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On the integrability of strictly convex billiard tables with boundaries close to ellipses with small eccentricities

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Abstract In this paper, we introduce a new notion of integrability for billiard tables, namely, integrability away from the boundary. One key feature of our notion is that the integrable region could be disjoint from the boundary with a positive distance. We prove that if a strictly convex billiard table, whose boundary is a small perturbation of an ellipse with small eccentricity, is integrable in this sense, then its boundary must be itself an ellipse.

Keywords Birkhoff conjecture, Billiard tables, Integrability

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1 Introduction

The *mathematical billiard*, a model of dynamical system first proposed by G. D. Birkhoff ([4]), has long been a popular object of study in the field. It consists by the inertial motions of a point mass inside a fixed domain and the elastic reflections at the boundary. Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain, whose boundary $\partial\Omega$ is C^r (for some $r \geqslant 3$) and has non-vanishing curvature everywhere. The phase space $\mathcal{M}$ of the induced billiard system is a (topological) cylinder formed by the pair (x, v), with x being a point on $\partial\Omega$ and v being an inward unit vector. The billiard map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ takes (x, v) to (x', v'), where x' is the position on the boundary $\partial\Omega$, where the trajectory of the point mass starting at x with velocity v first hits, and v' is the reflected velocity, according to the standard law of reflection of light: the angle of incidence is equal to the angle of reflection. Despite the seemly simple (local) dynamics, the qualitative dynamical properties of billiard maps are extremely non-local. This global influence of the geometry structure of the boundary on the dynamics leads to serval intriguing rigidity phenomena, which are at the heart of several open questions (see, e.g. [9]). In this work, we focus on the question on the integrability of a billiard system, also known as *Birkhoff conjecture.*

A smooth convex curve $\Gamma \subset \Omega$ is called a *caustic*, if whenever a trajectory is tangent to it, then they remain tangent after each reflection. Note that each convex caustic Γ corresponds to an invariant curve of the associated billiard map $\mathcal{F}$ in the phase space and, hence, has a well-defined rotation number. If the union of all the caustics form a set with non-empty interior, then we call the billiard table *integrable*. The famous *Birkhoff conjecture* ([4, 14]) claims that

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every integrable billiard table has a circle or an ellipse as its boundary. Despite its long history and the considerable amount of attention it has attracted, Birkhoff conjecture remains open and only a few progresses have been made. In [2], Bialy proved that if the phase space of the billiard map is globally foliated by continuous invariant curves that are not null-homotopic, then it is induced by a disk. Innami showed in [12] that if a billiard system that induced by a strictly convex domain admits a sequence of smooth convex caustics with rotation numbers ρ_n such that $\rho_n \neq \frac{1}{2}$, and $\rho_n \to \frac{1}{2}$, then the boundary of the domain has to be an ellipse. After Innami's work, the study on Birkhoff conjecture has been mainly focus on integrability near the boundary. So far, only local results are available. In [7], De Carvalho and Ramirez-Ros showed that resonant caustics generally do not persist for perturbed elliptic billiards. Kaloshin et al. proved in [1, 13] a local version of Birkhoff conjecture, namely, if a smooth strictly convex domain is close enough to an ellipse and the induced billiard system has smooth convex caustics with rotation numbers $\frac{1}{p}$, for all integers $p \geqslant 3$, then the domain itself is an ellipse. In [11], we proved a similar statement with integrability assumption inside a smaller neighborhood of the boundary for domains with boundaries close to ellipses with small eccentricities. For general strictly convex domains, a very partial attempt was made in [10], showing that the integrable deformations of an integrable domain must be tangent to a finite dimensional space. Recently, Bialy and Mironov [3] show that the Birkhoff conjecture is true if the domain is centrally symmetric and the neighborhood of the boundary is foliated by convex caustics of rotation numbers in the interval $(0, \frac{1}{4}]$. In this work, we consider a notion of integrability different from those in [1, 11, 13]. In our assumption here, *the region forms by the union of the caustics could have non-zero distance away from the boundary.*

Definition 1.1

1. *We say Γ is an integrable rational caustic for the billiard map in Ω if the corresponding non-contractible invariant curve in the phase space consists of periodic points; in particular, the corresponding rotation number is rational.*

2. *For any integer $P \geqslant 3$ and a real number $\delta \in (0, \frac{1}{6})$, if the billiard map induced by Ω admits integrable rational caustics*

   (a) of rotation numbers $\frac{p}{q}$ for all $\frac{p}{q} \in \mathbb{Q} \cap [\frac{1}{P}, \frac{1}{P} + \delta)$,

   (b) and of rotation numbers $\frac{1}{q}$, $q \in \{3, 4, \dots, N_\delta := \lceil \frac{P}{\delta} \rceil\} \setminus P\mathbb{N}$,

   then we say Ω is (δ, P)-rationally integrable.

Remark 1.2

1. By the definition, the (δ, P)-integrability requires the existence of caustics with rational rotation numbers whose divisors take full range (with reasonable multiplicity) in the whole set $\mathbb{N} \setminus \{0, 1, 2\}$. In this sense, it is similar to those in [1, 13]. In fact, the key requirement here is that we can choose a sequence of rational rotation numbers ρ_n whose divisors can run over all the positive integer larger than 2 and approximating the number $\frac{1}{P}$ with speed as least $O(\frac{1}{n})$.

2. A simple sufficient condition for the rational (δ, P)-integrability is the following (see [1, Lemma 1]). Let $\mathcal{C}_\Omega$ denote the union of all smooth convex caustics of the billiard in Ω; if the interior of $\mathcal{C}_\Omega$ contains caustics of rational rotation number p/q for all $\frac{1}{P} \leqslant p/q < \frac{1}{P} + \delta$ and for $\frac{1}{j}$, $j \in \{3, \dots, N_\delta\} \setminus P\mathbb{N}$, then Ω is (δ, P)-rationally integrable. The key

novelty of the notion (δ, P)-integrability is that under its assumption, it might happen that $dist(C_\Omega, \partial\Omega) > 0$.

We show the following statement in this paper.

Theorem 1.3 *Let $P \geqslant 3$ be an integer and $\delta \in (0, \frac{1}{6})$. There exists $e_0 = e(P, \delta) > 0$ such that for every $M > 0$, there exists $\epsilon_0 = \epsilon(e_0, P, M) > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, if the C^{16}-domain Ω is (δ, P)-integrable and its boundary is C^{16}-M-close and C^1-ϵ-close to an ellipse with eccentricity $e \leqslant e_0$, then the boundary of the domain Ω is an ellipse.*

Our approach here is a combination of those from [1] and [8]. For the proof, we do not need the existence of the rational integrable caustics for all the rational numbers in $[\frac{1}{P}, \frac{1}{P} + \delta)$, but those of the form $\frac{m}{mP-r}$. More precisely, the strategy we used proceeds as follows:

Step 1: Derive a quantitative necessary condition for the preservation of an integrable caustic of a given rational rotation number (see [1, Theorem 3] or Proposition 2.2 below).

Step 2: Define a set of deformed Fourier modes $\{\mathcal{C}_k, k \in \mathbb{Z}\}$. The first five of them are just the standard Fourier modes $\{\sin(k\cdot),\ \cos(k\cdot),\ k = 0, 1, 2\}$, and the remaining are constructed through the action-angle parametrization of the boundary of the ellipse, associated to the caustics with rotation numbers in $\{\frac{m}{mP-r} \in [\frac{1}{P}, \frac{1}{P} + \delta), r, m \in \mathbb{N}\} \cup \{3, 4, \ldots, N_\delta\} \setminus P\mathbb{N}$. The (P, δ)-integrability reduces to certain annihilation conditions for L^2-inner product with the deformed Fourier modes (See Lemma 3.5).

Step 3: We then construct a linear operator $\mathcal{I}$ from a Banach subspace $\mathcal{X} \subset L^2(\mathbb{T})$ to the infinite sequence space l^∞ with suitable weighted l^∞-norm (these function spaces were introduced in [8]). Each component of the operator $\mathcal{I}$ is just the linear functional defined by the L^2-inner product with the function $\mathcal{C}_k$, $k \in \mathbb{Z}$. We show (in Theorem 3.6) that the resulting linear operator $\mathcal{I}$ is invertible when the eccentricity is small.

Step 4: Finally, in Section 4, we prove the theorem with the approximating argument from [1].

It might be possible to get rid of the assumption (b) in the definition of (δ, P)-integrability and prove a similar result with the computer-assisted approach in [11]. Namely, one might try to derive the higher order condition for the existence of caustics with different rational rotation numbers, whose divisors are the same, then deduce the needed complementary data from certain non-degeneracy argument. However, we do not pursue in this direction here.

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2 Preliminary

2.1 The Elliptic-polar coordinates

Let us consider the ellipse

$$\mathcal{E}_{e,c} = \left\{(x, y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\},$$

centered at the origin and with semiaxes of lengths, respectively, $0 < b \leqslant a$; in particular e denotes its eccentricity, given by $e = \sqrt{1 - \frac{b^2}{a^2}} \in [0, 1)$ and $c = \sqrt{a^2 - b^2}$ the semi-focal distance. Observe that when $e = 0$, then $c = 0$ and $\mathcal{E}_{0,0}$ degenerates to a 1-parameter family of circles centered at the origin.

The family of confocal elliptic caustics in $\mathcal{E}_{e,c}$ is given by:

$$C_\lambda = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2-\lambda^2} + \frac{y^2}{b^2-\lambda^2} = 1\right\} \qquad 0 < \lambda < b. \tag{2.1}$$

Observe that the boundary corresponds to $\lambda = 0$, while the limit case $\lambda = b$ corresponds to the the two foci $(\pm\sqrt{a^2-b^2}, 0)$. Clearly, for $e = 0$ we have a family of concentric circles.

Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. A more convenient coordinate frame for addressing our question is provided by the so-called *elliptic-polar coordinates* (or, simply, *elliptic coordinates*) $(\mu, \varphi) \in \mathbb{R}_{\geq 0} \times \mathbb{T}$, given by:

$$\begin{cases} x = c\cosh\mu\,\cos\varphi \\ y = c\sinh\mu\,\sin\varphi, \end{cases}$$

where $c = \sqrt{a^2-b^2} > 0$ represents the semi-focal distance (in the case $e = 0$, this parametrisation degenerates to the usual polar coordinates). Observe that for each $\mu_* > 0$, the equation $\mu \equiv \mu_*$ represents a confocal ellipse.

Therefore, in these elliptic polar coordinates $\mathcal{E}_{e,c}$ becomes:

$$\mathcal{E}_{e,c} = \{(\mu_0, \varphi),\ \varphi \in \mathbb{T}\},$$

where $\mu_0 = \mu_0(e) := \cosh^{-1}(1/e)$.Then, any smooth perturbation Ω of the ellipse $\mathcal{E}_{e,c}$ can be written in this elliptic coordinate frame as

$$\partial\Omega = \{(\mu_0 + \mu(\varphi), \varphi) : \varphi \in \mathbb{T}\},$$

where $\mu(\varphi)$ is a small smooth 2π-periodic function; hereafter, we shall adopt this shorthand notation and write

$$\partial\Omega = \mathcal{E}_{e,c} + \mu(\varphi).$$

2.2 The integrable dynamics induced by an ellipse

Now we provide a more precise description of the billiard dynamics in $\mathcal{E}_{e,c}$.

Let $0 \leqslant k < 1$, we define elliptic integrals and Jacobi Elliptic functions:

- Incomplete elliptic integral of the first kind:

$$F(\varphi; k) := \int_0^\varphi \frac{1}{\sqrt{1-k^2\sin^2\tau}}d\tau.$$

- Complete elliptic integral of the first kind:

$$K(k) = F\left(\frac{\pi}{2}; k\right).$$

- Jacobi Elliptic functions are obtained by inverting incomplete elliptic integrals of the first kind. Precisely, if

$$u = F(\varphi; k) = \int_0^\varphi \frac{1}{\sqrt{1-k^2\sin^2\tau}}\,d\tau,$$

then by definition

$$\varphi := \mathrm{am}(u; k),$$

and the Jacobi elliptic functions are defined as:

$$\begin{aligned}\mathrm{sn}\,(u;k) &:= \sin(\mathrm{am}(u;k)) = \sin(\varphi),\\ \mathrm{cn}\,(u;k) &:= \cos(\mathrm{am}(u;k)) = \cos(\varphi).\end{aligned}$$

The following result has been proven in [5] (see also [6, Lemma 2.1]).

Proposition 2.1 *Let $\lambda \in (0,b)$ and let*

$$k_\lambda^2 := \frac{a^2-b^2}{a^2-\lambda^2} = e^2\frac{1}{1-\frac{\lambda^2}{a^2}} \quad \text{and} \quad \delta_\lambda := 2\,F(\arcsin(\lambda/b);k_\lambda).$$

Let us denote, in cartesian coordinates,

$$q_\lambda(t) := (a\,\mathrm{cn}\,(t;k_\lambda), b\,\mathrm{sn}\,(t;k_\lambda)).$$

Then, for every $t \in [0, 4K(k_\lambda))$ the segment joining $q_\lambda(t)$ and $q_\lambda(t+\delta_\lambda)$ is tangent to the caustic C_λ, defined in (2.1).

Observe that:

- $k_\lambda = e\frac{1}{\sqrt{1-\frac{\lambda^2}{a^2}}}$ respesents the eccentricity of the caustic C_λ; it is a strictly increasing function of $\lambda \in (0,b)$; in particular $k_\lambda \to e$ as $\lambda \to 0^+$, while $k_\lambda \to 1$ as $\lambda \to b^-$.
- δ_λ is also a strictly increasing function of $\lambda \in (0,b)$; in fact, $F(\varphi;k)$ is clearly strictly increasing in both φ and $k \in [0,1)$. Moreover, $\delta_\lambda \to 0$ as $\lambda \to 0^+$, and $\delta_\lambda \to +\infty$ as $\lambda \to b^-$.

Let us now cnosider the parametrization of the boundary induced by the dynamics of the caustic C_λ:

$$Q_\lambda : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2,\ \theta \mapsto q_\lambda\Big(\frac{4K(k_\lambda)}{2\pi}\theta\Big).$$

Clearly, the rotation number associated to the caustic C_λ is

$$\omega_\lambda := \frac{\delta_\lambda}{4K(k_\lambda)} = \frac{F(\arcsin(\lambda/b);k_\lambda)}{2K(k_\lambda)}. \tag{2.2}$$

In particular, ω_λ is strictly increasing as a function of λ and $\omega_\lambda \to 0$ as $\lambda \to 0^+$, while $\omega_\lambda \to \frac{1}{2}$ as $\lambda \to b^-$. We write the inverse of ω_λ with respect to λ as λ_ω.

It is easy to deduce from the above expressions that, in elliptic coordinates (μ, φ), the boundary parametrization induced by the caustic with rotation number ω, that is, C_{λ_ω}, is given by

$$S_\omega(\theta) := (\mu_\omega(\theta), \varphi_\omega(\theta)) = \Big(\mu_0, \mathrm{am}\,\Big(\frac{4K(k_{\lambda_\omega})}{2\pi}\theta; k_{\lambda_\omega}\Big)\Big). \tag{2.3}$$

That is, the orbit starting at $S_\omega(\theta)$ and tangent to C_{λ_ω} hits the boundary at $S_\omega(\theta + 2\pi\omega)$.

2.3 Necessary condition for the preservation of the integrable rational caustics

We continue to discuss the perturbations of an ellipse, for which the induced billiard dynamics admits integrable rational caustics with rational rotation numbers.

Let us consider a perturbed domain

$$\partial\Omega_\epsilon = \mathcal{E}_e + \mu_{\epsilon,M}(\varphi),$$

where $\mu_{\epsilon,M}(\varphi) \in C(\mathbb{T})$,

$$\|\mu_{\epsilon,M}\|_{C^1} \leqslant \epsilon, \quad \|\mu_{\epsilon,M}\|_{C^m} \leqslant M,$$

where $M > 0$ is an arbitrary fixed number, and $\epsilon > 0$ is chosen to be sufficient small. The following necessary condition for the preservation of the integrable rational caustics is proven in [7, Cor.9 and Prop. 11].

Proposition 2.2 *Assume that the billiard map induced by the domain Ω_ϵ has a rationally integrable caustics with rotation number, in the lowest term, $p/q \in (0, \frac{1}{2})$. If we denote by $\{\varphi^k_{p/q}\}_{k=0}^q$ the periodic orbit of the billiard map in $\mathcal{E}_{e,c}$ with rotation number p/q and starting at $\varphi^0_{p/q} = \varphi$ (these orbits are all tangent to the caustic $C_{\lambda_{p/q}}$), then*

$$L_1(\varphi) := 2\lambda_{p/q} \sum_{k=1}^{q} \mu_\epsilon(\varphi^k_{p/q}) = c_{p/q} + O_{e,M}(q^8 \|\mu_\epsilon\|^2_{C^1}),$$

where the constant $c_{p/q}$ depending only on p/q.

Remark 2.3 The estimate $O_{M,e}(q^8\|\mu_\epsilon\|^2_{C^1})$ for the error term was established in [1, Lemma 13].

3 The deformed Fourier modes and the operator of integrability

In this section, we define a set of deformed Fourier modes through the action-angle parameterization of an ellipse. Then we introduce an operator between two Banach spaces, adapted to the integrability condition from the last section. We show that the operator is invertible for small enough eccentricity e.

Let (μ, φ) be the elliptic-polar coordinates associated to the ellipse $\mathcal{E}_e$,

$$\mathcal{E}_e = \{(\mu_0, \varphi) : \varphi \in [0, 2\pi)\}.$$

Let ξ be the action-angle parametrization of the boundary of the ellipse, induced by the caustic with rotation number $\frac{1}{P}$, that is, the caustic $C_{\lambda_{1/P}}$, and $\Phi_P(\xi)$ the change of parametrization from ξ to φ (see (2.3)). Then we have the following lemma.

Lemma 3.1 *For each $n \in \mathbb{N}$, there exists $C_{n,P}$ such that for $e \in [0, \frac{1}{2}]$,*

$$\|\Phi_P(\xi) - \xi\|_{C^r} \leqslant C_{n,P} e^2.$$

Let us denote for $q \geqslant 3$,

$$(q|P) := \begin{cases} \frac{1}{P}, & \text{if } q \in P\mathbb{N}, \\ \frac{N}{NP-r}, & \text{if } q \notin P\mathbb{N}, \text{ and } q > N_\delta,\ q = NP - r, \\ \frac{1}{q}, & \text{if } q \in \{3, \dots, N_\delta\} \setminus P\mathbb{N}. \end{cases}$$

Now let us introduce the change of variables from the action-angle parametrization θ of the boundary, induced by the caustic with rotation number $(q|P)$, that is $C_{\lambda_{(q|P)}}$, to the action-angle parametrization ξ, *i.e*,

$$\xi = X_q(\theta) := \Phi_P^{-1}\Big(\varphi_{(q|P)}(\theta)\Big),$$

where the function $\varphi_{(q|P)}(\theta)$ is defined in (2.3). Then we have,

Lemma 3.2 *For each $n \in \mathbb{N}$, there exists $C_{n,P,\delta} > 0$ such that for $e \in [0, \frac{1}{2}]$, we have*

$$\|X_q(\xi) - \xi\|_{C^n} \leqslant C_{n,P,\delta} e^2 \frac{1}{q}.$$

The proof of Lemmas 3.1 and 3.2 are straightforward. The reader is kindly referred to [13, Appendix A] for the details.

Let us denote $L^2(\mathbb{T})$ the L^2-space of 2π-periodic functions, with the trigonometric basis $\{v_k\}_{k\in\mathbb{Z}}$, where

$$v_0 = \frac{1}{2},\ v_k(\xi) = \cos k\xi,\ v_{-k} = \sin k\xi,\ k = 1, 2, \dots, .$$

For any $f \in L^2(\mathbb{T})$, let

$$f(\xi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\xi + a_{-k} \sin k\xi), \tag{3.1}$$

be its Fourier series. For $\gamma \geqslant 1$, we define a subspace $\mathcal{X}^\gamma \subset L^2(\mathbb{T})$ as

$$\mathcal{X}^\gamma = \{f \in L^2(\mathbb{T}) : \|f\|_{\mathcal{X}^\gamma} < +\infty\},$$

with

$$\|f\|_{\mathcal{X}^\gamma} = |a_0| \wedge \Big(\sup_{j\geqslant 1} j^\gamma (|a_j| \wedge |a_{-j}|) \Big),$$

where $a \wedge b = \max\{a, b\}$. Then the space $(\mathcal{X}^\gamma, \|\cdot\|_{\mathcal{X}^\gamma})$ is a (separable) Banach space. Clearly, $C^1(\mathbb{T}) \subset \mathcal{X}^1$. For the inverse, we have the following simple lemma whose proof is obvious,

Lemma 3.3 *For $\gamma > 2$, we have $\mathcal{X}^\gamma \subset C^1(\mathbb{T})$, that is,*

$$\|f\|_{C^1} \leqslant C_\gamma \|f\|_{\mathcal{X}^\gamma}, \gamma > 2.$$

Consider the set of deformed Fourier modes $\{\mathcal{C}_k\}_{k\in\mathbb{Z}}$, where,

$$\mathcal{C}_0(\xi) = \frac{1}{2}, \mathcal{C}_k(\xi) = v_k(\xi),\ k = \pm 1, \pm 2,$$

and for $k \geqslant 3$,

$$\begin{cases} \mathcal{C}_{\pm k}(\xi) = v_{\pm k}(\xi), & \text{if } k \in P\mathbb{N}, \\ \mathcal{C}_k(\xi) = \dfrac{\cos\big(kX_k^{-1}(\xi)\big)}{X_k'\big(X_k^{-1}(\xi)\big)},\ \mathcal{C}_{-k} = \dfrac{\sin\big(kX_k^{-1}(\xi)\big)}{X_k'\big(X_k^{-1}(\xi)\big)}, & \text{if } k \notin P\mathbb{N}. \end{cases}$$

The name-deformed Fourier modes-is justified by the following lemma.

Lemma 3.4 *For any $u \in C^n(\mathbb{T})$, $n \geqslant 1$, we have*

$$\Big| \int_0^{2\pi} u(\xi)\mathcal{C}_k(\xi) d\xi \Big| \leqslant \frac{(1 + C_{n,P,\delta} e^2)|\|u\|_{C^n}}{|k|^n}, \quad |k| \geqslant 1.$$

Proof We consider the case $k \geqslant 1$, the other case is similar. From the definition of $\mathcal{C}_k(\xi)$, we have that

$$\int_0^{2\pi} u(\xi)\mathcal{C}_k(\xi) d\xi = \int_0^{2\pi} u(\xi) \cos kX_k^{-1}(\xi) dX_k^{-1}(\xi) = \int_0^{2\pi} u(X_k(\theta)) \cos k\theta d\theta.$$

By Lemma 3.2, we have

$$\|u(X_K(\cdot))\|_{C^n} \leqslant (1 + C_{n,P,\delta} e^2)\|u\|_{C^n}.$$

Then, the assertion of the lemma follows from integration by parts.

We have the following necessary condition for the (δ, P)-integrability, which is a direct consequence of Proposition 2.2.

Lemma 3.5 *Consider a domain Ω, whose boundary $\partial\Omega$ is close to the ellipse $\mathcal{E}_e$, written in the elliptic-polar coordinates associated to $\mathcal{E}_e$ as*

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi),$$

where $\|\mu\|_{C^n} \leqslant M$ and $\|\mu\|_{C^1}$ is small enough. If Ω is (P,δ)-integrable, then

$$\int_0^{2\pi} \Gamma(\xi)\mathcal{C}_k(\xi)d\xi = O_{e,M}(|k|^8\|\mu\|_{C^1}^2), \quad |k| \geqslant 3,$$

where $\Gamma(\xi) = \mu(\Phi_P(\xi))$.

Proof If $u(x)$ denotes either $\cos x$ or $\sin x$, then by Proposition 2.2, we have that

$$\int_0^{2\pi} \Gamma(X_k(\theta))u(k\theta)d\theta = O_{e,M}(|k|^8\|\mu\|_{C^1}^2), \quad k \geqslant 3,$$

where $\Gamma(\xi) = \mu(\Phi_P(\xi))$. Note that

$$\int_0^{2\pi} \Gamma(X_k(\theta))u(k\theta)d\theta = \int_0^{2\pi} \Gamma(\xi)\frac{u(kX_k^{-1}(\xi))}{X_k'(X_k^{-1}(\xi))}d\xi,$$

the assertion of the lemma follows.

We introduce another (separable) Banach space $h^\gamma \subset \mathbb{R}^\infty$,

$$h^\gamma := \{c = (c_0, c_1, c_{-1}, c_2, c_{-2}, \dots) \in \mathbb{R}^\infty : \|c\|_{h^\gamma} < +\infty\},$$

equipped with the norm

$$\|c\|_{h^\gamma} = |c_0| \wedge \Big(\sup_{j\geqslant 1}\big(j^\gamma(|c_j| \wedge |c_{-j}|)\big)\Big).$$

For $\gamma > 2$, we introduce the following linear operator,

$$\mathcal{I}^{\gamma,e} : \mathcal{X}^\gamma \to h^\gamma, \quad f \mapsto \big(c_0, c_1, c_{-1}, \dots\big), \tag{3.2}$$

where

$$c_k = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\mathcal{C}_k(\xi)d\xi, \quad k \in \mathbb{Z}.$$

Then we have the following statement.

Theorem 3.6 *For any $\gamma > 2$, there exists $e_0 := e_0(P,\gamma,\delta) > 0$ suct that for every $e \in [0, e_0]$, we have that the linear operator*

$$\mathcal{I}^{\gamma,e} : \mathcal{X}^\gamma \to h^\gamma,$$

is invertible and there exists $C(e,P,\delta,\gamma) > 0$ such that for any $c \in h^\gamma$,

$$\|(I^{\gamma,e})^{-1}(c)\|_{\mathcal{X}^\gamma} \leqslant C(e,P,\delta,\gamma)\|c\|_{h^\gamma}.$$

Moreover, $C(e,P,\delta,\gamma) \to 1$ as $e \to 0$.

The proof of the theorem is based on the following technical lemma.

Lemma 3.7 *There exist $C(e) := C(e,P,\delta) > 0$ and $\rho(e) := \rho(e,P,\delta) > 0$ such that*

$$|l_{k,j} - \delta_{k,j}| \leqslant C(e)e^{-\rho(e)\big||k|-|j|\big|}, \quad \forall k, j \in \mathbb{Z},$$

where

$$\delta_{k,j} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j, \end{cases} \quad \text{and} \quad l_{k,j} = \frac{1}{\pi}\int_0^{\pi} \mathcal{C}_k(\xi) v_j(\xi) d\xi, \quad \forall k, j \in \mathbb{Z}.$$

Moreover, one can choose $C(e) \to 0$ and $\rho(e) \to \infty$ as $e \to 0$.

Proof Let $\Delta_q(\xi) = X_q^{-1}(\xi) - \xi$. Clearly, $\Delta_q(\xi)$ is an analytic function in ξ with certain strip of analyticity. From [13, Appendix E], we have that there exist $C(e) := C(e, P, \delta) > 0$ and $\rho := \rho(e, P, \delta) > 0$ such that

$$q\|\Delta_q\|_\rho,\ q\|\Delta_q'\|_\rho \leqslant C(e),\ q \geqslant 3, \tag{3.3}$$

where $\|\cdot\|_\rho$ denotes the analytic norm of the function in the strip $\{|\text{Im } \xi| \leqslant \rho\}$ (namely, the sup-norm on this closed strip of its complex extension). Observe that $X_q^{-1}(\xi) = \xi = \varphi$ when $e = 0$. Hence, we could choose $C(e) \to 0$ and $\rho(e) \to \infty$ as $e \to 0$. We only consider the cases with $k, j \geqslant 3$. The other cases can be dealt with similarly. Recalling the definition of $\mathcal{C}_k(\xi)$ and $v_j(\xi)$, we have that

$$\begin{aligned} l_{k,j} &= \frac{1}{\pi}\int_0^{2\pi} \frac{\cos\big(kX^{-1}(\xi)\big)}{X_k'(X_k^{-1}(\xi))} \cos j\xi d\xi \\ &= \frac{1}{\pi}\int_0^{2\pi} \cos\big(k(\xi + \Delta_k(\xi))\big)\cos(j\xi)\big(1 + \Delta_k'(\xi)\big) d\xi \\ &= \frac{1}{\pi}\int_0^{2\pi} \Big(\cos k\xi \cos k\Delta_k(\xi) - \sin k\xi \sin k\Delta_k(\xi)\Big)\cos(j\xi)\big(1 + \Delta_k'(\xi)) d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Big[\Big(\cos(k+j)\xi + \cos(k-j)\xi\Big)\cos k\Delta_k(\xi) \\ &\qquad + \big(\sin(j-k)\xi - \sin(j+k)\xi\big)\sin k\Delta_k(\xi)\Big](1 + \Delta_k'(\xi)) d\xi. \end{aligned}$$

Note that when $k \neq j$, the terms in the above equality are just $|k-j|$-th and $|k+j|$-th Fourier coefficients of the functions $(1 + \Delta_k')\cos k\Delta_k$ and $(1 + \Delta_k')\sin k\Delta_k$. Then the assertion of the lemma follows from (3.3) for this case. When $k = j$, we have that

$$\cos(k-j)\xi \cos(k\Delta_k(\xi))(1 + \Delta_k'(\xi)) - 1 = \cos k\Delta_k - 1 + \Delta_k' + (\cos k\Delta_k - 1)\Delta_k'.$$

Then the assertion of the lemma follows again from (3.3).

Now we continue to prove Theorem 3.6.

Proof [Proof of Theorem 3.6] The operator $\mathcal{I}^{\gamma,e}$ can be identified as the infinite matrix $(l_{k,j})_{j,k\in\mathbb{Z}}$, where $l_{k,j}$ are defined in Lemma 3.7. Then we have the norm of $\mathcal{I}^{\gamma,e}$ as a linear operator from $\mathcal{X}^\gamma$ to h^γ is

$$\|\mathcal{I}^{\gamma,e}\|_\gamma = \sup_{q\geqslant 0}(1 \wedge q^\gamma)\Big[\sum_{j\geqslant 0}(1 \wedge j)^{-\gamma} \max\{|l_{q,j}| + |l_{q,-j}|, |l_{-q,j}| + |l_{-q,j}|\}\Big].$$

To show that $\mathcal{I}^{\gamma,e}$ is an invertible operator, we consider the operator $\mathcal{I}^{\gamma,e} - \mathbb{I}$, where

$$\mathbb{I}(f) = (a_0, a_1, b_2, \ldots, a_k, b_k, \ldots),$$

if f is given as in (3.1). Clearly, if we show for e_0 small enough and $0 \leqslant e \leqslant e_0$, we have

$$\|\mathcal{I}^{\gamma,e} - \mathbb{I}\|_\gamma < 1,$$

then $\mathcal{I}^{\gamma,e}$ is an invertible linear operator from $\mathcal{X}^\gamma$ to h^γ. By Lemma 3.7, we have that

$$\|\mathcal{I}^{\gamma,e} - \mathbb{I}\|_\gamma \leqslant \sup_{q\geqslant 3} q^\gamma \sum_{j=1}^{\infty} C(e) j^{-\gamma} e^{-\rho(e)|q-j|} \leqslant C'(e)C(\gamma).$$

where $C'(e) \to 0$ as $e \to 0$. Thus, the assertion of the theorem follows.

4 Proof of Theorem 1.3

Consider a C^m-smooth domain Ω, which is a C^n-perturbation of the ellipse $\mathcal{E}_e$, *i.e.*, in the elliptic coordinates associated to $\mathcal{E}_e$,

$$\partial\Omega = \mathcal{E}_e + \mu(\varphi),$$

where

$$\|\mu\|_{C^n} \leqslant \varepsilon, \quad \text{and} \quad \|\mu\|_{C^m} \leqslant M.$$

Here $\varepsilon > 0$ is a small parameter to be determined below and $M > 0$ is a fixed constant. We assume $0 < e < \frac{4}{5}e_0$ with e_0 to be determined later, and the domain Ω is (δ, P)-integrable.

The proof consists of two steps:

- Find an ellipse $\mathcal{E}''$, close to $\mathcal{E}_e$, which best approximates Ω.
- Show that $\Omega = \mathcal{E}''$.

Let us denote $\mathbb{E}_\varepsilon = \mathbb{E}_\varepsilon(\mathcal{E}_e)$ the set of ellipses whose C^0-Hausdorff distance to $\mathcal{E}_e$ is not greater than 2ε, *i.e.*,

$$\mathbb{E}_\varepsilon := \{\mathcal{E}' \subset \mathbb{R}^2 : \ \mathrm{dist}_H(\mathcal{E}',\ \mathcal{E}_e) \leqslant 2\varepsilon\}.$$

Clearly, $\mathbb{E}_\varepsilon$ is a compact set in any C^r-topology (it is completely determined by 5 parameters). We could choose ε small enough so that the eccentricities of all the ellipses in $\mathbb{E}_\varepsilon$ are between $4e/5$ and $5e/4$. For each $\mathcal{E}' \in \mathbb{E}_\varepsilon$, we can write the domain Ω in the elliptic coordinate frame associated to $\mathcal{E}'$, as

$$\partial\Omega = \mathcal{E}' + \mu_{\mathcal{E}'}(\varphi).$$

Choosing smaller ε if necessary, assuming $\|\mu_{\mathcal{E}'}\|_{C^m} \leqslant 2M$, $\forall \mathcal{E}' \in \mathbb{E}_\varepsilon$, from Lemma 0.2, we know that $\|\mu_{\mathcal{E}'}\|_{C^1}$ changes continuously with respect to $\mathcal{E}'$.

The proof is by contradiction. Assume that the statement of the theorem was not true – namely, $\partial\Omega$ was not an ellipse – since $\mathbb{E}_\varepsilon$ is compact, then we choose $\mathcal{E}'' \in \mathbb{E}$ such that

$$\|\mu_{\mathcal{E}''}\|_{C^1} = \min\{\mathcal{E}' \in \mathbb{E} : \ \|\mu_{\mathcal{E}'}\|_{C^1}\} > 0.$$

We have that

$$\|\mu_{\mathcal{E}''}\|_{C^m} \leqslant 2M \quad \text{and} \quad \|\mu_{\mathcal{E}''}\|_{C^1} \leqslant \|\mu_{\mathcal{E}'}\|_{C^1}.$$

Lemma 4.1 *There exists an ellipse $\bar{\mathcal{E}} \in \mathbb{E}_\varepsilon$ such that in the elliptic coordinate frame associated to $\bar{\mathcal{E}}$*

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^1} \leqslant \frac{1}{2}\|\mu_{\mathcal{E}''}\|_{C^1},$$

Note that this contradicts the minimality of $\|\mu_{\mathcal{E}''}\|_{C^1}$ and the assertion of the theorem follows.

Proof of Lemma 4.1: By Lemma 0.3, there exists an ellipse $\bar{\mathcal{E}} \in \mathbb{E}_\varepsilon$ such that in the elliptic coordinate frame associated to $\bar{\mathcal{E}}$, the domain Ω reads as

$$\partial\Omega = \bar{\mathcal{E}} + \mu_{\bar{\mathcal{E}}}(\varphi),$$

with

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^m} \leqslant 2M, \quad \|\mu_{\bar{\mathcal{E}}}\|_{C^1} \leqslant 2\|\mu_{\mathcal{E}''}\|_{C^1}.$$

and if we writes $\mu_{\bar{\mathcal{E}}}$ as Fourier series, *i.e.*,

$$\mu_{\bar{\mathcal{E}}}(\varphi) := \sum_{k=0}^{+\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

then there exists $C > 0$, independent of the eccentricity, such that

$$|a_i|, |b_i| \leqslant Ce^2\|\mu_{\mathcal{E}''}\|_{C^1}, \quad i = 0, 1, 2. \tag{4.1}$$

Let us denote $U(\xi) := \mu_{\bar{\mathcal{E}}}(\Phi_P(\xi))$. Then by Lemma 3.1, we have

$$(1 - C_{k,P,\delta}e^2)\|\mu_{\bar{\mathcal{E}}}\|_{C^k} \leqslant \|U(\xi)\|_{C^k} \leqslant (1 + C_{k,P,\delta}e^2)\|\mu_{\bar{\mathcal{E}}}\|_{C^k}, \quad k = 1, \dots, m.$$

We consider the operator $\mathcal{I}^{3,e}$, which is defined in (3.2), and denote

$$\mathcal{I}^{3,e}(U(\xi)) = (c_0, c_1, c_{-1}, \dots) \in \mathbb{R}^\infty.$$

By Lemma 3.1 and (4.1), we have

$$|c_k| \leqslant Ce^2\|\mu_{\mathcal{E}''}\|_{C^1}, \quad k = 0, 1, 2.$$

By Lemma 3.5, , we have

$$|c_k| \leqslant C_{e,M}\|\mu_{\bar{\mathcal{E}}}\|^2_{C^1} \leqslant 4C_{e,M}\|\mu_{\mathcal{E}''}\|^2_{C^1}, \quad 3 \leqslant |k| \leqslant Q_0,$$

where Q_0 is an integer to be determined later. By Lemma 3.4, we have

$$|c_k| \leqslant \frac{2M}{q^m}(1 + C_{m,P,\delta}e^2)^2, \quad |k| > Q_0.$$

Therefore, we have

$$A := \|\mathcal{I}^{3,e}(U(\cdot))\|_{h^3} \leqslant 2^3Ce^2\|\mu_{\mathcal{E}''}\|_{C^1} + 4Q_0^{11}C_{e,M}\|\mu_{\mathcal{E}''}\|^2_{C^1} + Q_0^{-m+3}2M(1 + C_{m,P,\delta}e^2)^2.$$

Choose $Q_0 = \lfloor \|\mu_{\mathcal{E}''}\|_{C^1}^{\frac{-1}{12}} \rfloor$, with $m = 16$, we have

$$A \leqslant 6Ce^2\|\mu_{\mathcal{E}''}\|_{C^1} + C_{e,M}\|\mu_{\mathcal{E}''}\|_{C^1}^{\frac{13}{12}}.$$

By Theorem 3.6,

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^1} \leqslant \frac{1}{1 - C_{e,P,\delta}e^2}C(e)A \leqslant C'(e)e^2\|\mu_{\mathcal{E}''}\|_{C^1} + C'_{e,M}\|\mu_{\mathcal{E}''}\|^{\frac{13}{12}}.$$

Clearly if $C'(e_0)e_0^2 < \frac{1}{4}$ and $\epsilon^{1/12}C'_{e,M} < \frac{1}{4}$, we have

$$\|\mu_{\bar{\mathcal{E}}}\|_{C^1} \leqslant \frac{1}{2}\|\mu_{\mathcal{E}''}\|_{C^1}.$$

This finishes the proof of Lemma 4.1, so does Theorem 1.3.

APPENDIX

0 Elliptic Polar Coordinates

Consider an ellipse

$$\mathcal{E} = \Big\{(x, y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\Big\}, \quad a > b > 0.$$

Associated to $\mathcal{E}$, there exists an elliptic coordinate frame (μ, φ) given by the relations

$$x = c\,\cosh(\mu_0 + \mu)\,\cos\varphi \quad , \quad y = c\,\sinh(\mu_0 + \mu)\,\sin\varphi,$$

where $c = \sqrt{a^2 - b^2}$ is the semifocal distance of $\mathcal{E}$, e denotes its eccentricity and $\mu_0 = \cosh^{-1}(e^{-1})$. Observe that $\mathcal{E}$ in this elliptic coordinates is represented by

$$\{(\mu_0, \varphi) : \ \varphi \in [0, 2\pi)\},$$

Then, any (small) smooth perturbation Ω of the ellipse $\mathcal{E}$ can be written in elliptic coordinates as

$$\partial\Omega = \{(\mu_0 + \mu(\varphi), \varphi) : \ \varphi \in [0, 2\pi)\}.$$

where $\mu(\varphi)$ is a 2π-periodic smooth functions. We will denote

$$\partial\Omega = \mathcal{E} + \mu(\varphi).$$

Lemma 0.2 *[13, Lemma 35] Let $\mathcal{E}_{e_0,c} = \mathcal{E}(0, 0, c, \mu_0, 0)$ be an ellipse of eccentricity $e_0 = 1/\cosh\mu_0$ and semi-focal distance c, and suppose that Ω is a perturbation of $\mathcal{E}_{e_0,c}$, which can be written (in the elliptic coordinate frame (μ, φ) associated to $\mathcal{E}_{e_0,c}$) as $\Omega = \mathcal{E}_{e_0,c} + \mu_\Omega(\varphi)$. Consider another ellipse $\bar{\mathcal{E}}$ sufficiently close to $\mathcal{E}_{e_0,c}$, which can be written (in elliptic coordinates frame associated to $\mathcal{E}_{e_0,c}$) as*

$$\bar{\mathcal{E}} = \mathcal{E}_{e_0,c} + \mu_{\bar{\mathcal{E}}}.$$

If $\bar{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0,c}$, we can write (in the elliptic coordinate frame $(\bar{\mu}, \bar{\varphi})$ associated to $\bar{\mathcal{E}}$) $\Omega = \bar{\mathcal{E}} + \bar{\mu}_\Omega(\bar{\varphi})$, for some function $\bar{\mu}_\Omega$. Then, there exists $C = C(e_0)$ such that

$$\|\mu_\Omega(\varphi) - (\mu_{\bar{\mathcal{E}}}(\varphi) + \bar{\mu}_\Omega(\varphi))\|_{C^1} \leqslant C\|\mu_{\bar{\mathcal{E}}}\|_{C^1}\|\|\mu_\Omega - \mu_{\bar{\mathcal{E}}}\|_{C^1}. \tag{A1}$$

In particular, for any $C' > 1$, if $\bar{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0}$ then we have

$$\frac{1}{C'}\|\mu_\Omega - \mu_{\bar{\mathcal{E}}}\|_{C^1} \leqslant \|\bar{\mu}_\Omega\|_{C^1} \leqslant C'\|\mu_\Omega - \mu_{\bar{\mathcal{E}}}\|_{C^1}. \tag{A2}$$

Lemma 0.3 *([13, Proposition 23], [11, Lemma B.1]) Let $\mathcal{E}_e$ be an ellipse of eccentricity $e \in (0, \frac{1}{2})$, and Ω be a small perturbation of $\mathcal{E}_e$, which written in the elliptic coordinates associated to $\mathcal{E}$ as*

$$\partial\Omega = \mathcal{E} + \mu(\varphi).$$

Assume $\|\mu\|_{C^1}$ small enough. Then there exists $C > 0$, independent of the eccentricity, and an ellipse $\bar{\mathcal{E}}$ such that

$$\partial\Omega = \bar{\mathcal{E}} + \bar{\mu}(\varphi),$$

such that $\|\bar{\mu}\|_{C^1} \leqslant 2\|\mu\|_{C^1}$ and if we write $\bar{\mu}(\varphi)$ in Fourier series,

$$\bar{\mu}(\varphi) = \sum_{k=0}^{\infty} a_k \cos k\varphi + b_k \sin k\varphi,$$

then

$$|a_i|, |b_i| \leqslant Ce^2\|\mu\|_{C^1}, \quad i = 0, 1, 2.$$

Remark 0.4 As it had been discussed in [1, 11, 13], the first five harmonics in the Fourier series are closely relative to the so-called elliptic motions that preserve the elliptic shape of the domain. That is, they correspond to homothety, translation, rotation and hyperbolic rotation, respectively.

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