KdV equation under periodic boundary conditions and its perturbations

HUANG Guan[1] **and Sergei KUKSIN**[2]

[1] C.M.L.S, École Polytechnique, Palaiseau, France
[2] I.M.J, Université Paris Diderot-Paris 7, Paris, France

Abstract. In this paper we discuss properties of the KdV equation under periodic boundary conditions, especially those which are important to study perturbations of the equation. Next we review what is known now about long-time behaviour of solutions for perturbed KdV equations.

Contents

0. Introduction

The famous Korteweg-de Vries (KdV) equation

$$u_t = -u_{xxx} + 6uu_x, \quad x \in \mathbb{R},$$

was first proposed by Joseph Boussinesq [17] as a model for shallow water wave propagation. It became famous later when two Dutch mathematicians, Diederik Korteweg and Gustav De Vries [41], used it to explain the existence of a soliton water wave, previously observed by John Russel in physical experiments. Their work was so successful that this equation is now named after them. Since the mid-sixties of the 20th century the KdV equation received a lot of attention from mathematical and physical communities after the numerical results of Kruskal and Zabusky [43] led to the discovery that its solitary wave solutions interact in an integrable way. It turns out that in some suitable setting, the KdV equation can be viewed as an integrable infinite dimensional hamiltonian system.

In his “New Methods of Celestial Mechanics”, Poincaré calls the task of studying perturbations of integrable systems the “General Problem of Dynamics”. The great scientist was motivated by the celestial mechanics, where perturbed integrable systems play a very important role‡. For a similar reason his maxim is true for mathematical physics, where many important processes are described by suitable perturbations of an integrable PDE, while the unperturbed integrable equations correspond to idealisation of physical reality. In particular, no physical process may be exactly described by the KdV equation.

In this paper§, we focus on the KdV equation with zero mean value periodic boundary condition. It is known since the works of Novikov, Lax, Marcênko, Its-Matveev and McKean-Trubowitz that this system is integrable ([60, 62, 55, 69]). All of its solutions are periodic, quasi-periodic or almost periodic in time. In Section 1 we discuss the KdV equation in the framework of infinite-dimensional hamiltonian systems, in Section 2 we present some normal form results for finite-dimensional integrable hamiltonian systems and in Section 3 — recent results on KdV which may be regarded as infinite-dimensional versions of those in Section 2. Finally in Section 4 we discuss the long-time behaviour of solutions for the perturbed KdV equations, under hamiltonian and non-hamiltonian perturbations. Results presented there are heavily based on theorems from Section 3.

As indicated in the title of our work, we restrict our study to the periodic boundary conditions. In this case the KdV equation behaves as a hamiltonian system with countably-many degrees of freedom and the method of Dynamical Systems may be used for its study (the same is true for other hamiltonian PDEs in finite volume, e.g. see [48]). The KdV is a good example of an integrable PDE in the sense that properties of many other integrable equations with self-adjoint Lax operators, e.g. of the defocusing Zakharov-Shabat equation (see [28]), and of their perturbations are very similar to those of KdV and its perturbations, while the equations with non-selfadjoint Lax operators, e.g. the Sine-Gordon equation, are similar to KdV when we study their small-amplitude solutions (and the KAM-theory for such equation is similar to the KAM theory for KdV without the smallness assumption, see [47]).

‡ For example, the solar system, regarded as a system of 8 interacting planets rotating around the Sun, is a small perturbation of the Kaplerian system. The latter is integrable

§ Based on the courses, given by the second author in Saint-Etienne de Tinée in February 2012 and in the High School for Economics (Moscow) in April 2013.

When considered on the whole line with "zero at infinity" boundary condition, due to the effect of radiation, the KdV equation and its perturbations behave differently, and people working on these problems prefer to call them "dispersive systems". Discussion of the corresponding results should be the topic of another work.

The number of publications, dedicated to KdV and its perturbations is immense, and our bibliography is hopelessly incomplete.

1. KdV under periodic boundary conditions as a hamiltonian system

Consider the KdV equation under zero mean value periodic boundary condition:

$$u_t + u_{xxx} - 6uu_x = 0, \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_{\mathbb{T}} u dx = 0. \tag{1.1}$$

(Note that the mean-value $\int_{\mathbb{T}} u dx$ of a space-periodic solution u is a time-independent quantity; to simplify presentation we choose it to be zero.) To fix the setup, for any integer $p \geqslant 0$, we introduce the Sobolev space of real valued functions on $\mathbb{T}$ with zero mean-value:

$$H^p = \Big\{ u \in L^2(\mathbb{T}, \mathbb{R}) : \ ||u||_p < +\infty, \ \int_{\mathbb{T}} u dx = 0 \Big\}, \quad ||u||_p^2 = \sum_{k \in \mathbb{N}} |2\pi k|^{2p} (|\hat{u}_k|^2 + |\hat{u}_{-k}|^2).$$

Here $\hat{u}_k$, $\hat{u}_{-k}$, $k \in \mathbb{N}$, are the Fourier coefficients of u with respect to the trigonometric base

$$e_k = \sqrt{2} \cos 2\pi k x, \quad k > 0 \quad \text{and} \quad e_k = \sqrt{2} \sin 2\pi k x, \quad k < 0, \tag{1.2}$$

i.e.

$$u = \sum_{k \in \mathbb{N}} \hat{u}_k e_k + \hat{u}_{-k} e_{-k}. \tag{1.3}$$

In particular, H^0 is the space of L^2-functions on $\mathbb{T}$ with zero mean-value. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in H^0 (i.e. the L^2-scalar product).

For a C^1-smooth functional F on some space H^p, we denote by ∇F its gradient with respect to $\langle \cdot, \cdot \rangle$, i.e.

$$dF(u)(v) = \langle \nabla F(u), v \rangle,$$

if u and v are sufficiently smooth. So $\nabla F(u) = \frac{\delta F}{\delta u(x)} + const$, where $\frac{\delta F}{\delta u}$ is the variational derivative, and the constant is chosen in such a way that the mean-value of the r.h.s vanishes. See [47, 34] for details. The initial value problem for KdV on the circle $\mathbb{T}$ is well posed on every Sobolev space H^p with $p \geqslant 1$, see [72, 13]. The regularity of KdV in function spaces of lower smoothness was studied intensively, see [18, 37] and references in these works; also see [18] for some qualitative results concerning the KdV flow in these spaces. We avoid this topic.

It was observed by Gardner [25] that if we introduce the Poisson bracket which assigns to any two functionals $F(u)$ and $G(u)$ the new functional $\{F, G\}$,

$$\Big\{ F, G \Big\}(u) = \int_{\mathbb{T}} \frac{d}{dx} \nabla F(u(x)) \nabla G(u(x)) dx \tag{1.4}$$

(we assume that the r.h.s is well defined, see [47, 48, 34] for details), then KdV becomes a hamiltonian PDE. Indeed, to a differentiable hamiltonian function F this bracket assigns a vector filed $\mathcal{V}_F$, such that

$$\langle \mathcal{V}_F(u), \nabla G(u) \rangle = \{F, G\}(u)$$

for any differentiable functional G. From this relation we see that $\mathcal{V}_F(u) = \frac{\partial}{\partial x}\nabla F(u)$. So the KdV equation takes the hamiltonian form

$$u_t = \frac{\partial}{\partial x}\nabla\mathcal{H}(u), \tag{1.5}$$

with the KdV Hamiltonian

$$\mathcal{H}(u) = \int_{\mathbb{T}}(\frac{u_x^2}{2} + u^3)dx. \tag{1.6}$$

The Gardner bracket (1.4) corresponds to the symplectic structure, defined in H^0 (as well as in any space H^p, $p \geqslant 0$) by the 2-form

$$\omega_2^G(\xi,\eta) = \big\langle(-\frac{\partial}{\partial x})^{-1}\xi,\eta\big\rangle \quad \text{for} \quad \xi,\eta \in H^0. \tag{1.7}$$

Indeed, since $\omega_2^G(\mathcal{V}_F(u),\xi) \equiv -\langle\nabla F(u),\xi\rangle$, then the 2-form ω_2^G also assigns to a Hamiltonian F the vector field $\mathcal{V}_F$ (see [3, 34, 47, 48]).

We note that the bracket (1.4) is well defined on the whole Sobolev spaces $H^p(\mathbb{T}) = H^p \oplus \mathbb{R}$, while the symplectic form ω_2^G is not, and the affine subspaces $\{u \in H^p(\mathbb{T}) : \int_{\mathbb{T}} u dx = const\} \simeq H^p$ are symplectic leaves for this Poisson system. We study the equation only on the leaf $\int_{\mathbb{T}} u dx = 0$, but on other leaves it may be similarly studied.

Writing a function $u(x) \in H^0$ as in (1.3) we see that $\omega_2^G = \sum_{k=1}^{\infty} k^{-1} d\hat{u}_k \wedge d\hat{u}_{-k}$ and that $\mathcal{H}(u) = H(\hat{u}) := \Lambda(\hat{u}) + G(\hat{u})$ with

$$\Lambda(\hat{u}) = \sum_{k=1}^{+\infty}(2\pi k)^2\big(\frac{1}{2}\hat{u}_k^2 + \frac{1}{2}\hat{u}_{-k}^2\big), \qquad G(\hat{u}) = \sum_{k,l,m\neq 0, k+l+m=0} \hat{u}_k\hat{u}_l\hat{u}_m.$$

Accordingly, the KdV equation may be written as the infinite chain of hamiltonian equations

$$\frac{d}{dt}\hat{u}_j = -2\pi j\frac{\partial H(\hat{u})}{\partial\hat{u}_{-j}}, \quad j = \pm 1,\ \pm 2, \dots .$$

2. Finite dimensional integrable systems

Classically, integrable systems are particular hamiltonian systems that can be integrated in quadratures. It was observed by Liouville that for a hamiltonian system with n degrees of freedom to be integrable, it has to possess n independent integrals in involution. This assertion can be understood globally and locally. Now we recall corresponding finite-dimensional definitions and results.

2.1. Liouville-integrable systems

Let $Q \subset \mathbb{R}^{2n}_{(p,q)}$ be a $2n$-dimensional domain. We provide it with the standard symplectic form $\omega_0 = dp \wedge dq$ and the corresponding Poisson bracket

$$\{f,g\} = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g,$$

where $g, f \in C^1(Q)$ and " $\cdot$ " stands for the Euclidean scalar product in $\mathbb{R}^n$ (see [3]). If $\{f,g\} = 0$, the functions f and g are called *commuting*, or *in involution*. If $h(p,q)$ is a C^1-function on Q, then the hamiltonian system with the Hamiltonian h is

$$\dot{p} = -\nabla_q h, \quad \dot{q} = \nabla_p h. \tag{2.1}$$

Definition 2.1 *(Liouville-integrability). The hamiltonian system (2.1) is called integrable in the sense of Liouville if its Hamiltonian h admits n independent integrals in involution $h_1, \ldots, h_n$. That is, $\{h, h_i\} = 0$ for $1 \leqslant i \leqslant n$; $\{h_i, h_j\} = 0$ for $1 \leqslant i,\ j \leqslant n$, and $dh_1 \wedge \cdots \wedge dh_n \neq 0$.*

A nice structure of an Liouville-integrable system is assured by the celebrated Liouville-Arnold-Jost theorem (see [3, 65]). It claims that if an integrable system is such that the level sets $T_c = \{(p, q) \in Q : h_1(p, q) = c_1, \ldots, h_n(p, q) = c_n\}$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ are compact, then each non-empty set T_c is an embedded n-dimensional torus. Moreover for a suitable neighbourhood O_{T_c} of T_c in Q there exists a symplectomorphism

$$\Theta :\ O_{T_c} \to O \times \mathbb{T}^n = \{(I, \varphi)\}, \quad O \subset \mathbb{R}^n,$$

where the symplectic structure in $O \times \mathbb{T}^n$ is given by the 2-form $dI \wedge d\varphi$. Finally, there exists a function $\bar{h}(I)$ such that $h(p, q) = \bar{h}(\Theta(p, q))$. This result is true both in the smooth and analytic categories.

The coordinates (I, φ) are called the *action-angle variables* for (2.1). Using them, the hamiltonian system may be written as

$$\dot{I} = 0, \quad \dot{\varphi} = \nabla_I \bar{h}(I). \tag{2.2}$$

Accordingly, in the original coordinates (p, q), solutions of the system are

$$(p, q)(t) = \Theta^{-1}(I_0, \varphi_0 + \nabla_I \bar{h}(I_0)t).$$

On $O \times \mathbb{T}^n$, consider the 1-form $Id\varphi = \sum_{j=1}^n I_j d\varphi_j$, then $d(Id\varphi) = dI \wedge d\varphi$. For any vector $I \in O$, and for $j = 1, \ldots, n$, denote by $C_j(I)$ the cycle $\{(I, \varphi) \in O \times \mathbb{T}^n : \varphi_j \in [0, 2\pi] \text{ and } \varphi_i = const, \text{ if } i \neq j\}$. Then

$$\frac{1}{2\pi} \int_{C_j} Id\varphi = \frac{1}{2\pi} \int_{C_j} I_j d\varphi_j = I_j.$$

Consider a disc $D_j \subset O \times \mathbb{T}^n$ such that $\partial D_j = C_j$. For any 1-form ω_1 satisfying $d\omega_1 = dI \wedge d\varphi$, we have

$$\frac{1}{2\pi} \int_{C_j} (Id\varphi - \omega_1) = \frac{1}{2\pi} \int_{D_j} d(Id\varphi - \omega_1) = 0.$$

So

$$I_j = \frac{1}{2\pi} \int_{C_j(I)} \omega_1, \quad \text{if} \quad d\omega_1 = dI \wedge d\varphi. \tag{2.3}$$

This is the *Arnold formula for actions.*

2.2. Birkhoff Integrable systems

We denote by $\mathbb{J}$ the standard symplectic matrix $\mathbb{J} = \text{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$, operating in any $\mathbb{R}^{2n}$ (e.g. in $\mathbb{R}^2$). Assume that the origin is an elliptic critical point of a smooth Hamiltonian h, i.e. $\nabla h(0) = 0$ and that the matrix $\mathbb{J}\nabla^2 h(0)$ has only pure imagine eigenvalues. Then there exists a linear symplectic change of coordinates which puts h to the form

$$h = \sum_{i=1}^n \lambda_i (p_i^2 + q_i^2) + h.o.t, \qquad \lambda_j \in \mathbb{R} \quad \forall j.$$

If the frequencies $(\lambda_1, \ldots, \lambda_n)$ satisfy some non-resonance conditions, then this normalisation process can be carried out to higher order terms. The result of this normalisation is known as the *Birkhoff normal form for the Hamiltonian h.*

Definition 2.2 *The frequencies* $\lambda_1, \dots, \lambda_n$ *are non-resonant up to order* $m \geqslant 1$ *if* $\sum_{i=1}^n k_i \lambda_i \neq 0$ *for each* $k \in \mathbb{Z}^n$ *such that* $1 \leqslant \sum_{i=1}^n |k_i| \leqslant m$. *They are called non-resonant if* $k_1\lambda_1 + \ldots + k_n\lambda_n = 0$ *with integers* $k_1, \dots, k_n$ *only when* $k_1 = \ldots = k_n = 0$.

Theorem 2.3 *(Birkhoff normal form, see [64, 65]) Let* $H = N_2 + \cdots$ *be a real analytic Hamiltonian in the vicinity of the origin in* $(\mathbb{R}^{2n}_{(p,q)}, dp \wedge dq)$ *with the quadratic part* $N_2 = \sum_{i=1}^n \lambda_i (q_i^2 + p_i^2)$. *If the (real) frequencies* $\lambda_1, \dots, \lambda_n$ *are non-resonant up to order* $m \geqslant 3$, *then there exists a real analytic symplectic transformation* $\Psi_m = Id + \cdots$, *such that*

$$H \circ \Psi_m = N_2 + N_4 + \ldots + N_m + h.o.t.$$

Here N_i *are homogeneous polynomials of order* i, *which are actually smooth functions of variables* $p_1^2 + q_1^2, \dots, p_n^2 + q_n^2$. *If the frequencies are non-resonant, then there exists a formal symplectic transformation* $\Psi = Id + \cdots$, *represented by a formal power series, such that* $H \circ \Psi = N_2 + N_4 + \cdots$ *(this equality holds in the sense of formal series).*

If the transformation, converting H to the Birkhoff normal form, was convergent, then the resulting Hamiltonian would be integrable in a neighbourhood of the origin with the integrals $p_1^2 + q_1^2, \dots, p_n^2 + q_n^2$. These functions are not independent when $p_i = q_i = 0$ for some i. So the system is not integrable in the sense of Liouville. But it is integrable in a weaker sense:

Definition 2.4 *Functions* $f_1, \dots, f_k$ *are functionally independent if their differentials* $df_1, \dots, df_k$ *are linearly independent on a dense open set. A* $2n$*-dimensional Hamiltonian is called Birkhoff integrable near an equilibrium* $m \in \mathbb{R}^{2n}$, *if it admits* n *functionally independent integrals in involution in the vicinity of* m.

Birkhoff normal form provides a powerful tool to study the dynamics of hamiltonian PDEs, e.g. see [53, 7] and references in [7] . However, in this paper we shall not discuss its version for KdV, since for that equation there exists a stronger normal form. Now we pass to its counterpart in finite dimension.

2.3. Vey theorem

The results of this section hold both in the C^∞-smooth and analytic categories.

Definition 2.5 *Consider a Birkhoff integrable system, defined near an equilibrium* $m \in \mathbb{R}^{2n}$, *with independent commuting integrals* $F = (F_1, \dots, F_n)$. *Its Poisson algebra is the linear space* $\mathcal{A}(F) = \Big\{ G : \{G, F_i\} = 0,\ i = 1, \dots, n \Big\}$.

Note that, although the integrals of an integrable system are not defined in a unique way, the corresponding algebra $\mathcal{A}(F)$ will be.

Definition 2.6 *A Poisson algebra* $\mathcal{A}(F)$ *is said to be non-resonant at a point* $m \in \mathbb{R}^{2n}$, *if it contains a Hamiltonian with a non-resonant elliptic critical point at* m *(i.e., around* m *one can introduce symplectic coordinates* (p, q) *such that the quadratic part of that Hamiltonian at* m *is* $\sum \lambda_j (p_j^2 + q_j^2)$, *where the real numbers* λ_j *are non-resonant).*

It is easy to verify that if some $F_1 \in \mathcal{A}(\mathcal{F})$ is elliptic and non-resonant at the equilibrium m, then all other functions in $\mathcal{A}(\mathcal{F})$ are elliptic at m as well.

Theorem 2.7 *(Vey's theorem). Let $F = (F_1, \dots, F_n)$ be n functionally independent functions in involution in a neighbourhood of a point $m \in \mathbb{R}^{2n}$. If the Poisson algebra $\mathcal{A}(F)$ is non-resonant at m, then one can introduce around m symplectic coordinates (p, q) so that $\mathcal{A}(F)$ consists of all functions, which are actually functions of $p_1^2 + q_1^2, \dots, p_n^2 + q_n^2$.*

Example. Let $F = (f_1, \dots, f_n)$ be a system of smooth commuting Hamiltonians, defined in the vicinity of their joint equilibrium $m \in \mathbb{R}^{2n}$, such that the hessians $\nabla^2 f_i(m)$, $1 \leqslant i \leqslant n$, are linear independent. Then the theorem above applies to the Poisson algebra $\mathcal{A}(F)$.

In [73], Vey proved the theorem in the analytic case with an additional non-degeneracy condition, which was later removed by Ito in [31]. The results in [73, 31] also apply to non-elliptic cases. The smooth version of Theorem 2.7 was demonstrated by Eliasson [23]. There exists an infinite dimensional extension of the theorem, see [51].

3. Integrability of KdV

The KdV equation (1.1) admits infinitely many integrals in involution, and there are different ways to obtain them, see [25, 63, 26, 55, 75]. Below we present an elegant way to construct a set of Poisson commuting integrals by considering the spectrum of an associated Schrödinger operator, due to Peter Lax [55] (see [56] for a well written presentation of the theory).

3.1. Lax pairs

Let $u(x)$ be a L^2-function on $\mathbb{T}$. Consider the differential operators L_u and B_u, acting on 2-periodic functions‖

$$L_u = -\frac{d^2}{dx^2} + u(x), \quad B_u = -4\frac{d^3}{dx^3} + 3u(x)\frac{d}{dx} + 3\frac{d}{dx}u(x),$$

where we view $u(x)$ as a multiplication operator $f \mapsto u(x)f$. The operators B_u and L_u are called the *Lax pair* for KdV. Calculating the commutator $[B_u, L_u] = B_u L_u - L_u B_u$, we see that most of the terms cancel and the only term left is $-u_{xxx} + 6uu_x$. Therefore if $u(t, x)$ is a solution of (1.1), then the operators $L(t) = L_{u(t,\cdot)}$ and $B(t) = B_{u(t,\cdot)}$ satisfy the operator equation

$$\frac{d}{dt}L(t) = [B(t), L(t)]. \tag{3.1}$$

Note that $B(t)$ is a skew-symmetric operator, $B(t)^* = -B(t)$. Let $U(t)$ be the one-parameter family of unitary operators, defined by the differential equation

$$\frac{d}{dt}U = B(t)U, \quad U(0) = \mathrm{Id}.$$

Then $L(t) = U^{-1}(t)L(0)U(t)$. Therefore, the operator $L(t)$ is unitary conjugated to $L(0)$. Consequently, its spectrum is independent of t. That is, the spectral data of the operator L_u provide a set of conserved quantities for the KdV equation (1.1). Since L_u is the Sturm–Liouville operator with a potential $u(x)$, then in the context of this theory functions $u(x)$ are called *potentials*.

‖ note the doubling of the period.

It is well known that for any L^2-potential u, the spectrum of the Sturm-Liouville operator L_u – regarded as an unbounded operator in $L^2(\mathbb{R}/2\mathbb{Z})$ – is a sequence of simple or double eigenvalues $\{\lambda_j : j \geqslant 0\}$, tending to infinity:

$$\mathrm{spec}(u) = \{\lambda_0 < \lambda_1 \leqslant \lambda_2 < \cdots \nearrow \infty\}.$$

Equality or inequality may occur in every place with a "$\leqslant$" sign (see [60, 34]). The segment $[\lambda_{2j-1}, \lambda_{2j}]$ is called the *n-th spectral gap*. The asymptotic behaviour of the periodic eigenvalues is

$$\lambda_{2n-1}(u),\ \lambda_{2n}(u) = n^2\pi^2 + [u] + l^2(n),$$

where $[u]$ is the mean value of u, and $l^2(n)$ is the n-th number of an l^2 sequence. Let $g_n(u) = \lambda_{2n}(u) - \lambda_{2n-1}(u) \geqslant 0$, $n \geqslant 1$. These quantities are conserved under the flow of KdV. We call g_n the *n-th gap-length* of the spectrum. The n-th gap is called *open* if $g_n > 0$, otherwise it is *closed*. However, the periodic eigenvalues and the gap-lengths are not satisfactory integrals from the analytic point of view since λ_n is not a smooth function of the potential u when $g_n = 0$. Fortunately, the squared gap lengths $g_n^2(u)$, $n \geqslant 1$, are real analytic functions on L^2, which Poisson commute with each other (see [61, 56, 34]). Moreover, together with the mean value, the gap lengths determine uniquely the periodic spectrum of a potential, and their asymptotic behaviour characterises the regularity of a potential in exactly the same way as its Fourier coefficients [60, 27].

This method applies to integrate other hamiltonian systems in finite or infinite dimension. It is remarkably general and is referred to as the *method of Lax pairs.*

3.2. Action-angle coordinates

We denote by $\mathrm{Iso}(u_0)$ the isospectral set of a potential $u_0 \in H^0$:

$$\mathrm{Iso}(u_0) = \Big\{u \in H^0 : \quad \mathrm{spec}(u) = \mathrm{spec}(u_0)\Big\}.$$

It is invariant under the flow of KdV and may be characterised by the gap lengths

$$\mathrm{Iso}(u_0) = \Big\{u \in H^0 : \quad g_n(u) = g_n(u_0),\ n \geqslant 1\Big\}.$$

Moreover, for any $n \geqslant 1$, $u_0 \in H^n$ if, and only if, $\mathrm{Iso}(u_0) \subset H^n$.

In [61], McKean and Trubowitz showed that the $\mathrm{Iso}(u_0)$ is homemorphic to a compact torus whose dimension equals the number of open gaps. So the phase space H^0 is foliated by a collection of KdV-invariant tori of different dimensions, finite or infinite. A potential $u \in H^0$ is called *finite-gap* if only a finite number of its spectral gaps are open. The finite-dimensional KdV-invariant torus $\mathrm{Iso}(u_0)$ is called a *finite-gap torus.* For any $n \in \mathbb{N}$ let us set

$$\mathcal{J}^n = \Big\{u \in H^0 : \ \ g_j(u) = 0 \ \text{ if } \ j > n\Big\}. \tag{3.2}$$

We call the sets $\mathcal{J}^n$, $n \in \mathbb{N}$, the finite-gap manifolds.

Theorem 3.1 *For any $n \in \mathbb{N}$, the finite gap manifold $(\mathcal{J}^n, \omega_2^G)$ is a smooth symplectic $2n$-manifold, invariant under the flow of KdV (1.1), and*

$$T_0\mathcal{J}^n = \Big\{u \in H^0 : \ \ \hat{u}_k = 0 \ \ \textit{if} \ \ |k| \geqslant n+1\Big\},$$

(see (1.3)). Moreover, the square gap lengths $g_k^2(u)$, $k = 1, \ldots, n$, form n commuting analytic integrals of motions which are non-degenerated everywhere on the dense domain $\mathcal{J}_0^n = \{u \in \mathcal{J}^n : g_1(u), \ldots, g_n(u) > 0\}$.

Therefore, the Liouville-Arnold-Jost theorem applies everywhere on $\mathcal{J}_0^n$, $n \in \mathbb{N}$. Furthermore, the union of the finite gap manifolds $\cup_{n\in\mathbb{N}}\mathcal{J}^n$ is dense in each space H^s (see [60]). This hints that on the spaces H^s, $s \geqslant 0$, it may be possible to construct global action-angle coordinates for KdV. In [24], Flaschka and McLaughlin used the Arnold formula (2.3) to get an explicit formula for action variables of KdV in terms of the 2-period spectral data of L_u. To explain their construction, denote by $y_1(x,\lambda,u)$ and $y_2(x,\lambda,u)$ the standard fundamental solutions of the equation $-y'' + uy = \lambda y$, defined by the initial conditions

$$y_1(0,\lambda,u) = 1, \quad y_2(0,\lambda,u) = 0,$$
$$y_1'(0,\lambda,u) = 0, \quad y_2'(0,\lambda,u) = 1.$$

The quantity $\triangle(\lambda,u) = y_1(1,\lambda,u) + y_2'(1,\lambda,u)$ is called the *discriminant*, associated with this pair of solutions. The periodic spectrum of u is precisely the zero set of the entire function $\triangle^2(\lambda,u) - 4$, for which we have the explicit representation (see e.g. [75, 61])

$$\triangle^2(\lambda,u) - 4 = 4(\lambda_0 - \lambda)\prod_{n\geqslant 1}\frac{(\lambda_{2n}-\lambda)(\lambda_{2n-1}-\lambda)}{n^4\pi^4}.$$

This function is a spectral invariant. We also need the spectrum of the differential operator $L_u = -\frac{d^2}{dx^2} + u$ under Dirichlet boundary conditions on the interval $[0,1]$. It consists of an unbounded sequence of single Dirichlet eigenvalues

$$\mu_1(u) < \mu_2(u) < \ldots \nearrow \infty,$$

which satisfy $\lambda_{2n-1}(u) \leqslant \mu_n(u) \leqslant \lambda_{2n}(u)$, for all $n \in \mathbb{N}$. Thus, the n-th Dirichlet eigenvalue μ_n is always contained in the n-th spectral gap. The Dirichlet spectrum provides coordinates on the isospectral sets $\mathrm{Iso}(u)$ (see [61, 60, 34]). For any $z \in \mathbb{T}$, denote by $\{\mu_j(u,z),\ j \geqslant 1\}$ the spectrum of the operator L_u under the shifted Dirichlet boundary conditions $y(z) = y(z+1) = 0$ (so $\mu_j(u,0) = \mu_j(u)$); still $\lambda_{2n-1} \leqslant \mu_n(u,z) \leqslant \lambda_{2n}(u)$. Jointly with the spectrum $\{\lambda_j\}$, it defines the potential $u(x)$ via the remarkable *trace formula* (see [75, 22, 34, 61]):

$$u(z) = \lambda_0(u) + \sum_{j=1}^{\infty}(\lambda_{2j-1}(u) + \lambda_{2j}(u) - 2\mu_j(u,z)).$$

Define

$$f_n(u) = 2\log(-1)^n y_2'(1,\mu_n(u),u), \quad \forall n \in \mathbb{N}.$$

Flaschka and McLaughlin [24] observed that the quantities $\{\mu_n, f_n\}_{n\in\mathbb{N}}$ form canonical coordinates of H^0, i.e.

$$\{\mu_n,\mu_m\} = \{f_n,f_m\} = 0, \quad \{\mu_n,f_m\} = \delta_{n,m}, \quad \forall n,m \in \mathbb{N}.$$

Accordingly, the symplectic form ω_2^G (see (1.7)) equals $d\omega_1$, where ω_1 is the 1-form $\sum_{n\in\mathbb{N}} f_n d\mu_n$. Now the KdV action variables are given by the Arnold formula (2.3), where C_n is a circle on the invariant torus $\mathrm{Iso}(u)$, corresponding to $\mu_n(u)$. It is shown in [24] that

$$I_n = \frac{2}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}} \lambda\frac{\dot{\triangle}(\lambda)}{\sqrt{\triangle^2(\lambda)-4}}d\lambda, \quad \forall n \in \mathbb{N}.$$

The analytic properties of the functions $u \mapsto I_n$ and of the mapping $u \mapsto I = (I_1, I_2, \ldots)$ were later studied by Kappeler and Korotyaev (see references in [34, 39] and below). In particular, it was shown that $I_n(u)$, $n \in \mathbb{N}$, are real analytic functions on H^0 of the form $I_n = g_n^2+$ *higher order terms*, and $I_n = 0$, if and only if, $g_n = 0$ (see in [34]). For any vector $I = (I_1, I_2, \dots)$ with non-negative components we will denote

$$T_I = \{u(x) \in H^0 : I_n(u) = I_n \quad \forall n\}. \tag{3.3}$$

The angle-variables φ^n on the finite-gap manifolds $\mathcal{J}^n$ were found in the 1970's by Soviet mathematicians who constructed them from the Dirichlet spectrum $\{\mu_j(u)\}$ by means of the Abel transform, associated with the Riemann surface of the function $\sqrt{\triangle^2 - 4}$ (see [22, 60, 75], and see [32, 21, 42, 8] for the celebrated explicit formulas for angle-variables φ^n and for finite-gap solutions of KdV in terms of the theta-functions).

In [45] and [47], Section 7, the action-angle variables (I^n, φ^n) on a finite-gap manifold $\mathcal{J}^n$ and the explicit formulas for solutions of KdV on manifolds $\mathcal{J}^N$, $N \geq n$, from the works [21, 42, 8] were used to obtain an analytic symplectic coordinate system (I^n, φ^n, y) in the vicinity of $\mathcal{J}^n$ in H^p. The variable y belongs to a ball in a subspace $Y \subset H^p$ of co-dimention $2n$, and in the new coordinates the KdV Hamiltonian (1.6) reads

$$\mathcal{H} = \text{const} + h^n(I^n) + \langle A(I^n)y, y\rangle + O(y^3). \tag{3.4}$$

The selfadjoint operator $A(I^n)$ is diagonal in some fixed symplectic basis of Y. The nonlinearity $O(y^3)$ defines a hamiltonian operator of order one. That is, the KdV's linear operator, which is an operator of order three, mostly transforms to the linear part of the new hamiltonian operator and "does not spread much" to its nonlinear part. This is the crucial property of (3.4). The normal form (3.4) is instrumental for the purposes of the KAM-theory, see below Section 4.1.

McKean and Trubowitz in [61, 62] extended the construction of angles on finite-gap manifolds to the set of all potentials, thus obtaining angle variables $\varphi = (\varphi_1, \varphi_2, \dots)$ on the whole space H^p, $p \geqslant 0$. The angles $(\varphi_k(u), k \geq 1)$ are well defined Gâteaux-analytic functions of u outside the locus

$$\eth = \{u(x) : g_j(u) = 0 \text{ for some } j\}, \tag{3.5}$$

which is dense in each space H^p. The action-map $u \mapsto I$ was not considered in [61, 62], but it may be shown that outside $\eth$, in a certain weak sense, the variables (I, φ) are KdV's action-angles (see the next section for a stronger statement). This result is elegant, but insufficient to study perturbations of KdV since the transformation to the variables (I, φ) is singular at the dense locus $\eth$.

3.3. Birkhoff coordinates and nonlinear Fourier transform

In a number of publications (see in [34]), Kappeler with some collaborators proved that the Birkhoff coordinates $v = \{v_n,\ n = \pm 1, \pm 2, \dots\}$, associated with the action-angles variables (I, φ) by the relations

$$v_n = \sqrt{2I_n}\cos(\varphi_n), \quad v_{-n} = \sqrt{2I_n}\sin(\varphi_n), \quad \forall n \in \mathbb{N}, \tag{3.6}$$

are analytic on the whole of H^0. In that space the Birkhoff coordinates define a global coordinate system, in which the KdV Hamiltonian (1.6) is a function exclusively of the actions. This remarkable result significantly specifies the normal form (3.4). To formulate the result more specifically, for any $p \in \mathbb{R}$, we introduce the Hilbert space h^p,

$$h^p := \Big\{v = (\mathbf{v}_1, \mathbf{v}_2, \cdots) : |v|_p^2 = \sum_{j=1}^{+\infty} (2\pi j)^{2p+1} |\mathbf{v}_j|^2 < \infty,\ \mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2,\ j \in \mathbb{N}\Big\},$$

and the weighted l^1-space h_I^p,

$$h_I^p := \Big\{ I = (I_1, \dots) \in \mathbb{R}^\infty : |I|_p^\sim = 2\sum_{j=1}^{+\infty} (2\pi j)^{2p+1} |I_j| < +\infty \Big\}.$$

Define the mappings

$$\pi_I : \ h^p \to h_I^p, \quad v \mapsto I = (I_1, I_2, \dots), \ \text{ where } \ I_k = \frac{1}{2}|\mathbf{v}_k|^2 \quad \forall\, k,$$

$$\pi_\varphi : \ h^p \to \mathbb{T}^\infty, \quad v \mapsto \varphi = (\varphi_1, \varphi_2, \dots), \ \text{ where } \ \varphi_k = \arctan(\frac{v_{-k}}{v_k})$$
$$\text{if } \ \mathbf{v}_k \neq 0, \ \text{ and } \ \varphi_k = 0 \ \text{ if } \ \mathbf{v}_k = 0.$$

Since $|\pi_I(v)|_p^\sim = |v|_p^2$, then π_I is continuous. Its image $h_{I+}^p = \pi_I(h^p)$ is the positive octant in h_I^p. When there is no ambiguity, we write $I(v) = \pi_I(v)$.

Consider the mapping

$$\Psi : u(x) \mapsto v = (\mathbf{v}_1, \mathbf{v}_2, \dots), \quad \mathbf{v}_n = (v_n, v_{-n})^t \in \mathbb{R}^2,$$

where $v_{\pm n}$ are defined by (3.6) and $\{I_n(u)\}$, $\{\varphi_n(u)\}$ are the actions and angles as in Section 3.2. Clearly $\pi_I \circ \Psi(u) = I(u)$ and $\pi_\varphi \circ \Psi(u) = \varphi(u)$. Below we refer to Ψ as to the *nonlinear Fourier transform*.

Theorem 3.2 *(see [34, 33]) The mapping Ψ defines an analytical symplectomorphism $\Psi : (H^0, \omega_2^G) \to (h^0, \sum_{k=1}^\infty dv_k \wedge dv_{-k})$ with the following properties:*

(i) For any $p \in [-1, +\infty)$, it defines an analytic diffeomorphism $\Psi : H^p \mapsto h^p$.

(ii) (Percival's identity) If $v = \Psi(u)$, then $|v|_0 = ||u||_0$.

(iii) (Normalisation) The differential $d\Psi(0)$ is the operator $\sum u_s e_s \mapsto v$, where $v_s = |2\pi s|^{-1/2} u_s$ for each s.

(iv) The function $\hat{H}(v) = \mathcal{H}(\Psi^{-1}(v))$ has the form $\hat{H}(v) = H_K(I(v))$, where the function $H_K(I)$ is analytic in a suitable neighbourhood of the octant h_{I+}^1 in h_I^1, such that a curve $u \in C^1(0, T; H^0)$ is a solution of KdV if and only if $v(t) = \Psi(u(t))$ satisfies the equations

$$\dot{\mathbf{v}}_j = \mathbb{J}\frac{\partial H_K}{\partial I_j}(I)\mathbf{v}_j, \quad \mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2, \ j \in \mathbb{N}. \tag{3.7}$$

Assertion (iii) normalises Ψ in the following sense. For any $\theta = (\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$ denote by Φ_θ the operator

$$\Phi_\theta v = v', \quad \mathbf{v}'_j = \bar{\Phi}_{\theta_j} \mathbf{v}_j, \quad \forall j \in \mathbb{N}, \tag{3.8}$$

where $\bar{\Phi}_\alpha$ is the rotation of the plane $\mathbb{R}^2$ by the angle α. Then $\Phi_\theta \circ \Psi$ satisfies all assertions of the theorem except (iii). But the properties (i)-(iv) jointly determine Ψ in a unique way.

The theorem above can be viewed as a global infinite dimensional version of the Vey's Theorem 2.7 for KdV, and eq. (3.7) – as a global Birkhoff normal form for KdV. Note that in finite dimension, a global Birkhoff normal form exists only for very exceptional integrable equations, which were found during the boom of activity in integrable systems, provoked by the discovery of the method of Lax pairs.

Remark 3.3 *The map Ψ simultaneously transforms all Hamiltonians of the KdV hierarchy to the Birkhoff normal form. The KdV hierarchy is a collection of hamiltonian functions $\mathcal{J}_l$, $l \geqslant 0$, commuting with the KdV Hamiltonian, and having the form*

$$\mathcal{J}_l(u) = \int \Big(\frac{1}{2}(u^{(l)})^2 + J_{l-1}(u)\Big)dx.$$

Here $J_{-1} = 0$ and $J_{l-1}(u)$, $l \geq 1$, is a polynomial of $u, \dots, u^{(l-1)}$. The functions from the KdV hierarchy form another complete set of KdV integrals. E.g. see [22, 34, 56].

Properties of the nonlinear Fourier transform Ψ may be specified in two important respects. One of this specifications – the quasilinearity of Ψ – is presented in the theorem below. Another one – its behaviour at infinity – is discussed in the next section.

The nonlinear Fourier transform Ψ is quasi-linear. Precisely,

Theorem 3.4 *If $m \geqslant 0$, then the map $\Psi - d\Psi(0) : H^m \to h^{m+1}$ is analytic.*

That is, the non-linear part of Ψ is 1-smoother than its linearisation at the origin. See [51] for a local version of this theorem, applicable as well to other integrable infinite-dimensional systems, and see [35, 36] for the global result. We note that the transformation to the normal form (3.4) is also known to be quasi-linear, see [45, 47].

Problem 3.5 *Does the mapping $\Psi - d\Psi(0)$ analytically maps H^m to $h^{m+1+\gamma}$ with $\gamma > 0$?*

In [51] we proved that the answer to this question is negative if $\gamma > 1$, and conjectured that it also is negative if $\gamma > 0$.

3.4. Behaviour of Ψ near infinity and large solutions of KdV

By the assertion (ii) of Theorem 3.2, $|\Psi(u)|_0 = ||u||_0$. It was established by Korotyaev in [39] that higher order norms of u and $v = \Psi(u)$ are related by both-sides polynomial estimates:

Theorem 3.6 *For any $m \in \mathbb{N}$, there are polynomials $\mathcal{P}_m(y)$ and $\mathcal{Q}_m(y)$ such that if $u \in H^m$ and $v = \Psi(u)$, then*

$$|v|_m \leqslant \mathcal{P}_m(||u||_m), \quad ||u||_m \leqslant \mathcal{Q}_m(|v|_m).$$

The polynomials $\mathcal{P}_m$ and $\mathcal{Q}_m$ are constructed in [39] inductively. From a personal communication from Korotyaev we know that one can take

$$\mathcal{P}_m(y) = C_m y(1+y)^{\frac{2(m+2)}{3}}. \tag{3.9}$$

Estimating a potential $u(x)$ via its actions¶ is more complicated. Corresponding polynomials $\mathcal{Q}_m$ may be chosen of the form

$$\mathcal{Q}_m(y) = C'_m y(1+y)^{a_m}, \tag{3.10}$$

where $a_1 = \frac{5}{2}$, $a_2 = 3$ and a_m has a factorial growth as $m \to \infty$.

¶ Note that any $|v|_m^2$ is a linear combination of the actions I_j, $j \geqslant 1$.

Problem 3.7 *Prove that there exist polynomials $\mathcal{P}_m^1$ and $\mathcal{Q}_m^1$, $m \in \mathbb{N}$, such that for any $u \in H^m$, we have*

$$||d\Psi(u)||_{m,m} \leqslant \mathcal{P}_m^1(||u||_m), \quad ||d\Psi^{-1}(v)||_{m,m} \leqslant \mathcal{Q}_m^1(||v||_m),$$

where $v = \Psi(u)$. Prove similar polynomial bounds for the norms of higher differentials of Ψ and Ψ^{-1}.

It seems that to solve the problem a new proof of Theorem 3.2 has to be found (note that the existing proof is rather bulky and occupies half of the book [34]).

The difficulty in resolving the problem above streams from the fact that $\Psi(u)$ is constructed in terms of spectral characteristics of the Strum-Liouville operator L_u, and their dependence on large potentials $u(x)$ is poorly understood. Accordingly, the following question seems to be very complicated:

Problem 3.8 *Let $u(t,x) = u(t,x;\lambda)$ be a solution of (1.1) such that $u(0,x) = \lambda u_0(x)$, where $u_0 \not\equiv 0$ is a given smooth function with zero mean-value, and $\lambda > 1$ is a large parameter. The task is to study the behaviour of $u(t,x;\lambda)$ when $\lambda \to \infty$.*

Let $u(t,x)$ be as above. Then $||u(0,x)||_m = \lambda C(m,u_0)$. By Theorem 3.6, $|v(0)|_m \leqslant \mathcal{P}_m(\lambda C(m,u_0))$. Since $|v(t)|_m$ is an integral of motion, then, yet again using the theorem, we get that

$$||u(t)||_m \leqslant \mathcal{Q}_m\Big(\mathcal{P}_m(\lambda C(m,u_0))\Big).$$

In particular, by (3.9) and (3.10) we have $||u(t)||_1 \leqslant C(1+\lambda)^{21/2}$. A lower bound for the Sobolev norms comes from the fact that $||u(t)||_0$ is an integral of motion. So

$$||u(t)||_m \geqslant ||u(0)||_0 = \lambda C(0,u_0).$$

We have demonstrated:

Proposition 3.9 *Let $u(t,x)$ be a solution of (1.1) such that $u(0,x) = \lambda u_0(x)$, where $0 \not\equiv u_0 \in C^\infty \cap H^0$ and $\lambda \geqslant 1$. Then for $m \geq 1$ we have*

$$1 + (\lambda C(m,u_0))^{A_m} \geqslant \limsup_{t\to\infty} ||u(t)||_m \geqslant \liminf_{t\to\infty} ||u(t)||_m \geqslant c(u_0)\lambda, \qquad (3.11)$$

for a suitable $A_m > 1$. E.g. $A_1 = 21/2$.

The third estimate in (3.11) is optimal up to a constant factor as

$$\liminf_{t\to\infty} ||u(t)||_m \leqslant \lambda C(m,u_0),$$

since the curve $u(t)$ is almost periodic. The first estimate with the exponent A_m – which follows from Theorem 3.6 – is certainly non-optimal. But the assertion that $\limsup_{t\to\infty} ||u(t)||_m$ grows with λ as λ^{A_m}, where A_m goes to infinity with m, is correct. It follows from our next result:

Theorem 3.10 *Let $k \geqslant 4$. Then there exists $\alpha > 0$ and, for any $\lambda > 1$, there exists $t_* = t_*(u_0,\lambda)$ such that*

$$||u(t_*)||_k \geqslant c'_{u_0}\lambda^{1+\alpha k}. \qquad (3.12)$$

In [46] (see there Theorem 3 and Appendix 2), the theorem is proved for a class of non-linear Schrödinger equations which includes the defocusing Zakharov-Shabat equation. The proof applies to KdV. See [9], where the argument is applied to the multidimensional Burgers equation, similar to KdV for the proof of this result. We mention that (unlike the majority of the results in this work) the assertion of Theorem 3.10 remains true for other boundary conditions.

Problem 3.8 may be scaled as the non-dispersive limit for KdV. Indeed, let us substitute $u = \lambda w$ and pass to fast time $\tau = \lambda t$. Then the function $w(\tau, x; \lambda)$ satisfies

$$w_\tau + \lambda^{-1} w_{xxx} - 6 w w_x = 0, \quad w(0,x) = u_0(x), \tag{3.13}$$

and we are interested in $w(\lambda t, x; \lambda)$ when $\lambda \to \infty$. For $\lambda = \infty$ the equation above becomes the Hopf equation. Since $u_0(x)$ is a periodic non-constant function, then the solution of $(3.13)_{\lambda=\infty}$ develop a shock at time τ_*, $0 < \tau_* < \infty$. Accordingly, the elementary perturbation theory allows to study solutions of (3.13) when $\lambda \to \infty$ for $\tau < \tau_*$, but not for $\tau \geqslant \tau_*$. The problem to study this limit for $\tau \geqslant \tau_*$ is addressed by the Lax-Levermore theory (mostly for the case when $x \in \mathbb{R}$ and $u_0(x)$ vanishes at infinity). There is vast literature on this subject, e.g. see [57, 20] and references in [20]. The existing Lax-Levermore theory does not allow to study solutions $w(\tau, x)$ for $\tau \sim \lambda^{-1}$, as is required by Problem 3.8.

3.5. Frequency map properties

Let us denote

$$W(I) = (W_1(I),\ W_2(I), \ldots), \quad W_i(I) = \frac{\partial H_K}{\partial I_i}, \quad i \in \mathbb{N}. \tag{3.14}$$

This is the *frequency map for KdV.* By Theorem 3.2 each of its component is an analytic function, defined in the vicinity of h^1_{I+} in h^1_I.

Lemma 3.11 *a) For $i, j \geq 1$ we have $\partial^2 W(0)/\partial I_i \partial I_j = -6\delta_{i,j}$.*
b) For any $n \in \mathbb{N}$, if $I_{n+1} = I_{n+2} = \ldots = 0$, then

$$\det\Big(\big(\frac{\partial W_i}{\partial I_j}\big)_{1 \leqslant i,j \leqslant n} \Big) \neq 0.$$

For a) see [10, 34, 47]. For a proof of b) and references to the original works of Krichever and Bikbaev-Kuksin see Section 3.3 of [47].

Let l_i^∞, $i \in \mathbb{Z}$, be the Banach spaces of all real sequences $l = (l_1, l_2, \ldots)$ with norms

$$|l|_i^\infty = \sup_{n \geqslant 1} n^i |l_n| < \infty.$$

Denote $\boldsymbol{\kappa} = (\kappa_n)_{n \in \mathbb{N}}$, where $\kappa_n = (2\pi n)^3$. For the following result see [34], Theorem 15.4.

Lemma 3.12 *The normalised frequency map $I \mapsto W(I) - \boldsymbol{\kappa}$ is real analytic as a mapping from h^1 to l^∞_{-1}.*

From these two lemmas we know that the Hamiltonian $H_K(I)$ of KdV is non-degenerated in the sense of Kolmogorov and its nonlinear part is more regular than its linear part. These properties are very important to study perturbations of KdV.

3.6. Hamiltonian $H_K(I)$ convexity

By Theorem 3.2, the dynamics of KdV are determined by the Hamiltonian $H_K(I)$. To understand the properties of the latter is an important step toward the study of perturbations of KdV.

Denote by P_j the moments of the actions, given by

$$P_j = \sum_{i\geqslant 1}(2\pi n)^j I_i, \quad j \in \mathbb{Z}.$$

Due to Theorem 3.2, the linear part of $H_K(I)$ at the origin $dH_K(0)(I)$ equals to $\frac{1}{2}\int_{\mathbb{T}}(\frac{\partial}{\partial x}(\Psi^{-1}v))^2 dx = P_3$. So we can write $H_K(I)$ as

$$H_K(I) = P_3(I) - V(I), \quad V(I) = \mathcal{O}(|I|_1^2).$$

(The minus-sign here is convenient since, as we will see, $V(I)$ is non-negative.) For any $N \geqslant 1$, denote $\tilde{l}^N \subset l^2$ the N-dimensional subspace

$$\tilde{l}^N = \{l = (l_1, \dots) \in \mathbb{R}^\infty : l_n = 0, \ \forall n > N\},$$

and set $\tilde{l}^\infty = \cup_{N\in\mathbb{N}}\tilde{l}^N$. Clearly the function V is analytic on each octant $\tilde{l}^N_+$. So it is Gâteaux-analytic on the octant $\tilde{l}^\infty_+$. That is, it is analytic on every interval $\{(ta + (1-t)c) \in \tilde{l}^\infty_+ : 0 \leq t \leq 1\}$, where $a, c \in \tilde{l}^\infty_+$.

By Lemma 3.11 a), $d^2V(0)(I) = 6||I||_2^2$. This suggests that the Hilbert space l^2 rather than the Banach space h^1_I (which is contained in l^2) is a distinguished phase space for the Hamiltonian $H_K(I)$. This guess is justified by the following result:

Theorem 3.13 *(see [40]). (i) The function $V : \tilde{l}^\infty_+ \to \mathbb{R}$ extends to a non-negative continuous function on the l^2-octant l^2_+, such that $V(I) = 0$ for some $I \in l^2_+$ iff $I = 0$. Moreover $0 \leqslant V(I) \leqslant 8P_1P_{-1}$.*

(ii) For any $I \in l^2_+$, the following estimates hold true:

$$\frac{\pi}{10}\frac{||I||_2^2}{1+2P_{-1}^{1/2}} \leqslant V(I) \leqslant (8^3(1+P_{-1}^{1/2})^{1/2}P_{-1}^2 + 6\pi e^{P_{-1}^{1/2}/2}||I||_2)||I||_2.$$

(iii) The function $V(I)$ is convex on l^2_+.

Note that the assertion (iii) follows from (i) and Lemma 3.11. Indeed, since $V(I)$ is analytic on $\tilde{l}^N_+$, then Lemma 3.11 assures that the Hessian $\{\frac{\partial^2 V}{\partial I_i \partial I_j}\}_{1\leqslant i,j\leqslant N}$ is positive definite on $\tilde{l}^N_+$. Thus V is convex on $\tilde{l}^N_+$, for each $N \in \mathbb{N}$. Then the assertion (iii) is deduced from the fact the $\tilde{l}^\infty_+ = \cup_{N\in\mathbb{N}}\tilde{l}^N_+$ is dense in l^2_+, where $V(I)$ is continuous.

Assertion ii) of the theorem shows that l^2 is the biggest Banach space on which $V(I)$ is continuous[+]. Jointly with the convexity of $V(I)$ on l^2, this hints that l^2 is the natural space to study the long-time dynamics of actions $I(u(t))$ for solutions $u(t)$ of a perturbed KdV.

This theorem and the analyticity of the KdV Hamiltonian (1.6) do not leave many doubts that $V(I)$ is analytic on l^2_+. If so, then by Lemma 3.11, this function is strictly convex in a neighbourhood of an origin in l^2. Most likely, it is strictly convex everywhere on l^2_+.

Unlike $V(I)$, the total Hamiltonian $H_K(I)$ is not continuous on l^2_+ since its linear part $P_3(I)$ is an unbounded linear functional. But $P_3(I)$ contributes to equation (3.7)

[+] That is, if $V(I)$ continuously extends to a Banach space $\mathcal{B}$ of the sequences $(I_1, I_2, \dots)$, then $\mathcal{B}$ may be continuously embedded in l^2.

the linear rotation $\dot{\mathbf{v}}_k = \mathbb{J}(2\pi n)^3 \mathbf{v}_k$, $k \in \mathbb{N}$. So the properties of equation (3.7) are essentially determined by the component $V(I)$ of the Hamiltonian. Note that since $P_3(I)$ is a bounded linear functional on the space $h^1 \subset l^2$, then the Hamiltonian $H_K(I)$ is concave in h^1.

Problem 3.14 *Is it true that the function $V(I) - 3|I|^2_{l^2}$ extends analytically (or continuously) to a space, bigger than l^2?*

4. Perturbations of KdV

In the theory of integrable systems in finite dimension, there are two types of perturbative results concerning long-time stability of solutions. The first type is the KAM theory. It essentially states that among a family of invariant tori of the unperturbed system, given by the Liouville-Arnold-Jost theorem, there exists (under generic assumptions) a large set of tori which survive under sufficiently small hamiltonian perturbations, deforming only slightly. In particular, the perturbed system admits plenty of quasi-periodic solutions [4, 1]. Results of the second type are obtained by the techniques of averaging which apply to a larger class of dynamical systems, characterised by the existence of fast and slow variables. This method has a much longer history which dates back to the epoch of Lagrange and Laplace, who applied it to the problems of celestial mechanics without proper justifications. Only in the last fifty years has rigorous mathematical justification of the principle been obtained (see in [66, 4, 59]). If the unperturbed system is hamiltonian integrable, then for the slow-fast variables one can choose the action-angle variables. The averaging theorems say that under appropriate assumptions, the action variables, calculated for solutions of the perturbed system, can be well approximated by solutions of a suitable averaged vector field, over an extended time interval. If the perturbation is hamiltonian, then the averaged vector field vanishes. The strongest result in this direction belongs to Nekhoroshev [68, 58], who proved that in the hamiltonian case the action variables vary only a little over exponentially long time intervals.

Concerning instability of solutions, also two types of phenomena are known. One is called the Arnold diffusion. In [2], Arnold observed that despite most of the phase space of near integrable hamiltonian systems with more than two degrees of freedom is foliated by invariant KAM tori, there can still exist solutions such that their actions admit increments of order one in a sufficiently long time span. Arnold conjectured that this phenomenon is generic. Though there have been many developments in this direction in the last ten years, the mechanism of the Arnold diffusion is still far from being well understood. Another instability mechanism is known as the capture in resonance. The essence of this phenomenon is that a solution of a perturbed system reaches a resonant zone and begins drifting along it in such a way that the resonance condition approximately holds. Therefore, solutions of the original perturbed system and the averaged one diverge by a quantity of order one in a time interval of order ϵ^{-1} (see e.g. [67]).

Now return to PDEs. Two types of perturbations of the KdV equation (1.1) have been considered: when the boundary condition is perturbed but the equation is not, and the other way around. Problems of the first type lie outside the scope of our work, and very few results (if any) are rigorously proven there, see [42] for discussion and some related statement. Problems of the second type are much closer to the finite-dimensional situation and are discussed below. Several attempts were made to establish stability results for perturbed integrable PDEs, e.g. for perturbed KdV, analogous to those in finite dimension. Among them, the KAM theory was the most successful [19]. The first results in this direction are due to Kuksin [44, 45] and Wayne [74]. Although there are no rigorously proven instability results for the perturbed KdV, we mention our belief that to study the instability, Theorem 3.2, 3.4 and 3.13 should be important.

4.1. KAM theorem for perturbed KdV

Consider the hamiltonian perturbation of KdV, corresponding to a Hamiltonian $H_\epsilon = \mathcal{H}(u) + \epsilon F(u)$:

$$\dot{u} + u_{xxx} - 6uu_x - \epsilon \frac{\partial}{\partial x} \nabla F(u) = 0, \quad u(0) \in H^1. \tag{4.1}$$

Here $F(u)$ is an analytic functional on H^1 and ∇F is its L^2-gradient in u. Let $n \in \mathbb{N}$, and $\Gamma \subset \mathbb{R}^n_+$ be a compact set of positive Lebesgue measure. Consider a family of the n-gap tori:

$$\mathcal{T}_\Gamma = \cup_{I \in \Gamma} \mathcal{T}^n_I \subset \mathcal{J}^n, \quad \mathcal{T}^n_I = T_{(I,0,\ldots)},$$

where $I = (I_1, \ldots, I_n)$, see (3.2) and (3.3). It turns out that most of them persist as invariant tori of the perturbed equation (4.1):

Theorem 4.1 *For some $M \geqslant 1$, assume that the Hamiltonian F analytically extends to a complex neighbourhood U^c of $\mathcal{T}_\Gamma$ in $H^M \otimes \mathbb{C}$, where it satisfies the regularity condition*

$$\nabla F : U^c \to H^M \otimes \mathbb{C}, \quad \sup_{u \in U^c} (|F(u)| + ||\nabla F(u)||_M) \leqslant 1.$$

Then, there exists an $\epsilon_0 > 0$ and for $\epsilon < \epsilon_0$ there exist

(i) a nonempty Cantor set $\Gamma_\epsilon \subset \Gamma$ with $mes(\Gamma \setminus \Gamma_\epsilon) \to 0$ as $\epsilon \to 0$;

(ii) a Lipschitz mapping $\Xi : \mathbb{T}^n \times \Gamma_\epsilon \to U \cap H^M$, such that its restriction to each torus $\mathbb{T}^n \times I$, $I \in \Gamma_\epsilon$, is an analytical embedding;

(iii) a Lipschitz map $\chi : \Gamma_\epsilon \to \mathbb{R}^n$, $|\chi - Id| \leqslant Const \cdot \epsilon$.

These three objects are such that for every $(\varphi, I) \in \mathbb{T}^n \times \Gamma_\epsilon$, the curve $u(t) = \Xi(\varphi + \chi(I)t, I)$ is an quasi-periodic solution of (4.1) winding around the invariant torus $\Xi(\mathbb{T}^n \times \{I\})$. Moreover, these solutions are linearly stable.

Proof: In the coordinates v as in Theorem 3.2, the Hamiltonian H_ϵ becomes $H_K(I) + \epsilon F(v)$. For any $I_0 \in h^M_I$, using Taylor's formula, we write

$$\begin{aligned} H_K(I_0 + I) &= H_K(I_0) + \sum_{i \geqslant 1} \frac{\partial H_K}{\partial I_i}(I_0) I_i + \int_0^1 (1-t) \sum \frac{\partial^2 H_K}{\partial I_i \partial I_j} I_i I_j dt \\ &:= const + \sum_{i \geqslant 1} W_i(I_0) I_i + Q(I_0, I), \end{aligned} \tag{4.2}$$

where W is the frequency map, see (3.14). Now we introduce the symplectic polar coordinates around the tori in the family $\mathcal{T}_\Gamma$. Namely, for each $\xi_0 \in \Gamma$, we set

$$\begin{cases} v_i = \sqrt{\xi_0 + y_i} \cos \varphi, \quad v_{-i} = \sqrt{\xi_0 + y_i} \sin \varphi, & 1 \leqslant i \leqslant n, \\ b_i = v_i, \quad b_{-i} = v_{-i}, & i \geqslant n+1. \end{cases}$$

Denote $b = (b_{n+1}, b_{-n-1}, \ldots)$. The transformation above is real analytic and symplectic on the domain

$$D(s,r) = \{|\mathrm{Im}\,\varphi| < s\} \times \{|y| < r^2\} \times \{|b|_M < r\},$$

if $s, r > 0$ are small enough. Using the expansion (4.2), setting $I_0 = (\xi_0, 0)$ and neglecting an irrelevant constant we see that the integrable Hamiltonian in new coordinates is given by

$$H_K = N + Q = N(y, \xi_0, b) + Q(y, \xi_0, b),$$

where $N = \sum_{1\leqslant i\leqslant n} W_i(\xi_0) y_i + \frac{1}{2}\sum_{i\geqslant 1} W_{n+i}(b_{n+i}^2 + b_{-n-i}^2)$, with $I_i = y_i$ for $1 \leqslant i \leqslant n$, and $2I_i = b_i^2 + b_{-i}^2$ for $i > n$. Then the whole Hamiltonian of the perturbed equation (4.1) can be written as

$$H_\epsilon = N + Q + \epsilon F. \tag{4.3}$$

We consider the new perturbation term $P = Q + \epsilon F$. By Lemma 3.12 and the regularity assumption, if $r^2 = \sqrt{\epsilon}$, then

$$\sup_{D(s,r)} ||X_P||_{M-1} \leqslant c\sqrt{\epsilon}.$$

Using the non-degeneracy Lemma 3.11, we can take the vector $\{W_i(\xi_0), 1 \le i \le n\}$ for a free n-dimensional parameter of the problem and apply an abstract KAM theorem (see Theorem 8.3 in [47] and Theorem 18.1 in [34]) to obtain the statements of the theorem. For complete proof, see [47, 34]. □

Example. The theorem applies if (4.1) is the hamiltonian PDE with the local Hamiltonian

$$\mathcal{H}(u) + \epsilon \int_{\mathbb{T}} g(u(x), x)\, dx = \int_{\mathbb{T}} \big(\frac{u_x^2}{2} + u^3 + \epsilon g(u(x), x)\big)\, dx,$$

where $g(u, x)$ is a smooth function, periodic in x and analytic in u. In this case in (4.1) we have $\frac{\partial}{\partial x}\nabla F(u(\cdot))(x) = \frac{\partial}{\partial x} g_u(u(x), x)$. □

It is easy to see that for the proof of Theorem 4.1, explained above, the normal form (3.4) (weaker and much simpler than that in Theorem 3.2) is sufficient, see [45, 47]. Normal forms, similar to (3.4), exist for integrable PDEs for which a normal form as in Theorem 3.2 does not hold, e.g, for the Sine-Gordon equations, see [47].

Theorem 4.1 establishes the KAM-persistence most of the finite-gap KdV solutions under small hamiltonian perturbations of the equation with Hamiltonians $\epsilon \int g(u(x), x)\, dx$ (see example above). Similar one can establish the KAM-persistence most of small-amplitude finite-gap solutions under higher order perturbations with Hamiltonians $\int g_4(u(x), x)\, dx$, where g_4 is an analytic function such that $g_4(u, x) = O(|u|^4)$, see [53, 34]. Recently Baldi, Berti and Montalto [5] proved that most of small-amplitude finite-gap solutions of KdV persist under perturbations of the equation with Hamiltonians $\int g_5(u, \nabla u)\, dx$, where $g_5 = O(|u| + |\nabla u|)^5$ is a sufficiently smooth function. Note that the perturbation of the KdV equation, corresponding to the Hamiltonian $\int g_5$, contains three x-derivatives, i.e. as much as the linear part of KdV.

Theorem 4.1 allows to perturb a set of KdV-solutions which form a null-set for any reasonable measure in the function space (in difference with the finite-dimensional KAM theory which insures the persistence of a set of almost-periodic solutions which occupy the phase space up to a set of Lebesgue measure $\lesssim \epsilon^\gamma$, $\gamma > 0$). It gives rise to a natural question:

Problem 4.2 *Do typical tori T_I as in (3.3), where $I = (I_1, I_2, \dots) \in h_I^p$, $p < \infty$, persist in the perturbed equation (4.1)? If they do not, what happens to them?*

Though there are KAM-theorems for perturbations of infinite-dimensional invariant tori, e.g. see [70, 15], they are not applicable to the problem above since, firstly, those works do not apply to KdV due to the strong nonlinear effects and long-range coupling between the modes and, secondly, they only treat invariant tori corresponding to actions $I_1, I_2, \dots$ which decay very fast (faster than exponentially).

4.2. Averaging for perturbed KdV

Compared to KAM, averaging type theorems for perturbed KdV are more recent and less developed. Their stochastic versions, which we discuss below in Section 4.3, are significantly stronger than the corresponding deterministic statements in Section 4.4. We will explain the reason for that later on. Let us start with the 'easiest' case, where KdV is stabilised by small dissipation:

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon u_{xx}, \quad u(0) \in H^3. \tag{4.4}$$

A simple calculation shows that a solution $u(t)$ satisfies

$$||u(t)||_0 \leqslant e^{-\epsilon t}||u(0)||_0.$$

So $u(t)$ becomes negligible for $t \gg \epsilon^{-1}$. But what happens during time intervals of order ϵ^{-1}? Let us pass to the slow time $\tau = \epsilon t$ and apply to equation (4.4) the nonlinear Fourier transform Ψ, denoting for $k = 1, 2, \dots$, $\Psi_k(u) = \mathbf{v}_k$ if $\Psi(u) = v = (\mathbf{v}_1, \dots)$:

$$\frac{d}{d\tau}\mathbf{v}_k = \epsilon^{-1}\mathbb{J}W_k(I)\mathbf{v}_k + d\Psi_k(\Psi^{-1}(v))(\triangle\Psi^{-1}(v)) =: \epsilon^{-1}\mathbb{J}W_k(I)\mathbf{v}_k + P_k(v), \quad k \in \mathbb{N}. \tag{4.5}$$

Since $I_k = \frac{1}{2}|\mathbf{v}_k|^2$ is an integral of motion for KdV, then

$$\frac{d}{d\tau}I_k = \big(P_k(v), \mathbf{v}_k\big) := F_k(v), \quad k \in \mathbb{N}, \tag{4.6}$$

where $(\cdot, \cdot)$ stands for the Euclidian scalar product in $\mathbb{R}^2$. Using (4.5) we get

$$\frac{d}{d\tau}\varphi_k = \epsilon^{-1}W_k(I) + \langle\text{term of order } 1\rangle, \text{ if } \mathbf{v}_k \neq 0,\ k \in \mathbb{N}. \tag{4.7}$$

We have written equation (4.4) in the action-angle variables (I, φ). Consider the *averaged equation for actions:*

$$\frac{d}{d\tau}J_k = \langle F_k\rangle(J), \quad \langle F_k\rangle(J) = \int_{\mathbb{T}^\infty} F_k(J, \varphi)d\varphi, \quad k \in \mathbb{N}, \quad J(0) = I(u(0))\,, \tag{4.8}$$

where $F_k(I, \varphi) = F_k(v(I, \varphi))$, $k \in \mathbb{N}$, and $d\varphi$ is the Haar measure on the infinite dimensional torus $\mathbb{T}^\infty$. The main problem of the averaging theory is to see if the following holds true:

Averaging principle: Fix any $T > 0$. Let $(I(\tau), \varphi(\tau))$ be a solution of (4.6), (4.7), and $J(\tau)$ be a solution of (4.8). Then (either for all or, for 'typical' initial data $u(0)$) we have

$$||I(\tau) - J(\tau)|| \leqslant \rho(\epsilon), \quad \forall\, 0 \leqslant \tau \leqslant T,$$

where $||\cdot||$ is a suitable norm, and $\rho(\epsilon) \to 0$ with $\epsilon \to 0$.

The main obstacles to prove this for the perturbed equation (4.4) are the following:

(1) The KdV-dynamics on some tori are resonant.

(2) The well-posedness of the averaged equation (4.8) is unknown.

(3) Equations (4.7) are singular at the locus $\eth$ (see (3.5)).

To handle the third difficulty observe that eq. (4.7) with a specific k is singular if $v_k = 0$. But when v_k is small, the k-th mode does not much affect the dynamics, so the equation for φ_k may be excluded from the consideration. Concerning the second difficulty, in [49] the second author of this work established that the averaged equation (4.8) may be lifted to a regular system in the space h^p, which is well posed, at least locally. More specifically, note that equation (4.8) may be written as follows:

$$\frac{d}{d\tau} J_k = \langle F_k \rangle = \int_{\mathbb{T}^\infty} (\bar{\Phi}_{\theta_k} \mathbf{v}_k,\ P_k(\Phi_\theta v)) d\theta = (\mathbf{v}_k,\ R_k(v)), \tag{4.9}$$
$$R_k = \int_{\mathbb{T}^\infty} \bar{\Phi}_{-\theta_k} P_k(\Phi_\theta v) d\theta,$$

where the maps Φ_θ and $\bar{\Phi}_{\theta_k}$ are defined in (3.8). Now consider equation

$$\frac{d}{d\tau} v = R(v). \tag{4.10}$$

Then relation (4.9) implies:

Lemma 4.3 *If $v(\tau)$ satisfies (4.10), then $I(v(\tau))$ satisfies (4.8).*

Equation (4.10) is called the *effective equation* for the perturbed KdV equation (4.4). It is rotation-invariant: if $v(\tau)$ is a solution of (4.10), then for each $\theta \in \mathbb{T}^\infty$, $\Phi_\theta(v(\tau))$ is also a solution. Since the map Ψ is quasilinear by Theorem 3.4, we may write $R(v)$ more explicitly. Namely, denote by $\hat{\triangle}$ the Fourier-image of the Laplacian, $\hat{\triangle} = \mathrm{diag}\{-k^2,\ k \in \mathbb{N}\}$, and set

$$L = d\Psi(0),\ \ \Psi_0 = \Psi - L,\ \ G = \Psi^{-1} = L^{-1} + G_0.$$

Then $G_0 : h^s \to H^{s+1}$ is analytic for any $s \geqslant 0$, and direct calculation shows that

$$R(v) = \hat{\triangle} v + R_0(v),$$

where

$$R_0(v) = \int_{\mathbb{T}^\infty} [\Phi_{-\theta} L \triangle (G_0 \Phi_\theta v) + \Phi_{-\theta} d\Psi_0(G\Phi_\theta v) \triangle (G\Phi_\theta v)] d\theta.$$

Hence, $R_0(v)$ is an operator of order one and the effective equation (4.10) is a Fourier transform of a quasi-linear heat equation with a non-local nonlinearity of first order. Such equations are locally well posed. Due to the direct relation between the effective equation and the averaged equation, the former can be used to study the latter.

The first difficulty is serious. Sometimes it cannot be overcome, and then the averaging fails. One way to handle it is discussed in the next section.

4.3. Stochastic averaging.

A way to handle the first obstacle, mentioned in the previous section, – the resonant tori – is to add to the perturbed equation (4.4) a random force which would shake solutions $u(t)$ off a resonant torus (as well as off any other fixed invariant torus T_I). So let us consider a randomly perturbed KdV:

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon u_{xx} + \sqrt{\epsilon}\, \eta(t,x), \quad u(0) = u_0 \in C^\infty \cap H^0, \tag{4.11}$$
$$\eta(t,x) = \frac{\partial}{\partial t} \sum_{j \in \mathbb{Z}_0} b_j \beta_j(t) e_j(x).$$

Here $\mathbb{Z}_0$ is the set of all non-zero integers, $\{e_j(x)\}_{j \in \mathbb{Z}_0}$ is the basis (1.2) and

- all $b_j > 0$ and decay fast when $|j| \to \infty$.
- $\{\beta_j(t)\}_{j\in\mathbb{Z}_0}$ are independent standard Wiener processes (so $\beta_j(t) = \beta_j^\omega(t)$, where ω is a point in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$).

The scaling factor $\sqrt{\epsilon}$ in the r.h.s is natural since only with this scaling do solutions of equation (4.11) remain of order one when $t \to \infty$ and $\epsilon \to 0$. To simplify formulas we assume that $b_j = b_{-j}$ for all j.

In [52, 49], Kuksin and Piatnitski justified the averaging principle for the stochastic equation (4.11). To explain their result we pass to the slow time $\tau = \epsilon t$ and use Itô's formula (e.g. see in [38]) to write the corresponding equation for the vector of actions $I(u(\epsilon^{-1}\tau)) = I^\omega(\tau)$:

$$\frac{dI_k}{d\tau} = F_k(I,\varphi) + K_k(I,\varphi) + \sum_j G_k^j(I,\varphi)\frac{\partial}{\partial\tau}\beta_j(\tau), \quad k \geq 1. \tag{4.12}$$

Here F is defined as in (4.6), K is the Itô term

$$K_k = \frac{1}{2}\sum_{j\in\mathbb{Z}_0} b_j^2\big((d^2\Psi_k(u)[e_j,e_j], \mathbf{v}_k) + |d\Psi_k(u)e_j|^2\big),$$

and G is the dispersion matrix, $G_k^j = b_j d\Psi_k(u)e_j$.

Let us average the equation above:

$$\frac{dJ}{d\tau} = \langle F\rangle(J) + \langle K\rangle(J) + \sum_j \langle G^j\rangle(J)\frac{\partial}{\partial\tau}\beta_j(\tau). \tag{4.13}$$

Here $\langle F\rangle$ is the same as in (4.8), $\langle K\rangle$ is the average of K and $\langle G^j\rangle(J)$, $j \in \mathbb{Z}_0$, are column-vectors, forming an infinite matrix $\langle G\rangle(J)$. The latter is defined as a square root of the averaged diffusion matrix $\int_{\mathbb{T}^\infty} G(J,\varphi)G^T(J,\varphi)d\varphi$, where $G(J,\varphi)$ is formed by the columns $G^j(J,\varphi)$. Similar to Section 4.2, equation (4.11) also admits an effective equation of the form

$$\frac{d\mathbf{v}_k}{d\tau} = -k^2\mathbf{v}_k + R'_k(v) + \sum_j (R'')_k^j(v)\frac{\partial}{\partial t}\beta_j(\tau), \qquad k \geq 1, \tag{4.14}$$

where $R'(v)$ is an operator of first order, $R''(v)$ is a Hilbert-Schmidt matrix, which is an analytic function of v, and $\{\beta_j(\cdot), j \geqslant 1\}$ are standard independent Wiener processes. This is a quasilinear stochastic heat equation with a non-local nonlinearity, written in the Fourier coordinates. Construction of the effective stochastic equation (4.14) is more involved then that of the deterministic effective equation (4.10) since the diffusion in (4.14) cannot be written down explicitly and is obtained by some indirect construction. Equation (4.14) is well posed in the spaces $h^p, p \geq 1$. Similar to above, if $v(\tau)$ is a solution of (4.9), then $I(v(\tau))$ is a weak solution of (4.13). See [49] for details.

We recall (e.g. see [38]) that a random process $J = J^\omega(\tau)$ is a *weak solution (in the sense of stochastic analysis)* of equation (4.13), if for almost every ω it satisfies the integrated in time version of equation (4.13), where the processes β_j's are replaced by some other independent standard Brownian motions $\hat{\beta}_j$'s:

$$J^\omega(\tau) = \int_0^\tau (\langle F\rangle + \langle K\rangle)(J^\omega(s))ds + \int_0^\tau \sum_j \langle G^j\rangle(J^\omega(s))d\hat{\beta}_j^\omega(s), \quad \forall\tau \in [0,T].$$

Fix any $T > 0$. Let $u^\epsilon(t)$, $0 \leqslant t \leqslant \epsilon^{-1}T$, be a solution of (4.11). Introduce slow time $\tau = \epsilon t$ and denote $I^\epsilon(\tau) = I(u^\epsilon(\epsilon^{-1}\tau))$. Consider the distribution of this random process. This is a measure in the space $C([0,T], h_I^p)$. We assume $p \geqslant 3$.

Theorem 4.4 *(i) The limiting measure* $\lim_{\epsilon\to 0}\mathcal{D}(I^\epsilon(\cdot))$ *exists. It is the law of a weak solution* $I^0(\tau)$ *of (4.13) with the initial data* $I^0(0) = I(u_0)$.

(ii) The law $\mathcal{D}(I^0(\tau))$ *equals to that of* $I(v(\tau))$, *where* $v(\tau)$ $(0 \leqslant \tau \leqslant T)$ *is a regular solution of the corresponding effective system (4.9) with the initial data* $v_0 = \Psi(u_0)$.

(iii) Let $f \geqslant 0$ *be a continuous function such that* $\int_0^T f(s)ds = 1$. *Then the measure* $\int_0^T f(\tau)\mathcal{D}(I^\epsilon(\tau), \varphi^\epsilon(\tau))d\tau$ *on the space* $h_I^p \times \mathbb{T}^\infty$ *weakly converges, as* $\epsilon \to 0$, *to the measure* $\Big(\int_0^T f(\tau)\mathcal{D}(I^0(\tau))ds\Big) \times d\varphi$. *In particular, the measure* $\int_0^T f(\tau)\mathcal{D}(\varphi^\epsilon(\tau))ds$ *weakly converges to* $d\varphi$, *where* $d\varphi$ *is the Haar measure on the infinite dimensional torus* $\mathbb{T}^\infty$.

(iv) Every sequence $\epsilon'_j \to 0$ *contains a subsequence* $\epsilon_j \to 0$ *such that the double limit* $\lim_{\epsilon_j\to 0}\lim_{t\to\infty}\mathcal{D}\big(\Psi(u^\epsilon(t))\big)$ *exists for any solution* $u^\epsilon(t)$ *and is a stationary measure* for the effective equation (4.9).*

For the proof see [52, 49]. For the last assertion also see [50], Section 4. The proof of the theorem applies to other stochastic perturbations of KdV. In particular, assertions (i)-(iii) hold for equations

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon g(u(x), x) + \sqrt{\epsilon}\,\eta,$$

where η is the same as in (4.11) and g is a smooth function, periodic in x, which has at most a linear growth in u, and is such that $g(u(\cdot), x) \in H^p$ if $u \in H^p$ (this notably holds if $g(u,x)$ is even in u and odd in x).

The key to the proof of Theorem 4.4 is the following result (see Lemma 5.2 in [52]), where for any $m \in \mathbb{N}$, $K > 0$ and $\delta > 0$ we denote

$$\begin{aligned}\Omega(\delta, m, K) := \big\{ I : |W_1(I)k_1 + \ldots + W_m(I)k_m| < \delta,\\ \text{for some } k \in \mathbb{Z}^m \text{ such that } 1 \leqslant |k| \leqslant K \big\}.\end{aligned} \tag{4.15}$$

Lemma 4.5 *For any* $m \in \mathbb{N}$, $K > 0$, $T > 0$ *and* $\delta > 0$ *we have*

$$\int_0^T \mathbf{P}\{I(\tau) \in \Omega(\delta, m, K)\}d\tau \leqslant \kappa(\delta, K, m, T),$$

where $\kappa(\delta, K, m, T)$ *goes to zero with* δ, *for any fixed* K, m *and* T.

The assertion follows from the analyticity of the frequency map W (Lemma 3.12) and its non degeneracy, stated in Lemma 3.11.

Lemma 4.5 assures that in average, solutions of (4.11) do not spend much time in the vicinity of resonant tori. The stochastic nature of the equation is crucial for this result.

* For this notion and its discussion see [54]. We are certain that eq. (4.11) has a unique stationary measure. When this is proven, it would imply that the convergence in (iv) holds as $\epsilon \to 0$.

4.4. Deterministic averaging.

It is plausible that the averaging principle also holds for equation (4.4). But without randomness, it is unclear how to assure that solutions of (4.4) 'pass the resonant zone quickly' (in analogy with Lemma 4.5). This naturally leads to the question: for which deterministic perturbations of KdV it is possible to prove the property of fast crossing the resonant zones and verify the averaging principle? Some results in this direction are obtained by the first author in [29, 30]. Now we discuss them.

Consider a deterministically perturbed KdV equation:

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon f(u), \quad x \in \mathbb{T}, \ u \in H^p, \tag{4.16}$$

where $p \geqslant 3$ and the perturbation $f(u) = f(u(\cdot))$ may be non-local. I.e., $f(u)(x)$ may depend on values of $u(y)$, where $|y-x| \geq \varkappa > 0$. We are going to discuss solutions of (4.16) on time-intervals of order ϵ^{-1}. Accordingly we fixe some $\zeta_0 \leq 0$, $p \geqslant 3$, $T > 0$ and make the following assumption:

Assumption A. *(i) There exists $p' = p'(p) < p$, such that for any $q \in [p', p]$ the perturbation in (4.16) defines an analytic mapping of order ζ_0:*

$$H^q \to H^{q-\zeta_0}, \quad u(\cdot) \mapsto f(u(\cdot)).$$

(ii) For any $u_0 \in H^p$, there exists a unique solution $u(t) \in H^p$ of (4.16) with $u(0) = u_0$. For $0 \leq t \leqslant T\epsilon^{-1}$ its norm satisfies $||u(t)||_p \leqslant C(T, p, ||u_0||_p)$.

It will be convenient for us to discuss equation (4.16) in the v-variables. With some abuse of notation we will denote by S_t, $0 \leqslant t \leqslant T\epsilon^{-1}$, the flow-maps of (4.16), both in the u- and in the v-variables. We denote

$$B_p(M) = \{v \in h^p : |v|_p \leqslant M\}.$$

Definition 4.6 *1) A Borelian measure μ on h^p is called regular if for any analytic function $g \not\equiv 0$ on h^p, we have $\mu(\{v \in h^p : \ g(v) = 0\}) = 0$.*

2) A measure μ on h^p is said to be ϵ-quasi-invariant for equation (4.16) if it is regular and for any $M > 0$ there exists a constant $C(T, M)$ such that for every Borel set $A \subset B_p(M)$ we have ♯

$$e^{-\epsilon t C(T,M)}\mu(A) \leqslant \mu(S^t(A)) \leqslant e^{\epsilon t C(T,M)}\mu(A), \quad \forall\, 0 \leq t \leqslant \epsilon^{-1}T. \tag{4.17}$$

Similarly, these definitions can be carried to measures on the space H^p and the flow maps of equation (4.16) on H^p.

For an ϵ-perturbed finite-dimensional hamiltonian system the Lebesgue measure is ϵ-quasi-invariant by the Liouville theorem. This fact is crucial for the Anosov approach to justify the classical averaging principle (see in [4, 59]). In infinite dimension there is no Lebesgue measure, and the existence of an ϵ-quasi-invariant measure is a serious restriction.

If equation (4.16) has an ϵ-quasi-invariant measure μ, then the argument, invented by Anosov for the finite dimensional averaging, insures the required analogy of Lemma 4.5 for equation (4.16). Indeed, let us define the resonant subset $\mathcal{B}$ of $h^p \times \mathbb{R}$ as

$$\mathcal{B} := \big\{(v,t) : v \in B_p(M), \ \ t \in [0, \epsilon^{-1}T] \ \text{ and } \ S^t v \in \Omega(\delta, m, K)\big\}$$

♯ This specifies the usual definition of a quasi-invariant measure. We recall that if a flow $\{S_t\}$ of some equation exists for all $t \geq 0$ (or for all $t \in \mathbb{R}$), then a measure m is called quasi-invariant for this equation if the measures $S_t \circ m$ are absolutely continuous with respect to m for all $t \geq 0$ (respectively for all $t \in \mathbb{R}$).

(see (4.15)), and consider the measure $\boldsymbol{\mu}$ on $h^p \times \mathbb{R}$, where $d\boldsymbol{\mu} = d\mu dt$. Then by (4.17) we have

$$\boldsymbol{\mu}(\mathcal{B}) = \int_0^{\epsilon^{-1}T} \mu\Big(B_p(M) \cap S^{-t}\big(\Omega(\delta, m, K)\big)\Big) dt \leqslant \epsilon^{-1} T e^{C(T,M)} \mu(\Omega(\delta, m, K)).$$

For any $v \in B_p(M)$, define $Res(v)$ as the set of resonant instants of time for a trajectory, which starts from v:

$$Res(v) = \{\tau \in [0, \epsilon^{-1}T] : \quad S^t(v) \in \Omega(\delta, m, K)\}.$$

Then

$$\boldsymbol{\mu}(\mathcal{B}) = \int_{B_p(M)} mes\big(Res(v)\big) d\mu(v)$$

by the Fubini theorem, where $mes(\cdot)$ stands for the Lebesgue measure on $[0, \epsilon^{-1}T]$. If for $\rho > 0$ we denote

$$\mathcal{V}\mathrm{Res}(\rho) := \{v \in B_p(M) : mes\,(Res(v)) > \epsilon^{-1}\rho\}\,,$$

then in view of the Chebyshev inequality we have

$$\mu(\mathcal{V}\mathrm{Res}(\rho)) \leqslant \frac{\epsilon}{\rho} \boldsymbol{\mu}(\mathcal{B}) \leqslant \frac{T e^{C(M,T)}}{\rho} \mu(\Omega(\delta, m, K)).$$

By Theorem 3.2 the functions $v \mapsto W_1(I(v))k_1 + \ldots + W_m(I(v))k_m$, $k \in \mathbb{Z}^m \setminus \{0\}$, are analytic on h^p. Since they do not vanish identically by Lemma 3.11 and the measure μ is regular, then $\mu(\Omega(0, m, K)) = 0$. Accordingly $\mu(\Omega(\delta, m, K))$ goes to zero with δ, and

$$\mu(\mathcal{V}\mathrm{Res}(\rho)) \to 0 \quad \text{as} \quad \delta \to 0, \tag{4.18}$$

for any ρ. This gives us a sought for analogy of Lemma 4.5 for deterministic perturbations of KdV which have ϵ-quasi-invariant measures.

The averaged equation for actions, corresponding to (4.16), reads

$$\frac{dJ_k}{d\tau} = \langle F_k \rangle (J), \quad k = 1, 2, \ldots, \tag{4.19}$$

where $F_k = \big(d\Psi_k(\Psi^{-1}(v))(f(\Psi^{-1}(v)(\cdot)), \mathbf{v}_k\big)$ (cf. (4.6)). Due to item (i) of Assumption A, the r.h.s. of (4.19) defines a Lipschitz vector filed on h_I^p, so the averaged equation is well posed locally on h_I^p. We denote by $J_{I_0}(\tau)$ a solution of (4.19) with an initial data $J_{I_0}(0) = I_0 \in h_I^p$. It is shown in [29, 30] that relation (4.18) and the well-posedness of the averaged equation jointly allow to establish an averaging theorem for equation (4.16), which has an ϵ-quasi-invariant measure.

In the statement below $u^\epsilon(t)$ stand for solutions of equation (4.16) and $v^\epsilon(\tau)$ – for these solutions, written using the v-variables and slow time $\tau = \epsilon t$. By Assumption A, for $\tau \in [0, T]$ we have $|I(v^\epsilon(\tau))|_p^\sim \leqslant C_1\big(T, |I(v^\epsilon(0))|_p^\sim\big)$. Denote

$$\tilde{T}(I_0) := \min\{\tau \in \mathbb{R}_+ : |J_{I_0}(\tau)|_p^\sim \geqslant C_1(T, |I_0|_p^\sim) + 1\}.$$

Theorem 4.7 *Fix some $M > 0$. Suppose that Assumption A holds and equation (4.16) has an ϵ-quasi-invariant measure μ on h^p such that $\mu(B_p(M)) > 0$. Then*

(i) For any $\rho > 0$ and any $q < p - \frac{1}{2}\max\{\zeta_0, -1\}$, there exist $\epsilon_{\rho,q} > 0$ and a Borel subset $\Gamma^\epsilon_{\rho,q} \subset B_p(M)$, satisfying $\lim_{\epsilon\to 0} \mu(B_p(M) \setminus \Gamma^\epsilon_{\rho,q}) = 0$, with the following property: if $\epsilon \leqslant \epsilon_{\rho,q}$ and $v^\epsilon(0) \in \Gamma^\epsilon_{\rho,q}$, then

$$|I(v^\epsilon(\tau)) - J_{I_0^\epsilon}(\tau)|_q^\sim \leqslant \rho \quad \textit{for} \quad 0 \leqslant \tau \leqslant \min\{T, \tilde{T}(I_0^\epsilon)\}, \tag{4.20}$$

where $I_0^\epsilon = I(v^\epsilon(0))$.

(ii) Let $\lambda_\epsilon^{v_0}$ be a probability measure on $\mathbb{T}^\infty$, defined by the relation

$$\int_{\mathbb{T}^\infty} f(\varphi)\, d\lambda_\epsilon^{v_0}(d\varphi) = \frac{1}{T}\int_0^T f(\varphi(v^\epsilon(\tau))d\tau, \quad \forall f \in C(\mathbb{T}^\infty),$$

where $v_0 := v^\epsilon(0) \in B_p(M)$. Then the measure $\mu\big(B_p(M)\big)^{-1}\int_{B_p(M)} \lambda_\epsilon^{v_0} d\mu(v_0)$ converges weakly, as $\epsilon \to 0$ to the Haar measure $d\varphi$ on $\mathbb{T}^\infty$††

Proposition 4.8 *If Assumption A holds with $\zeta_0 < 0$, then for $\rho < 1$, $p \leqslant q$ and $\epsilon \leqslant \epsilon_{\rho,q}$, we have $\tilde{T} := \tilde{T}\big(I(v^\epsilon(0))\big) > T$ for $v^\epsilon(0) \in \Gamma^\epsilon_{\rho,q}$. So (4.20) holds for $0 \le \tau \le T$.*

Proof: Assume that $\tilde{T} \leqslant T$. By (4.20) for $0 \le \tau \le \tilde{T}$ we have $|I^\epsilon(\tau) - J_{I_0^\epsilon}(\tau)|_p^\sim \leqslant \rho$. Therefore $|J_{I_0^\epsilon}(\tau)|_p^\sim \leqslant C_1(T, |I_0^\epsilon|_p^\sim) + \rho < C_1(T, |I_0^\epsilon|_p^\sim) + 1$. This contradicts the definition of $\tilde{T}$, so $\tilde{T} > T$. $\square$

Remark. Assume that ϵ-quasi-invariant measure μ depends on ϵ, i.e., $\mu = \mu_\epsilon$, and

a) $\mu_\epsilon(\Omega(\delta, m, K))$ goes to zero with δ uniformly in ϵ,

b) the constants $C(T, M)$ in (4.17) are bounded uniformly in ϵ.

Then assertion (i) holds with μ replaced by μ_ϵ. For assertion (ii) to hold, more restrictions should be imposed, see [30].

Theorem 4.7 gives rise to the questions:

Problem 4.9 *Does a version of the averaging theorem above hold without assuming the existence of an ϵ-quasi-invariant measure?*

Problem 4.10 *Which equations (4.16) have ϵ-quasi-invariant measures?*

See the next subsection for some results in this direction.

Problem 4.11 *Find an averaging theorem for equations (4.16), where the nonlinearity defines an unbounded operator, i.e. in Assumption A(i) we have $\zeta_0 > 0$ (note that in the equation from Example in Section 4.1 we have $\zeta_0 = 1$, and in equation (4.4) $\zeta_0 = 2$).*

It is unlikely that the assertion of Theorem 4.7 holds for all initial data, and we believe that the phenomenon of capture in resonance happens for some solutions of (4.16):

Problem 4.12 *Prove that (4.20) does not hold for some solutions of (4.16).*

4.5. Existence of ϵ-quasi-invariant measures

Clearly every regular measure, invariant for equation (4.16), is ϵ-quasi-invariant. Gibbs measures for some equations of the KdV type are regular and invariant. They were studied by a number of different people (e.g., see [14, 76]). However, for generic hamiltonian perturbations of KdV it is difficult – probably impossible – to construct invariant measures in higher order Sobolev spaces due to the lack of high order conservation laws. Below, we give some examples of ϵ-quasi-invariant measures for smoothing perturbations of KdV, which are Gibbs measures of KdV (so they are KdV-invariant). Note that some of these perturbed equations do not have non-trivial

††In [29] a stronger assertion was claimed. Namely, that the measure $\lambda_\epsilon^{v_0}$ converges to $d\varphi$ for μ-a.a. v_0 in $B_p(M)$. Unfortunately, the proof in [29] contains a gap which we still cannot fix.

invariant measures. For example, our argument applies to equation which in the v-variables read as

$$\dot{\mathbf{v}}_j = \mathbb{J}W_j(I)\mathbf{v}_j - \epsilon j^{-\rho}\mathbf{v}_j\,, \qquad j \in \mathbb{N},$$

where $\rho > 1$. Since all trajectories of this equation converge to zero, then its only invariant measure is the δ-measure at the origin. That is, for averaging in the perturbed KdV equations (4.16) (various) Gibbs measures of KdV are universal objects which play a role, similar to that of the Lebesgue for the classical averaging.

Definition 4.13 *For any $\zeta_0' < -1$, a Gaussian measure μ_0 on the Hilbert space h^p is called ζ_0'-admissible if it has zero mean value and a diagonal correlation operator $(\mathbf{v}_1,\dots) \mapsto (\sigma_1\mathbf{v}_1,\dots)$, where $0 < j^{\zeta_0'}/\sigma_j \leqslant Const$ for each j.*

For any $\zeta_0' < -1$, a ζ_0'-admissible measure μ_0 is a well-defined probability measure on h^p, which can be formally written as

$$\mu_0 = \prod_{j=1}^{\infty} \frac{(2\pi j)^{1+2p}}{2\pi\sigma_j} \exp\{-\frac{(2\pi j)^{1+2p}|\mathbf{v}_j|^2}{2\sigma_j}\}d\mathbf{v}_j, \tag{4.21}$$

where $d\mathbf{v}_j$, $j \geqslant 1$, is the Lebesgue measure on $\mathbb{R}^2_{\mathbf{v}_j}$. It is known that (4.21) is a well-defined measure on h^p, if and only if, $\sum \sigma_j < \infty$ (see [11]). It is regular and non-degenerate in the sense that its support equals h^p (see [11, 12]). Writing KdV in the v-variables, we see that μ_0 is invariant under the KdV flow. Note that μ_0 is a Gibbs measure for KdV (written in the form (3.7)) since it may be formally written as $\mu_0 = Z^{-1}\exp\{-\langle Qv, v\rangle\}dv$, where $\langle Qv, v\rangle = \sum c_j|\mathbf{v}_j|^2$ is an integrals of motion for KdV (the statistical sum $Z = \infty$, so this is indeed a formal expression).

For a perturbed KdV (4.16) we define $\mathcal{P}(v) = d\Psi(u)(f(u))$, where $u = \Psi^{-1}(v)$. A non-complicated calculation (see in [29]) shows that:

Theorem 4.14 *If Assumption A holds and*
$(i)'$ the operator $\mathcal{P}$ analytically maps the space h^p to $h^{p-\zeta_0'}$ with some $\zeta_0' < -1$,
then every ζ_0'-admissible Gaussian measure on h^p is ϵ-quasi-invariant for equation (4.16) on the space h^p.

However, due to the complexity of the nonlinear Fourier transform Ψ, it is not easy to verify the condition $(i)'$ of Theorem 4.14 for specific equation (4.16). Now we will give other examples of ϵ-quasi-invariant measures on the space H^p, by strengthening the restrictions in Assumption A. Suppose that there $p \in \mathbb{N}$. Let μ_p be the centred Gaussian measure on H^p with the correlation operator $\triangle^{-1}$. Since $\triangle^{-1}$ is an operator of the trace type, then μ_p is a well-defined probability measure on H^p.

We recall (see Remark 3.3) that KdV has infinitely many conservation laws $\mathcal{J}_n(u)$, $n \geqslant 0$, of the form $\mathcal{J}_n = \frac{1}{2}||u||_n^2 + J_{n-1}(u)$, where $J_{-1}(u) = 0$ and for $n \geqslant 1$,

$$J_{n-1}(u) = \int_{\mathbb{T}} \big\{c_n u(\partial_x^{n-1}u)^2 + \mathcal{Q}_n(u,\dots,\partial_x^{n-2}u)\big\}dx\,. \tag{4.22}$$

Here c_n are real constants and $\mathcal{Q}_n$ are polynomial in their arguments. From (4.22), we know that the functional J_p is bounded on bounded sets in H^p. We consider a Gibbs measure η_p for KdV, defined by its density against μ_p,

$$\eta_p(du) = e^{-J_p(u)}\mu_p(du).$$

It is regular and non-degenerated in the sense that its support contains the whole space H^p (see [11]). Moreover, it is invariant for KdV [76]. The following theorem was shown in [30]:

Theorem 4.15 *If Assumption A holds with $\zeta_0 \leqslant -2$, then the measure η_p is ϵ-quasi-invariant for perturbed KdV (4.16) on the space H^p.*

Corollary 4.16 *If $\zeta_0 \leqslant -2$, then the assertions of Theorem 4.7 hold with $\mu = \eta_p$, and we have $\tilde{T} > T$.*

In particular, this corollary applies to the equation

$$\dot{u} + u_{xxx} - 6uu_x = \epsilon f(x), \quad x \in \mathbb{T}, \ u \in H^p,$$

where $f(x)$ is a smooth function with zero mean-value. This equation may be viewed as a model for shallow water wave propagation under small external force. Note that the KAM-Theorem 4.1 also applies to it.

Problem 4.17 *Besides the class of ζ_0'-admissible Gaussian measures and Gibbs measure η_p, there are many other KdV-invariant measures. How to check if a measure like that is ϵ-quasi-invariant for a given ϵ-perturbation of KdV?*

4.6. Nekhoroshev type results (long-time stability)?

In the finite dimensional case, the strict convexity of the unperturbed integrable Hamiltonian assures the long-time stability of solutions for perturbed hamiltonian equations ([68, 58, 71]). Theorem 3.13 tells us that the KdV Hamiltonian $H_K(I)$ is convex in l^2 and hints that it is strictly convex (at least) in a neighbourhood of the origin in l^2. This suggests that an Nekhoroshev type stability may hold for perturbed KdV under hamiltonian perturbations (see equation (4.1)), at least for initial data in a neighbourhood of the origin, where the strict convexity should hold. But at the moment of writing no exact statement is available.

There are several *ad hoc* quasi-Nekhoroshev theorems for hamiltonian PDEs, see [6, 16] and references therein. However, these results only apply in a small neighbourhood of the origin of the size of the perturbation. Nonetheless, we believe that the corresponding technique and the results in Theorem 3.13 will lead to results on long-time stability (at least in time interval of order $\epsilon^{-p}, p \geq 1$,) for some solutions of perturbed KdV (4.1). Note that Theorem 4.7 and Corollary 4.16 imply such a stability for $p = 1$ and for typical initial data, if the perturbation f is smoothing. Stability on time-intervals of order ϵ^{-2} seems to be a much harder question.

Problem 4.18 *Is it possible to prove a long-time stability result for perturbed KdV under hamiltonian perturbations, e.g. for equation (4.1), which holds for all 'smooth' initial data?*

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