Pseudo-Néron Model and Restriction of Sections

with an appendix by Jason Michael Starr

Santai Qu

Abstract

We introduce the notion of pseudo-Néron model and give new examples of varieties admitting pseudo-Néron models other than Abelian varieties. As an application of pseudo-Néron models, given a scheme admitting a finite morphism to an Abelian scheme over a positive-dimensional base, we prove that for a very general genus-0, degree-d curve in the base with d sufficiently large, every section of the scheme over the curve is contained in a unique section over the entire base.

Contents

1 Introduction

The main results of this article are Theorem 1.3 and Theorem 1.11 in subsection 1.1. We compare our results with other theorems in literatures in subsection 1.2. The geometric idea to prove Theorem 1.3 is given in subsection 1.3.

1.1 Main results

The starting point of this work is a theorem in [7], where Tom Graber and Jason Michael Starr prove the theorem of restriction of sections for families of Abelian varieties (see Theorem 1.2). To state their theorems, we cite the following definition from [7].

Definition 1.1 ([7], p.312)**.** Let k be an algebraically closed field. Fix a generically finite, generically unramified morphism $u_0 : S \to \mathbb{P}^n_k$. We define

- an u_0-*line* is a curve in S of the form $S \times_{\mathbb{P}^n_k} L$ for a line $L \subset \mathbb{P}^n_k$;
- an u_0-*conic* is a curve in S of the form $S \times_{\mathbb{P}^n_k} C$ for a plane conic $C \subset \mathbb{P}^n_k$;
- an u_0-*line-pair* is a curve in S of the form $S \times_{\mathbb{P}^n_k} L$, where $L = L_1 \cup L_2$ for a pair of incident lines in $\mathbb{P}^n_k$;
- an u_0-*smooth-curve* is an irreducible smooth curve in S of the form $S \times_{\mathbb{P}^n_k} C_0$ for a smooth curve $C_0 \subset \mathbb{P}^n_k$;
- an u_0-*curve-pair* of degree-$(d + 2)$ is a connected curve in S of the form $S \times_{\mathbb{P}^n_k} C$, where $C = C_0 \cup C_1$ is a pair of curves in $\mathbb{P}^n_k$ intersecting transversally at a single closed point such that C_0 is a genus zero, smooth curve of degree d, and C_1 is a smooth conic;
- an u_0-*planar surface* is a surface in S of the form $S \times_{\mathbb{P}^n_k} \Sigma$ for a 2-plane $\Sigma \subset \mathbb{P}^n_k$.

Note that, by Bertini's theorem, for a sufficiently general line, conic, and plane, the corresponding u_0-line, u_0-conic, and u_0-planar surface will be smooth. By abuse of notations, we will just say line, conic, line-pair, curve-pair, planar surface, and smooth curve in S instead of u_0-line, u_0-conic, u_0-line-pair, u_0-curve-pair, u_0-planar surface, and u_0-smooth-curve.

Let k be an uncountable algebraically closed field. We say a subset of a scheme is *general*, resp. *very general*, if the subset contains an open dense subset, resp. the intersection of a countable collection of open dense subsets. We say that a property of points in a scheme holds at a *general point*, resp. at a *very general point*, if the set where the property holds is general, resp. very general.

Now, we state the main theorem in [7] as following.

Theorem 1.2 ([7], Theorem 1.3, p.312)**.** *Let k be an uncountable algebraically closed field. Let S be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let A be an Abelian scheme over S. For a very general line-pair C in S, the restriction map of sections*

$$\text{Sections}(A/S) \to \text{Sections}(A_C/C)$$

is a bijection. The theorem also holds with C a very general planar surface in S. If $\operatorname{char} k = 0$, this also holds with C a very general conic in S.

In this article, we prove that there exists a broader class of varieties for which Theorem 1.2 holds for higher order curve-pairs and smooth curves as in the following theorem.

Theorem 1.3. *Let k be an uncountable algebraically closed field of characteristic zero. Let S be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let X be a smooth S-scheme admitting a finite morphism $f : X \to A$ to an Abelian scheme A over S. Let e be the fiber dimension of $\text{Iso}(A)$ where $\text{Iso}(A)$ is the isotrivial factor of A (see Definition-Lemma 3.15). Let $d \geq 0$ be an even integer.*

Then, for $d > 2e - 2$, every section of X_C over a very general genus-0 and degree-$(d+2)$ curve-pair or a very general genus-0, degree-$(d+2)$ smooth curve C is the restriction of a unique global section of X over S.

Remark 1.4. Let $X \to S$ be a finite type morphism of locally Noetherian schemes where S is integral. Denote by K the function field of S. Then a rational point of the generic fiber X_K is the same as a rational section of $X \to S$. When $X \to S$ is an Abelian scheme, every rational section is a section on the whole S, which also holds when X admits a finite morphism to an Abelian scheme over S. Thus, the Theorems above claim that we can detect rational points of X_K over the function field K by restricting $X \to S$ to a very general curve in S.

In [7], the authors first prove the theorem of restriction of sections for line-pairs. To prove that the result also holds for smooth conics, Néron model is applied to deform line-pairs to smooth conics. We also need this technique in our Theorem 1.3, so we give the definition of Néron models as following.

By a Dedekind scheme, we always mean an irreducible, Noetherian and normal scheme of dimension one. Let S be a Dedekind scheme with function field K. Let X_K be a smooth and separated K-scheme of finite type. We say that X is an *S-model* of X_K if X is an S-scheme with generic fiber isomorphic to X_K. A Néron model of X_K is an S-model satisfying a universal property of extending morphisms. This extends the smooth variety X_K to a family of smooth varieties over S. The precise definition is the following.

Definition 1.5 ([2], Def.1.2/1, p.12)**.** Let X_K be a smooth and separated K-scheme of finite type. A *Néron model* of X_K is an S-model X which is *smooth*, separated, and of finite type, and which satisfies the following universal property, called the *Néron mapping property*:

For each smooth S-scheme Y and each K-morphism $u_K : Y_K \to X_K$ there is a unique S-morphism $u : Y \to X$ extending u_K.

From the uniqueness of the morphism extension, it is easy to see that a Néron model is unique as soon as it exists. If X_K is an Abelian variety over K, then the existence of Néron model is proved in the survey book [2]. However, the Néron model of an Abelian variety is not necessarily an Abelian scheme over the Dedekind scheme S (see [2], Theorem 1.4/3, p.19).

Theorem 1.6 ([2], Theorem 1.4/3, p.19)**.** *Let X_K be an Abelian variety over K. Then X_K admits a Néron model X over S.*

The main application of Néron models in the proof of Theorem 1.2 is Lemma 4.13 in [7], p.323. However, going over the proof, it is easy to see that only the existence of extensions of morphisms is needed, and this is also the case for many other applications of Néron models. This leads us to weaken the definition of Néron mapping property, and consider a weak version of Néron model.

Definition 1.7. Let X be a flat scheme of finite type over S. We say X has the *weak extension property* if for every smooth morphism $Z \to S$ and every K-morphism $u_K : Z_K \to X_K$, there exists an S-morphism $u : Z \to X$ extending u_K.

Definition 1.8. Let X_K be a smooth, finite type K-scheme. Suppose that X is a flat and finite type scheme over S with generic fiber X_K. We say that X is a *pseudo-Néron model of its generic fiber* if X satisfies the weak extension property.

Remark 1.9. In Definition 1.7, the extension u of u_K is not necessarily unique; however, if X is separated, then the extension is unique. In Definition 1.8, we do not require that X is normal or regular since after an étale base change $T \to S$, X_T is not necessarily normal or regular. And, we stress that, unlike Néron models, pseudo-Néron model is usually not unique since it can be not smooth over S.

By [2] Proposition 1.2/8, we know that every Abelian scheme over S satisfies the weak extension property. Moreover, from Theorem 1.6, every Abelian variety has a Néron model, and hence a pseudo-Néron model.

If a smooth variety X_K admits a finite morphism to an Abelian variety A_K, then X_K admits a pseudo-Néron model as we will prove in Theorem 2.5. Besides the application of pseudo-Néron model to prove Theorem 1.3, it is natural to ask:

Question 1.10. *Is there any other class of varieties, besides Abelian schemes, Abelian varieties and finite covers of Abelian varieties, satisfying the weak extension property or admitting pseudo-Néron models?*

In the first part of this article (Section 2), we give a positive answer to this question. It turns out that the existence of pseudo-Néron models is closely related to the non-existence of rational curves on the variety (Corollary 2.8). Our new example of pseudo-Néron models is the following, which will be proved as Corollary 2.16.

Theorem 1.11 (New Examples). *Let k be an uncountable algebraically closed field. Let S be a Dedekind scheme of finite type over k with function field K (e.g., S is a smooth curve over k). Let d be an integer prime to* $\operatorname{char} k$. *Suppose that $H \subset \mathbb{P}^n_k$ is a very general smooth hypersurface of degree $d \geq 2n-1$. Let X_K be a smooth K-variety admitting a finite morphism to $H \times_{\operatorname{Spec} k} \operatorname{Spec} K \subset \mathbb{P}^n_K$. Then X_K has a pseudo-Néron model over S. In particular, every smooth K-subvariety of $H \times_{\operatorname{Spec} k} \operatorname{Spec} K$ has a pseudo-Néron model.*

In the second part of this article (Section 3), we will use pseudo-Néron models to restate the Lemma 4.13 in [7] in a more general set up and prove the main result, Theorem 1.3.

1.2 Review of theorems about sections

Now, we give a brief review of theorems about sections in literatures and compare our main result, Theorem 1.3, with these results.

A complex variety V is said to be *rationally connected* if two general points of V can be joined by a rational curve ([14], Definition 3.2, p.199). In [8], it is proved that an one-parameter family of rationally connected complex varieties has a section.

Theorem 1.12 ([8], Theorem 1.1, p.57). *Let $f : X \to B$ be a proper morphism of complex varieties with B a smooth curve. If the general fiber of f is rationally connected, then f has a section.*

Definition 1.13 ([9], Def.1.2, p.672). Let $\pi : X \to B$ be an arbitrary morphism of complex varieties. By a *pseudosection* of π we will mean a subvariety $Z \subset X$ such that the restriction $\pi|_Z : Z \to B$ is dominant with rationally connected general fiber.

In [9], the authors prove the converse of Theorem 1.12 as following.

Theorem 1.14 ([9], Theorem 1.3, p.672, [7], Theorem 1.1, p.311). *Let $\pi : X \to B$ be a proper morphism of complex varieties. If π admits a section when restricted to a very general sufficiently positive curve in B, then there exists a pseudosection of π.*

However, the theorem asserts only the existence of a pseudosection of π; it does not claim any direct connection between the sections of $X_C \to C$ over very general curves C and the pseudosection. So the following question is asked in [7] and [9].

Question 1.15 ([7], Conjecture 1.2, p.311, [9], Section 7.1, p.689). *If $\pi : X \to B$ is a morphism of complex varieties, then for a very general, sufficiently positive curve $C \subset B$, does every section of the restricted family $X_C = \pi^{-1}(C) \to C$ take values in a pseudosection?*

On the other hand, in Theorem 1.14, the genus and degree of the very general curve depend on the relative dimension of π (see the statement of theorem in

[9], Theorem 1.3, p.672). The genus and degree can grow enormously fast with respect to the relative dimension of $\pi : X \to B$. So it is natural to ask the following question.

Question 1.16 ([9], Section 7.3, p.689)**.** *Can we eliminate the dependence of the family of curves on the relative dimension of π in Theorem 1.14?*

The answer of this question is "no". The detailed proof can be found in [22]. A sketch of the argument could also be found in [9], Section 7.3. Then, a further question is the following.

Question 1.17. *If the dependence in Theorem 1.14 can not be eliminated, how fast do the genus and degree of the family of curves grow?*

One extreme special case of Question 1.15 and Question 1.16 is that X is an Abelian scheme over B. In this case, since the fibers contain no rational curves, every pseudosection is a rational section, and every rational section is everywhere defined. Then, Theorem 1.2 gives positive answers to both Question 1.15 and Question 1.16 when X is an Abelian scheme over B.

When X is a smooth scheme admitting a finite morphism to an Abelian scheme A over B, Theorem 1.3 gives a positive answer to Question 1.15 and Question 1.17. The genus of curves is zero as in Theorem 1.2. And, the degree of the curves grows at a *linear* rate with respect to the relative dimension of the isotrivial factor of the Abelian scheme.

1.3 Idea of the proof

The idea to prove Theorem 1.3 is quite geometric. For a very general point $b \in S$ and a very general genus-0, degree-d, smooth curve m containing b, there will be a subset $\mathcal{B}_{d,b}$ in A_b (we actually take $\mathcal{B}_{d,b}$ in $\mathrm{Iso}(A)_b$, see subsection 3.6) such that Theorem 1.3 does not hold for sections over m that map b to points in $\mathcal{B}_{d,b}$. We call this subset the *bad set*, see subsection 3.3.3 and subsection 3.6 for precise definitions. However, if we attach a very general conic ℓ to m at a very general point on m, the bad set $\mathcal{B}_{d+2,b}$ for the curve-pair $m \cup \ell$ has dimension strictly less than the dimension of $\mathcal{B}_{d,b}$. Therefore, if we increase the degree of the curve by attaching more conic curves, the bad set will be empty, and so Theorem 1.3 holds for every section over the very general curve.

Acknowledgement: The author is very grateful to his PhD thesis advisor Professor Jason Michael Starr for introducing this problem, his consistent support during the proof and his writing for the Appendix A. The author especially wants to thank Professors Aise Johan de Jong and Tom Graber for helpful comments in earlier drafts of this article. The author also thanks the anonymous referees for many helpful suggestions to improve this paper.

2 Pseudo-Néron Model

2.1 Basic properties

In this section, we will assume that S is a *Nagata* Dedekind scheme and K is its function field. Recall that every scheme of finite type over a field is Nagata. By a *variety*, we mean a scheme that is finite type over a field.

Lemma 2.1. *Suppose that $S = \operatorname{Spec} R$ is an affine Nagata Dedekind scheme. Let Y_K be a smooth, finite type K-scheme and Y be a normal pseudo-Néron model over S. Let X_K be a smooth K-scheme with a finite K-morphism $f : X_K \to Y_K$. Then there exists a flat normal S-scheme X admitting a finite morphism $g : X \to Y$ which extends f.*

Proof. It suffices to prove the case when f is surjective and $Y = \operatorname{Spec} A$ is affine. Denote by $\operatorname{Spec} A_K$ the generic fiber Y_K where $A_K = A \otimes_R K$, and $\operatorname{Spec} B_K = f^{-1}(\operatorname{Spec} A_K)$. We claim that there exists a finite A-algebra C such that $B_K = C \otimes_R K$. The field $\operatorname{Frac}(B_K)$ is a finite field extension of $\operatorname{Frac}(A_K)$ since B_K is finite over A_K. Let B' be the integral closure of A in $\operatorname{Frac}(B_K)$. We have the ring A is Nagata ([16], Prop.8.2.29(b), p.340, and Def.8.2.30, p.341), and hence B' is finite over A ([16], Def.8.2.27, p.340). Now, take $C = B_K \cap B'$. Then since A is Noetherian we have C is finite over A, and by construction C is the integral closure of A in B_K. It is easy to check that $B_K = C \otimes_R K$. $\square$

Lemma 2.2. *Let Y be a separated, flat S-scheme of finite type satisfying the weak extension property. Suppose that X is an integral S-scheme with a finite S-morphism $f : X \to Y$. Then X satisfies the weak extension property.*

Remark 2.3. The Dedekind scheme S does not have to be Nagata in this lemma.

Proof. We can assume that S, X and Y are affine. Let Z be an irreducible smooth S-scheme with generic fiber Z_K and a K-morphism $u_K : Z_K \to X_K$. Since Y satisfies the weak extension property, $f_K \circ u_K$ extends to an S-morphism $g : Z \to Y$. Denote by ζ one of the generic points of codimension one irreducible subsets of Z. Then, since Z is normal, $\mathcal{O}_{Z,\zeta}$ is a discrete valuation ring. And we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K(Z) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}_{Z,\zeta} \arrow[r] \arrow[ur, dashed, "u_\zeta"] & Y \end{tikzcd}$$

Moreover, by the properness of the morphism f, there exists a unique morphism $u_\zeta : \operatorname{Spec} \mathcal{O}_{Z,\zeta} \to X$ making the diagram commute. Since Z is locally of finite type over S, the morphism u_ζ can be extended to a neighborhood V of ζ in Z. We denote this morphism by $u_V : V \to X$. Checking every open affine $\operatorname{Spec} C$ in V, since Z is an integral scheme, we have that the generic fiber of the morphism from $\operatorname{Spec} C$ to X is the same as the restriction of u_K. Moreover, suppose that there are two codimension one points ζ_1 and ζ_2, and they are giving two extensions $V_1 \to X$ and $V_2 \to X$ respectively. Since f is separated, it is easy to see that these two morphisms agree on every open affine in the overlap $V_1 \cap V_2$ because they have the same generic fiber. Therefore, the morphism u_K can be extended to a rational map defined over every condimension one point on Z. Hence, by [2] Lemma 4.4/2, since X is affine, this rational map is actually defined everywhere. ☐

Remark 2.4. Let $\mathcal{C}$ denote the category of normal S-schemes with finite morphisms. Then the lemma above asserts that normal S-schemes with the weak extension property form a fully faithful subcategory of $\mathcal{C}$.

Theorem 2.5. *Let S be a Nagata Dedekind scheme with generic point* $\operatorname{Spec} K$. *Let X_K be a smooth scheme admitting a finite K-morphism to a smooth, separated, finite type K-scheme Y_K which has a normal pseudo-Néron model Y over S. Then X_K has a normal pseudo-Néron model X over S.*

Proof. Take an affine cover of S. By Lemma 2.1, we can construct models of X_K over these affine bases. We can glue these models to be an S-model X of X_K over the whole S because the process of taking integral closure is unique up to a unique isomorphism and compatible with localization. Then, since X admits a finite S-morphism to Y, Lemma 2.2 shows that X satisfies the weak extension property. ☐

This theorem gives us a strategy. Suppose that S is a Nagata Dedekind scheme. Then every time we have a class of smooth varieties admitting normal pseudo-Néron models, by considering smooth varieties with finite morphisms to the varieties in this class, we will get a new class of varieties admitting normal pseudo-Néron models. As a first result, we know that all smooth varieties admitting finite morphisms to Abelian varieties have normal pseudo-Néron models. In particular, every smooth subvariety of an Abelian variety has a normal pseudo-Néron model.

2.2 Application of rational curves

In [15], Qing Liu and Jilong Tong prove theorems about Néron models of smooth proper curves of positive genus, see [15], Theorem 1.1, p.7019, for details. In our situation, their result ([15], Prop.4.13, p.7031) in the higher dimensional case can be used to construct new examples of varieties admitting pseudo-Néron models. We start with the basic notion of rational curves as following.

Definition 2.6 ([15], p.7031)**.** Let V be a variety over an algebraically closed field k. We say that V *contains a rational curve* if there is a locally closed subscheme of V which is isomorphic to an open dense subscheme of $\mathbb{P}^1_k$.

If V is proper over k, then every morphism from an open dense subset of $\mathbb{P}^1_k$ can be extended to the whole $\mathbb{P}^1_k$. Moreover, by Lüroth's theorem, our definition is the same as the existence of a nonconstant morphism from $\mathbb{P}^1_k$ to V ([14], Definition 2.6, p.105).

Proposition 2.7 ([15], Prop.4.13, p.7031)**.** *Let S be a Dedekind scheme with function field K. Let X_K be a smooth proper variety over K. Suppose X_K has a proper regular S-model X such that no geometric fiber $X_{\overline{s}}$, $s \in S$, contains a rational curve. Then the smooth locus X_{sm} of X is the Néron model of X_K.*

Note that the regularity of X in the proposition above is only used to apply [15] Cor.3.2. Thus, in the case of pseudo-Néron models, the same proof gives the following corollary.

Corollary 2.8. *Let S be a Dedekind scheme with function field K. Let X_K be a smooth proper variety over K. Suppose X_K has a proper and flat S-model X such that no geometric fiber $X_{\overline{s}}$, $s \in S$, contains a rational curve. Then X satisfies the weak extension property, i.e., X is a pseudo-Néron model of X_K.*

The following lemma is well-known.

Lemma 2.9. *Let k be an algebraically closed field. Let $f : X \to Y$ be an étale surjective k-morphism of proper k-varieties. If X does not contain any rational curve, then Y does not contain any rational curve.*

Lemma 2.9 gives an immediate application of Corollary 2.8 as following.

Corollary 2.10. *Let X be a proper pseudo-Néron S-model of X_K such that no geometric fiber contains rational curves, as in Corollary 2.8. Suppose that Y is a proper S-scheme with smooth generic fiber Y_K and there exists an étale surjective S-morphism $f : X \to Y$. Then Y is a pseudo-Néron model of Y_K.*

There are many varieties which do not contain any rational curve. One of the typical examples is very general hypersurfaces of large degrees. We cite the following result.

Theorem 2.11 ([20], Theorem 1.2)**.** *Let k be an uncountable algebraically closed field. For $d \geq 2n-1$, a very general hypersurface $X \subset \mathbb{P}^n_k$ of degree d contains no rational curves, and moreover, the locus of hypersurfaces that contain rational curves will have codimension at least $d - 2n + 2$.*

We will apply the following lemma to prove Theorem 2.14.

Lemma 2.12. *Let R be a Nagata DVR with fraction field K and residue field k. Suppose that X is a proper scheme over R with nonempty fibers. If $X_{\overline{k}}$ contains no rational curves, then $X_{\overline{K}}$ also contains no rational curves.*

Definition 2.13. Let R be a DVR with fraction field K and $E = R^\times \cup \{0\}$. Let H be a hypersurface in $\mathbb{P}^n_K$. We say that (H, f) is a *unitary hypersurface* if the defining equation f of H has coefficients in E.

Theorem 2.14. *Let R be a Nagata DVR with fraction field K. Suppose that the residue field k is uncountable and algebraically closed, and d is an integer prime to* $\operatorname{char} k$*. Then, there exist unitary hypersurfaces of degree $d \geq 2n-1$ in $\mathbb{P}^n_K$ admitting Néron models.*

Proof. Suppose that $X_K = V_+(f)_K \subset \mathbb{P}^n_K$ is a unitary hypersurface defined by an irreducible homogeneous polynomial f of degree d in $n+1$ variables. Since all the nonzero coefficients of f are in the group of units of K, there is no term in f vanishing in the residue field of R, so the specialization $X_k = V_+(f)_k$ is a hypersurface of degree d in $\mathbb{P}^n_k$. Conversely, every hypersurface of degree d in $\mathbb{P}^n_k$ arises as a specialization of some unitary hypersurface of degree d in $\mathbb{P}^n_K$.

Set $N = \binom{n+d}{d} - 1$. Let M be the space of unitary hypersurfaces in $\mathbb{P}^n_K$. The argument above gives a surjective map of parameter spaces $F : M \to \mathbb{P}^N_k$ by sending (X_K, f) to its specialization X_k. Let U be an intersection of countably many open dense subsets of $\mathbb{P}^N_k$ such that every member in U is smooth without rational curves. Take $X_K \in F^{-1}(U)$ a K-point. By Lemma 2.12, there is no rational curve on $X_{\overline{K}}$. Let $X = V_+(f) \subset \mathbb{P}^n_R$ be the R-model of X_K.

Since f is irreducible, X is an integral hypersurface. Thus, X is flat over $\operatorname{Spec} R$ and every nonempty fiber is irreducible of dimension $n-1$([16], Cor.4.3.10, p.137). By [10] Théorème (12.2.4) (in [10], tome 28, p.183), also X_K is smooth over $\operatorname{Spec} K$ since X_k is smooth. Consequently, every fiber of X is smooth and X is flat over R, then X is a smooth R-scheme ([2], Prop.2.4/8, p.53). Therefore, by Lemma 2.12 and Proposition 2.7, X is the Néron model of X_K. $\square$

This theorem gives a direct corollary in the geometric setting as following.

Corollary 2.15. *Let k be an uncountable algebraically closed field. Let S be a Dedekind scheme of finite type over k with function field K (for example, S is a smooth curve over k). Let d be an integer prime to* $\operatorname{char} k$. *Then, a very general smooth hypersurface of degree $d \geq 2n-1$ defined over k in $\mathbb{P}^n_K$ has a Néron model. In particular, the Néron model of such a hypersurface is the constant family over S.*

Note that we say a K-scheme X is defined over k if there exists a k-scheme Y such that X is isomorphic to $Y \times_{\operatorname{Spec} k} \operatorname{Spec} K$ (see [14], Definition 1.15, p.19). Combining Theorem 2.5 and Corollary 2.15, we get the following corollary.

Corollary 2.16. *Keep the notations of Corollary 2.15. Let X_K be a smooth K-variety admitting a finite morphism to a very general smooth hypersurface of degree $d \geq 2n-1$ defined over k in $\mathbb{P}^n_K$. Then X_K has a normal pseudo-Néron model over S. In particular, every smooth subvariety of a very general hypersurface of degree $d \geq 2n-1$ in $\mathbb{P}^n_K$, where the hypersurface is defined over k, has a normal pseudo-Néron model.*

From this Corollary, we see that there are many smooth varieties admitting normal pseudo-Néron models over a smooth curve defined over an uncountable algebraically closed field. In the situation of Corollary 2.16, we cannot control the regularity of other fibers except X_K. It is a normal model of X_K, but in general not a Néron model in the sense of Definition 1.5. Moreover, the variety X_K is not necessarily defined over k, unlike the constant case in Corollary 2.15.

2.3 Base change properties

In this section, we collect some base change properties of pseudo-Néron models; the proofs are straightforward and left to the readers. The next lemma shows that pseudo-Néron models commute with étale extension of the base scheme, which is the analogue of [2] Prop.1.2/2 (c) for Néron models.

Lemma 2.17. *Let S be a Dedekind scheme with function field K and X_K be a smooth K-variety with pseudo-Néron model X over S. Suppose that S' is another Dedekind scheme with $S' \to S$ étale and the function field of S' is K'. Let $X_{S'} = X \times_S S'$ and $X_{K'} = X_K \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ be its generic fiber. Then $X_{S'}$ is a pseudo-Néron model of $X_{K'}$.*

The following lemma is an analogue of [2] Prop.1.2/4. However, since a pseudo-Néron model is not unique, we can not have the converse direction as in [2] Prop.1.2/4.

Lemma 2.18. *Let S be a Dedekind scheme with function field K, X finite type over S and it is a pseudo-Néron model of its generic fiber. Then, for each closed point $s \in S$, the $\mathcal{O}_{S,s}$-scheme $X_s = X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is a pseudo-Néron model of its generic fiber.*

Proof. Use limit arguments ([2], Lemma 1.2/5) and Lemma 2.17. □

Definition 2.19. Let S be a Dedekind scheme and let X be an S-scheme satisfying the weak extension property. We say that X *satisfies the weak extension property universally* if for any S' a Dedekind scheme and for any finite type morphism $S' \to S$, $X \times_S S' \to S'$ also satisfies the weak extension property.

Definition 2.20. Let S be a Dedekind scheme with function field K. Let X_K be a smooth, finite type K-scheme, and let X be a pseudo-Néron model of X_K. We say that X is a *universal pseudo-Néron model* of X_K if X satisfies the weak extension property universally.

Lemma 2.21. *Keep the notations and hypotheses as in Corollary 2.8. Then, X is a universal pseudo-Néron model of X_K.*

Lemma 2.22. *Let S be a Nagata Dedekind scheme with function field K. Let X_K be a smooth and separated scheme of finite type over $\operatorname{Spec} K$. If X_K has a proper S-model satisfying the weak extension property universally, then $X_{\overline{K}}$ contains no rational curve.*

Proof. We can assume that $S = \operatorname{Spec} R$ for some discrete valuation ring R. If $X_{\overline{K}}$ contains a rational curve, limit argument shows that there exists a discrete valuation ring T in $\overline{K}$ dominating R with fraction field L and residue field l, and a finite morphism $f_L : \mathbb{P}^1_L \to X_L$. Since X is a universal pseudo-Néron model, X_T also satisfies the weak extension property. Then f_L extends to a finite T-morphism $f_T : \mathbb{P}^1_T \to X_T$. Thus, $\mathbb{P}^1_T$ satisfies the weak extension property, which is a contradiction by [2], Example 5, p.75. □

Remark 2.23. Let X_K be a smooth, separated, finite type K-scheme, and let X be a proper S-model of X_K. The results above give us the following picture:

(i) no rational curve in any geometric fiber,

(ii) universal pseudo-Néron model,

(iii) no rational curve in the geometric generic fiber.

Then, (i)⇒(ii)⇒(iii).

3 Theorem of Restriction of Sections

3.1 Higher dimensional pseudo-Néron model

We will need the notion of higher dimensional pseudo-Néron model which generalizes Definition 4.10 in [7].

Definition 3.1. ([7], Definition 4.10) Let S be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. A *flat*, finite type morphism $X \to S$ has the *weak extension property* if for every triple $(Z \to S, U, s_U)$ of

(i) a smooth morphism $Z \to S$,

(ii) a dense, open subset $U \subset S$,

(iii) and an S-morphism $s_U : Z \times_S U \to X_U$,

there exists a pair (V, s_V) of

(i) an open subset $V \subset S$ containing U and all codimension 1 points of S,

(ii) and an S-morphism $s_V : Z \times_S V \to X$ whose restriction to $Z \times_S U$ is equal to s_U.

Definition 3.2. Let S be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. Let K be the function field of S, and X_K be a smooth, finite type K-scheme. A flat, finite type S-scheme X is called a *pseudo-Néron model* of X_K if X_K is isomorphic to its generic fiber and X satisfies the weak extension property as in Definition 3.1.

By a limit argument, it is easy to see that Definition 3.1 (resp. Definition 3.2) agrees with Definition 1.7 (resp. Definition 1.8) when S is a Dedekind scheme and X is separated over S. Now, we prove the corresponding results for Lemma 2.1, Lemma 2.2 and Theorem 2.5.

Lemma 3.3. *Suppose that S is an integral, regular, separated, Noetherian Nagata scheme of dimension $b \geq 1$ with function field K. Let Y_K be a smooth K-variety and Y be its normal pseudo-Néron model over S. Let X_K be a smooth K-variety with a finite K-morphism $f : X_K \to Y_K$. Then, there exists a flat, normal S-scheme X admitting a finite morphism $g : X \to Y$ which extends f.*

Proof. Since S is Noetherian, we can cover S by finitely many affine open subsets. As we assume that S is Nagata, the same proof of Lemma 2.1 gives the extension as claimed. $\square$

Lemma 3.4. *Keep the same hypotheses of S as in Lemma 3.3. Let Y be a separated, flat S-scheme of finite type satisfying the weak extension property. Suppose that X is an integral S-scheme with a finite S-morphism $f : X \to Y$. Then X satisfies the weak extension property.*

Proof. Let U be a dense open subset of S, and let Z be a smooth S-scheme with a U-morphism $t_U : Z_U \to X_U$. Composing this morphism with f_U gives a U-morphism $s_U : Z_U \to Y_U$. Since Y satisfies the weak extension property, there exists an open dense subset V in S containing all the codimension one points, and an extension $s_V : Z_V \to Y_V$ of s_U. Up to replacing S by V, we can assume that V is the whole S. Cover S and Y by open affines as in Lemma 2.2, then the same proof as in Lemma 2.2 completes the proof. $\square$

Therefore, combining the two lemmata above and the proof of Theorem 2.5, we get the following theorem.

Theorem 3.5. *Keep the same hypotheses of S as in Lemma 3.3. Let X_K be a smooth K-scheme admitting a finite K-morphism to a smooth, separated variety Y_K which has a separated, normal pseudo-Néron model Y over S. Then X_K has a normal pseudo-Néron model X over S.*

The result also holds if X_K and Y_K are replaced by some X_U and Y_U defined over a dense open subset U of S.

Moreover, Corollary 4.12 in [7] and Theorem 3.5 give us the following corollary.

Corollary 3.6. *Let W be an integral, regular, separated, Nagata Noetherian scheme of dimension $b \geq 1$. Let S be a dense open subset of W, and let X be a scheme admitting a finite morphism to an Abelian scheme over S. Then, there exists an open subset $\widetilde{S}$ of W containing S and all codimension one points of W such that there exists a normal pseudo-Néron model $\widetilde{X}$ over $\widetilde{S}$ whose restriction over S equals X.*

3.2 Bertini's theorems for higher order curves

Recall that a scheme X is called *algebraically simply connected* if for every connected scheme Y, and every surjective finite étale morphism $f : Y \to X$, the morphism f is an isomorphism ([4], p.97). In this section, by a scheme over a field k or a k-scheme, we mean a scheme that is of finite type over k.

Theorem 3.7 ([17], Proposition 3.1)**.** *Let k be a field of characteristic zero. Let X be a smooth, algebraically simply connected variety over k. Let N be a normal, connected and quasi-projective k-scheme. Let $h : N \to X$ be a projective k-morphism. If the closed subscheme N_h of N where h is not smooth has codimension at least 2, then the geometric generic fiber of h is connected.*

We include the proof here for completeness.

Proof. Let $u : \widetilde{N} \to X$ be the finite part of the Stein factorization of h. Since N_h has codimension at least two, $\widetilde{N}_u$ also has codimension at least two. Since X is smooth, u is étale by the Purity Theorem ([11], X, Section 3). Since X is algebraically simply connected, u is an isomorphism. Therefore, h has connected fibers. $\square$

Proposition 3.8. *Let k be a field of characteristic zero. Let X be a smooth, irreducible k-scheme that is algebraically simply connected. Let Y be an irreducible quasi-projective k-scheme. Let M be a normal, irreducible, quasi-projective k-scheme. Let $(h, g) : M \to X \times_k Y$ be a k-morphism such that h is projective and surjective, and g is dominant and flat with irreducible geometric generic fiber. Let Z be an irreducible k-scheme. Let $f : Z \to Y$ be a finite, surjective k-morphism. Denote by $\nu : N \to Z \times_Y M$ the normalization of the fiber product $Z \times_Y M$. Denote by $h' : N \to X$ the composition of h and the projection from N to M. If the closed subscheme of N where h' is not smooth has codimension at least 2, then the geometric generic fiber of h' is connected.*

$$\begin{tikzcd}
N \arrow[r, "\nu"] \arrow[dr] \arrow[ddrr, bend right, "h'"'] & Z \times_Y M \arrow[rr] \arrow[d] & & Z \arrow[d, "f"] \\
& M \arrow[r, "{(h,g)}"] \arrow[dr, "h"'] & X \times_k Y \arrow[r] \arrow[d, "\mathrm{pr}_1"] & Y \\
& & X &
\end{tikzcd}$$

Proof. Since the geometric generic fiber of g is irreducible and Z is irreducible, also the normalization N is irreducible, see the proof of [23], Proposition 3.1. Then the statement reduces to Theorem 3.7 because h' is projective. $\square$

Definition 3.9. Let Y be a regular locally Noetherian scheme. Let $f : Z \to Y$ be a finite surjective morphism that is generically étale. The closed subscheme R inside Z where f is not étale is called *the ramification locus* of f. The closed subscheme $B = f(R)$ of Y is called *the branch locus* of f.

Note that, by Zariski's purity theorem ([16], Exercise 8.2.15(c), p.347), the branch locus B of f is either empty or pure of codimension one if f generically separable. In particular, this holds when $\mathrm{char}\, k = 0$ and Y is a smooth k-scheme.

Theorem 3.10. *Let k be an algebraically closed field of characteristic zero. Let Z be a normal k-scheme. Let $f : Z \to \mathbb{P}^n_k$ be a finite surjective morphism that is generically étale. Then, for a general smooth curve $C \subset \mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is a smooth curve.*

Proof. This follows from Kleiman-Bertini's Theorem ([12], Theorem III.10.8). $\square$

Theorem 3.11. *Let k be an algebraically closed field of characteristic zero. Let $f : Z \to \mathbb{P}^n_k$ be a finite surjective morphism from a normal irreducible variety. Then, for a general genus-0, degree-d smooth curve C in $\mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is connected.*

Proof. Keep the notations in Proposition 3.8. Let Y be the projective space $\mathbb{P}^n_k$. Let X be the non-stacky locus inside the stack of genus-0, degree-d stable maps from nonsingular curves to $\mathbb{P}^n_k$. In other words, X is the maximal open subscheme of this stack. Denote by $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ the scheme of morphisms of degree d from $\mathbb{P}^1_k$ to $\mathbb{P}^n_k$, which is an open subset of the projective space $P = \mathbb{P}((S^d k^{\oplus 2})^{n+1})$, see [4], p.38-39. Also $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ has codimension two in P, and hence $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k)$ is algebraically simply connected. Also, there is a canonical morphism $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^n_k) \to X$ that dominates X. A general fiber of this morphism is isomorphic to $PGL_2(k)$, which is connected. Consequently, the scheme X is algebraically simply connected. Let M be the universal family of curves over X, and let h be the universal morphism. Denote by g the projection from M to Y. Then, the geometric generic fiber of g is irreducible.

To complete the proof, we need to prove that the singular locus of h' inside N has codimension at least 2. Since the singular locus of Z has codimension at least two in Z, we can assume that Z is nonsingular. Let $D \subset X$ be the subset that parameterizes genus-0, degree-d stable maps whose images in $\mathbb{P}^n_k$ are not transversal to the branch locus of f. Then D has codimension one in X. Thus, away from D, the fibers of h' are everywhere smooth (by looking at the tangent spaces). Moreover, for a genus-0, degree-d nonsingular curve C that is not transversal to the branch locus, the singularities of the fiber of h' over $[C]$ occur only over the intersection points of the curve with the branch locus, and this is codimension one in the fiber. Thus, the total codimension of singular locus of h' in N is at least two. By Proposition 3.8, for a general genus-0, degree-d nonsingular curve C in $\mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is connected. □

Corollary 3.12. *Let k be an algebraically closed field of characteristic zero. Let $f : Z \to \mathbb{P}^n_k$ be a finite surjective morphism from a normal irreducible variety. Then, for a general genus-0, degree-d smooth curve C in $\mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is smooth and irreducible.*

Theorem 3.13. *Let k be an algebraically closed field of characteristic zero. Let Z be a normal k-scheme that is not necessarily connected. Let $f : Z \to S$ be a finite surjective morphism to a smooth, connected, quasi-projective k-scheme S where S admits a finite, generically étale morphism to an open dense subset of $\mathbb{P}^n_k$, $u_0 : S \to \mathbb{P}^n_k$. Then, for a general genus-0, degree-d smooth curve C in S (see Definition 1.1), the restriction map of sections*

$$\mathrm{Sections}(Z/S) \to \mathrm{Sections}(Z_C/C)$$

is bijective.

Proof. Shrinking S if necessary, we can assume that f is étale. Also, we can assume that Z is connected. By taking a normal projective compactification of

Z, Corollary 3.12 shows that for a general genus-0, degree-d curve C the inverse image $f^{-1}(C)$ is an irreducible and smooth curve. If $\deg(f)$ is strictly greater than one, then there is no section for f ([24], Prop.5.3.1, p.165). Since f is flat, $\deg(f|_{f^{-1}(C)})$ equals $\deg(f)$ ([16], Exercise 5.1.25(a), p.176). Thus, there is no section for $f|_{f^{-1}(C)}$ neither. If $\deg(f)$ is one, then the restriction map of sections is trivially bijective. □

3.3 Notations and Set up

In the rest of this paper, we will assume that S is a smooth, quasi-projective k-variety of dimension≥ 2, where k is an uncountable algebraically closed field of characteristic zero. And we fix a generically finite dominant morphism $u_0 : S \to \mathbb{P}^n_k$ so that we can talk about lines and line-pairs, or curves and curve-pairs in S (see Definition 1.1). Note that, without changing any of the results, we can shrink S to a dense open subset and assume further that u_0 is a finite, étale morphism onto a dense open subset of $\mathbb{P}^n_k$.

3.3.1 Isotrivial quotient and spaces of sections

Recall that an Abelian scheme over S is defined as a proper and smooth S-group scheme with connected fibers. Theorem 1.2 gives the result for restriction of sections over line-pairs for families of Abelian varieties. We hope to generalize Theorem 1.2 to schemes X admitting a finite morphism to some Abelian scheme A over S. Unfortunately, in this situation, the trick of taking boundaries fails to apply on X (cf. [7], Lemma 4.3, Lemma 4.4 and Lemma 4.5). So the isotrivial factor of A gives moduli of sections (see Remark 3.46), and hence we have to consider curves of higher degrees instead of line-pairs. The process of the proof will involve the application of pseudo-Néron models.

We first fix some notations to clarify the situation. Let A be an Abelian scheme over S, and $f : X \to A$ a finite S-morphism. There exists an open dense subset $V \subset S$ and a finite étale Galois cover $p : V' \to V$ such that the pullback of A to V' is isogenous to a product of a strongly nonisotrivial family of Abelian varieties and a trivial family (see the proof of Theorem 4.7 in [7]). We can assume that $V = S$, and denote V' by S'.

Notation 3.14. By [7], Corollary 3.7, p.317, there exist

- an Abelian k-variety A_0, an S'-morphism of Abelian schemes,

$$v_0 : S' \times_k A_0 \to S' \times_S A,$$

  such that (A_0, v_0) is a Chow S'/k-trace of $S' \times_S A$,

- a strongly nonisotrivial Abelian scheme Q over S' with $v_Q : Q \to S' \times_S A$ an S'-morphism of Abelian schemes, and

- an isogeny of Abelian schemes over S',

$$\rho_{iso} := v_0 \times v_Q : (S' \times_k A_0) \times_{S'} Q \to S' \times_S A.$$

Since $A_0 \times_k S'$ is projective over S', the Weil restriction $\mathfrak{R}_{S'/S}(A_0 \times_k S')$ exists ([2], Theorem 7.6/4, p.194). Moreover, since S is a normal scheme, it is geometrically unibranch. Thus, A is projective over S ([21], Théorème XI 1.4). Therefore, the Weil restriction $\mathfrak{R}_{S'/S}(A \times_S S')$ also exists. We can check that the functorial morphism

$$\mathfrak{R}_{S'/S}(v_0) : \mathfrak{R}_{S'/S}(A_0 \times_k S') \to \mathfrak{R}_{S'/S}(A \times_S S')$$

is a closed immersion (see the discussion after Theorem 6.2 in [3]).

Definition-Lemma 3.15. Let $A \to \mathfrak{R}_{S'/S}(A \times_S S')$ be the functorial morphism of S-schemes, which is a closed immersion since A is separated over S ([2], p.197). The *isotrivial factor* $\mathrm{Iso}(A)$ of the Abelian scheme A over S is the fiber product of A and $\mathfrak{R}_{S'/S}(A_0 \times_k S')$ over $\mathfrak{R}_{S'/S}(A \times_S S')$. In other words, the following diagram is Cartesian

$$\begin{tikzcd}
\mathrm{Iso}(A) \arrow[r] \arrow[d] & \mathfrak{R}_{S'/S}(A_0 \times_k S') \arrow[d, "\mathfrak{R}_{S'/S}(v_0)"] \\
A \arrow[r] & \mathfrak{R}_{S'/S}(A \times_S S').
\end{tikzcd}$$

It is easy to check that $\mathrm{Iso}(A)$ is a closed Abelian subgroup scheme of A over S, and the fiber dimension of $\mathrm{Iso}(A) \to S$ is just $\dim A_0$.

Remark 3.16. By Poincaré's complete reducibility theorem, there exists a morphism of S-schemes $\pi : A \to \mathrm{Iso}(A)$ such that the composition

$$\mathrm{Iso}(A) \longrightarrow A \xrightarrow{\pi} \mathrm{Iso}(A)$$

is an isogeny on the generic fiber of $\mathrm{Iso}(A) \to S$. We call such a morphism π an *isotrivial quotient* of the Abelian scheme A. For the rest of this article, we fix an isotrivial quotient $\pi : A \to \mathrm{Iso}(A)$. Denote by $\rho : \mathrm{Iso}(A) \to S$ the structure morphism of $\mathrm{Iso}(A)$. If b is a point in S, we will denote the fiber of $\mathrm{Iso}(A)$ over b by $\mathrm{Iso}(A)_b$.

Notation 3.17. Let $\mathrm{Sections}_b^p(A/S)$ (resp. $\mathrm{Sections}_b^p(X/S)$) be the set of sections of A (resp. X) over S such that every section in the set maps $b \in S$ to $p \in \mathrm{Iso}(A)$ via $\pi : A \to \mathrm{Iso}(A)$ (resp. $\pi \circ f$).

Notation 3.18. Take a smooth, projective compactification W of S. Let $\widetilde{A} \to W_0$ be the Néron model of $A \to S$ where W_0 is an open dense subset of W which contains all the codimension one points of W. Let $\overline{A} \to W$ be a projective morphism whose restriction over W_0 equals $\widetilde{A}$.

Every section of A over S gives a unique rational section of $\overline{A} \to W$, whose maximal domain of definition contains all the codimension one points of W since A has a Néron model. Conversely, let τ be a rational section of $\overline{A} \to W$. The maximal domain of definition of τ contains S since every geometric fiber of $A \to S$ does not contain any rational curve ([6], Proposition 6.2, p.1234). Thus, the set of rational sections of $\overline{A} \to W$ bijectively corresponds to the set of sections of $A \to S$.

There is a Chow variety parameterizing cycles in $\overline{A}$. The Chow variety has countably many irreducible components. There is an open subset of this Chow variety parameterizing cycles $Z \subset \overline{A}$ such that $Z \to W$ is birational. Then, this variety is the parameter space for rational sections of $\overline{A} \to W$. By the correspondence of rational sections of $\overline{A} \to W$ and sections of $A \to S$ as above, also this is the parameter space for sections of $A \to S$. Denote by this parameter space $\mathbf{Sec}(A/S)$. Every irreducible component of $\mathbf{Sec}(A/S)$ is quasi-projective over k.

The fiber product $\mathbf{Sec}(A/S) \times_k S$ parameterizes the pairs $([\sigma], b)$ where $\sigma \in \mathrm{Sections}(A/S)$ and $b \in S$. Let $\mathrm{pr}_{\mathrm{ev}} : \mathbf{Sec}(A/S) \times_k S \to S$ be the projection to S, i.e., mapping $([\sigma], b)$ to b. Denote by $\mathbf{Sec}_b(A/S)$ the product $\mathbf{Sec}(A/S) \times \{b\}$. Let $\Sigma : \mathbf{Sec}(A/S) \times_k S \to A$ be the universal section over $\mathbf{Sec}(A/S)$. Then the morphism $\Sigma_b := \pi \circ (\Sigma|_{\mathbf{Sec}_b(A/S)}) : \mathbf{Sec}_b(A/S) \to \mathrm{Iso}(A)$ maps each $[\sigma]$, a section σ of A over S, to $(\pi \circ \sigma)(b) \in \mathrm{Iso}(A)$. Therefore, $\mathbf{Sec}_b(A/S)$ is also a space parameterizing pairs $([\sigma], p)$, a closed point $p \in \mathrm{Iso}(A)$ and a section of A over S, $\sigma \in \mathrm{Sections}(A/S)$, such that $\sigma(b) = p$. Denote by $\mathbf{Sec}_b^p(A/S)$ the fiber of Σ_b over a point $p \in \mathrm{Iso}(A)_b$. Similarly, we can define $\mathbf{Sec}_b(X/S)$ and $\mathbf{Sec}_b^p(X/S)$.

3.3.2 Curves and curve-pairs

Take a smooth, projective compactification W of S such that there is a projective, surjective morphism $W \to \mathbb{P}_k^n$ extending $u_0 : S \to \mathbb{P}_k^n$. Let X be a scheme over S such that it admits a pseudo-Néron model $\widetilde{X}$ over an open subset $\widetilde{S}$, containing S, and the complement of $\widetilde{S}$ in W has codimension at least two, e.g., X admits a finite morphism to an Abelian scheme A over S, cf. Corollary 3.6.

Definition 3.19. An *1-pointed, genus 0, degree* $d{+}2$*, k-curve-pair* $C = (s, [m], t, [\ell])$ in W is a projective, connected, reduced, nodal curve of arithmetic genus 0 and degree $d + 2$ (see Definition 1.1) such that

- $C = m \cup \ell$ where m is a smooth curve of genus 0, degree d, ℓ is a smooth conic, and m intersects with ℓ transversally at a closed point t,
- the marked point $s \in m$ is a nonsingular closed point of C.

Notation 3.20. Let $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$ be the stack over k:

$$T \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of flat, projective families } \mathcal{A}_T \to T \text{ of} \\ \text{genus zero, degree } d+2, \text{ 1-pointed, reduced, embedded curves} \\ \text{in } W, \text{ at worst curve-pairs, see Definition 3.19} \end{array} \right\}$$

where T is a k-scheme. Moreover, if the marked point on every curve is a fixed point $b \in W$, we denote the stack by $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$.

Remark 3.21. Our notations are motivated by [1] and [5], but the curves we consider have at most two irreducible components. Also, if $d > 2$, the marked point is always on the irreducible component that is not a smooth conic.

Notation 3.22. Denote by $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$ (resp. $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$) the substack of $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$ (resp. $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$) of 1-pointed *curve-pairs*.

Keep the notations in Definition 3.19 and Notation 3.20, the marked point $s \in C$ defines an evaluation morphism

$$\rho_{\mathrm{ev}} : \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2) \to W, (s, [m], t, [\ell]) \mapsto s$$

whose fiber over $b \in W$ is just $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$.

Notation 3.23. Denote by $\mathcal{M}_{0,1}(W, d)$ the stack over k:

$$T \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of flat, projective families } \mathcal{B}_T \text{ over } T \\ \text{of pairs } (s, [m]) \text{ of genus zero, degree } d\text{, smooth, embedded} \\ \text{curve } m \subset W \text{ with a marked point } s \in m \end{array} \right\}$$

where T is a k-scheme. If the marked point is a fixed closed point $b \in W$, we denote the stack by $\mathcal{M}_{0,1}(W, d; b)$. Similarly, denote by $\mathcal{M}_{0,0}(W, 2)$ the stack of smooth embedded conic curves; note that there is no marked points on conics.

There are forgetful morphisms from $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$:

$$\delta_1 : \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2) \to \mathcal{M}_{0,1}(W, d), (s, [m], t, [\ell]) \mapsto (s, [m]),$$
$$\delta_2 : \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2) \to \mathcal{M}_{0,0}(W, 2), (s, [m], t, [\ell]) \mapsto [\ell].$$

Notation 3.24. As substacks of $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$, denote the open locus of genus-0, degree-$(d+2)$ curves or curve-pairs contained in $\widetilde{S}$ (resp. S) by $\overline{\mathcal{M}}_{0,1}^{\le 2}(\widetilde{S}, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(\widetilde{S}, d+2)$ (resp. $\overline{\mathcal{M}}_{0,1}^{\le 2}(S, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(S, d+2)$). Similarly, we can define the substacks of $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$ for curves or curve-pairs contained in $\widetilde{S}$ and S.

Remark 3.25. Taking the maximal open subschemes of the stacks in Notation 3.24, we can assume that they are all schemes over k. Then, the scheme $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$ is an integral and smooth k-scheme and the locus $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$ is an integral, smooth divisor in $\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2)$.

Notation 3.26. Suppose that $\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(S, d+2)^{\mathrm{RS}} \subset \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(S, d+2)$ is the subset parameterizing very general genus zero, degree-$(d+2)$ curve-pairs C in S such that the restriction of sections

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}((A \times_S C)/C) \qquad (\dagger)$$

is bijective. Denote by $\mathcal{M}_{0,1}(S,d)^{\mathrm{RS}} \subset \mathcal{M}_{0,1}(S,d)$ the subset parameterizing very general genus-0, degree-d smooth curves such that (†) is bijective. We will prove the existence of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2)^{\mathrm{RS}}$ and $\mathcal{M}_{0,1}(S,d)^{\mathrm{RS}}$ for $d \geq 2$ an even integer in Corollary 3.41. Similarly, for a fixed very general closed point $b \in S$, we can define $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2;b)^{\mathrm{RS}}$ and $\mathcal{M}_{0,1}(S,d;b)^{\mathrm{RS}}$.

3.3.3 Bad sets in parameter spaces

Now we define the bad set for sections and curve-pairs. The fiber product $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2) \times_{\rho_{\mathrm{ev}},S,\mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$, see the notation at the end of Notation 3.18 and the notation after Notation 3.22, parameterizes the pairs of sections and curve-pairs $((b,[m],t,[\ell]),([\sigma],p))$ where σ is a section of A over S mapping b to p and the marked point on the curve m is a point $b \in S$.

For a pair of closed points $b \in S$, $p \in \mathrm{Iso}(A)_b$, let σ be a section of A over S that maps b to p. Denote by $C = m \cup \ell$ a curve-pair in S where m is a smooth curve, ℓ is a conic. Consider the following two properties,

(i) $\mathrm{Sections}_b^p(A/S) \to \mathrm{Sections}_b^p((A \times_S C)/C)$ is bijective;

(ii) $\mathrm{Sections}_b^p((X \times_{A,\sigma} S)/S) \to \mathrm{Sections}_b^p((X \times_{A,\sigma} S \times_S \ell)/\ell)$ is bijective,

where the maps of the sets of sections are restrictions and the fiber product $X \times_{A,\sigma} S$ comes from the section σ from S to A mapping b to p.

Definition 3.27. The subset

$$\mathcal{B}\mathrm{ad}(d+2) \subset \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2) \times_{\rho_{\mathrm{ev}},S,\mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$$

such that either (i) is false or (ii) is false is called *the bad set of sections and curve-pairs*. Denote by $\mathcal{B}\mathrm{ad}(d+2;b)$ the fiber of $\mathcal{B}\mathrm{ad}(d+2)$ over $b \in S$, and we call $\mathcal{B}\mathrm{ad}(d+2;b)$ *the bad set of sections and curve-pairs marked by b.*

Remark 3.28. The subset $\mathcal{B}\mathrm{ad}(d+2;b)$ is contained in $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2;b) \times_k \mathbf{Sec}_b(A/S)$. Denote by ϕ_1 and ϕ_2 the projections from $\mathcal{B}\mathrm{ad}(d+2;b)$ to $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S,d+2;b)$ and $\mathbf{Sec}_b(A/S)$. The composition of ϕ_2 and Σ_b (see Notation 3.18) gives

$$\phi_3 : \mathcal{B}\mathrm{ad}(d+2;b) \to \mathrm{Iso}(A)_b, \text{ by } \{(b,[m],t,[\ell]),([\sigma],p)\} \mapsto p.$$

Definition 3.29. Denote by $\mathcal{B}\mathrm{ad}(d+2;b,p)$ the fiber of ϕ_3 over $p \in \mathrm{Iso}(A)_b$. If

$$\{(b,[m],t,[\ell]),([\sigma],p)\} \in \mathcal{B}\mathrm{ad}(d+2;b,p),$$

p is called a *bad point* for the curve-pair $(b,[m],t,[\ell])$, and σ is called a *bad section* for $(b,[m],t,[\ell])$. Otherwise, $(b,[m],t,[\ell])$ is called *good for the section* $\sigma \in \mathrm{Sections}_b^p(A/S)$. If $(b,[m],t,[\ell])$ is good for every section in $\mathrm{Sections}_b^p(A/S)$, the point p is called a *good point* for $(b,[m],t,[\ell])$.

Remark 3.30. Fix a closed point $b \in S$, and a closed point $p \in \mathrm{Iso}(A)_b$. By Theorem 3.13, there is a subset $\mathcal{U} \subset \mathcal{M}_{0,0}(S, 2)$ parameterizing very general points in $\mathcal{M}_{0,0}(S, 2)$ such that for each $[\ell] \in \mathcal{U}$ and each $\sigma \in \mathrm{Sections}_b^p(A/S)$ the property (ii) holds. Then, $\mathcal{B}\mathrm{ad}(d + 2; b, p)$ is contained in the complement of the subset

$$(\delta_2^{-1}(\mathcal{U}) \cap \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d + 2; b)^{\mathrm{RS}}) \times_{\mathrm{Spec}\, k} \mathbf{Sec}_b^p(A/S).$$

Definition 3.31. An irreducible smooth curve C with a marked point $b \in S$ is called *good for a section σ in* $\mathrm{Sections}_b^p(A/S)$ if the following two properties hold

(i) $\mathrm{Sections}_b^p(A/S) \to \mathrm{Sections}_b^p((A \times_S C)/C)$ is bijective,

(ii) $\mathrm{Sections}_b^p((X \times_{A,\sigma} S)/S) \to \mathrm{Sections}_b^p((X \times_{A,\sigma} S \times_S C)/C)$ is bijective,

where the maps of the sets of sections are restrictions and the fiber product $X \times_{A,\sigma} S$ comes from the section σ from S to A. And p is a *good point* for $(b, [C])$ if $(b, [C])$ is good for every section in $\mathrm{Sections}_b^p(A/S)$.

The same construction as for curve-pairs shows that there is a subset $\mathcal{B}\mathrm{ad}(d)$ in the fiber product of $\mathcal{M}_{0,1}(S, d)$ and $\mathbf{Sec}(A/S)\times_k S$ over S via evaluation maps such that $\mathcal{B}\mathrm{ad}(d)$ parameterizes smooth curves and sections making either (i) false or (ii) false. Also, for fixed closed points $b \in S$ and $p \in \mathrm{Iso}(A)_b$, $\mathcal{B}\mathrm{ad}(d; b, p)$ is contained in a countable union of closed subsets.

Remark 3.32. Strictly speaking, we should use different notations for bad sets of curve-pairs and smooth curves since they are in different parameter spaces. Since in our proof of Theorem 1.3 every family of curves is clear from context, we apply the same notation $\mathcal{B}\mathrm{ad}$ and just indicate the degrees of the curves.

3.3.4 Universal curves and universal sections

Notation 3.33. Let $\mathcal{C}_{\overline{\mathcal{M}}(W)} \subset \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d + 2; b) \times_k W$ be the universal family of curves over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d + 2; b)$. Denote the open subset of universal family of curves in $\widetilde{S}$ (resp. S) by $\mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})}$ (resp. $\mathcal{C}_{\overline{\mathcal{M}}(S)}$).

Notation 3.34. Fix a closed point $b \in S$ and $p \in \mathrm{Iso}(A)_b$. Denote by H_b^p the scheme parameterizing sections γ of $X_C \to C$ where C is a curve parameterized by $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d + 2; b)$ *and γ maps b to p via $\pi \circ f$.*

Equivalently, we have the following diagram, where Φ is the composition of the structure morphism of H_b^p over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d + 2; b)$ and the open immersion $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d + 2; b) \to \overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d + 2; b)$ and all squares are Cartesian.

$$\begin{tikzcd}
H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \times_{\widetilde{S}} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{X} \times_{\widetilde{S}} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \arrow[r] \arrow[d] & \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \arrow[r] \arrow[d] & \widetilde{S} \\
H_b^p \arrow[r, "\Phi"] & \overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b) &
\end{tikzcd}$$

We note that H_b^p is locally of finite type over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$, and hence over k, but may have infinitely many irreducible components. Every irreducible component of H_b^p is quasi-projective over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$.

Notation 3.35. There is a universal section of $H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \times_{\widetilde{S}} \widetilde{X} \to H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})}$. We compose it with the top row of the diagram above, and denote the morphism by

$$\varrho : H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \to \widetilde{X},$$

which factors through the inclusion $X \to \widetilde{X}$.

3.4 Restrictions of Sections for Abelian Schemes

3.4.1 Main pseudo-Néron model theorem

In this subsection, we prove the key theorem that applies pseudo-Néron models to the problem of restriction of sections. The proof is similar to Lemma 4.13 of [7], but we prove a relative version of this result, i.e., for fixed $b \in S$ and $p \in \mathrm{Iso}(A)_b$. The proof of the following lemma is left to the readers.

Lemma 3.36. *Let $X \to S$ be a morphism locally of finite type of regular Noetherian schemes. Let Z be a codimension one regular closed subscheme of X, and suppose that $Z \to S$ is smooth. Then, there exists an open subset U of X that contains Z such that $U \to S$ is smooth.*

Recall that W is a smooth, projective compactification of S.

Theorem 3.37. *Suppose that:*

- *X is smooth, projective over S,*
- *X has a pseudo-Néron model $\widetilde{X}$ over an open dense subset $\widetilde{S} \subset W$, where $\widetilde{S}$ has codimension at least two in W, and*
- *every geometric fiber $X_{\overline{s}}$ for $s \in S$ does not contain any rational curve.*

Then, sections of $X \to S$ over genus-0, degree-$(d+2)$ smooth curves specialize to sections over genus-0, degree-$(d+2)$ curve-pairs. More precisely, any irreducible component H_0 of H_b^p which dominates $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$ also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$. That is, the intersection of the image $\Phi(H_0)$ with $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$ contains a dense open subset of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$.

Proof. As in [7], we consider the diagram:

$$\begin{tikzcd}
\mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \arrow[r] \arrow[d] & \mathcal{C}_{\overline{\mathcal{M}}(W)} \arrow[r] \arrow[d] & W \\
\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) \arrow[r] & \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) &
\end{tikzcd}$$

where $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ is the universal family of curve-pairs with nodes of curve-pairs deleted so that $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \to \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$ is smooth. Also, the morphisms $\mathcal{C}_{\overline{\mathcal{M}}(W)} \to W$ and the composition $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \to W$ are smooth.

Since H_0 is quasi-projective over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$, we can choose a compactification $\overline{H}$ of H_0 such that $\overline{H}$ is normal and Φ extends to a proper surjection $\overline{\Phi} : \overline{H} \to \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$. Thus, $D := \overline{\Phi}^{-1}(\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b))$ is nonempty, and we may assume that D is irreducible. Since $\overline{H}$ is normal, it is regular at a general point of D. Set D_{red} as the reduced structure of D. Denote by $\overline{H}^{reg}$ the regular locus of $\overline{H}$. By the generic smoothness theorem, there is a dense open $V \subset D_{\mathrm{red}} \cap \overline{H}^{reg}$ such that $\overline{\Phi} : D_{\mathrm{red}} \to \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$ is smooth on V.

Denote by $\mathcal{C}_{\overline{H}}$ the base change $\mathcal{C}_{\overline{\mathcal{M}}(W)} \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(W,d+2;b)} \overline{H}$. Then, $\mathcal{C}_V \to W$ is smooth; $\mathcal{C}_V \to \mathcal{C}_{\overline{H}}$ is an immersion and V is contained in the regular locus of $\overline{H}$. We summarize the objects in the following diagram (cf. [7], p.324).

$$\begin{tikzcd}
\mathcal{C}_V \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \mathcal{C}_{\overline{H}} \arrow[r, dashed, "\varrho"] \arrow[dl] \arrow[ddd] & \widetilde{X} \arrow[d] \\
& \mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \arrow[r] \arrow[d] & \mathcal{C}_{\overline{\mathcal{M}}(W)} \arrow[rr, dotted] \arrow[d] & & W \\
& \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) \arrow[r, hook] & \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) & & \\
V \arrow[ur, "\overline{\Phi}"] \arrow[rr] & & D \arrow[ul, "\overline{\Phi}"'] \arrow[r, hook] & \overline{H} \arrow[ul, "\overline{\Phi}"'] &
\end{tikzcd}$$

By Lemma 3.36, there exists a maximal open subset U of $\mathcal{C}_{\overline{H}}$ containing $\mathcal{C}_V$ such that $U \to W$ is smooth. The universal section ϱ (see Notation 3.35) gives a rational map $U \dashrightarrow X$, which is marked as the dashed arrow from $\mathcal{C}_{\overline{H}}$ to $\widetilde{X}$ in the diagram above. Denote this rational map also by ϱ.

Now, let $\widetilde{W} \subset W$ be the image of $U \to W$. By construction, $\widetilde{W}$ contains an open dense subset of the image of $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ in W. Shrinking S if necessary, also we can assume that $\widetilde{W}$ contains S. Since $\widetilde{X}$ is a pseudo-Néron model of X over $\widetilde{S}$, $U \dashrightarrow \widetilde{X}$ is well-defined outside a codimension two subset W^c of $\widetilde{W}$ by the weak extension property. Also, by construction, the closed point $b \in S$ is not contained in W^c. Therefore, W^c can be avoided by a general smooth curve or curve-pair in S that passes through $b \in S$. This gives an open dense subset U' of U such that U' contains an open dense subset of $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ and ϱ is well-defined on U'. As a consequence, every curve-pair intersecting an open subset of $\mathcal{C}_{\partial\overline{\mathcal{M}}(S)}$ with nodes deleted can be lifted. Moreover, since there is no rational curve on every geometric fiber of $X \to S$, any morphism from a punctured curve-pair to X can be extended to the node by [7] Remark 4.14, p.323. Since H_b^p parameterizes the space of genus zero, degree-$(d+2)$ curves that can be lifted (Notation 3.34), an open subset of V is contained in H_0, i.e., H_0 also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$. □

Remark 3.38. Technically, the assumption that the characteristic of k is zero is required to apply the generic smoothness theorem.

3.4.2 Inductive pseudo-Néron deformation step

Now we prove the existence of the subset $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)^{\mathrm{RS}}$ of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)$, see Notation 3.26. That is, the subset parameterizing very general genus zero, degree-$(d+2)$ curve-pairs C such that the restriction

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}((A \times_S C)/C)$$

is bijective. In [7] (see Theorem 1.2), it was proved that this subset exists if C is a very general line-pair. And since we work over a field of characteristic zero, the same is true for a very general smooth conic by using the pseudo-Néron model (Theorem 1.2). The following inductive step gives the relative version of Theorem 1.2 for very general genus-0, degree-$(d+2)$ curves, and curve-pairs with fixed $b \in S$, $p \in \mathrm{Iso}(A)_b$.

Corollary 3.39. *Fix $b \in S$ and $p \in \mathrm{Iso}(A)_b$ closed points. Suppose that for a very general genus-0, degree-$(d+2)$ curve-pair $C = m \cup \ell$, where ℓ is a smooth conic, every section in $\mathrm{Sections}_b^p(X_C/C)$ is the restriction of a unique section in $\mathrm{Sections}_b^p(X/S)$, see Notation 3.17. Then, for a very general genus-0, degree-$(d+2)$ irreducible smooth curve containing b, every section over this curve mapping b to p is the restriction of a unique section of X over S.*

Proof. Since k has characteristic zero, $f : X \to A$ is generically unramified. Thus, up to shrinking S, we assume that f is finite and unramified. Let $\mathcal{Y}$ be the scheme parameterizing sections of X over S mapping b to p. Then, because f is unramified, H_b^p has reduced fibers over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$. Moreover, there is a $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$-morphism $\Upsilon : \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b) \times_k \mathcal{Y} \to H_b^p$ that maps $\{[C_1]\} \times \{[\sigma]\}$ to $[\sigma|_{C_1}]$ for each $[C_1] \in \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ and $[\sigma] \in \mathcal{Y}$. The image of this morphism is a union of irreducible components of H_b^p.

Suppose that there exists an irreducible component of H_b^p dominating $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ such that a generic section parameterized by this component is not a restriction of a section of X over S. By Theorem 3.37, this irreducible component also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$. But then by hypothesis this component intersects with the image of Υ, which contradicts that the fibers of $H_b^p \to \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ are reduced. □

Lemma 3.40. *Suppose that every section of A over a very general genus-0, degree-d, irreducible, smooth curve is contained in a unique section of A over S. Then, for very general points b' and b'' in S and for a very general conic ℓ containing b' and a very general genus-0, degree-d curve m containing b'' such that ℓ and m intersect at a very general point b, every section in* $\mathrm{Sections}(A_{m\cup\ell}/m\cup\ell)$ *is the restriction of a unique section in* $\mathrm{Sections}(A/S)$.

Proof. Fix very general points c' and c'' in S. Denote by M' the space of conics $\ell \subset S$ passing through c', and M'' the space of genus-0, degree-d curves $m \subset S$ containing c''. Let p', resp. p'', be a closed point in the fiber $A_{c'}$, resp. $A_{c''}$. Let γ be a section in $\mathrm{Sections}(A_{m\cup\ell}/m \cup \ell)$ that maps (c', c'') to (p', p'').

Denote by C', resp. C'', the universal family of curves over M', resp. M''. Denote by $u'_S : C' \to S$ and $u''_S : C'' \to S$ the cycle morphisms. There are only countably many sections in $\mathrm{Sections}_{c'}^{p'}(A/S)$, resp. $\mathrm{Sections}_{c''}^{p''}(A/S)$, see Lemma 3.43. Denote by $(\Sigma'_{i'})_{i'\in I'}$, resp. $(\Sigma''_{i''})_{i''\in I''}$, the images of these countably many sections. For each $i' \in I'$, resp. $i'' \in I''$, the fiber product $\Omega'_{i'} := C' \times_{u'_S,S} \Sigma'_{i'}$, resp. $\Omega''_{i''} := C'' \times_{u''_S,S} \Sigma''_{i''}$, is a (C', u'_S)-multisection, resp. (C'', u''_S)-multisection, of A over S, see Definition A.3. Set $r = 2\dim(S) - 2$ as the transversal dimension, see Definition A.4 (vii). Fix an embedding from M', resp. M'', into a projective space. The transversal Grassmannian G' is the Grassmannian parameterizing r-dimensional linear subvarieties N' of M', and similarly define G'' for M'', see Definition A.4 (vii). Then, for every $(N', b') \in G' \times_{\mathrm{Spec}\, k} S$, $(\Omega'_{i'})_{i'\in I'}$ gives a countable collection of multisections $(\Omega'_{M',N',b',i'})_{i'\in I'}$ of A over S, see Notation A.8 and Lemma A.11. Similarly, for each $(N'', b'') \in G'' \times_{\mathrm{Spec}\, k} S$, the multisections $(\Omega''_{M'',N'',b'',i''})_{i''\in I''}$ are defined.

For each pair $(N', N'') \in G' \times_{\mathrm{Spec}\, k} G''$, consider the 2-pointed bi-gon $(C'_{t'} \cup_b C''_{t''}, b', b'')$ parameterized by a point $t' \in N'$, resp. $t'' \in N''$, whose curve $C'_{t'}$ contains b and b', resp. whose curve $C''_{t''}$ contains b and b'' (see the statement of Lemma A.11). Take $m \cup \ell$ as a 2-pointed bi-gon $(C'_{t'} \cup_b C''_{t''}, b', b'')$. Then, by Theorem 1.2 and the Bi-gon Lemma (Lemma A.11), for a very general pair (N', N'', b', b, b'') in $G' \times_{\mathrm{Spec}\, k} G'' \times_{\mathrm{Spec}\, k} S \times_{\mathrm{Spec}\, k} S \times_{\mathrm{Spec}\, k} S$, γ comes from a unique section of A over S. □

Corollary 3.41. *Let S be a smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let A be an Abelian scheme over S. For a very general curve-pair*

$C = m \cup \ell$ in S such that the degree of m is even, ℓ is a smooth conic, the restriction map of sections

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}(A_C/C)$$

is a bijection. This also holds for C a very general genus-0, irreducible, smooth curve of even degree in S.

Proof. Take $X = A$ in Corollary 3.39. In Lemma 3.40, take m as a conic curve. Then by Theorem 1.2 and Lemma 3.40 the result holds for a very general $m \cup \ell$. Next, use Corollary 3.39 to deform $m \cup \ell$ to a very general genus-0, irreducible smooth curve of degree 4. Attach a very general conic to this curve at a very general point and apply Lemma 3.40 and Corollary 3.39 again. Then, the result follows by induction. $\square$

3.5 Moduli of bad points caused by $\mathrm{Iso}(A)$

Lemma 3.42. *Let A and B be two Abelian varieties over an algebraically closed field k. Then, there are at most countably many homomorphism of Abelian varieties from A to B.*

Proof. Take a very ample sheaf $\mathcal{L}$ on $A \times_k B$. For every homomorphism $u : (A, 0) \to (B, 0)$, the graph G_u in $A \times_k B$ has a Hilbert polynomial $P(t)$ with respect to $\mathcal{L}$. Let $\mathrm{Hom}_k^P(A, B)$ be the scheme parameterizing homomorphisms from A to B with Hilbert polynomial $P(t)$. By [18], (iii), p.39, the Zariski tangent space of $\mathrm{Hom}_k^P(A, B)$ at $[u]$ is isomorphic to the k-vector space of global sections of $\mathcal{E} := \mathrm{Hom}_k(T_{B,0}^{\vee}, k) \otimes_k \mathcal{I}_0$, where $\mathcal{I}_0$ is the ideal sheaf defining the origin $0 \in A$. Therefore, $\mathrm{H}^0(A, \mathcal{E}) = 0$ and $[u]$ is an isolated point of $\mathrm{Hom}_k^P(A, B)$. $\square$

Lemma 3.43. *Let C be a smooth curve in S. Then, for fixed $b \in C$ and $p \in \mathrm{Iso}(A)_b$ closed points, there are at most countably many sections of X (resp. A, resp. $\mathrm{Iso}(A)$) over C that map b to p. And there are at most countably many sections of X (resp. A, resp. $\mathrm{Iso}(A)$) over S that map b to p.*

Proof. It suffices to prove the statement for the Abelian scheme A since $X \to A$ is finite and $\mathrm{Iso}(A)$ is a closed subscheme of A. First suppose that $A = A_0 \times_k S$ is a constant family for some Abelian variety A_0 over k. By taking the Jacobian $\mathrm{Jac}(C)$ of C, the result follows from Lemma 3.42. If A is not a constant family of Abelian varieties, then by a finite, étale and Galois base change $S' \to S$ we have an isogeny of Abelian S'-schemes,

$$\rho_{iso} : (A_0 \times_k S') \times_{S'} Q \to A \times_S S'$$

where $A_0 \times_k S'$ is a constant family of Abelian varieties over S' and Q is a strongly nonisotrivial Abelian scheme over S'. Recall that there are at most countably many sections of Q over S' ([7], Lemma 3.6, p.316). Combine the constant family case and the strongly nonisotrivial case, the result follows from diagrams chasing. Finally, by replacing the Jacobian of a smooth curve by the Albanese variety of S ([19], Theorem 5.7.13, p.141), the result for $\mathrm{Sections}_b^p(X/S)$, resp. $\mathrm{Sections}_b^p(A/S)$, resp. $\mathrm{Sections}_b^p(\mathrm{Iso}(A)/S)$ follows immediately. $\square$

Lemma 3.44. *Fix closed points $b \in S$, $p \in \mathrm{Iso}(A)_b$. Let σ be a section in* $\mathrm{Sections}_b^p(A/S)$. *Recall Definition 3.29 and Definition 3.31 for good curves for sections.*

(1). For a very general smooth conic ℓ and a very general genus-0, degree-d curve m containing b with $d \geq 2$ such that ℓ and m intersect at a very general point, every section in $\mathrm{Sections}_b^p(X_{m\cup\ell}/m \cup \ell)$ *that maps to $\sigma|_{m\cup\ell}$ is the restriction of a unique section in* $\mathrm{Sections}_b^p(X/S)$ *that maps to σ if $m \cup \ell$ is good for σ.*

(2). Let C be a very general genus-0, degree-d, irreducible smooth curve marked by b with $d \geq 2$. Then, every section in $\mathrm{Sections}_b^p(X_C/C)$ *that maps to $\sigma|_C$ is the restriction of a unique section in* $\mathrm{Sections}_b^p(X/S)$ *that maps to σ if C is good for σ.*

(3). Conversely, if p is a bad point for a very general irreducible smooth curve C, then there exists a section in $\mathrm{Sections}_b^p(X_C/C)$ *that cannot be extended uniquely.*

Proof. (1). Let $f_\sigma : X \times_{A,\sigma} S \to S$ be the finite morphism arising from base change of f by σ. Denote by X_S the fiber product $X \times_{A,\sigma} S$. Denote $g_\sigma : X_S \to X$ to be the base change of σ by f. For any two different sections in $\mathrm{Sections}_b^p(X/S)$, the intersection in X maps in S to a proper closed subset of S. Moreover, there are at most countably many sections in $\mathrm{Sections}_b^p(X_m/m)$. For any two distinct such sections, the intersection in X maps, via the structure morphism of X, to a proper closed subset of S. Denote by $S_0 \subset S$ the complement of all these closed subsets.

Take $c \in S_0$ as the intersection point of m and ℓ. Let $\gamma \in \mathrm{Sections}_b^p(X_{m\cup\ell}/m\cup\ell)$ such that $f \circ \gamma$ is contained in the section σ in $\mathrm{Sections}_b^p(A/S)$. Then form the following diagram.

$$\begin{tikzcd}
m\cup\ell \arrow[dr, "\gamma_0"] \arrow[drr, bend left] \arrow[ddr, bend right, "\gamma"'] & & & \\
& X_S \arrow[r, "f_\sigma"'] \arrow[d, "g_\sigma"] & S \arrow[d, "\sigma"] & \\
& X \arrow[r, "f"'] & A \arrow[r, "\pi"] \arrow[d] & \mathrm{Iso}(A) \arrow[dl, "\rho"] \\
& & S &
\end{tikzcd}$$

Since X_S is the fiber product of X and S via f and σ, every section of f_σ over S (resp. over $m\cup\ell$) arises from a unique section of X over S (resp. over $m\cup\ell$). Thus, γ gives a section, γ_0, of f_σ over $m \cup \ell$ such that $g_\sigma \circ \gamma_0 = \gamma$. Since $m \cup \ell$ is *good* for σ, $\gamma_0|_\ell$ is contained in a unique section of f_σ over S, say, τ. Then, $g_\sigma \circ \tau$ is a section of X over S such that $f \circ g_\sigma \circ \tau = \sigma$. By construction, $(g_\sigma \circ \tau)|_\ell$ equals $g_\sigma \circ (\gamma_0|_\ell)$, which is $\gamma|_\ell$. Let γ_1 be the restriction of $g_\sigma \circ \tau$ on m. Suppose that $\gamma_1 \neq \gamma|_m$, then $\gamma_1(c) = g_\sigma \circ \tau(c) = \gamma(c)$ is in the intersection of the images of γ_1 and $\gamma|_m$. Since γ_1 is also a section of X over m mapping b to p, c is in the complement of S_0, contradicting the choice of c. Therefore, γ_1 equals $\gamma|_m$, and γ extends to a unique section of X over S, which is $g_\sigma \circ \tau$. The statement (2) follows from the same proof as (1).

(3). Suppose that p is a bad point for C. For every two distinct sections in $\mathrm{Sections}^p_b(A/S)$, the intersection of their images in A maps to a proper closed subset of S. Remove these countably many closed subsets from S. Denote this subset by S°. Take C such that $S^\circ \cap C \neq \emptyset$ and a section $\tau \in \mathrm{Sections}^p_b(A/S)$ such that C is bad for $([\tau], p)$. Since p is a bad point, there exists a section γ of X over C mapping b to p such that either γ_0 cannot be extended, the extension is not unique, or $f \circ \gamma$ cannot be extended. If $f \circ \gamma$ cannot be extended, γ does not have an extension. If γ_0 cannot be extended, then γ cannot be extended. If the extension is not unique, these different extensions of γ_0 give distinct extensions of γ as in the proof of the first part. $\square$

Corollary 3.45. *Fix a very general point $b \in S$ and a point p in $\mathrm{Iso}(A)_b$. Then, for a very general conic ℓ and a very general genus-0, degree-d curve m containing b with d even and $d \geq 2$ such that ℓ and m intersect at a very general point, every section in $\mathrm{Sections}^p_b(X_{m\cup\ell}/m \cup \ell)$ is the restriction of a unique section in $\mathrm{Sections}^p_b(X/S)$.*

Proof. Consider the subset S_0 as in Lemma 3.44. For a very general b, and very general $m \cup \ell$, the restriction of sections

$$\mathrm{Sections}^p_b(A/S) \to \mathrm{Sections}^p_b(A_{m\cup\ell}/m \cup \ell)$$

is bijective by Corollary 3.41.

For every σ in $\mathrm{Sections}^p_b(A/S)$, there is a general family of smooth conic curves $\mathcal{N}_2(\sigma)$ such that every section of f_σ over ℓ extends to a unique section of f_σ by Bertini's theorem (Theorem 3.13). Take a very general conic ℓ that is contained in $\mathcal{N}_2(\sigma)$ for every $\sigma \in \mathrm{Sections}^p_b(A/S)$. Also, take the smooth conic ℓ such that it intersects with m at some $c \in m \cap S_0$. Now, for every section $\gamma \in \mathrm{Sections}^p_b(X_{m\cup\ell}/m \cup \ell)$, $f \circ \gamma$ is contained in a unique section $\sigma \in \mathrm{Sections}^p_b(A/S)$. Therefore, by Lemma 3.44 (1), γ is contained in a unique section of $\mathrm{Sections}^p_b(X/S)$. $\square$

Remark 3.46. For a fixed $p \in \mathrm{Iso}(A)_b$, Corollary 3.45 claims the existence of good curve-pairs for sections in $\mathrm{Sections}^p_b(A/S)$. However, such a good curve-pair might be bad for other choices $p_0 \in \mathrm{Iso}(A)_b$. And as we vary the point p_0, the bad sets $\mathcal{B}\mathrm{ad}(d+2; b, p_0)$ might sweep out the moduli space $\partial\overline{\mathcal{M}}^{\leq 2}_{0,1}(S, d+2) \times_{\rho_{\mathrm{ev}}, S, \mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$, see Definition 3.27. To resolve this problem, we have to increase the degree of curve-pairs (Theorem 1.3).

3.6 Proof of the Main Theorem

Now, we can give the proof of Theorem 1.3.

Proof. Recall that the integer e is the fiber dimension of $\mathrm{Iso}(A) \to S$. If e equals zero, then Theorem 1.3 holds for a very general smooth conic, i.e., a genus-0, degree-$(d+2)$ curve-pair with $d=0$. This follows immediately from [7], Lemma 3.6, p.316, and the same proof of Lemma 3.44 (2). So, in the rest of the proof, we assume that $e \geq 1$. Let C_1 be a very general conic curve containing a very general point $b_1 \in S$. Let $\mathcal{B}_{2,b_1}$ be the image $\phi_3(\mathcal{B}\mathrm{ad}(2;b_1))$, i.e., the set of bad points $p_1 \in \mathrm{Iso}(A)_{b_1}$ for C_1. By Corollary 3.45, $\mathcal{B}_{2,b_1}$ is a proper subset of $\mathrm{Iso}(A)_{b_1}$. Take C_2 a very general conic intersecting with C_1 at a very general point c_2. Take a very general point b_2 on C_2. Denote by $\Delta_2'(C_1 \cup C_2, b_1)$ the union of the images of $C_1 \cup C_2$ under bad sections σ of A over S mapping b_1 to some $p \in \mathcal{B}_{2,b_1}$. Let $\Delta_2(C_1 \cup C_2, b_1)$ be the image of $\Delta_2'(C_1 \cup C_2, b_1)$ in $\mathrm{Iso}(A)$ under π. Then, $\Delta_2(C_1 \cup C_2, b_1)$ is contained in $\rho^{-1}(C_1 \cup C_2)$ and $\Delta_2(C_1 \cup C_2, b_1) \cap \mathrm{Iso}(A)_{b_1}$ equals $\mathcal{B}_{2,b_1}$. Define $\Delta_2(C_1 \cup C_2, b_2)$ in the same way for points in $\mathcal{B}_{2,b_2}$.

By choosing C_2, c_2 and b_2 very generally, $\Delta_2(C_1 \cup C_2, b_2) \cap \mathrm{Iso}(A)_{b_1}$ will intersect $\mathcal{B}_{2,b_1}$ transversally. Moreover, since $C_1 \cup_{c_2} C_2$ is very general, we may assume that

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}(A_{C_1 \cup_{c_2} C_2}/C_1 \cup_{c_2} C_2)$$

is bijective by Corollary 3.41.

Let p be a point in $\mathcal{B}_{2,b_1}$, but not in $\Delta_2(C_1 \cup C_2, b_2)$. Let σ be a section in $\mathrm{Sections}^p_{b_1}(A/S)$. If $\sigma(b_2)$ does not belong to $\mathcal{B}_{2,b_2}$, $(b_1, [C_1], c_2, [C_2])$ is good for σ. If $\sigma(b_2)$ is in $\mathcal{B}_{2,b_2}$, then $\sigma(b_1)$ is in $\Delta_2(C_1 \cup C_2, b_2)$, which contradicts the choice of p. Thus, $(b_1, [C_1], c_2, [C_2])$ is good for every section in $\mathrm{Sections}^p_{b_1}(A/S)$, and p is a good point for this marked curve-pair. Now, take a point p in the set $\Delta_2(C_1 \cup C_2, b_2) \cap \mathrm{Iso}(A)_{b_1}$, but not in $\mathcal{B}_{2,b_1}$. Denote by γ a section of X over $C_1 \cup C_2$ mapping b_1 to p. Let p_2 be the image of $\gamma(b_2)$ in $\mathrm{Iso}(A)$. Let σ be a section of A over S extending $f \circ \gamma$. Since p is not in $\mathcal{B}_{2,b_1}$, $(b_2, [C_2], c_2, [C_1])$ is good for $([\sigma], p_2)$. By Lemma 3.44 (1), γ extends to a unique section of X over S mapping b_2 to p_2 and b_1 to p. Denote by $\mathcal{B}'_{4,b_1}$ the set of points in $\mathrm{Iso}(A)_{b_1}$ such that for each $p \in \mathcal{B}'_{4,b_1}$ the restriction of sections

$$\mathrm{Sections}^p_{b_1}(X/S) \to \mathrm{Sections}^p_{b_1}(X_{C_1 \cup_{c_2} C_2}/C_1 \cup_{c_2} C_2)$$

is not a bijection. By the argument above, $\mathcal{B}'_{4,b_1}$ is contained in the intersection of $\Delta_2(C_1 \cup C_2, b_2)$ and $\mathcal{B}_{2,b_1}$. Therefore, $\dim \mathcal{B}'_{4,b_1}$ is strictly less than $\dim \mathcal{B}_{2,b_1}$.

Let $C_{1,2}$ be a very general, genus-0, degree-4, irreducible, smooth curve containing b_1. By Corollary 3.39 and Lemma 3.44 (3), the bad set of points $\mathcal{B}_{4,b_1}$ for $(b_1, [C_{1,2}])$ is contained in $\mathcal{B}'_{4,b_1}$. Attach a very general smooth conic C_3 to $C_{1,2}$ at a very general point c_3. Then inductively, we get a decreasing sequence of dimensions

$$\dim \mathcal{B}_{2,b_1} > \dim \mathcal{B}'_{4,b_1} \geq \dim \mathcal{B}_{4,b_1} > \dim \mathcal{B}'_{6,b_1} \geq \dim \mathcal{B}_{6,b_1} > \cdots.$$

Then, for $d > 2e-2$ an even number, $\mathcal{B}'_{d+2,b_1}$ for $(b_1, [m], c, [\ell])$, a very general genus-0, degree-$(d+2)$ curve-pair, is empty, and hence every section of X over $m \cup \ell$ is the restriction of a unique section. And, by Corollary 3.39, this is also true for very general irreducible smooth curves of genus-0, degree-$(d+2)$. $\square$

Appendix A The Bi-gon Lemma

Let k be an algebraically closed field. In the statement of the Bi-gon Lemma, also k will be uncountable. Let B be an irreducible, quasi-projective k-scheme of dimension ≥ 2.

Definition A.1. For every k-morphism of locally finite type k-schemes, $f : R \to S$, for every integer $\delta \geq 0$, the **δ-locus of** f, $E_{f,\geq\delta} \subseteq R$, is the union of all irreducible components of fibers of f that have dimension $\geq \delta$. The **δ-image of** f, $F_{f,\geq\delta} \subseteq S$, is the image under f of $E_{f,\geq\delta}$.

Lemma A.2 ([12], Exercise II.3.22, p.95)**.** *For every locally finite type morphism f and every integer $\delta \geq 0$, the subset $E_{f,\geq\delta}$ of R is closed. If also f is quasi-compact, resp. proper, then the subset $F_{f,\geq\delta}$ of S is constructible, resp. closed.*

Definition A.3. For every proper, surjective morphism $\rho : Y \to B$, for every pair (T, w) of an integral scheme T and a dominant, finite type morphism,

$$w : T \to B,$$

a (T, w)-**multisection of** ρ is a pair (Ω, v) of an irreducible scheme Ω and a proper morphism $v = (v_Y, v_T)$,

$$v : \Omega \to Y \times_B T, \quad v_Y : \Omega \to Y, \quad v_T : \Omega \to T,$$

such that v_T is surjective and generically finite. Since v is proper, also the image $(v(\Omega), v(\Omega) \hookrightarrow Y \times_B T)$ is a (T, w)-multisection of ρ. This is the **image multisection** of (Ω, v). If w equals Id_B, we just say (Ω, v) is a **multisection of** ρ.

For every pair $((\Omega', v'), (\Omega'', v''))$ of (T, w)-multisections, denote the fiber product of v' and v'' by

$$(\pi' : P_{v',v''} \to \Omega', \quad \pi'' : P_{v',v''} \to \Omega''), \quad v' \circ \pi' = v'' \circ \pi''.$$

The **special subset** $S_{v',v''}$ **of the pair** is the closed image in T of $P_{v',v''}$.

Definition A.4. (i). For an integral, quasi-projective k-scheme M, a **family of smooth, proper, connected curves over** M is a smooth, proper morphism,

$$\overline{u}_M : \overline{C} \to M,$$

whose geometric fibers are connected curves.

(ii). For every open immersion

$$\iota : C \to \overline{C}$$

whose image is dense in every fiber of $\overline{u}_M$, the composite morphism $u_M = \overline{u}_M \circ \iota$ is a **family of smoothly compactifiable curves over** M.

(iii). A **family of curves to** B is a pair (M, u) of an integral, quasi-projective k-scheme M and a proper morphism $u = (u_B, u_M)$,

$$u : C \to B \times_{\mathrm{Spec}\ k} M, \quad u_B : C \to B, \quad u_M : C \to M,$$

such that u_M is a family of smoothly compactifiable curves over M.

(iv). The family of curves to B is **connecting**, resp. **minimally connecting**, if the following induced k-morphism is dominant, resp. dominant and generically finite,

$$u^{[2]} : C \times_M C \to B \times_{\mathrm{Spec}\ k} B, \quad \mathrm{pr}_i \circ u^{[2]} = u_B \circ \mathrm{pr}_i, \quad i = 1, 2.$$

By definition, both $u_M \circ \mathrm{pr}_1$ and $u_M \circ \mathrm{pr}_2$ are equal as morphisms from $C \times_M C$ to M; denote this common morphism by $\widetilde{u}_M$. Denote by $\widetilde{u}^{[2]}$ the induced morphism

$$(u^{[2]}, \widetilde{u}_M) : C \times_M C \to B \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} M.$$

(v). A connecting family of curves to B is a **Bertini family** if for every integral k-scheme $\widetilde{B}$ and for every finite, surjective k-morphism $\phi : \widetilde{B} \to B$, the induced morphism $\widetilde{B} \times_B C \to M$ has integral geometric generic fiber. Denote by M_ϕ the maximal open subscheme of M over which $\widetilde{B} \times_B C$ has integral geometric fibers.

(vi). An integral closed subvariety N of M is **transversal** if the following family of curves to B is minimally connecting,

$$(N, u \times \mathrm{Id}_N : C \times_M N \to B \times_{\mathrm{Spec}\ k} M \times_M N).$$

Such a subvariety is ϕ-**Bertini** if N intersects the open subscheme M_ϕ of M.

(vii). The **transversal dimension** is $d := 2\dim(B) - 2$. Fix an embedding $M \to \mathbb{P}^n_k$ of the quasi-projective k-scheme M. For every integer e with $0 \le e \le d$, the **transversal Grassmannian** G_e is the open subscheme of the Grassmannian parameterizing linear subvarieties H of $\mathbb{P}^n_k$ such that $N = M \cap H$ is a nonempty, e-dimensional linear section of M.

Lemma A.5. *For every connecting family of curves (M, u), for every integer e with $0 < e \le d$, a general point of G_e parameterizes a linear section N of M that is geometrically integral.*

Proof. This follows from a Bertini Connectedness Theorem, [13], Théorème 6.10. ☐

Lemma A.6. *For every connecting family (M, u), for every integer e with $0 \le e \le d$, a general point of G_e parameterizes a linear section N of M such that the induced morphism $u_N^{[2]}$ is generically finite, where u_N is the morphism $u \times Id_N$. Assume further that (M, u) is a connecting, Bertini family of curves to B. Then, for every finite, surjective k-morphism $\phi : \widetilde{B} \to B$, every general $N \in G_d$ is transversal and ϕ-Bertini.*

Proof. Generic finiteness of $u_N^{[2]}$ is proved by induction on e. The base case is when $e = 0$. Since the family is connecting, the morphism u is generically finite to its image. Thus, the morphism $\widetilde{u}^{[2]}$ is generically finite to its image. The 1-relative locus E of $\widetilde{u}^{[2]}$ is a proper, closed subset of $C \times_M C$. For the induced morphism,

$$\widetilde{u}_M|_E : E \to M,$$

the 2-relative locus $E_{\geq 2}$ of this morphism is a closed subset of E. Since E is a proper closed subset of $C \times_M C$, and since the geometric generic fiber of $\widetilde{u}_M$ is irreducible, the proper closed subset $E_{\geq 2}$ is disjoint from this fiber. Thus, the image $F_{\geq 2}$ of $E_{\geq 2}$ in M is a constructible subset that does not contain the generic point, i.e., it is not Zariski dense. Denote by M^o the open subset of M that is the complement of the closure of the image of $F_{\geq 2}$. For every singleton N of a closed point of M^o, the restrictions to N of $\widetilde{u}^{[2]}$ and $u^{[2]}$ are equal; refer to this common restriction by $u_N^{[2]}$. Since $\widetilde{u}^{[2]}$ is generically finite on the fiber over N by construction, also $u_N^{[2]}$ is generically finite. This establishes the base case.

For the induction step, assume that the result is proved for an integer e satisfying $0 \leq e < 2\dim(B) - 2$. Then for a general linear subvariety N of M of dimension e, the image of $u_N^{[2]}$ has dimension $e + 2 < 2\dim(B)$. Thus, the image is not Zariski dense. Since $u^{[2]}$ is dominant, a general point of $B \times_{\text{Spec } k} B$ is contained in the image of $u^{[2]}$ over a general point of M, say m. Let N' be the intersection of M with the span of N and m. Then N' is a linear subvariety of M of dimension $e + 1$. By Lemma A.5, for N general and for m general, the linear section N' is geometrically integral. Thus, the image of $u_{N'}^{[2]}$ is a geometrically integral scheme that is strictly larger than the image of $u_N^{[2]}$. Thus, the dimension of the image of $u_{N'}^{[2]}$ is strictly larger than the dimension of the image of $u_N^{[2]}$. Since $u_N^{[2]}$ is generically finite, and since N' has dimension precisely 1 larger than the dimension of N, also the image of $u_{N'}^{[2]}$ has dimension precisely 1 larger than the dimension of the image of $u_N^{[2]}$. Thus, also $u_{N'}^{[2]}$ is generically finite to its image.

In particular, for e equals d, since $u_N^{[2]}$ is generically finite and the domain and target both have the same dimension, the image of $u_N^{[2]}$ contains a nonempty Zariski open subset of $B \times_{\text{Spec } k} B$. By hypothesis, $B \times_{\text{Spec } k} B$ is integral. Thus, the image contains a dense Zariski open subset of $B \times_{\text{Spec } k} B$. Therefore (N, u_N) is a minimally connecting family, i.e., N is transversal.

Finally, if the connecting family (M, u) is also Bertini, the open subscheme M_ϕ contains the generic point of M, and hence this open subscheme is dense. Therefore, a general N intersects M_ϕ. □

Lemma A.7. *For every minimally connecting family of curves to B,*

$$(M, u : C \to B \times_{\text{Spec } k} M), \quad u_B : C \to B, \quad u_M : C \to M,$$

for every (C, u_B)-multisection (Ω, v) of ρ, see Definition A.3, the scheme $\Omega \times_M C$ is irreducible, and the following composition $\widetilde{v}_{\Omega,M}$ is a multisection of ρ relative *to* $B \times_{Spec\ k} B \xrightarrow{pr_1} B$*,*

$$\Omega \times_M C \xrightarrow{v \times Id_C} Y \times_B C \times_M C \xrightarrow{Id_Y \times u^{[2]}} Y \times_B B \times_{Spec\ k} B.$$

Proof. Since the morphism

$$u_M : C \to M$$

is flat with integral geometric fibers, the following base change morphism is flat with integral geometric fibers,

$$\mathrm{pr}_\Omega : \Omega \times_M C \to \Omega.$$

Since Ω is irreducible, and since pr_Ω is flat with integral geometric generic fiber, also $\Omega \times_M C$ is irreducible.

Denote by $v_C : \Omega \to C$ and $v_Y : \Omega \to Y$ the morphisms such that $v = (v_C, v_Y) : \Omega \to C \times_B Y$. Since v_C is surjective and generically finite, and since u_M is flat, also the following morphism is surjective and generically finite,

$$v_C \times \mathrm{Id}_C : \Omega \times_M C \to C \times_M C.$$

Since (M, u) is minimally connecting, the following morphism is dominant and generically finite,

$$u^{[2]} : C \times_M C \to B \times_{\mathrm{Spec}\ k} B.$$

Thus, the composition is dominant and generically finite. This composition equals the composition of $\widetilde{v}_{\Omega,M}$ with the morphism

$$\rho \times \mathrm{Id}_B : Y \times_{\mathrm{Spec}\ k} B \to B \times_{\mathrm{Spec}\ k} B.$$

Thus, the morphism $\widetilde{v}_{\Omega,M}$ is a multisection of ρ. $\square$

Notation A.8. Keep the notations in Lemma A.7. Fix an embedding from M to a projective space. Suppose that N is a transversal linear section of M.

(1). Denote by C_N (resp. Ω_N) the fiber product $C \times_M N$ (resp. $\Omega \times_M N$).

(2). The morphism $u_B : C \to B$ (resp. $u_M : C \to M$) induces a morphism $u_{B,N} : C_N \to B$ (resp. $u_{M,N} : C_N \to N$). Denote by u_N the morphism $u \times \mathrm{Id}_N$.

(3). Denote by $(\Omega_N, v_N : \Omega_N \to Y \times_B C_N)$ the $(C_N, u_{B,N})$-multisection of ρ induced by the (C, u_B)-multisection (Ω, v) of ρ.

(4). As in Lemma A.7, the $(C_N, u_{B,N})$-multisection (Ω_N, v_N) of ρ gives a multisection of ρ relative to $\mathrm{pr}_1 : B \times_{\mathrm{Spec}\ k} B \to B$. We denote this multisection of ρ relative to $\mathrm{pr}_1 : B \times_{\mathrm{Spec}\ k} B \to B$ by $\widetilde{v}_{\Omega,M,N}$.

(5). For each $b \in B$, denote by $\widetilde{v}_{\Omega,M,N,b}$ the restriction of $\widetilde{v}_{\Omega,M,N}$ to the fiber of $\mathrm{pr}_2 : B \times_{\mathrm{Spec}\ k} B \to B$ over b. Denote by $\Omega_{M,N,b}$ the image of $\widetilde{v}_{\Omega,M,N,b}$ in Y.

For every pair of connecting families of curves to B,

$$(M', u' : C' \to B \times_{\mathrm{Spec}\ k} M'),\quad u'_B : C' \to B,\quad u'_{M'} : C' \to M',$$
$$(M'', u'' : C'' \to B \times_{\mathrm{Spec}\ k} M''),\quad u''_B : C'' \to B,\quad u''_{M''} : C'' \to M'',$$

denote by G', resp. by G'', the open subscheme of the Grassmannian parameterizing d-dimensional linear sections N' of M', resp. N'' of M'', where $d = 2\dim(B) - 2$; by Lemma A.6, there is a dense open subscheme of G', resp. of G'', parameterizing linear sections of M', resp. of M'', that are transversal.

Lemma A.9. (a). *For every pair of connecting families of curves to B as above that are Bertini families, for every pair*

$$(\Omega', v'),\quad (\Omega'', v'')$$

of a (C', u'_B)-multisection of ρ and a (C'', u''_B)-multisection of ρ, for a general pair $(N', N'') \in G' \times_{\mathrm{Spec}\ k} G''$, the families $(N', u'_{N'})$ and $(N'', u''_{N''})$ are minimally connecting families of curves to B.

(b). *Also, for a general pair $(b', b'') \in B \times_{\mathrm{Spec}\ k} B$, the family $(N', u'_{N'})$, resp. $(N'', u''_{N''})$, is a Bertini family for each irreducible component of $\Omega''_{M'',N'',b''}$, resp. of $\Omega'_{M',N',b'}$.*

Proof. The first assertion follows from Lemma A.6. For (b), by Lemma A.7, each of $(\Omega'_{N'} \times_{N'} C'_{N'}, \widetilde{v}_{\Omega',M',N'})$ and $(\Omega''_{N''} \times_{N''} C''_{N''}, \widetilde{v}_{\Omega'',M'',N''})$ is a pr_1-multisection of ρ. Thus, for general $(b', b'') \in B \times_{\mathrm{Spec}\ k} B$, the restriction of the pr_1-multisection $\Omega'_{N'} \times_{N'} C'_{N'}$, resp. $\Omega''_{N''} \times_{N''} C''_{N''}$, to the pr_2-fiber over b', resp. over b'', maps dominantly and generically finitely to B, i.e., each of the finitely many irreducible components of the restriction is a multisection of ρ. By Lemma A.6, for N'' general applied to the finitely many irreducible components of $\Omega'_{M',N',b'}$, the family $(N'', u''_{N''})$ is transversal and Bertini relative to $\Omega'_{M',N',b'}$. Similarly, for N' general, the family $(N', u'_{N'})$ is transversal and Bertini relative to $\Omega''_{M'',N'',b''}$. ☐

Lemma A.10. *Assume that k is algebraically closed and uncountable. With the same hypotheses as in Lemma A.9, for a countable family of (C', u'_B)-multisections, $(\Omega'_{i'}, v'_{i'})_{i' \in I'}$, with pairwise distinct images in $Y \times_B C'$, resp. for a countable family of (C'', u''_B)-multisections, $(\Omega''_{i''}, v''_{i''})_{i'' \in I''}$, with pairwise distinct images in $Y \times_B C''$, if $(N', N'') \in G' \times_{\mathrm{Spec}\ k} G''$ and $(b', b'') \in B \times_{\mathrm{Spec}\ k} B$ are very general, then for every $(i', i'') \in I' \times I''$, the conclusion in Lemma A.9 holds for $(\Omega'_{i'}, v'_{i'})$ and $(\Omega''_{i''}, v''_{i''})$.*

Proof. For each (i', i''), by Lemma A.9, there exists a dense open $W_{i',i''}$ of $G' \times_{\mathrm{Spec}\ k} G'' \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B$ parameterizing (N', N'', b', b'') such that Lemma A.9 holds. Thus, for every (N', N'', b', b'') in the countable intersection $\cap_{(i',i'')} W_{i',i''}$, the conclusion of the lemma holds for every $(\Omega'_{i'}, v'_{i'})$ and $(\Omega''_{i''}, v''_{i''})$. ☐

Lemma A.11 (The Bi-gon Lemma)**.** *With hypotheses as in Lemma A.10, for a very general (N', N'', b', b, b'') in $G' \times_{\mathrm{Spec}\ k} G'' \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B$, for* *a very general* 2*-pointed bi-gon* $(C = C'_{t'} \cup_b C''_{t''}, b', b'')$ *parameterized by a point* $t' \in N'$*, resp.* $t'' \in N''$*, whose curve* $C'_{t'}$ *contains* b *and* b'*, resp. whose curve* $C''_{t''}$ *contains* b *and* b''*, the only sections* σ *of* ρ *over* C *whose restriction to* $C'_{t'}$ *is in an irreducible component* $\Omega'_{M',N',b',i',0}$ *of some* $\Omega'_{M',N',b',i'}$ *and whose restriction to* $C''_{t''}$ *is in an irreducible component* $\Omega''_{M'',N'',b'',i'',0}$ *of some* $\Omega''_{M'',N'',b'',i''}$ *are those that come from global sections* $\Omega'_{M',N',b',i',0} = \Omega = \Omega''_{M'',N'',b'',i'',0}$ *over* B*.*

Proof. Let W denote the countable intersection of $W_{i',i''}$ inside $G' \times_{\mathrm{Spec}\ k} G'' \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B$ as in the proof of Lemma A.10. Let (N', N'', b', b'') be an element of W. Consider the countable collection of images $(\Omega'_{M',N',b',i'})_{i' \in I'}$ and $(\Omega''_{M'',N'',b'',i''})_{i'' \in I''}$ of ρ as closed subschemes of Y. Without loss of generality, we assume that every $\Omega'_{M',N',b',i'}$, resp. every $\Omega''_{M'',N'',b'',i''}$, is irreducible. For every pair (i'_1, i'_2) of distinct elements of I', the special subset $S_{i'_1,i'_2}$ associated to Ω'_{M',N',b',i'_1} and Ω'_{M',N',b',i'_2} is a proper closed subset of B, and similarly for the special subset $S_{i''_1,i''_2}$ associated to every pair (i''_1, i''_2) of distinct elements of I''. Finally, for every $i' \in I'$ and every $i'' \in I''$, the special subset $S_{i',i''}$ associated to $\Omega'_{M',N',b',i'}$ and $\Omega''_{M'',N'',b'',i''}$ is a proper closed subset except in those cases where $\Omega'_{M',N',b',i'}$ equals $\Omega''_{M'',N'',b'',i''}$.

Choose b to be a very general point of B that is contained in none of these special subsets that are proper closed subsets of B. The matching condition at b for a section σ implies that $\sigma(C)$ is contained in $\Omega'_{M',N',b',i'} = \Omega''_{M'',N'',b'',i''}$ for unique $\Omega'_{M',N',b',i'}$ and $\Omega''_{M'',N'',b'',i''}$ in their respective countable multisections.

By Lemma A.10, for every $i' \in I'$, if the restriction of $\Omega_{M',N',b',i'}$ over $C''_{t''}$ has a section, then the multisection $\Omega_{M',N',b',i'}$ is a global section. Similarly, for every $i'' \in I''$, if the restriction of $\Omega_{M'',N'',b'',i''}$ over $C'_{t'}$ has a section, then the multisection $\Omega_{M'',N'',b'',i''}$ is a global section. Thus, for every section σ as in the previous paragraph, $\Omega'_{M',N',b',i'} = \Omega''_{M'',N'',b'',i''}$ is a global section. □

References

[1] K. Behrend and Yu. Manin, *Stacks of Stable Maps and Gromov-Witten Invariants*, Duke Mathematical Journal, Vol. 85, No. 1, 1-60.

[2] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. **21**, Springer-Verlag, Berlin, 1990.

[3] Brian Conrad, *Chow's K/k-image and K/k-trace, and the Lang-Néron theorem*, Enseign. Math. (2) **52** (2006), no. 1-2, 37-108. MR2255529 (2007e: 14068).

[4] Olivier Debarre, *Higher-Dimensional Algebraic Geometry*, Universitext, Springer-Verlag New York, Inc (2001).

[5] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in *Algebraic geometry - Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45-96. Amer. Math. Soc., Providence, RI, 1997.

[6] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Mathematical Journal, Vol. **164**, No. 7 (2015), 1187-1270.

[7] Tom Graber and Jason Starr, *Restriction of Sections for Families of Abelian Varieties*, in *A Celebration of Algebraic Geometry: A Conference in Honor of Joe Harris' 60th Birthday*, Clay Mathematics Proceedings, volume **18**, 2013, p.311-327.

[8] Tom Graber, Joe Harris and Jason Starr, *Families of Rationally Connected Varieties*, Journal of the American Mathematical Society, Volume 16, Number 1, pages 57-67, 2003.

[9] Tom Graber, Joe Harris, Barry Mazur and Jason Starr, *Rational Connectivity and Sections of Families over Curves*, Ann. Scient. Éc. Norm. Sup., 4^{e} série, t. 38, 2005, p. 671 à 692.

[10] Alexander Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas.*, Inst. Hautes Études Sci. Publ. Math. **20** (1964), 101355; ibid. **24** (1965), 5-231; ibid. **28** (1966), 5-255; ibid. (1967), no. **32**, 5-361. MR0173675 (30:3885), MR0199181 (33:7330), MR0217086 (36:178), MR0238860 (39:220).

[11] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Documents Mathématiques (Paris) [Mathematical Documents (Pairs)], 4, Société Mathématique de France, Paris, 2005, Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original. MR 2171939.

[12] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[13] Jean-Pierre Jouanolou, *Théorèmes de Bertini et Applications*, Progress in Mathematics, vol. **42**, Birkhäuser Boston Inc., Boston, MA, 1983. MR725671 (86b:13007)

[14] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. **32**, Springer-Verlag, Berlin, 1996. MR1440180 (98c:14001)

[15] Qing Liu and Jilong Tong, *Néron Models of Algebraic Curves*, Transactions of the American Mathematical Society, Volume 368, Number 10, October 2016, pages 7019-7043.

[16] Qing Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, volume **6**.

[17] Christian Minoccheri, *On the Arithmetic of Weighted Complete Intersections of Low Degree*, PhD thesis, available at `https://arxiv.org/abs/1608.01703v1`.

[18] David Mumford, *Abelian Varieties, with Appendices by C. P. Ramanujam and Yuri Manin*, Tata Institute of Fundamental Research Studies in Mathematics, Tata Institute of Fundamental Research Publications, Volume: 13; 2012, reprinting of the revised second edition (1974).

[19] Bjorn Poonen, *Rational Points on Varieties*, Graduate Studies in Mathematics, Volume 186, 2017.

[20] Eric Riedl and Matthew Woolf, *Rational Curves on Complete Intersections in Positive Characteristic*, Journal of Algebra **494** (2018) 28-39.

[21] Michel Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics, volume **119**.

[22] Jason Starr, *Explicit computations related to "Rational connectivity ..." by Graber, Harris, Mazur and Starr*, available at `http://www.math.stonybrook.edu/~jstarr/papers/explicit2.ps`.

[23] Jason Starr, *Degenerations of rationally connected varieties and PAC fields*, in *A Celebration of Algebraic Geometry: A Conference in Honor of Joe Harris' 60th Birthday*, Clay Mathematics Proceedings, volume **18**, 2013, p.577-589.

[24] Tamás Szamuely, *Galois Groups and Fundamental Groups*, Cambridge Studies in Advanced Mathematics, no. **117**, Cambridge University Press; 1st edition (2009).

Mathematics Department, Stony Brook University, Stony Brook, NY, 11794

Email: `santai.qu@stonybrook.edu`

Current address: Yau Mathematical Sciences Center, Tsinghua University, Beijing, China, 100084
Email: `stqumat@mail.tsinghua.edu.cn`