* at v , if the image $\alpha(\text{Star}(v))$ intersects all strata $M_{[\Theta]}$ adjacent to $M_{[\Theta]}$. The following is Case 1 of Step 3 in the proof of [CHT20, Theorem 4.6]:

Lemma 4.2. *Let $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ be the tropicalization of a family of parametrized curves $\mathcal{X} \dashrightarrow S_\Delta$ with respect to $\mathcal{X}^0 \rightarrow B^0$. Let $v \in \Lambda$ be a vertex corresponding to an irreducible component \tilde{B}_v of the reduction \tilde{B} . Suppose that the moduli map $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ induced by the family $\mathcal{X}^0 \rightarrow B^0$ contracts \tilde{B}_v . Then the induced map $\alpha: \Lambda \rightarrow M_{g,n,\nabla}^{\text{trop}}$ is harmonic at v .*

Sketch of the proof. If $\chi(\tilde{B}_v) = p \in \overline{M}_{g,n+|\nabla|}$, then every fiber of $\mathcal{X}^0 \rightarrow B^0$ over \tilde{B}_v has the same dual graph \mathbb{G} . This implies that the combinatorial type Θ of $h: \Gamma_\Lambda \rightarrow \Lambda$ is locally constant around v , and thus α lifts to a map $\text{Star}(v) \rightarrow M_\Theta$. As explained in § 2.4, we may view M_Θ naturally as a subset of $\mathbb{R}^{|E(\mathbb{G})|+2|V(\mathbb{G})|}$, and harmonicity can be checked coordinate-wise.

We first consider the γ -coordinate for an edge $\gamma \in E(\mathbb{G})$. The locus in $\overline{M}_{g,n+|\nabla|}$ in which the node z corresponding to γ persists, is cut out locally at p by a single function. Since we can pull this function back from $\overline{M}_{g,n+|\nabla|}$ to B^0 , the node z is given by $xy = g_z$ in the total space of \mathcal{X}^0 , where g_z is a regular function in a neighborhood of \tilde{B}_v in B^0 (and not just in a neighborhood of a closed point of \tilde{B}_v as it is in general). By Remark 4.1, the slopes of α at v in the γ -coordinate are the orders of vanishing of a scaled reduction of g_z at \tilde{B}_v , and hence their sum vanishes.

Let $u \in V(\mathbb{G})$ be a vertex, and consider the slopes of α at v in the $h(u)$ -coordinates. Let $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ be the surface corresponding to u . Since the model is split and fibers of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$

are isomorphic, we have (after a finite base change in \tilde{B}_v) $\tilde{\mathcal{X}}_u \simeq \mathcal{C}_u \times \tilde{B}_v$ for some curve \mathcal{C}_u . Fix a general section $\{c\} \times \tilde{B}_v$. By Remark 4.1, the slopes of $h(u)(m)$ for $m \in M$ are given by the orders of pole along the $\tilde{\mathcal{X}}_{s,u} \simeq \{s\} \times \tilde{B}_v$ of a scaled reduction of $f^*(x^m)$ at $\tilde{\mathcal{X}}_u \simeq \mathcal{C}_u \times \tilde{B}_v$ for the special points $s \in \tilde{B}_v$. The latter coincide with the orders of pole of the restriction of the scaled reduction to $\{c\} \times \tilde{B}_v$ at the points $\{c\} \times \{s\}$, and hence again have to sum up to zero for any $m \in M$. \square

Remark 4.3. Suppose in the setting of Lemma 4.2 that $v \in \Lambda$ is a vertex, such that the fiber Γ_v has weightless and 3-valent underlying graph \mathbb{G} . Since the stratum $M_{\mathbb{G}}$ of curves with dual graph \mathbb{G} in $\overline{M}_{g,n+|\nabla|}$ is a single point, the lemma in this case implies that α is harmonic at v .

Recall that a stratum M_{Θ} is called *nice* if it is regular (that is, of the expected dimension), and the underlying graph of Θ is weightless and 3-valent; M_{Θ} is a *simple wall* if it is regular, and the underlying graph is weightless and 3-valent except for a unique 4-valent vertex. Note that each simple wall M_{Θ} is contained in the closure of exactly 3 strata, all of them nice. They correspond to the three ways of splitting the 4-valent vertex into two 3-valent vertices. The following lemma is a part of [CHT20, Theorem 4.6] and applies in particular to simple walls and nice strata:

Lemma 4.4. *Let $h: \Gamma_{\Lambda} \rightarrow N_{\mathbb{R}}$ be the tropicalization of a family of parametrized curves $\mathcal{X} \dashrightarrow S_{\Delta}$ with respect to $\mathcal{X}^0 \rightarrow B^0$. Let $v \in \Lambda$ be a vertex with $\alpha(v) \in M_{[\Theta]}$, where the underlying graph of Θ is weightless and 3-valent except for at most one 4-valent vertex. Then the induced map $\alpha: \Lambda \rightarrow M_{g,n,\nabla}^{\text{trop}}$ is either harmonic or locally combinatorially surjective at v .*

Sketch of the proof. Consider the moduli map $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ induced by the family $\mathcal{X}^0 \rightarrow B^0$. If χ contracts \tilde{B}_v , then α is harmonic at v by Lemma 4.2. Otherwise, the underlying graph \mathbb{G} of Θ must contain a unique 4-valent vertex, since strata $M_{\mathbb{G}}$ in $\overline{M}_{g,n+|\nabla|}$ corresponding to weightless, 3-valent dual graphs \mathbb{G} have dimension 0. Thus, $M_{\mathbb{G}}$ is 1-dimensional, and χ maps \tilde{B}_v surjectively onto the closure $\overline{M}_{\mathbb{G}} \subset \overline{M}_{g,n+|\nabla|}$. Now one notes that $M_{[\Theta]}$ is contained in the closure of 3 strata in $M_{g,n,\nabla}^{\text{trop}}$, which correspond to the 3-splittings of the 4-valent vertex, and $\overline{M}_{\mathbb{G}}$ contains as its boundary the 3 strata parametrizing curves with the corresponding dual graphs; hence local combinatorial surjectivity follows in this case. \square

4.3. Flattened cycles. The first case we need, that was not covered in [CHT20], is a vertex v of Λ , such that the fiber over v contains two 4-valent vertices that form the endpoints of a flattened cycle. In this case, we do not get local combinatorial surjectivity of α at v , as will become clear in the proof of the next lemma; instead, we get that α is either harmonic at v or $\alpha(\text{Star}(v))$ intersects precisely the maximal adjacent strata in $M_{g,n,\nabla}^{\text{trop}}$.

Lemma 4.5. *Let $h: \Gamma_{\Lambda} \rightarrow N_{\mathbb{R}}$ be the tropicalization of a family of parametrized curves $\mathcal{X} \dashrightarrow S_{\Delta}$ with respect to $\mathcal{X}^0 \rightarrow B^0$. Let $v \in \Lambda$ be a vertex with $\alpha(v) \in M_{[\Theta]}$, where the underlying graph of Θ is weightless and 3-valent, except for the two endpoints of a flattened cycle, which are 4-valent. Assume that for some choice of coordinates, the flattened cycle satisfies the three axioms of Definition 3.5 for a basic floor-to-floor elevator. Then either α is harmonic at v ; or for every $M_{[\Theta']}$ whose closure contains $M_{[\Theta]}$ and where the underlying graph of Θ' is 3-valent, there is an edge/leg e in $\text{Star}(v)$ such that $\alpha(e) \cap M_{[\Theta']} \neq \emptyset$.*

Proof. Let $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ be the moduli map induced by the family $\mathcal{X}^0 \rightarrow B^0$. If χ contracts \tilde{B}_v , then α is harmonic at v by Lemma 4.2. Thus, we assume χ does not contract \tilde{B}_v . Let u, u' be the endpoints of the flattened cycle E in Θ ; that is, $h(E)$ is the line segment between $h(u)$ and $h(u')$. Any combinatorial type Θ' as in the claim of the lemma is obtained by splitting u and u' each into two 3-valent vertices.

Since $\mathcal{X}^0 \rightarrow B^0$ is split, there are irreducible components $\widetilde{\mathcal{X}}_u \rightarrow \widetilde{B}_v$ and $\widetilde{\mathcal{X}}_{u'} \rightarrow \widetilde{B}_v$ of $\widetilde{\mathcal{X}}$ that restrict over non-special points of \widetilde{B}_v to the irreducible component of the fiber corresponding to u and u' , respectively. Furthermore, both $\widetilde{\mathcal{X}}_u \rightarrow \widetilde{B}_v$ and $\widetilde{\mathcal{X}}_{u'} \rightarrow \widetilde{B}_v$ have 4 marked points, corresponding to the 4 adjacent edges/legs to u and u' , respectively. Thus, we get maps $\chi_u: \widetilde{B}_v \rightarrow \overline{M}_{0,4}$ and $\chi_{u'}: \widetilde{B}_v \rightarrow \overline{M}_{0,4}$ induced by the two families. Since χ does not contract \widetilde{B}_v and a non-special fiber of $\mathcal{X}^0 \rightarrow B^0$ over \widetilde{B}_v has moduli only at the components corresponding to u and u' , at least one of the maps χ_u and $\chi_{u'}$ is non-constant, and hence surjective. We assume this to be the case for χ_u .

Surjectivity of χ_u implies that there is an edge or leg $e \in \text{Star}(v)$ such that $\alpha(e) \cap M_{[\Theta'']} \neq \emptyset$ for any Θ'' obtained by splitting u : each splitting of u corresponds to a boundary point of $\overline{M}_{0,4}$. Let γ_1, γ_2 be the two edges adjacent to u that are contained in E and γ'_1, γ'_2 the two edges adjacent to u' contained in E (possibly identical with the γ_i); see Figure 4. If γ_1 and γ_2 are not adjacent to the same vertex of the underlying graph of Θ'' , also u' has to split into two 3-valent vertices in Θ'' by balancing, and the claim of the lemma follows for such Θ'' .

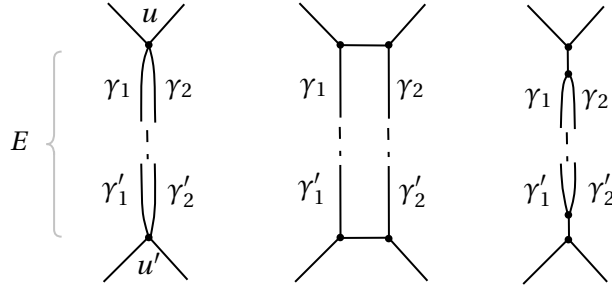


FIGURE 4. On the left, the flattened cycle in a curve of combinatorial type Θ . In the middle, a splitting of u in which γ_1 and γ_2 are not adjacent to the same vertex. On the right, the splitting in which they remain adjacent.

Thus, it remains to show the following. If γ_1, γ_2 are adjacent to the same vertex in the underlying graph of Θ'' , then the vertex u' necessarily splits in Θ'' . If not, then the curves of type Θ'' are not realizable by Proposition 3.8, and therefore can not appear in the tropicalization of a family of parametrized curves. \square

4.4. Elliptic components and contracted elliptic tails. We next describe, how to develop a loop or a contracted edge from a 2-valent, weight 1 vertex u in the tropicalization $\Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ of a family of parametrized curves. If Θ is the combinatorial type containing such a vertex u , we will consider the following combinatorial types obtained from Θ (see also Definition 3.7 and Figure 2): let Θ' be obtained from Θ by developing a loop based at u , replacing u with a weightless vertex u' ; let Θ'' be obtained from Θ by developing a contracted edge adjacent to a 1-valent vertex u'' of weight 1; and let Θ''' be obtained from Θ'' by developing a loop at u'' , replacing u'' with a weightless vertex.

Lemma 4.6. *Let $\Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ be the tropicalization of a family of parametrized curves $\mathcal{X} \dashrightarrow S_\Delta$ with respect to $\mathcal{X}^0 \rightarrow B^0$. Let $v \in \Lambda$ be a vertex with $\alpha(v) \in M_{[\Theta]}$, where Θ is 3-valent and weightless, except for one 2-valent vertex u of weight 1. Then, with Θ', Θ'' and Θ''' as above the following holds.*

- (1) *Suppose $\text{char}(\widetilde{K})$ does not divide the multiplicity of h along the two edges adjacent to u . Then α is harmonic at v , or there is an edge/leg e in $\text{Star}(v)$ such that $\alpha(e^\circ) \subset M_{[\Theta']}$.*

- (2) Suppose the multiplicity of h along the two edges adjacent to u is a power of $\text{char}(\tilde{K})$. Then α is harmonic at v , or there is an edge/leg e in $\text{Star}(v)$ such that $\alpha(e^\circ) \subset M_{[\Theta^v]}$.
- (3) Let $w \in \Lambda$ be a vertex. If $\alpha(w) \in M_{[\Theta^w]}$ or $\alpha(w) \in M_{[\Theta^m]}$, then α is harmonic at w . If $\alpha(w) \in M_{[\Theta^v]}$, then α is harmonic at w , or there is an edge/leg e in $\text{Star}(w)$ such that $\alpha(e^\circ) \subset M_{[\Theta^m]}$.

Proof. As in the proof of Lemma 4.5, let $\tilde{\mathcal{X}}_u$ be the irreducible component of $\tilde{\mathcal{X}}$ corresponding to u . Let σ_1 and σ_2 be the two sections of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ corresponding to the two edges γ_1 and γ_2 adjacent to u (as before, they are sections since $\mathcal{X}^0 \rightarrow B^0$ is split). Let $\tilde{\gamma}_i$ be the orientation of γ_i away from u ; then by balancing we have $\frac{\partial h}{\partial \tilde{\gamma}_1} = -\frac{\partial h}{\partial \tilde{\gamma}_2}$. Let $n \in N$ be a primitive vector for the line spanned by $\frac{\partial h}{\partial \tilde{\gamma}_1}$; that is, $\frac{\partial h}{\partial \tilde{\gamma}_i}$ is $\pm n$ times the multiplicity $\kappa := \mathfrak{w}(h, \gamma_i)$ of h along the γ_i . Let $m \in M$ be a primitive lattice element such that $(m, n) = 0$, where (\cdot, \cdot) is the pairing $M \times N \rightarrow \mathbb{Z}$ defining M and N as dual lattices.

We consider the rational function

$$G := (\lambda_{m,u} f^*(x^m))|_{\tilde{\mathcal{X}}_u}$$

on $\tilde{\mathcal{X}}_u$, where $\lambda_{m,u} \in K$ is chosen such that G is regular and invertible at the generic point of $\tilde{\mathcal{X}}_u$. Over non-special points $s \in \tilde{B}_v$, G restricts to a rational function $G|_{\tilde{\mathcal{X}}_{s,u}}$ on the fiber $\tilde{\mathcal{X}}_{s,u}$. We have that $G|_{\tilde{\mathcal{X}}_{s,u}}$ is regular and invertible away from the image of s under the two sections σ_1 and σ_2 . More precisely, we get

$$\text{div}(G|_{\tilde{\mathcal{X}}_{s,u}}) = \kappa \sigma_1(s) - \kappa \sigma_2(s).$$

Viewing $\tilde{\mathcal{X}}_{s,u}$ as an elliptic curve with identity $\sigma_2(s)$, we thus get that $\sigma_1(s)$ is a point of order dividing κ . Possibly replacing κ by a number that divides it, we may assume that $\sigma_1(s)$ has order κ . Note that this replacement preserves the divisibility condition – $\text{char}(\tilde{K}) \nmid \kappa$ or $\kappa = \text{char}(\tilde{K})^l$ – in the assumptions of the lemma. Since $\sigma_2(s) \neq \sigma_1(s)$ and u is of weight 1, we must have $\kappa > 1$. Furthermore, this order is constant along smooth, non-supersingular fibers of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$. Let $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ as before be the moduli map induced by the family $\mathcal{X}^0 \rightarrow B^0$. If χ contracts \tilde{B}_v , then α is harmonic at v by Lemma 4.2. Thus we assume χ does not contract \tilde{B}_v . Since the underlying graph of Θ is 3-valent and weightless except for u , this implies that the family $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ contains non-isomorphic fibers.

For the first claim assume $\text{char}(\tilde{K}) \nmid \kappa$. We will show that in this case $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ necessarily contains a singular, irreducible fiber, which immediately implies that there is an $e \in \text{Star}(v)$ satisfying the assertion of the first claim. Let $X_1(\kappa) \rightarrow \text{Spec}(\tilde{K})$ be the compactification of the modular curve $Y_1(\kappa)$ as constructed by Deligne and Rapoport [DR73], where $Y_1(\kappa)$ parametrizes elliptic curves together with a point of exact order κ . Let $\mathcal{C}_1(\kappa) \rightarrow X_1(\kappa)$ be the universal curve. The fibers of $\mathcal{C}_1(\kappa)$ over boundary points of $X_1(\kappa)$ are Néron polygons; they are semistable curves with dual graph a weightless cycle with an additional group structure. In the case of $X_1(\kappa)$, the Néron polygons that appear as fibers of $\mathcal{C}_1(\kappa)$ are precisely the ones for which the length of the dual graph divides κ (this is an immediate consequence of [DR73, Construction IV.4.14, p. 224]). In particular, there are fibers of $\mathcal{C}_1(\kappa)$ that are Néron polygons of length 1, or in other words, $\mathcal{C}_1(\kappa)$ contains a singular irreducible fiber.

By what we showed above, $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ induces a rational moduli map $\chi_{u,\kappa}: \tilde{B}_v \dashrightarrow X_1(\kappa)$, defined at points where the fibers of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ are smooth. Furthermore, over this locus $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ is the pullback of $\mathcal{C}_1(\kappa)$. Recall next that because $\text{char}(\tilde{K}) \nmid \kappa$, $X_1(\kappa)$ is irreducible by [DR73, Corollaire IV.5.6, p. 227]. Since $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ contains non-isomorphic fibers and $X_1(\kappa)$ has dimension 1, this implies that the moduli map $\chi_{u,\kappa}$ is dominant. The curve \tilde{B}_v is smooth and $X_1(\kappa)$ is proper.

Thus, after replacing \tilde{B}_v with a finite covering, the map $\chi_{u,\kappa}$ extends to the whole curve. To ease the notation, we will assume that $\chi_{u,\kappa}$ is defined on all of \tilde{B}_v .

Let $b \in \tilde{B}_v$ be any point such that the fiber C_b of $\mathcal{C}_1(\kappa) \rightarrow X_1(\kappa)$ over $\chi_{u,\kappa}(b)$ is singular and irreducible. Since $\mathcal{C}_1(\kappa) \rightarrow X_1(\kappa)$ is a family of stable curves over a neighborhood of $\chi_{u,\kappa}(b)$, so is its pullback to \tilde{B}_v . But the family $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ is also stable, and coincides with the pullback over a dense open subset of \tilde{B}_v . Therefore, its fiber over b is C_b by the uniqueness of the stable model. Hence, the family $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ contains irreducible singular fibers as claimed.

For the second claim, assume now that $\kappa = \text{char}(\tilde{K})^l$. As before, we obtain a rational map $\chi_{u,\kappa}: \tilde{B}_v \dashrightarrow Y_1(\kappa)$, defined at points where the fibers of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ are smooth; it dominates an irreducible component of $Y_1(\kappa)$. Since for any irreducible component of $Y_1(\kappa)$ the forgetful morphism to $M_{1,1}$ is surjective, there is in particular $s \in \tilde{B}_v$ such that $\chi_{u,\kappa}(s)$ parametrizes a supersingular elliptic curve. Since $\kappa = \text{char}(\tilde{K})^l$, the only point of order κ on such a supersingular curve is the origin. Thus the images of the two points $\sigma_1(s)$ and $\sigma_2(s)$ in the universal family over $Y_1(\kappa)$ need to coincide. Since $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ is a family of stable curves and the σ_i sections, we thus get that the fiber of $\tilde{\mathcal{X}}_u \rightarrow \tilde{B}_v$ over s has two irreducible components: a rational curve containing $\sigma_1(s)$ and $\sigma_2(s)$, which is attached to an elliptic curve along a single node. This is clearly equivalent to the second claim.

For the third claim, note first that the underlying graph of Θ''' is weightless and 3-valent; thus if $\alpha(w) \in M_{[\Theta''']}$, then α is harmonic at w by Remark 4.3. We next consider Θ' . Recall that u' denotes the 4-valent vertex of the underlying graph of Θ' that is adjacent to the loop, and let $w \in \Lambda$ be a vertex such that $\alpha(w) \in M_{[\Theta']}$. As before, let $\tilde{\mathcal{X}}_{u'} \rightarrow \tilde{B}_w$ be the irreducible component of $\tilde{\mathcal{X}} \rightarrow \tilde{B}$ corresponding to u' . Up to an isomorphism, we may assume that the general fiber of $\tilde{\mathcal{X}}_{u'} \rightarrow \tilde{B}_w$ is \mathbb{P}^1 with 0 and ∞ identified, and one of its two marked points is 1. Arguing as for the first claim, we get that the second marked point needs to be a κ -th root of unity in \mathbb{P}^1 . In particular, all fibers of $\tilde{\mathcal{X}}_{u'} \rightarrow \tilde{B}_w$ are isomorphic. Since general fibers of $(\tilde{\mathcal{X}})|_{\tilde{B}_w}$ have by assumption moduli only at $\tilde{\mathcal{X}}_{u'}$, this implies that the global moduli map $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ contracts \tilde{B}_w . Thus α is harmonic at w by Lemma 4.2. Finally, we consider Θ'' . Recall that u'' denotes the vertex of weight and valence 1. As before, u'' corresponds to a family $\tilde{\mathcal{X}}_{u''} \rightarrow \tilde{B}_w$ of elliptic curves and induces a moduli map $\chi_{u''}: \tilde{B}_w \rightarrow \overline{M}_{1,1}$. If this map has constant image, the global moduli map $\chi: B^0 \rightarrow \overline{M}_{g,n+|\nabla|}$ contracts \tilde{B}_w , since general fibers of $(\tilde{\mathcal{X}})|_{\tilde{B}_w}$ have by assumption moduli only at $\tilde{\mathcal{X}}_{u''}$. In this case α is harmonic at w by Lemma 4.2. Otherwise, $\chi_{u''}$ is dominant, and the closure of its image contains the boundary point of $\overline{M}_{1,1}$. Again by uniqueness of the stable model, this implies that the family $\tilde{\mathcal{X}}_{u''} \rightarrow \tilde{B}_w$ contains a singular, irreducible fiber, and the claim follows. \square

5. DEGENERATION OF CURVES ON POLARIZED TORIC SURFACES

In this section, we prove the main result, Theorem 5.6. We work over an arbitrary algebraically closed field K and consider h -transverse polygons Δ . For a line bundle \mathcal{L} on S_Δ , recall that the Severi varieties $V_{g,\mathcal{L}}$ and $V_{g,\mathcal{L}}^{\text{irr}}$ are the following loci of the linear system $|\mathcal{L}|$:

$$V_{g,\mathcal{L}} = \{[C] \in |\mathcal{L}| \mid C \text{ is reduced, torically transverse, and } \text{p}_g(C) = g\},$$

where $\text{p}_g(C)$ denotes the geometric genus of C , and *torically transverse* means that C contains no zero-dimensional orbits of the torus action. Similarly,

$$V_{g,\mathcal{L}}^{\text{irr}} = \{[C] \in V_{g,\mathcal{L}} \mid C \text{ is irreducible}\}.$$

When $\mathcal{L} = \mathcal{L}_\Delta$, we will write $V_{g,\Delta} := V_{g,\mathcal{L}_\Delta}$ and $V_{g,\Delta}^{\text{irr}} := V_{g,\mathcal{L}_\Delta}^{\text{irr}}$.

Remark 5.1. The definition of Severi varieties given here is slightly more restrictive than the one in [CHT20, Definition 2.1] since we require the curves to avoid *all* zero-dimensional orbits in S_Δ . This restriction is necessary to obtain well-behaved tangency profiles, which will be introduced below. One can check that the closures of the loci defined in the two different ways coincide, thus the modification is harmless for our purposes.

5.1. Tangency conditions. In Theorem 5.6 we will allow for some prescribed tangencies with the toric boundary of S_Δ , and we establish the setup necessary for this statement next. Denote by $D_1, \dots, D_s \subset S_\Delta$ for $i = 1, 2, \dots, s$ the closures of the codimension-one orbits in S_Δ corresponding to the sides $\partial\Delta_i$ of Δ . A *tangency profile* of (S_Δ, \mathcal{L}) is a collection of multisets

$$\underline{d} := \left(\{d_{i,j}\}_{1 \leq j \leq a_i} \right)_{1 \leq i \leq s}$$

with $a_i \in \mathbb{Z}_{\geq 0}$ and $d_{i,j} \in \mathbb{Z}_{>0}$ such that $D_i \cdot \mathcal{L} = \sum_{1 \leq j \leq a_i} d_{i,j}$. We say that a parametrized curve $f: X \rightarrow S_\Delta$ with $f(X)$ torically transverse has tangency profile \underline{d} , if for all i we have $f^*(D_i) = \sum_{1 \leq j \leq a_i} d_{i,j} p_{i,j}$ as an effective divisor on X , where $p_{i,j}$ are distinct points.

Definition 5.2. For a tangency profile \underline{d} of (S_Δ, \mathcal{L}) , we define $V_{g,\underline{d},\mathcal{L}}$ (respectively, $V_{g,\underline{d},\mathcal{L}}^{\text{irr}}$) to be the locus of $V_{g,\mathcal{L}}$ (respectively, $V_{g,\mathcal{L}}^{\text{irr}}$) consisting of curves C such that the normalization $C^v \rightarrow C$ induces a parametrized curve $f: C^v \rightarrow S_\Delta$ with tangency profile \underline{d} .

As before, we set $V_{g,\underline{d},\Delta} := V_{g,\underline{d},\mathcal{L}_\Delta}$ and $V_{g,\underline{d},\Delta}^{\text{irr}} := V_{g,\underline{d},\mathcal{L}_\Delta}^{\text{irr}}$ for the line bundle \mathcal{L}_Δ associated to Δ . Notice that in this case $D_i \cdot \mathcal{L} = \sum_{1 \leq j \leq a_i} d_{i,j}$ is the integral length of $\partial\Delta_i$. We say that a tangency profile \underline{d} is *trivial* on D_i if $d_{i,j} = 1$ for all $1 \leq j \leq a_i$. If a tangency profile is trivial on all orbits D_i we simply call it trivial, and denote it by \underline{d}_0 ; that is, \underline{d}_0 is the tangency profile for (S_Δ, \mathcal{L}) with $a_i = D_i \cdot \mathcal{L}$ and $d_{i,j} = 1$ for all i, j .

Remark 5.3. If $S_\Delta = \mathbb{P}^2$ and \underline{d} is trivial on two of the codimension-one orbits, then both $V_{g,\underline{d},\Delta}$ and $V_{g,\underline{d},\Delta}^{\text{irr}}$ are *generalized Severi varieties* in the sense of [CH98a]. On the other hand, in *loc. cit.* the authors also discuss the loci obtained by fixing the point of given tangency with the toric boundary, which we do not consider here.

Remark 5.4. Clearly we have $V_{g,\underline{d}_0,\mathcal{L}}^{\text{irr}} \subset V_{g,\mathcal{L}}^{\text{irr}}$. On the other hand, $V_{g,\mathcal{L}}^{\text{irr}} \subset \overline{V}_{g,\underline{d}_0,\mathcal{L}}^{\text{irr}}$ by [CHT20, Proposition 2.7]. Therefore $\overline{V}_{g,\mathcal{L}}^{\text{irr}} = \overline{V}_{g,\underline{d}_0,\mathcal{L}}^{\text{irr}}$ and we are free to switch between $V_{g,\underline{d}_0,\mathcal{L}}^{\text{irr}}$ and $V_{g,\mathcal{L}}^{\text{irr}}$ in the study of degenerations of irreducible curves in S_Δ .

We write $|\underline{d}| := \sum_i a_i$; that is, $|\underline{d}|$ denotes the number of distinct points in the preimage under f of the boundary divisor of S_Δ for any parametrized curve $f: X \rightarrow S_\Delta$ with tangency profile \underline{d} .

Proposition 5.5. *Let \underline{d} be a tangency profile of (S_Δ, \mathcal{L}) . Then $V_{g,\underline{d},\mathcal{L}}$ and $V_{g,\underline{d},\mathcal{L}}^{\text{irr}}$ are either empty or constructible subsets of $|\mathcal{L}|$ of pure dimension $|\underline{d}| + g - 1$.*

Proof. By [CHT20, Lemma 2.6], the Severi variety $V_{g,\mathcal{L}}$ is a locally closed subset in the linear system $|\mathcal{L}|$, and by [CHT20, Proposition 2.7], it has pure dimension $-\mathcal{L} \cdot K_{S_\Delta} + g - 1$. To estimate the codimension of $V_{g,\underline{d},\mathcal{L}}$ in $V_{g,\mathcal{L}}$ we follow the approach of [CH98b].

Let $V \subseteq V_{g,\mathcal{L}}$ be an irreducible component, $\eta \in V$ its generic point, $\overline{\eta}$ the corresponding geometric point, and $C_{\overline{\eta}}$ the corresponding curve. Since the geometric genus of $C_{\overline{\eta}}$ is g , it follows that there exists an irreducible finite covering $F_V: U \rightarrow V$ such that the geometric genus of the fiber over $F^{-1}(\eta)$ is g . Therefore, the pullback $C_U \rightarrow U$ of the universal curve to U is equinormalizable by [CHL06, Theorem 4.2], since normalization commutes with localization. Set W to

be the disjoint union of U 's for different irreducible components. Then $F: W \rightarrow V_{g,\mathcal{L}}$ is a finite surjective map, and the pullback $C_W \rightarrow W$ of the universal curve to W is equinormalizable.

Consider the relative Hilbert scheme of points $H \rightarrow W$ on the normalized family $C_W^v \rightarrow W$ of degree $\mathcal{L} \cdot K_{S_\Delta}$, and the natural section $\psi: W \rightarrow H$ that associates to $w \in W$ the pullback of the boundary divisor $\sum D_i$ under the map $C_W^v \rightarrow C_w \rightarrow S_\Delta$. Since the (relative) Hilbert scheme of points on a smooth (relative) curve is smooth, the standard dimension-theoretic arguments provide a lower bound on the dimensions of the irreducible components of $V_{g,\underline{d},\mathcal{L}}$ as follows. The locus $Z \subset H$ of subschemes of the form $\sum_{i,j} d_{i,j} p_{i,j}$ is locally closed of pure codimension $\sum_{i,j} (d_{i,j} - 1)$. Therefore, $\psi^{-1}(Z)$ is locally closed and has codimension at most

$$\sum_{i,j} (d_{i,j} - 1) = -\mathcal{L} \cdot K_{S_\Delta} - |\underline{d}|$$

at any point. Finally, since $V_{g,\underline{d},\mathcal{L}}$ is a union of components of $F \circ \psi^{-1}(Z)$ and F is finite, it follows that $V_{g,\underline{d},\mathcal{L}}$ is constructible and the dimension of any of its components is at least

$$(-\mathcal{L} \cdot K_{S_\Delta} + g - 1) - (-\mathcal{L} \cdot K_{S_\Delta} - |\underline{d}|) = |\underline{d}| + g - 1.$$

On the other hand, let V be an irreducible component of $V_{g,\underline{d},\mathcal{L}}$, and $[C] \in V$ a general point. Suppose C has m irreducible components C_1, \dots, C_m . Set $g_k := \text{pg}(C_k)$, $\mathcal{L}_k := \mathcal{O}_{S_\Delta}(C_k)$, and let \underline{d}_k be the tangency profile of the parametrized curve induced by normalizing C_k . Consider the map $\prod_{k=1}^m V_{g_k, \underline{d}_k, \mathcal{L}_k} \rightarrow V_{g, \underline{d}, \mathcal{L}}$. Its fiber over $[C]$ is finite, and the map is locally surjective. Thus, $\dim_{[C]}(V) = \sum_{k=1}^m \dim_{[C_k]}(V_{g_k, \underline{d}_k, \mathcal{L}_k})$. However, $\dim_{[C_k]}(V_{g_k, \underline{d}_k, \mathcal{L}_k}) \leq |\underline{d}_k| + g_k - 1$ by [Tyo13, Theorem 1.2 (1)], and hence

$$\dim_{[C]}(V) \leq \sum_{k=1}^m (|\underline{d}_k| + g_k - 1) = |\underline{d}| + g - 1.$$

Thus, V has dimension $|\underline{d}| + g - 1$, which completes the proof for $V_{g,\underline{d},\mathcal{L}}$. The assertion for $V_{g,\underline{d},\mathcal{L}}^{\text{irr}}$ now follows too, since $V_{g,\underline{d},\mathcal{L}}^{\text{irr}}$ is a union of components of $V_{g,\underline{d},\mathcal{L}}$. \square

5.2. The degeneration result. We are now ready to state our main result. Denote by Δ° the interior of the polygon Δ and recall that we denote by $w(\Delta)$ the width of Δ . Recall furthermore, that a smooth curve in the linear system $|\mathcal{L}_\Delta|$ has genus $\#\Delta^\circ \cap M|$.

Theorem 5.6. *Let Δ be an h -transverse polygon and suppose $\text{char}(K) = 0$ or $\text{char}(K) > w(\Delta)/2$. Let $0 \leq g \leq \#\Delta^\circ \cap M|$ be an integer and \underline{d} a tangency profile of $(S_\Delta, \mathcal{L}_\Delta)$, which is trivial on toric divisors corresponding to non-horizontal sides of Δ . Then $V_{0,\underline{d},\Delta}^{\text{irr}} \subset V$ for every irreducible component V of $\overline{V}_{g,\underline{d},\Delta}^{\text{irr}}$. In particular, each irreducible component of $\overline{V}_{g,\underline{d},\Delta}^{\text{irr}}$ contains $V_{0,\Delta}^{\text{irr}}$.*

The proof of Theorem 5.6 occupies much of the remainder of this section. It proceeds by induction on g . In Lemma 5.12, we prove that $V_{0,\underline{d},\Delta}^{\text{irr}}$ is non-empty and irreducible. To prove the induction step, we show that V must contain an irreducible component of the Severi variety $V_{g-1,\Delta}^{\text{irr}}$. To do so, we impose $\dim(V) - 1$ general point constraints on curves in V and reduce its dimension to 1. We consider a one-parameter family of parametrized curves $\mathcal{X} \rightarrow B$ associated to an irreducible component of the obtained locus and analyze its tropicalization $\Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ by investigating the induced map α from Λ to the moduli space of parametrized tropical curves.

Lemma 5.10 provides us with good control over the image of α . In particular, it is used in Lemma 5.11 to show that V must contain a codimension-one locus of curves $[C]$, whose tropicalization contains a cycle O satisfying combinatorial properties, that are tailored to be able to develop a contracted edge or loop from it. We complete the proof in § 5.2.3, by showing

that for an appropriate choice of point constraints, Λ necessarily contains a leg parametrizing tropical curves with a fixed image in $N_{\mathbb{R}}$ and having a contracted edge or loop, whose length grows to infinity. Furthermore, one of the components of the complement of this edge/loop is a parametrized tropical curve of genus $g - 1$ satisfying $\dim(V) - 1$ general point constraints. This allows us to prove that V must contain a codimension-one locus of curves of genus $g - 1$.

To use tropicalizations, we have to work over a valued field. To this end, note that the statement of Theorem 5.6 is compatible with base field extension. Hence, by passing to the algebraic closure of $K((t))$, we work in the proof over the algebraic closure of a complete DVR, whose residue field is algebraically closed and has characteristic either 0 or greater than $w(\Delta)/2$.

5.2.1. Point constraints. Let ∇ be a tropical degree dual to Δ ; see § 2.3. Then ∇ induces a tangency profile \underline{d} of $(S_{\Delta}, \mathcal{L}_{\Delta})$ as follows. For each $1 \leq i \leq s$, let a_i be the number of slopes in ∇ corresponding to the side $\partial_i \Delta \subset \Delta$, and $\{d_{i,1}, \dots, d_{i,a_i}\}$ the multiset of multiplicities of these slopes. Vice versa, \underline{d} determines a tropical degree ∇ dual to Δ uniquely up-to ordering of the slopes. Notice also that if $f: X \rightarrow S_{\Delta}$ is a parametrized curve of tangency profile \underline{d} , then the degree ∇ of its tropicalization induces the tangency profile \underline{d} by the very construction of the tropicalization. We set once and for all

$$n := |\underline{d}| + g - 1 = |\nabla| + g - 1.$$

Denote by $\text{ev}: M_{g,k,\nabla}^{\text{trop}} \rightarrow N_{\mathbb{R}}^k$ the evaluation map

$$\text{ev}: (\Gamma, h) \mapsto (h(l_1), \dots, h(l_k)),$$

where $h: \Gamma \rightarrow N_{\mathbb{R}}$ is a parametrized tropical curve with contracted legs l_1, \dots, l_k . If Θ is a combinatorial type with k contracted legs, we define ev_{Θ} to be the composition $\overline{M}_{\Theta} \rightarrow M_{g,k,\nabla}^{\text{trop}} \rightarrow N_{\mathbb{R}}^k$. Then for k points q_1, \dots, q_k in $N_{\mathbb{R}}$, the preimage $\text{ev}_{\Theta}^{-1}(q_1, \dots, q_k)$ is a polyhedron cut out in \overline{M}_{Θ} by an affine subspace.

Definition 5.7. For $k \in \mathbb{Z}_{>0}$ fix points $p_1, \dots, p_k \in S_{\Delta}$ and $q_1, \dots, q_k \in N_{\mathbb{R}}$. Denote by $H_i \subset |\mathcal{L}_{\Delta}|$ the hyperplane parametrizing curves that pass through the point p_i .

- (1) Let $[C] \in V_{g,\underline{d},\Delta}^{\text{irr}}$ pass through all of the p_i 's. We say that the p_i 's are in *general position with respect to $[C]$* , if the intersection $V_{g,\underline{d},\Delta}^{\text{irr}} \cap (\bigcap_{i=1}^k H_i)$ has dimension $n - k$ at $[C]$.
- (2) Let $(\Gamma, h) \in M_{g,k,\nabla}^{\text{trop}}$. We say that the q_i 's are in *tropical general position with respect to (Γ, h)* if the preimage $\text{ev}^{-1}(q_1, \dots, q_k)$ has dimension $n - k$ at (Γ, h) .

Remark 5.8. Let $f: X \rightarrow S_{\Delta}$ be a parametrized curve with $C := f(X)$ passing through points p_1, \dots, p_k and $[C] \in V_{g,\underline{d},\Delta}^{\text{irr}}$. Assume that its tropicalization (Γ, h) is weightless, 3-valent, and immersed except at contracted legs. Set $q_i := \text{trop}(p_i)$. If q_1, \dots, q_k are in general position with respect to (Γ, h) , then so are p_1, \dots, p_k with respect to $[C]$. Indeed, let M_{Θ} be the stratum of $M_{g,0,\nabla}^{\text{trop}}$ that contains the curve obtained from (Γ, h) by forgetting the contracted legs and stabilizing. Then M_{Θ} is maximal and has dimension n by [Mik05, Proposition 2.23]. Thus we may choose $n - k$ points q_{k+1}, \dots, q_n contained in $h(\Gamma)$, such that (Γ, h) is an isolated point in $\text{ev}^{-1}(q_1, \dots, q_n) \subset M_{g,n,\nabla}^{\text{trop}}$. Let $p_{k+1}, \dots, p_n \in C$ be $n - k$ points such that $\text{trop}(p_i) = q_i$. We claim that then also $[C]$ is an isolated point in $V_{g,\underline{d},\Delta}^{\text{irr}} \cap (\bigcap_{i=1}^n H_i)$, and therefore $V_{g,\underline{d},\Delta}^{\text{irr}} \cap (\bigcap_{i=1}^k H_i)$ has dimension $n - k$ at $[C]$, since $V_{g,\underline{d},\Delta}^{\text{irr}}$ is equidimensional of dimension n . Indeed, any irreducible curve $B \subseteq V_{g,\underline{d},\Delta}^{\text{irr}} \cap (\bigcap_{i=1}^n H_i)$ containing $[C]$ gives rise to a 1-dimensional family of parametrized tropical curves $\Gamma_{\Lambda} \rightarrow N_{\mathbb{R}}$ (cf. Construction 5.9 below). The family $\Gamma_{\Lambda} \rightarrow N_{\mathbb{R}}$ then induces a map

$\alpha: \Lambda \rightarrow M_{g,n,\nabla}^{\text{trop}}$ satisfying $(\Gamma, h) \in \alpha(\Lambda) \subset \text{ev}^{-1}(q_1, \dots, q_n)$. Since the curves over B dominate S_Δ , their tropicalizations cover a dense subset of $N_{\mathbb{R}}$ and α is not constant. Since Λ and hence also $\alpha(\Lambda)$ is connected, this contradicts the fact that (Γ, h) is an isolated point in $\text{ev}^{-1}(q_1, \dots, q_n)$.

Let $V \subset \overline{V}_{g,d,\Delta}^{\text{irr}}$ be an irreducible component, $[C] \in V$ a curve, and $p_1, \dots, p_{n-1} \in S_\Delta$ points in general position with respect to $[C]$. Pick an irreducible component $Z \subseteq V \cap (\bigcap_{i=1}^{n-1} H_i)$ containing $[C]$, and let us briefly explain how to associate a family of parametrized curves to Z , which we then can study by investigating its tropicalization. We refer to Step 2 of the proof of [CHT20, Theorem 5.1] for technical details.

Construction 5.9. First, we pull back the tautological family of curves from Z to the normalization Z^\vee . After replacing Z^\vee with a finite covering, we may assume that the resulting family is generically equinormalizable by [dJ96, §5.10]. Normalizing its total space, and restricting the obtained family to an open dense subset of the base, we get a family $f: \mathcal{X} \rightarrow S_\Delta$ of genus g parametrized curves over the complement of finitely many points τ_\bullet in a smooth projective base curve B . Next, after replacing B with a finite covering, we may assume that the marked points σ_\bullet of \mathcal{X} are the points that are mapped to $\{p_i\}_{i=1}^{n-1}$ and to the boundary divisors of S_Δ . Furthermore, we may assume that the family $\mathcal{X} \rightarrow B \setminus \tau_\bullet$ extends to a split stable model $\mathcal{X}^0 \rightarrow B^0$, where B^0 is a prestable model of (B, τ_\bullet) over K^0 . In particular, for $b \in B$ that gets mapped to a general $[C] \in Z$ under the natural projection $B \rightarrow Z$, the fiber \mathcal{X}_b of $\mathcal{X} \rightarrow B$ is the normalization of C , equipped with the natural map to S_Δ .

Recall from § 4.2 that we consider two types of regular strata M_Θ in $M_{g,n-1,\nabla}^{\text{trop}}$: nice strata and simple walls. The former have the expected dimension $2n-1$, and the latter $2n-2$. Since both are automorphism free, we have $M_\Theta = M_{[\Theta]}$ and may consider M_Θ as a subset of $M_{g,n-1,\nabla}^{\text{trop}}$.

Lemma 5.10. *Let $\alpha: \Lambda \rightarrow M_{g,n-1,\nabla}^{\text{trop}}$ be the moduli map induced by the tropicalization of the family $f: \mathcal{X} \dashrightarrow S_\Delta$ given by imposing $n-1$ general point constraints p_i on V as in Construction 5.9. Set $q_i = \text{trop}(p_i)$. Then α is not constant and the following hold:*

- (1) *Suppose α maps a vertex $v \in V(\Lambda)$ to a simple wall M_Θ , and $M_\Theta \cap \text{ev}_\Theta^{-1}(q_1, \dots, q_{n-1}) = \alpha(v)$ is a single point. Then there exists a vertex $w \in V(\Lambda)$ such that $\alpha(w) = \alpha(v)$ and the map α is locally combinatorially surjective at w .*
- (2) *Suppose $M_{\Theta'}$ is a nice stratum, $\alpha(\Lambda) \cap M_{\Theta'} \neq \emptyset$ and $\dim(M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})) = 1$. Then $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1}) = \alpha(\Lambda) \cap M_{\Theta'}$ is an interval, whose boundary in $\overline{M}_{\Theta'}$ is disjoint from $M_{\Theta'}$.*

Proof. By construction, α is obtained by tropicalizing a family of parametrized curves $f: \mathcal{X} \dashrightarrow S_\Delta$ over a curve B . Since f is dominant onto S_Δ , the image of $\Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ covers a dense subset of $N_{\mathbb{R}}$. Hence the image of Λ in $M_{g,n,\nabla}^{\text{trop}}$ is not constant.

(1) Since α is not constant and Λ connected, we can find $w \in V(\Lambda)$ such that $\alpha(w) = \alpha(v)$ and α is non-constant at w . Since $\alpha(\Lambda) \subset \text{ev}^{-1}(q_1, \dots, q_{n-1})$, we have $\alpha(\Lambda) \cap M_\Theta = \alpha(v)$. It follows that α does not induce a map $\text{Star}(w) \rightarrow M_\Theta$. By definition, α is thus not harmonic at w . Hence by Lemma 4.4, α is locally combinatorially surjective at w .

(2) Since $M_{\Theta'}$ is a nice stratum of $M_{g,n-1,\nabla}^{\text{trop}}$, the underlying graph \mathbb{G} of Θ' in particular is weightless and 3-valent. Thus by Remark 4.3, α is harmonic at vertices of Λ that map to $M_{\Theta'}$. On the other hand, since $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$ is the intersection of finitely many (open) half-spaces and hyperplanes, it is an interval. Since $\alpha(\Lambda) \cap M_{\Theta'}$ is contained in $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$, the

harmonicity of α shows that $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1}) = \alpha(\Lambda) \cap M_{\Theta'}$, and its boundary in $\overline{M}_{\Theta'}$ is disjoint from $M_{\Theta'}$. Compare to [CHT20, Lemma 3.9 (3)]. \square

5.2.2. *Cycles in floor decomposed curves.* We specify some notation that is convenient for the later proofs. Given a stable, 3-valent, and floor decomposed curve (Γ, h) , and a cycle O in Γ , we denote by $n(O)$ the number of basic floor-to-floor elevators of Γ that are contained in O , and call it the *vertical complexity* of O . Let $n(\Gamma)$ be the minimal vertical complexity among all cycles of Γ , which we call the *vertical complexity* of Γ . A point in $h(\Gamma)$ is called *special* if it is the image of a vertex or a contracted leg on a floor.

Recall that V denotes an irreducible component of $\overline{V}_{g, \underline{d}, \Delta}^{\text{irr}}$. Since curves $[C]$ in $V_{g, \underline{d}, \Delta}^{\text{irr}}$ are torically transverse, we can view the map $C^v \rightarrow S_{\Delta}$ from the normalization of C as a parametrized curve by marking the preimage of the toric boundary. We denote by $\text{trop}(C) = (\Gamma_C, h_C)$ the tropicalization of this parametrized curve. Let $V^{FD} \subset V \cap V_{g, \underline{d}, \Delta}^{\text{irr}}$ be the locus of curves $[C]$ such that the tropicalization $\text{trop}(C)$ is floor decomposed, weightless, 3-valent, and immersed except at contracted legs (in particular, contained in a nice stratum in the moduli of parametrized tropical curves by [Mik05, Proposition 2.23]). Given $[C] \in V^{FD}$, we define the *vertical complexity* of C to be the vertical complexity of $\text{trop}(C)$. Note that by the construction of V^{FD} , all basic floor-to-floor elevators of $\text{trop}(C)$ have genus 0, and h_C is injective on any basic floor-to-floor elevator of Γ_C away from contracted legs. The following lemma provides the first part of the degeneration we construct in order to prove Theorem 5.6.

Lemma 5.11. *Suppose V is a component of $\overline{V}_{g, \underline{d}, \Delta}^{\text{irr}}$ with $g \geq 1$ and \underline{d} is a tangency profile trivial on all toric divisors of S_{Δ} corresponding to non-horizontal sides of Δ as in Theorem 5.6. Then there exists $[C] \in V^{FD}$ whose tropicalization $\text{trop}(C) = (\Gamma_C, h_C)$ contains two elevators E and E' adjacent to the same floor F and contained in the same cycle O of Γ_C such that either*

- (1) $\frac{\partial h_C}{\partial \vec{E}} = -\frac{\partial h_C}{\partial \vec{E}'}$, where \vec{E} and \vec{E}' are oriented outwards from F ; or
- (2) O has vertical complexity 2.

Proof. We first show that V^{FD} is not empty. Since Δ is h -transverse, for any n vertically stretched points $q'_1, \dots, q'_n \in N_{\mathbb{R}}$ that are in general position, all curves in $\text{ev}^{-1}(q'_1, \dots, q'_n) \subset M_{g, n, \nabla}^{\text{trop}}$ are weightless, 3-valent, floor decomposed and immersed by [BM09, §5.1]. Note that while in *loc. cit.* only tropical degrees ∇ associated to trivial \underline{d} are considered, the argument there works also in our case. For the coordinate-wise tropicalization map $\text{trop}: T \rightarrow N_{\mathbb{R}}$ on the dense torus $T \subset S_{\Delta}$, the preimage $\text{trop}^{-1}(q'_i)$ is Zariski dense (though not algebraic) in T . Thus, we can choose p'_i in S_{Δ} , such that $\text{trop}(p'_i) = q'_i$ and there are curves in $V \cap V_{g, \underline{d}, \Delta}^{\text{irr}}$ that pass through all of the p'_i . Since any such curve is contained in V^{FD} , it follows in particular that V^{FD} is not empty.

Pick $[C] \in V^{FD}$ with minimal vertical complexity; we will show that C satisfies the claim of the lemma. Let h be the height of Δ , namely $h = \max_{q \in \Delta} y_q - \min_{q \in \Delta} y_q$. Denote the floors of the tropicalization $\text{trop}(C)$ from the bottom to the top by F_1, \dots, F_h . Pick points $q_1, \dots, q_n \in N_{\mathbb{R}}$ such that (1) each floor and elevator of $\text{trop}(C)$ contains exactly one of the q_i in its image and (2) the preimage $x_i = h_C^{-1}(q_i) \in \Gamma_C$ is a single point contained in the interior of an edge or leg.

For each $1 \leq i \leq n$, let Θ'_i be the combinatorial type in $M_{g, n-1, \nabla}^{\text{trop}}$ obtained from $\text{trop}(C)$ by adding one contracted leg at x_j for each $j \neq i$. Note that each contracted leg is either on a floor or adjacent to two elevators, and $M_{\Theta'_i}$ is a nice stratum. In this way, we can view $[\text{trop}(C)]$ as a point in $M_i := M_{\Theta'_i} \cap \text{ev}_{\Theta'_i}^{-1}(q_1, \dots, \hat{q}_i, \dots, q_n)$. Since $\Gamma_C \setminus \{x_i\}_{1 \leq i \leq n}$ is a disjoint union of trees, each

containing exactly one non-contracted leg, the points $(q_1, \dots, \hat{q}_i, \dots, q_n)$ are in general position with respect to $\text{trop}(C)$ and $\dim M_i = 1$ (compare to [Mik05, Lemma 4.20]).

Step 1. The reduction to the case of vertically stretched points. For each $1 \leq i \leq n$, pick $p_i \in C$ such that $\text{trop}(p_i) = q_i$. It follows from Remark 5.8 that, for each i , the points $\{p_j\}_{j \neq i}$ are in general position with respect to $[C] \in V$. Let $\alpha_i: \Lambda_i \rightarrow M_{g,n-1,\nabla}^{\text{trop}}$ be induced by the family of parametrized tropical curves obtained from Construction 5.9 with respect to the $n-1$ points $(p_1, \dots, \hat{p}_i, \dots, p_n)$. We have $M_i \subset \alpha_i(\Lambda_i)$ by Lemma 5.10 (2), and the curves in M_i are obtained from $\text{trop}(C)$ by either moving the image of a floor vertically or moving the image of an elevator horizontally within a suitable distance (until some two adjacent vertices collapse); see Figure 5. Since $\text{trop}(C)$ is floor decomposed, the curves in M_i have the same vertical complexity as $\text{trop}(C)$.

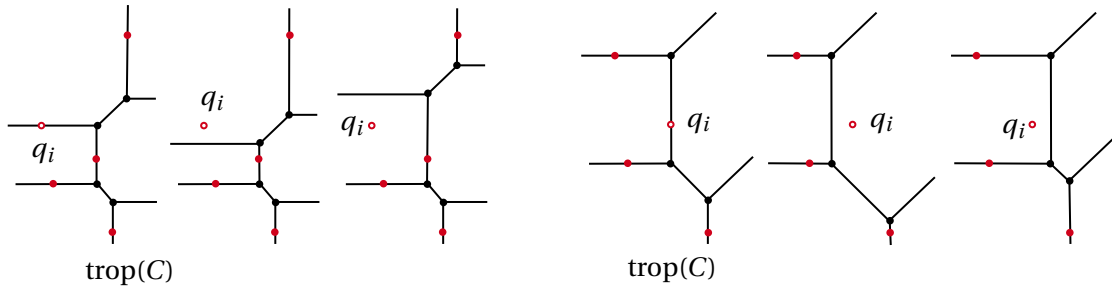


FIGURE 5. The curves in M_i . The left is when q_i is contained in the image of a floor of $\text{trop}(C)$, while on the right q_i is contained in the image of an elevator.

Assume now that q_i is contained in the image of F_i for $1 \leq i \leq h$. We deform all the points $\{q_i\}_{1 \leq i \leq n}$ so that they are vertically stretched and in general position with respect to a curve that has the same vertical complexity as $\text{trop}(C)$, as follows. We start with the top floor F_h . On any upward elevator adjacent to F_h , we may move the point q_j it contains up the elevator for any distance; that is, any point $q'_j = q_j + t(0, 1)$ with $t \in \mathbb{R}_{\geq 0}$ is still contained in $h_C(\Gamma_C)$. By first moving the q_j up an appropriate distance, we then can move up F_h any distance, without changing the combinatorial type of $\text{trop}(C)$; that is, we consider the family of curves, starting with $\text{trop}(C)$, that is cut out by replacing q_h with $q_h + t(0, 1)$ for varying $t \in \mathbb{R}_{\geq 0}$. By definition, the curves obtained in this way are contained in M_j or M_h . Repeating this construction for each floor F_i , with i going down from h to 1 , we see that we can increase the vertical distance between any two points in $\{q_i\}_{1 \leq i \leq n}$ arbitrarily while preserving their horizontal distance. In particular, we can move the $\{q_i\}_{1 \leq i \leq n}$ until they are vertically stretched.

Since the curves obtained by moving the q_i 's as above remain in M_i and $M_i \subset \alpha_i(\Lambda_i)$, the parametrized tropical curve we obtain after stretching the q_i is still the tropicalization of a curve C' in V^{FD} . Furthermore, C' has the same vertical complexity as C , and the image $h_{C'}(\Gamma_{C'})$ contains n vertically stretched points q_i that are in general position with respect to $\text{trop}(C')$. To ease notation, we assume that C already satisfies these properties.

Step 2. The curve C satisfies the assertion of the Lemma. Recall that q_i is contained in the image of F_i for $1 \leq i \leq h$. Let O be a cycle in $\text{trop}(C)$ with minimal vertical complexity. Let F_k be the highest floor containing an edge in O and pick an elevator $E \subseteq O$ that is adjacent to F_k . Let F_s be the other floor adjacent to E . Suppose O is oriented from E towards an edge in F_s , and let $E' \subseteq O$ be the first elevator adjacent to F_s that appears after E ; see Figure 6. Reordering $\{q_i\}_{i=1}^n$ if necessary, we may assume that E contains q_n . As before, pick $p_1, \dots, p_{n-1} \in C$ such

that $\text{trop}(p_i) = q_i$, and consider $\text{trop}(C)$ as a curve in the family of parametrized tropical curves $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ obtained from Construction 5.9 for the points p_1, \dots, p_{n-1} . Let Θ' be the combinatorial type of $\text{trop}(C)$. Then $M_{\Theta'}$ is a nice stratum of $M_{g,n-1,\nabla}^{\text{trop}}$ by assumption.

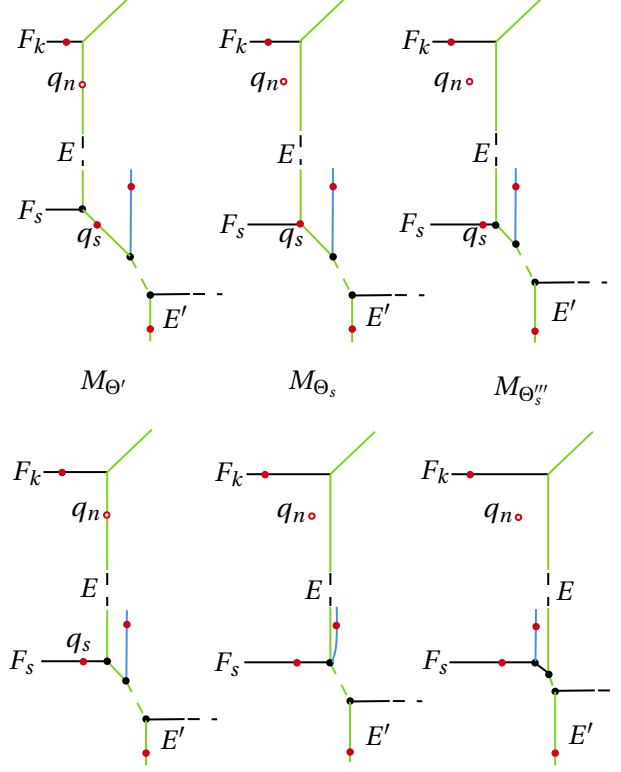


FIGURE 6. Moving E towards E' . The left column is $\text{trop}(C)$. In the top row passing through q_s , in the bottom row through a vertex in F_s . The case of passing through q_k or a vertex in F_k is analogous. The edges in O are colored green.

We claim that, by moving E towards E' and passing through the special points on the way, the problem reduces to the case where there is no special point in $F_k \cup F_s$ with x -coordinate between the x -coordinates of E and E' . To prove this, assume without loss of generality that the x -coordinate of E is less than that of E' , and there are exactly r special points in $F_k \cup F_s$ with x -coordinates between those of E and E' . We will proceed by induction on r . The base case is trivial, so we assume $r > 0$. We will describe the inductive step only for the case that, among the r special points, the one with x -coordinate closest to that of E is the marked point $q_s \in F_s$, the image of a contracted leg l_s . The argument for the other cases is similar, see Figure 6.

Recall that $\alpha: \Lambda \rightarrow M_{g,n-1,\nabla}^{\text{trop}}$ is the map induced by the family of parametrized tropical curves $\Gamma_\Lambda \rightarrow N_{\mathbb{R}}$. Moving E horizontally towards E' , until its lower vertex hits q_s , we get a tropical curve represented by a boundary point of $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$, which is contained in a simple wall M_{Θ_s} such that E is adjacent to the leg l_s contracted to q_s . By Lemma 5.10 (2), $\alpha(\Lambda)$ contains $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$, hence also contains this boundary point. Let $\Theta_s''' \neq \Theta'$ be the other nice stratum in the star of M_{Θ_s} such that E and the leg l_s contracted to q_s are not adjacent. Hence in Θ_s''' the x -coordinate of E is greater than that of q_s (whereas in Θ' it is the other way around). By Lemma 5.10 (1), there is a vertex w of Λ such that $\alpha(w) \in M_{\Theta_s} \cap \text{ev}_{\Theta_s}^{-1}(q_1, \dots, q_{n-1})$ and α

is locally combinatorially surjective at w . Thus, there is an edge e of Λ such that $\alpha(e^\circ) \subset M_{\Theta_s'''} \cap \text{ev}_{\Theta_s'''}^{-1}(q_1, \dots, q_{n-1})$, where e° denotes the interior of e . Now we may replace $\text{trop}(C)$ with any curve represented by a point in $\alpha(e^\circ)$, which contains $r - 1$ special points on $F_k \cup F_s$ with x -coordinates between those of E and E' , and has the same vertical complexity as $\text{trop}(C)$ by construction. This completes the inductive step.

Now assume that there is no special point of $\text{trop}(C)$ on $F_k \cup F_s$ with x -coordinate between those of E and E' . Let q_j be the marked point contained in $h_C(E')$, and E'' the second elevator for which $q_j \in h_C(E'')$. Let $F_{s'}$ be the floor adjacent to E'' . Since there is no special point in $F_k \cup F_s$ with x -coordinate between those of E and E' , there is a boundary point of $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$ that is contained in the simple wall M_Θ in which E and E' get adjacent to the same 4-valent vertex u . There are two cases to consider:

- (1) $s' < s$, i.e., E' lies below F_s ; and
- (2) $s' > s$, i.e., E' lies above F_s .

See Figure 7 and Figure 8, respectively, for an illustration. In both cases, let $M_{\Theta'''}$ denote the

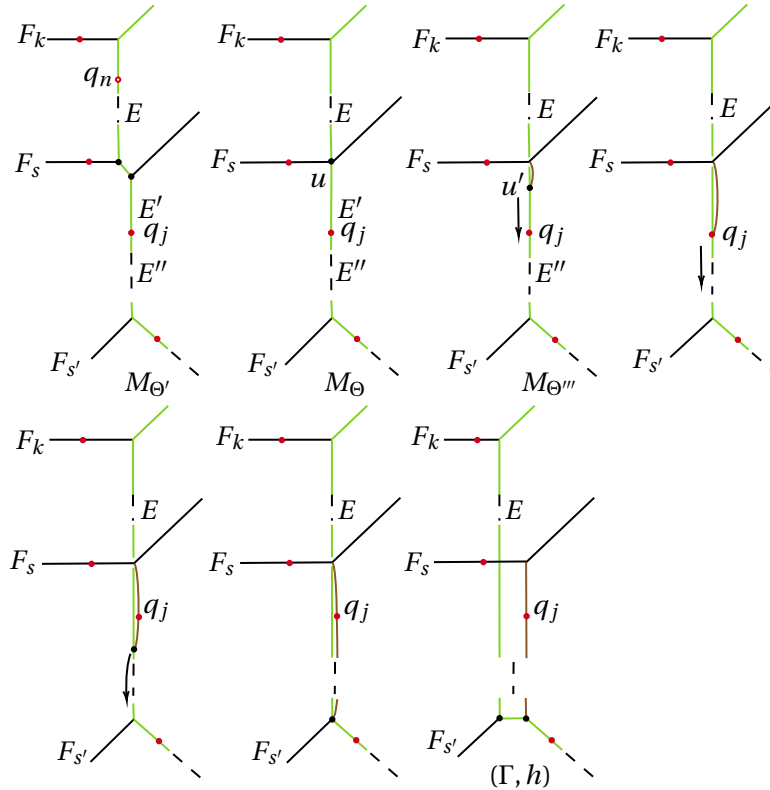


FIGURE 7. Moving E down the elevators E' and E'' with $\mathfrak{w}(E') > \mathfrak{w}(E)$ (Case 1 in the proof of Lemma 5.11). The edges in the cycle O are colored green.

nice stratum in the star of M_Θ in which the elevators E and E' get attached to the same 3-valent vertex u' obtained from splitting u into two 3-valent vertices. By Lemma 5.10 (1), there is a vertex $w \in V(\Lambda)$ and an edge $e \in E(\Lambda)$ adjacent to w , such that $\alpha(w) \in M_\Theta$ and $\alpha(e^\circ) \subset M_{\Theta'''}$.

Case 1: $s' < s$. If $\mathfrak{w}(E) = \mathfrak{w}(E')$, then assertion (1) is satisfied and we are done. Otherwise, we may assume, without loss of generality, that $\mathfrak{w}(E) < \mathfrak{w}(E')$.

It follows from the balancing condition that u' lies below F_s , see Figure 7. We keep using Lemma 5.10, to deduce that all curve types illustrated in Figure 7 belong to the image of α . Starting with the nice cone $M_{\Theta''}$ and applying Lemma 5.10 (2), we conclude that there is a vertex of Λ corresponding to the simple wall illustrated on the top right of Figure 7 and parametrizing curves with a 4-valent vertex u' attached to E, E', E'' , and to the contracted leg corresponding to q_j . As before, by Lemma 5.10 (1), there exists an edge $e' \in E(\Lambda)$ parametrizing curves illustrated on the bottom left of Figure 7, and therefore, there also exists a vertex of Λ corresponding to the simple wall illustrated on the bottom middle of Figure 7 and parametrizing curves with a 4-valent vertex u' attached to E, E' , and the two edges on the floor $F_{s'}$. Finally, applying once again Lemma 5.10 (1), we conclude that the image of α contains curves (Γ, h) of the desired type illustrated on the bottom right of Figure 7.

The cycle in (Γ, h) has vertical complexity $n(O) - 1$ and is obtained from O by replacing the elevators $\{E, E', E''\}$ in $\text{trop}(C)$ with the elevator E in Γ (note that E' and E'' together form a basic floor-to-floor elevator in $\text{trop}(C)$, and thus contribute 1 to the vertical complexity). This gives a contradiction, since we chose the curve C with minimal vertical complexity in V^{FD} .

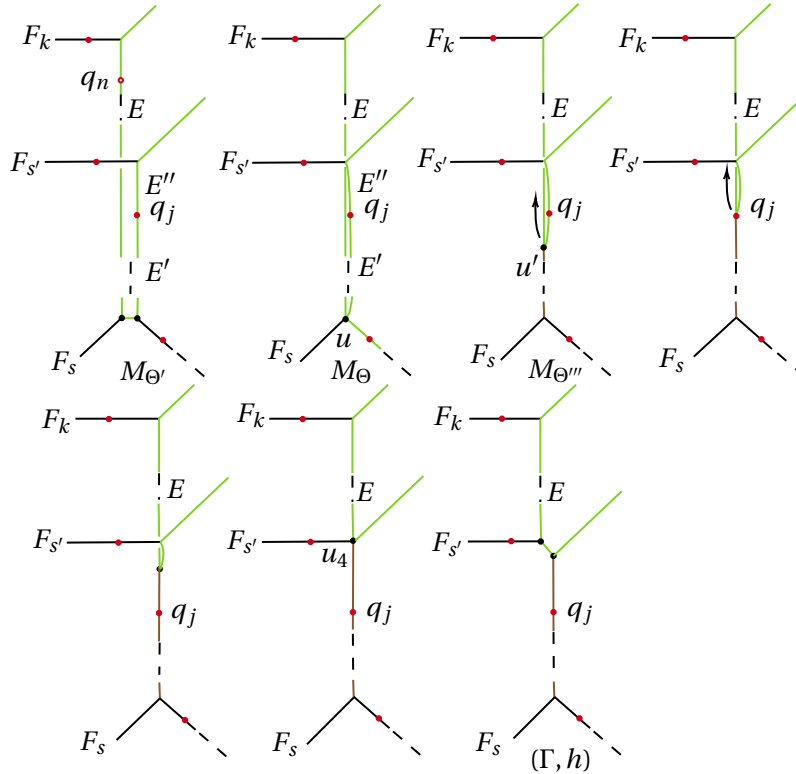


FIGURE 8. Moving E up through the elevators E' and E'' (Case 2 in the proof of Lemma 5.11). The edges in the cycle O are colored green.

Case 2: $s' > s$. If $s' = k$, then $\{E, E', E''\}$ induces a cycle of vertical complexity 2, which is assertion (2) and we are done. Otherwise, by a similar argument as in Case 1, we get a tropical curve in $\alpha(\Lambda)$ with vertical complexity less than $\text{trop}(C)$. The only difference is that we are moving up E along E' and E'' instead of moving down. See Figure 8 for an illustration. This provides a contradiction and we are done. \square

5.2.3. *The proof of Theorem 5.6.* We now complete the proof of Theorem 5.6 by induction on g . The base of induction follows from the first assertion of the following Lemma.

Lemma 5.12. *Let Δ and \underline{d} be as in Theorem 5.6, and $[C] \in V_{0,\underline{d},\Delta}^{\text{irr}}$. Pick a parametrization of the normalization $\mathbb{P}^1 \simeq C^\vee$, and let $f: \mathbb{P}^1 \rightarrow S_\Delta$ be the composition $\mathbb{P}^1 \rightarrow C^\vee \rightarrow C \rightarrow S_\Delta$. Then,*

- (1) *The Severi variety $V_{0,\underline{d},\Delta}^{\text{irr}}$ is non-empty and irreducible, regardless of the characteristic. Furthermore, if $[C]$ is general, then f is injective on the preimage of the boundary of S_Δ ;*
- (2) *If $[C]$ is general and either $\text{char}(K) = 0$ or $\text{char}(K) > w(\Delta)$, then f is locally an immersion. In particular, the singular locus of C is contained in the maximal torus.*

Proof. (1) Let $D_i \subset S_\Delta$ be the irreducible toric divisors for $1 \leq i \leq s$, and $n_i \in N$ the primitive inner normal vectors to the corresponding sides $\partial\Delta_i$ of Δ . Let $p_{i,j} \in \mathbb{P}^1$ be the points such that $f^*(D_i) = \sum_{1 \leq j \leq a_i} d_{i,j} p_{i,j}$. Without loss of generality, we may assume that $p_{i,j} \neq \infty$ for all i and j . Let t be the coordinate on $\mathbb{P}^1 \setminus \{\infty\}$, and set $c_{i,j} := t(p_{i,j})$. Since C is torically transverse of tangency profile \underline{d} , it follows that the pullback under f of the monomial functions is given by

$$f^*(x^m) = \chi(m) \prod_{i,j} (t - c_{i,j})^{d_{i,j}(n_i, m)},$$

where (\cdot, \cdot) is the pairing $M \times N \rightarrow \mathbb{Z}$ on the dual lattices and $\chi: M \rightarrow K^\times$ is a character.

Vice versa, for any choice of a character χ and distinct points $p_{i,j}$ in $\mathbb{P}^1 \setminus \{\infty\}$, the formula above defines a morphism $f: \mathbb{P}^1 \rightarrow S_\Delta$, whose image is torically transverse. It remains to show that for a general choice of the $p_{i,j}$'s, the map $f: \mathbb{P}^1 \rightarrow S_\Delta$ is injective on the preimage of the boundary of S_Δ . Indeed, by the assumption, the tangency profile \underline{d} is trivial on the non-horizontal sides of Δ . Therefore, the injectivity of f over the boundary of S_Δ implies that f is birational onto its image, and hence the image of f belongs to $V_{0,\underline{d},\Delta}^{\text{irr}}$. Thus, $V_{0,\underline{d},\Delta}^{\text{irr}}$ is non-empty and is dominated by an open subset of $\text{Hom}(M, K^\times) \times (\mathbb{P}^1)^{|\underline{d}|}$. The irreducibility now follows.

Let us show the asserted injectivity. Since $f^*(D_k) = \sum_{1 \leq j \leq a_k} d_{k,j} p_{k,j}$, it is sufficient to show that for each k and each $j_1 \neq j_2$, there exists $m \in M$ such that $f^*(x^m)(c_{k,j_1}) \neq f^*(x^m)(c_{k,j_2})$. Let $m_k \in M$ be a primitive integral vector along the side $\partial\Delta_k$. Then $(n_k, m_k) = 0$, and hence the function $f^*(x^{m_k})$ is a non-trivial rational function in t that does not depend on the parameters c_{k,j_1} and c_{k,j_2} . Therefore, for a general choice of $c_{k,j}$'s, $f^*(x^m)(c_{k,j_1}) \neq f^*(x^m)(c_{k,j_2})$ as claimed.

(2) To prove that f is locally an immersion, we shall analyze the differential of f . For any $m \in M$, set

$$\partial_m \log(f) := \frac{1}{f^*(x^m)} \frac{d(f^*(x^m))}{dt} = \sum_{1 \leq i \leq s} \sum_{1 \leq j \leq a_i} \frac{d_{i,j} \cdot (n_i, m)}{t - c_{i,j}}. \quad (5.1)$$

to be the log-derivative of $f^*(x^m)$.

Let us first show that for each k , the log-derivative $\partial_{m_k} \log(f)$ is a non-constant rational function in t that does not depend on the $c_{k,j}$'s. Indeed, since $(n_k, m_k) = 0$, the function $\partial_{m_k} \log(f)$ contains no term(s) with $\{c_{k,j}\}_{1 \leq j \leq a_k}$. On the other hand, we claim that we can pick a side $\partial\Delta_l$ such that $0 < |(n_l, m_k)| \leq w(\Delta)$. If this is true, then by the assumptions on the characteristic of K , we get in particular, that (n_l, m_k) is not divisible by $\text{char}(K)$, and neither is $d_{l,j}$. It follows that $\partial_{m_k} \log(f)$ contains a non-trivial summand $\frac{d_{l,j} \cdot (n_l, m_k)}{t - c_{l,j}}$ in expansion (5.1) for any $1 \leq j \leq a_l$. Thus, $\partial_{m_k} \log(f)$ depends non-trivially on the $c_{l,j}$'s, and hence, for a general choice of the $c_{i,j}$'s, it is a non-constant rational function in t .

Let us now explain how to choose the side $\partial\Delta_l$ as above. If $\partial\Delta_k$ is horizontal, then any non-horizontal side works. Otherwise, let $y = a$ be a horizontal line in $M_{\mathbb{R}}$ with non-integral y -coordinate that intersects $\partial\Delta_k$. Let $\partial\Delta_{k'}$ be the other side of Δ that intersects this line. If $\partial\Delta_{k'}$

is parallel to $\partial\Delta_k$, let $\partial\Delta_l$ be a side of Δ that is adjacent to $\partial\Delta_k$; otherwise let $\partial\Delta_l := \partial\Delta_{k'}$. Then $|(n_l, m_k)|$ is twice the area of the lattice triangle with vertices $(0, 0)$, m_k and m_l , which is non-zero in each case. Furthermore, since Δ is h -transverse, this lattice triangle has a horizontal edge, it is of height 1, and can be embedded into Δ via a translation; hence its area does not exceed $w(\Delta)/2$. Therefore, $|(n_l, m_k)| \leq w(\Delta)$. This justifies our choice of $\partial\Delta_l$.

Finally, we are ready to prove that f is locally an immersion. Let $t_0 \in \mathbb{P}^1$ be any point. We shall show that there exists an $m \in M$ such that $\partial_m \log(f)(t_0) \neq 0$. Assume first, that $t_0 = c_{k,j}$. Since the $c_{i,j}$'s are general and the function $\partial_{m_k} \log(f)$ is non-constant and does not depend on $c_{k,j}$, it follows that $t = c_{k,j}$ is not a zero of $\partial_{m_k} \log(f)$ as needed. Assume next, that $t_0 \neq c_{i,j}$ for all i and j . In particular, $f^*(x^m)$ is defined at t_0 , and $f^*(x^m)(t_0) \neq 0$ for all $m \in M$. Pick $1 \leq k \leq s$, and let $m'_k \in M$ be such that $(n_k, m'_k) = 1$. We will show that either $\partial_{m_k} \log(f)(t_0) \neq 0$ or $\partial_{m'_k} \log(f)(t_0) \neq 0$. We have seen above that $\partial_{m_k} \log(f)$ is not identically zero, and does not depend on the $c_{k,j}$'s. Thus, the zero set $Z \subset \mathbb{P}^1$ of $\partial_{m_k} \log(f)$ is finite and is independent of the choice of the $c_{k,j}$'s. On the other hand, the log derivative $\partial_{m'_k} \log(f)$ contains a non-trivial summand $\frac{d_{k,j} \cdot (n_k, m'_k)}{t - c_{k,j}}$ for all $1 \leq j \leq a_k$. Thus, no point of Z is a zero of $\partial_{m'_k} \log(f)$ since the $c_{k,j}$'s are general. Hence either $\partial_{m_k} \log(f)(t_0) \neq 0$ or $\partial_{m'_k} \log(f)(t_0) \neq 0$, and we are done. \square

Proof of Theorem 5.6. Recall that we assume that K is the algebraic closure of a complete DVR whose residue field \tilde{K} is algebraically closed and has characteristic either 0 or greater than $w(\Delta)/2$. As mentioned, we prove Theorem 5.6 by induction on g . The base case $g = 0$ follows from Lemma 5.12 (1), and we can assume $g > 0$ in the sequel. By the inductive hypothesis, it is enough to show that each irreducible component V of $\overline{V}_{g, \underline{d}, \Delta}^{\text{irr}}$ contains an irreducible component of $\overline{V}_{g-1, \underline{d}, \Delta}^{\text{irr}}$, which further reduces to showing that the locus of curves of geometric genus $g-1$ in V has dimension $n-1 = |\underline{d}| + g - 2$, the dimension of $\overline{V}_{g-1, \underline{d}, \Delta}^{\text{irr}}$ by Proposition 5.5.

Let $[C] \in V^{FD}$ be a curve as in Lemma 5.11. As in Step 1 of the proof of Lemma 5.11, after replacing $[C]$ with another curve in V^{FD} , we can find vertically stretched points $q_1, \dots, q_n \in h_C(\Gamma_C)$ that are in general position with respect to $\text{trop}(C)$; here we view $\text{trop}(C)$ as an element of $M_{g, n, \nabla}^{\text{trop}}$ by adding contracted legs over the q_i , as before. Pick $p_1, \dots, p_n \in C$ such that $\text{trop}(p_i) = q_i$. Then p_1, \dots, p_n are in general position with respect to C by Remark 5.8. Let E and E' be the elevators of $\text{trop}(C)$ as in Lemma 5.11, and assume without loss of generality that $q_n \in h_C(E)$.

Let $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ be the family of parametrized tropical curves over Λ obtained from Construction 5.9 for the points p_1, \dots, p_{n-1} and the curve C . Let $\alpha: \Lambda \rightarrow M_{g, n-1, \nabla}^{\text{trop}}$ be the induced map. Recall that this family is obtained from tropicalizing a family of parametrized curves $f: \mathcal{X} \rightarrow S_\Delta$ over a base (B, τ_\bullet) , where B admits a finite cover to a component $Z \ni [C]$ of the locus in V of curves passing through p_1, \dots, p_{n-1} . To prove the theorem, it suffices to show that there is a $\tau \in B(K)$ such that $[f(\mathcal{X}_\tau)] \in V$ is an integral curve of geometric genus $g-1$, since varying the p_1, \dots, p_{n-1} then produces an $n-1$ dimensional locus of such curves in V .

By the definition of a family of parametrized tropical curves (cf. § 2.4), for any edge e of Λ the underlying graph of the fiber of Γ_Λ over any point in the interior e° of e is a fixed graph \mathbb{G}_e .

Claim 5.13. Λ contains a leg l such that the map h is constant on all vertices of \mathbb{G}_l along l and one of the following holds:

- (1) The lengths of all but one edge γ of \mathbb{G}_l are constant in the family Γ_Λ , and the graph obtained from \mathbb{G}_l by removing γ is connected; or
- (2) the length of all edges of \mathbb{G}_l is constant in the family Γ_Λ , except for at least one edge of a contracted elliptic tail O .

Let l be as in Claim 5.13, and $\tau \in B(K)$ the marked point corresponding to l . We will show that τ is the desired point, namely that $[f(\mathcal{X}_\tau)] \in V$ is an integral curve of geometric genus $g - 1$. Set $D_{\mathcal{X}^0} := \widetilde{\mathcal{X}} \cup (\bigcup_i \mathcal{X}_{\tau_i}) \cup (\bigcup_j \sigma_j)$, where τ_i are the marked points of B and σ_j are the marked points of $\mathcal{X} \rightarrow B$. First, let us show that f is defined on \mathcal{X}_τ .

To see that the rational map f is defined at the generic points of \mathcal{X}_τ , consider the pullbacks $f^*(x^m)$ for $m \in M$. Let u be a vertex of \mathbb{G}_l such that the corresponding component of $\mathcal{X}_{\tilde{\tau}}$ belongs to the closure of a given generic point $\eta \in \mathcal{X}_\tau$. By Remark 4.1, the order of pole of $f^*(x^m)$ at η is equal to the slope $\frac{\partial h(u)(m)}{\partial l}$, which has to vanish since $h(u)$ is constant along l . Therefore, f is defined at the generic points of \mathcal{X}_τ and maps them to the dense orbit of S_Δ . Next, let us show that f is defined on all of \mathcal{X}_τ . Notice that $f^*(x^m)$ is regular and invertible away from $D_{\mathcal{X}^0}$, and it is regular and invertible in codimension one on \mathcal{X}_τ . Since \mathcal{X}^0 is normal, it follows that f is defined on $\mathcal{X}_\tau \setminus (\bigcup_j \sigma_j)$. Pick any σ_j , and let us show that f is defined at $\sigma_j(\tau) \in \mathcal{X}_\tau$, too. Let $S_j \subseteq S_\Delta$ be the affine toric variety consisting of the dense torus orbit and the orbit of codimension at most one containing the image of $\sigma_j(b)$ for a general $b \in B$. Then the pullback of any regular monomial function $x^m \in \mathcal{O}_{S_j}(S_j)$ is regular in codimension one in a neighborhood of $\sigma_j(\tau)$, and hence regular at $\sigma_j(\tau)$ by normality of \mathcal{X} . Thus, f is defined at $\sigma_j(\tau)$, and maps it to S_j .

To complete the proof, assume that assertion (2) of Claim 5.13 is satisfied. The case of assertion (1) can be treated in a similar way. Let $\gamma \in E(\mathbb{G}_l)$ be an edge that corresponds to a node $z \in \mathcal{X}_{\tilde{\tau}}$. Then \mathcal{X}^0 is given étale locally near z by $xy = g_z$ for some regular function g_z defined on a neighbourhood of $\tilde{\tau} \in B^0$. If the length of γ varies over l , then g_z vanishes along τ by Remark 4.1, and thus z does not get smoothed in \mathcal{X}_τ .

If the only edge of varying length in O is a loop, then \mathcal{X}_τ is irreducible, since \mathbb{G}_l is still connected even if we remove this loop. By the argument above, f is defined on \mathcal{X}_τ and maps its generic point to the dense orbit. Moreover, the image of \mathcal{X}_τ intersects the boundary divisor at $|\underline{d}|$ distinct points, and some of them are points of simple intersection by the assumption on \underline{d} . Thus, $f|_{\mathcal{X}_\tau}$ is birational onto its image, and $[f(\mathcal{X}_\tau)] \in V$ is an integral curve of geometric genus $g - 1$. Assume next that O contains a non-loop edge of varying length. Then \mathcal{X}_τ is a union of a smooth irreducible component of genus $g - 1$ and an irreducible component of arithmetic genus 1, and without marked points. Denote the components by \mathcal{X}'_τ and E , respectively. Then $f(E)$ is disjoint from the toric boundary, and therefore f contracts E to a point in the dense orbit of S_Δ . In particular, $f(\mathcal{X}_\tau) = f(\mathcal{X}'_\tau)$ is irreducible. On the other hand, we get arguing as above, that $f|_{\mathcal{X}'_\tau}$ is birational onto its image. Thus, the normalization of $f(\mathcal{X}_\tau)$ is equal to \mathcal{X}'_τ , and $[f(\mathcal{X}_\tau)] \in V$ is an integral curve of geometric genus $g - 1$. \square

Proof of Claim 5.13. Recall that C satisfies the assertion of Lemma 5.11. The points q_1, \dots, q_n are in general position with respect to $\text{trop}(C) = (\Gamma_C, h_C)$ and vertically stretched. The points $p_1, \dots, p_n \in C$ are points such that $\text{trop}(p_i) = q_i$. The edges E and E' are the elevators of $\text{trop}(C)$ contained in a cycle O and adjacent to a floor F as in Lemma 5.11, and $q_n \in h_C(E)$. Moreover, $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ is the family of parametrized tropical curves obtained from Construction 5.9 for p_1, \dots, p_{n-1} , and $\alpha: \Lambda \rightarrow M_{g,n-1,\nabla}^{\text{trop}}$ the induced map.

As before, we may consider $\text{trop}(C)$ as a curve in the family $h: \Gamma_\Lambda \rightarrow N_{\mathbb{R}}$ by marking the preimages of q_1, \dots, q_{n-1} in Γ_C . Then E contains no marked point and E' contains one marked point which we denote by q_j . As in the proof of Lemma 5.11, we may assume that on the floors adjacent to E there is no special point that has x -coordinate between those of E and E' . Let $M_{\Theta'} \subset M_{g,n-1,\nabla}^{\text{trop}}$ be the nice stratum that contains $\text{trop}(C)$. According to the two conditions of Lemma 5.11, there are two cases to consider.

Case 1: $\frac{\partial h_C}{\partial \vec{E}} = -\frac{\partial h_C}{\partial \vec{E}'}$, where \vec{E} and \vec{E}' are oriented away from F . As in the proof of Lemma 5.11, we move E towards E' till we hit a simple wall M_Θ that parametrizes curves (Γ, h) containing a 4-valent vertex u adjacent to both E and E' . See Figure 9 for an illustration. As before, all parametrized tropical curves along this deformation belong to $\alpha(\Lambda)$, and hence $[(\Gamma, h)] = \alpha(w)$ for a vertex w of Λ . Furthermore, by Lemma 5.10 (1), we may assume that α is locally combinatorially surjective at w . Let $M_{\Theta''}$ be the nice stratum in the star of M_Θ in which E and E' are adjacent to the same vertex u' of the splitting of u into two 3-valent vertices. Let $e \in \text{Star}(w)$ be an edge such that $\alpha(e^\circ) \subset M_{\Theta''}$. Since $\mathfrak{w}(E) = \mathfrak{w}(E')$, it follows from the balancing condition that the third edge adjacent to u' gets contracted by the parametrization map. Thus, the locus $M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1})$ is an *unbounded* interval. Furthermore, the images of all curves in this locus agree with $h(\Gamma)$.

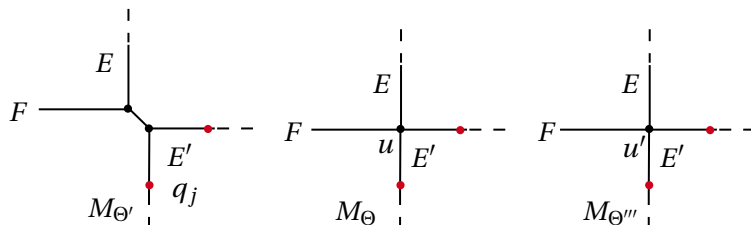


FIGURE 9. Developing a contracted edge in Case 1 of the proof of Claim 5.13.

By Lemma 5.10 (2), $M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1}) = \alpha(\Lambda) \cap M_{\Theta''}$, and thus there exists a leg l of Λ such that $\alpha(l) \subset M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1})$ is not bounded. The leg l satisfies the assertion of the claim: the graph \mathbb{G}_l is the underlying graph of Θ'' , and the edge γ in the claim is just the contracted edge adjacent to u' as above. The connectivity of $\mathbb{G}_l \setminus \gamma$ follows from the fact that E and E' are contained in the same cycle, and hence assertion (1) of Claim 5.13 is satisfied.

Case 2: O has vertical complexity 2. See left picture in Figure 10. We may assume that E'

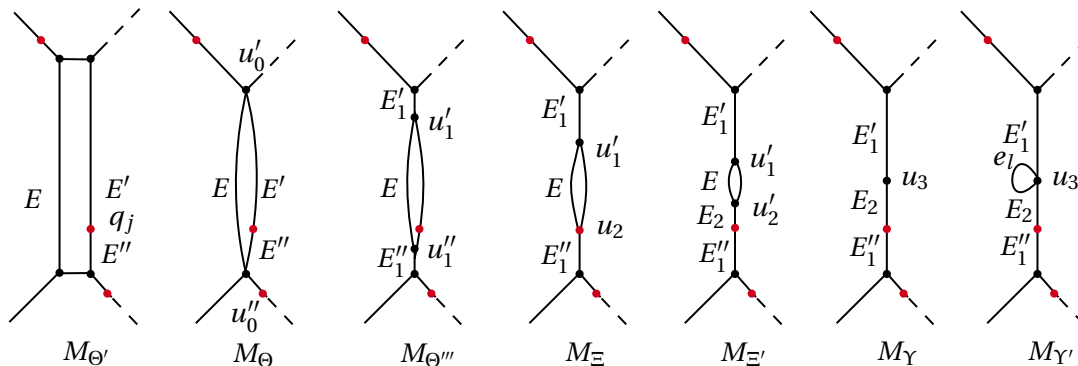


FIGURE 10. Developing a contracted loop in Case 2 of the proof of Claim 5.13.

lies above q_j . Let E'' be the other elevator that contains q_j . By moving E towards E' as before, we find a boundary point (Γ', h') of $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$ that is contained in a stratum M_Θ . In this case, M_Θ is not a simple wall. Instead, (Γ', h') is weightless and 3-valent except for two 4-valent vertices u'_0 and u''_0 , with u'_0 adjacent to E and E' , and u''_0 adjacent to E and E'' ; in other words, E, E', E'' form a flattened cycle between u'_0 and u''_0 . By Lemma 5.10 (2), $\alpha(\Lambda)$ contains $M_{\Theta'} \cap \text{ev}_{\Theta'}^{-1}(q_1, \dots, q_{n-1})$, hence also contains (Γ', h') .

We keep deforming the obtained tropical curve by shrinking the flattened cycle, cf. Figure 10: let $M_{\Theta''}$ be the stratum in the star of M_{Θ} in which both u'_0 and u''_0 split into two 3-valent vertices, such that each pair of elevators adjacent to u'_0 (resp., u''_0) remains adjacent to vertices u'_1 (resp., u''_1); that is, the flattened cycle is preserved. Denote the new edges adjacent to u'_1 and u''_1 by E'_1 and E''_1 , respectively. Then $\alpha(\Lambda) \cap M_{\Theta''}$ is non-empty by Lemma 4.5. Note that, unlike before, $M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1})$ has dimension 2: one can perturb the images of u'_1 and u''_1 independently. Nevertheless, the locus of realizable curves in $M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1})$ is one-dimensional by Proposition 3.8; more precisely, we get that $\alpha(\Lambda) \cap M_{\Theta''}$ is contained in the sublocus $L_{\Theta''}$ of $M_{\Theta''} \cap \text{ev}_{\Theta''}^{-1}(q_1, \dots, q_{n-1})$ consisting of curves such that $\ell(E'_1) = \ell(E''_1)$, where $\ell(\bullet)$ denotes the edge lengths. Plainly, $L_{\Theta''}$ is a line segment in $M_{\Theta''}$. Since the underlying graph of Θ'' is weightless and 3-valent, α is harmonic at vertices of Λ that are mapped to $M_{\Theta''}$ by Remark 4.3. It follows that $\alpha(\Lambda)$ contains $L_{\Theta''}$.

By the balancing condition, we have $\mathfrak{w}(E'_1) = \mathfrak{w}(E''_1)$. This means that, along $L_{\Theta''}$, the images of u'_1 and u''_1 are moving (one downwards and the other upwards) with the same speed. We may assume that q_j lies strictly below the center of the flattened cycle. When u''_1 reaches q_j , we obtain a boundary point of $L_{\Theta''}$ that is not contained in $M_{\Theta''}$; namely, a curve which contains a 4-valent vertex u_2 with the following adjacent edges/legs: E''_1 , the leg l_j contracted to q_j , and the two elevators that lie above it. Let Ξ be the combinatorial type of this curve. Note that this parametrized tropical curve has a non-trivial automorphism induced by switching the two elevators above u_2 when these two elevators have the same multiplicity.

Let Ξ' be the combinatorial type obtained from Ξ by splitting u_2 into two 3-valent vertices, such that l_j and E''_1 remain incident. This shrinks the flattened cycle further, and we denote by u'_2 its lower endpoint and by E_2 the downward elevator adjacent to u'_2 . By Lemma 4.4, $\alpha(\Lambda) \cap M_{[\Xi']}$ $\neq \emptyset$, and, arguing as above, $\alpha(\Lambda)$ contains (the image in $M_{[\Xi']}$ of) the line segment $L_{\Xi'} \subset M_{\Xi'} \cap \text{ev}_{\Xi'}^{-1}(q_1, \dots, q_{n-1})$ consisting of curves with $\ell(E'_1) = \ell(E_2) + \ell(E''_1)$.

The second boundary point of $L_{\Xi'}$, obtained by shrinking the flattened cycle to a point, contains a 2-valent vertex u_3 of weight 1 adjacent to E'_1 and E_2 . Let Y be its combinatorial type. Recall that $\text{char}(\tilde{K})$ is either 0 or greater than $\mathfrak{w}(\Delta)/2$.¹ By Lemma 3.4, the multiplicity of the edge E'_1 (and, by balancing, of E_2) is at most $\mathfrak{w}(\Delta)$. Thus either the multiplicity is not divisible by $\text{char}(\tilde{K})$ or equal to $\text{char}(\tilde{K})$. In the first case we develop a contracted loop, and in the second case a contracted elliptic tail as follows.

First, assume that $\mathfrak{w}(E'_1)$ is not divisible by $\text{char}(\tilde{K})$. Let $M_{[\Upsilon']}$ be a stratum in the star of $M_{[\Upsilon]}$ such that Y' is obtained from Y by adding a contracted loop at u_3 and setting the weight of u_3 to 0. By Lemma 4.6 (1) we have that $\alpha(\Lambda) \cap M_{[\Upsilon']}$ is non-empty. By Proposition 3.8, we have that $\alpha(\Lambda) \cap M_{[\Upsilon']}$ is contained in (the image of) the ray $L_{\Upsilon'} \subset M_{\Upsilon'} \cap \text{ev}_{\Upsilon'}^{-1}(q_1, \dots, q_{n-1})$ consisting of curves such that $\ell(E'_1) = \ell(E_2) + \ell(E''_1)$. By Lemma 4.6 (3), α is harmonic at vertices of Λ that are mapped to $M_{[\Upsilon']}$, and hence, similar as above, $L_{\Upsilon'}$ is contained in $\alpha(\Lambda)$. Thus, there is a leg l of Λ such that $\alpha(l) \subset L_{\Upsilon'}$ is not bounded. Then l satisfies the assertion of the claim: the graph \mathbb{G}_l of the claim is the underlying graph of Y' , and the edge γ of the claim is the contracted loop at u_3 (in particular, $\mathbb{G}_l \setminus \gamma$ is connected). Thus assertion (1) of Claim 5.13 is satisfied.

Finally, suppose $\mathfrak{w}(E'_1) = \text{char}(\tilde{K})$. In this case, let Y'' be the combinatorial type obtained from Y by adding a contracted edge adjacent to u_3 , now 3-valent and of weight 0, and u'_3 , a new vertex of weight 1 and valence 1; let Y''' be the combinatorial type obtained from Y'' by developing a loop at u'_3 (that is, u_3 , u'_3 , the edge between them and possibly the loop form a contracted elliptic

¹What follows is the only point in the proof of Theorem 5.6 at which we use the assumption on the characteristic, and we need it to apply Lemma 4.6.

tail; cf. Figure 2). By Lemma 4.6 (2) we have that $\alpha(\Lambda) \cap M_{[\gamma'']}$ is non-empty. By Lemma 4.6 (3), for a vertex $w \in \Lambda$ with $\alpha(w) \in M_{[\gamma'']}$, α is harmonic at w , or there is an edge/leg e adjacent to w such that $\alpha(e^\circ) \subset M_{[\gamma'']}$. If α is harmonic at all vertices mapped to $M_{[\gamma'']}$, we argue as above to show that assertion (2) of Claim 5.13 is satisfied; that is, we deduce from Proposition 3.8 that $\alpha(\Lambda)$ contains the ray $L_{\gamma''}$, defined analogous to $L_{\gamma'}$. Otherwise, we get that $\alpha(\Lambda) \cap M_{[\gamma'']}$ is not empty. Since α is harmonic at vertices mapped to $M_{[\gamma'']}$ by Lemma 4.6 (3), we can argue again as above to obtain that assertion (2) of Claim 5.13 is satisfied; that is, we deduce from Proposition 3.8 that $\alpha(\Lambda)$ contains a ray in the intersection of $M_{[\gamma'']}$ with the plane given by the condition $\ell(E'_1) = \ell(E_2) + \ell(E''_1)$. \square

5.3. A counterexample in low characteristic. We conclude our discussion of degenerations on S_Δ with an example that shows that Theorem 5.6 may fail without the assumption on the characteristic of K . Recall that we defined kites in Example 3.2 following [LT20] as polygons with vertices $(0, 0)$, $(k, \pm 1)$ and $(k + k', 0)$ for non-negative integers k, k' with $k' \geq k$ and $k' > 0$. As we noted above, kites are h -transverse, and by definition we have $w(\Delta) = k + k'$.

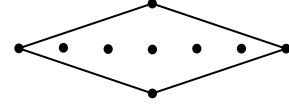


FIGURE 11. $k = k' = 3$.

Proposition 5.14. *Suppose $\text{char}(K) = p > 0$ and let Δ be the kite with*

$$k = k' = \begin{cases} 3 & \text{if } p = 2 \\ p & \text{otherwise} \end{cases}.$$

Consider $V_{1,\Delta}^{\text{irr}}$ parametrizing irreducible curves of geometric genus 1 on S_Δ with trivial tangency profile. Then there is an irreducible component V of $\overline{V}_{1,\Delta}^{\text{irr}}$, such that $V \cap V_{0,\Delta}^{\text{irr}} = \emptyset$.

Proof. We assume $k = k' = p > 2$, the argument for $p = 2$ and $k = k' = 3$ is completely analogous. We first follow the description in [LT20, §3] to obtain some properties of the curves parametrized by $V_{1,\Delta}^{\text{irr}}$. For a general point $[C] \in V_{1,\Delta}^{\text{irr}}$, let $f: E \rightarrow S_\Delta$ be the parametrized curve given by normalizing C . Let O, P, Q, R be the preimages of the toric divisors D_i corresponding to the sides of Δ with outer normals $(-1, -p)$, $(-1, p)$, $(1, -p)$, $(1, p)$, respectively. Then the divisors on E of the pullbacks of the monomials x and y are given by $\text{div}(f^*(y)) = -pO + pP - pQ + pR$ and $\text{div}(f^*(x)) = -O - P + Q + R$. Using these equations, we get in the group law of the elliptic curve (E, O) :

$$P = R + Q \quad \text{and} \quad 2pR = 0. \tag{5.2}$$

Conversely, any smooth genus 1 curve E and choice of distinct points O, P, Q, R satisfying the two conditions above gives an element in $V_{1,\Delta}^{\text{irr}}$.

For $l \in \mathbb{N}_{\geq 2}$ let $X_1(l)$ denote the modular curve parametrizing generalized elliptic curves together with an ample Drinfeld $\mathbb{Z}/l\mathbb{Z}$ -structure as in [Con07, Definitions 2.4.1 and 2.4.6]; see also [DR73] and [KM85]. Denote by $Y_1(l)$ the open subset of $X_1(l)$ over which the elliptic curves are smooth. We will consider the forgetful map from $V_{1,\Delta}^{\text{irr}}$ to $Y_1(2p)$ induced by (E, O, R) . More precisely, by what we discussed above, we may choose a point $[C] \in V_{1,\Delta}^{\text{irr}}$ such that for the triple (E, O, R) , R is a point of exact order $2p$ – that is, the subgroup scheme generated by multiples of R is isomorphic to $\mathbb{Z}/2p\mathbb{Z}$. Denote by V° the irreducible component of $V_{1,\Delta}^{\text{irr}}$ containing $[C]$. Possibly after pulling back along a finite surjective map, we may assume that the universal family over V° is equinormalizable by [dJ96, §5.10]. Thus the normalization of the total space of the universal family induces a map $V^\circ \rightarrow Y_1(2p)$, whose image is contained in an irreducible component Y° of $Y_1(2p)$. Denote by $Y \subset X_1(2p)$ its closure.

Now let $[C_0]$ be a point in the boundary of $V = \overline{V^\circ}$ and assume contrary to the claim that $[C_0] \in V_{0,\Delta}^{\text{irr}}$. Choose a discretely valued field F with valuation ring F^0 and a map $\text{Spec}(F^0) \rightarrow V$ such that the generic point of $\text{Spec}(F^0)$ maps to a point in V° , and the special point to $[C_0]$. Pulling back the normalization of the total space of the universal family over V° , we get a smooth genus 1 curve $\mathcal{E} \rightarrow \text{Spec}(F)$ with the four marked points O, P, Q, R . Up to a finite extension of F , we may assume that this family extends to a stable model $\mathcal{E}^0 \rightarrow \text{Spec}(F^0)$. Then f extends on the central fiber $\tilde{\mathcal{E}}$ of \mathcal{E}^0 to a map $\tilde{f}: \tilde{\mathcal{E}} \rightarrow S_\Delta$ with $C_0 = \tilde{f}(\tilde{\mathcal{E}})$. Since we assume that C_0 is torically transverse, C_0 intersects the toric boundary in 4 distinct points. Since $\tilde{\mathcal{E}}$ is stable of arithmetic genus 1, and C_0 is irreducible, this implies the following:

(†) at least 3 marked points lie on an irreducible component E_0 of $\tilde{\mathcal{E}}$ of geometric genus 0.

On the other hand, consider the map $\varphi^\circ: \text{Spec}(F) \rightarrow Y^\circ$ induced by (\mathcal{E}, O, R) as above. Since $X_1(2p)$ is proper, the map extends uniquely to a map $\varphi: \text{Spec}(F^0) \rightarrow Y$. We denote by $[X_0] \in Y$ the image of the special point under this extension, corresponding to a generalized elliptic curve X_0 . Let O_0, P_0, Q_0 and R_0 denote the points on X_0 that extend, respectively, $\varphi^\circ(O), \varphi^\circ(P), \varphi^\circ(Q)$, and $\varphi^\circ(R)$ to X_0 . Recall that we have two natural maps $\varphi_2: X_1(2p) \rightarrow X_1(p)$ and $\varphi_p: X_1(2p) \rightarrow X_1(2)$, both over $\overline{M}_{1,1}$. They map a level $2p$ structure to a level p structure (respectively, a level 2 structure) via multiplication by 2 (respectively, p); on a singular curve, they contract irreducible components, that no longer intersect the level structure, see [Con07, Lemma 4.2.3 (ii)].

Suppose first X_0 is smooth. Since $p \nmid 2$, the torsion points parametrized by $Y_1(2)$ all are distinct from the origin on the elliptic curves. Since R has exact order $2p$, this implies that the map on universal families induced by φ_p maps R_0 and O_0 to distinct points. In particular, R_0 and O_0 are distinct points on X_0 . On the other hand, by (5.2), we have $Q_0 = O_0 + Q_0$ and $P_0 = R_0 + Q_0$. Thus also Q_0 and P_0 are distinct points on X_0 . This implies that the stable model of the curve with marked points $(\varphi^\circ(\mathcal{E}), \varphi^\circ(O), \varphi^\circ(P), \varphi^\circ(Q), \varphi^\circ(R))$ cannot contain a rational component with 3 marked points; by uniqueness of the stable model, this contradicts (†). Now suppose X_0 is singular. By [DR73, Proposition 2.3, p. 246], $X_1(p)$ has 2 irreducible components and one of them, Y' , parametrizes, away from supersingular curves, generalized elliptic curves together with a fixed inclusion of $\mathbb{Z}/p\mathbb{Z}$ as a subgroup scheme. Since the multiplicative torus \mathbb{G}_m does not contain a subgroup scheme isomorphic to $\mathbb{Z}/p\mathbb{Z}$ in characteristic p , the universal family over Y' does not contain singular, irreducible fibers. Since we chose R such that it generates a subgroup scheme isomorphic to $\mathbb{Z}/2p\mathbb{Z}$, we have that $\varphi_2(Y) \subset Y'$. Thus X_0 is not irreducible. By the definition of an ample Drinfeld $\mathbb{Z}/2p\mathbb{Z}$ structure, it follows that O_0 and R_0 lie on different irreducible components of X_0 . Arguing similar as before, we obtain that also Q_0 and P_0 lie on different irreducible components of X_0 by (5.2). This contradicts (†) by the uniqueness of the stable model. \square

6. ZARISKI'S THEOREM

As a consequence of Theorem 5.6, we prove in this section Zariski's Theorem for S_Δ . Recall that we denote the sides of Δ by $\partial\Delta_i$ and the corresponding toric divisors of S_Δ by D_i , where $1 \leq i \leq s$.

Theorem 6.1 (Zariski's Theorem). *Let Δ be an h -transverse polygon and suppose either $\text{char}(K) = 0$ or $\text{char}(K) > w(\Delta)$. Let $0 \leq g \leq \#\Delta^\circ \cap M$ be an integer and \underline{d} a tangency profile of $(S_\Delta, \mathcal{L}_\Delta)$, which is trivial on toric divisors corresponding to non-horizontal sides of Δ . Let $V \subset V_{g, \underline{d}, \Delta}$ be a subvariety. Then:*

(1) $\dim V \leq |\underline{d}| + g - 1$, and

(2) if $\dim V = |\underline{d}| + g - 1$, then for a general $[C] \in V$, the curve C is nodal, and its singular locus is contained in the maximal torus.

Lemma 6.2. *Under the assumptions of Theorem 6.1, suppose that Δ has height 2 and no horizontal sides. Then for a general point $[C] \in V_{0,\Delta}^{\text{irr}}$, the curve C is nodal.*

Proof. After a translation and a change of coordinates, we may assume that Δ has vertices $(0,0)$, $(a,1)$, $(a+b,0)$, $(0,-1)$, where $a, b \geq 0$ and $a+b > 0$. We may further assume that $w(\Delta) \geq 2$ as otherwise C is smooth. Therefore, $\text{char}(K) > 2$. Let us label the sides of Δ as illustrated in Figure 12. Note that the case where Δ is a triangle is included by setting $a = 0$.

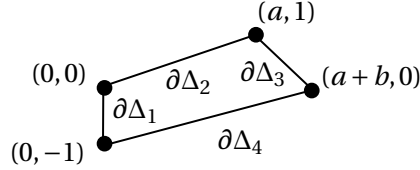


FIGURE 12

As in the proof of Lemma 5.12, a general curve $[C] \in V_{0,\Delta}^{\text{irr}}$ is represented as the image of a map $f: \mathbb{P}^1 \rightarrow S_\Delta$. We may assume that $f^{-1}(D_1) = c$, $f^{-1}(D_2) = 1$, $f^{-1}(D_3) = 0$ and $f^{-1}(D_4) = \infty$. Then

$$f^*(x) = \alpha \cdot \frac{t}{(t-c)(t-1)} \text{ and } f^*(y) = \beta \cdot t^b (t-1)^a,$$

where $\alpha, \beta \in K^\times$.

By Lemma 5.12, the curve C has no unibranch singularities and is smooth on the boundary divisor of S_Δ . On the other hand, since $f^*(x)$ is the quotient of two polynomials of degree at most 2, each singularity of C has multiplicity at most 2. Thus it suffices to show that a general curve $[C] \in V_{0,\Delta}^{\text{irr}}$ does not contain a (generalized) tacnode that lies in the maximal torus.

Suppose to the contrary that $p \in (K^\times)^2$ is a tacnode of C , and $f^{-1}(p) = \{t_1, t_2\}$. Then $f^*(x)(t_1) = f^*(x)(t_2)$, from which we get $t_1 t_2 = c$. On the other hand, since $\left(\frac{df^*(x)}{dt}\Big|_{t_i}, \frac{df^*(y)}{dt}\Big|_{t_i}\right)$ gives the same tangent direction for $i = 1, 2$, we have

$$\frac{df^*(x)}{dt}\Big|_{t_1} \cdot \frac{df^*(y)}{dt}\Big|_{t_2} = \frac{df^*(x)}{dt}\Big|_{t_2} \cdot \frac{df^*(y)}{dt}\Big|_{t_1}. \quad (6.1)$$

Let us compute the derivatives:

$$\frac{df^*(x)}{dt} = f^*(x) \left(\frac{1}{t} - \frac{1}{t-c} - \frac{1}{t-1} \right) \text{ and } \frac{df^*(y)}{dt} = f^*(y) \left(\frac{a}{t-1} + \frac{b}{t} \right).$$

Substituting these in equation (6.1), we get

$$(a+b) \cdot (t_1-c)(t_2-c) + a(c-1) \cdot t_1 t_2 + bc \cdot (t_1-1)(t_2-1) = 0.$$

Since $t_1 t_2 = c \neq 0$, we can rearrange terms to obtain

$$(a+2b) \cdot (t_1+t_2) = (2a+2b) \cdot c + 2b.$$

As c is general and $\text{char}(K) > \max\{2, a+b\}$, this equation can not hold if $a+2b = 0$. Hence we may assume $a+2b \neq 0$, which gives

$$t_1 + t_2 = \frac{(2a+2b) \cdot c + 2b}{a+2b}.$$

It follows that t_1 and t_2 are the two roots of the quadratic polynomial

$$F(t) := t^2 - \frac{(2a+2b) \cdot c + 2b}{a+2b} t + c.$$

On the other hand, since $f^*(y)(t_1) = f^*(y)(t_2)$, t_1 and t_2 are also two roots of

$$G_\mu(t) := t^b(t-1)^a - \mu$$

for some $\mu \in K^\times$. Therefore, in order to get a contradiction, it suffices to show that for a general $c \in \mathbb{P}^1$, there is no $\mu \in K^\times$ such that $F(t)$ divides $G_\mu(t)$.

Let $R(t)$ be the residue of $t^b(t-1)^a$ divided by $F(t)$. We can write $R(t) = A(c) \cdot t + B(c)$ where A and B are polynomials in c . Now $F(t)$ not dividing $G_\mu(t)$ for any $\mu \in K$ (including $\mu = 0$) is equivalent to $A(c) \neq 0$. It is therefore sufficient to show that A is a non-zero polynomial, which further reduces to showing that $F(t)$ does not divide $G_\mu(t)$ for any $\mu \in K$ when $c = 0$. In this case the two roots of $F(t)$ are 0 and $2b/(a+2b)$. If $a = 0$, then $t = 0$ and $t = 2b/(a+2b) = 1$ can not be the roots of $G_\mu = t^b - \mu$ at the same time. If $a \neq 0$ and $b = 0$ then $F = t^2$ and there is a term $\pm at$ in G_μ , hence F does not divide G_μ . If $a, b \neq 0$ and $\mu = 0$ then $t = 2b/(a+2b) \neq 0, 1$ is not a root of $G_\mu = t^b(t-1)^a$; if $a, b, \mu \neq 0$ then $t = 0$ is not a root of G_μ . In each case G_μ is not divisible by F , hence we are done. \square

Proof of Theorem 6.1. The first part of the claim follows immediately from Proposition 5.5. We need to show that if V is an irreducible component of $V_{g, \underline{d}, \Delta}$, then for a general point $[C] \in V$, the curve C is nodal, and all its singular points are contained in the maximal torus. As in the proof of Proposition 5.5, suppose C has m irreducible components C_1, \dots, C_m . Set $g_k := \mathfrak{p}_g(C_k)$, $\mathcal{L}_k := \mathcal{O}_{S_\Delta}(C_k)$, and let \underline{d}_k be the tangency profile of the parametrized curve induced by the normalization of C_k . Then V is dominated by an irreducible component $\prod_{k=1}^m V_k$ of $\prod_{k=1}^m \overline{V}_{g_k, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$, where V_k is an irreducible component of $\overline{V}_{g_k, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$.

Let $\Delta_k \subset M_{\mathbb{R}}$ be the polygon that defines the line bundle \mathcal{L}_k . If Δ_k is degenerate, then $g_k = 0$, \underline{d}_k is trivial, $|\underline{d}_k| = 1$, and $V_k = \overline{V}_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}} = |\mathcal{L}_k|$. Otherwise it is easy to check that Δ_k is still h -transverse, and we have a natural map $\pi_k: S_\Delta \rightarrow S_{\Delta_k}$ obtained from a sequence of toric blow-ups. Note that π_k induces a natural map $\pi'_k: V_k \rightarrow \overline{V}_{g_k, \underline{d}_k, \Delta_k}^{\text{irr}}$. By Proposition 5.5, we have $\dim V_k = \dim \overline{V}_{g_k, \underline{d}_k, \Delta_k}^{\text{irr}}$. Since a general point of V_k represents a curve in S_Δ that does not intersect the exceptional divisors of π_k , the map π'_k maps V_k birationally to an irreducible component V'_k of $\overline{V}_{g_k, \underline{d}_k, \Delta_k}^{\text{irr}}$. Now by Theorem 5.6, V'_k contains $V_{0, \underline{d}_k, \Delta_k}^{\text{irr}}$, hence V_k contains $V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$, and $\prod_{k=1}^m V_k$ contains $\prod_{k=1}^m V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$. Since the locus of nodal curves is open in $\prod_{k=1}^m V_k$ by [Sta20, Tag 0DSC], and similarly the locus of curves with singular locus contained in the maximal torus of S_Δ is open, it remains to prove the following claim. \square

Claim 6.3. Let $C = \cup_{1 \leq k \leq m} C_k$ be a curve such that $[C_k] \in V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$ is general. Then C is nodal, and its singular locus is contained in the maximal torus.

Proof. According to Lemma 5.12, the singular locus of each C_k is contained in the maximal torus of S_Δ . On the other hand, as each $V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$ admits a torus action, we can move C_k so that it misses the points of intersection of the other components of C with the boundary of S_Δ . As a result, the intersections of components of C are also contained in the maximal torus, hence the singular locus of C is contained in the maximal torus. It remains to show that C is nodal.

Let $\underline{d} = \left(\{d_{i,j}\}_{1 \leq j \leq a_i} \right)_{1 \leq i \leq s}$ and $C \cap D_i = \{p_{i,1}, \dots, p_{i,a_i}\}$ for all $1 \leq i \leq s$. Assume to the contrary that C has a singular point p that is not a node. By Lemma 5.12, each C_i contains no unibranch singularities, and thus there are the following two cases to consider.

Case 1: *there are at least three branches of C that pass through p .* Again, since the variety $V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$ admits a torus action, and C_k is general, these branches must come from the same component of C , say C_1 . Then Δ_1 is non-degenerate and has height at least 2. In particular, we have $|\underline{d}_1| \geq 4$ as \underline{d}_1 is trivial on toric divisors corresponding to non-horizontal sides of Δ . Let $U \subset |\mathcal{L}_\Delta|$ be the locus of curves $C' = \cup_{1 \leq k \leq m} C'_k$ such that $[C'_k] \in V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$ for all k and $p \in C'_1$ is a singular point with at least three branches. Then

$$\dim U \geq \dim V_{0, \underline{d}_1, \mathcal{L}_1}^{\text{irr}} - 2 + \sum_{k=2}^m \dim V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}} = \sum_{k=1}^m (|\underline{d}_k| - 1) - 2.$$

Let $\pi_p: S \rightarrow S_\Delta$ be the surface obtained from S_Δ by blowing up p . Let E be the exceptional divisor, and \tilde{C} (respectively, \tilde{C}_k) be the strict transform of C (respectively, C_k). It follows that

$$K_S \sim \pi_p^* K_{S_\Delta} + E \text{ and } \tilde{C}_1 \sim \pi^* C_1 - aE,$$

where $a \geq 3$, and $\tilde{C}_k \sim \pi^* C_k$ for $k \geq 2$. Taking strict transforms in S gives a rational map $U \dashrightarrow V_{1-m, \sigma_S(\tilde{C})}$, since $1 - m = p_g(C)$. It is an inclusion on the domain and we denote by U' its image. Then $[\tilde{C}] \in U'$ is general, and the normalization $f: C^\vee \rightarrow S$ of \tilde{C} is immersed. Hence the normal sheaf \mathcal{N}_f is a line bundle on C^\vee , and its degree on each component C_k^\vee of C^\vee is $-K_S \cdot \tilde{C}_k + \deg K_{C_k^\vee}$. In other words, \mathcal{N}_f has degree $-K_{S_\Delta} \cdot C_1 - 2 - a$ on C_1^\vee , and degree $-K_{S_\Delta} \cdot C_k - 2$ on C_k^\vee for $k \geq 2$.

Now since $\text{char}(K) > w(\Delta)$, each $d_{i,j}$ is not divisible by $\text{char}(K)$, and the space of first order deformations of f (with fixed target S_Δ) that preserves the tangency with the boundary of S_Δ is identified with $H^0\left(C^\vee, \mathcal{N}_f\left(-\sum_{i=1}^s \sum_{j=1}^{a_i} (d_{i,j} - 1)p_{i,j}\right)\right)$. Since $|\underline{d}_1| \geq 4$, we have

$$\begin{aligned} \dim T_{[C]}U &= \dim T_{[\tilde{C}]}U' \leq h^0\left(C^\vee, \mathcal{N}_f\left(-\sum_{i=1}^s \sum_{j=1}^{a_i} (d_{i,j} - 1)p_{i,j}\right)\right) = \\ &= \max\{|\underline{d}_1| - 1 - a, 0\} + \sum_{k=2}^m (|\underline{d}_k| - 1) < \dim U. \end{aligned}$$

This provides a contradiction.

Case 2: *there are two branches of C passing through p with non-trivial tangency at p .* Denote by σ the tangent direction of these branches at p . Let $\pi_p: S \rightarrow S_\Delta$ be the surface obtained by blowing up p as in Case 1, and $\pi_\sigma: S' \rightarrow S$ be obtained by blowing up the point in the exceptional divisor E of S corresponding to the direction σ . Let \tilde{C} and \tilde{C}_k be the strict transforms of C and C_k in S' respectively, and $f': C^\vee \rightarrow S'$ the map induced by normalizing \tilde{C} . Let F_1 be the pullback of E to S' , and F_2 the exceptional divisor of π_σ . Then $K_{S'} = \pi_\sigma^* \pi_p^* K_{S_\Delta} + F_1 + F_2$

Case 2.1: *the two branches at p come from two different components, say C_1 and C_2 .* Then at least one of Δ_1 and Δ_2 is non-degenerate. By the argument above we can assume that p is contained in the maximal torus. By assumption, we have a rational map from $\prod_{1 \leq k \leq m} V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$ to the incidence variety P of a point contained in a line in the maximal torus of S_Δ ; namely, it is given by mapping C to the point p and the tangent line of C_1 at p (in direction σ). Let W be the fiber of this map for our fixed p and σ . Since P has dimension 3, we have

$$\dim W \geq \sum_{k=1}^m \dim V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}} - 3 = \sum_{k=1}^m (|\underline{d}_k| - 1) - 3.$$

Moreover, if Δ_1 or Δ_2 is degenerate, then $\dim W \geq \sum_{k=1}^m (|\underline{d}_k| - 1) - 2$ since the choice of the tangent direction σ is unique, and the image of the map to P has dimension at most 2.

Similar as in Case 1, passing to strict transforms in S' gives a rational inclusion $W \dashrightarrow V_{g, \mathcal{O}_{S'}}(\tilde{C})$. Let W' be the image of this map. By construction, we have

$$\tilde{C}_k \sim \pi_\sigma^* \pi_p^* C_k - F_1 - F_2 \text{ for } k = 1, 2, \text{ and } \tilde{C}_k \sim \pi^* C_k \text{ for } k > 2.$$

Hence $\mathcal{N}_{f'}$ has degree $-K_{S_\Delta} \cdot C_k - 4$ on C_k^v for $k = 1, 2$, and degree $-K_{S_\Delta} \cdot C_k - 2$ on C_k^v for $k \geq 3$. Thus

$$\begin{aligned} \dim T_{[C]} W &= \dim T_{[\tilde{C}]} W' \leq h^0 \left(C^v, \mathcal{N}_{f'} \left(-\sum_{i=1}^s \sum_{j=1}^{a_i} (d_{i,j} - 1) p_{i,j} \right) \right) = \\ &= \sum_{k=1}^2 \max \{ |\underline{d}_k| - 3, 0 \} + \sum_{k=3}^m (|\underline{d}_k| - 1) < \dim W. \end{aligned}$$

This again provides a contradiction.

Case 2.2: *the two branches at p come from the same component, say C_1 .* Then C_1 is singular and Δ_1 has height at least 2 as in Case 1. By Lemma 6.2, we have $|\underline{d}_1| \geq 5$. Similar to W in Case 2.1, let $Z \subset |\mathcal{L}_\Delta|$ be the locus of curves $C' = \cup_{1 \leq k \leq m} C'_k$ such that $[C'_k] \in V_{0, \underline{d}_k, \mathcal{L}_k}^{\text{irr}}$, $p \in C'_1$ is a tacnode, and the tangent direction of C'_1 at p is σ . Arguing as in Case 2.1, we get

$$\dim Z \geq \sum_{k=1}^m (|\underline{d}_k| - 1) - 3.$$

Let Z' be the image of the rational inclusion $Z \dashrightarrow V_{g, \mathcal{O}_{S'}}(\tilde{C})$. We have

$$\tilde{C}_1 \sim \pi^* C_1 - 2F_1 - 2F_2 \text{ and } C_k \sim \pi^* C_k \text{ for } k > 2.$$

Therefore, $\mathcal{N}_{f'}$ has degree $-K_{S_\Delta} \cdot C_1 - 6$ on C_1^v , and degree $-K_{S_\Delta} \cdot C_k - 2$ on C_k^v for $k \geq 2$. It follows that

$$\begin{aligned} \dim T_{[C]} Z &= \dim T_{[\tilde{C}]} Z' \leq h^0 \left(C^v, \mathcal{N}_{f'} \left(-\sum_{i=1}^s \sum_{j=1}^{a_i} (d_{i,j} - 1) p_{i,j} \right) \right) = \\ &= \max \{ |\underline{d}_1| - 5, 0 \} + \sum_{k=2}^m (|\underline{d}_k| - 1) < \dim Z. \end{aligned}$$

This is again a contradiction. \square

We conclude this section with an example showing that some bound on $\text{char}(K)$ is necessary in order to obtain generically nodal curves in Theorem 6.1.

Example 6.4. Suppose Δ is the parallelogram with vertices $(0, 0)$, $(0, -1)$, $(q, 0)$ and $(q, 1)$, where q is a positive power of the characteristic of K . Then according to [Tyo14, Theorem 4.2], for any $[C] \in V_{0, \Delta}^{\text{irr}}$, all singularities of C are unibranch. Moreover, for a general curve $[C'] \in V_{0, \Delta}^{\text{irr}}$, $C' \cap C$ consists of two points, both of multiplicity q . In particular, the torus action on C' cannot be used to change the tangency relation between C and C' .

7. IRREDUCIBILITY OF SEVERI VARIETIES

In this section, we use a result of Lang [Lan20] on the monodromy action on the nodes of a general rational curve in $V_{0, \Delta}^{\text{irr}}$ to prove the irreducibility of $V_{g, \Delta}^{\text{irr}}$ in certain cases. Let $\{m_i\}_{1 \leq i \leq s}$ be

the set of primitive integral vectors along the sides of Δ . Set

$$l := 5 \max_{1 \leq i \leq s} \|m_i\|^2,$$

where $\|m_i\|$ denotes the Euclidean length of m_i . As in [Lan20], we denote by $\mathcal{C}_{\geq l}(S_\Delta)$ the set of ample line bundles \mathcal{L} on S_Δ such that $\deg(\mathcal{L}|_C) \geq l$ for any curve $C \subset S_\Delta$. Let $N_\Delta \subseteq N$ be the sublattice generated by the primitive normals of the sides of Δ .

Theorem 7.1. *Suppose $\text{char}(K) = 0$. Let Δ be an h -transverse polygon. If $\mathcal{L}_\Delta \in \mathcal{C}_{\geq l}(S_\Delta)$ and $N_\Delta = N$, then $V_{g,\Delta}^{\text{irr}}$ is irreducible for all $0 \leq g \leq \#\Delta^\circ \cap M$.*

Proof. By Lefschetz's principle, we may assume $K = \mathbb{C}$. According to Theorem 6.1, a general curve $[C] \in V_{0,\Delta}^{\text{irr}}$ is nodal and has $\#\Delta^\circ \cap M$ nodes. As in the planar case, $\overline{V}_{g,\Delta}^{\text{irr}}$ thus has $\binom{\#\Delta^\circ \cap M}{g}$ smooth branches through $[C]$, corresponding to smoothing out subsets of g nodes of C . Since $\mathcal{L}_\Delta \in \mathcal{C}_{\geq l}(S_\Delta)$, according to [Lan20, Theorem 1 and Proposition 7.4], the monodromy group of $V_{0,\Delta}^{\text{irr}}$ acts on C^{sing} as the group of deck transformations of a map from C^{sing} to a quotient of the group $\text{Hom}(M_\Delta/M, \mathbb{C})$, where M_Δ is the dual lattice of N_Δ . As $N_\Delta = N$, the group $\text{Hom}(M_\Delta/M, \mathbb{C})$ is trivial and hence the monodromy group acts as the full symmetric group on C^{sing} . In particular, the action on the set of subsets of g nodes is transitive. Consequently, there is a unique irreducible component of $\overline{V}_{g,\Delta}^{\text{irr}}$ containing $[C]$. On the other hand, by Theorem 5.6 any irreducible component of $\overline{V}_{g,\Delta}^{\text{irr}}$ contains $[C]$. Thus $V_{g,\Delta}^{\text{irr}}$ is irreducible. \square

Example 7.2. Suppose Δ is a kite as in Example 3.2 and \mathcal{L} a sufficiently ample line bundle on S_Δ . If $N_\Delta = N$, e.g., $(k, k') = (1, 2)$ as in Figure 1, then the Severi varieties $V_{g,\mathcal{L}}^{\text{irr}}$ are either empty or irreducible for all $g \geq 0$ by Theorem 7.1. On the other hand, if $N_\Delta \neq N$, e.g., $(k, k') = (3, 3)$ as in Figure 11, and $\mathcal{L} = \mathcal{L}_\Delta^{\otimes l} = \mathcal{L}_{l\Delta}$, then $V_{g,\mathcal{L}}^{\text{irr}}$ is known to be reducible for $g = 1$ and all $l \geq 1$ by [LT20, Theorem A] applied to the polygon $l\Delta$.

In Theorem 7.1 we do not allow for non-trivial tangency profiles. Our final example shows that imposing tangency conditions requires a separate analysis of the monodromy action:

Example 7.3. Let Δ be the h -transverse polygon with vertices $(0, 0)$, $(3, 1)$, $(3, -1)$, $(6, 1)$, $(6, -1)$ and $(9, 0)$. Suppose $\text{char}(K) = 0$ and let \underline{d} be the tangency profile given by imposing a point of order 3 along each of the 2 toric divisors that correspond to horizontal sides of Δ . Using a similar argument as in the proof of [LT20, Proposition 3.1], one can check that $V_{1,\Delta}^{\text{irr}}$ is irreducible, whereas $V_{1,\underline{d},\Delta}^{\text{irr}}$ contains 2 irreducible components.

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