A NOTE ON ARITHMETIC BREUIL-KISIN-FARGUES MODULES

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Abstract. Let K be a discrete valuation field, we combine the construction of Fargues-Fontaine of G_K-equivariant modifications of vector bundles over the Fargues-Fontaine curve X_{FF} using weakly admissible filtered (φ, N, G_K)-modules over K, with Scholze and Fargues' theorems that relate modifications of vector bundles over the Fargues-Fontaine curve with mixed characteristic shtukas and Breuil-Kisin-Fargues modules. We give a characterization of Breuil-Kisin-Fargues modules with semilinear G_K-actions that produced in this way and compare those Breuil-Kisin-Fargues modules with Kisin modules.

1. Introduction

1.1. **Review of the work of Fargues-Fontaine and Scholze.** Fargues and Fontaine in [FF] construct a complete abstract curve X_{FF}, the Fargues-Fontaine curve (constructed using the perfectoid field $\mathbb{C}_p^\flat$ and p-adic field $\mathbb{Q}_p$). For any p-adic field K, they show $\mathcal{O}_X = \mathcal{O}_{X_{FF}}$ carries an action of G_K, and they define $\mathcal{O}_X$-representations of G_K as vector bundles over X_{FF} that carries a continuous $\mathcal{O}_X$-semilinear action of G_K. They can show $\mathcal{O}_X$-representations of G_K are related to p-adic representations of G_K in many aspects. For example, Fargues-Fontaine show that X_{FF} is complete in the sense that there is a Harder-Narasimhan theorem holds for coherent $\mathcal{O}_X$-modules over X_{FF}, and they prove that the category of $\mathcal{O}_X$-representations of G_K such that the underlying vector bundles over X_{FF} are semistable of pure slope 0 is equivalence to the category of p-adic representations of G_K over $\mathbb{Q}_p$. Moreover, they give an explicit construction of slope 0 $\mathcal{O}_X$-representations from weakly admissible filtered (φ, N)-modules D over K. Their construction is that: first using D and the (φ, N)-structure, they construct an $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N)$ of G_K whose underlying vector bundle is not semistable in general, then using the filtration structure of D_K, they constructed a G_K-equivariant modification $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ of $\mathcal{E}(D, \varphi, N)$ along a special closed point called ∞ on X_{FF}. They can show if D is weakly admissible, then $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ is pure of slope 0, and the $\mathbb{Q}_p$ representation corresponds to $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ is nothing but the log-crystalline representation corresponds to the data $(D, \varphi, N, \mathrm{Fil}^\bullet)$. By going through such a construction, they give new proofs of some important theorems in p-adic Hodge theory, for instance, they give a lovely proof of the fact that being admissible is the same as being weakly admissible.

The abstract curve X_{FF} also plays a role in Scholze's work. In his Berkeley lectures on p-adic geometry[SW], Scholze defined a mixed characteristic analog of shtukas with legs. To be more precise, he introduced the functor $\mathrm{Spd}(\mathbb{Z}_p)$ which plays a similar role of a proper smooth curve in the equal characteristic story, and for any perfectoid space S in characteristic p, he was able to define shtukas over S with legs. If we restrict us to the case that when $S = \mathrm{Spa}(C)$ is just a point, with $C = \mathbb{C}_p^\flat$ an algebraically closed perfectoid field in characteristic p, and assume there is only one leg which corresponds to the untilt $\mathbb{C}_p$, then he can realize shtukas over S as modifications of vector bundles over X_{FF} along ∞. Here ∞ is the same closed point on X_{FF} as we mentioned in the work of Fargues-Fontaine. Fargues and Scholze also show that those shtukas can be realized using some commutative algebra data, called the (free) Breuil-Kisin-Fargues modules, which are modules over $A_{\mathrm{inf}} = W(\mathcal{O}_C)$ with some additional structures.

1.2. **Arithmetic Breuil-Kisin-Fargues modules and our main results.** If we combine the construction of Fargues and Fontaine of modifications of vector bundles over X_{FF} from log-crystalline representations and the work of Fargues and Scholze that relates modifications of vector bundles over X_{FF} with shtukas and Breuil-Kisin-Fargues modules, one can expect that if starting with a weakly admissible filtered (φ, N)-module over K, one can produce a free Breuil-Kisin-Fargues module (actually only up to isogeny if we do not specify an integral structure of the log-crystalline representation) using the modification constructed by Fargues-Fontaine. Moreover, since the modification is G_K-equivariant and all the correspondences of Fargues and Scholze we have mentioned are functorial, we have the Breuil-Kisin-Fargues module produced in this way carries a semilinear G_K-action that commutes with all other structures of it. In this paper, we will give a naive generalization of Fargues-Fontaine's construction of G_K-equivariant modifications of vector bundles over X_{FF} when the inputs are weakly admissible filtered (φ, N, G_K)-modules over K, which is also mentioned in the work of Fargues-Fontaine but using descent. Fix a p-adic field K, and we make the following definition.

Definition 1. A Breuil-Kisin-Fargues G_K-module is a free Breuil-Kisin-Fargues module with a semilinear action of G_K that commutes with all its other structures. A Breuil-Kisin-Fargues G_K-module is called *arithmetic* if, up to isogeny, it comes from the construction mentioned above using a weakly admissible filtered (φ, N, G_K)-module.

If $\mathfrak{M}^{\mathrm{inf}}$ is a free Breuil-Kisin-Fargues module, then one defines its étale realization $T(\mathfrak{M}^{\mathrm{inf}})$ as

$$T(\mathfrak{M}^{\mathrm{inf}}) = (\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(C))^{\varphi=1}$$

which is a finite free $\mathbb{Z}_p$-module, and recall the following theorem of Fargues and Scholze-Weinstein:

Theorem 1. [SW] *The functor*

$$\mathfrak{M}^{\inf} \to (T(\mathfrak{M}^{\inf}), \mathfrak{M}^{\inf} \otimes_{A_{\inf}} B_{\mathrm{dR}}^{+})$$

defines an equivalence of the following categories

$$\{\textit{free Breuil-Kisin-Fargues modules over } A_{\inf}\}$$

and

$$\{(T, \Xi) \mid T : \textit{finite free } \mathbb{Z}_p\textit{-modules, } \Xi : \textit{full } B_{\mathrm{dR}}^{+}\textit{-lattice in } T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\}$$

The category of pairs (T, Ξ) is what Fargues called Hodge-Tate modules in [Far]. Note that if $\mathfrak{M}^{\inf}$ is a Breuil-Kisin-Fargues G_K-module, then the Hodge-Tate module corresponds to $\mathfrak{M}^{\inf}$ carries a G_K action by functoriality. Using the Hodge-Tate module description of Breuil-Kisin-Fargues modules, we give the following easy characterization of arithmetic Breuil-Kisin-Fargues modules.

Proposition 1. *A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\inf}$ is arithmetic if and only if its étale realization $T = T(\mathfrak{M}^{\inf})$ is potentially log-crystalline as a representation of G_K over $\mathbb{Z}_p$, and there is a G_K-equivariant isomorphism of*

$$(T(\mathfrak{M}^{\inf}), \mathfrak{M}^{\inf} \otimes_{A_{\inf}} B_{\mathrm{dR}}^{+}) = (T, (T \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \otimes_K B_{\mathrm{dR}}^{+}).$$

Moreover, if the isogeny class of $\mathfrak{M}^{\inf}$ corresponds the G_K-equivariant modification coming from a weakly admissible filtered (φ, N, G_K)-module D over K, then $T(\mathfrak{M}^{\inf}) \otimes \mathbb{Q}_p$ is the potentially log-crystalline representation of G_K corresponds to the weakly admissible filtered (φ, N, G_K)-module D.

Remark 1.

(1) We want to remind the readers that p-adic monodromy theorem for de Rham representations, which was first proved in the work of Berger[Ber], tells us that if a p-adic representation is de Rham, then it is potentially log-crystalline (and the converse is true and actually much easier to prove). We will use the equivalence of being de Rham and potentially log-crystalline through out this paper.

(2) From the work of [BMS], we know there is a large class of Breuil-Kisin-Fargues G_K-modules comes from geometry: start with a proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, and let $\overline{\mathfrak{X}}$ be its base change to $\mathcal{O}_{\mathbb{C}_p}$. Then there is a $A_{\inf}$-cohomology theory attaches to $\overline{\mathfrak{X}}$ which is functorial in $\overline{\mathfrak{X}}$, so all the $A_{\inf}$-cohomology groups $H^i_{A_{\inf}}(\overline{\mathfrak{X}})$ carry natural semi-linear G_K-actions that commute with all other structures. If we take the maximal free quotients of the cohomology groups, then they are all arithmetic automatically from the étale-de Rham comparison theorem. So being arithmetic is the same as to ask an abstract Breuil-Kisin-Fargues G_K-module to satisfy étale-de Rham comparison theorem.

(3) The terminology of being *arithmetic* was first introduced in the work of Howe in [How, §4], the above proposition shows our definition are the same. The advantage of our definition is that it enables us to see how arithmetic Breuil-Kisin-Fargues behavior over $\mathrm{Spa}(A_{\mathrm{inf}})$ instead of only look at the stalks at closed points $\mathcal{O}_C$ and $\mathcal{O}_{\mathbb{C}_p}$.

As we mentioned in the above remark, if we study the behavior of arithmetic Breuil-Kisin-Fargues modules at the closed point corresponds to $W(\overline{k})$ of $\mathrm{Spa}(A_{\mathrm{inf}})$, we will have:

Corollary 1. *Let $\mathfrak{M}^{\mathrm{inf}}$ be an arithmetic Breuil-Kisin-Fargues module $\mathfrak{M}^{\mathrm{inf}}$ then*

(1) $T(\mathfrak{M}^{\mathrm{inf}})$ *is log-crystalline if and only if the inertia subgroup I_K of G_K acts trivially on $\overline{\mathfrak{M}^{\mathrm{inf}}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\overline{k})$.*
(2) $T(\mathfrak{M}^{\mathrm{inf}})$ *is potentially crystalline if and only if there is a φ and G_K equivariant isomorphism*

$$\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{cris}} \cong \overline{\mathfrak{M}^{\mathrm{inf}}} \otimes_{W(\overline{k})} B^+_{\mathrm{cris}}.$$

(3) $T(\mathfrak{M}^{\mathrm{inf}})$ *is crystalline if and only if it satisfies the conditions in* (1) *and* (2)*.*

Remark 2. We continue the discussion in Remark 1 (2), in [BMS, Theorem 14.1] one can see the A_{inf}-crystalline comparison theorem should give us condition (1) in Corollary 1. And Proposition 13.21. of *loc.cit.* shows that there is a canonical isomorphism

$$H^i_{\mathrm{crys}}(\overline{\mathfrak{X}}_{\mathcal{O}_{\mathbb{C}_p}/p}/A_{\mathrm{cris}})[\frac{1}{p}] \cong H^i_{\mathrm{crys}}(\overline{\mathfrak{X}}_{\overline{k}}/W(\overline{k})) \otimes_{W(\overline{k})} B_{\mathrm{cris}},$$

which means the condition in Corollary 1 (2) is always satisfied. And we know their étale realizations are crystalline since [BMS] is in the good reduction case. A_{inf}-crystalline comparison theorem were extended to the case of semistable reduction by Česnavičius-Koshikawa in [CK] and Zijian Yao in [Yao], and one can show (1) in Corollary 1 is satisfied in the semistable reduction case.

Another natural question is if one starts with a potentially log-crystalline representation T of G_K over $\mathbb{Z}_p$, one can construct a Hodge-Tate module

$$(T, (T \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \otimes B^+_{\mathrm{dR}})$$

which corresponds to an arithmetic Breuil-Kisin-Fargues module $\mathfrak{M}^{\mathrm{inf}}(T)$ as in Proposition 1. On the other hand, the theory of Kisin in [Kis] shows that if L is a finite Galois extension of K such that $T|_{G_L}$ is log-crystalline, fixing a Kummer tower L_∞ over L, then there is a finite free module $\mathfrak{M}(T)$ over a subring $\mathfrak{S}$ of A_{inf} with a φ-structure that captures many properties of T. We have the following proposition that enables us to compare $\mathfrak{M}(T)$ with $\mathfrak{M}^{\mathrm{inf}}(T)$.

Proposition 2. *Let $\mathfrak{M}_0^{\inf}(T) = \mathfrak{M}(T) \otimes_{\mathfrak{S},\varphi} A_{\inf}$, then $\mathfrak{M}_0^{\inf}(T)$ is a Breuil-Kisin-Fargues module with a semilinear action of G_{L_∞} by construction. There is a unique way to extend the G_{L_∞}-action to G_K such that the étale realization of $\mathfrak{M}_0^{\inf}(T)$ is exactly the G_K-representation T. Moreover, $\mathfrak{M}_0^{\inf}(T)$ with such G_K action is isomorphic to $\mathfrak{M}^{\inf}(T)$. In particular, $\mathfrak{M}_0^{\inf}(T)$ is arithmetic.*

The above proposition is related to one of the main results in [GL], and we will give another proof in this paper. We want to emphasize that the above proposition can be proved nothing but by comparing the construction of the $\mathfrak{S}$-module by Kisin, and the construction of arithmetic Breuil-Kisin-Fargues module by Fargues-Fontaines. One easy consequence of this proposition is that one observe that $\mathfrak{M}^{\inf}(T)$ is defined only using the datum T while the $\mathfrak{M}(T)$ is defined with respect to the choice of the Kummer tower L_∞, so $\mathfrak{M}_0^{\inf}(T)$ is independent of the choice of the tower L_∞. In [EG], they use the terminology of *admitting all descents over K* for a Breuil-Kisin-Fargues G_K-module, and they can prove that the condition of *admitting all descents over K* is equivalence to that the étale realization of the Breuil-Kisin-Fargues G_K-module is log-crystalline. To compare with their result, we have the following proposition:

Proposition 3. *A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\inf}$ admits all descents over K if and only if it is arithmetic and satisfies the condition* (1) *in Corollary 1.*

We want to mention that in the proof of [GL, Theorem F.11], they use the following criterion of the author about arithmetic Breuil-Kisin-Fargues modules.

Lemma 1. [GL, F.13.] *A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\inf}$ is arithmetic if and only if $\mathfrak{M}^{\inf} \otimes_{A_{\inf}} B_{\mathrm{dR}}^+/(\xi)$ has a G_K-stable basis.*

Remark 3.
(1) The terminology of *admitting all descents over a K*, as been mentioned in [EG], is very likely to be related to the prismatic-$A_{\inf}$ comparison theorem of Bhatt-Scholze[BS], while our condition seems only relates to $A_{\inf}$-cohomology as we mentioned in Remark 1.
(2) It is natural to ask if one can make sense of "moduli space of arithmetic shtukas", and compare it with the Emerton-Gee stack defined in [EG].

1.3. **Structure of the paper.** In section 2, we will first review Scholze's definition of shtukas in mixed characteristic and how to relate shtukas with Breuil-Kisin-Fargues modules. Then we will give a brief review on how to realize Scholze's shtukas using Fargues-Fontaine curve. For readers familiar with their theories, they can skip this section. In section 3, we will give an explicit construction of G_K-equivariant modifications of vector bundles over X_{FF} using data come from weakly admissible filtered (φ, N, G_K)-modules following Fargues-Fontaine's method, and we will give a characterization of Breuil-Kisin-Fargues G_K-modules come from modifications constructed in this way. In section 4, we will study the relationship between arithmetic Breuil-Kisin-Fargues modules and Kisin modules, and also prove some propositions relate to the work of [EG].

1.4. **Notions and conventions.** Throughout this paper, k_0 will be a perfect field in characteristic p and $W(k_0)$ the ring of Witt vectors over k_0. Let K be a finite extension of $W(k_0)[\frac{1}{p}]$. Let $\mathcal{O}_K$ be the ring of integers of K, ϖ be any uniformizer and let $k = k_K = \mathcal{O}_K/(\varpi)$ be the residue field. Define $K_0 = W(k)[\frac{1}{p}]$. By a compatible system of p^n-th roots of ϖ, we mean a sequence of elements $\{\varpi_n\}_{n\geq 0}$ in $\overline{K}$ with $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$ for all n.

Define $\mathbb{C}_p$ as the p-adic completion of $\overline{K}$, there is a unique valuation v on $\mathbb{C}_p$ extending the p-adic valuation on K. Let $\mathcal{O}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) \geq 0\}$ and let $\mathfrak{m}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) > 0\}$. We will have $\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} = \overline{k}$.

Let C be the tilt of $\mathbb{C}_p$, then by the theory of perfectoid fields, C is algebraically closed of characteristic p, and complete with respect to a nonarchimedean norm. Let $\mathcal{O}_C$ be the ring of the integers of C, then $\mathcal{O}_C = \mathcal{O}_{\mathbb{C}_p}^\flat = \varprojlim_\varphi \mathcal{O}_{\mathbb{C}_p}/p$. Define $A_{\text{inf}} = W(\mathcal{O}_C)$, there is a Frobenius $\varphi_{A_{\text{inf}}}$ acts on A_{inf}. There is a surjection $\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ whose kernel is principal and let ξ be a generator of $\text{Ker}(\theta)$. Let $\tilde{\xi} = \varphi(\xi)$ as in [BMS]. There is a G_K action on A_{inf} via its action on $C = \mathbb{C}_p^\flat$, one can show θ is G_K-equivariant.

In this paper, we will use the notion *log-crystalline representations* instead of semistable representations to make a difference to the semistability of vector bundles over complete regular curves.

A filtered (φ, N)-module over K is a finite dimensional K_0-vector space D equipped with two maps

$$\varphi, N : D \to D$$

such that

(1) φ is semi-linear with respect to the Frobenius φ_{K_0}.
(2) N is K_0-linear.
(3) $N\varphi = p\varphi N$.

And a decreasing, separated and exhaustive filtration on the K-vector space $D_K = K \otimes_{K_0} D$.

Let L be a finite Galois extension of K and let $L_0 = W(k_L)_{\mathbb{Q}}$. A filtered $(\varphi, N, \text{Gal}(L/K))$-module over K is a filtered (φ, N)-module D' over L together with a semilinear action of $\text{Gal}(L/K)$ on the L_0 vector space D', such that:

(1) The semilinear action is defined by the action of $\text{Gal}(L/K)$ on L_0 via $\text{Gal}(L/K) \twoheadrightarrow \text{Gal}(k_L/k) = \text{Gal}(L_0/K_0)$.
(2) The semilinear action of $\text{Gal}(L/K)$ commutes with φ and N.
(3) Define diagonal action of $\text{Gal}(L/K)$ on $D' \otimes_{L_0} L$, then the filtration on $D' \otimes_{L_0} L$ is stable under this action.

If L' is another finite Galois extension of K containing L, then one can show there is a fully faithful embedding of the category of filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules into the category of filtered $(\varphi, N, \mathrm{Gal}(L'/K))$-modules. One defines the category of filtered (φ, N, G_K)-modules to be the limit of filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules over all finite Galois extensions L of K.

2. Shtukas in mixed characteristic, Breuil-Kisin-Fargues modules and modifications of vector bundles

2.1. **Shtukas in mixed characteristic and Breuil-Kisin-Fargues modules.** In this subsection we briefly review Scholze's definition of shtukas (with one leg) in mixed characteristic and its relation with Breuil-Kisin-Fargues modules following Scholze's Berkeley notes[SW] and Kedlaya's AWS notes[Ked1].

Definition 2. [SW, Definition 11.4.1] Let **Pfd** be the category of perfectoid spaces of characteristic p, for any $S \in \mathbf{Pfd}$, a shtuka with one leg over S is the following data:

- A morphism $x : S \to \mathrm{Spd}(\mathbb{Z}_p)$. (the leg)
- $\mathcal{E}$ a vector bundle over $\mathrm{Spd}(\mathbb{Z}_p) \times S$ together with
$$\varphi_{\mathcal{E}} : \mathrm{Fr}_S^*(\mathcal{E})|_{\mathrm{Spd}(\mathbb{Z}_p)\times S\backslash\Gamma_x} \dashrightarrow \mathcal{E}|_{\mathrm{Spd}(\mathbb{Z}_p)\times S\backslash\Gamma_x}$$
Here Γ_x denotes the graph of x and $\dashrightarrow$ means $\varphi_{\mathcal{E}}$ is an isomorphism over $(\mathrm{Spd}(\mathbb{Z}_p) \times S)\backslash\Gamma_x$ and meromorphic along Γ_x.

The most revolutionary part in Scholze's definition is he came up with the object "$\mathrm{Spa}\mathbb{Z}_p$" (as well as the $\mathrm{Spd}(\mathbb{Z}_p)$ we use in the definition) which he used as the replacement of the curve $\mathcal{C}/\mathbb{F}_p$ in the equal characteristic case. Instead of go into the details in the definition, we will unpack concepts in this definition when $S = \mathrm{Spa}(C)$ is just a point, i.e., we assume C is a perfectoid field in characteristic p. Then for the first datum in the definition of shtukas, we have:

Lemma 2. [SW, Proposition 11.3.1] *For $S = \mathrm{Spa}(C)$ is a perfectoid field in characteristic p, the following sets are naturally identified:*

- *A morphism $x : S \to \mathrm{Spd}(\mathbb{Z}_p)$.*
- *The set of isomorphism classes of untilts of S, or more precisely of pairs (F, ι) in which F is a perfectoid field and $\iota : (S^{\#})^{\flat} \to S$ is an isomorphism and isomorphism classes are taken from $F \simeq F'$ that commutes with ι and ι'.*
- *Sections of $S^{\diamond} \times \mathrm{Spd}\mathbb{Z}_p \to S^{\diamond}$*

Here $S^{\diamond}$ denotes the diamond associated with S by identifying S with the functor it represents as a pro-étale sheaf of sets. Again, instead of go into Scholze's definition of diamonds, we unpack the concept with the following lemma in the case of $S = \mathrm{Spa}(C)$ a perfectoid field.

Lemma 3. [Ked1, Lemma 4.3.6.] *For $S = \mathrm{Spa}(C)$, let $\overline{\varpi}$ be a (nonzero) topological nilpotent element in C and put $A_{\mathrm{inf}} = W(\mathcal{O}_C)$. Let*

$$\mathcal{U}_S = \{v \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) | v([\varpi]) \neq 0\},$$

Then for any $Y \in \mathbf{Pfd}$, there is a natural indentifacation of

- *morphisms of $Y \to S^{\diamond} \times \mathrm{Spd}\mathbb{Z}_p$,*
- *pairs of (X, f) in which X is an isomorphism class of untilts of Y and $f : X \to \mathcal{U}_S$ is a morphism of adic spaces.*

Remark 4. By tilt equivalence in the relative setting, we have that if S is a perfectoid space (not necessary in **Pfd**), and $Y \in \mathbf{Pfd}$, then $S^{\flat}(Y)$ is naturally isomorphic to pairs (X, f) where X is an isomorphism class of untilts of Y and $f : X \to S$ is a morphism of perfectoid spaces. Scholze generalize this notion and define $S^{\diamond}$ for any analytic adic spaces S on which p is topologically nilpotent as be the functor that for $Y \in \mathbf{Pfd}$,

$$S^{\flat}(Y) = \{(X, f) | X \text{ is an isomorphism class of untilts of } Y \text{ and } f : X \to S\}$$

Using the terminology in the above remark, Lemma 3 says $(S^{\diamond} \times \mathrm{Spd}\mathbb{Z}_p)(Y)$ is naturally isomorphic to $\mathcal{U}_S^{\diamond}(Y)$, so if we go back to the definition of shtuka, the second data can be taken as a vector bundle over $\mathcal{U}_S$, note that over $\mathcal{U}_S$ there is a natural Frobenius induced for the Frobenius on $\mathcal{O}_C$.

Now let's restrict to the case that the leg of the shtuka correspondence to an untilt of C in characteristic 0, then we will have the following lemma:

Lemma 4. [Ked1, Lemma 4.5.14.] *For $S = \mathrm{Spa}(C)$ be a perfectoid space in characteristic p, and let*

$$\mathcal{Y}_S = \{v \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) | v([\varpi]) \neq 0, v(p) \neq 0\}.$$

Then a shtukas over S with leg x correspondence to an untilt of C in characteristic 0, is the same as the following data

$$\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$$

- *$\mathcal{E}_0$ is a φ-equivariant vector bundle over $\mathcal{U}_S$,*
- *$\mathcal{E}_1$ is a φ-equivariant vector bundle over $\mathcal{Y}_S$.*
- *$\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ is an isomorphism of $\mathcal{E}_0$ with $\mathcal{E}_1$ over $\mathcal{Y}_S \setminus \cup_{n \in \mathbb{Z}} \Gamma_{\varphi^n(x)}$*

Remark 5. Here Γ_x is Cartier divisor correspondence to the leg x, and the assumption x correspondence to an untilt of C in characteristic 0 is the same as Γ_x is inside $\mathcal{Y}_S$. Instead of proving the lemma, we just recall that $\mathcal{E}_0$(resp. $\mathcal{E}_1$) comes from restricting $\mathcal{E}$ in Definition 2 to a "neighborhood" of $V(p)$(resp.$V([\varpi])$), and use the Frobenius to extend it to $\mathcal{U}_S$(resp. $\mathcal{Y}_S$).

If we further assume $S = \mathrm{Spa}(C)$ with C an algebraically closed non-archimedean field in characteristic p, then we have the following lemma:

Lemma 5. [Ked1, Lemma 4.5.17.] *Let $S = \mathrm{Spa}(C)$ with C an algebraically closed non-archimedean field in characteristic p, and let*

$$\mathcal{Y}_S^{+} = \{v \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) | v([\varpi]) \neq 0, v(p) \neq 0\}.$$

Then any φ-equivariant vector bundle over $\mathcal{Y}_S$ extends uniquely to a φ-equivariant vector bundle over $\mathcal{Y}_S^+$.

Let x_k be the unique closed point of $\mathrm{Spa}(A_{\mathrm{inf}})$, then combine all the lemmas, we will have the following theorem of Fargues and Schozle-Weinsterin:

Theorem 2. [SW, Theorem 14.1.1][Ked1, Theorem 4.5.18] *Let $S = \mathrm{Spa}(C)$ with C an algebraically closed non-archimedean field in characteristic p, the following categories are canonically equivalent:*

(1) *a shtuka with one leg x over S such that x corresponds to an untilt in characteristic 0,*
(2) *a vector bundle $\mathcal{E}$ over $\mathrm{Spa}(A_{\mathrm{inf}})\backslash\{x_k\}$ together with an isomorphism $\varphi_{\mathcal{E}} : \varphi^*(\mathcal{E}) \cong \mathcal{E}|_{(\mathrm{Spa}(A_{\mathrm{inf}})\backslash\{x_k\})\backslash(x)}$,*
(3) *a finite free module $\mathcal{M}$ over A_{inf} together with $\varphi_{\mathcal{M}} : (\varphi^*\mathcal{M})[\frac{1}{z}] \simeq \mathcal{M}[\frac{1}{z}]$, where z generate the primitive ideal in A_{inf} correspondence to the untilt x.*

Proof. (sketch) For the equivalence between (1) and (2), by Lemma 4, we have a shtukas over S with leg x correspondence to an untilt of C in characteristic is the same as $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$, then we construct a vector bundle over $\mathrm{Spa}(A_{\mathrm{inf}})\backslash\{x_k\}$ by gluing $\mathcal{E}_0$ with $\mathcal{E}_1^+$ over a rational subdomain "between x and $\varphi(x)$", where $\mathcal{E}_1^+$ is the unique φ-equivariant vector bundle over $\mathcal{Y}_S^+$ extending $\mathcal{E}_1$ under Lemma 5. The "gluing" process makes sense because of the fact the presheaf over $\mathrm{Spa}(A_{\mathrm{inf}})\backslash\{x_k\}$ defined by rational subspaces and their Tate algebras is actually a sheaf.

For the equivalence between (2) and (3), one refers to the following theorem of Kedlaya. □

Theorem 3. [Ked2, Theorem 3.6] *There is an equivalence of categories between:*

(1) *Finite free modules $\mathcal{M}$ over A_{inf},*
(2) *Vector bundle $\mathcal{E}$ over $\mathrm{Spa}(A_{\mathrm{inf}})\backslash\{x_k\}$*

2.2. **Fargues-Fontaine curve and Breuil-Kisin-Fargues modules.** In this subsection, we will review Fargues-Fontaine's construction of the p-adic fundamental curve, and its relation to Scholze's definition of Shtukas in mixed characteristic.

Keep all the notions as in 1.4. Let $S = \mathrm{Spa}(C)$ with $C = \mathbb{C}_p^\flat$, we also fix the leg $x = \varphi^{-1}(x_{\mathbb{C}_p})$ of shtukas over S to make it corresponds to the untilt $\theta \circ \varphi^{-1} : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{C_p}$. Recall we have a Frobenius φ acts on the space $\mathcal{Y}_S$, we define:

Definition 3. Let $B = H^0(\mathcal{Y}_S, \mathcal{O}_{\mathcal{Y}_S})$ and the schematic Fargues-Fontaine curve is the scheme:
$$X_{FF} = \mathrm{Proj} \oplus_{n\geq 0} B^{\varphi = p^n}.$$
We also definite the adic Fargues-Fontaine curve to be quotient:
$$\mathcal{X}_{FF} = \mathcal{Y}_S/\varphi^{\mathbb{Z}}.$$

We have θ induces a map $B \to \mathbb{C}_p$, and this defines a closed point ∞ on X_{FF}.

Theorem 4. [FF, Fargues-Fontaine]

(1) *X_{FF} is a regular noetherian scheme of Krull dimension 1, or an abstract regular curve in the sense of Fargues and Fontaine.*
(2) *The set of closed points of X_{FF} is identified with the set of characteristic 0 untilts of C modulo Frobenius equivalence. Under this identification, ∞ sends to the untilt $\mathbb{C}_p$ of C. The stalk of X_{FF} at ∞ is isomorphic to B_{dR}^+.*
(3) *$X_e = X_{FF} \backslash \{\infty\}$ is an affine scheme $SpecB_e$ with $B_e = B^{\varphi=1}$ being a principal ideal domain.*
(4) *Vector bundles $\mathcal{E}$ over X_{FF} are equivalence to B-pairs $(M_e, M_{\mathrm{dR}}^+, \iota)$ in the sense of Berger. Here $M_e = \Gamma(X_e, \mathcal{E})$ are finite projective modules over B_e and M_{dR}^+ are the completion of $\mathcal{E}$ at ∞ which are finite free over B_{dR}^+, ι is an isomorphism of M_e and M_{dR}^+ over B_{dR}.*
(5) *The abstract curve X_{FF} is also complete in the sense that $deg(f) := \sum_{x \in |X_{FF}|} v_x(f)$ is 0 for all rational functions on X_{FF}.*

Fargues-Fontaine shows that there is a Dieudonné-Manin classification for vector bundles over X_{FF}.

Theorem 5. [FF] *Let (D, φ) be an isocrystal over k, then (D, φ) defines a vector bundle $\mathcal{E}(D, \varphi)$ over X_{FF} which associated with the graded module*

$$\oplus_{n \geq 0} (D \otimes_{K_0} B)^{\varphi = p^n}.$$

Moreover, every vector bundle over X_{FF} is isomorphic to $\mathcal{E}(D, \varphi)$ for some (D, φ).

Let $\mathcal{E}$ be a vector bundle over X_{FF}, assume $\mathcal{E} \cong \mathcal{E}(D, \varphi)$ under the above theorem, and if $\{-\lambda_i\}$ are the slopes of (D, φ) in the Dieudonné-Manin classification theorem, then λ_i are called the slopes of $\mathcal{E}$. Moreover $\mathcal{E}$ is called semistable of slope λ if and only if $\mathcal{E}$ corresponds a semisimple isocrystal of slope $-\lambda$. We define $\mathcal{O}(n) = \mathcal{E}(K_0, p^{-n})$, one can show $\mathcal{O}(1)$ is a generator of the Picard group of X_{FF} (which is isomorphic to $\mathbb{Z}$). A simple corollary of Dieudonné-Manin classification is:

Corollary 2. *The category of finite-dimensional $\mathbb{Q}_p$-vector spaces is equivalent to the category of vector bundles over X_{FF} that are semistable of slope 0 under the functor*

$$V \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_X.$$

The inverse of this functor is given by:

$$\mathcal{E} \to H^0(X_{FF}, \mathcal{E}).$$

Note there is a morphism of locally ringed spaces from $\mathcal{X}_{FF} \to X_{FF}$, and pullback along this morphism induces a functor from the category of vector bundles over X_{FF} to vector bundles over $\mathcal{X}_{FF}$, we have the following GAGA theorem for the Fargues-Fontaine curve.

Theorem 6. *Vector bundles over X_{FF} and vector bundles over $\mathcal{X}_{FF}$ are equivalent under the above functor.*

We have, by the definition of $\mathcal{X}_{FF}$, vector bundles over $\mathcal{X}_{FF}$ is the same as φ-equivariant vector bundles over $\mathcal{Y}_S$. So by the theorem of GAGA one can make sense of slopes of φ-equivariant vector bundles over $\mathcal{Y}_S$. We have the following theorem of Kedlaya:

Theorem 7. [KL, Theorem 8.7.7] *A φ-equivariant vector bundle $\mathcal{F}$ over $\mathcal{Y}_S$ is semistable of slope 0 if and only if it can be extended to a φ-equivariant vector bundle over $\mathcal{U}_S$. The set of such extensions is the same as the set of $\mathbb{Z}_p$-lattices inside the $\mathbb{Q}_p$-vector space $H^0(X_{FF}, \mathcal{E})$, where $\mathcal{E}$ is the vector bundle over X_{FF} corresponds to $\mathcal{F}$ under GAGA.*

Remark 6. One can also rewrite the above theorem in terms of (étale) φ-modules over B and B^+, where $B^+ = H^0(\mathcal{Y}_S^+)$, one can show that (when C is algebraically closed) the category of vector bundles over X_{FF} is equivalence to all the following categories [FF, Section 11.4]:

- φ-modules over B.
- φ-modules over B^+.
- φ-modules over B^+_{cris}.
- Vector bundles over X_{FF}.

Combine Lemma 4, Theorem 6 and Theorem 7 we have:

Theorem 8. *Let $S = \mathrm{Spa}(C)$ with $C = \mathbb{C}_p^\flat$, then a shtuka with one leg $\varphi^{-1}(x_{\mathbb{C}_p})$ over S which corresponds to the untilt $\theta \circ \varphi^{-1} : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{C_p}$ is the same as a quadruple $(\mathcal{F}_0, \mathcal{F}_1, \beta, T)$ where*

- *$\mathcal{F}_0$ is a vector bundle over X_{FF} that is semistable of slope 0,*
- *$\mathcal{F}_1$ is a vector bundle over X_{FF},*
- *β is an isomorphism of $\mathcal{F}_0$ and $\mathcal{F}_1$ over $X_{FF}\backslash\{\infty\}$,*
- *T is a $\mathbb{Z}_p$-lattice of the $\mathbb{Q}_p$ vector space $H^0(X_{FF}, \mathcal{E})$.*

Using part (4) of Theorem 4, we have the first three data in the above theorem is the same as a $\mathbb{Q}_p$ vector space V together with a B^+_{dR} lattice inside $V \otimes B_{\mathrm{dR}}$. Note also we have $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} = T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ for any $\mathbb{Z}_p$-lattice T inside V. So we have:

Corollary 3. *Let $S = \mathrm{Spa}(C)$ and x as above, then a shtuka with one leg x over S is the same as a pair (T, Ξ) where T is a $\mathbb{Z}_p$-lattice and Ξ is a B^+_{dR} lattice inside $T \otimes B_{\mathrm{dR}}$.*

Definition 4. The pair (T, Ξ) as above is called a Hodge-Tate module. And we define a free Breuil-Kisin-Fargues module as a finite free module $\mathfrak{M}^{\mathrm{inf}}$ over A_{inf} with an isomorphism

$$\varphi_{\mathfrak{M}^{\mathrm{inf}}} : \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}}[\frac{1}{\tilde{\xi}}] \simeq \mathfrak{M}^{\mathrm{inf}}[\frac{1}{\tilde{\xi}}]$$

Where $\tilde{\xi} = \varphi(\xi)$ as we defined in 1.4.

Corollary 4. *(Fargues, Scholze-Weinstein) Let $S = \mathrm{Spa}(C)$ and $\varphi^{-1}(x_{\mathbb{C}_p})$ as above, then the following categories are equivalence:*

- *Shtukas with one leg $\varphi^{-1}(x_{\mathbb{C}_p})$ over S.*
- *Hodge-Tate modules.*
- *Free Breuil-Kisin-Fargues modules.*

3. p-adic representations and vector bundles on the curve

Now let us briefly recall how Fargues-Fontaine construct G_K-equivariant modifications of vector bundles over X_{FF} from potentially log-crystalline representations of G_K in [FF, §10.3.2].

Keep the notions as in 1.4, let D' be a filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules, Fargues-Fontaine first define the $\mathcal{O}_X$-representation $\mathcal{E}(D', \varphi, G_K)$ of G_K whose underlying vector bundle is $\mathcal{E}(D', \varphi)$(as we defined in Theorem 5) and the semilinear G_K-action coming from the diagonal action of G_K on $D' \otimes_{L_0} B$. Note that this construction is functorial, so the relation $N\varphi = p\varphi N$ tells that N defines a G_K-equivariant map

$$\mathcal{E}(D, \varphi, G_K) \to \mathcal{E}(D, p\varphi, G_K) = \mathcal{E}(D', \varphi, G_K) \otimes \mathcal{O}(-1).$$

Let $\varpi \in C$ be any element such that $v(\varpi) = 1$, and for any $\sigma \in G_K$ define $\log_{\varpi,\sigma} = \sigma(\log[\varpi]) - \log[\varpi]$. One can show $(\sigma \mapsto \log_{\varpi,\sigma})$ defines an element in $Z^1(G_K, B^{\varphi=p}) = Z^1(G_K, H^0(\mathcal{O}(1)))$. So we know the composition:

$$\mathcal{E}(D', \varphi) \xrightarrow{N} \mathcal{E}(D', \varphi) \otimes \mathcal{O}(-1) \xrightarrow{\mathrm{Id} \otimes \log_{\varpi,\sigma}} \mathcal{E}(D', \varphi) \otimes \mathcal{O} = \mathcal{E}(D', \varphi),$$

defines an element in $Z^1(G_K, \mathrm{End}(\mathcal{E}(D', \varphi)))$ whose image actually lies in the nilpotent elements of $\mathrm{End}(\mathcal{E}(D', \varphi))$. So we can define

$$\alpha = (\alpha_\sigma)_\sigma = (\exp(-\mathrm{Id} \otimes \log_{\varpi,\sigma} \circ N))_\sigma \in Z^1(G_K, Aut(\mathcal{E}(D', \varphi))),$$

Fargues-Fontaine define the G_K-equivariant vector bundle associated with a (φ, N, G_K)-module D' to be the vector bundle:

$$\mathcal{E}(D', \varphi, N, G_K) = \mathcal{E}(D', \varphi, G_K) \wedge \alpha,$$

i.e., $\mathcal{E}(D', \varphi, N, G_K)$ is isomorphic to $\mathcal{E}(D', \varphi)$ as vector bundle, and the G_K action on $\mathcal{E}(D', \varphi, N, G_K)$ is given by twisting the G_K-action of $\mathcal{E}(D', \varphi, G_K)$ with the 1-cocycle α.

Lemma 6. *We have*

1. *α becomes trivial when completes at ∞.*
2. *If the data (D', φ, N, G_K) comes from a potentially log-crystalline representation V of G_K, then the completion of $\mathcal{E}(D', \varphi, N, G_K)$ at ∞ together with its G_K-action is isomorphic to $D_{\mathrm{dR}}(V) \otimes_K B^+_{\mathrm{dR}}$.*
3. *If we rewrite the above construction in terms of φ-modules over B^+ as in Remark 6, then α becomes trivial after the base change $B^+ \to W(\overline{k})[\frac{1}{p}]$.*

Proof. (1) is [FF, Proposition 10.3.18, Remark 10.3.19]. For (3), if we rewrite the above construction in terms of φ-modules over B^+, then $\mathcal{E}(D',\varphi)$ corresponds to the φ-module $D' \otimes_{L_0} B^+$ and the φ-equivariant map

$$N \otimes \log_{\varpi,\sigma} : D' \otimes_{L_0} B^+ \to D' \otimes_{L_0} B^+$$

corresponds to the map

$$\mathcal{E}(D',\varphi) \xrightarrow{N} \mathcal{E}(D',\varphi) \otimes \mathcal{O}(-1) \xrightarrow{\mathrm{Id}\otimes\log_{\varpi,\sigma}} \mathcal{E}(D',\varphi) \otimes \mathcal{O} = \mathcal{E}(D',\varphi),$$

In order to show α becomes trivial after the base change $B^+ \to W(\overline{k})[\frac{1}{p}]$, one just need to show that $\log_{\varpi,\sigma} \in B^+$ maps to 0 under $B^+ \to W(\overline{k})[\frac{1}{p}]$, while this can be seen from the fact $\log_{\varpi,\sigma} \in (B^+)^{\varphi=p}$, but $W(\overline{k})[\frac{1}{p}]^{\varphi=p} = 0$.

For (2), if D' is a weakly admissible filtered (φ, N, G_K)-module and we are assuming $D' = (V \otimes B_{\mathrm{st}})^{G_L}$ with V a potentially log-crystalline representation that becomes log-crystalline over the finite Galois extension L over K. Since α becomes trivial at the stalk ∞, one has

$$\mathcal{E}(D',\varphi,N,G_K)_\infty = \mathcal{E}(D',\varphi,G_K)_\infty = (V \otimes B_{\mathrm{st}})^{G_L} \otimes_{L_0} B_{\mathrm{dR}}^+$$

with the diagonal action of G_K. We have $V|_{G_L}$ is log-crystalline representation of G_L, so it is de Rham and satiesfies:

$$(V \otimes B_{\mathrm{st}})^{G_L} \otimes_{L_0} L = (V \otimes B_{\mathrm{dR}})^{G_L}$$

And since V is a de Rham representation of G_K. We have:

$$(V \otimes B_{\mathrm{dR}})^{G_L} = (V \otimes B_{\mathrm{dR}})^{G_K} \otimes_K L.$$

Tensoring everything with B_{dR}^+ we get what we want to prove. □

From now on, we will always assume the data (D',φ,N,G_K) comes from a potentially log-crystalline representation V that becomes log-crystalline over a finite Galois extension L over K. Using the G_K-equivariant filtration on D_L, Fargues-Fontaine construct a G_K-equivariant modification $\mathcal{E}(D',\varphi,N,\mathrm{Fil}^\bullet,G_K)$ of $\mathcal{E}(D',\varphi,N,G_K)$ by letting:

$$\mathcal{E}(D',\varphi,N,\mathrm{Fil}^\bullet,G_K)|_{X_{FF}\backslash\infty} = \mathcal{E}(D',\varphi,N,G_K)|_{X_{FF}\backslash\infty}$$

and

$$\mathcal{E}(D',\varphi,N,\mathrm{Fil}^\bullet,G_K)_\infty = \mathrm{Fil}^0(\mathcal{E}(D',\varphi,N,G_K)_\infty) = \mathrm{Fil}^0(D_L \otimes_L B_{\mathrm{dR}})$$

where the filtration on $D_L \otimes_L B_{\mathrm{dR}}$ is given by

$$\mathrm{Fil}^k(D_L \otimes B_{\mathrm{dR}}) = \sum_{i+j=k} \mathrm{Fil}^i(D_L) \otimes \mathrm{Fil}^j(B_{\mathrm{dR}}).$$

Proposition 4. *If the filtered (φ,N,G_K)-module D' is weakly admissible, then $\mathcal{E}(D',\varphi,N,\mathrm{Fil}^\bullet,G_K)$ is semistale of slope 0. Moreover, there is a G_K-equivalraint isomorphism*

$$V = H^0(X_{FF}, \mathcal{E}(D',\varphi,N,\mathrm{Fil}^\bullet,G_K)),$$

where V is the potentially log-crystalline representation corresponds to the data $(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$.

Proof. This is stated as [FF, §10.5.3, Remark 10.5.8] and the proof also works for potentially log-crystalline representations. Actually, let V be the potentially log-crystalline representation corresponds to $(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$ and let $\mathcal{E}_V = V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$ be corresponded slope 0 $\mathcal{O}_X$-representation of G_K. The theorem is equivalence to show

$$\mathcal{E}_V = \mathcal{E}(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$$

and we can prove it by comparing the B-pairs of $\mathcal{E}(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$ and $\mathcal{E}_V$ by (4) of Theorem 4. While we have the B_e-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is the same as the B_e-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D', \varphi, N, G_K)$ by construction, and which is equal to

$$(D' \otimes_{L_0} B_{st})^{\varphi=1, N=0}$$

by Corollary 10.3.17 of *loc.cit.*. The B_{dR}^+-part of $\mathcal{E}(D', \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is the B_{dR}^+-representation

$$\mathrm{Fil}^0(D'_L \otimes_L B_{\mathrm{dR}})$$

by definition.

On the other hand, the B-pair correspond to $\mathcal{E}_V$ is

$$(V \otimes_{\mathbb{Q}_p} B_e, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+).$$

Since V is potentially log-crystalline, so we have $V \otimes_{\mathbb{Q}_p} B_e$ is potentially log-crystalline as a B_e-representation, which means there is a G_K-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_e = \Big(\big((V \otimes_{\mathbb{Q}_p} B_e) \otimes_{B_e} B_{st} \big)^{G_L} \otimes_{L_0} B_{st} \Big)^{\varphi=1, N=0}$$

by Proposition 10.3.20 of *loc.cit.*. And $\big((V \otimes_{\mathbb{Q}_p} B_e) \otimes_{B_e} B_{st}\big)^{G_L}$ is nothing but D'. Since V is de Rham, so the B_{dR}^+-representation $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$ is *generically flat* in the sense of Definition 10.4.1 of *loc.cit.*, so Proposition 10.4.4 of *loc.cit*. shows that there is a G_K-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ = \mathrm{Fil}^0(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}).$$

The proposition follows from the fact $D_{\mathrm{dR}}(V) \otimes_K L = D'_L$, and the G_K-equivariant filtration on D'_L descents to the filtration $D_{\mathrm{dR}}(V)$ by the definition of weakly admissibility filtered (φ, N, G_K)-modules. □

Definition 5. If the filtered (φ, N, G_K)-module D' is weakly admissible, and let V be the corresponded potentially log-crystalline representation, then the G_K-equivariant modification

$$\mathcal{E}(D', \varphi, N, \mathrm{Fil}^\bullet, G_K) \dashrightarrow \mathcal{E}(D', \varphi, N, G_K)$$

together with a G_K-stable lattice T inside V defines a shtuka with one leg at $\varphi^{-1}(x_{\mathbb{C}_p})$ over $S = \mathrm{Spa}(C)$. Moreover, since the correspondence in

Theorem 8 is functorial, the shtuka constructed in this way carries a natural G_K-action.

A shtuka (resp. Hodge-Tate module or Breuil-Kisin-Fargues module) is called arithmetic if it (resp. the corresponded shtuka with one leg at $\varphi^{-1}(x_{\mathbb{C}_p})$ over $\mathrm{Spa}(C)$) carries a G_K-action arisen as above.

Proposition 5. *A Hodge-Tate module (T, Ξ) together with a semilinear G_K-action is arithmetic if and only if T is de Rham as a G_K-representation over $\mathbb{Z}_p$ and there is a G_K-equivalence isomorphism*

$$(T, \Xi) \cong (T, (T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{G_K} \otimes_K B_{\mathrm{dR}}^+).$$

A Breuil-Kisin-Fargues module $\mathfrak{M}^{\inf}$ is arithmetic if and only if its étale realization $T = (\mathfrak{M}^{\inf} \otimes W(C))^{\varphi=1}$ is de Rham as a G_K-representation over $\mathbb{Z}_p$ and $\mathfrak{M}^{\inf} \otimes B_{\mathrm{dR}}^+$ has a B_{dR}^+-basis fixed by the G_K-action.

Proof. First, $(T, (T \otimes B_{\mathrm{dR}})^{G_K} \otimes B_{\mathrm{dR}}^+)$ is a Hodge-Tate module since T is potentially log-crystalline so de Rham, we have $(T \otimes B_{\mathrm{dR}})^{G_K} = D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p)$ is of full rank.

The rest of the proposition comes from Lemma 6 (2) and the correspondence in Corollary 4. □

Corollary 5. *Let $\mathfrak{M}^{\inf}$ be an arithmetic Breuil-Kisin-Fargues module $\mathfrak{M}^{\inf}$ then*

1. *$T(\mathfrak{M}^{\inf})$ is log-crystalline if and only if the inertia subgroup I_K of G_K acts trivially on $\overline{\mathfrak{M}^{\inf}} = \mathfrak{M}^{\inf} \otimes_{A_{\inf}} W(\overline{k})$.*
2. *$T(\mathfrak{M}^{\inf})$ is potentially crystalline if and only if there is a φ and G_K equivariant isomorphism*

$$\mathfrak{M}^{\inf} \otimes_{A_{\inf}} B^+ \cong \overline{\mathfrak{M}^{\inf}} \otimes_{W(\overline{k})} B^+.$$

3. *$T(\mathfrak{M}^{\inf})$ is crystalline if it satisfies the conditions in (1) and (2).*

Proof. Keep the notions as in the proof of Lemma 6. For (1), recall we have the fact that T is log-crystalline if and only if $T|_{I_K}$ is log-crystalline, so we can assume $\overline{k}$ is algebraically closed and $L_0 = K_0$. And we have since α becomes trivial after $B^+ \to W(\overline{k})[\frac{1}{p}] = K_0$ from Lemma 6 (3), we have $\overline{\mathfrak{M}^{\inf}}[\frac{1}{p}]$ with its G_K-action is nothing but D' with the G_K-action coming from the G_K-structure of the filtered (φ, N, G_K)-module D', and it is log-crystalline if and only if the filtered (φ, N, G_K)-module D' is a filtered (φ, N)-module D', i.e. G_K acts trivially on D'.

For (2), again we can restrict our statement to I_K, and (2) means the shtuka comes from just modifying the vector bundle $\mathcal{E}(D', \varphi, G_K)$ (there is not a twist by α), i.e., $N = 0$. □

4. Kisin modules and arithmetic shtukas

Let K and $\mathcal{O}_K$ as before, fix a uniformizer ϖ of $\mathcal{O}_K$ and $k = \mathcal{O}_K/\varpi$ the residue field. We also fix a compatible system $\{\varpi_n\}$ of p^n-th roots of ϖ, we define $K_\infty = \cup_{n=1}^\infty K(\varpi_n)$. The compatible system $\{\varpi_n\}$ also defines an element $(\overline{\varpi}_n)$ in $\mathcal{O}_C = \varprojlim_\varphi \mathcal{O}_{\mathbb{C}_p}/p$ and so an element $u = [(\overline{\varpi}_n)]$ in A_{inf}. We have $k = \mathcal{O}_K/\varpi$ and we have $k = \varprojlim_\varphi \mathcal{O}_K/\varpi \xrightarrow{\sim} \varprojlim_\varphi \mathcal{O}_{\mathbb{C}_p}/\varpi = \varprojlim_\varphi \mathcal{O}_{\mathbb{C}_p}/p = \mathcal{O}_C$ by [BMS, Lemma 3.2], so $W(k)$ is a subring of A_{inf}. Define $\mathfrak{S}$ as the sub-$W(k)$-algebra of A_{inf} generated by u. One can check $\varphi_{A_{\text{inf}}}(u) = u^p$, so in particular $\mathfrak{S}$ in stable under $\varphi_{A_{\text{inf}}}$, let $\varphi_{\mathfrak{S}} = \varphi_{A_{\text{inf}}}|_{\mathfrak{S}}$. We also have G_{K_∞} fix u so G_{K_∞} acts trivially on $\mathfrak{S}$. And we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\theta|_{\mathfrak{S}}} & \mathcal{O}_K \\ \downarrow & & \downarrow \\ A_{\text{inf}} & \xrightarrow{\theta} & \mathcal{O}_{\mathbb{C}_p} \end{array}$$

the vertical arrows are faithful flat ring extensions by [BMS] and moreover, $\theta|_{\mathfrak{S}}$ is surjective and the kernel is generated by $E(u)$, which is an Eisenstein polynomial. All arrows in this diagram are G_{K_∞}-equivalent.

Let T be a log-crystalline representation of G_K over $\mathbb{Z}_p$ with nonnagative Hodge-Tate weights, then Kisin in [Kis] can associate T with a free Kisin module, i.e. a finite free $\mathfrak{S}$-module $\mathfrak{M}$ together a $\varphi_{\mathfrak{S}}$-semilinear endomorphism $\varphi_{\mathfrak{M}}$ such that the cokernal of the $\mathfrak{S}$-linearization $1 \otimes \varphi_{\mathfrak{M}} : \varphi_{\mathfrak{S}}^* \mathfrak{M} \to \mathfrak{M}$ is killed by a power of $E(u)$. Moreover, if we define $\mathfrak{M}^{\text{inf}}(T) = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\text{inf}}$ then one can show $\mathfrak{M}^{\text{inf}}(T)$ is a Breuil-Kisin-Fargues module as in Definition 4, it carries a natural G_{K_∞}-semilinear action. We claim that there is an unique way to define a G_K-semilinear action on $\mathfrak{M}^{\text{inf}}(T)$ commutes with $\varphi_{\mathfrak{M}^{\text{inf}}(T)}$ extending the G_{K_∞}-semilinear action such that

$$(\mathfrak{M}^{\text{inf}}(T) \otimes W(C))^{\varphi=1} = T.$$

In the case when T is a potentially log-crystalline representation of G_K over $\mathbb{Z}_p$ with nonnagative Hodge-Tate weights, and assume L/K is a finite Galois extension such that $T|_{G_L}$ is log-crystalline. Then as before, $\mathfrak{M}^{\text{inf}}(T|_{G_L})$ carries a natural G_{L_∞}-semilinear action for a choices of L_∞. We make the claim:

Proposition 6. *There is a unique way to extends the G_{L_∞}-semilinear action on $\mathfrak{M}^{\text{inf}}(T) := \mathfrak{M}^{\text{inf}}(T|_{G_L})$ to an action of G_K, such that it commutes with $\varphi_{\mathfrak{M}^{\text{inf}}(T)}$ and*

$$(\mathfrak{M}^{\text{inf}}(T) \otimes W(C))^{\varphi=1} = T.$$

Proof. As we mentioned in the introduction, we will prove this proposition by comparing the construction of Kisin of the $\mathfrak{S}$-module and Fargues-Fontaine's construction.

First, we need a brief review of the construction of Kisin module from log-crystalline representation: let T is a potentially log-crystalline representation of G_K over $\mathbb{Z}_p$, L/K be a finite Galois extension such that $T|_{G_L}$ becomes log-crystalline, and let $L_0 = W(k_L)[\frac{1}{p}]$ and define $D' = (T \otimes B_{\text{st}})^{G_L}$

as the filtered (φ, N, G_K)-module associated with $T \otimes \mathbb{Q}_p$. Then we obtain a filtered (φ, N)-module D' over L or $(D', \varphi, N, \mathrm{Fil}^\bullet)$ by forgetting the G_K-action. D' corresponds to the log-crystalline representation $T \otimes \mathbb{Q}_p|_{G_L}$. Now let $\mathcal{O}$ be the ring of rigid analytic functions over the open unit disc over L_0 in the variable u. Let $\mathfrak{S} = W(k_L)[[u]]$, then one has $\mathfrak{S}[\frac{1}{p}] \subset \mathcal{O}$ and there is a $\varphi_{\mathcal{O}}$ extending $\varphi_{\mathfrak{S}}$. Fix $(\varpi_{L,n})$ any choice of compatible system of p^n-th roots of a uniformizer $\varpi_{L,0}$ of L, then one can easily show that the the inclusion $\mathfrak{S}[\frac{1}{p}] \to A_{\mathrm{inf}}[\frac{1}{p}]$ with $u \mapsto [(\overline{\varpi_{L,n}})]$ extends to an inclusion $\mathcal{O} \to B^+$. Geometrically, $\mathcal{O}$ (resp. B^+) is the locus $\{p \neq 0\}$ of $\mathrm{Spa}(\mathfrak{S})$ (resp. $\mathrm{Spa}(A_{\mathrm{inf}})$), and restrict the covering map $\mathrm{Spa}(A_{\mathrm{inf}}) \to \mathrm{Spa}(\mathfrak{S})$ to these loci will give $\mathcal{O} \to B^+$.

Given $(D', \varphi, N, \mathrm{Fil}^\bullet)$, Kisin constructs a finite free module $\mathcal{M}(D')$ over $\mathcal{O}$ together with a $\varphi_{\mathcal{O}}$-semilinear action[Kis, §1.2].

To be more precise, for every $n \geq 0$ consider the composition:

$$\theta_n : \mathcal{O} \longrightarrow B^+ \xrightarrow{\varphi^{-n}} B^+ \xrightarrow{\theta} \mathbb{C}_p$$

and let x_n be the closed points on the rigid open unit disc defined by θ_n. Now define

$$\log u = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(u/\varpi_{L,0} - 1)^i}{i}$$

One can check $\log u \in B_{\mathrm{dR}}^+$ and let $\mathcal{O}_{st} = \mathcal{O}[\log u]$, the $\mathcal{O}$-algebra generated by $\log u$ in side B_{dR}^+. And extend the φ action to $\log u$ by $\varphi(\log u) = p \log u$ and define a $\mathcal{O}$-derivation N on $\mathcal{O}_{st}^+$ by letting $N(\log u) = -\lambda$ for some $\lambda \in \mathcal{O}$. Kisin defines $\mathcal{M}(D')$ as a modification of the vector bundle

$$(\mathcal{O}[\log u] \otimes_{L_0} D)^{N=0}$$

over $\mathcal{O}$ along all the stalks at x_n for $n \geq 0$. And the modifications are defined using the filtration on D_L. As a result, the stalks of $\mathcal{M}(D')$ away from $\{x_n\}$ are isomorphic to those of $(\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0}$. If we base change $\mathcal{M}(D')$ to B^+, and consider the closed points x_{-1} corresponds to

$$B^+ \xrightarrow{\varphi} B^+ \xrightarrow{\theta} \mathbb{C}_p.$$

We know the completion of B^+ at x_{-1} is isomorphic to B_{dR}^+ and the above arguments tell us that:

$$\mathcal{M}(D') \otimes_{\mathcal{O},\varphi} B_{\mathrm{dR}}^+ = (\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0} \otimes_{\mathcal{O},\varphi} B_{\mathrm{dR}}^+.$$

Moreover, Kisin shows there is a natural isomorphism[Kis, Proposition 1.2.8.]:

$$(\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0} = (L_0[\log u] \otimes_{L_0} D')^{N=0} \otimes_{L_0} \mathcal{O} \xrightarrow{\eta \otimes \mathrm{id}} D' \otimes_{L_0} \mathcal{O}.$$

This tells us the stalk $(\mathcal{M}(D') \otimes_{\mathcal{O}} B^+) \otimes_{B^+,\varphi} B_{\mathrm{dR}}^+$ is isomorphic to

$$(D' \otimes_{L_0,\varphi} L_0) \otimes_{L_0} B_{\mathrm{dR}}^+ \cong D' \otimes_{L_0} B_{\mathrm{dR}}^+.$$

Moreover, using the theory of slope of Kedlaya, Kisin was able to prove that the φ-module $\mathcal{M}(D')$ over $\mathcal{O}$ descents to a Kisin module $\mathfrak{M}$ over $\mathfrak{S}$ when D' is weakly admissible, i.e., $\mathfrak{M} \otimes \mathcal{O} = \mathcal{M}(D')$. So we have

$$\mathfrak{M} \otimes_{\mathfrak{S},\varphi} B^+ = \mathcal{M}(D') \otimes_{\mathcal{O},\varphi} B^+.$$

In particular, one has:

$$\mathfrak{M} \otimes_{\mathfrak{S},\varphi} B_{\mathrm{dR}}^+ = \mathcal{M}(D') \otimes_{\mathcal{O},\varphi} B_{\mathrm{dR}}^+ = D' \otimes_{L_0} B_{\mathrm{dR}}^+.$$

A theorem of Fontaine says that the ways of descent $\mathcal{M}(D')$ to $\mathfrak{M}$ are canonically corresponded with G_{L_∞}-stable $\mathbb{Z}_p$-lattices in $T \otimes \mathbb{Q}_p$, where $L_\infty = \cup_{n=1}^{\infty} L(\varpi_{L,n})$. Then Kisins define $\mathfrak{M}$ to be the $\mathfrak{S}$-module descents $\mathcal{M}(D')$ using the lattice $T|_{G_{L_\infty}}$. This is the same as saying that

$$(\mathfrak{M} \otimes_{\mathfrak{S}} W(C))^{\varphi=1} = T|_{G_{L_\infty}}.$$

Note that

$$(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} W(C))^{\varphi=1} = (\mathfrak{M} \otimes_{\mathfrak{S}} W(C))^{\varphi=1}$$

since $W(C)^{\varphi=1} = \mathbb{Z}_p$. Now if we let $\mathfrak{M}^{\mathrm{inf}}(T) = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$, then the Hodge-Tate module of $\mathfrak{M}^{\mathrm{inf}}(T)$ is

$$(T|_{G_{L_\infty}}, D' \otimes_{L_0} B_{\mathrm{dR}}^+).$$

To finish the proof, we let $\mathfrak{M}_c^{\mathrm{inf}}$ be the Breuil-Kisin-Fargues module corresponds to the constant Hodge-Tate module $(T, T \otimes B_{\mathrm{dR}}^+)$, then $\mathfrak{M}_c^{\mathrm{inf}}$ is equipped with an unique semilinear G_K-action coming from the action on T. It is enough to prove that there is an injection $\mathfrak{M}^{\mathrm{inf}}(T) \to \mathfrak{M}_c^{\mathrm{inf}}$ such that $\mathfrak{M}^{\mathrm{inf}}(T)$ is stable under G_K. From the construction in Corollary 4, it is enough to show the Hodge-Tate module corresponds to $\mathfrak{M}^{\mathrm{inf}}(T)$ injects to $(T, T \otimes B_{\mathrm{dR}}^+)$ and stable under G_K (the functor is left exact). But we have computed the the Hodge-Tate module corresponds to $\mathfrak{M}^{\mathrm{inf}}(T)$. Using the fact

$$D' \otimes_{L_0} L = D_{st}(T \otimes \mathbb{Q}_p|_{G_L}) \otimes_{L_0} L = D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p|_{G_L}),$$

And

$$D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p) = D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p|_{G_L})^{G_K} = (D' \otimes_{L_0} L)^{G_K}.$$

Then the proposition follows from that $D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p) \otimes B_{\mathrm{dR}}^+$ injects into $T \otimes B_{\mathrm{dR}}^+$. And when we extends the G_{L_∞} action on $T|_{G_{L_\infty}}$ to G_K by T, $D' \otimes_{L_0} B_{\mathrm{dR}}^+$ which equals to $D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p) \otimes_K B_{\mathrm{dR}}^+$ is automatically stable under G_K inside $T \otimes B_{\mathrm{dR}}^+$. □

From the proof of the above proposition, we have

Corollary 6. *Let T be a log-crystalline representation of G_K over $\mathbb{Z}_p$ with nonnagative Hodge-Tate weights, and let $\mathfrak{M}^{\mathrm{inf}}(T)$ be the Breuil-Kisin-Fargues module with the semilnear G_K-action described as in the pervious proposition. Then $\mathfrak{M}^{\mathrm{inf}}(T)$ is arithmetic. Moreover, $\mathfrak{M}^{\mathrm{inf}}(T)$ corresponds to the shtuka associate with the G_K-equivariant modification:*

$$\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K) \dashrightarrow \mathcal{E}(D, \varphi, N, G_K)$$

together with the G_K-stable $\mathbb{Z}_p$-lattice T, where $(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is the filtered (φ, N, G_K)-module corresponds to $T \otimes \mathbb{Q}_p$.

Proof. We have showed that the Hodge-Tate module of $\mathfrak{M}^{\inf}(T)$ together with the G_K-action defined in the previous proposition is isomorphic to

$$(T, D_{\mathrm{dR}}(T \otimes \mathbb{Q}_p) \otimes_K B_{\mathrm{dR}}^+).$$

Then use Proposition 5. □

Definition 6. [GL, F.7. Definition] Let $\mathfrak{M}^{\inf}$ be a Breuil-Kisin-Fargues G_K-module. Then we say that $\mathfrak{M}^{\inf}$ admits all descents over K if the following conditions hold.

(1) For any choice ϖ of uniformaizer of $\mathcal{O}_K$ and any compatible system $\varpi^\flat = (\varpi_n)$ of p^n-th roots of ϖ, there is a Breuil-Kisin module $\mathfrak{M}_{\varpi^\flat}$ defined using $\varpi^\flat$ such that $\mathfrak{M}_{\varpi^\flat} \otimes_{\mathfrak{S},\varphi} A_{\inf}$ is isomorphic to $\mathfrak{M}^{\inf}$ and $\mathfrak{M}_{\varpi^\flat}$ is fixed by $G_{K_{\varpi^\flat,\infty}}$ under the above isomorphism, where $K_{\varpi^\flat,\infty} = \cup_n K(\varpi_n)$
(2) Let $u_{\varpi^\flat} = [(\overline{\varpi_n})]$, then $\mathfrak{M}_{\varpi^\flat} \otimes_{\mathfrak{S},\varphi} (\mathfrak{S}/u_{\varpi^\flat}\mathfrak{S})$ is independent of the choice of ϖ and $\varpi^\flat$ as a $W(k)$-submodule of $\mathfrak{M}^{\inf} \otimes_{A_{\inf}} W(\overline{k})$.
(3) $\mathfrak{M}_{\varpi^\flat} \otimes_{\mathfrak{S},\varphi} (\mathfrak{S}/E(u_{\varpi^\flat})\mathfrak{S})$ is independent of the choice of ϖ and $\varpi^\flat$ as a $\mathcal{O}_K$-submodule of $\mathfrak{M}^{\inf} \otimes_{A_{\inf}} (A_{\inf}/\xi A_{\inf})$.

Remark 7. From Corollary 6, and if we further assume that the étale realization T of a arithmetic Breuil-Kisin-Fargues module $\mathfrak{M}^{\inf}$ is log-crystalline, we observe

$$\mathfrak{M}^{\inf} = \mathfrak{M}^{\inf}(T) = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\inf}.$$

We have $\mathfrak{M}$ depends on the choice of $\{\varpi_n\}$, while the left hand side only depends on T from Corollary 6.

Proposition 7. *Let $\mathfrak{M}^{\inf}$ be a Breuil-Kisin-Fargues G_K-module. Then $\mathfrak{M}^{\inf}$ admits all descents over K if and only if $\mathfrak{M}^{\inf}$ is arithmetic and satisfies the condition* (1) *in Corollary 5, i.e., the inertia subgroup I_K of G_K acts trivially on $\overline{\mathfrak{M}^{\inf}} = \mathfrak{M}^{\inf} \otimes_{A_{\inf}} W(\overline{k})$.*

Proof. The if part of the proposition comes from Corollary 5 (1), Corollary 6 and Remark 7.

For the only if part of the proposition, one uses the lemma about Kummer extensions as stated in [GL, F.15. Lemma.]. Then it will imply that the submodules defined in (2) and (3) of Definition 6 are G_K-stable.

Then (3) in Definition 6 together with Lemma 1 will imply that $\mathfrak{M}^{\inf}$ is arithmetic. And similarly, (2) in Definition 6 will force I_K acts trivially on $\overline{\mathfrak{M}^{\inf}} = \mathfrak{M}^{\inf} \otimes_{A_{\inf}} W(\overline{k})$. □

Remark 8. As been mentioned in [GL, F.12. Remark.], it is plausible that (2) and (3) in Definition 6 are actually consequences of (1). And one observes in the proof of Proposition 7, (2) + (3) implies $\mathfrak{M}^{\inf}$ is arithmetic and $T(\mathfrak{M}^{\inf})$ is log-crystalline, so by Remark 7, $\mathfrak{M}^{\inf}$ satisfies (1).

Corollary 7. (2) + (3) *implies* (1) *in Definition 6.*

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