

ON THE FINITE DIMENSIONALITY OF INTEGRABLE DEFORMATIONS OF STRICTLY CONVEX INTEGRABLE BILLIARD TABLES

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ABSTRACT. In this paper, we show that any smooth one-parameter deformations of a strictly convex integrable billiard table Ω_0 preserving the integrability near the boundary have to be tangent to a finite dimensional space passing through Ω_0 .

Dedicated to Yulij Ilyashenko on his 75th birthday

1. INTRODUCTION

A billiard system ([3]) consists by the inertial motions of a point mass inside a fixed domain and the elastic reflections at the boundary. Let Ω be a strictly convex domain in \mathbb{R}^2 with C^r boundary $\partial\Omega$, with $r \geq 3$. The phase space M of the induced billiard system is a (topological) cylinder formed by the pair (x, v) , with x being a foot point on $\partial\Omega$ and v being an inward unit vector. The billiard ball map $f : M \rightarrow M$ takes (x, v) to (x', v') , where x' is the position on the boundary $\partial\Omega$, where the trajectory of the point mass starting at x with velocity v first hits, and v' is the reflected velocity, according to the standard reflection law of light: the angle of incidence is equal to the angle of reflection. For a systematic introduction to the billiard dynamics, see e.g. [7, 14, 15].

A smooth closed convex curve $\Gamma \subset \Omega$ is called a *caustic*, if whenever a trajectory is tangent to it, then both remain tangent after each reflection. Notice that each convex caustic Γ corresponds to an invariant curve of the associated billiard map f and, hence, has a well-defined rotation number. If the union of all the caustics forms a set with non-empty interior, then we call the billiard table *integrable*. The famous Birkhoff conjecture ([12]) claims that every integrable billiard table has a circle or an ellipse as its boundary. Though much attention it has attracted, this conjecture remains open, and only a few partial progresses were obtained. As far as our understanding of integrable billiards is concerned, the most important related results are 1) a theorem ([2]) by Bialy which asserts that if the phase space of a billiard map is almost everywhere foliated by non-null homotopic invariant curves, then the corresponding billiard table is a disk; 2) a result ([9]) by Innami, in which he showed that if a strictly

convex billiard table admits a sequence of smooth convex caustics with rotation numbers converge to $1/2$, then its boundary has to be an ellipse; 3) a result ([5]) by Delshams and Ramírez-Ros in which they study entire perturbations of elliptic billiards and prove that any nontrivial symmetric perturbation of the elliptic billiard is not integrable (see also [4, 13]); and 4) the more recent works ([1, 8, 10]) by Kaloshin et al., justifying a perturbative version of the Birkhoff conjecture for the billiard table with boundary close to an ellipse, assuming integrability near the boundary.

In this work, we study the deformation of a strictly convex integrable billiard table, whose boundary may not be close to an ellipse. Let us introduce some notions in this paper.

Definition 1.1. (i) We say that $\Gamma \subset \Omega$ is an integrable rational caustic for the billiard system in Ω if the corresponding (non-contractible) invariant curve consists of periodic points; in particular, the corresponding rotation number is rational.

(ii) Let $q_0 \geq 2$. If the billiard system induced by Ω admits integrable rational caustic of rotation number p/q for all $0 < p/q < 1/q_0$, we say that Ω is q_0 -rationally integrable.

Remark 1.2. Let \mathcal{C}_Ω denote the union of all smooth convex caustics of the billiard in Ω ; if the interior of \mathcal{C}_Ω contains caustics of rotation numbers p/q for all $0 < p/q < 1/q_0$, then Ω is q_0 -rationally integrable. See [1, Lemma1].

The main result of this work is the following:

Theorem 1.3. *Let Ω_0 be a strictly convex C^r -smooth ($r \geq 8$) domain that is q_* -rationally integrable. Then there is $q_0 = q_0(\Omega) \geq q_*$, $d = 2q_0 + 1$, and, in the space of strictly convex C^r -smooth domains, a d -dimensional space $\mathcal{T}(\Omega_0)$ passing through Ω_0 such that any smooth deformation $\{\Omega_t\}_t$ with Ω_t being q_0 -rationally integrable is tangent to \mathcal{T} .*

Remark 1.4. For the proof, we only need the preservation of integrable caustics with rotation numbers $1/q$, $q = q_0, q_0 + 1, \dots$. Our approach here is inspired by those in [6]. Namely, we first derive the necessary annihilation conditions (see Proposition 2.1) for the infinitesimal deformation function (see (2.1)) of the integrable deformation, then using these constraints we construct an operator (see (2.6)) which is invertible for suitable q_0 (see Theorem 2.5), and finally from the invertibility of the operator, we conclude that the infinitesimal deformation function must belong to a certain linear space of finite dimension.

Remark 1.5. The most illuminating example of this theorem is the integrable deformations of a domain with an ellipse as its boundary. The domain enclosed by an ellipse is 2-rationally integrable. Due to [10], any smooth one parameter family of deformations of this domain, preserving the 2-integrability, is a family of domains with ellipses as their boundaries, belonging to a 5-dimensional space.

Theorem 1.3 can be viewed as a finite-dimensional reduction for integrable deformations. More explicit bounds on the dimension d is subjected to future development.

2. NECESSARY CONDITIONS FOR THE PRESERVATION OF CAUSTICS

From now on, we restrict ourself to strictly convex C^r -domains with $r \geq 8$. Consider a one-parameter smooth deformation Ω_τ , $\tau \in [-1, 1]$, of the strictly convex domain Ω_0 , preserving the existence of an integrable caustic with rotation number $\frac{1}{q}$, $q > 2$. Let $\Gamma(\tau, \xi)$ be a parametrization of $\partial\Omega_\tau$. As in [6], we define the infinitesimal deformation function

$$(2.1) \quad n_\Gamma(\tau, \xi) = \langle \partial_\tau \Gamma(\tau, \xi), N_\Gamma(\tau, \xi) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^2 and $N_\Gamma(\tau, \xi)$ is the outgoing unit normal vector to $\partial\Omega_\tau$ at the point $\Gamma(\tau, \xi)$. Note that n_Γ is continuous in τ , and $n_\Gamma(\tau, \cdot) \in C^r(\mathbb{T}^1, \mathbb{R})$ for each $\tau \in [-1, 1]$.

Let $S_q(\tau, \xi) = (\xi_{\tau,k}^q, \varphi_{\tau,k}^q)_{k=0, \dots, q-1}$ be a periodic orbit of the billiard map induced by Ω_τ , where the starting point is $\Gamma(\tau, \xi)$, that is $\xi_{\tau,0} = \xi$, and $\varphi_{\tau,k}^q$ is the angle between the trajectory and the tangent line of Ω_τ at $\xi_{\tau,k}$. For any C^1 -smooth function $\nu : \mathbb{T}^1 \rightarrow \mathbb{R}$, we define

$$L_{\Gamma_\tau}^q(\nu)(\xi) = \sum_{k=0}^{q-1} \nu(\xi_{\tau,k}^q) \sin \varphi_{\tau,k}^q.$$

Proposition 2.1. *The function $L_{\Gamma_\tau}^q(n_\Gamma(\tau, \cdot))(\xi)$ is a constant with respect to ξ . In particular,*

$$\frac{d}{d\xi} L_{\Gamma_\tau}^q(n_\Gamma(\tau, \cdot))(\xi) = 0.$$

Proof. Let $\mathcal{L}^q(\tau, \xi)$ be the perimeter of the periodic orbit $S_q(\tau, \xi)$. From [6, Proposition 4.6] we have that

$$\partial_\tau \mathcal{L}^q(\tau, \xi) = L_{\Gamma_\tau}^q(n_\Gamma(\tau, \cdot))(\xi).$$

Since Ω_τ is a one-parameter family preserving the existence of an integrable caustic with rotation number $\frac{1}{q}$, we have that $\mathcal{L}^q(\tau, \xi)$ is a function independent of ξ . So does $L_{\Gamma_\tau}^q(n_\Gamma(\tau, \cdot))(\xi)$. Moreover, $\frac{d}{d\xi} L_{\Gamma_\tau}^q(n_\Gamma(\tau, \cdot))(\xi) = 0$. □

The above statement is true for any parametrization of the boundary $\partial\Omega_\tau$. Now we fix $\tau = 0$ and choose the *Lazutkin parametrization* x ([11]), which is particularly convenient when dealing with nearly glancing orbits (e.g. periodic orbits with rotation number $\frac{1}{q}$, when q is large). Let s be the length parameter of the boundary $\partial\Omega_0$ and $\rho(s)$ be the radius of curvature of $\partial\Omega_0$ at s . Note that

ρ is C^{r-2} , since $\partial\Omega_0$ is C^r . Then the Lazutkin parametrization of the boundary $\partial\Omega_0$ is given as follows:

$$x(s) = C_{\Omega_0} \int_0^s \rho^{-2/3}(\tau) d\tau, \quad \text{and} \quad C_{\Omega_0}^{-1} = \int_0^{|\partial\Omega_0|} \rho^{-2/3}(s) ds.$$

We introduce the *Lazutkin density*:

$$\mu(x) = \frac{1}{2C_{\Omega_0}\rho(x)^{1/3}},$$

where we denote $\rho(x) = \rho(s(x))$, the radius of curvature in the Lazutkin parametrization. The following statement was obtained in [6, Appendices A and B].

Lemma 2.2. *Assume $r \geq 8$. There exist constant $C = C(\Omega_0)$ and 1-periodic functions $\alpha(x)$ and $\beta(x)$ such that for each $q \geq 2$, there exist 1-periodic functions $\gamma_{a, \frac{k}{q}}(x)$, $\gamma_{b, \frac{k}{q}}(x)$, $\gamma_{c, \frac{k}{q}}$, $k = 1, \dots, q$ such that*

$$\|\alpha\|_{C^{r-4}}, \|\beta\|_{C^{r-4}}, \|\gamma_{a, \bullet}\|_{C^{r-6}}, \|\gamma_{b, \bullet}\|_{C^{r-6}}, \|\gamma_{c, \bullet}\|_{C^{r-6}} \leq C,$$

and for any periodic orbit $x = x_q^0, \dots, x_q^{q-1}(x)$ with rotation number $\frac{1}{q}$, we have

$$x_q^k(x) = x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{a, \frac{k}{q}}(x).$$

Moreover, if $\varphi_q^k(x)$ denotes the angle of reflection of the trajectory at the k -th collision, we have

$$\varphi_q^k(x) = \frac{\mu(x_q^k)}{q} \left(1 + \frac{\beta(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{b, \frac{k}{q}}(x) \right).$$

and

$$\frac{\sin \varphi_q^k(x)}{\mu(x_q^k(x))} = \frac{1}{q} \left(1 + \frac{\beta(x + \frac{k}{q})}{q^2} + S_q(x + \frac{k}{q}) + \frac{1}{q^4} \gamma_{c, k/q}(x) \right),$$

where

$$(2.2) \quad S_q(x) = \frac{\sin(\mu(x)/q)}{\mu(x)/q} - 1.$$

Remark 2.3. (i) Notice that $S_q(x) = \frac{(\mu(x)/q)^2}{6}(-1 + O(q^{-2}))$.

(ii) The functions $\alpha(x)$ and $\beta(x)$ satisfy

$$\alpha'(x) = -4\beta(x) + \frac{\mu^2(x)}{6} + C_B,$$

with $C_B = \int_0^1 \frac{\rho^{1/3}(x)\rho''(x)}{3C_\Omega^2} dx - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx$, and

$$\begin{aligned} \beta(x) &= \int_0^1 \frac{\rho^{\frac{1}{3}}\rho''(x)}{15C_\Omega^2} dx + \frac{1}{5} \left(\frac{\rho^{-2/3}(x)}{24C_\Omega^2} - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx \right) \\ &\quad + \frac{1}{5} \left(\frac{\rho^{1/3}(x)\rho''(x)}{36C_\Omega^2} - \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2} \right). \end{aligned}$$

See Lemma B.1.

Now denoting $\nu_0(x) = n_\Gamma(0, x)\mu(x)$, we use Lemma 2.2 to write the quantity

$$\frac{d}{dx} L_{\Gamma_0}^q(n_\Gamma(0, \cdot))(x)$$

more explicitly,

$$\begin{aligned} \frac{d}{dx} L_{\Gamma_0}^q(n_\Gamma(0, \cdot))(x) &= \frac{d}{dx} \sum_{k=0}^{q-1} n_\Gamma(0, x_q^k(x)) \mu(x_q^k(x)) \frac{\sin \varphi_q^k(x)}{\mu(x_q^k(x))} \\ &= \sum_{k=0}^{q-1} \nu'_0 \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{a,k/q}(x) \right) \times \left(1 + \frac{\alpha'(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma'_{a,k/q}(x) \right) \\ &\quad \times \frac{1}{q} \left(1 + \frac{\beta(x + \frac{k}{q})}{q^2} + S_q(x + \frac{k}{q}) + \frac{1}{q^4} \gamma_{c,k/q}(x) \right) \\ &\quad + \sum_{k=0}^{q-1} \nu_0 \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{a,k/q}(x) \right) \\ &\quad \times \frac{1}{q} \left(\frac{\beta'(x + \frac{k}{q})}{q^2} + S'_q(x + \frac{k}{q}) + \frac{1}{q^4} \gamma'_{c,k/q}(x) \right). \end{aligned}$$

For any C^1 -function $f : \mathbb{T}^1 \rightarrow \mathbb{R}$, we consider two linear operators:

$$\begin{aligned} \mathbb{L}_q^1[f](x) &= \frac{1}{2\pi q^2} \sum_{k=0}^{q-1} f' \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{a, \frac{k}{q}}(x) \right) \\ (2.3) \quad &\quad \times \left(1 + \frac{\alpha'(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma'_{a,k/q}(x) \right) \\ &\quad \times \left(1 + \frac{\beta(x + \frac{k}{q})}{q^2} + S_q(x + \frac{k}{q}) + \frac{1}{q^4} \gamma_{c,k/q}(x) \right), \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} \mathbb{L}_q^2[f](x) &= \frac{1}{2\pi q^2} \sum_{k=0}^{q-1} f\left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} + \frac{1}{q^4} \gamma_{a,k/q}(x)\right) \\ &\quad \times \left(\frac{\beta'(x + \frac{k}{q})}{q^2} + S'_q(x + \frac{k}{q}) + \frac{1}{q^4} \gamma'_{c,k/q}(x) \right). \end{aligned}$$

By definition, we have

$$(2.5) \quad \frac{d}{dx} L_{\Gamma_0}^q(n_\Gamma(0, \cdot))(x) = 2\pi q \left(\mathbb{L}_q^1[\nu_0](x) + \mathbb{L}_q^2[\nu_0](x) \right).$$

For any $f \in C^1(\mathbb{T})$, let

$$f(x) = a_0 + \sum_{k=1}^{+\infty} \left(a_k \cos 2\pi kx + b_k \sin 2\pi kx \right),$$

be its Fourier series. For $2 < \gamma < 3$, define a subspace $X^\gamma \subset C^0(\mathbb{T})$ as

$$X^\gamma = \{f \in C^0(\mathbb{T}) : \|f\|_{X^\gamma} < +\infty\},$$

with

$$\|f\|_{X^\gamma} = |a_0| \wedge \left(\sup_{j \geq 1} (j^\gamma (|a_j| \wedge |b_j|)) \right),$$

where $a \wedge b = \max\{a, b\}$. The space $(X^\gamma, \|\cdot\|_{X^\gamma})$ is a (separable) Banach space.

Remark 2.4. We have $C^3(\mathbb{T}) \subset X^\gamma \subset C^1(\mathbb{T})$, since $2 < \gamma < 3$. So the linear operators \mathbb{L}_q^1 and \mathbb{L}_q^2 are well defined on X^γ .

Let us introduce another (separable) Banach space $h^\gamma \subset \mathbb{R}^\infty$,

$$h^\gamma := \{c = (c_0, c_1, d_1, c_2, d_2, \dots) \in \mathbb{R}^\infty : \|c\|_{h^\gamma} < +\infty\},$$

equipped with the norm

$$\|c\|_{h^\gamma} = |c_0| \wedge \left(\sup_{j \geq 1} (j^\gamma (|c_j| \wedge |d_j|)) \right).$$

For $\bar{q} \geq 3$, we define the following linear map,

$$(2.6) \quad \begin{aligned} \mathcal{I}^{\bar{q}} : X^\gamma &\rightarrow \mathbb{R}^\infty, \\ f &\mapsto \left(a_0, \dots, a_{\bar{q}-1}, b_{\bar{q}-1}, \right. \\ &\quad \left. \mathbb{L}_{\bar{q}}[f]\left(\frac{3}{4\bar{q}}\right), \mathbb{L}_{\bar{q}}[f](0), \dots, \mathbb{L}_q[f]\left(\frac{3}{4q}\right), \mathbb{L}_q[f](0), \dots \right), \end{aligned}$$

where

$$\mathbb{L}_q[f](x) = \mathbb{L}_q^1[f](x) + \mathbb{L}_q^2[f](x).$$

Let γ_0 be the number¹ such that

$$\sum_{k=2}^{+\infty} \frac{1}{k^{\gamma_0-1}} = 0.9.$$

Observe that $\frac{8}{3} < \gamma_0 < 3$. Our main theorem is implied by the following statement:

Theorem 2.5. *For any γ such that $\gamma_0 \leq \gamma < 3$, there exists $q_0 = q_0(\Omega_0, \gamma)$ such that for each $\bar{q} \geq q_0$, we have that the linear map*

$$\mathcal{I}^{\bar{q}}(\cdot) : X^\gamma \rightarrow h^\gamma$$

is invertible.

Proof of Theorem 1.3. Fix $\gamma \geq \gamma_0$. Let $\{\Omega_\tau\}$ be a smooth one-parameter \bar{q} -rationally integrable deformations of the domain Ω_0 , with $\bar{q} \geq q_0(\Omega_0, \gamma)$, and $n_\Gamma(0, \cdot)$ be the corresponding deformation function defined as in (2.1). Then, with $\nu_0 = \mu n_\Gamma$, by Theorem 2.5 and Proposition 2.1 we have that

$$\nu_0 \in \mathcal{I}_d = [\mathcal{I}^{\bar{q}}]^{-1}(h_{\bar{q}}^\gamma) \subset C^1(\mathbb{T}),$$

where $h_{\bar{q}}^\gamma = \{c \in h^\gamma : c_j = d_j = 0, \forall j \geq \bar{q}\}$ is a finite dimensional subspace of h^γ . Since μ is a nowhere vanishing function because of the fact that Ω_0 is strictly convex, we have that the infinitesimal deformation function $n_\Gamma(0, \cdot)$ belongs to the set $\mathcal{I} = \mu^{-1}\mathcal{I}_d$, which is a finite dimensional linear subspace of $C^1(\mathbb{T})$. Hence the assertion of Theorem 1.3 follows. \square

In the rest of this paper, we focus on the proof of Theorem 2.5. In Section 3, we obtain estimates for the linear operators \mathbb{L}_q^1 and \mathbb{L}_q^2 , applying to the base functions $\cos 2\pi p\xi$ and $\sin 2\pi p\xi$. Then in Section 4 we complete the proof of Theorem 2.5.

3. ESTIMATES FOR THE OPERATORS \mathbb{L}_q^1 AND \mathbb{L}_q^2

In this section we study the linear operators \mathbb{L}_q^1 and \mathbb{L}_q^2 , which are defined in (2.3) and (2.4). The main statement is the following lemma:

Lemma 3.1. *For any $p, q \in \mathbb{N}$, let*

$$f_{p,c}(\xi) = \cos(2\pi p\xi) \quad \text{and} \quad f_{p,s}(\xi) = \sin(2\pi p\xi).$$

Then

$$(1) \text{ If } p = 0, \text{ we have } \mathbb{L}_q^1(f_{0,c}) = \mathbb{L}_q^1(f_{0,s}) = 0.$$

¹ $\gamma_0 \approx 2.78831\dots$

(2) If $p = q$, we have

$$\mathbb{L}_q^1[f_{p,c}](x) = -\sin 2\pi qx + O\left(\frac{1}{q}\right),$$

$$\mathbb{L}_q^1[f_{p,s}](x) = \cos 2\pi qx + O\left(\frac{1}{q}\right).$$

(3) If $p = qk$ for some $1 < k \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{L}_q^1[f_{p,c}](x) &= \frac{-kq}{q} \left(\left(\int_0^1 \sin 2\pi kq \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi kqx \right. \\ &\quad \left. + \left(- \int_0^1 \cos 2\pi kq \frac{\alpha(x)}{q^2} dx \right) \sin 2\pi kqx \right) \\ &\quad + O\left(\frac{1}{q^3} + \frac{p^2}{q^5} + \frac{p^3}{q^7} + \frac{p}{q^3}\right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{L}_q^1[f_{p,s}](x) &= \frac{kq}{q} \left(\left(\int_0^1 \cos 2\pi kq \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi kqx \right. \\ &\quad \left. + \left(\int_0^1 \sin 2\pi \frac{kq\alpha(x)}{q^2} dx \right) \sin 2\pi kqx \right) \\ &\quad + O\left(\frac{1}{q^3} + \frac{p^2}{q^5} + \frac{p^3}{q^7} + \frac{p}{q^3}\right). \end{aligned}$$

(4) If $p = nq + r$ for some $n \in \mathbb{N}$ and $1 \leq r \leq q - 1$,

$$\mathbb{L}_q^1[f_{p,c}](x) = O\left(\frac{1}{q^3} + \frac{p^2}{q^5} + \frac{p^3}{q^7}\right) + O\left(\left(\frac{p^2}{q^3} + \frac{p^3}{q^7}\right)\left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right),$$

and

$$\mathbb{L}_q^1[f_{p,s}](x) = O\left(\frac{1}{q^3} + \frac{p^2}{q^5} + \frac{p^3}{q^7}\right) + O\left(\left(\frac{p^2}{q^3} + \frac{p^3}{q^7}\right)\left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right).$$

Proof. We first consider $\mathbb{L}_q^1[f_{p,c}](x)$ with $f_{p,c}(\xi) = \cos(2\pi p\xi)$. Then,

$$\begin{aligned} \mathbb{L}_q^1[f_{p,c}](x) &= \frac{1}{q^2} \sum_{k=0}^{q-1} -p \sin \left(2\pi p \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} \right) \right) \\ &\quad \times \left(1 + \frac{\alpha'(x + \frac{k}{q}) + \beta(x + \frac{k}{q})}{q^2} + S_q \left(x + \frac{k}{q} \right) \right) + O\left(\frac{p^2}{q^5}\right) \\ &= \frac{1}{q^2} \sum_{k=0}^{q-1} -p \sin 2\pi p \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} \right) \\ &\quad \times \left(1 + \frac{\alpha'(x + \frac{k}{q}) + \beta(x + \frac{k}{q})}{q^2} - \frac{\mu^2(x + \frac{k}{q})}{6q^2} \right) + O\left(\frac{p^2}{q^5}\right), \end{aligned}$$

where we have used $S_q(x) = \frac{\mu^2(x)}{6q^2}(-1 + O(\frac{1}{q^2}))$.

Let us set

$$\Lambda_1^q[f_{p,c}](x) = \frac{1}{q^2} \sum_{k=0}^{q-1} -p \sin 2\pi p \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} \right).$$

and

$$\Lambda_2^q[f_{p,c}](x) = \frac{1}{q^4} \sum_{k=0}^{q-1} -p \sin 2\pi p \left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2} \right) \left(\alpha' \left(x + \frac{k}{q} \right) + \beta \left(x + \frac{k}{q} \right) - \frac{\mu^2 \left(x + \frac{k}{q} \right)}{6} \right).$$

Clearly,

$$\mathbb{L}_q^1[f_{p,c}](x) = \Lambda_1^q[f_{p,c}](x) + \Lambda_2^q[f_{p,c}](x) + O\left(\frac{p^2}{q^5}\right).$$

For Λ_2^q , the following simple inequality is enough for our purpose here,

$$\left| \Lambda_2^q[f_{p,c}](x) \right| \leq C_1 \frac{p}{q^3},$$

where C_1 depends only on the C^0 -norm of α' , β and μ .

Now let us study $\Lambda_1^q[f_{p,c}](x)$. If $p = q$, we have that

$$\left| \Lambda_1^q[f_{p,c}](x) + \sin 2\pi qx \right| \leq C_2 \frac{1}{q}.$$

where C_2 depends only on the C^1 -norm of α .

For $p \neq q$, let us consider the function

$$F_q^p(x) = \sin 2\pi p \left(x + \frac{\alpha(x)}{q^2} \right),$$

and its Fourier series

$$F_q^p(x) = a_0^{p,q} + \sum_{k=1}^{+\infty} \left(a_k^{p,q} \cos 2\pi kx + b_k^{p,q} \sin 2\pi kx \right).$$

Then

$$\Lambda_1^q(\cos 2\pi px) = \frac{-p}{q} a_0^{p,q} + \frac{-p}{q} \sum_{j=1}^{\infty} \left(a_{jq}^{p,q} \cos 2\pi jqx + b_{jq}^{p,q} \sin 2\pi jqx \right).$$

Now we estimate each quantity. Let us begin with $a_0^{p,q}$,

$$\begin{aligned} (3.1) \quad a_0^{p,q} &= \int_0^1 \sin 2\pi p \left(x + \frac{\alpha(x)}{q^2} \right) dx \\ &= \int_0^1 \left(\sin 2\pi px \cos 2\pi p \frac{\alpha(x)}{q^2} + \cos 2\pi px \sin 2\pi p \frac{\alpha(x)}{q^2} \right) dx. \end{aligned}$$

For the first term, using integration by parts, we have

$$\begin{aligned}
& \int_0^1 \sin 2\pi p x \cos 2\pi \frac{p\alpha(x)}{q^2} dx \\
&= \int_0^1 -\frac{\cos 2\pi p x}{2\pi p} \left(\sin 2\pi \frac{p\alpha(x)}{q^2} \right) \frac{2\pi p \alpha'(x)}{q^2} dx \\
&= \int_0^1 \frac{\sin 2\pi p x}{2\pi p} \left(\left(\cos 2\pi \frac{p\alpha(x)}{q^2} \right) \frac{2\pi p (\alpha'(x))^2}{q^4} + \left(\sin 2\pi \frac{p\alpha(x)}{q^2} \right) \frac{\alpha''(x)}{q^2} \right) dx.
\end{aligned}$$

Similarly, for the second term in (3.1), we have

$$\begin{aligned}
& \int_0^1 \cos 2\pi p x \sin 2\pi \frac{p\alpha(x)}{q^2} dx \\
&= \int_0^1 \frac{\cos 2\pi p x}{2\pi p} \left(\left(\sin 2\pi \frac{p\alpha(x)}{q^2} \right) \frac{2\pi p (\alpha'(x))^2}{q^4} - \left(\cos 2\pi \frac{p\alpha(x)}{q^2} \right) \frac{\alpha''(x)}{q^2} \right) dx.
\end{aligned}$$

Therefore,

$$|a_0^{p,q}| \leq C_3 \left(\frac{1}{q^4} + \frac{1}{pq^2} \right),$$

where the constant $C_3 > 0$ depends only on C^2 -norm of α .

For the quantity $a_{kq}^{p,q}$,

$$\begin{aligned}
a_{kq}^{p,q} &= 2 \int_0^1 \sin 2\pi p \left(x + \frac{\alpha(x)}{q^2} \right) \cos 2\pi k q x dx \\
&= 2 \int_0^1 \left(\sin 2\pi p x \cos 2\pi p \frac{\alpha(x)}{q^2} + \cos 2\pi p x \sin 2\pi p \frac{\alpha(x)}{q^2} \right) \cos 2\pi k q x dx \\
&= 2 \int_0^1 \left(\cos 2\pi p \frac{\alpha(x)}{q^2} \right) \left(\frac{\sin 2\pi (kq + p)x}{2} - \frac{\sin 2\pi (kq - p)x}{2} \right) dx \\
&\quad + \left(\sin 2\pi p \frac{\alpha(x)}{q^2} \right) \left(\frac{\cos 2\pi (p + kq)x}{2} + \frac{\cos 2\pi (kq - p)x}{2} \right) dx.
\end{aligned}$$

Hence there exists $C_4 > 0$ depending only on C^2 -norm of α such that if $p \neq nq$ for all $n \in \mathbb{N}$, then

$$|a_{kq}^{p,q}| \leq C_4 \frac{1}{|kq - p|^2} \left(\frac{p}{q^2} + \frac{p^2}{q^4} \right),$$

and if $p = kq$ for some $k \in \mathbb{N}$, then

$$|a_{kq}^{p,q} - \int_0^1 \sin 2\pi p \frac{\alpha(x)}{q^2} dx| \leq C_4 \frac{1}{(kq + p)^2} \left(\frac{p}{q^2} + \frac{p^2}{q^4} \right).$$

So if $p = nq$ for some $n > 1$, then we have

$$\begin{aligned} & \sum_{k=1}^{\infty} a_{kq}^{p,q} \cos 2\pi kqx \\ &= \left(\int_0^1 \sin 2\pi p \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi nqx + O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \sum_{k=1}^{\infty} \frac{1}{k^2 q^2}\right) \\ &= \left(\int_0^1 \sin 2\pi p \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi nqx + O\left(\frac{p}{q^4} + \frac{p^2}{q^6}\right). \end{aligned}$$

If $p = nq + r$ for some $n \geq 0$ and $1 \leq r \leq q - 1$,

$$\begin{aligned} & \sum_{k=1}^{\infty} a_{kq}^{p,q} \cos 2\pi kqx \\ &= O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \sum_{k=1}^{\infty} \frac{1}{q^2(k + \frac{r}{q})^2} + \left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right) \\ &= O\left(\frac{p}{q^4} + \frac{p^2}{q^6}\right) + O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} b_{kq}^{p,q} &= 2 \int_0^1 \sin 2\pi p \left(x + \frac{\alpha(x)}{q^2}\right) \sin 2\pi kqx dx \\ &= \begin{cases} O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \frac{1}{|kq-p|^2}\right), & p \notin q\mathbb{N}, \\ -\int_0^1 \cos 2\pi p \frac{\alpha(x)}{q^2} dx + O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \frac{1}{(kq+p)^2}\right), & p \in q\mathbb{N}. \end{cases} \end{aligned}$$

Then, if $p = nq$ for some $n > 1$, we have

$$\begin{aligned} & \sum_{k=1}^{\infty} b_{kq}^{p,q} \cos 2\pi kqx \\ &= \left(-\int_0^1 \cos 2\pi p \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi nqx + O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \sum_{k=1}^{\infty} \frac{1}{k^2 q^2}\right) \\ &= \left(-\int_0^1 \cos 2\pi p \frac{\alpha(x)}{q^2} dx \right) \cos 2\pi nqx + O\left(\frac{p}{q^4} + \frac{p^2}{q^6}\right), \end{aligned}$$

and if $p = nq + r$ for some $n \geq 0$ and $1 \leq r \leq q - 1$, we have

$$\begin{aligned} & \sum_{k=1}^{\infty} b_{kq}^{p,q} \cos 2\pi kqx \\ &= O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \sum_{k=1}^{\infty} \frac{1}{q^2(k + \frac{r}{q})^2} + \left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right) \\ &= O\left(\frac{p}{q^4} + \frac{p^2}{q^6}\right) + O\left(\left(\frac{p}{q^2} + \frac{p^2}{q^4}\right) \left(\frac{1}{r^2} + \frac{1}{(q-r)^2}\right)\right). \end{aligned}$$

Hence, for $\mathbb{L}_q^1[f_{p,c}](x)$, we have obtained the assertion of the lemma. Now for $f_{p,s}(\xi) = \sin(2\pi p\xi)$. Then

$$\begin{aligned} \mathbb{L}_q^1[f_{p,s}](x) &= \frac{1}{q^2} \sum_{k=0}^{q-1} p \cos 2\pi p\left(x + \frac{k}{q} + \frac{\alpha(x + \frac{k}{q})}{q^2}\right) \\ &\quad \times \left(1 + \frac{\alpha'(x + \frac{k}{q}) + \beta(x + \frac{k}{q})}{q^2} - \frac{\mu^2(x + \frac{k}{q})}{6q^2}\right) + O\left(\frac{p^2}{q^5}\right) \end{aligned}$$

The statement for $\mathbb{L}_q^1[f_{p,s}](x)$ follows from similar argument. \square

We end this section with a straightforward estimate for the operator \mathbb{L}_q^2 .

Lemma 3.2. *There exists a constant $C_5 > 0$ which depends only on the C^1 -norms of β , μ and $\gamma_{c,k,q}$, such that for $u = \cos 2\pi px$ or $u = \sin 2\pi px$,*

$$|\mathbb{L}_q^2(u)(x)| \leq C_5/q^3.$$

4. PROOF OF THEOREM 2.5

In this section we prove Theorem 2.5, using the results in Section 3. Now we fix $\gamma \in (\gamma_0, 3)$.

The operator $\mathcal{I}^{\bar{q}}$ defined in (2.6) can be identified as a pair formed by two infinite matrices

$$\left(\begin{bmatrix} l_{qj}^{1,c} & l_{qj}^{1,s} \\ l_{qj}^{2,c} & l_{qj}^{2,s} \end{bmatrix} \right)_{qj} \quad \text{and} \quad ((g_k^1, g_k^2))_k$$

where for $q < \bar{q}$,

$$g_0 = 1, \quad g_q^1 = g_q^2 = l_{qj}^{1,s} = l_{qj}^{2,c} = 0 \quad \text{and} \quad l_{qj}^{1,c} = l_{qj}^{2,s} = \delta_{qj},$$

and for $q \geq \bar{q}$, $g_q^1 = \mathbb{L}_q[1](\frac{3}{4q})$, $g_q^2 = \mathbb{L}_q[1](0)$,

$$\begin{cases} l_{qj}^{1,c} = \mathbb{L}_q(\cos 2\pi jx)(\frac{3}{4q}), & l_{qj}^{1,s} = \mathbb{L}_q(\sin 2\pi jx)(\frac{3}{4q}), \\ l_{qj}^{2,c} = \mathbb{L}_q(\cos 2\pi jx)(0), & l_{qj}^{2,s} = \mathbb{L}_q(\sin 2\pi jx)(0). \end{cases}$$

For

$$(4.1) \quad f = a_0 + \sum_{k=1}^{+\infty} a_k \cos 2\pi kx + b_k \sin 2\pi kx,$$

we have that

$$\mathcal{I}^{\bar{q}}(f) = (c_0, c_1, d_1, \dots) \in \mathbb{R}^\infty,$$

with

$$c_0 = a_0, \quad c_k = a_k, \quad d_k = b_k, \quad k = 1, \dots, \bar{q} - 1,$$

and

$$c_q = g_q^1 a_0 + \sum_{j=1}^{\infty} a_k l_{qj}^{1,c} + b_k l_{qj}^{1,s}, \quad d_q = g_q^2 a_0 + \sum_{j=1}^{\infty} a_k l_{qj}^{2,c} + b_k l_{qj}^{2,s}, \quad q = \bar{q}, \dots$$

Then the norm of $\mathcal{I}^{\bar{q}}$ as a linear operator from X^γ to h^γ is

$$\|\mathcal{I}^{\bar{q}}\|_\gamma = \sup_q q^\gamma \left[\sum_{j \geq 1} j^{-\gamma} \max\{|l_{qj}^{1,c}| + |l_{qj}^{1,s}|, |l_{qj}^{2,c}| + |l_{qj}^{2,s}|\} + \max\{|g_q^1|, |g_q^2|\} \right].$$

To show that $\mathcal{I}^{\bar{q}}$ is an invertible operator, we consider the operator $\mathcal{I}^{\bar{q}} - \mathbb{I}$, where

$$\mathbb{I}(f) = (a_0, a_1, b_2, \dots, a_k, b_k, \dots),$$

if f is given as in (4.1). Clearly, if we show for \bar{q} large enough,

$$\|\mathcal{I}^{\bar{q}} - \mathbb{I}\| < 1,$$

then $\mathcal{I}^{\bar{q}}$ is an invertible linear operator from X^γ to h^γ .

From Lemmata 3.1 and 3.2, we know that for $q \geq \bar{q}$,

$$g_q^1, g_q^2 = O\left(\frac{1}{q^3}\right),$$

and there exists $C > 0$ such that if $j = q$,

$$|l_{qq}^{1,c} - 1| \leq \frac{C}{q}, \quad |l_{qq}^{1,s}| \leq \frac{C}{q}, \quad |l_{qq}^{2,c}| = \frac{C}{q}, \quad |l_{qq}^{2,s} - 1| \leq \frac{C}{q},$$

if $j = kq$, $k = 2, 3, \dots$,

$$\begin{aligned} & |l_{qj}^{1,c}| + |l_{qj}^{1,s}| \leq \\ & \frac{j}{q} \left(\left| \int_0^1 \cos \frac{2\pi k\alpha(x)}{q} dx \right| + \left| \int_0^1 \sin \frac{2\pi k\alpha(x)}{q} dx \right| \right) + C \left(\frac{j+1}{q^3} + \frac{j^2}{q^5} + \frac{j^3}{q^7} \right), \\ & |l_{qj}^{2,c}| + |l_{qj}^{2,s}| \leq \\ & \frac{j}{q} \left(\left| \int_0^1 \cos \frac{2\pi k\alpha(x)}{q} dx \right| + \left| \int_0^1 \sin \frac{2\pi k\alpha(x)}{q} dx \right| \right) + C \left(\frac{j+1}{q^3} + \frac{j^2}{q^5} + \frac{j^3}{q^7} \right), \end{aligned}$$

and if $j = kq + r$ for some $r, k \in \mathbb{N}$ and $0 < r < q$,

$$|l_{qj}^{1,c}| + |l_{qj}^{1,s}|, |l_{qj}^{2,c}| + |l_{qj}^{2,s}| \leq C \left(\frac{1}{q^3} + \frac{j^2}{q^5} + \frac{j^3}{q^7} + \left(\frac{j^2}{q^3} + \frac{j^3}{q^7} \right) \left(\frac{1}{r^2} + \frac{1}{(q-r)^2} \right) \right).$$

Now, letting $B \geq 1$ be a constant to be determined later and using for $j \geq q^{B+1}$ the simple estimate

$$|\mathbb{L}_q[u](x)| \leq C \frac{j}{q}, \quad u = \cos 2\pi jx, \text{ or } u = \sin 2\pi jx,$$

we have that

$$\begin{aligned} & \|\mathcal{I}^{\bar{q}} - \mathbb{I}\|_{\gamma} \\ & \leq \sup_{q \geq \bar{q}} q^{\gamma} \left[\sum_{k=2}^{+\infty} (kq)^{-\gamma} \frac{kq}{q} \left(\left| \int_0^1 \cos \frac{2\pi k\alpha(x)}{q} dx \right| + \left| \int_0^1 \sin \frac{2\pi k\alpha(x)}{q} dx \right| \right) \right. \\ & \quad + C \left(\frac{1}{q^3} + \frac{q^{-\gamma}}{q} + \sum_{k=2}^{q^B} (kq)^{-\gamma} \left(\frac{kq+1}{q^3} + \frac{(kq)^2}{q^5} + \frac{(kq)^3}{q^7} \right) \right) \\ & \quad + C \sum_{r=1}^{q-1} \sum_{k=0}^{q^B} \frac{\left(\frac{1}{q^3} + \frac{(kq+r)^2}{q^5} + \frac{(kq+r)^3}{q^7} + \frac{(kq+r)^2}{q^3} \left(\frac{1}{r^2} + \frac{1}{(q-r)^2} \right) \right)}{(kq+r)^{\gamma}} \\ & \quad \left. + C \sum_{j=q^{B+1}}^{\infty} j^{-\gamma} \frac{j}{q} \right]. \end{aligned}$$

We simplify the right hand side of the above inequality and get

$$\begin{aligned} \|\mathcal{I}^{\bar{q}} - \mathbb{I}\|_{\gamma} & \leq \sum_{k=2}^{\infty} \frac{1}{k^{\gamma-1}} + \sup_{q \geq \bar{q}} \sum_{k=2}^{\infty} \left(\frac{1}{k^{\gamma-1}} \left| \int_0^1 \sin \left(\frac{2\pi k\alpha(x)}{q} \right) dx \right| \right) \\ & \quad + \sup_{q \geq \bar{q}} C \sum_{j=q^{B+1}}^{\infty} q^{\gamma-1} j^{-\gamma+1} \\ (4.2) \quad & \quad + \sup_{q \geq \bar{q}} C \left(\frac{1}{q} + q^{\gamma-3} + q^{\gamma} \sum_{j=1}^{q^{B+1}} j^{-\gamma} \left(\frac{1}{q^3} + \frac{j}{q^3} + \frac{j^2}{q^5} + \frac{j^3}{q^7} \right) \right. \\ & \quad \left. + q^{\gamma} \sum_{r=1}^{q-1} \sum_{k=0}^{q^B} \frac{(kq+r)^{2-\gamma}}{q^3} \left(\frac{1}{r^2} + \frac{1}{(q-r)^2} \right) \right). \end{aligned}$$

Now we estimate each term on the right hand side.

For the first term, since $\gamma \geq \gamma_0$, we have that

$$\sum_{k=2}^{\infty} \frac{1}{k^{\gamma-1}} \leq 0.9.$$

For the second term, since

$$\left| \sin\left(\frac{2\pi k\alpha(x)}{q}\right) \right| \leq \min\left\{\frac{2k\pi\|\alpha\|_{C^0}}{q}, 1\right\},$$

we have

$$(4.3) \quad \sum_{k=2}^{\infty} \left(\frac{1}{k^{\gamma-1}} \left| \int_0^1 \sin\left(\frac{2\pi k\alpha(x)}{q}\right) dx \right| \right) \leq C(\gamma) \frac{1}{q^{\gamma-2}}.$$

For third term, we have

$$(4.4) \quad \sum_{j=q^{B+1}}^{\infty} q^{\gamma-1} j^{-\gamma+1} \leq C'(\gamma) q^{(B+1)(2-\gamma)+\gamma-1}.$$

For the fourth term,

$$(4.5) \quad \begin{aligned} & q^\gamma \sum_{j=1}^{q^{B+1}} j^{-\gamma} \left(\frac{1}{q^3} + \frac{j}{q^3} + \frac{j^2}{q^5} + \frac{j^3}{q^7} \right) \\ & \leq q^{\gamma-3} \sum_{j=1}^{\infty} \left(\frac{1}{j^\gamma} + \frac{1}{j^{\gamma-1}} \right) + C(\gamma) \left(q^{\gamma-5} q^{(B+1)(3-\gamma)} + q^{\gamma-7} q^{(B+1)(4-\gamma)} \right). \end{aligned}$$

For the fifth term,

$$(4.6) \quad \begin{aligned} & q^\gamma \sum_{r=1}^{q-1} \sum_{k=0}^{q^B} \frac{(kq+r)^{2-\gamma}}{q^3} \left(\frac{1}{r^2} + \frac{1}{(q-r)^2} \right) \\ & \leq C(\gamma) q^{B(3-\gamma)-1} \sum_{k=1}^{\infty} \frac{1}{k^2} \leq C'(\gamma) q^{B(3-\gamma)-1}. \end{aligned}$$

We want the exponents on q of the right hand sides of (4.3), (4.4), (4.5) and (4.6) to be negative, that is,

$$\begin{cases} (B+1)(2-\gamma) + \gamma - 1 < 0, \\ 2 < \gamma_0 \leq \gamma < 3, \\ (B+1)(3-\gamma) + \gamma - 5 < 0, \\ (B+1)(4-\gamma) + \gamma - 7 < 0, \\ B(3-\gamma) - 1 < 0. \end{cases}$$

Therefore, we need

$$(4.7) \quad \begin{cases} B > \frac{-1}{2-\gamma}, \\ 2 < \gamma_0 \leq \gamma < 3, \\ B < \frac{2}{3-\gamma}, \\ B < \frac{3}{4-\gamma}, \\ B < \frac{1}{3-\gamma}. \end{cases}$$

For

$$\frac{8}{3} < \gamma_0 \leq \gamma < 3,$$

we have

$$1 < \frac{-1}{2-\gamma} < \frac{3}{4-\gamma} < 3 < \frac{1}{3-\gamma} < \frac{2}{3-\gamma}.$$

Then with the choice of B ,

$$1 < \frac{-1}{2-\gamma} < B < \frac{3}{4-\gamma} < 3,$$

all the inequalities in (4.7) are satisfied, that is, all the exponents on q are negative. Hence, we have that for each $\gamma_0 \leq \gamma < 3$, there exists $\sigma(\gamma) > 0$,

$$\|\mathcal{I}^{\bar{q}} - \mathbb{I}\|_{\gamma} \leq 0.9 + C(\gamma) \frac{1}{\bar{q}^{\sigma(\gamma)}}.$$

So for \bar{q} large enough,

$$\|\mathcal{I}^{\bar{q}} - \mathbb{I}\|_{\gamma} < 0.95,$$

which implies the invertibility of $\mathcal{I}^{\bar{q}}$ as an operator from X^{γ} to h^{γ} . Thus, we finish the proof of Theorem 2.5.

APPENDIX A. HIGHER ORDER CONSTRAINTS

In this section we derive additional constraints on the infinitesimal deformation function when the smooth deformation preserves both the integrable caustics with rotation numbers $\frac{1}{p}$ and $\frac{2}{p}$ (p is an odd number). The following lemma is a slight modification of the result in [6, Appendices A and B].

Lemma A.1. *There exists $C = C(\Omega_0) > 0$ such that for each odd number $p \geq 5$, there exist 1-periodic functions $\gamma_{a, \frac{2k}{p}}(x)$, $\gamma_{b, \frac{2k}{p}}(x)$, $\gamma_{c, \frac{2k}{p}}$, $k = 1, \dots, p-1$, such that*

$$\|\gamma_{a, \bullet}\|_{C^{r-6}}, \|\gamma_{b, \bullet}\|_{C^{r-6}}, \|\gamma_{c, \bullet}\|_{C^{r-6}} \leq C,$$

and for any periodic orbit $x = x_{2,p}^0, \dots, x_{2,p}^{p-1}$ with rotation number $\frac{2}{p}$, we have

$$x_{2,p}^k = x + \frac{2k}{p} + \frac{4\alpha(x + \frac{2k}{p})}{p^2} + \frac{\gamma_{a, \frac{2k}{p}}(x)}{p^4}.$$

Moreover, if $\varphi_{2,p}^k$ denotes the angle of reflection of the trajectory at the k -th collision, we have

$$\varphi_{2,p}^k = \frac{2\mu(x_{2,p}^k(x))}{p} \left(1 + \frac{4\beta(x + \frac{2k}{p})}{p^2} + \frac{\gamma_{b, \frac{2k}{p}}(x)}{p^4} \right),$$

and

$$\frac{\sin \varphi_{2,p}^k(x)}{\mu(x_{2,p}^k(x))} = \frac{2}{p} \left[1 + \frac{4\beta(x + \frac{2k}{p})}{p^2} + S_{2,p}(x + \frac{2k}{p}) + \frac{\gamma_{c, \frac{2k}{p}}(x)}{p^4} \right].$$

Here the function α and β are the same as in Lemma 2.2, and

$$S_{2,p}(x) = \frac{\sin(2\mu(x)/p)}{2\mu(x)/p} - 1 = \frac{(2\mu(x)/p)^2}{6} \left(-1 + O\left(\frac{1}{p^2}\right) \right).$$

Then from Proposition 2.1, we have

$$\begin{aligned} 0 &= \frac{d}{dx} L_{\Gamma_0}^{2,p}(n_{\Gamma}(0, \cdot))(x) = \frac{d}{dx} \sum_{k=0}^{p-1} n_{\Gamma}(0, x_{2,p}^k(x)) \mu(x_{2,p}^k(x)) \frac{\sin \varphi_{2,p}^k(x)}{\mu(x_{2,p}^k(x))} \\ &= \sum_{k=0}^{p-1} \nu_0' \left(x + \frac{2k}{p} + \frac{4\alpha(x + \frac{2k}{p})}{p^2} + \frac{\gamma_{a, 2k/p}(x)}{p^4} \right) \\ &\quad \times \left(1 + \frac{4\alpha'(x + \frac{2k}{p})}{p^2} + \frac{\gamma'_{a, 2k/p}(x)}{p^4} \right) \\ &\quad \times \frac{2}{p} \left[1 + \frac{4\beta(x + \frac{2k}{p})}{p^2} + S_{2,p}(x + \frac{2k}{p}) + \frac{\gamma_{c, 2k/p}(x)}{p^4} \right] \\ &+ \sum_{k=0}^{p-1} \nu_0 \left(x + \frac{2k}{p} + \frac{4\alpha(x + \frac{2k}{p})}{p^2} + \frac{\gamma_{a, 2k/p}(x)}{p^4} \right) \\ &\quad \times \frac{2}{p} \left[\frac{4\beta'(x + \frac{2k}{p})}{p^2} + S'_{2,p}(x + \frac{2k}{p}) + \frac{\gamma'_{c, 2k/p}(x)}{p^4} \right]. \end{aligned}$$

Now we consider the quantity

$$\mathbb{L}_{2,p}(n_{\Gamma}(0, \cdot))(x) = \frac{d}{dx} L_{\Gamma_0}^{2,p}(n_{\Gamma}(0, \cdot))(x) - 2 \frac{d}{dx} L_{\Gamma_0}^p(n_{\Gamma}(0, \cdot))(x).$$

Then we have

$$\begin{aligned}
0 &= \mathbb{L}_{2,p} \left(n_{\Gamma}(0, \cdot) \right) (x) \\
&= \sum_{k=0}^{p-1} \frac{2}{p} \left[\nu_0' \left(x + \frac{2k}{p} \right) - \nu_0' \left(x + \frac{k}{p} \right) \right. \\
&\quad + \frac{1}{p^2} \left(4\nu_0' \left(x + \frac{2k}{p} \right) \left(\alpha' + \beta - \frac{\mu^2}{6} \right) \left(x + \frac{2k}{p} \right) - \nu_0' \left(x + \frac{k}{p} \right) \left(\alpha' + \beta - \frac{\mu^2}{6} \right) \left(x + \frac{k}{p} \right) \right) \\
&\quad + \frac{1}{p^2} \left(4\nu_0'' \left(x + \frac{2k}{p} \right) \alpha \left(x + \frac{2k}{p} \right) - \nu_0'' \left(x + \frac{2k}{p} \right) \alpha \left(x + \frac{k}{p} \right) \right) \\
&\quad + \frac{1}{p^2} \left(4\nu_0 \left(x + \frac{2k}{p} \right) \left(\beta' \left(x + \frac{2k}{p} \right) - \frac{(\mu^2)' \left(x + \frac{2k}{p} \right)}{6} \right) \right. \\
&\quad \left. - \nu_0 \left(x + \frac{k}{p} \right) \left(\beta' \left(x + \frac{k}{p} \right) - \frac{(\mu^2)' \left(x + \frac{k}{p} \right)}{6} \right) \right) \left. \right] + O\left(\frac{\|\nu_0\|_{C^3}}{p^4}\right)
\end{aligned}$$

Therefore, we have

$$0 = \sum_{k=0}^{p-1} \left[\frac{6}{p^3} \left(\nu_0'' \alpha + \nu_0' \left(\alpha' + \beta - \frac{\mu^2}{6} \right) + \nu_0 \beta' - \nu_0 (\mu^2)' / 6 \right) \left(x + \frac{k}{p} \right) \right] + O\left(\frac{\|\nu_0\|_{C^3}}{p^4}\right).$$

Notice that

$$\nu_0'' \alpha + \nu_0' \alpha' + \nu_0' \beta - \nu_0' \frac{\mu^2}{6} + \nu_0 \beta' - \nu_0 (\mu^2)' / 6 = (\nu_0' \alpha)' + (\nu_0 \beta)' - (\nu_0 \mu^2)' / 6$$

Then we obtain a new constraint on the infinitesimal deformation function

$$\text{(A.1)} \quad 0 = \sum_{k=0}^{p-1} \left[\frac{6}{p^3} \left((\nu_0' \alpha)' + (\nu_0 \beta)' - (\nu_0 \mu^2)' / 6 \right) \left(x + \frac{k}{p} \right) \right] + O\left(\frac{\|\nu_0\|_{C^3}}{p^4}\right).$$

In the same way, for even numbers of the form $q = 3\bar{q} + 2$, with \bar{q} also being an even number, we could study the higher order constraints from the preservation of integrable caustics with rotation numbers $\frac{1}{q}$ and $\frac{3}{q}$, and obtain similar equalities. These additional conditions on the infinitesimal deformation function $\mu^{-1}\nu_0$ might lead to further reduction of dimension.

APPENDIX B. RELATION BETWEEN α , β , AND THE RADII OF THE CURVATURE ρ

In this section we sketch the proof of the explicit relations for the functions α , β and the radius of the curvature of the boundary, as mentioned in Remark 2.3. Though these formulae are not used in this work, we hope it would be useful for future research.

Lemma B.1. *The following relations between $\alpha(x)$, $\beta(x)$ and the radii of the curvature hold true:*

(1)

$$\alpha'(x) = -4\beta(x) + \frac{\rho^{-2/3}(x)}{24C_\Omega^2} + C_B,$$

(2)

$$\begin{aligned} \beta(x) &= \int_0^1 \frac{\rho^{1/3}\rho''(x)}{15C_\Omega^2} dx + \frac{1}{5} \left(\frac{\rho^{-2/3}(x)}{24C_\Omega^2} - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx \right) \\ &\quad + \frac{1}{5} \left(\frac{\rho^{1/3}(x)\rho''(x)}{36C_\Omega^2} - \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2} \right), \end{aligned}$$

where $C_\Omega = (\int_{\partial\Omega} \rho^{-2/3}(s) ds)^{-1}$, and

$$C_B = \int_0^1 \frac{\rho^{1/3}(x)\rho''(x)}{3C_\Omega^2} dx - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx.$$

Proof. Consider the billiard map in the Lazutkin coordinates (see, e.g. [11])

$$x = C_\Omega \int_0^s \rho^{-2/3}(s) ds, \quad y = 4C_\Omega \rho^{1/3}(s) \sin \frac{\varphi}{2},$$

where $C_\Omega^{-1} = \int_{\partial\Omega} \rho^{-2/3}(s) ds$. Then the billiard map is written as

$$\begin{aligned} f_L : (x, y) &\rightarrow (\bar{x}, \bar{y}) = (x + y + y^3 f(x, y), y + y^4 g(x, y)) \\ &= (x + y + y^3 f_0(x) + O(y^4), y + y^4 g_0(x) + O(y^5)), \end{aligned}$$

where $f_0(x) = f(x, 0)$ and $g_0 = g(x, 0)$. The map f_L preserves the area form $|y| dx dy$. Therefore, for $y > 0$,

$$\bar{y} d\bar{x} d\bar{y} = y dx dy.$$

Compute the determinant of the linearization:

$$\det Df_L = 1 + y^3(4g_0 + f'_0) + O(y^4)$$

Substitute

$$\begin{aligned} 1 &= \frac{\bar{y} d\bar{x} d\bar{y}}{y dx dy} = (1 + y^3(4g_0 + f'_0) + O(y^4))(1 + y^3 g_0 + O(y^4)) \\ &= 1 + y^3(5g_0 + f'_0) + O(y^4). \end{aligned}$$

This implies that

$$(B.1) \quad -5g_0(x) = f'_0(x).$$

The billiard ball map $(s, \varphi) \mapsto (\bar{s}, \bar{\varphi})$ in Taylor expansion is ([11])

$$\bar{s} = s + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + O(\varphi^4),$$

with

$$a_1 = 2\rho(s), \quad a_2 = \frac{4}{3}\rho(s)\rho'(s), \quad a_3 = \frac{2}{3}\rho^2(s)\rho''(s) + \frac{4}{9}\rho(s)\rho'^2(s).$$

Through straightforward calculations, in the Lazutkin parametrization, with $x = x(s)$ and $\bar{x} = x(\bar{s})$ we have

$$\begin{aligned} \bar{x} - x &= C_\Omega \int_s^{\bar{s}} \rho^{-2/3}(\tau) d\tau \\ &= C_\Omega \left[\rho^{-2/3} (2\rho\varphi + \frac{4}{3}\rho\rho'\varphi^2 + (\frac{2}{3}\rho^2\rho'' + \frac{4}{9}\rho(\rho')^2)\varphi^3) \right. \\ &\quad - \frac{1}{2!} \frac{2}{3} \rho^{-5/3} \rho' (2\rho\varphi + \frac{4}{3}\rho\rho'\varphi^2)^2 \\ &\quad \left. + \frac{1}{3!} [\frac{10}{9}\rho^{-8/3}(\rho')^2 - \frac{2}{3}\rho^{-5/3}\rho''] (2\rho\varphi)^3 + O(\varphi^4) \right] \\ &= C_\Omega \left[2\rho^{1/3}\varphi + \left(\frac{2}{3}\rho^{4/3}\rho'' + \frac{4}{9}\rho^{1/3}(\rho')^2 - \frac{16}{9}\rho^{1/3}(\rho')^2 \right. \right. \\ &\quad \left. \left. + \frac{40}{27}\rho^{1/3}(\rho')^2 - \frac{8}{9}\rho^{4/3}\rho'' \right) \varphi^3 + O(\varphi^4) \right] \\ &= C_\Omega \left[2\rho^{1/3}(s)\varphi + \left(-\frac{2}{9}\rho^{4/3}(s)\rho''(s) + \frac{4}{27}\rho^{1/3}(s)\rho'^2(s) \right) \varphi^3 + O(\varphi^4) \right]. \end{aligned}$$

Since $y(s, \varphi) = 4C_\Omega\rho^{1/3}(s)\sin\varphi/2$, we have

$$\bar{x} - x = y + \left(\frac{\rho^{-2/3}(x)}{96C_\Omega^2} - \frac{\rho^{1/3}\rho''(x)}{36C_\Omega^2} + \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2} \right) y^3 + O(y^4).$$

where $\rho(x)$ is read as $\rho(s(x))$. Therefore we have

$$(B.2) \quad f_0(x) = \frac{1}{96C_\Omega^2}\rho^{-2/3}(x) - \frac{\rho^{1/3}\rho''(x)}{36C_\Omega^2} + \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2}.$$

From Lemma 2.2, for the periodic orbit $(x_q^k, \varphi_q^k)_{k=0, \dots, q-1}$ with rotation number $\frac{1}{q}$, we have that for $k = 0, \dots, q-1$,

$$(B.3) \quad \begin{cases} x_q^k = x_q^0 + \frac{k}{q} + \frac{\alpha(x_q^0 + \frac{k}{q})}{q^2} + O(\frac{1}{q^4}), \\ \varphi_q^k = \frac{\mu(x_q^k)}{q} \left(1 + \frac{\beta(x_q^0 + \frac{k}{q})}{q^2} + O(\frac{1}{q^4}) \right). \end{cases}$$

Then

$$(B.4) \quad \begin{aligned} x_q^{k+1} - x_q^k &= \frac{1}{q} + \frac{\alpha(x_q^0 + \frac{k+1}{q}) - \alpha(x_q^0 + \frac{k}{q})}{q^2} + O(\frac{1}{q^4}) \\ &= \frac{1}{q} + \frac{\alpha'(x_q^0 + \frac{k}{q})}{q^3} + O(\frac{1}{q^4}). \end{aligned}$$

For the y -component, recalling that $\mu(x) = \frac{1}{2C_\Omega \rho(x)^{1/3}}$, and using (B.3), we have

$$y_q^k = 4C_\Omega \rho^{1/3}(x_q^k) \sin \frac{\varphi_q^k}{2} = \frac{1}{q} + \frac{\beta(x_q^0 + \frac{k}{q}) - \frac{1}{24}\mu^2(x_q^0 + \frac{k}{q})}{q^3} + O\left(\frac{1}{q^5}\right).$$

Denote

$$B(x) := \beta(x) - \frac{1}{24}\mu^2(x) = \beta(x) - \frac{1}{96C_\Omega^2 \rho^{2/3}(x)}.$$

Therefore,

$$\begin{aligned} y_q^{k+1} - y_q^k &= \frac{B(x_q^0 + \frac{k+1}{q}) - B(x_q^0 + \frac{k}{q})}{q^3} + O\left(\frac{1}{q^5}\right) \\ &= \frac{B'(x_q^0 + \frac{k}{q})}{q^4} + O\left(\frac{1}{q^5}\right). \end{aligned}$$

From the form of f_L and (B.1), we get

$$B'(x) = g_0 = -\frac{1}{5}f'_0.$$

So

$$(B.5) \quad f_0(x) = -5B(x) + C_B.$$

For the x -component, we have

$$\begin{aligned} x_q^{k+1} - x_q^k &= y_q^k + (y_q^k)^3 f_0(x_q^k) + O(|y_q^k|^4) = \frac{1}{q} + \frac{B(x_q^0 + \frac{k}{q}) + f_0(x_q^0 + \frac{k}{q})}{q^3} + O\left(\frac{1}{q^4}\right). \\ &= \frac{1}{q} + \frac{B(x_q^0 + \frac{k}{q})}{q^3} + \frac{f_0(x_q^0 + \frac{k}{q})}{q^3} + O\left(\frac{1}{q^4}\right). \end{aligned}$$

This together with (B.4) leads to the following equality

$$\alpha'(x) = B(x) + f_0(x).$$

With (B.2), we obtain

$$(B.6) \quad \alpha'(x) = \beta(x) + \frac{-\rho^{1/3}(x)\rho''(x)}{36C_\Omega^2} + \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2}.$$

Integrating both sides, we have

$$\int_0^1 \beta(x) dx = \int_0^1 \frac{\rho^{1/3}(x)\rho''(x)}{12C_\Omega^2} dx.$$

With (B.5), we also have

$$(B.7) \quad \alpha'(x) = -4\beta(x) + \frac{\mu^2(x)}{6} + C_B.$$

Integrating both sides, we have

$$C_B = 4 \int_0^1 \beta(x) dx - \int_0^1 \frac{\mu^2(x)}{6} dx = \int_0^1 \frac{\rho^{1/3}(x)\rho''(x)}{3C_\Omega^2} dx - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx.$$

Comparing (B.6) and (B.7), we have

$$\begin{aligned} \beta(x) &= \int_0^1 \frac{\rho^{\frac{1}{3}}\rho''(x)}{15C_\Omega^2} dx + \frac{1}{5} \left(\frac{\rho^{-2/3}(x)}{24C_\Omega^2} - \int_0^1 \frac{\rho^{-2/3}(x)}{24C_\Omega^2} dx \right) \\ &\quad + \frac{1}{5} \left(\frac{\rho^{1/3}(x)\rho''(x)}{36C_\Omega^2} - \frac{\rho^{-2/3}(x)(\rho'(x))^2}{54C_\Omega^2} \right). \end{aligned}$$

Hence, we have obtained the statement of the lemma. \square

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