$\mathbb{A}_{\mathrm{inf}}$ HAS UNCOUNTABLE KRULL DIMENSION

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ABSTRACT. Let R be a non-discrete rank one valuation ring of characteristic p and let $\mathcal{O}_E$ be any discrete valuation ring, we prove the ring of $\mathcal{O}_E$-Witt vectors over R has uncountable Krull dimension without assuming the axiom of existence of prime ideals for general commutative unitary rings.

1. INTRODUCTION

Let R be a perfect non-discrete valuation ring over $\mathbb{F}_p$ and let $\mathcal{O}_E$ be a discrete valuation ring, define $\mathbb{A} = W_{\mathcal{O}_E}(R)$, the ring of $\mathcal{O}_E$-Witt vectors over R as in [FF, §1.2], i.e., elements in $\mathbb{A}$ can be regarded as holomorphic functions in variable π, for any uniformizer π of $\mathcal{O}_E$. The main result of this paper is:

Theorem 1. *$\mathbb{A}$ has uncountable Krull dimension.*

Note that the result of the above theorem for the equal characteristic case, i.e., $\mathbb{A} = R[[T]]$ with R a rank one non-discrete valuation ring, is due to Kang-Park [KP, Theorem 10]. The result that $\mathbb{A}$ is infinite-dimensional is due to Arnold [Arn] for the equal characteristic case. In the mixed characteristic case, there are conjectures on the Krull dimension of $\mathbb{A}$ by [Ked] and [Bha, Warning 2.24], then proved to be infinite by Lang-Ludwig [LL]. We want to note that all proofs we mentioned above make use of the existence of prime ideals for general commutative unitary rings, whose proof relies on Zorn's Lemma, or equivalently, the axiom of choice. In fact, it is known that the existence of prime ideals for general commutative unitary rings is strictly weaker than the axiom of choice in Zermelo-Fraenkel axiomatic system by [Hal]. Our approach to Theorem 1 is different; we will give an explicit construction of an uncountable chain of prime ideals in $\mathbb{A}$ without assuming the axiom of the existence of prime ideals.

The main inputs in our proof are the functions $s \mapsto v_s(f)$ for elements f in $\mathbb{A}$. Here $\{v_s\}_{s\geq 0}$ is a collection of valuations on $\mathbb{A}$ associated to a family of Gauss norms on $\mathbb{A}$, and those Gauss norms are crucial for the study of the (adic) geometry of $\mathbb{A}$. When R is perfectoid, the geometry of $\mathbb{A}$ is one of the main topics in recent developments in p-adic Hodge theory, and has been studied by Fargues-Fontaine[FF], Kedalaya-Liu[KL], Scholze[Sch], Bhatt-Scholze-Morrow[BMS], etc. The function $s \mapsto v_s(f)$ we use in this paper was studied in the work of Fargues and Fontaine, and they can show for a fixed $f \in \mathbb{A}$, the function $s \mapsto v_s(f)$ is a piecewise-linear concave increasing function with integer slopes. For some f, there could be a subtle complexity of the function $s \mapsto v_s(f)$ when s approaches to 0. We are going to define a chain of prime ideals by resolving part of the subtlety; we are going to group elements f in $\mathbb{A}$ by the rate of convergence of the functions $s \mapsto v_s(f)$ at 0.

We will review some basic facts of the function $s \mapsto v_s(f)$ and its relation with the Newton polygon of f in section 2. In section 3, we will give a construction of a chain of prime ideals in $\mathbb{A}$ using the function $s \mapsto v_s(f)$. Then we will show the prime ideals we defined are properly ordered by the totally ordered set $[0,1]$. To show this, we need to construct elements f in $\mathbb{A}$ and estimate the rate of convergence of $s \mapsto v_s(f)$ at 0 using a lemma we prove in section 2. In the end of this paper, we show all prime ideals we construct are fixed by the Frobenius action on $\text{Spec}(\mathbb{A})$.

Acknowledgements. We thank Jaclyn Lang and Judith Ludwig for their paper on the related topic. The author got the idea of this paper when he prepared a student colloquium talk based on their paper. We also thank the organizer Zachary Letterhos and all audiences in his talk. We thank Pavel Čoupek, Tong Liu, Linquan Ma, Dongming She, and Yifu Wang for their interests and reading an early draft of the paper.

2. Family of valuations on $\mathbb{A}$

Fix a perfect non-discrete valuation ring R in characteristic p, and let v be the valuation map to $\mathbb{R} \cup \{\infty\}$. Fix a discrete valuation ring $\mathcal{O}_E$, let $\mathbb{A} = W_{\mathcal{O}_E}(R)$ be the ring of $\mathcal{O}_E$-Witt vectors over R. For any uniformizer $\pi \in \mathcal{O}_E$, one can show the projection $\mathbb{A} \to R = \mathbb{A}/(\pi)$ admits a unique multiplicative section $[-]$, which is independent of the choice of π. Moreover, use the theory of strict π-rings, one can show that every elements $f \in \mathbb{A}$ has an unique π-expansion:

$$f = \sum_{i \geq 0} [a_i] \pi^i.$$

Remark 1. (1) We will allow $\mathcal{O}_E = \mathbb{F}_p[[T]]$ and in this case $\mathbb{A} = R[[T]]$. Actually, as been mentioned in [LL, Remark 3.3], all methods we are going to use work for $\mathbb{A} = R[[T]]$ for any non-discrete valuation ring R . In fact, we will only use the Frobenius structure on $\mathbb{A}$ in Proposition 5.

(2) We will not assume R to be complete with respect to the valuation, since to establish the properties we need, we can always regard elements in $\mathbb{A}$ as elements in $W_{\mathcal{O}_E}(\widehat{R})$ via $\mathbb{A} \subset W_{\mathcal{O}_E}(\widehat{R})$.

Fix a positive real number s, for any element $f = \sum_{i \geq 0} [a_i] \pi^i \in \mathbb{A}$, define

$$v_s(f) = \inf_i \{v(a_i) + is\}.$$

One can show that for f in $\mathbb{A}$, and $t \geq s > 0$, then $v_t(f) \geq v_s(f) \geq 0$, and $v_s(f) = \infty$ if and only if $f = 0$. Define

$$v_0(f) = \lim_{s \to 0} v_s(f).$$

Proposition 1 ([FF] §1.4)**.** *For $s \geq 0$, we have*

$$v_s(fg) = v_s(f) + v_s(g), \quad v_s(f+g) \geq \min\{v_s(f), v_s(g)\}$$

for all $f, g \in \mathbb{A}$. And we have for $f = \sum_{i \geq 0} [a_i]\pi^i \in \mathbb{A}$,

$$v_0(f) = \inf_i v(a_i).$$

2.1. **Relation with Newton polygons.** Fix $f \in \mathbb{A}$, let $\mathcal{N}(f)$ be the Newton polygon of f. Recall that $\mathcal{N}(f)$ is defined to be the nonnegative convex piecewise-linear decreasing functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R} \cup \{\infty\}$ determined by the boundary of the decreasing convex hull of the set $\{i, v(a_i)\}$. For a convex piecewise-linear decreasing function $\mathcal{F}$ from $\mathbb{R}_{\geq 0}$ to $\mathbb{R} \cup \{\infty\}$, we say x is a node of $\mathcal{F}$ if $\mathcal{F}(x) < \infty$ and $\mathcal{F}$ is not differentiable at x, i.e., either $\lim_{t \to x^-} \mathcal{F}(t) = \infty$ or $\partial_- \mathcal{F}(x) \neq \partial_+ \mathcal{F}(x)$, where $\partial_- \mathcal{F}(x), \partial_+ \mathcal{F}(x)$ are the left and right differentials of $\mathcal{F}$ at x. Note that it is easy to see that if n is a node of $\mathcal{N}(f)$ then $\mathcal{N}(f)(n) = v(x_n)$.

For a convex piecewise-linear function $\mathcal{F}$ from $\mathbb{R}_{\geq 0}$ to $\mathbb{R} \cup \{\infty\}$ that is not identically equal to ∞, we define its Legendre transform $\mathscr{L}(\mathcal{F})$ to be

$$\begin{aligned} \mathscr{L}(\mathcal{F}) : \mathbb{R}_{\geq 0} &\to \mathbb{R} \cup \{-\infty\} \\ t &\mapsto \inf\{\mathcal{F}(x) + tx \mid x \in \mathbb{R}_{\geq 0}\} \end{aligned}$$

It is easy to see that $\mathscr{L}(\mathcal{F})$ is also piecewise-linear. And when $\mathcal{F}$ is nonnegative and decreasing, we have the infimum in the above definition can be taken over the set of nodes of $\mathcal{F}$. In particular, fix a nonzero $f \in \mathbb{A}$, we have

$$\begin{aligned} \mathscr{L}(\mathcal{N}(f))(t) &= \inf\{\mathcal{N}(f)(x) + tx \mid x \in \mathbb{R}_{\geq 0}\} \\ &= \inf\{\mathcal{N}(f)(x) + tx \mid x \in \mathbb{N}\} \\ &= v_t(f). \end{aligned}$$

Moreover, by studying the nodes of $\mathcal{N}(f)$, one can show:

Proposition 2 ([FF]§1.5)**.** *Fix a nonzero f in $\mathbb{A}$, the function $t \mapsto v_t(f)$ is equal to the Legendre transform of $\mathcal{N}(f)$. More explicitly, let $\{n_i\}$ be the set of nodes of $\mathcal{N}(f)$ and let $-s_i$ be the slope of $\mathcal{N}(f)$ on the interval (n_i, n_{i+1}) (with the convention that $s_m = 0$ if there are only finitely many nodes and n_m is the maximal node). Then $\mathscr{L}(\mathcal{N}(f))$ is the unique piecewise-linear function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$ such that*

(1) $\mathscr{L}(\mathcal{N}(f))(0) = v_0(f)$,
(2) $\mathscr{L}(\mathcal{N}(f))$ *has slope n_{i+1} on the interval (s_{i+1}, s_i),*
(3) $\mathscr{L}(\mathcal{N}(f))$ *has slope n_1 on the interval (s_1, ∞).*

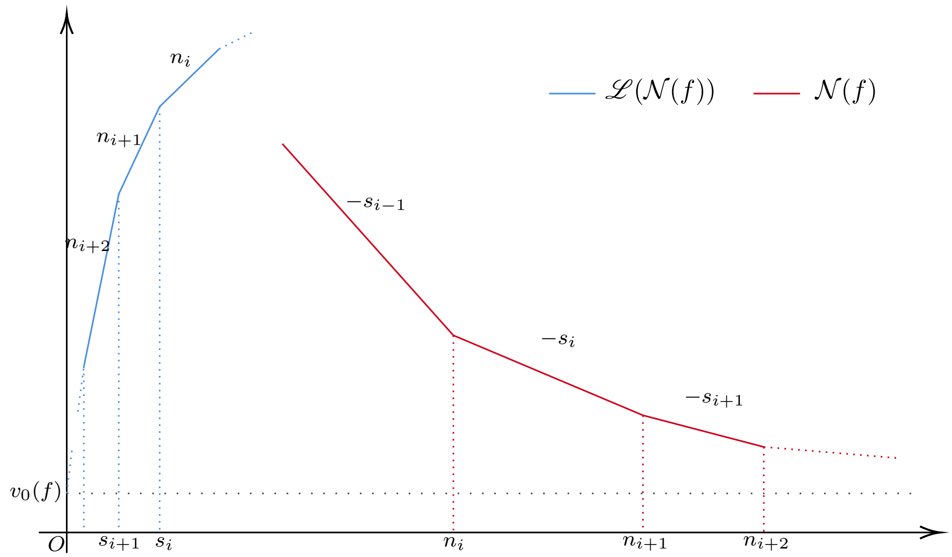

FIGURE 1. $\mathcal{N}(f)$ and its Legendre transform $\mathscr{L}(\mathcal{N}(f))$. $\mathscr{L}(\mathcal{N}(f))$ can be very complex when $s \to 0$.

Corollary 1. *Under the notions in Proposition 2, and further assume that* $\lim_{i\to\infty} s_i n_{i+1} = 0$ *or there are only finitely many nodes, then*

$$\mathcal{N}(f)(n_i) = -s_i n_i + \mathscr{L}(\mathcal{N}(f))(s_i).$$

Proof. From last proposition or Figure 1, we have

$$\begin{aligned}
\mathcal{N}(f)(n_i) &= \sum_{j\geq i} s_j(n_{j+1} - n_j) + v_0(f) \\
&= \sum_{j\geq i}^{m-1} s_j(n_{j+1} - n_j) + v_0(f) \quad \text{if } n_m \text{ is the maximal node,} \\
\mathscr{L}(\mathcal{N}(f))(s_i) &= \sum_{j\geq i} (s_j - s_{j+1}) n_{j+1} + v_0(f) \\
&= \sum_{j\geq i}^{m-1} (s_j - s_{j+1}) n_{j+1} + v_0(f) \quad \text{if } n_m \text{ is the maximal node.}
\end{aligned}$$

From Abel's lemma on summation by parts, we have

$$\sum_{j=i}^{m} s_j(n_{j+1} - n_j) = (s_m n_{m+1} - s_i n_i) - \sum_{j=i}^{m-1} (s_{j+1} - s_j) n_{j+1}.$$

When n_m is the maximal node, the above equation make sense since $s_m = 0$ under our convention, and the left hand side equals to $\mathcal{N}(f)(n_i) - v_0(f)$. When there are infinitely many nodes, let m go to infinity, we get the formula. □

Remark 2. (1) For smooth functions, Legendre transform is related to integration by part, and Corollary 1 can be regarded as a discrete version of that.
(2) There are counterexamples that $\lim_{i\to\infty} s_i n_{i+1} = 0$ is not satisfied.

3. CONSTRUCTION OF A CHAIN OF PRIMES

Define $\mathfrak{p} = \{f \in \mathbb{A} \mid v_0(f) > 0\}$ and $\mathfrak{m} = \{\sum_{i\geq 0}[a_i]\pi^i \mid v(a_i) > 0 \text{ for all } i\}$. It is easy to see $\mathfrak{p} \subset \mathfrak{m}$ and both are prime ideals using Proposition 1.

Lemma 1. *We have*

$$\mathfrak{m} = \{f \mid \limsup_{t\to 0^+} \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t} = \infty\}.$$

Proof. We are going to show their compliments are the same. If $f \notin \mathfrak{m}$, then $\mathcal{N}(f) \equiv 0$ for $t \gg 0$, so $\mathcal{N}(f)$ has only finitely many nodes and slopes. From Proposition 2, we have there is a neighborhood of 0 where $\mathscr{L}(\mathcal{N}(f))$ is linear and $\lim_{t\to 0} \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t}$ converge to the slope.

On the other hand, from Proposition 2, we have $\frac{\mathscr{L}(\mathcal{N}(f))(t)}{t} = n_i + \frac{b_i}{t}$ on the interval (s_i, s_{i-1}), where b_i is the y-intercepts of the linear functions on each interval. Because $\mathscr{L}(\mathcal{N}(f))$ is concave, we have $b_i > 0$. $\limsup_{t\to 0^+} \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t} \geq \limsup_i n_i$. Then if we assume $\limsup_{t\to 0^+} \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t}$ to be finite, then $\limsup_i n_i$ is finite which means there can be only finite many nodes and slopes. Besides, $\mathscr{L}(\mathcal{N}(f))(0) = v_0(f)$ has to be 0. We have in this case there is a node n such that $v(a_n) = v_0(f) = 0$, in particular, f is not in $\mathfrak{m}$. □

Definition 1. For any real number $\lambda \in (0, 1]$, let

$$\mathfrak{p}_\lambda = \{f \mid \limsup_{t\to 0^+} \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t^\lambda} = \infty\},$$

we have $\mathfrak{p}_1 = \mathfrak{m}$ by the previous lemma and we set $\mathfrak{p}_0 = \mathfrak{p}$.

Proposition 3. *All $\mathfrak{p}_\lambda$ defined as above are prime ideals, and if $1 \geq \mu > \lambda \geq 0$, we have $\mathfrak{p}_\lambda \subset \mathfrak{p}_\mu$.*

Proof. Can be deduced easily from the fact $\mathscr{L}(\mathcal{N}(f))(t) = v_t(f)$ and properties in Proposition 1. We only show the compliments of $\mathfrak{p}_\lambda$ are multiplicative closed here and leave the readers to check $\mathfrak{p}_\lambda$ are ideals and $\mathfrak{p}_\lambda \subset \mathfrak{p}_\mu$.
Let f, g are two elements in $\mathbb{A}$ and not in $\mathfrak{p}_\lambda$. Then we have both $\frac{\mathscr{L}(\mathcal{N}(f))(t)}{t^\lambda}$ and $\frac{\mathscr{L}(\mathcal{N}(g))(t)}{t^\lambda}$ have finite supremum limit at 0. Since we have

$$\frac{\mathscr{L}(\mathcal{N}(fg))(t)}{t^\lambda} = \frac{\mathscr{L}(\mathcal{N}(f))(t)}{t^\lambda} + \frac{\mathscr{L}(\mathcal{N}(g))(t)}{t^\lambda}$$

from $\mathscr{L}(\mathcal{N}(f))(t) = v_t(f)$ and Proposition 1. So $\limsup_{t\to 0^+} \frac{\mathscr{L}(\mathcal{N}(fg))(t)}{t^\lambda}$ can not be infinity when t approaches to 0.

□

4. A constructive proof of the main theorem

From the construction in the last section, to show there is an uncountable chain of primes in $\mathbb{A}$ between $\mathfrak{p}$ and $\mathfrak{m}$, it is enough to show for all $\lambda < \mu$ between 0 and 1, we have $\mathfrak{p}_\lambda \subsetneq \mathfrak{p}_\mu$.

For any real number $a > 1$, let $\mathcal{F}_a$ be the piecewise-linear function on $\mathbb{R}_{\geq 0}$, such that $\mathcal{F}_a(i) = \sum_{j \geq i} j^{-a}$ for all $i \in \mathbb{N}$ and has nodes at every positive integer. Since the valuation group of R is non-discrete, we can find a $f_a \in \mathbb{A}$, such that $\mathcal{N}(f_a)(0) = \infty$ and $|\mathcal{N}(f_a)(i) - \mathcal{F}_a(i)| < e^{-i}$ for $i \in \mathbb{N}_{>0}$. Moreover, we can choose f_a so that $\mathcal{N}(f_a)$ has nodes at all positive integers, and let $-s_i$ be the slope of $\mathcal{N}(f_a)$ on the interval $(i, i+1)$. Since s_i converge to 0 as i goes to infinity, we have for any $\lambda \in (0,1)$,

$$\limsup_{t \to 0^+} \frac{\mathscr{L}(\mathcal{N}(f_a))(t)}{t^\lambda} \geq \limsup_{i \to \infty} \frac{\mathscr{L}(\mathcal{N}(f_a))(s_i)}{s_i^\lambda}. \tag{I}$$

On the other hand, for $t \in [s_{i+1}, s_i]$, we have

$$\mathscr{L}(\mathcal{N}(f_a))(t) \leq \mathscr{L}(\mathcal{N}(f_a))(s_i) \quad \text{and} \quad t^\lambda \geq s_{i+1}^\lambda.$$

In particular,

$$\limsup_{t \to 0^+} \frac{\mathscr{L}(\mathcal{N}(f_a))(t)}{t^\lambda} \leq \limsup_{i \to \infty} \frac{\mathscr{L}(\mathcal{N}(f_a))(s_i)}{s_{i+1}^\lambda}. \tag{II}$$

For $a > 1$, we have the estimation:

$$|s_i - i^{-a}| < 2e^{-i}$$

and use a standard estimation of $\sum_{j \geq i} j^{-a}$, we have

$$(a-1)^{-1} i^{1-a} - e^{-i} < \mathcal{N}(f_a)(i) < (a-1)^{-1}(i-1)^{1-a} + e^{-i}.$$

In particular, one can use these to check $\lim_{i \to \infty} s_i(i+1) = 0$ and $v_0(f_a) = 0$. So we can apply Corollary 1 to f_a to get

$$\mathscr{L}(\mathcal{N}(f_a))(s_i) = i s_i + \mathcal{N}(f_a)(i). \tag{III}$$

Proposition 4. *For all $\lambda < \mu$ between 0 and 1, choose $a > 1$ satisfying $\frac{a-1}{a} = \lambda$, then we have $f_a \in \mathfrak{p}_\mu$ but $f_a \notin \mathfrak{p}_\lambda$.*

Proof. From (I)(II)(III) and the above estimations, for any $\nu \in (0,1)$, we have

$$\limsup_{i \to \infty} \frac{a i^{a\nu + 1 - a}}{a-1} \leq \limsup_{t \to 0^+} \frac{\mathscr{L}(\mathcal{N}(f_a))(t)}{t^\nu}$$

and

$$\limsup_{t \to 0^+} \frac{\mathscr{L}(\mathcal{N}(f_a))(t)}{t^\nu} \leq \limsup_{i \to \infty} \frac{i^{1-a} + (a-1)^{-1}(i-1)^{1-a}}{(i+1)^{-a\nu}}.$$

Let $\nu = \lambda$ and μ respectively, and the result follows from a direct computations of the two limits we use to bound $\limsup_{t \to 0^+} \frac{\mathscr{L}(\mathcal{N}(f_a))(t)}{t^\nu}$. ☐

The Frobenius structure. Recall $\mathbb{A}$ has an automorphism φ coming from the Frobenius on R which will induce an automorphism φ^* on $\text{Spec}(\mathbb{A})$.

Proposition 5. *The primes ideals* $\{\mathfrak{p}_\lambda\}_{\lambda\in[0,1]}$ *are fixed by* φ^*.

Proof. We have

$$v_t(\varphi^{-1}(f)) = \inf_i\{\frac{1}{p}v(a_i) + it\} = \frac{1}{p}\inf_i\{v(a_i) + ipt\} = \frac{1}{p}v_{pt}(f).$$

This is the same as

$$\mathscr{L}(\mathcal{N}(\varphi^{-1}(f)))(t) = \frac{1}{p}\mathscr{L}(\mathcal{N}(f))(pt).$$

Then from Definition 1, we have $\varphi^{-1}(\mathfrak{p}_\lambda) = \mathfrak{p}_\lambda$ □

References

[Arn] Jimmy T. Arnold, Krull dimension in power series rings. Trans. Amer. Math. Soc., 177:299304, 1973

[Bha] Bhargav Bhatt. Specializing varieties and their cohomology from characteristic 0 to characteristic p. In Algebraic geometry: Salt Lake City 2015, vol. 97 of Proc. Sympos. Pure Math.(2018), 4388.

[BMS] Bhatt, B., Morrow, M., Scholze, P.: Integral p-adic Hodge theory. Publ. Math. de l'IHS 128 (2018), 219–397.

[FF] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique, Astérisque, 406(2018).

[Hal] J. D. Halpern. The independence of the axiom of choice from the Boolean prime ideal theorem. Ibid., vol. 55 (1964), pp. 57-66

[Ked] Kiran Kedlaya. Some ring-theoretic properties of $\mathbb{A}_{\text{inf}}$. arXiv: 1602.09016v3, 201

[KL] K. S. Kedlaya and R. Liu, Relative p-adic Hodge theory: foundations, Astérisque (2015), no. 371, 239

[KP] G.Kang, M.H.Park, Krull-dimension of the power series ring over a nondiscrete valuation domain is uncountable, Journal of Pure and Applied Algebra, Vol 217, pages 254-258

[LL] J. Lang and J. Ludwig, $\mathbb{A}_{\text{inf}}$ is infinite dimensional, arXiv:1906.03583v1 (2019)

[Sch] Peter Scholze, p-adic geometry, Proceedings of the ICM 2018.

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