

ON LONG TIME DYNAMICS OF PERTURBED KDV EQUATIONS

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ABSTRACT. Consider a perturbed KdV equation:

$$u_t + u_{xxx} - 6uu_x = \epsilon f(u(\cdot)), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_{\mathbb{T}} u(x, t) dx = 0,$$

where the nonlinear perturbation defines analytic operators $u(\cdot) \mapsto f(u(\cdot))$ in sufficiently smooth Sobolev spaces. Assume that the equation has an ϵ -quasi-invariant measure μ and satisfies some additional mild assumptions. Let $u^\epsilon(t)$ be a solution. Then on time intervals of order ϵ^{-1} , as $\epsilon \rightarrow 0$, its actions $I(u^\epsilon(t, \cdot))$ can be approximated by solutions of a certain well-posed averaged equation, provided that the initial datum is μ -typical.

0. INTRODUCTION

The KdV equation on the circle, perturbed by smoothing perturbations, was studied in [7]. There an averaging theorem that describes the long-time behavior for solutions of the perturbed KdV equation was proved. In this work, we suggest an abstract theorem which applies to a large class of ϵ -perturbed KdV equations which have ϵ -quasi-invariant measures; the latter notion is explained in the main text. We show that the systems considered in [7], satisfy this condition, and believe that it may be verified for many other perturbations of KdV. More exactly, we consider a perturbed KdV equation with zero mean-value periodic boundary condition:

$$u_t + u_{xxx} - 6uu_x = \epsilon f(u(\cdot)), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_{\mathbb{T}} u(x, t) dx = 0, \quad (0.1)$$

where ϵf is a nonlinear perturbation, specified below. For any $p \in \mathbb{R}$ we introduce the Sobolev space of real valued function on \mathbb{T} with zero mean-value:

$$H^p = \left\{ u \in L^2(\mathbb{T}, \mathbb{R}) : \|u\|_p < +\infty, \int_{\mathbb{T}} u dx = 0 \right\}, \quad \|u\|_p^2 = \sum_{k \in \mathbb{N}} (2\pi k)^{2p} (|\hat{u}_k|^2 + \hat{u}_{-k}^2).$$

Here \hat{u}_k and \hat{u}_{-k} , $k \in \mathbb{N}$ are the Fourier coefficients of u with respect to the trigonometric base

$$e_k = \sqrt{2} \cos 2\pi k x, \quad k > 0 \quad \text{and} \quad e_k = \sqrt{2} \sin 2\pi k x, \quad k < 0,$$

i.e. $u = \sum_{k \in \mathbb{N}} \hat{u}_k e_k + \hat{u}_{-k} e_{-k}$. It is well known that KdV is integrable. It means that the function space H^p admits analytic coordinates $v = (\mathbf{v}_1, \mathbf{v}_2, \dots) = \Psi(u(\cdot))$, where $\mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2$, such that the quantities $I_j = \frac{1}{2} |\mathbf{v}_j|^2$ and $\varphi_j = \text{Arg } \mathbf{v}_j$, $j \geq 1$, are action-angle variables for KdV. In the (I, φ) -variables, KdV takes the integrable form

$$\dot{I} = 0, \quad \dot{\varphi} = W(I), \quad (0.2)$$

where $W(I) \in \mathbb{R}^\infty$ is the KdV frequency vector, see [9, 8]. For any $p \geq 0$ the integrating transformation Ψ (the nonlinear Fourier transform) defines an analytic diffeomorphism $\Psi : H^p \rightarrow h^p$, where

$$h^p = \left\{ v = (\mathbf{v}_1, \mathbf{v}_2, \dots) : |v|_p^2 = \sum_{j=1}^{\infty} (2\pi j)^{2p+1} |\mathbf{v}_j|^2 < \infty, \mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2 \right\}.$$

We introduce the weighted l^1 -space h_I^p ,

$$h_I^p = \left\{ I = (I_1, I_2, \dots) \in \mathbb{R}^\infty : |I|_p^\sim = 2 \sum_{j=1}^{+\infty} (2\pi j)^{2p+1} |I_j| < \infty \right\},$$

and the mapping π_I :

$$\pi_I : h^p \rightarrow h_I^p, \quad (\mathbf{v}_1, \dots) \mapsto (I_1, \dots), \quad I_j = \frac{1}{2} \mathbf{v}_j^t \mathbf{v}_j, \quad j \in \mathbb{N}. \quad (0.3)$$

Obviously, π_I is continuous, $|\pi_I(v)|_p^\sim = |v|_p^2$ and its image $h_{I+}^p = \pi_I(h^p)$ is the positive octant of h_I^p .

We wish to study the long-time behavior of solutions for equation (0.1). Accordingly, we fix some

$$\zeta_0 \geq 0, \quad p \geq 3, \quad T > 0,$$

and assume

Assumption A:

(i) For any $u_0 \in H^p$, there exists a unique solution $u(t) \in C([0, \epsilon^{-1}T], H^p)$ of (0.1) with $u(0) = u_0$. It satisfies

$$\|u\|_p \leq C(T, p, \|u_0\|_p), \quad 0 \leq t \leq T\epsilon^{-1}.$$

(ii) There exists a $p' = p'(p) < p$ such that for $q \in [p', p]$, the perturbation term defines an analytic mapping

$$H^q \rightarrow H^{q+\zeta_0}, \quad u(\cdot) \mapsto f(u(\cdot)).$$

We are mainly concerned with the behavior of the actions $I(u(t)) \in \mathbb{R}_+^\infty$ on time interval $[0, T\epsilon^{-1}]$. For this end, we write the perturbed KdV (0.1), using slow time $\tau = \epsilon t$ and the v -variables:

$$\frac{dv}{d\tau} = \epsilon^{-1} d\Psi(u)V(u) + P(v). \quad (0.4)$$

Here $V(u) = -u_{xxx} + 6uu_x$ is the vector field of KdV and $P(v)$ is the perturbation term, written in the v -variables. In the action-angle variables (I, φ) this equation reads:

$$\frac{dI}{d\tau} = F(I, \varphi), \quad \frac{d\varphi}{d\tau} = \epsilon^{-1} W(I) + G(I, \varphi). \quad (0.5)$$

Here $I \in \mathbb{R}^\infty$ and $\varphi \in \mathbb{T}^\infty$, where $\mathbb{T}^\infty := \{\theta = (\theta_j)_{j \geq 1}, \theta_j \in \mathbb{T}\}$ is the infinite-dimensional torus, endowed with the Tikhonov topology. The two functions $F(I, \varphi)$ and $G(I, \varphi)$ represent the perturbation term f , written in the action-angle variables, see below (1.3) and (1.4). We consider an averaged equation for the actions:

$$\frac{dJ}{d\tau} = \langle F \rangle(J), \quad \langle F \rangle(J) = \int_{\mathbb{T}^\infty} F(J, \varphi) d\varphi, \quad (0.6)$$

where $d\varphi$ is the Haar measure on \mathbb{T}^∞ . It turns out that $\langle F \rangle(J)$ defines a Lipschitz vector field in h^p_I (see (3.11) below). So equation (0.6) is well-posed, at least locally. We want to study the relation between the actions $I(\tau)$ of solutions for equation (0.5) and solutions $J(\tau)$ of equation (0.6), for $\tau \in [0, T]$.

Let S_ϵ^τ , $0 \leq \tau \leq T$, be the flow-maps of equation (0.4) on h^p and denote

$$B_p^v(M) = \{v \in h^p : |v|_p \leq M\}.$$

Definition 0.1. 1) A measure μ on h^p is called regular if for any analytic function g on h^p such that $g \not\equiv 0$, we have $\mu(\{v \in h^p : g(v) = 0\}) = 0$.

2) A measure μ on h^p is said to be ϵ -quasi-invariant for equation (0.4) on the ball $B_p^v(M)$ if it is regular, $0 < \mu(B_p^v(M)) < \infty$ and there exists a constant $C(T, M)$ such that for any Borel set $A \subset B_p^v(M)$, we have

$$e^{-\tau C(T, M)} \mu(A) \leq \mu(S_\epsilon^\tau(A)) \leq e^{\tau C(T, M)} \mu(A). \quad (0.7)$$

Similarly, these definitions can be carried to measures on the space H^p and the flow-maps of equation (0.1) on H^p .

The ϵ -quasi-invariant measure is needed to guarantee that a small 'bad' set which we have to prohibit for a solution of the perturbed equation at a time $t > 0$ corresponds to a small set of initial data.

The main result of this paper is the following theorem, in which $u^\epsilon(t)$ denotes solutions for equation (0.1), $v^\epsilon(\tau) = \Psi(u^\epsilon(\epsilon^{-1}\tau))$ denotes solutions for (0.4) and $I(v^\epsilon)$, $\varphi(v^\epsilon)$ are their action-angle variables. By Assumption A, for $\tau \in [0, T]$,

$$|I(v^\epsilon(\tau))|_p \lesssim C_1(|I(v^\epsilon(0))|_p).$$

Theorem 0.2. Fix any $M > 0$. Suppose that assumption A holds and equation (0.1) has an ϵ -quasi-invariant measure μ on $B_p^v(M)$. Then

(i) For any $\rho > 0$ and any $q < p + \frac{1}{2} \min\{\zeta_0, 1\}$, there exists $\delta_\rho > 0$, $\epsilon_{\rho, q} > 0$ and a Borel subset $\Gamma_{\rho, q}^\epsilon \subset B_p^v(M)$ such that

$$\lim_{\epsilon \rightarrow 0} \mu(B_p^v(M) \setminus \Gamma_{\rho, q}^\epsilon) = 0, \quad (0.8)$$

and for $\epsilon \leq \epsilon_{\rho, q}$, we have that if $v^\epsilon(0) \in \Gamma_{\rho, q}^\epsilon$, then

$$|I(v^\epsilon(\tau)) - J(\tau)|_q \lesssim \rho, \quad \text{for } 0 \leq \tau \leq \min\{T, \mathcal{T}(I_0^\epsilon)\}. \quad (0.9)$$

Here $I_0^\epsilon = I(v^\epsilon(0))$, $J(\cdot)$ is the unique solution of the averaged equation (0.6) with any initial data $J_0 \in h^p$, satisfying $|J_0 - I_0^\epsilon|_q \lesssim \delta_\rho$, and

$$\mathcal{T}(I_0^\epsilon) = \min \{ \tau : |J(\tau)|_p \gtrsim C_1(|I_0^\epsilon|_p) + 1 \}.$$

(ii) Let $\lambda_\epsilon^{v_0}$ be the probability measure on \mathbb{T}^∞ , defined by the relation

$$\int_{\mathbb{T}^\infty} f(\varphi) d\lambda_\epsilon^{v_0}(d\varphi) = \frac{1}{T} \int_0^T f(\varphi(v^\epsilon(\tau))) d\tau, \quad \forall f \in C(\mathbb{T}^\infty),$$

where $v_0 = v^\epsilon(0) \in B_p(M)$. Then the averaged measure

$$\lambda_\epsilon := \frac{1}{\mu(B_p(M))} \int_{B_p(M)} \lambda_\epsilon^{v_0} d\mu(v_0)$$

converges weakly, as $\epsilon \rightarrow 0$, to the Haar measure $d\varphi$ on \mathbb{T}^∞

Remark 0.3. 1) Assume that an ϵ -quasi-invariant measure μ depends on ϵ , i.e. $\mu = \mu_\epsilon$. Then the same conclusion holds with μ replaced by μ_ϵ , if μ_ϵ satisfies some consistency conditions. See Subsection 3.3.

2) Item (ii) of Assumption A is not needed if the perturbation is hamiltonian. See the end of Subsection 3.1.

Toward the existence of ϵ -quasi-invariant measures, following [7], consider a class of Gaussian measures μ_0 on the Hilbert space h^p :

$$\mu_0 := \prod_{j=1}^{\infty} \frac{(2\pi j)^{1+2p}}{2\pi\sigma_j} \exp\left\{-\frac{(2\pi j)^{1+2p}|\mathbf{v}_j|^2}{2\sigma_j}\right\} d\mathbf{v}_j, \quad (0.10)$$

where $d\mathbf{v}_j$, $j \geq 1$, is the Lebesgue measure on \mathbb{R}^2 . We recall that (0.10) is a well-defined probability measure on h^p if and only if $\sum \sigma_j < \infty$ (see [2]). It is regular in the sense of Definition 0.1 and is non-degenerated in the sense that its support equals to h^p (see [2, 3]). From (0.2), it is easy to see that this kind of measures are invariant for KdV.

For any $\zeta'_0 > 1$, we say the measure μ_0 is ζ'_0 -admissible if the σ_j in (0.10) satisfies $0 < j^{-\zeta'_0}/\sigma_j < \text{const}$ for all $j \in \mathbb{N}$. It was proved in [7] that if Assumption A holds and

(ii)' there exists $\zeta'_0 > 1$ such that the operator defined by $h^p \rightarrow h^{p+\zeta'_0}$, $v \mapsto P(v)$ (see (0.4)) is analytic,

then every ζ'_0 -admissible measure μ_0 is ϵ -quasi-invariant for equation (0.1) on h^p .

However, the conditions (ii)' is not easy to verify due to the complexity of the nonlinear Fourier transform. Fortunately, there exists another series of Gibbs-type measures (see (4.3) below) known to be invariant for KdV, explicitly constructed on the space H^p in [15]. We will show in Section 4 that these measures are ϵ -quasi-invariant for equation (0.1) if Assumption A holds with $\zeta_0 \geq 2$. We point out straight away that this condition is not optimal (see Remark 4.11).

Some recent results (see, e.g. [15, 4, 5]) show that the Gibbs measure is an efficient tool to study nonlinear partial differential equations. There are mainly two kinds of applications: the recurrent properties given by the Poincaré Recurrent Theorem and the almost sure (in the sense of the Gibbs measure) global well-posedness for 'rough' initial data. Here we give a new application of the Gibbs measure. We show that in the averaging theory for perturbed KdV, the Gibbs measure plays a role which the Lebesgue measure plays in the classic finite dimensional averaging theory. This indicates that the Gibbs measure is important not only for the study of the original PDEs but also for the study of its perturbations.

The paper is organized as follows: Section 1 is about some important properties of the nonlinear Fourier transform and the action-angle form of the perturbed KdV (0.1). We discuss the averaged equation in Section 2. The Theorem 0.2 is proved in Section 3. Finally we will discuss the existence of ϵ -quasi-invariant measures in Section 4.

Agreements. Analyticity of maps $B_1 \rightarrow B_2$ between Banach spaces B_1 and B_2 , which are the real parts of complex spaces $B_1^{\mathbb{C}}$ and $B_2^{\mathbb{C}}$, is understood in the sense of Fréchet. All analytic maps that we consider possess the following additional property: for any R , a map extends to a bounded analytical mapping in a complex ($\delta_R > 0$)-neighborhood of the ball $\{|u|_{B_1} < R\}$ in $B_1^{\mathbb{C}}$. We call such analytic maps uniformly analytic.

1. PRELIMINARIES ON THE KdV EQUATION

In this section we discuss integrability of the KdV equation (0.1) $_{\epsilon=0}$.

1.1. Nonlinear Fourier transform for KdV.

Theorem 1.1. (see [9]) *There exists an analytic diffeomorphism $\Psi : H^0 \mapsto h^0$ and an analytic functional K on h^1 of the form $K(v) = \tilde{K}(I(v))$, where the function $\tilde{K}(I)$ is analytic in a suitable neighborhood of the octant h_{I+}^1 in h_I^1 , with the following properties:*

(i) *For any $p \in [-1, +\infty)$, the mapping Ψ defines an analytic diffeomorphism $\Psi : H^p \mapsto h^p$.*

(ii) *The differential $d\Psi(0)$ is the operator $\sum u_s e_s \mapsto v, v_s = |2\pi s|^{-1/2} u_s$.*

(iii) *A curve $u \in C^1(0, T; H^0)$ is a solution of the KdV equation (0.1) $_{\epsilon=0}$ if and only if $v(t) = \Psi(u(t))$ satisfies the equation*

$$\dot{\mathbf{v}}_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \tilde{K}}{\partial I_j}(I) \mathbf{v}_j, \quad \mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2, \quad j \in \mathbb{N}. \quad (1.1)$$

The coordinates $v = \Psi(u)$ are called the *Birkhoff coordinates*, and the form (1.1) of KdV is its *Birkhoff normal form*

Since the maps Ψ and Ψ^{-1} are analytic, then for $m = 0, 1, 2, \dots$, we have

$$\|d^j \Psi(u)\|_m \leq P_m(\|u\|_m), \quad \|d^j \Psi^{-1}(v)\|_m \leq Q_m(\|v\|_m), \quad j = 0, 1, 2,$$

where P_m and Q_m are continuous functions.

A remarkable property of the nonlinear Fourier transform Ψ is its quasi-linearity. It means:

Theorem 1.2. (see [12, 10]) *If $p \geq 0$, then the map $\Psi - d\Psi(0) : H^p \rightarrow h^{p+1}$ is analytic.*

We denote

$$W(I) = (W_1, W_2, \dots), \quad W_k(I) = \frac{\partial \tilde{K}}{\partial I_k}(I), \quad k = 1, 2, \dots$$

This is the KdV-frequency map. It is non-degenerate:

Lemma 1.3. (see [11], appendix 6) *For any $n \in \mathbb{N}$, if $I_{n+1} = I_{n+2} = \dots = 0$, then*

$$\det\left(\left(\frac{\partial W_i}{\partial I_j}\right)_{1 \leq i, j \leq n}\right) \neq 0.$$

Let l_{-1}^∞ be the Banach space of all real sequences $l = (l_1, l_2, \dots)$ with the norm

$$\|l\|_{-1} = \sup_{n \geq 1} n^{-1} |l_n| < \infty.$$

Denote $\kappa = (\kappa_n)_{n \geq 1}$, where $\kappa_n = (2\pi n)^3$.

Lemma 1.4. (see [9], Theorem 15.4) *The normalized frequency map*

$$I \mapsto W(I) - \kappa$$

is real analytic as a map from h_{I+}^1 to l_{-1}^∞ .

1.2. Equation (0.1) in the Birkhoff coordinates. For $k = 1, 2, \dots$ we denote:

$$\Psi_k : H^m \rightarrow \mathbb{R}^2, \quad \Psi_k(u) = \mathbf{v}_k,$$

where $\Psi(u) = v = (\mathbf{v}_1, \mathbf{v}_2, \dots)$. Let $u(t)$ be a solution of equation (0.1). Passing to the slow time $\tau = \epsilon t$ and denoting $\dot{\cdot}$ to be $\frac{d}{d\tau}$, we get

$$\dot{\mathbf{v}}_k = d\Psi_k(u)(\epsilon^{-1}V(u)) + \mathcal{P}_k(v), \quad k \geq 1, \quad (1.2)$$

where $V(u) = -u_{xxx} + 6uu_x$ and $\mathcal{P}_k(v) = d\Psi_k(\Psi^{-1}(v))(f(\Psi^{-1}(v)))$. Since the action $I_k(v) = \frac{1}{2}|\Psi_k|^2$ is an integral of motion for the KdV equation (0.1) $_{\epsilon=0}$, we have

$$\dot{I}_k = (\mathcal{P}_k(v), \mathbf{v}_k) := F_k(v). \quad (1.3)$$

Here and below (\cdot, \cdot) indicates the scalar product in \mathbb{R}^2 .

For $k \geq 1$ defines the angle $\varphi_k = \arctan(\frac{v_{-k}}{v_k})$ if $\mathbf{v}_k \neq 0$ and $\varphi_k = 0$ if $\mathbf{v}_k = 0$. Using equation (1.2), we get

$$\dot{\varphi}_k = \epsilon^{-1}W_k(I) + |\mathbf{v}_k|^{-2}(d\Psi_k(u)f(x, u), \mathbf{v}_k^\perp), \quad \text{if } \mathbf{v}_k \neq 0, \quad (1.4)$$

where $\mathbf{v}_k^\perp = (-v_{-k}, v_k)$. Denoting for brevity, the vector field in equation (1.4) by $\epsilon^{-1}W_k(I) + G_k(v)$, we rewrite the equation for the pair $(I_k, \varphi_k)(k \geq 1)$ as

$$\begin{aligned} \dot{I}_k &= F_k(v) = F_k(I, \varphi), \\ \dot{\varphi}_k &= \epsilon^{-1}W_k(I) + G_k(v). \end{aligned} \quad (1.5)$$

We set

$$F(I, \varphi) = (F_1(I, \varphi), F_2(I, \varphi), \dots).$$

Denote

$$\bar{\zeta}_0 = \min\{1, \zeta_0\}.$$

For any $q \in [p', p]$, define a map \mathcal{P} as

$$\mathcal{P} : h^q \rightarrow h^{q+\bar{\zeta}_0}, \quad v \mapsto (\mathcal{P}_1(v), \dots).$$

Clearly, $\mathcal{P}(v) = d\Psi(\Psi^{-1}(v))(f(\Psi^{-1}(v)))$. Then Theorem 1.2 and Assumption A imply that the map \mathcal{P} is analytic. Using (1.3), for any $k \in \mathbb{N}$, we have

$$(2\pi k)^{2q+1+\bar{\zeta}_0}|F_k(v)| \leq (2\pi k)^{2q+1}|\mathbf{v}_k|^2 + (2\pi k)^{2q+1+2\bar{\zeta}_0}|\mathcal{P}_k(v)|^2.$$

Therefore,

$$|F(I, \varphi)|_{q+\bar{\zeta}_0/2} \lesssim |v|_q^2 + |\mathcal{P}(v)|_{q+\bar{\zeta}_0}^2 \leq C(|v|_q). \quad (1.6)$$

In the following lemma P_k and P_k^j are some fixed continuous functions.

Lemma 1.5. For $k, j \in \mathbb{N}$ and each $q \in [p', p]$, we have:

- (i) The function $F_k(v)$ is analytic in each space h^q .
- (ii) For any $\delta > 0$, the function $G_k(v)\chi_{\{I_k \geq \delta\}}$ is bounded by $\delta^{-1/2}P_k(|v|_q)$.
- (iii) For any $\delta > 0$, the function $\frac{\partial F_k}{\partial I_j}(I, \varphi)\chi_{\{I_j \geq \delta\}}$ is bounded by $\delta^{-1/2}P_k^j(|v|_q)$.
- (iv) The function $\frac{\partial F_k}{\partial \varphi_j}(I, \varphi)$ is bounded by $P_k^j(|v|_q)$, and for any $n \in \mathbb{N}$ and $(I_1, \dots, I_n) \in \mathbb{R}_+^n$, the function $\varphi \mapsto F_k(I_1, \varphi_1, \dots, I_n, \varphi_n, 0, \dots)$ is smooth on \mathbb{T}^n .

We denote

$$\begin{aligned} \Pi_I : h^p &\rightarrow h_I^p, \quad \Pi_I(v) = I(v), \\ \Pi_{I, \varphi} : h^p &\rightarrow h_I^p \times \mathbb{T}^\infty, \quad \Pi_{I, \varphi}(v) = (I(v), \varphi(v)). \end{aligned}$$

Abusing notation, we will identify v with $(I, \varphi) = \Pi_{I, \varphi}(v)$.

Definition 1.6. We say that a curve $(I(\tau), \varphi(\tau))$, $\tau \in [0, \tau_1]$, is a regular solution of equation (1.5), if there exists a solution $u(\cdot) \in H^p$ of equation (0.1) such that

$$\Pi_{I, \varphi}(\Psi(u(\epsilon^{-1}\tau))) = (I(\tau), \varphi(\tau)), \quad \tau \in [0, \tau_1].$$

Note that if $(I(\tau), \varphi(\tau))$ is a regular solution, then each $I_j(\tau)$ is a C^1 -function, while $\varphi_j(\tau)$ may be discontinuous at points τ , where $I_j(\tau) = 0$.

2. AVERAGED EQUATION

For a function f on a Hilbert space H , we write $f \in Lip_{loc}(H)$ if

$$|f(u_1) - f(u_2)| \leq P(R)\|u_1 - u_2\|, \quad \text{if } \|u_1\|, \|u_2\| \leq R, \quad (2.1)$$

for a suitable continuous function P which depends on f . By the Cauchy inequality, any analytic function on H belongs to $Lip_{loc}(H)$ (see Agreements). In particular, for any $k \geq 1$,

$$W_k(I) \in Lip_{loc}(h^q), \quad q \geq 1, \quad \text{and} \quad F_k(v) \in Lip_{loc}(h^q), \quad q \in [p'(p), p]. \quad (2.2)$$

Let $f \in Lip_{loc}(h^{p_0})$ for some $p_0 \geq 0$ and $v \in h^{p_1}$, $p_1 > p_0$. Denoting by Π^M , $M \geq 1$, the projection

$$\Pi^M : h^0 \rightarrow h^0, \quad (\mathbf{v}_1, \mathbf{v}_2, \dots) \mapsto (\mathbf{v}_1, \dots, \mathbf{v}_M, 0, \dots),$$

we have $|v - \Pi^M v|_{p_0} \leq (2\pi M)^{-(p_1 - p_0)}|v|_{p_1}$. Accordingly,

$$|f(v) - f(\Pi^M v)| \leq P(|v|_{p_1})(2\pi M)^{-(p_1 - p_0)}. \quad (2.3)$$

The torus \mathbb{T}^∞ acts on the space h^0 by the linear transformations Φ_θ , $\theta \in \mathbb{T}^\infty$, where $\Phi_\theta : (I, \varphi) \mapsto (I, \varphi + \theta)$. For a function $f \in Lip_{loc}(h^p)$, we define the averaging in angles as

$$\langle f \rangle(v) = \int_{\mathbb{T}^\infty} f(\Phi_\theta(v)) d\theta,$$

where $d\theta$ is the Haar measure on \mathbb{T}^∞ . Clearly, the average $\langle f \rangle$ is independent of φ . Thus $\langle f \rangle$ can be written as $\langle f \rangle(I)$.

Extend the mapping π_I to a complex mapping $h^p \otimes \mathbb{C} \rightarrow h_I^p \otimes \mathbb{C}$, using the same formulas (0.3). Obviously, if \mathcal{O} is a complex neighbourhood of h^p , then $\pi_I^c(\mathcal{O})$ is a complex neighbourhood of h_I^p .

Lemma 2.1. (See [13], Lemma 4.2) Let $f \in Lip_{loc}(h^p)$, then

(i) The function $\langle f \rangle(v)$ satisfy (2.1) with the same function P as f and take the same value at the origin.

(ii) This function is smooth (analytic) if f is. If $f(v)$ is analytic in a complex neighbourhood \mathcal{O} of h^p , then $\langle f \rangle(I)$ is analytic in the complex neighbourhood $\pi_I^c(\mathcal{O})$ of h_I^p .

For any $\bar{q} \in [p', p]$, we consider the mapping defined by

$$\langle F \rangle : h_I^{\bar{q}} \rightarrow h_I^{\bar{q} + \bar{c}_0/2}, \quad J \mapsto \langle F \rangle(J),$$

where $\langle F \rangle(J) = (\langle F_1 \rangle(J), \langle F_2 \rangle(J), \dots)$.

Corollary 2.2. For every $\bar{q} \in [p', p]$, the mapping $\langle F \rangle$ is analytic as a map from the space $h_I^{\bar{q}}$ to $h_I^{\bar{q} + \bar{c}_0/2}$.

Proof. The mapping $\mathcal{P}(v)$ extends analytically to a complex neighbourhood \mathcal{O} of $h^{\bar{q}}$ (see Agreements). Then by (1.3), the functions $F_j(v)$, $j \in \mathbb{N}$ are analytic in \mathcal{O} . Hence it follows from Lemma 2.1 that for each $j \in \mathbb{N}$, the function $\langle F_j \rangle$ is analytic in the complex neighbourhood $\pi_I^c(\mathcal{O})$ of $h_I^{\bar{q}}$. By (1.6), the mapping $\langle F \rangle$ is locally bounded on $\pi_I^c(\mathcal{O})$. It is well known that the analyticity of each coordinate function and the locally boundness of the maps imply the analyticity of the maps (see, e.g. [1]). This finishes the proof of the corollary. \square

We recall that a vector $\omega \in \mathbb{R}^n$ is called *non-resonant* if

$$\omega \cdot k \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}.$$

Denote by $C^{0+1}(\mathbb{T}^n)$ the set of all Lipschitz functions on \mathbb{T}^n . The following lemma is a version of the classical Weyl theorem (for a proof, see e.g. Lemma 2.2 in [7]).

Lemma 2.3. *Let $f \in C^{0+1}(\mathbb{T}^n)$ for some $n \in \mathbb{N}$. For any $\delta > 0$ and any non-resonant vector $\omega \in \mathbb{R}^n$, there exists $T_0 > 0$ such that if $T \geq T_0$, then*

$$\left| \frac{1}{T} \int_0^T f(x_0 + \omega t) dt - \langle f \rangle \right| \leq \delta,$$

uniformly in $x_0 \in \mathbb{T}^n$.

3. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 0.2.

Assume $u(0) = u_0 \in H^p$. So

$$\Pi_{I,\varphi}(\Psi(u_0)) = (I_0, \varphi_0) \in h_{I_+}^p \times \mathbb{T}^\infty. \quad (3.1)$$

We denote

$$B_p^I(M) = \{I \in h_{I_+}^p : |I|_p^\sim \leq M\}.$$

Without loss of generality, we assume that $T = 1$. Fix any $M_0 > 0$. Let

$$(I_0, \varphi_0) \in B_p^I(M_0) \times \mathbb{T}^\infty := \Gamma_0,$$

that is,

$$v_0 = \Psi(u_0) \in B_p^v(\sqrt{M_0}).$$

We pass to the slow time $\tau = \epsilon t$. Let $(I(\cdot), \varphi(\cdot))$ be a regular solution of the system (1.5) with $(I(0), \varphi(0)) = (I_0, \varphi_0)$. We will also write it as $(I^\epsilon(\cdot), \varphi^\epsilon(\cdot))$ when we want to stress the dependence on ϵ . Then by assumption A, there exists $M_1 \geq M_0$ such that

$$I(\tau) \in B_p^I(M_1), \quad \tau \in [0, 1]. \quad (3.2)$$

By (1.6), we know that

$$|F(I, \varphi)|_1^\sim \leq C_{M_1}, \quad \forall (I, \varphi) \in B_p^I(M_1) \times \mathbb{T}^\infty, \quad (3.3)$$

where the constant C_{M_1} depends only on M_1 .

We denote $I^m = (I_1, \dots, I_m, 0, 0, \dots)$, $\varphi^m = (\varphi_1, \dots, \varphi_m, 0, 0, \dots)$, and $W^m(I) = (W_1(I), \dots, W_m(I), 0, 0, \dots)$, for any $m \in \mathbb{N}$.

3.1. **Proof of assertion (i).** Fix any

$$n_0 \in \mathbb{N} \quad \text{and} \quad \rho > 0.$$

By (2.2), there exists $m_0 \in \mathbb{N}$ such that

$$|F_k(I, \varphi) - F_k(I^{m_0}, \varphi^{m_0})| \leq \rho, \quad \forall (I, \varphi) \in B_p^I(M_1) \times \mathbb{T}^\infty, \quad (3.4)$$

where $k = 1, \dots, n_0$.

From now on, we always assume that

$$(I, \varphi) \in \Gamma_1 := B_p^I(M_1) \times \mathbb{T}^\infty, \quad \text{i.e.} \quad v \in B_p^v(\sqrt{M_1}),$$

and identify $v \in h^p$ with $(I, \varphi) = \Pi_{I, \varphi}(v)$.

By Lemma 1.5, we have

$$\begin{aligned} |G_j(I, \varphi)| &\leq \frac{C_0(j, M_1)}{\sqrt{I_j}}, \\ \left| \frac{\partial F_k}{\partial I_j}(I, \varphi) \right| &\leq \frac{C_0(k, j, M_1)}{\sqrt{I_j}}, \\ \left| \frac{\partial F_k}{\partial \varphi_j}(I, \varphi) \right| &\leq C_0(k, j, M_1). \end{aligned} \quad (3.5)$$

From Lemma 1.4 and Lemma 2.1, we know that

$$\begin{aligned} |W_j(I) - W_j(\bar{I})| &\leq C_1(j, M_1)|I - \bar{I}|_1, \\ |\langle F_k \rangle(I) - \langle F_k \rangle(\bar{I})| &\leq C_1(k, j, M_1)|I - \bar{I}|_1. \end{aligned} \quad (3.6)$$

By (2.2) we get

$$|F_k(v^{m_0}) - F_k(\bar{v}^{m_0})| \leq C_2'(k, M_1)|v^{m_0} - \bar{v}^{m_0}|_p \leq C_2(k, m_0, M_1)|v^{m_0} - \bar{v}^{m_0}|_\infty, \quad (3.7)$$

where $|\cdot|_\infty$ is the l^∞ -norm.

We denote

$$C_{M_1}^{n_0, m_0} = m_0 \cdot \max\{C_0, C_1, C_2 : 1 \leq j \leq m_0, 1 \leq k \leq n_0\}.$$

Below we define a number of sets, depending on various parameters. All of them also depend on ρ , n_0 and m_0 , but this dependence is not indicated. For any $\delta > 0$ and $T_0 > 0$, we define a subset

$$E(\delta, T_0) \subset \Gamma_1$$

as the collection of all $(I, \varphi) \in \Gamma_1$ such that for every $T \geq T_0$, we have,

$$\left| \frac{1}{T} \int_0^T [F_k(I^{m_0}, \varphi^{m_0} + W^{m_0}(I)s) - \langle F_k \rangle(I^{m_0})] ds \right| \leq \delta, \quad \text{for } k = 1, \dots, n_0. \quad (3.8)$$

Let $\mathcal{S}_\epsilon^\tau$ be the flow generated by regular solutions of the system (1.5). We define two more groups of sets.

$$\Delta(\tau) = \Delta(\tau, \epsilon, \delta, T_0, I, \varphi) := \{\tau_1 \in [0, \tau] : \mathcal{S}_\epsilon^{\tau_1}(I, \varphi) \notin E(\delta, T_0)\}.$$

$$N(\gamma) = N(\gamma, \epsilon, \delta, T_0) := \{(I, \varphi) \in \Gamma_0 : \text{Mes}[\Delta(1, \epsilon, \delta, T_0, I, \varphi)] \leq \gamma\}.$$

Here and below $\text{Mes}[\cdot]$ stands for the Lebesgue measure in \mathbb{R} . We will indicate the dependence of the set $N(\gamma)$ on n_0 and ρ as $N_{n_0, \rho}(\gamma)$, when necessary.

By continuity, $E(\delta, T_0)$ is a closed subset of Γ_1 and $\Delta(\tau)$ is an open subset of $[0, \tau]$. Repeating a version of the classical averaging argument (cf. [14]), presented in the proof of Lemma 4.1 in [7], we have the following averaging lemma:

Lemma 3.1. For $k = 1, \dots, n_0$, the I_k -component of any regular solution of (1.5) with initial data in $N(\gamma, \epsilon, \delta, T_0)$ can be written as

$$I_k(\tau) = I_k(0) + \int_0^\tau \langle F_k \rangle(I(s)) ds + \Xi_k(\tau),$$

where the function $|\Xi_k(\tau)|$ is bounded on $[0, 1]$ by

$$\begin{aligned} & 4C_{M_1}^{n_0, m_0} \left\{ [2(\epsilon^{1/4} + 2T_0 C_{M_1} \epsilon)^{1/2}] (\epsilon T_0 + \epsilon \gamma + 1) \right. \\ & + [T_0 C_{M_1} \epsilon^{7/8} + T_0 C_{M_1} \epsilon + C_{M_1}^{n_0, m_0} (\frac{1}{2} T_0 \epsilon^{7/8} + \frac{1}{3} C_{M_1} T_0^2 \epsilon)] (\epsilon T_0 + \epsilon \gamma + 1) \\ & \left. + 4C_{M_1} \gamma + 2\rho + 2\delta + 2C_{M_1} T_0 \epsilon. \right. \end{aligned}$$

Corollary 3.2. For any $\bar{\rho} > 0$, with a suitable choice of $\rho, \delta, T_0, \gamma$, the function $|\Xi_k(\tau)|$ in Lemma 3.1 can be made smaller than $\bar{\rho}$, if ϵ is small enough.

Proof. We choose

$$T_0 = \epsilon^{-\sigma}, \quad \gamma = \frac{\bar{\rho}}{9C_{M_1}}, \quad \delta = \rho = \frac{\bar{\rho}}{9}$$

with $0 < \sigma < \frac{1}{2}$. Then for ϵ sufficiently small we have

$$|\Xi_k(\tau)| < \bar{\rho}.$$

□

Now let μ be a regular ϵ -quasi-invariant measure and $\{S_\epsilon^\tau, \tau \geq 0\}$ be the flow of equation (0.4) on h^p . Below we follow the arguments, invented by Anosov for the finite dimensional averaging (e.g. see in [14]).

Consider the measure $\mu_1 = d\mu dt$ on $h^p \times \mathbb{R}$. Define the following subset of $h^p \times \mathbb{R}$:

$$\mathcal{B}^\epsilon := \{(v, \tau) : v \in \Gamma_0, \tau \in [0, 1] \text{ and } S_\epsilon^\tau v \in \Gamma_1 \setminus E(\delta, T_0)\}.$$

Then by (0.7), there exists $C(M_1)$ such that

$$\mu_1(\mathcal{B}^\epsilon) = \int_0^1 \mu(\Gamma_0 \cap S_\epsilon^{-\tau}(\Gamma_1 \setminus E(\delta, T_0))) d\tau \leq e^{C(M_1)} \mu(\Gamma_1 \setminus E(\delta, T_0)).$$

For any $v \in \Gamma_0$, denote $\bar{\Delta}(v) = \Delta(1, \epsilon, \delta, T_0, I, \varphi)$, where $(I, \varphi) = \Pi_{I, \varphi}(v)$. Then by the Fubini theorem, we have

$$\mu_1(\mathcal{B}^\epsilon) = \int_{\Gamma_0} \text{Mes}(\bar{\Delta}(v)) d\mu(v).$$

Using Chebyshev's inequality, we obtain

$$\mu(\Gamma_0 \setminus N(\gamma, \epsilon, \delta, T_0)) \leq \frac{e^{C(M_1)}}{\gamma} \mu(\Gamma_1 \setminus E(\delta, T_0)). \quad (3.9)$$

By the definition of $E(\delta, T_0)$, we know that

$$E(\delta, T_0) \subset E(\delta, T'_0), \quad \text{if } T'_0 \geq T_0. \quad (3.10)$$

We set $E^\infty(\delta) := \cup_{T_0 \geq 1} E(\delta, T_0)$. Define

$$\mathcal{RES}(m_0) = \left\{ (I, \varphi) \in \Gamma_1 : \exists k \in \mathbb{Z}^{m_0} \text{ such that } k_1 W_1(I) + \dots + k_{m_0} W_{m_0}(I) = 0 \right\}.$$

Since the measure μ is regular, then by Lemma 1.3, we have that $\mu(\mathcal{RES}(m_0)) = 0$. If $(I', \varphi') \in \Gamma_1 \setminus \mathcal{RES}(m_0)$, then the vector $W^{m_0}(I') \in \mathbb{R}^{m_0}$ is non-resonant. Due to

Lemma 2.3, we know that there exists $T'_0 > 0$ such that for $T \geq T'_0$, the inequalities (3.8) hold. Therefore $(I', \varphi') \in E(\delta, T'_0) \subset E^\infty(\delta)$. Hence

$$\Gamma_1 \setminus E^\infty(\delta) \subset \mathcal{RES}(m_0).$$

So we have $\mu(E^\infty(\delta)) = \mu(\Gamma_1)$. Since $\mu(E^\infty(\delta)) = \lim_{T_0 \rightarrow \infty} \mu(E(\delta, T_0))$ due to (3.10), then for any $\nu > 0$, there exists $T'_0 > 0$ such that for each $T_0 \geq T'_0$, we have

$$\mu(E^\infty \setminus E(\delta, T_0)) \leq \nu.$$

So the r.h.s of the inequality (3.9) can be made arbitrary small if T_0 is large enough.

Fix some $0 < \sigma < 1/2$, we have proved the following lemma.

Lemma 3.3. *Fix any $\delta > 0$, $\bar{\rho} > 0$. Then for every $\nu > 0$ we can find $\epsilon(\nu) > 0$ such that, if $\epsilon < \epsilon(\nu)$, then*

$$\mu\left(\Gamma_0 \setminus N\left(\frac{\bar{\rho}}{9C_{M_1}}\right)\right) < \nu,$$

where $N\left(\frac{\bar{\rho}}{9C_{M_1}}\right) = N\left(\frac{\bar{\rho}}{9C_{M_1}}, \epsilon, \delta, \epsilon^{-\sigma}\right)$.

We now are in a position to prove assertion (i) of Theorem 0.2.

By Corollary 2.2, for each $q \in [p', p]$, there exists $C_3(q, M_1)$ such that for any $J_1, J_2 \in B_q^I(M_1 + 1)$ (see Corollary 2.2 and Agreements),

$$|\langle F \rangle(J_1) - \langle F \rangle(J_2)|_q \lesssim |\langle F \rangle(J_1) - \langle F \rangle(J_2)|_{q+\zeta_0/2} \lesssim C_3(\bar{q}, M_1) |J_1 - J_2|_q. \quad (3.11)$$

Since the mapping $\langle F \rangle : h_T^p \rightarrow h_T^p$ is locally Lipschitz by (3.11), then using Picard's theorem, for any $J_0 \in B_p^I(M_1)$ there exists a unique solution $J(\cdot)$ of the averaged equation (0.6) with $J(0) = J_0$. We denote

$$\mathcal{T}(J_0) := \inf\{\tau > 0 : |J(\tau)|_p > M_1 + 1\} \leq \infty.$$

For any $\bar{\rho} > 0$ and $q < p + \zeta_0$, there exist $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} |F(I, \varphi) - F^{n_1}(I, \varphi)|_q &< \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad (I, \varphi) \in B_p^I(M_1 + 1) \times \mathbb{T}^\infty, \\ |\langle F \rangle(J) - \langle F \rangle^{n_1}(J)|_q &< \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad J \in B_p^I(M_1 + 1). \end{aligned} \quad (3.12)$$

Here

$$C_3(M_1) = \begin{cases} C_3(p, M_1) & \text{if } q > p, \\ C_3(q, M_1) & \text{if } q \leq p. \end{cases}$$

Find ρ_0 from the relation

$$8 \sum_{j=1}^{n_1} j^{1+2q} \rho_0 = \bar{\rho} e^{-C_3(M_1)}.$$

By Lemma 3.1 and Corollary 3.2, there exists $\epsilon_{\bar{\rho}, q}$ such that if $\epsilon \leq \epsilon_{\bar{\rho}, q}$, then for initial data in the subset $\Gamma_{\bar{\rho}} = N_{n_1, \rho_0}\left(\frac{\bar{\rho}_0}{9C_{M_1}\epsilon}, \epsilon, \frac{\bar{\rho}_0}{9}, \epsilon^{-\sigma}\right)$ we have for $k = 1, \dots, n_1$,

$$I_k^\epsilon(\tau) = I_k^\epsilon(0) + \int_0^\tau \langle F_k \rangle(I^\epsilon(s)) ds + \xi_k(\tau), \quad |\xi_k(\tau)| < \rho_0, \quad \tau \in [0, 1], \quad (3.13)$$

Therefore, by (3.12) and (3.13), for $(I^\epsilon(0), \varphi^\epsilon(0)) \in \Gamma_{\bar{\rho}}$, $J(0) \in B_p(M_1 + 1)$ and $|\tau| \leq \min\{1, \mathcal{T}(J(0))\}$,

$$\begin{aligned} & |I^\epsilon(\tau) - J(\tau)|_q^\sim - |I^\epsilon(0) - J(0)|_q^\sim \\ & \leq \int_0^\tau |F(I^\epsilon(s)) ds - \int_0^\tau \langle F \rangle(J(s))|_q^\sim ds \\ & \leq \int_0^\tau |F^{n_1}(I^\epsilon(s)) - \langle F \rangle^{n_1}(J(s))|_q^\sim ds + \frac{\rho}{4} e^{-C_3(M_1)}. \\ & \leq \int_0^\tau |\langle F \rangle(I^\epsilon(s)) - \langle F \rangle(J(s))|_q^\sim ds + \frac{\rho}{2} e^{-C_3(M_1)}. \end{aligned}$$

Using (3.11), we obtain

$$|I^\epsilon(\tau) - J(\tau)|_q^\sim \leq |I^\epsilon(0) - J(0)|_q^\sim + \int_0^\tau C_3(M_1) |I^\epsilon(s) - J(s)|_q^\sim ds + \xi_0(\tau),$$

where $|\xi_0(\tau)| \leq \frac{\bar{\rho}}{2} e^{-C_3(M_1)}$. By Gronwall's lemma, if

$$|I^\epsilon(0) - J(0)|_q^\sim \leq \delta = e^{-C_3(M_1)} \bar{\rho},$$

then

$$|I(\tau) - J(\tau)|_q \leq 2\bar{\rho}, \quad |\tau| \leq \min\{1, \mathcal{T}(J(0))\}.$$

This establishes inequality (0.9). Assuming that $\bar{\rho} \ll 1$, we get from the definition of $\mathcal{T}(J(0))$ that $\mathcal{T}(J(0))$ is bigger than 1, if $\zeta_0 > 0$ and $q \geq p$. From Lemma 3.3 we know that for any $\nu > 0$, if ϵ small enough, then $\mu(\Gamma_0 \setminus \Gamma_{\bar{\rho}}) < \nu$. This completes the proof of the assertion (i) of Theorem 0.2.

Proof of statement (2) of Remark 0.3 If the perturbation is hamiltonian with Hamiltonian H , then $F = -\nabla_\varphi H$. Therefore the averaged vector field $\langle F \rangle = 0$. For any $\rho > 0$ and any $q < p$, there exists n_2 such that

$$|I - I^{n_2}|_q^\sim < \rho/4, \quad \forall I \in B_p(M).$$

By similar argument, we can obtain that, there exists a subset $\Gamma_{\rho, n_2}^\epsilon \subset \Gamma_0$, satisfying (0.8), such that for initial data $(I^\epsilon(0), \varphi^\epsilon(0)) \in \Gamma_{\rho, n_2}^\epsilon$, and for $\tau \in [0, 1]$, we have

$$|I^{\epsilon, n_2}(\tau) - I^{\epsilon, n_2}(0)|_q^\sim \leq \rho/4.$$

So

$$|I^\epsilon(\tau) - I^\epsilon(0)|_q^\sim \leq \rho \quad \text{for } \tau \in [0, 1].$$

In this argument we do not require $\zeta_0 \geq 0$. So item (ii) of Assumption A is not needed if the perturbation is hamiltonian.

3.2. Proof of the assertion (ii). We fix $\alpha < 1/4$. For any $(m, n) \in \mathbb{N}^2$ denote

$$\begin{aligned} \mathcal{B}_m(\epsilon) & := \left\{ (I, \varphi) \in \Gamma_1 : \inf_{k \leq m} |I_k| < \epsilon^\alpha \right\}, \\ \mathcal{R}_{m, n}(\epsilon) & := \bigcup_{|L| \leq n, L \in \mathbb{Z}^m \setminus \{0\}} \left\{ (I, \varphi) \in \Gamma_1 : |W(I) \cdot L| < \epsilon^\alpha \right\}. \end{aligned}$$

Then let

$$\Upsilon_{m, n}(\epsilon) = \left(\bigcup_{m_0 \leq m} \mathcal{R}_{m_0, n}(\epsilon) \right) \cup \mathcal{B}_m(\epsilon), \quad (3.14)$$

and for any $(I_0, \varphi_0) \in \Gamma_0$, denote

$$S(\epsilon, m, n, I_0, \varphi_0) = \{ \tau \in [0, 1] : (I^\epsilon(\tau), \varphi^\epsilon(\tau)) \in \Upsilon_{m, n}(\epsilon) \}.$$

Fix $m \in \mathbb{N}$, take a bounded Lipschitz function g defined on the torus \mathbb{T}^m such that $Lip(g) \leq 1$ and $|g|_{L^\infty} \leq 1$. Let $\sum_{s \in \mathbb{Z}^m} g_s e^{is \cdot \varphi}$ be its Fourier series. Then for any $\rho > 0$, there exists n , such that if we denote $\bar{g}_n = \sum_{|s| \leq n} g_s e^{is \cdot \varphi}$, then

$$\left| g(\varphi) - \bar{g}_n(\varphi) \right| < \frac{\rho}{2}, \quad \forall \varphi \in \mathbb{T}^m.$$

As the measure μ is regular and $\Upsilon_{m,n}(\epsilon_1) \subset \Upsilon_{m,n}(\epsilon_2)$ if $\epsilon_1 \leq \epsilon_2$, then

$$\mu(\Upsilon_{m,n}(\epsilon)) \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Since the measure μ is ϵ -quasi-invariant, then following the same argument that proves Lemma 3.3, we have there exists subset $\Lambda_\rho^\epsilon \subset \Gamma_0$ such that for initial data $(I_0, \varphi_0) \in \Lambda_\rho^\epsilon$, we have $\text{Mes}(S(\epsilon, m, n, I_0, \varphi_0)) \leq \rho/4$, and if ϵ is small enough, then $\mu(\Gamma_0 \setminus \Lambda_\rho^\epsilon) < \mu(\Gamma_0)\rho/4$. Due to Lemma 2.3, if $(I^\epsilon(\cdot), \varphi^\epsilon(\cdot))$ stays long enough time outside the subset $\Upsilon_{m,n}(\epsilon)$, then the time average of $\bar{g}(\varphi^{\epsilon,m}(\tau))$ can be well approximated by its space average. Following an argument of Anosov (see [14]), we obtain that for ϵ small enough and for initial data $(I_0, \varphi_0) \in \Lambda_\rho^\epsilon$, we have

$$\left| \int_{\mathbb{T}^m} \bar{g}(\varphi) d\lambda_\epsilon^{I_0, \varphi_0} - \int_{\mathbb{T}^m} \bar{g} d\varphi \right| = \left| \int_0^1 \bar{g}(\varphi^{\epsilon,m}(\tau)) d\tau - \int_{\mathbb{T}^m} \bar{g}(\varphi) d\varphi \right| < \rho/2. \quad (3.15)$$

(For a proof, see Lemma 4.1 in [7].) So if ϵ is small enough, then

$$\begin{aligned} & \left| \int_{\mathbb{T}^m} g(\varphi) \lambda_\epsilon - \int_{\mathbb{T}^m} g(\varphi) d\varphi \right| \\ & \leq \frac{1}{\mu(\Gamma_0)} \left\{ \left| \int_{(I_0, \varphi_0) \in \Lambda_\rho^\epsilon} \left[\int_{\mathbb{T}^m} g(\varphi) d\lambda_\epsilon^{I_0, \varphi_0} - \int_{\mathbb{T}^m} g(\varphi) d\varphi \right] d\mu(I_0, \varphi_0) \right| \right. \\ & \left. + \left| \int_{(I_0, \varphi_0) \in \Gamma_0 \setminus \Lambda_\rho^\epsilon} \left[\int_{\mathbb{T}^m} g(\varphi) d\lambda_\epsilon^{I_0, \varphi_0} - \int_{\mathbb{T}^m} g(\varphi) d\varphi \right] d\mu(I_0, \varphi_0) \right| \right\} \leq 2\rho. \end{aligned}$$

That is ,

$$\left| \int g(\varphi) \lambda_\epsilon - \int g(\varphi) d\varphi \right| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \quad (3.16)$$

for any Lipschitz function as above. Hence, the probability measure λ_ϵ converges weakly to the Haar measure $d\varphi$ (see [6]). This proves the assertion (ii).

3.3. Consistency conditions. Assume the ϵ -quasi-invariant measure μ depends on ϵ , i.e. $\mu = \mu_\epsilon$. Using again the Anosov arguments, we have for the measure μ_ϵ that

$$\mu_\epsilon(N(\tilde{T}, \epsilon, \delta, T_0)) \leq \frac{e^{C_\epsilon(M_1)}}{\tilde{T}} \mu_\epsilon(\Gamma_1 \setminus E(\delta, T_0)).$$

It is easy to see that assertion (i) of Theorem 0.2 holds, with μ replace by μ_ϵ , if the following consistency conditions are satisfied:

- 1) For any $\delta > 0$, $\mu_\epsilon(\Gamma_1 \setminus E(\delta, \epsilon^{-\sigma}))$ go to zero with ϵ .
- 2) $C_\epsilon(M_1)$ is uniformly bounded with respect to ϵ

In subsection 3.2, we can see that for assertion (ii) of Theorem 0.2 to hold, one more conditions should be added to the family $\{\mu_\epsilon\}_{\epsilon \in (0,1)}$. That is,

- 3) For any $m, n \in \mathbb{N}$, $\mu_\epsilon(\Upsilon_{m,n}(\epsilon))$ (see (3.14)) goes to zero with ϵ .

4. ON EXISTENCE OF ϵ -QUASI-INVARIANT MEASURES

In this section we prove that if Assumption A holds with $\zeta_0 \geq 2$, then there exist ϵ -quasi-invariant measures for the perturbed KdV (0.1) on the space H^p , where $p \geq 3$ is an integer. Through this section, we suppose that $\zeta_0 = 2$, $3 \leq p \in \mathbb{N}$ and $p' = 0$. Our presentation closely follows Chapter 4 of the book [15].

Let η_p be the centered Gaussian measure on H^p with correlation operator ∂_x^{-2} . Since ∂_x^{-2} is an operator of trace type, then η_p is a well-defined probability measure on H^p .

As is known, for solutions of KdV, there are countably many conservation laws $\mathcal{J}_n(u)$, $n \geq 0$ of the form $\mathcal{J}_n = \frac{1}{2}\|u\|_n^2 + J_{n-1}(u)$, where $J_{-1}(u) = 0$ and for $n \geq 1$,

$$J_{n-1}(u) = \int_{\mathbb{T}} \{c_n u (\partial_x^{n-1} u)^2 + \mathcal{Q}_n(u, \dots, \partial_x^{n-2} u)\} dx. \quad (4.1)$$

where c_n are real constants, and \mathcal{Q}_n are polynomial in their arguments (see, p.209 in [9] for exact form of the conservation laws). By induction we get from these relations that

$$\|u\|_n^2 \leq 2\mathcal{J}_n + C(\mathcal{J}_{n-1}, \dots, \mathcal{J}_0), \quad n \geq 1, \quad (4.2)$$

where C vanishes with $u(\cdot)$.

From (4.1) we know that the functional J_p is bounded on bounded sets in H^p . We consider the measure μ_p defined by

$$\mu_p(\Omega) = \int_{\Omega} e^{-J_p(u)} d\eta_p(u), \quad (4.3)$$

for every Borel set $\Omega \subset H^p$. This measure is regular in the sense of Definition 0.1 and non-degenerated in the sense that its support contains the whole space H^p (see, e.g. Chapter 9 in [2]). Moreover, it is invariant for KdV [15].

The main result of this section is the following theorem:

Theorem 4.1. *The measure μ_p is ϵ -quasi-invariant for perturbed KdV equation (0.1) on the space H^p .*

To prove this theorem, we follow a classical procedure based on finite dimensional approximation (see, e.g. [15]).

Let us firstly write equation (0.1) using the slow time $\tau = \epsilon t$,

$$\dot{u} = \epsilon^{-1}(-u_{xxx} + 6uu_x) + f(u), \quad (4.4)$$

where $\dot{u} = \frac{du}{d\tau}$. By Assumption A, for each $u_0 \in B_p^u(M)$, the equation (4.4) has a unique solution $u(\cdot) \in C([0, T], H^p)$ and $\|u(\tau)\|_p \leq C(\|u_0\|_p, T)$ for all $\tau \in [0, T]$.

Denote \mathbb{L}_m the subspace of H^p , spanned by the basis vectors $\{e_1, e_{-1}, \dots, e_m, e_{-m}\}$. Let \mathbb{P}_m be the orthogonal projection of H^p onto \mathbb{L}_m and $\mathbb{P}_m^\perp = Id - \mathbb{P}_m$. For any $u \in H^p$, denote $u^m = \mathbb{P}_m u \in \mathbb{L}_m$. We will identify \mathbb{P}_∞ with Id and u^∞ with u . Consider the problem

$$\dot{u}^m = \epsilon^{-1}[-u_{xxx}^m + 6\mathbb{P}_m(u^m u_x^m)] + \mathbb{P}_m(f(u^m)), \quad u^m(x, 0) = \mathbb{P}_m u_0(x). \quad (4.5)$$

Clearly, for each $u_0 \in H^p$ this problem has a unique solution $u^m(\cdot) \in C([0, T'], \mathbb{L}_m)$ for some $T' > 0$.

Proposition 4.2. *Let $u_0 \in H^p$ and $u_0^m \in \mathbb{L}_m$ such that u_0^m strongly converge to u_0 in H^p as $m \rightarrow +\infty$. Then as $m \rightarrow +\infty$,*

$$u^m(\cdot) \rightarrow u(\cdot) \quad \text{in } C([0, T], H^p),$$

where $u(\cdot)$ is the solution of equation (0.1) with initial datum $u(0) = u_0$ and $u^m(\cdot)$ is the solution of problem (4.5) with initial condition $u^m(0) = u_0^m$.

In this result, as well as in the Lemmas 4.5-4.7 below, the rate of convergence depends on the small parameter ϵ .

To prove this proposition, we start with several lemmas. For any $n, m \in \mathbb{N}$, we have for the solution $u^m(\tau)$ of problem(4.5)

$$\begin{aligned} \frac{d}{d\tau} \mathcal{J}_n(u^m(\tau)) &= \langle \nabla_u \mathcal{J}_n(u^m), \dot{u}^m(\tau) \rangle \\ &= \langle \nabla_u \mathcal{J}_n(u^m), \epsilon^{-1}[-u_{xxx}^m + \mathbb{P}_m(u^m u_x^m)] + \mathbb{P}_m[f(u^m)] \rangle \end{aligned}$$

Here ∇_u stands for the L_2 -gradient with respect to u . Since \mathcal{J}_n is a conservation law of KdV, then $\langle \nabla_u \mathcal{J}_n(u^m), -u_{xxx}^m + u^m u_x^m \rangle = 0$. So

$$\frac{d}{d\tau} \mathcal{J}_n(u^m) = -\epsilon^{-1} \langle \nabla_u \mathcal{J}_n(u^m), \mathbb{P}_m^\perp(u^m u_x^m) \rangle + \langle \nabla_u \mathcal{J}_n(u^m), \mathbb{P}_m[f(u^m)] \rangle. \quad (4.6)$$

We denote the first term in the right hand side by $\epsilon^{-1} \mathcal{E}_n(u^m)$ and the second term by $\mathcal{E}_n^f(u^m)$.

Lemma 4.3. *There exist continuous functions $\gamma_n(R, s)$ and $\gamma_n'(R, s)$ on $\mathbb{R}_+^2 = \{(R, s)\}$ such that they are non-decreasing in the second variable s , vanish if $s = 0$, and*

$$|\mathcal{E}_n^f(u^m)| \leq \gamma_n'(\|u^m\|_{n-1}, \|u^m\|_{n-1}), \quad (4.7)$$

$$|\mathcal{E}_n(u^m)| \leq \gamma_n \left(\|u^m\|_{n-1}, \max_{\substack{0 \leq i, j \leq n-1, \\ i+j \neq 2n-2}} \|\mathbb{P}_m^\perp[\partial_x^i u^m \partial_x^j u^m]\|_0 + \|\mathbb{P}_m^\perp(u^m u_x^m)\|_1 \right). \quad (4.8)$$

for all $n = 3, 4, \dots$. For $n = 2$ equality (4.7) still holds, and

$$|\mathcal{E}_2(u^m)| \leq C_2(\|u^m\|_1) \|u^m\|_2^2 + C_2'(\|u^m\|_1). \quad (4.9)$$

Proof. Since $f(u)$ is 2-smoothing, from (4.1) and (4.6) we know that

$$|\mathcal{E}_n^f(u^m)| \leq \gamma_n'(\|u^m\|_{n-1}, \|u^m\|_{n-1}),$$

where $\gamma_n'(\cdot, \cdot)$ is a continuous function satisfying the requirement in the statement of the lemma.

For the quantity $\mathcal{E}_n(u^m)$, by (4.1) and (4.6) we have

$$\begin{aligned}\mathcal{E}_n(u^m) &= \int_{\mathbb{T}} \left\{ 6(-1)^n (\partial_x^{2n} u^m) \mathbb{P}_m^\perp(u^m u_x^m) + 6c_n \mathbb{P}_m^\perp(u^m u_x^m) (\partial_x^{n-1} u^m)^2 \right. \\ &\quad + (-1)^{n-1} 12c_n \partial_x^{n-1} (u^m \partial_x^{n-1} u^m) \mathbb{P}_m^\perp(u^m u_x^m) \\ &\quad \left. + 6 \sum_{i=0}^{n-2} \frac{\partial \mathcal{Q}_n(u^m, \dots, \partial_x^{n-2} u^m)}{\partial(\partial_x^i u^m)} \partial_x^i \mathbb{P}_m^\perp(u^m u_x^m) \right\} dx \\ &= 0 + \int_{\mathbb{T}} \left\{ 6c_n \mathbb{P}_m^\perp(u^m u_x^m) (\partial_x^{n-1} u^m)^2 \right. \\ &\quad + 12c_n \mathbb{P}_m^\perp(u^m \partial_x^{n-1} u^m) \left[\partial_x \sum_{i=0}^{n-3} C_{n-2}^i \partial_x^{n-2-i} u^m \partial_x^{i+1} u^m \right] \\ &\quad \left. + 6 \sum_{i=0}^{n-2} \frac{\partial \mathcal{Q}_n(u^m, \dots, \partial_x^{n-2} u^m)}{\partial(\partial_x^i u^m)} \partial_x^i \mathbb{P}_m^\perp(u^m u_x^m) \right\} dx.\end{aligned}$$

Hence we prove the assertion of the lemma. \square

Lemma 4.4. *For every $u_0 \in H^p$, there exist $\tau_1(\|u_0\|_0) > 0$ and a continuous non-decreasing ϵ -depending function $\beta_p^\epsilon(s)$ on $[0, +\infty)$ such that the value $\|u^m(\tau)\|_p$ are bounded by the quantity $\beta_n^\epsilon(\|u_0\|_p)$, uniformly in $m = 1, 2, \dots$ and $\tau \in [0, \tau_1]$.*

Proof. Let $M = \max\{\|u_0\|_0, 1\}$. It is easy to verify that

$$\frac{d}{d\tau} \|u^m\|_0^2 = 2\langle u^m, \mathbb{P}_m(f(u^m)) \rangle \leq 2\|u^m\|_0^2 + C(2M),$$

if $\|u^m\|_0 \leq 2M$. Therefore for a suitable $\tau_1 = \tau_1(\|u_0\|_0) > 0$ and all $\tau \in [0, \tau_1]$, we have $\|u^m(\tau)\|_0 \leq 2M$. For the quantity $\mathcal{J}_1(u^m)$ and $\tau \in [0, \tau_1]$,

$$\frac{d}{d\tau} \mathcal{J}_1(u^m) = \langle \nabla_u \mathcal{J}_1(u^m), \mathbb{P}_m f(u^m) \rangle \leq C_1(2M).$$

Therefore $\mathcal{J}_1(u^m(\tau)) \leq C_1\tau + \mathcal{J}_1(u^m(0))$. So $\|u^m(\tau)\|_1 \leq \beta_1(\|u_0\|_1)$. Similarly, by Lemma 4.3 and inequality (4.2), we have for $\tau \in [0, \tau_1]$,

$$\frac{d}{d\tau} \mathcal{J}_2(u^m(\tau)) \leq \epsilon^{-1} C_2 [\beta_1(\|u_0\|_1)] \mathcal{J}(u^m(\tau)) + C_2''[\epsilon^{-1}, \beta_1(\|u_0\|_1)].$$

By Gronwall's lemma and relation (4.2), we obtain $\|u^m(\tau)\|_2 \leq \beta_2^\epsilon(\|u_0\|_2)$. In the view of Lemma 4.3, we have

$$\mathcal{J}_n(u^m(\tau)) \leq \mathcal{J}_n(u^m(0)) + \tau C_n[\epsilon^{-1}, \beta_{n-1}^\epsilon(\|u_0\|_{n-1})],$$

for $n = 3, \dots, p$. Hence $\max_{\tau \in [0, \tau_1]} \|u^m(\tau)\|_p \leq \beta_p^\epsilon(\|u_0\|_p)$. \square

Below, we will denote by τ_1 the quantity $\min\{\tau_1(\|u_0\|_0), T\}$.

Lemma 4.5. *As $m \rightarrow \infty$,*

$$\|u^m(\tau) - u(\tau)\|_{p-1} \rightarrow 0,$$

uniformly in $\tau \in [0, \tau_1]$.

Proof. Denote $w = u^m - u$. Using that $\langle \partial_x^j u^m, \mathbb{P}_m^\perp u' \rangle = 0$ for any j and each $u' \in H^0$, we get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \|w\|_{p-1}^2 \\ &= \left\langle \partial_x^{p-1} w, \partial_x^{p-1} [\epsilon^{-1} (-w_{xxx} + 6\mathbb{P}_m(u^m u_x^m) - 6uu_x) + \mathbb{P}_m(f(u^m)) - f(u)] \right\rangle \\ &= 3\epsilon^{-1} \left\langle \partial_x^{p-1} w, \partial_x^p [(u^m)^2 - u^2] \right\rangle + 3\epsilon^{-1} \left\langle \mathbb{P}_m^\perp(\partial_x^{p-1} u), \partial_x^p [(u^m)^2] \right\rangle \\ &\quad + \left\langle \partial_x^{p-1} w, \partial_x^{p-1} [\mathbb{P}_m(f(u^m)) - f(u)] \right\rangle. \end{aligned}$$

Using Sobolev embedding and integration by part, we have

$$\begin{aligned} \left\langle \partial_x^{p-1} w, \partial_x^p [(u^m)^2 - u^2] \right\rangle &= \sum_{i=0}^p C_p^i \int_{\mathbb{T}} \partial_x^{p-1} w \partial_x^{p-i} w \partial_x^i (u^m + u) dx \\ &\leq - \int_{\mathbb{T}} \partial_x (u^m + u) (\partial_x^{p-1} w)^2 dx + \sum_{i=1}^p C_p^i \|w\|_{p-1}^2 \|u^m + u\|_p \\ &\leq C(\|u\|_p, \|u^m\|_p) \|w\|_{p-1}^2 \end{aligned}$$

Therefore,

$$\frac{d}{d\tau} \|w\|_{p-1}^2 \leq C_1(\epsilon^{-1}, \|u^m\|_n) \|\mathbb{P}_m^\perp u\|_p + \|\mathbb{P}_m^\perp f(u)\|_p + C_2(\epsilon^{-1}, \|u\|_n, \|u^m\|_n) \|w\|_{p-1}^2.$$

Since $\|\mathbb{P}_m^\perp(u)\|_p$ and $\|\mathbb{P}_m^\perp(f(u))\|_p$ go to zero as $m \rightarrow \infty$ for each $\tau \in [0, \tau_1]$ and as they are uniformly bounded on $[0, \tau_1]$ by Lemma 4.4, we have for $\tau \in [0, \tau_1]$,

$$\|w\|_{p-1}^2(\tau) = \|w(0)\|_{p-1}^2 + \int_0^\tau C(\epsilon^{-1}, \|u_0\|_p) \|w\|^2 ds + a_m(\epsilon^{-1}, \tau),$$

where $a_m(\epsilon^{-1}, \tau) \rightarrow 0$ as $m \rightarrow \infty$. So the assertion of the lemma follows from Gronwall's lemma. \square

Lemma 4.6. *Let $\tau^m \in [0, \tau_1]$ such that $\tau^m \rightarrow \tau^0 \in [0, \tau_1]$, then*

$$\|u^m(\tau^m) - u(\tau_0)\|_p \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Proof. We firstly prove $\mathcal{J}_p(u^m(\tau^m)) \rightarrow \mathcal{J}_p(u(\tau_0))$ as $m \rightarrow \infty$. Indeed, for $m \leq +\infty$, by (4.6), we have

$$\mathcal{J}_p(u^m(\tau^m)) = \mathcal{J}_p(u^m(0)) + \int_0^{\tau^m} [\epsilon^{-1} \mathcal{E}_p(u^m(s)) + \mathcal{E}_p^f(u^m(s))] ds$$

Since $f(u)$ is 2-smoothing, the second term in the integrand is continuous in H^{p-1} . So, in the view of Lemma 4.5, we only need to prove that the first term goes to zero as $m \rightarrow \infty$. Due to Lemma 4.3, we only need to show that uniformly in $\tau \in [0, \tau_1]$,

$$\lim_{m \rightarrow \infty} \|\mathbb{P}_m^\perp(\partial_x^i u^m(\tau) \partial_x^j u^m(\tau))\|_0 + \lim_{m \rightarrow \infty} \|\mathbb{P}_m^\perp u^m(\tau) u^m(\tau)_x\|_1 = 0,$$

where $0 \leq i, j \leq p-1$ and $i+j \neq 2p-2$. For the first term in the left hand side, we have

$$\begin{aligned} & \|\mathbb{P}_m^\perp(\partial_x^i u^m(\tau) \partial_x^j u^m(\tau))\|_0 \\ & \leq \|\mathbb{P}_m^\perp(\partial_x^i u^m(\tau) \partial_x^j u^m(\tau) - \partial_x^i u(\tau) \partial_x^j u(\tau))\|_0 + \|\mathbb{P}_m^\perp(\partial_x^i u(\tau) \partial_x^j u(\tau))\|_0. \end{aligned} \quad (4.10)$$

By Lemma 4.5, the first term in the r.h.s of (4.10) goes to zero as $m \rightarrow \infty$, uniformly in $\tau \in [0, \tau_1]$. Since $u(\cdot) \in C([0, \tau_1], H^p)$, then $\partial_x^i u(\cdot) \partial_x^j u(\cdot) \in C([0, \tau_1], H^0)$. Therefore, the quantity $\|\mathbb{P}_m^\perp(\partial_x^i u(\tau) \partial_x^j u(\tau))\|_0 \rightarrow 0$ as $m \rightarrow \infty$, uniformly in $\tau \in [0, \tau_1]$.

In the same way $\lim_{m \rightarrow \infty} \|\mathbb{P}_m^\perp u^m u_x^m\|_1 = 0$. Therefore, we have

$$\lim_{m \rightarrow \infty} \mathcal{J}_p(u^m(\tau^m)) = \mathcal{J}_p(u(0)) + \int_0^{\tau^0} \langle \nabla_u \mathcal{J}_p(u(s)), f(u(s)) \rangle ds = \mathcal{J}_p(u(\tau^0)).$$

Since the quantity $\mathcal{J}_p(u) - \|u\|_p^2/2$ is continuous in H^{p-1} , we have

$$\lim_{m \rightarrow \infty} \|u^m(\tau^m)\|_p = \|u(\tau_0)\|_p.$$

The assertion of the lemma follows from the fact that weak convergence plus norm convergence imply strong convergence. \square

Lemma 4.7. *As $m \rightarrow \infty$, $\|u^m(\tau) - u(\tau)\|_p \rightarrow 0$ uniformly for $\tau \in [0, \tau_1]$.*

Proof. Assume the opposite. Then there exists $\delta > 0$ such that for each $m \in \mathbb{N}$, there exists $\tau^m \in [0, \tau_1]$ satisfying

$$\|u^m(\tau^m) - u(\tau^m)\|_p \geq \delta. \quad (4.11)$$

Take a subsequence $\{m_k\}$ such that $\tau^{m_k} \rightarrow \tau^0 \in [0, \tau_1]$ as $m_k \rightarrow \infty$. By Lemma 4.6, we have

$$\begin{aligned} & \lim_{m_k \rightarrow \infty} \|u^{m_k}(\tau^{m_k}) - u(\tau^{m_k})\|_p \\ &= \lim_{m_k \rightarrow \infty} (\|u^{m_k}(\tau^{m_k}) - u(\tau^0)\|_p + \|u(\tau^{m_k}) - u(\tau^0)\|_p) = 0. \end{aligned}$$

This contradicts the inequality (4.11). So the assertion of the Lemma holds. \square

If $T = \tau_1$, Proposition 4.2 is proved. Otherwise, we iterate the above procedure by letting the initial datum to be $u(\tau_1)$. This completes the proof of Proposition 4.2.

Apart from Proposition 4.2, we will need two more results to prove Theorem 4.1.

Proposition 4.8. *For each $u_0 \in H^p$ and any $\nu > 0$, there exists $\delta > 0$ such that*

$$\|u^m(\tau) - u_1^m(\tau)\|_p < \nu,$$

uniformly in $m = 1, 2, \dots$, $\tau \in [0, T]$ and for every solution $u_1^m(\cdot)$ of problem (4.5) with initial data $u_1^m(0)$, satisfying

$$\|u^m(0) - u_1^m(0)\|_p < \delta,$$

(here $u^m(\cdot)$ is the solution of (4.5) with initial data $\mathbb{P}_m u_0$).

Proof. Assume the contrary. Then there exists $\nu > 0$ such that for each $\delta > 0$, there exists $m \in \mathbb{N}$, $u_1 \in \mathbb{L}_m$ and $\tau^m \in [0, T]$ satisfying

$$\|u_1^m(\tau^m) - u^m(\tau^m)\|_p \geq \nu \quad \text{and} \quad \|u_1^m(0) - u^m(0)\|_p < \delta. \quad (4.12)$$

Hence there exists a subsequence $\{m_k\}$ such that $\|u_1^{m_k} - \mathbb{P}_{m_k} u_0\|_p \rightarrow 0$ as $m_k \rightarrow \infty$. Therefore $\lim_{m_k \rightarrow \infty} \|u_1^{m_k} - u_0\|_p = 0$. By Proposition 4.2, we know that

$$\|u_1^{m_k}(\tau^{m_k}) - u^{m_k}(\tau^{m_k})\|_p \leq \|u_1^{m_k}(\tau^{m_k}) - u(\tau^{m_k})\|_p + \|u^{m_k}(\tau^{m_k}) - u(\tau^{m_k})\|_p \rightarrow 0,$$

as $m_k \rightarrow \infty$. This contradicts the first inequality of (4.12). Proposition 4.8 is proved. \square

Lemma 4.9. *Let $u_0 \in H^p$. Then for any $\delta > 0$, there exist $r > 0$ and $m_0 > 0$ such that for each $m \geq m_0$ and $\bar{u}(0) \in \dot{B}_p^u(u_0, r)$, the quantity*

$$\epsilon^{-1} |\mathcal{E}_{p+1}(\bar{u}^m(\tau))| \leq \delta,$$

for all $\tau \in [0, T]$.

Proof. In the view of Lemma 4.3, we only need to show for each $\delta_\epsilon > 0$, there exist $r > 0$ and $m_0 > 0$ such that for every $\bar{u}_0 \in \dot{B}_p^u(u_0, r)$, and $m \geq m_0$, we have for $\tau \in [0, T]$,

$$\max_{0 \leq i, j \leq p, i+j \neq 2p} \|\mathbb{P}_m^\perp [\partial_x^i \bar{u}^m(\tau) \partial_x^j \bar{u}^m(\tau)]\|_0 + \|\mathbb{P}_m^\perp (\bar{u}^m(\tau) \bar{u}_x^m(\tau))\|_1 < \delta_\epsilon. \quad (4.13)$$

Here $\bar{u}^m(\tau)$ is the solution of problem (4.5) with initial datum $\bar{u}^m(0) = \mathbb{P}_m \bar{u}_0$.

For the first term, we have

$$\begin{aligned} & \|\mathbb{P}_m^\perp [\partial_x^i \bar{u}^m(\tau) \partial_x^j \bar{u}^m(\tau)]\|_0 \\ & \leq \|\partial_x^i u^m(\tau) \partial_x^j u^m(\tau) - \partial_x^i \bar{u}^m(\tau) \partial_x^j \bar{u}^m(\tau)\|_0 \\ & \quad + \|\partial_x^i u^m(\tau) \partial_x^j u^m(\tau) - \partial_x^i u(\tau) \partial_x^j u(\tau)\|_0 + \|\mathbb{P}_m^\perp [\partial_x^i u(\tau) \partial_x^j u(\tau)]\|_0. \end{aligned}$$

By Proposition 4.2 and the fact that $\partial_x^i u(\cdot) \partial_x^j u(\cdot) \in C([0, T], H^0)$, the second and the third terms on the right hand side of this inequality converge to zero as $m \rightarrow \infty$, uniformly in $\tau \in [0, T]$. From Proposition 4.8, we know that there exists $r > 0$ such that the first term is smaller than $\delta_\epsilon/2$ for all $\bar{u} \in \dot{B}_r^p(u_0)$ and uniformly in $m \in \mathbb{N}$ and $\tau \in [0, T]$. Estimating in this way the term $\|\mathbb{P}_m^\perp (\bar{u}^m \bar{u}_x^m)\|_1$, we obtain inequality (4.13). Hence we prove the assertion of the lemma. \square

We now begin to prove Theorem 4.1.

Consider the following Gaussian measure η_p^m on the subspaces $\mathbb{L}_m \subset H^p$:

$$\begin{aligned} d\eta_p^m & := \prod_{i=1}^m (2\pi)^{2p} i^{2p+1} \exp - \frac{(2\pi i)^{2p+2} (\hat{u}_i^2 + \hat{u}_{-i}^2)}{2} d\hat{u}_i d\hat{u}_{-i} \\ & = c(m) \exp \frac{-\|u^m\|_{p+1}^2}{2} d\hat{u}_1 d\hat{u}_{-1} \dots d\hat{u}_m d\hat{u}_{-m}, \end{aligned}$$

where $u^m := \sum_{i=1}^m (\hat{u}_i e_i + \hat{u}_{-i} e_{-i}) \in \mathbb{L}_m$ and $d\hat{u}_{\pm i}$, $i \in \mathbb{N}$, is the Lebesgue measure on \mathbb{R} . Obviously, η_p^m is a Borel measure on \mathbb{L}_m . Then we have obtained a sequence of Borel measure $\{\eta_p^m\}$ on H^p (see, e.g. Section 4.1 in [15]). We set

$$\mu_p^m(\Omega) = \int_\Omega e^{-J_p(u)} d\eta_p^m,$$

for every Borel set $\Omega \in H^p$. Then μ_p^m are well defined Borel measure on H^p . Clearly

$$d\mu_p^m = c(m) e^{-\mathcal{J}_{p+1}(u^m)} d\hat{u}_1 d\hat{u}_{-1} \dots d\hat{u}_m d\hat{u}_{-m}.$$

Lemma 4.10. *(see Lemma 4.1.10 in [15]) The sequence of Borel measures μ_p^m in H^p converges weakly to the measure μ_p as $m \rightarrow \infty$.*

Rewrite the system (4.5) in the variables

$$\hat{u}^m = (\hat{u}_1, \hat{u}_{-1}, \dots, \hat{u}_m, \hat{u}_{-m}),$$

where $u^m = \sum_j^m (\hat{u}_j e_j + \hat{u}_{-j} e_{-j})$:

$$\frac{d}{d\tau} \hat{u}_j = -2\pi j \epsilon^{-1} \frac{\partial \mathcal{J}_1(\hat{u}^m)}{\partial \hat{u}_{-j}} + f_j(\hat{u}^m), \quad j = \pm 1, \dots, \pm m. \quad (4.14)$$

where $\mathbb{P}_m f(\hat{u}^m) = \sum_{j=1}^m (f_j(\hat{u}^m)e_j + f_{-j}(\hat{u}^m)e_{-j})$. Let S_m^τ , $\tau \in [0, T]$, be the flow map of equation (4.14). For any Borel set $\Omega \subset H^p$, let $S_m^\tau(\Omega) = S_m^\tau(\mathbb{P}_m(\Omega))$. By the Liouville Theorem and (4.6), we have

$$\frac{d}{d\tau} \mu^m(S_m^\tau(\Omega)) = \int_{S_m^\tau(\Omega)} \left[\epsilon^{-1} \mathcal{E}_{p+1}(u^m) + \mathcal{E}_{p+1}^f(u^m) + \sum_{i=-m, i \neq 0}^m \frac{\partial f_i}{\partial \hat{u}_i} \right] d\mu^m. \quad (4.15)$$

Denote S_ϵ^τ , $\tau \in [0, T]$, to be the flow map of equation (4.4) on the space H^p . Fix any $M > 0$. By Assumption A, there exists M_1 such that

$$S_\epsilon^\tau(B_p^u(M)) \subset B_p^u(M_1).$$

Since $f(u)$ is 2-smoothing, then on the ball $B_p(2M_1)$ we have $|f_i(u)| \leq |i|^{-p-2} C(2M_1)$. By Cauchy inequality, we have $|\partial f_i / \partial \hat{u}_i| \leq C'(2M_1) i^{-2}$ on the ball $B_p(M_1)$. So we have

$$|\mathcal{E}_{p+1}^f(u^m) + \sum_{i=-m, i \neq 0}^m \frac{\partial f_i}{\partial \hat{u}_i}(u^m)| \leq C(M_1), \quad \forall m \in \mathbb{N} \quad \text{and} \quad \forall u^m \in B_p^u(M_1). \quad (4.16)$$

Now fix $\tau_0 \in [0, T]$. Take an open set $\Omega \subset B_p^u(M)$. For any $\delta > 0$, there exists a compact set $K \subset \Omega$ such that $\mu_p(\Omega \setminus K) < \delta$. Let $K_1 = S_\epsilon^{\tau_0}(K)$. Then the set K_1 also is compact and $K_1 \subset S_\epsilon^{\tau_0}(\Omega) = \Omega_1$. Define

$$\alpha = \min\{\text{dist}(K, \partial\Omega); \text{dist}(K_1, \partial\Omega_1)\},$$

where $\text{dist}(A, B) = \inf_{u \in A, v \in B} \|u - v\|_p$ and ∂A is the boundary of the set $A \subset H^p$.

Clearly $\alpha > 0$. By Proposition 4.8 and Lemma 4.9, for each $u_0 \in K$, there exists a $m_{u_0} > 0$ and an open ball $\dot{B}_p^u(u_0, r_{u_0})$ of radius $r_{u_0} > 0$ such that

$$\|u^m(s) - \bar{u}^m(s)\|_p \leq \alpha/3 \quad \text{and} \quad |\epsilon^{-1} \mathcal{E}_{p+1}(\bar{u}^m)| \leq C(M_1)/2, \quad (4.17)$$

for all $\bar{u} \in B_p^u(u_0, r_{u_0})$, $m \geq m_0$ and $s \in [0, \tau_0]$. Let B_1, \dots, B_l be the finite covering of the compact set K by such balls. Let

$$D = \cup_{i=1}^l B_i \quad \text{and} \quad \Omega_{\alpha/3} := \{u \in \Omega_1 \mid \text{dist}(u, \partial\Omega_1) \geq \alpha/3\}.$$

By Proposition 4.2,

$$S_m^{\tau_0}(D) \subset \Omega_{\alpha/3},$$

for all large enough $m \in \mathbb{N}$. From inequalities (4.15), (4.16) and (4.17), we know that if m is sufficiently large, then

$$e^{-3C(M_1)\tau_0/2} \mu_p^m(D) \leq \mu_p^m(S_m^{\tau_0}(D)) \leq e^{3C(M_1)\tau_0/2} \mu_p^m(D).$$

By Lemma 4.10, we have

$$\begin{aligned} \mu_p(\Omega) &\leq \mu_p(D) + \delta \leq \liminf_{m \rightarrow \infty} \mu_p^m(D) + \delta \\ &\leq \liminf_{m \rightarrow \infty} e^{3C(M_1)\tau_0/2} \mu_p^m(S_m^{\tau_0}(D)) + \delta \leq \limsup_{m \rightarrow \infty} e^{3C(M_1)\tau_0/2} \mu_p^m(\Omega_{\alpha/3}) + \delta \\ &\leq e^{3C(M_1)\tau_0/2} \mu_p(\Omega_1) + \delta. \end{aligned}$$

Here we have used the Portemanteau theorem. Since δ was chosen arbitrarily, it follows that

$$\mu_p(\Omega) \leq e^{3C(M_1)\tau_0/2} \mu_p(S_\epsilon^{\tau_0}(\Omega)).$$

Similarly, $\mu_p(S^{\tau_0}(\Omega)) \leq e^{3C(M_1)\tau_0/2} \mu_p(\Omega)$. As $\tau_0 \in [0, T]$ is fixed arbitrarily, Theorem 4.1 is proved.

Remark 4.11. *The measure μ_p is also ϵ -quasi-invariant for the following perturbed KdV equations on H^p :*

$$\dot{u} + \epsilon^{-1}(u_{xxx} - 6uu_x) = \partial_x u, \quad (4.18)$$

$$\dot{u} + \epsilon^{-1}(u_{xxx} - 6uu_x) = \partial_x^{-1} u. \quad (4.19)$$

Indeed, consider the following finite dimensional system corresponding to equation (4.18) as in problem (4.5):

$$\dot{u}^m = \epsilon^{-1}[-u_{xxx}^m + 6\mathbb{P}_m(u^m u_x^m)] + \partial_x u^m, \quad u^m(0) = \mathbb{P}_m u_0. \quad (4.20)$$

Let us investigate the quantity $\frac{d}{d\tau} \mathcal{J}_n(u^m)$, $n \geq 3$, for equation (4.20):

$$\frac{d}{d\tau} \mathcal{J}_n(u^m) = \epsilon^{-1} \mathcal{E}_n(u^m) + \langle \nabla_u \mathcal{J}_n(u^m), \partial_x u^m \rangle.$$

For the first term, see in Lemma 4.3. For the second term,

$$\begin{aligned} D_n := \langle \nabla_u \mathcal{J}_n(u^m), \partial_x u^m \rangle &= \int_{\mathbb{T}} \left\{ \partial_x^n u^m \partial_x^{n+1} u^m + c_n \partial_x u^m (\partial_x^{n-1} u^m)^2 \right. \\ &\quad + 2c_n u^m \partial_x^{n-1} u^m \partial_x^n u^m \\ &\quad \left. + \sum_{i=0}^{n-2} \frac{\partial \mathcal{Q}_n(u^m, \dots, \partial_x^{n-2} u^m)}{\partial (\partial_x^i u^m)} \partial^{i+1} u^m \right\} dx. \end{aligned}$$

Notice that the first term in right hand side vanishes. For the second and the third terms,

$$\int_{\mathbb{T}} c_n [\partial_x u^m (\partial_x^{n-1} u^m)^2 + 2u^m \partial_x^{n-1} u^m \partial_x^n u^m] dx = c_n \int_{\mathbb{T}} d[u^m (\partial_x^{n-1} u^m)^2] = 0.$$

So we have

$$|D_n| \leq C(\|u^m\|_{n-1}). \quad (4.21)$$

Note that equation (4.20) can be written as a Hamiltonian system in coordinates $\hat{u}^m = (\hat{u}_1, \hat{u}_{-1}, \dots, \hat{u}_m, \hat{u}_{-m})$:

$$\frac{d}{d\tau} \hat{u}_j = -2\pi j \epsilon^{-1} \frac{\partial H_1(\hat{u}^m)}{\partial \hat{u}_{-j}}, \quad j = \pm 1, \dots, \pm m, \quad (4.22)$$

where the Hamiltonian $H_1(u) = \mathcal{J}_1(u) - \frac{\epsilon}{2} \int_{\mathbb{T}} u^2 dx$. Therefore the divergence for the vector field of equation (4.22) is zero. This property and inequality (4.21) also hold for equation (4.19). Hence the same proof as in this section applies to equation (4.18) and (4.19), which justifies the claim in the Remark 4.11.

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