

COMPLETED PRISMATIC F -CRYSTALS AND CRYSTALLINE \mathbf{Z}_p -LOCAL SYSTEMS

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ABSTRACT. We introduce the notion of completed F -crystals on the absolute prismatic site of a smooth p -adic formal scheme. We define a functor from the category of completed prismatic F -crystals to that of crystalline étale \mathbf{Z}_p -local systems on the generic fiber of the formal scheme and show that it gives an equivalence of categories. This generalizes the work of Bhatt and Scholze, which treats the case of a complete discrete valuation ring with perfect residue field.

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1. INTRODUCTION

Let p be a prime. In [9], Bhatt and Scholze introduce the notion of prisms and the relative prismatic ringed site $((\mathfrak{X}/(A, I))_{\Delta}, \mathcal{O}_{\Delta})$ for a bounded prism (A, I) and a smooth p -adic formal scheme \mathfrak{X} over A/I . Surprisingly, the cohomology

$R\Gamma((\mathfrak{X}/(A, I))_{\Delta}, \mathcal{O}_{\Delta})$ gives a good integral p -adic cohomology of \mathfrak{X} : it recovers the crystalline cohomology of the special fiber as well as the étale cohomology of the generic fiber. The prismatic formalism also gives a site-theoretic construction of the A_{inf} -cohomology and the Breuil–Kisin cohomology when $(A, I) = (A_{\text{inf}}(\mathcal{O}_{\mathbf{C}_p}), \ker \theta)$ and $(\mathfrak{S}, (E))$, respectively (see [9, Ex. 1.9] for the details).

Another advantage of this site-theoretic approach is that it provides a natural framework of the coefficient theory. In the case of the relative prismatic site, Tian [45] studies the cohomology of prismatic crystals when \mathfrak{X} is proper over A/I .

One can study crystals on the absolute prismatic site as well. Let \mathcal{O}_K be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k , and let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K . In the subsequent paper [7], Bhatt and Scholze study sheaves on the absolute prismatic site \mathfrak{X}_{Δ} . Recall that the site \mathfrak{X}_{Δ} has a sheaf \mathcal{O}_{Δ} of rings equipped with a Frobenius φ and an ideal sheaf \mathcal{I}_{Δ} (see Definition 3.2 for the details). They introduce the category $\text{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ of prismatic F -crystals of vector bundles on $(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$ as well as the category $\text{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1}$ of so-called Laurent F -crystals on \mathfrak{X} .

The main theorem [7, Thm. 1.2] of Bhatt and Scholze states that $\text{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta})$ is equivalent to the category of lattices in crystalline representations of K . They also show that, for general \mathfrak{X} , $\text{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1}$ is equivalent to the category of \mathbf{Z}_p -local systems on the generic fiber of \mathfrak{X} . Part of their work is reproved or generalized by Du–Liu [17], Wu [48], and Min–Wang [34]. For other works on the prismatic site, we refer the reader to a recent survey [4].

The present article studies the relationship between lattices in crystalline representations and suitable F -crystals on the absolute prismatic site in the relative situation. For this, we need to enlarge the category $\text{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ of prismatic F -crystals on \mathfrak{X} . To explain the enlarged category, we first focus on the small affine case. More precisely, let R_0 be the p -adic completion of an integral domain that is étale over $W(k)[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$ for some $d \geq 0$ and set $R := R_0 \otimes_{W(k)} \mathcal{O}_K$. We also fix a uniformizer π of \mathcal{O}_K with monic minimal polynomial $E(u)$ over $W(k)$. We consider the following type of sheaves on the absolute prismatic site R_{Δ} .

Definition 1.1 (Definition 3.16, Remark 3.17). A *completed prismatic F -crystal* on R is a sheaf \mathcal{F} of \mathcal{O}_{Δ} -modules on R_{Δ} together with a morphism $1 \otimes \varphi_{\mathcal{F}}: \varphi^* \mathcal{F} \rightarrow \mathcal{F}$ that satisfies the following properties:

- (1) for each $(A, I) \in R_{\Delta}$, the A -module $\mathcal{F}_A := \mathcal{F}(A, I)$ is finitely generated and classically (p, I) -complete;
- (2) for any morphism $(A, IA) \rightarrow (B, IB)$ of bounded prisms over R , the map $B \widehat{\otimes}_A \mathcal{F}_A \rightarrow \mathcal{F}_B$ is an isomorphism;
- (3) for the Breuil–Kisin prism $(\mathfrak{S} = R_0[[u]], (E(u))) \in R_{\Delta}$ (see Example 3.4), $\mathcal{F}_{\mathfrak{S}}$ is torsion free, $\mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge}$ is finite projective over $\mathfrak{S}[E^{-1}]_p^{\wedge}$, and $\mathcal{F}_{\mathfrak{S}} = \mathcal{F}_{\mathfrak{S}}[p^{-1}] \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge}$;
- (4) the cokernel of $1 \otimes \varphi_{\mathcal{F}_{\mathfrak{S}}}: \varphi^* \mathcal{F}_{\mathfrak{S}} \rightarrow \mathcal{F}_{\mathfrak{S}}$ is killed by E^r for a non-negative integer r .

We write $\text{CR}^{\wedge, \varphi}(R_{\Delta})$ for the category of completed prismatic F -crystals on R .

Condition (3) in the definition is technical but plays a crucial role in our theory. The category $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ contains, as a full subcategory, the category $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_{\Delta})$ of effective prismatic F -crystals of vector bundles on R_{Δ} in the sense of [7, Def. 4.1]. We note that there exists a completed prismatic F -crystal which is not a prismatic F -crystal of vector bundles (see below and Example 3.35), and thus $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ is strictly larger than $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_{\Delta})$ in general.

The main goal of this paper is to describe lattices in crystalline representations of the Galois group \mathcal{G}_R of $R[p^{-1}]$ in terms of completed prismatic F -crystals. Here is our main result:

Theorem 1.2 (Theorem 3.28). *There is a contravariant equivalence of categories*

$$T: \mathrm{CR}^{\wedge, \varphi}(R_{\Delta}) \xrightarrow{\cong} \mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$$

from the category of completed prismatic F -crystals on R to the category of crystalline \mathbf{Z}_p -representations of $R[p^{-1}]$ with non-negative Hodge–Tate weights.

Following [7], we call T the *étale realization functor*. Since T is compatible with Breuil–Kisin and Tate twists, one can further enlarge $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ and obtain an equivalence of categories with the category of *all* crystalline \mathbf{Z}_p -representations of $R[p^{-1}]$ (see Example 3.27 and Remark 3.29).

Once Theorem 1.2 is obtained, we can globalize our equivalence to the one for a smooth p -adic formal scheme over \mathcal{O}_K .

Theorem 1.3 (Theorem 3.45). *Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K . Then there is a natural equivalence of categories*

$$T: \mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\cong} \mathrm{Loc}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathfrak{X}_{\eta})$$

between the category of completed prismatic F -crystals on \mathfrak{X} and the category of crystalline \mathbf{Z}_p -local systems with non-negative Hodge–Tate weights on the adic generic fiber \mathfrak{X}_{η} of \mathfrak{X} (see § 3.6 for the precise definitions).

The main theorems give a prismatic description of crystalline \mathbf{Z}_p -representations in the relative case. Note that when $R = \mathcal{O}_K$, Kisin [26] gave a description of lattices in crystalline representations of K in terms of Breuil–Kisin modules. His work was generalized by Brinon–Trihan [13] to the case of complete discrete valuation rings with *imperfect* residue field with a finite p -basis. Furthermore, Kim [25] introduced the notion of Kisin \mathfrak{S} -modules over R as a generalization of Breuil–Kisin modules of E -height ≤ 1 in the relative case. Kim attached to a p -divisible group over a general R a Kisin \mathfrak{S} -module and showed that the category of p -divisible groups over R is equivalent to the category of Kisin \mathfrak{S} -modules when $p \geq 3$ [25, Cor. 3]. However, it has not yet been known how to describe crystalline \mathbf{Z}_p -representations of $R[p^{-1}]$ with non-negative Hodge–Tate weights in terms of suitable Breuil–Kisin type modules in the relative case. In fact, even a suitable description of rational crystalline representations of $R[p^{-1}]$ has not been given yet in general: while crystalline \mathbf{Q}_p -representations of K can be classified by weakly admissible filtered φ -modules [14], the correct weakly admissibility has not been found in the relative case. We hope that the notion of completed prismatic F -crystals clarifies these complications.

Examples of crystalline \mathbf{Z}_p -representations of $R[p^{-1}]$ with Hodge–Tate weights in $[0, 1]$ arise from p -divisible groups over R . Anschütz and Le Bras [2] developed the prismatic Dieudonné theory. It follows from their work that the category of p -divisible groups over R is equivalent to the category of effective prismatic F -crystals of vector bundles of \mathcal{I}_Δ -height ≤ 1 (see § 3.5 for the details). It is easy to see that their formulation is compatible with ours. Following [47], we will also provide an example of a completed prismatic F -crystal over R that does not arise from a p -divisible group over R (Example 3.35). This implies that our category $\mathrm{CR}^{\wedge, \varphi}(R_\Delta)$ is strictly larger than the subcategory of effective prismatic F -crystals of vector bundles on R and that the former category is necessary to describe crystalline \mathbf{Z}_p -representations in the relative case. It is an interesting question whether a completed prismatic F -crystal on R becomes a prismatic F -crystal of vector bundles on \mathfrak{X} by the pullback along an admissible blow-up $\mathfrak{X} \rightarrow \mathrm{Spf} R$. A related question is whether a crystalline \mathbf{Z}_p -representation of $R[p^{-1}]$ with Hodge–Tate weights in $[0, 1]$ comes from a p -divisible group on \mathfrak{X} for some admissible blow-up $\mathfrak{X} \rightarrow \mathrm{Spf} R$. We note that admissible blow-ups $\mathfrak{X} \rightarrow \mathrm{Spf} R$ usually yield *non-smooth* p -adic formal schemes \mathfrak{X} and thus these questions diverge from our current work.

Let us now explain the construction of the étale realization functor T in Theorem 1.2 (see Proposition 3.26). This will be explained best in the following commutative diagram:

$$\begin{array}{ccc}
 \mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_\Delta) & \longrightarrow & \mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1} \xrightarrow{\cong} \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R) \\
 \downarrow & \nearrow \mathcal{F} \mapsto \mathcal{F}_{\mathrm{ét}} & \nearrow T^\vee \\
 \mathrm{CR}^{\wedge, \varphi}(R_\Delta) & &
 \end{array}$$

Here $\mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1}$ denotes the category of Laurent F -crystals, namely, prismatic F -crystals of vector bundles of $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge$ -modules on R , and $\mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R)$ denotes the category of finite free \mathbf{Z}_p -representations of the Galois group \mathcal{G}_R of $R[p^{-1}]$. The functor $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_\Delta) \rightarrow \mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1}$ is the scalar extension functor. Bhatt–Scholze [7] and Min–Wang [34] show the (covariant) equivalence of categories $\mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1} \cong \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R)$. Hence, to define the contravariant functor $T: \mathrm{CR}^{\wedge, \varphi}(R_\Delta) \rightarrow \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R)$ or its dual T^\vee , it suffices to show that the functor $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_\Delta) \rightarrow \mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1}$ extends to a functor

$$\mathrm{CR}^{\wedge, \varphi}(R_\Delta) \rightarrow \mathrm{Vect}(R_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1}, \quad \mathcal{F} \mapsto \mathcal{F}_{\mathrm{ét}}.$$

The construction of the latter functor uses the following fact on the Breuil–Kisin prism $(\mathfrak{S}, (E(u)))$: it covers the final object of $\mathrm{Shv}(R_\Delta)$, and thus a sheaf on R_Δ is described by a descent datum involving the self-product $(\mathfrak{S}^{(2)}, (E(u)))$ and the self-triple-product $(\mathfrak{S}^{(3)}, (E(u)))$ of the Breuil–Kisin prism. In particular, completed prismatic F -crystals are described by the following data:

Proposition 1.4 (Proposition 3.25). *The association $\mathcal{F} \mapsto \mathcal{F}_\mathfrak{S}$ gives rise to an equivalence of categories $\mathrm{CR}^{\wedge, \varphi}(R_\Delta) \xrightarrow{\cong} \mathrm{DD}_\mathfrak{S}$. Here $\mathrm{DD}_\mathfrak{S}$ consists of triples $(\mathfrak{M}, \varphi_\mathfrak{M}, f)$ where*

- (1) \mathfrak{M} is a finite \mathfrak{S} -module satisfying condition (3) of Definition 1.1 in place of $\mathcal{F}_{\mathfrak{S}}$;
- (2) $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semi-linear endomorphism such that the cokernel of $1 \otimes \varphi_{\mathfrak{M}}: \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by E^r for a non-negative integer r ;
- (3) $f: \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$ is an isomorphism of $\mathfrak{S}^{(2)}$ -modules that is compatible with Frobenii and satisfies the cocycle condition over $\mathfrak{S}^{(3)}$.

Since $\mathrm{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1}$ has a similar description in terms of descent data involving $\mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge}$ and $\mathfrak{S}^{(3)}[E^{-1}]_p^{\wedge}$, the base change along the map $\mathfrak{S}^{(2)} \rightarrow \mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge}$ yields the desired functor $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \mathrm{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})^{\varphi=1}$. To put things together, the contravariant functor T is explicitly given by

$$T(\mathcal{F}) := ((\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R})), (\xi)))^{\varphi_{\mathcal{F}_{\text{ét}}}=1}{}^{\vee}.$$

See Example 3.7 for the definition of the \mathbf{A}_{inf} -prism $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$. Once T is defined, it is not difficult to see that T is fully faithful and that $T(\mathcal{F})[p^{-1}]$ is a crystalline \mathbf{Q}_p -representation of \mathcal{G}_R with non-negative Hodge–Tate weights.

The hardest part of the proof of Theorem 1.2 concerns the essential surjectivity of T , and § 4 is devoted to proving it. For this, take $T_0 \in \mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$. We will attach to T_0 an object $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f) \in \mathrm{DD}_{\mathfrak{S}}$.

To explain the outline of the construction, let us introduce several rings. Let $\mathcal{O}_{\mathcal{E}}$ denote the p -adic completion $\mathfrak{S}[E^{-1}]_p^{\wedge}$ of $\mathfrak{S}[E^{-1}]$. We write \mathcal{O}_{L_0} (resp. \mathcal{O}_L) for the p -adic completion of the localization of R_0 at the prime (p) (resp. the localization of R at the prime (π)): \mathcal{O}_{L_0} is an absolutely unramified complete discrete valuation ring with imperfect residue field having a finite p -basis, and $\mathcal{O}_L = \mathcal{O}_{L_0} \otimes_{W(k)} \mathcal{O}_K$. We set $\mathfrak{S}_L := \mathcal{O}_{L_0}[[u]]$ and $\mathcal{O}_{\mathcal{E}_L} := \mathfrak{S}_L[E^{-1}]_p^{\wedge}$.

On the one hand, the theory of étale φ -modules attaches to T_0 a finite free $\mathcal{O}_{\mathcal{E}}$ -module \mathcal{M} together with a Frobenius and a descent datum. On the other hand, Brinon–Trihan’s theory [13] of Breuil–Kisin modules associates with $T_0|_{\mathrm{Gal}(\overline{L}/L)}$ a finite free \mathfrak{S}_L -module \mathfrak{M}_L with a Frobenius. We set $\mathfrak{M} := \mathcal{M} \cap \mathfrak{M}_L$ inside $\mathcal{O}_{\mathcal{E}_L} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M} = \mathcal{O}_{\mathcal{E}_L} \otimes_{\mathfrak{S}_L} \mathfrak{M}_L$. Naturally, \mathfrak{M} is equipped with a Frobenius $\varphi_{\mathfrak{M}}$. With careful study of the structures, we are able to show that the pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ satisfies conditions (1) and (2) of Proposition 1.4. Finally, the connection on $D_{\mathrm{cris}}(V)$ equips $\mathfrak{M}[p^{-1}]$ with a descent datum. Combined with the descent datum on \mathcal{M} , it yields a descent datum f on \mathfrak{M} and thus an object $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f) \in \mathrm{DD}_{\mathfrak{S}}$. The associated completed prismatic F -crystal \mathcal{F} satisfies $T(\mathcal{F}) \cong T_0$.

Organization of the paper. Section 2 reviews basic concepts in relative p -adic Hodge theory. In § 2.1, we explain the assumptions on our base ring R and objects attached to R , which we will use throughout this article. We review crystalline representations developed by Brinon [12] in § 2.2, and étale φ -modules in § 2.3. The topics in the latter two subsections are standard, and the reader may skip them.

Section 3 introduces the notion of completed prismatic F -crystals and states the main theorems. In § 3.1, we define the absolute prismatic site of a p -adic formal scheme and explain key examples of prisms in the small affine case. In § 3.2, we

define finitely generated completed prismatic crystals and completed prismatic F -crystals in the small affine case. Then we describe the category of completed prismatic F -crystals in terms of descent data in § 3.3. Section 3.4 introduces the étale realization functor $T: \mathrm{CR}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$ and states the main theorem in the small affine case (Theorem 3.28). We also prove part of the main theorem that T is fully faithfully and $T(\mathcal{F})$ is crystalline for $\mathcal{F} \in \mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ in this subsection. In § 3.5, we consider the height one case and compare the étale realization functor with prismatic Dieudonné theory by Anschütz–Le Bras [2]. We also present an example of a completed prismatic F -crystal that is not an effective prismatic F -crystal of vector bundles (Example 3.35). In § 3.6, we define the notion of completed prismatic F -crystals on a general smooth p -adic formal scheme and the étale realization functor. We end the subsection with the main theorem in this general case (Theorem 3.45), which is a direct consequence of Theorem 3.28.

Section 4 is devoted to the proof of the remaining part of the main theorem (the essential surjectivity of Theorem 3.28). In § 4.1, we define the notion of quasi-Kisin modules and show that such an object yields a rational Kisin descent datum. Section 4.2 proves the general fact that for a finite torsion free φ -module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of finite E -height over \mathfrak{S} , $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ (Proposition 4.13). In § 4.3, we consider the CDVR case. With these preparations, we attach a quasi-Kisin module to a lattice in a crystalline representation in § 4.4 and 4.5, and complete the proof of Theorem 3.28 in § 4.6.

Appendix A follows the work of Tan and Tong [43] and defines the notion of crystalline local systems on the generic fiber of a smooth p -adic formal scheme.

Notation and conventions. Let p be a prime and let k be a perfect field of characteristic p . Write $W = W(k)$ and let K be a finite totally ramified extension of $K_0 := W[p^{-1}]$. Fix a uniformizer π of K and let $E(u) \in W[u]$ denote the monic minimal polynomial of π .

For derived completions and relevant concepts, we refer the reader to [9, §1.2]. In this article, most rings are classically p -complete, and we also call them *p -adically complete*. Similarly, a *p -adically completed étale map* from a p -adically complete ring A refers to the (classical) p -adic completion of an étale map from A .

We also follow [9] for the definitions of δ -rings and prisms. However, to avoid confusion, we say that a map of prisms $(A, I) \rightarrow (B, J)$ is *(p, I) -completely (faithfully) flat* if the map $A \rightarrow B$ is (p, I) -completely faithfully flat (compare [9, Def. 3.2]).

We write $W\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ for the p -adic completion of the Laurent polynomial ring $W[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$. In this article, the braces $\{\cdot \cdot \cdot\}$ denote the p -adically completed divided power polynomial, and for a fixed prism (A, I) , the notation $\{\cdot \cdot \cdot\}_{\delta}^{\wedge}$ stands for adjoining elements in the category of derived (p, I) -complete simplicial δ - A -algebras.

For an element a of a \mathbf{Q} -algebra A and $n \geq 0$, write $\gamma_n(a)$ for the element $\frac{a^n}{n!} \in A$.

Our convention is that the cyclotomic character $\mathbf{Z}_p(1) := T_p(\mu_{p^\infty})$ has Hodge–Tate weight one.

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2. REVIEW OF CRYSTALLINE REPRESENTATIONS AND ÉTALE φ -MODULES

2.1. **Base ring.** In this subsection, we will introduce our base ring R .

Definition 2.1. A p -adically complete \mathcal{O}_K -algebra is called *small and smooth* (or *small* for short) if it is p -adically completed étale over $\mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ for some $d \geq 0$.

Remark 2.2. Let R be small over \mathcal{O}_K . Since $R/\pi R$ is étale over $k[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$, there exists a subalgebra $R_0 \subset R$ such that R_0 is p -adically completed étale over $W\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ and $R = R_0 \otimes_W \mathcal{O}_K$.

Let R' be p -adically completed étale over R . If one fixes a subring $R_0 \subset R$ as above, then the étale map $R_0/pR_0 = R/\pi R \rightarrow R'/\pi R'$ lifts uniquely to a p -adically completed étale map $R_0 \rightarrow R'_0$. Moreover, $R'_0 \otimes_W \mathcal{O}_K$ is isomorphic to R' as R -algebras.

In this paper, we use the crystalline period rings developed in [12]. For this, we consider the following class of p -adic rings that contains connected small \mathcal{O}_K -algebras:

Set-up 2.3. A connected p -adically complete \mathcal{O}_K -algebra R is said to be a *base ring* if it is of the form $R := R_0 \otimes_W \mathcal{O}_K$, where R_0 is an integral domain obtained from $W\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ by a finite number of iterations of the following operations:

- p -adic completion of an étale extension;
- p -adic completion of a localization;
- completion with respect to an ideal containing p .

Remark 2.4. Brinon [12, p. 7] further assumes that $W[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \rightarrow R_0$ has geometrically regular fibers and that $k \rightarrow R \otimes_{\mathcal{O}_K} k$ is geometrically integral. By [1, Prop. 5.12] and the fact that any ideal-adic completion of an excellent ring is excellent [28, Main Thm. 2], we see that $W[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \rightarrow R_0$ has geometrically regular fibers for a ring R_0 as in Set-up 2.3. We also note that the latter assumption can be dropped. Indeed, if we let k' be the integral closure of k inside $\text{Frac}(R \otimes_{\mathcal{O}_K} k)$, then $R \otimes_{\mathcal{O}_K} k$ is geometrically connected (and thus geometrically integral) over k' , and R is an $\mathcal{O}_K \otimes_W W(k')$ -algebra. The claim now follows since Brinon's period rings for R are defined without any reference to \mathcal{O}_K . Finally, we note that if R is a base ring, then it satisfies Brinon's good reduction condition (BR) in [12, p. 9].

Let R be a base ring as defined in Set-up 2.3. In the rest of this subsection, we introduce basic objects attached to R that we will use throughout this article.

Let $\varphi: R_0 \rightarrow R_0$ denote the lift of the Frobenius on R_0/pR_0 with $\varphi(T_i) = T_i^p$; this uniquely determines φ . Let $\widehat{\Omega}_{R_0}$ denote the module of continuous Kähler differentials $\varprojlim_n \Omega_{(R_0/p^n R_0)/(W/p^n W)}$. By [12, Prop. 2.0.2], we have $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 \cdot d \log T_i$.

Let \overline{R} denote the union of finite R -subalgebras R' of a fixed algebraic closure of $\text{Frac } R$ such that $R'[p^{-1}]$ is étale over $R[p^{-1}]$. Set

$$\mathcal{G}_R := \text{Gal}(\overline{R}[p^{-1}]/R[p^{-1}]).$$

Let $\text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_R)$ denote the category of finite-dimensional \mathbf{Q}_p -vector spaces with continuous \mathcal{G}_R -action. We call its objects \mathbf{Q}_p -representations of \mathcal{G}_R for short. Similarly,

let $\text{Rep}_{\mathbf{Z}_p}(\mathcal{G}_R)$ denote the category of finite \mathbf{Z}_p -modules equipped with continuous \mathcal{G}_R -action and let $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R)$ denote the full subcategory consisting of finite free objects.

Remark 2.5. Assume that R is of topologically finite type over \mathcal{O}_K (for example, R is small over \mathcal{O}_K). If we equip R with p -adic topology, then $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R)$ (resp. $\text{Rep}_{\mathcal{O}_p}(\mathcal{G}_R)$) is equivalent to the category of \mathbf{Z}_p -local systems (resp. isogeny \mathbf{Z}_p -local systems) on the étale site of the adic space $\text{Spa}(R[p^{-1}], R)$ by [22, Ex. 1.6.6 ii)] and [24, Rem. 1.4.4]. Note also that $\text{Spa}(R[p^{-1}], R)$ is the adic generic fiber of $\text{Spf } R$.

Let \overline{R}^\wedge be the p -adic completion of \overline{R} and let \overline{R}^\flat be its tilt $\varprojlim_{\varphi} \overline{R}/p\overline{R}$. Set $\mathbf{A}_{\text{inf}}(\overline{R}) := W(\overline{R}^\flat)$. The first projection $\overline{R}^\flat \rightarrow \overline{R}/p\overline{R}$ lifts uniquely to a surjective W -algebra homomorphism $\theta: \mathbf{A}_{\text{inf}}(\overline{R}) \rightarrow \overline{R}^\wedge$.

Notation 2.6. Let $\mathfrak{S} = \mathfrak{S}_R := R_0[[u]]$ equipped with the Frobenius given by $\varphi(u) = u^p$. Let \mathcal{O}_ε be the p -adic completion of $\mathfrak{S}[u^{-1}]$, equipped with the Frobenius φ extending that on \mathfrak{S} . Note that E is invertible in \mathcal{O}_ε and the map $\mathfrak{S}[E^{-1}] \rightarrow \mathcal{O}_\varepsilon$ induces an isomorphism $\mathfrak{S}[E^{-1}]_p^\wedge \xrightarrow{\cong} \mathcal{O}_\varepsilon$.

We recall a result about the Frobenius on \mathfrak{S} .

Lemma 2.7. *The map $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ is classically faithfully flat. Moreover, \mathfrak{S} as a module over itself via φ is finite free.*

Proof. By [12, Lem. 7.1.5], $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ is classically flat. Let $\mathfrak{q} \subset \mathfrak{S}$ be any maximal ideal. Since \mathfrak{S} is p -adically complete, we have $p \in \mathfrak{q}$. Thus, $\varphi(\mathfrak{q}) \subset \mathfrak{q}$, which implies $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ is classically faithfully flat.

For the second part, consider \mathfrak{S} as a module over itself via φ . Note that $\mathfrak{S}/(p)$ has a finite p -basis. By Nakayama's lemma, a lift of a p -basis to \mathfrak{S} generates \mathfrak{S} . There cannot be any non-trivial relation among such a lift, since \mathfrak{S} is p -torsion free. \square

Notation 2.8. Note that (p) (resp. (π)) is a prime ideal of R_0 (resp. R). We let \mathcal{O}_{L_0} (resp. \mathcal{O}_L) denote the p -adic completion of the localization $(R_0)_{(p)}$ (resp. $R_{(\pi)}$). Then \mathcal{O}_{L_0} and \mathcal{O}_L are complete discrete valuation rings (CDVR's for short) with the same residue field $\text{Frac}(R_0/(p)) = \text{Frac}(R/(\pi))$. Set $L_0 := \mathcal{O}_{L_0}[p^{-1}]$ and $L := \mathcal{O}_L[p^{-1}]$. The Frobenius φ on R_0 extends to $\varphi: \mathcal{O}_{L_0} \rightarrow \mathcal{O}_{L_0}$. Note that \mathcal{O}_L is also a base ring. When we work on \mathcal{O}_L for a fixed base ring R , we simply write $R = \mathcal{O}_L$ by abuse of notation.

Define $\mathcal{O}_{K_{0,g}}$ to be the p -adic completion of $\varinjlim_{\varphi} \mathcal{O}_{L_0}$ and let $\mathcal{O}_{K_g} := \mathcal{O}_{K_{0,g}} \otimes_W \mathcal{O}_K$. Set $K_{0,g} := \mathcal{O}_{K_{0,g}}[p^{-1}]$ and $K_g := \mathcal{O}_{K_g}[p^{-1}]$. Note that there is a unique φ -compatible isomorphism $\mathcal{O}_{K_{0,g}} \cong W(k_g)$ that reduces to the identity modulo p , where k_g denotes $\varinjlim_{\varphi} \text{Frac}(R_0/(p))$. Hence $\mathcal{O}_{K_{0,g}}$ and \mathcal{O}_{K_g} are CDVR's with the same *perfect* residue field k_g . Note that the structure map $R_0 \rightarrow \mathcal{O}_{K_{0,g}}$ factors through $\mathcal{O}_{L_0} \rightarrow \mathcal{O}_{K_{0,g}} = W(k_g)$.

In § 3, we will consider the absolute prismatic site on a p -adic formal scheme. In the affine case, we usually make the following additional assumption:

Assumption 2.9. The base ring R is small over \mathcal{O}_K or $R = \mathcal{O}_L$. We equip R with p -adic topology. In particular, $\mathrm{Spf} R$ is a smooth p -adic formal scheme over \mathcal{O}_K (or \mathcal{O}_L in the second case). Note that $\mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R)$ is equivalent to the category of étale \mathbf{Z}_p -local systems on the adic generic fiber $\mathrm{Spa}(R[p^{-1}], R)$ of $\mathrm{Spf} R$ (cf. Remark 2.5).

2.2. Crystalline representations. Let R be a base ring. In this subsection, we will review the crystalline period ring $\mathbf{OB}_{\mathrm{cris}}(\overline{R})$ and the notion of crystalline representations of the Galois group \mathcal{G}_R of $R[p^{-1}]$ developed in [12, Chap. 6].

Recall the surjective W -algebra homomorphism $\theta: \mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\overline{R}^\flat) \rightarrow \overline{R}^\wedge$. Define $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ to be the p -adic completion of the divided power envelope of $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to $\mathrm{Ker} \theta$. Choose a non-trivial compatible p -power roots of unity: $\varepsilon_n \in \overline{R}$ with $\varepsilon_0 = 1, \varepsilon_1 \neq 1$, and $\varepsilon_n = \varepsilon_{n+1}^p$. Set $\varepsilon = (\varepsilon_n)_n \in \overline{R}^\flat$ and $t := \log[\varepsilon] \in \mathbf{A}_{\mathrm{cris}}(\overline{R})$. Define $\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathbf{A}_{\mathrm{cris}}(\overline{R})[p^{-1}, t^{-1}]$.

Extend the map θ to $\theta_{R_0}: R_0 \otimes_W \mathbf{A}_{\mathrm{inf}}(\overline{R}) \rightarrow \overline{R}^\wedge$. Define $\mathbf{OA}_{\mathrm{cris}}(\overline{R})$ to be the p -adic completion of the divided power envelope of $R_0 \otimes_W \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to $\mathrm{Ker} \theta_{R_0}$. Define $\mathbf{OB}_{\mathrm{cris}}(\overline{R}) := \mathbf{OA}_{\mathrm{cris}}(\overline{R})[p^{-1}, t^{-1}]$.

Remark 2.10.

- (1) Our period rings $\mathbf{B}_{\mathrm{cris}}(\overline{R})$ and $\mathbf{OB}_{\mathrm{cris}}(\overline{R})$ are written as $\mathbf{B}_{\mathrm{cris}}^\nabla(R)$ and $\mathbf{B}_{\mathrm{cris}}(R)$ respectively in [12].
- (2) When K is absolutely unramified and R is of topologically finite type over $\mathcal{O}_K = W$, Tan and Tong define the crystalline period sheaves $\mathbb{B}_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ on the pro-étale site of $\mathrm{Spa}(R[p^{-1}], R)$ [43, Def. 2.4, 2.9]. In this case, $U := \mathrm{Spa}(\overline{R}^\wedge[p^{-1}], \overline{R}^\wedge)$ is an affinoid perfectoid object of the pro-étale site. We then have $\mathbf{B}_{\mathrm{cris}}(\overline{R}) = \mathbb{B}_{\mathrm{cris}}(U)$ and $\mathbf{OB}_{\mathrm{cris}}(\overline{R}) = \mathcal{O}\mathbb{B}_{\mathrm{cris}}(U)$. See Proposition A.4.
- (3) The ring $\mathcal{O}_L = (R_{(\pi)})^\wedge$ is also a base ring. In this case, $\mathbf{B}_{\mathrm{cris}}(\overline{\mathcal{O}_L})$ and $\mathbf{OB}_{\mathrm{cris}}(\overline{\mathcal{O}_L})$ are studied in [11] and written as $\mathbf{B}_{\mathrm{cris}}^\nabla$ and $\mathbf{B}_{\mathrm{cris}}$, respectively. The notation $\mathbf{B}_{\mathrm{cris}}$ is also used in [13].

The crystalline period ring $\mathbf{OB}_{\mathrm{cris}}(\overline{R})$ has a natural \mathcal{G}_R -action and a Frobenius endomorphism φ extending those on $R_0 \otimes_W \mathbf{A}_{\mathrm{inf}}(\overline{R})$, and there is a natural $\mathbf{B}_{\mathrm{cris}}(\overline{R})$ -linear integrable connection $\nabla: \mathbf{OB}_{\mathrm{cris}}(\overline{R}) \rightarrow \mathbf{OB}_{\mathrm{cris}}(\overline{R}) \otimes_{R_0} \widehat{\Omega}_{R_0}$. Moreover, $R \otimes_{R_0} \mathbf{OB}_{\mathrm{cris}}(\overline{R})$ is equipped with a filtration by $R[p^{-1}]$ -modules, which is compatible with the natural PD-filtration on $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. See [12, Chap. 6] for the detail of these structures.

The following result on the crystalline period ring will be used later:

Lemma 2.11 ([12, Prop. 6.1.5]). *Choose a compatible system $(T_{i,n})$ of p -power roots of T_i in \overline{R} with $T_{i,0} = T_i$, and let $T_i^\flat \in \overline{R}^\flat$ denote the corresponding element. The map $X_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat]$ induces an $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ -linear isomorphism*

$$\mathbf{A}_{\mathrm{cris}}(\overline{R})\{X_1, \dots, X_d\} \cong \mathbf{OA}_{\mathrm{cris}}(\overline{R}),$$

where the former ring denotes the p -adically completed divided power polynomial with variables X_i and coefficients in $\mathbf{A}_{\mathrm{cris}}(\overline{R})$.

Let us recall the definition of crystalline representations.

Definition 2.12. For $V \in \text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_R)$, set

$$D_{\text{cris}}(V) := (\mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_R} \quad \text{and} \quad D_{\text{cris}}^\vee(V) := \text{Hom}_{\mathcal{G}_R}(V, \mathbf{OB}_{\text{cris}}(\overline{R})).$$

Then $D_{\text{cris}}(V)$ is a finite projective $R_0[p^{-1}]$ -module of rank at most $\dim_{\mathbf{Q}_p} V$ equipped with a natural φ and ∇ structure induced from $\mathbf{OB}_{\text{cris}}(\overline{R})$, and $R \otimes_{R_0} D_{\text{cris}}(V)$ has a filtration induced from $R \otimes_{R_0} \mathbf{OB}_{\text{cris}}(\overline{R})$. The natural map

$$\alpha_{\text{cris}}(V): \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{R_0[p^{-1}]} D_{\text{cris}}(V) \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V$$

is injective by [12, Prop. 8.3.5]. We say that V is R_0 -crystalline if $\alpha_{\text{cris}}(V)$ is an isomorphism. By [12, Prop. 8.3.5], this notion depends only on R , not on R_0 . Hence we simply say that V is crystalline from now on.

By [12, Thm. 8.4.2], V is crystalline if and only if V^\vee is crystalline. Note also that $D_{\text{cris}}^\vee(V) = D_{\text{cris}}(V^\vee) = \text{Hom}_{R_0[p^{-1}]}(D_{\text{cris}}(V), R_0[p^{-1}])$. We will mainly use $D_{\text{cris}}^\vee(V)$ in this paper.

A finite free \mathbf{Z}_p -representation $T \in \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R)$ is called *crystalline* if the associated \mathbf{Q}_p -representation $T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is crystalline.

Finally, let us explain the functoriality. Let $R' = R'_0 \otimes_W \mathcal{O}_K$ be another base ring and assume that there exists a φ -equivariant ring homomorphism $g: R_0 \rightarrow R'_0$ that extends to $g: \overline{R} \rightarrow \overline{R}'$. By the change of paths for étale fundamental groups, g induces a continuous group homomorphism $\mathcal{G}_{R'} \rightarrow \mathcal{G}_R$ and thus a natural \otimes -functor $\text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_R) \rightarrow \text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_{R'})$. The map g also induces a ring homomorphism $\mathbf{OB}_{\text{cris}}(\overline{R}) \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}')$, and the latter is compatible with Frobenii and Galois actions.

Lemma 2.13. *With the notation as above, for $V \in \text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_R)$, the natural morphism $R'_0[p^{-1}] \otimes_{R_0[p^{-1}]} \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}') \otimes_{\mathbf{Q}_p} V$ induces a φ -equivariant morphism of $R'_0[p^{-1}]$ -modules*

$$(2.1) \quad R'_0[p^{-1}] \otimes_{R_0[p^{-1}]} D_{\text{cris}}(V) \rightarrow D_{\text{cris}}(V|_{\mathcal{G}_{R'}}).$$

Moreover, if V is crystalline, then $V|_{\mathcal{G}_{R'}}$ is crystalline and the above map is an isomorphism.

Proof. The first assertion is obvious. Now assume that V is crystalline. Consider the composite of $\mathbf{OB}_{\text{cris}}(\overline{R}')$ -linear maps

$$\begin{aligned} \alpha: \mathbf{OB}_{\text{cris}}(\overline{R}') \otimes_{R'_0[p^{-1}]} (R'_0[p^{-1}] \otimes_{R_0[p^{-1}]} D_{\text{cris}}(V)) &\rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}') \otimes_{R'_0[p^{-1}]} D_{\text{cris}}(V|_{\mathcal{G}_{R'}}) \\ &\xrightarrow{\alpha_{\text{cris}}(V|_{\mathcal{G}_{R'}})} \mathbf{OB}_{\text{cris}}(\overline{R}') \otimes_{\mathbf{Q}_p} V. \end{aligned}$$

Observe that α is the base change of $\alpha_{\text{cris}}(V)$ along the map $\mathbf{OB}_{\text{cris}}(\overline{R}) \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}')$. Since V is crystalline, α is an isomorphism. Moreover, the second map $\alpha_{\text{cris}}(V|_{\mathcal{G}_{R'}}$ in α is injective. Hence $\alpha_{\text{cris}}(V|_{\mathcal{G}_{R'}})$ is an isomorphism and thus $V|_{\mathcal{G}_{R'}}$ is crystalline. We also see that the first map in α is an isomorphism. Since the map $R'_0[p^{-1}] \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}')$ is faithfully flat by [12, Thm. 6.3.8], the morphism (2.1) is an isomorphism. \square

2.3. Étale φ -modules. The classical theory of étale φ -modules and Galois representations is generalized to our relative setting in [25]. We briefly review some necessary facts discussed in [25] and [33]. Recall Notation 2.6: $\mathfrak{S} = \mathfrak{S}_R := R_0[[u]]$ equipped with the Frobenius given by $\varphi(u) = u^p$; $\mathcal{O}_\mathcal{E}$ is the p -adic completion of $\mathfrak{S}[u^{-1}]$, equipped with the Frobenius φ extending that on \mathfrak{S} .

Definition 2.14. An étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module is a pair $(\mathcal{M}, \varphi_\mathcal{M})$ where \mathcal{M} is a finitely generated $\mathcal{O}_\mathcal{E}$ -module and $\varphi_\mathcal{M}: \mathcal{M} \rightarrow \mathcal{M}$ is a φ -semi-linear endomorphism such that $1 \otimes \varphi_\mathcal{M}: \varphi^*\mathcal{M} \rightarrow \mathcal{M}$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module is *projective* (resp. *torsion*) if the underlying $\mathcal{O}_\mathcal{E}$ -module \mathcal{M} is projective (resp. p -power torsion).

Let $\text{Mod}_{\mathcal{O}_\mathcal{E}}$ denote the category of étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules whose morphisms are $\mathcal{O}_\mathcal{E}$ -linear maps compatible with Frobenii. Let $\text{Mod}_{\mathcal{O}_\mathcal{E}}^{\text{pr}}$ and $\text{Mod}_{\mathcal{O}_\mathcal{E}}^{\text{tor}}$, respectively, denote the full subcategories of projective and torsion objects. Note that we have a natural notion of tensor products for étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules, and duals are defined for projective and torsion objects.

We use étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules to study certain Galois representations as follows. We refer the reader to [39] for definitions and facts on perfectoid algebras. Recall that π denotes a uniformizer in \mathcal{O}_K . For integers $n \geq 0$, compatibly choose $\pi_n \in \overline{K}$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$, and let K_∞ be the p -adic completion of $\bigcup_{n \geq 0} K(\pi_n)$. Then K_∞ is a perfectoid field, and $(\overline{R}^\wedge[p^{-1}], \overline{R}^\wedge)$ is a perfectoid affinoid K_∞ -algebra. Let K_∞^\flat denote the tilt of K_∞ , and set $\pi^\flat := (\pi_n) \in K_\infty^\flat$.

Let $E_{R_\infty}^+ = \mathfrak{S}/p\mathfrak{S}$, and let $\tilde{E}_{R_\infty}^+$ be the u -adic completion of $\varinjlim_\varphi E_{R_\infty}^+$. Let $E_{R_\infty} = E_{R_\infty}^+[u^{-1}]$ and $\tilde{E}_{R_\infty} = \tilde{E}_{R_\infty}^+[u^{-1}]$. By [39, Prop. 5.9], $(\tilde{E}_{R_\infty}, \tilde{E}_{R_\infty}^+)$ is a perfectoid affinoid K_∞^\flat -algebra, and we have a natural injective map $(\tilde{E}_{R_\infty}, \tilde{E}_{R_\infty}^+) \hookrightarrow (\overline{R}^\flat[(\pi^\flat)^{-1}], \overline{R}^\flat)$ given by $u \mapsto \pi^\flat$.

Consider

$$(2.2) \quad \tilde{R}_\infty := W(\tilde{E}_{R_\infty}^+) \otimes_{W(K_\infty^\flat), \theta} \mathcal{O}_{K_\infty}.$$

By [39, Rem. 5.19], $(\tilde{R}_\infty[p^{-1}], \tilde{R}_\infty)$ is a perfectoid affinoid K_∞ -algebra whose tilt is $(\tilde{E}_{R_\infty}, \tilde{E}_{R_\infty}^+)$. Furthermore, we have a natural injective map $(\tilde{R}_\infty[p^{-1}], \tilde{R}_\infty) \hookrightarrow (\overline{R}^\wedge[p^{-1}], \overline{R}^\wedge)$ whose tilt is $(\tilde{E}_{R_\infty}, \tilde{E}_{R_\infty}^+) \hookrightarrow (\overline{R}^\flat[(\pi^\flat)^{-1}], \overline{R}^\flat)$. If we write $\mathcal{G}_{\tilde{R}_\infty}$ for $\pi_1^{\text{ét}}(\text{Spec } \tilde{R}_\infty[p^{-1}])$, we then have a continuous map of Galois groups $\mathcal{G}_{\tilde{R}_\infty} \rightarrow \mathcal{G}_R$, which is a closed embedding by [19, Prop. 5.4.54]. By [39, Thm. 7.12], we can canonically identify $\overline{R}^\flat[(\pi^\flat)^{-1}]$ with the π^\flat -adic completion of the affine ring of a pro-universal covering of $\text{Spec } \tilde{E}_{R_\infty}$. Let $\mathcal{G}_{\tilde{E}_{R_\infty}}$ be the Galois group corresponding to the pro-universal covering. Then we have a canonical isomorphism $\mathcal{G}_{\tilde{E}_{R_\infty}} \cong \mathcal{G}_{\tilde{R}_\infty}$.

There exists a unique $W(k)$ -linear map $R_0 \rightarrow W(\overline{R}^\flat)$ which maps T_i to $[T_i^\flat]$ and is compatible with Frobenii (see Lemma 2.11 for the definition of $[T_i^\flat]$). This induces a φ -equivariant embedding $\mathfrak{S} \rightarrow W(\overline{R}^\flat)$ given by $u \mapsto [\pi^\flat]$, which further extends to an embedding $\mathcal{O}_\mathcal{E} \rightarrow W(\overline{R}^\flat[(\pi^\flat)^{-1}])$. Let $\mathcal{O}_\mathcal{E}^{\text{ur}}$ be the integral closure of $\mathcal{O}_\mathcal{E}$ in $W(\overline{R}^\flat[(\pi^\flat)^{-1}])$, and let $\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$ be its p -adic completion. We also define

$\widehat{\mathfrak{S}}^{\text{ur}} := \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \cap W(\overline{R}^{\flat}) \subset W(\overline{R}^{\flat}[\pi^{\flat}]^{-1})$. Since $\mathcal{O}_{\mathcal{E}}$ is normal, we have $\text{Aut}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}}^{\text{ur}}) \cong \mathcal{G}_{E_{R_{\infty}}} := \pi_1^{\text{ét}}(\text{Spec } E_{R_{\infty}})$, and by [19, Prop. 5.4.54] and [39, Lem. 7.5], we have $\mathcal{G}_{E_{R_{\infty}}} \cong \mathcal{G}_{\widehat{E}_{R_{\infty}}} \cong \mathcal{G}_{\widehat{R}_{\infty}}$. This induces $\mathcal{G}_{\widehat{R}_{\infty}}$ -action on $\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$. The following is proved in [25].

Lemma 2.15 (cf. [25, Lem. 7.5 and 7.6]). *We have $(\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}})^{\mathcal{G}_{\widehat{R}_{\infty}}} = \mathcal{O}_{\mathcal{E}}$ and the same holds modulo p^n . Furthermore, there exists a unique $\mathcal{G}_{\widehat{R}_{\infty}}$ -equivariant ring endomorphism φ on $\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ lifting the p -th power Frobenius on $\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}/(p)$ and extending φ on $\mathcal{O}_{\mathcal{E}}$. The inclusion $\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \hookrightarrow W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])$ is φ -equivariant where the latter ring is given the Witt vector Frobenius.*

Let $\text{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\widehat{R}_{\infty}})$ denote the category of finite \mathbf{Z}_p -modules equipped with continuous $\mathcal{G}_{\widehat{R}_{\infty}}$ -action, and let $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{\widehat{R}_{\infty}})$ and $\text{Rep}_{\mathbf{Z}_p}^{\text{tor}}(\mathcal{G}_{\widehat{R}_{\infty}})$, respectively, denote the full subcategories of free and torsion objects. For $\mathcal{M} \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}$ and $T \in \text{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\widehat{R}_{\infty}})$, define

$$T(\mathcal{M}) := (\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M})^{\varphi=1} \quad \text{and} \quad \mathcal{M}(T) := (\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \otimes_{\mathbf{Z}_p} T)^{\mathcal{G}_{\widehat{R}_{\infty}}}.$$

For a torsion étale φ -module $\mathcal{M} \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{tor}}$, we define its *length* to be the length of $(\mathcal{O}_{\mathcal{E}})_{(p)} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M}$ as an $(\mathcal{O}_{\mathcal{E}})_{(p)}$ -module. The following equivalence is proved in [25] (see also [33, Prop. 2.5]).

Proposition 2.16 (cf. [25, Prop. 7.7], [33, Prop. 2.5]). *The assignments $T(\cdot)$ and $\mathcal{M}(\cdot)$ are exact equivalences (inverse of each other) of \otimes -categories between $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}$ and $\text{Rep}_{\mathbf{Z}_p}(\mathcal{G}_{\widehat{R}_{\infty}})$. Moreover, $T(\cdot)$ and $\mathcal{M}(\cdot)$ restrict to rank-preserving equivalence of categories between $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{pr}}$ and $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{\widehat{R}_{\infty}})$ and length-preserving equivalence of categories between $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{tor}}$ and $\text{Rep}_{\mathbf{Z}_p}^{\text{tor}}(\mathcal{G}_{\widehat{R}_{\infty}})$. In both cases, $T(\cdot)$ and $\mathcal{M}(\cdot)$ commute with taking duals.*

For $\mathcal{M} \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{pr}}$ and $T \in \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{\widehat{R}_{\infty}})$, we can consider the contravariant functors

$$T^{\vee}(\mathcal{M}) := \text{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(\mathcal{M}, \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}) \quad \text{and} \quad \mathcal{M}^{\vee}(T) := \text{Hom}_{\mathcal{G}_{\widehat{R}_{\infty}}}(T, \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}).$$

By Proposition 2.16, these contravariant functors give equivalences of categories between $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{pr}}$ and $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{\widehat{R}_{\infty}})$, and we have natural isomorphisms

$$T^{\vee}(\mathcal{M}) \cong T(\mathcal{M}^{\vee}) \quad \text{and} \quad \mathcal{M}^{\vee}(T) \cong \mathcal{M}(T^{\vee}).$$

We now explain certain functoriality of above constructions. Let R'_0 be another base ring over $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ as in § 2.1 equipped with Frobenius, and suppose we have a φ -equivariant injective map $R_0 \hookrightarrow R'_0$ of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ -algebras. Consider the induced \mathcal{O}_K -linear extension $R = R_0 \otimes_{W(k)} \mathcal{O}_K \hookrightarrow R' := R'_0 \otimes_{W(k)} \mathcal{O}_K$. By choosing a common geometric point, we have an injective map $\overline{R} \hookrightarrow \overline{R}'$, and this induces an embedding $\tilde{R}_{\infty} \hookrightarrow \tilde{R}'_{\infty}$ by the constructions given in Equation (2.2). Hence, the corresponding map of Galois groups $\mathcal{G}_{R'} \rightarrow \mathcal{G}_R$ restricts to $\mathcal{G}_{\tilde{R}'_{\infty}} \rightarrow \mathcal{G}_{\tilde{R}_{\infty}}$. Let $\mathfrak{S}_{R'} = R'_0[[u]]$ and let $\mathcal{O}_{\mathcal{E}, R'}$ be the p -adic completion of $\mathfrak{S}_{R'}[u^{-1}]$. Let $\mathcal{M}_{R'}(\cdot)$ be the functor for the base ring R' constructed similarly as above. If $T \in \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{\tilde{R}_{\infty}})$, then T can be also considered as a $\mathcal{G}_{\tilde{R}'_{\infty}}$ -representation via the map $\mathcal{G}_{\tilde{R}'_{\infty}} \rightarrow \mathcal{G}_{\tilde{R}_{\infty}}$,

and we have the natural isomorphism $\mathcal{O}_{\mathcal{E}, R'} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M}(T) \cong \mathcal{M}_{R'}(T)$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}, R'})$ -modules by Proposition 2.16. We will use this functoriality for the maps of base rings $R \rightarrow \mathcal{O}_L$ and $R \rightarrow \mathcal{O}_{K_g}$ as in Notation 2.8 in later sections.

3. COMPLETED PRISMATIC F -CRYSTALS AND CRYSTALLINE REPRESENTATIONS

This section introduces the notion of completed prismatic F -crystals on the absolute prismatic site of R and formulates the main theorem. In § 3.1, we recall the definition of absolute prismatic site and consider some important examples of prisms. In § 3.2-3.3, we define completed prismatic F -crystals and study their basic properties in the small affine case. In § 3.4, we study the étale realization and formulate our main theorem. In § 3.5, we consider the special case where crystalline representations have Hodge–Tate weights in $[0, 1]$ and study the relation to p -divisible groups. Finally, we globalize the étale realization functor and the main theorem in § 3.6.

We will frequently use the following well-known lemma on flat modules.

Lemma 3.1. *Let A be a ring, M a flat A -module, and N_1, N_2 submodules of an A -module N . Then as submodules of $M \otimes_A N$, we have*

$$M \otimes_A (N_1 \cap N_2) = (M \otimes_A N_1) \cap (M \otimes_A N_2).$$

3.1. The absolute prismatic site. We first recall the definition of the absolute prismatic site from [9] and [7]. Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K (or a CDVR of mixed characteristic $(0, p)$ such as \mathcal{O}_L in Notation 2.8).

Definition 3.2. ([7, Def. 2.3]) The *absolute prismatic site* \mathfrak{X}_{Δ} of \mathfrak{X} consists of the pairs $((A, I), \mathrm{Spf} A/I \rightarrow \mathfrak{X})$, where (A, I) is a bounded prism and $\mathrm{Spf} A/I \rightarrow \mathfrak{X}$ is a morphism of p -adic formal schemes. For simplicity, we often omit the structure map $\mathrm{Spf} A/I \rightarrow \mathfrak{X}$ and simply write (A, I) for an object of \mathfrak{X}_{Δ} . The morphisms are the opposite of morphisms of bounded prisms over \mathfrak{X} , i.e., the ones compatible with the structure morphisms to \mathfrak{X} . We equip \mathfrak{X}_{Δ} with the topology given by (p, I) -completely faithfully flat maps of prisms $(A, I) \rightarrow (B, J)$ over \mathfrak{X} . If $\mathfrak{X} = \mathrm{Spf} R$ is affine, then we also write R_{Δ} for \mathfrak{X}_{Δ} . Note that the associated topos is replete by [7, Rem. 2.4].

The prismatic site \mathfrak{X}_{Δ} has a sheaf \mathcal{O}_{Δ} of rings defined by $\mathcal{O}_{\Delta}(A, I) = A$ and an ideal sheaf $\mathcal{I}_{\Delta} \subset \mathcal{O}_{\Delta}$ given by $\mathcal{I}_{\Delta}(A, I) = I$ (cf. [9, Cor. 3.12]). A similar argument shows that for each $n \geq 1$, the association $(A, I) \mapsto A/(p, I)^n$ defines a sheaf $\mathcal{O}_{\Delta, n}$ on \mathfrak{X}_{Δ} . Moreover, we have $\mathcal{O}_{\Delta} \xrightarrow{\cong} \varprojlim_n \mathcal{O}_{\Delta, n} \cong \mathrm{Rlim} \mathcal{O}_{\Delta, n}$. Finally, the δ -structure on each $(A, I) \in \mathfrak{X}_{\Delta}$ induces a ring endomorphism $\varphi: \mathcal{O}_{\Delta} \rightarrow \mathcal{O}_{\Delta}$.

Let us explain the functoriality of the prismatic topoi. Let $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of smooth p -adic formal schemes over \mathcal{O}_K . Then f induces a cocontinuous functor

$$\mathfrak{Y}_{\Delta} \rightarrow \mathfrak{X}_{\Delta}, ((A, I), \iota: \mathrm{Spf} A/I \rightarrow \mathfrak{Y}) \mapsto ((A, I), f \circ \iota: \mathrm{Spf} A/I \rightarrow \mathfrak{X}).$$

Hence we have a morphism of topoi

$$f_{\Delta} = (f_{\Delta}^{-1}, f_{\Delta, *}) : \mathrm{Shv}(\mathfrak{Y}_{\Delta}) \rightarrow \mathrm{Shv}(\mathfrak{X}_{\Delta}).$$

Observe that if f is an open immersion, then $\mathrm{Shv}(\mathfrak{Y}_{\Delta})$ is an open subtopos of $\mathrm{Shv}(\mathfrak{X}_{\Delta})$ by f_{Δ} .

Lemma 3.3. *Let $(B, IB) \xleftarrow{b} (A, I) \xrightarrow{c} (C, IC)$ be a diagram of maps of bounded prisms over \mathfrak{X} with b being (p, I) -completely faithfully flat, and let $(B \otimes_A C)_{(p, I)}^\wedge$ denote the classical (p, I) -completion of $B \otimes_A C$. Then the pushout of b along c is represented by the map $(C, IC) \rightarrow ((B \otimes_A C)_{(p, I)}^\wedge, I(B \otimes_A C)_{(p, I)}^\wedge)$ and is (p, IC) -completely faithfully flat.*

Proof. By the proof of [9, Cor. 3.12], the pushout of the diagram is represented by the derived (p, I) -completion $\widehat{B \otimes_A^\mathbb{L} C}$ of $B \otimes_A^\mathbb{L} C$. Moreover, it is discrete and classically (p, I) -complete, and the map from C is (p, IC) -completely faithfully flat. By the proof of [48, Prop. 3.2], we also have $H^0(\widehat{B \otimes_A^\mathbb{L} C}) = (B \otimes_A C)_{(p, I)}^\wedge$, i.e., $\widehat{B \otimes_A^\mathbb{L} C}$ is nothing but the classical (p, I) -completion of $B \otimes_A C$. \square

Let R be small over \mathcal{O}_K or $R = \mathcal{O}_L$ (Assumption 2.9). We now explain several objects of R_Δ that we will use later.

Example 3.4 (The Breuil–Kisin prism and its self-products). Consider the pair $(\mathfrak{S}, (E))$ where $\mathfrak{S} = R_0[[u]]$ and $E = E(u)$ is the Eisenstein polynomial for $\pi \in \mathcal{O}_K$ over W as before. Equip \mathfrak{S} with the δ -structure defined by extending the fixed Frobenius φ on R_0 to \mathfrak{S} via $\varphi(u) = u^p$. Then $(\mathfrak{S}, (E)) \in R_\Delta$, where the structure map $R \rightarrow \mathfrak{S}/(E)$ is given by the natural isomorphism $R \cong \mathfrak{S}/(E)$. We call $(\mathfrak{S}, (E))$ the *Breuil–Kisin prism* attached to π and R_0 .

The self-product of $(\mathfrak{S}, (E))$ exists in R_Δ as follows. Consider the p -adically complete tensor-product $\widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}}$ equipped with the induced \otimes -product Frobenius. We have a projection $d: \widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}} \rightarrow R$ given by the composite of the natural projections $\widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}} \rightarrow \mathfrak{S}$ and $\mathfrak{S} \rightarrow \mathfrak{S}/(E) \cong R$. Let J be the kernel of d , and let

$$\mathfrak{S}^{(2)} := (\widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}}) \left\{ \frac{J}{E} \right\}_\delta^\wedge.$$

Here $\{\cdot\}_\delta^\wedge$ means adjoining elements in the category of derived (p, E) -complete simplicial δ - \mathfrak{S} -algebras, where $\widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}}$ is regarded as an \mathfrak{S} -algebra via $a \mapsto a \otimes 1$. By [8, Cor. 3.14], $(\mathfrak{S}^{(2)}, (E))$ is a (p, E) -completely flat prism over $(\mathfrak{S}, (E))$. Furthermore, $(\mathfrak{S}^{(2)}, (E))$ is bounded by [9, Lem. 3.7 (2)], so $(\mathfrak{S}^{(2)}, (E)) \in R_\Delta$. Let $(B, I) \in R_\Delta$. If we are given maps $f_1, f_2: (\mathfrak{S}, (E)) \rightarrow (B, I)$ such that two maps $R \cong \mathfrak{S}/(E) \rightarrow B/I$ induced by f_1 and f_2 agree, then we have a natural induced map $f_1 \otimes f_2: \widehat{\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}} \rightarrow B$ of δ -rings, and $(f_1 \otimes f_2)(J) \subset I$. Thus, by the universal property of prismatic envelope ([8, Cor. 3.14]), we obtain a map $(\mathfrak{S}^{(2)}, (E)) \rightarrow (B, I)$ in R_Δ uniquely determined by f_1, f_2 . So $(\mathfrak{S}^{(2)}, (E))$ is the self-product of $(\mathfrak{S}, (E))$ in R_Δ . Similarly, the self-triple-product of $(\mathfrak{S}, (E))$ exists in R_Δ . Write p_1, p_2 (resp. q_1, q_2, q_3) for the maps from $(\mathfrak{S}, (E))$ to $(\mathfrak{S}^{(2)}, (E))$ (resp. to $(\mathfrak{S}^{(3)}, (E))$).

A little more explicit description is given in [17, § 4.1] as follows. Recall that R_0 is a $W\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ -algebra. Let $B^{\widehat{\otimes} 2}$ denote the completion of $\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}$ with respect to the ideal $\text{Ker}(\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S} \rightarrow \mathfrak{S})$. We have two natural maps $p_1, p_2: \mathfrak{S} \rightarrow B^{\widehat{\otimes} 2}$ and regard $B^{\widehat{\otimes} 2}$ as an \mathfrak{S} -algebra via p_1 . If we set $s_j := p_2(T_j)$ and $y := p_2(u)$, then

$$\mathfrak{S}^{\widehat{\otimes} 2} := \mathfrak{S}[[y - u, s_1 - T_1, \dots, s_d - T_d]]$$

can be naturally considered as an \mathfrak{S} -subalgebra of $B^{\widehat{\otimes}2}$. Similarly, let $B^{\widehat{\otimes}3}$ denote the completion of $\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S}$ with respect to the ideal $\text{Ker}(\mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S} \otimes_{\mathbf{Z}_p} \mathfrak{S} \rightarrow \mathfrak{S})$. We have maps $q_1, q_2, q_3: \mathfrak{S} \rightarrow \mathfrak{S}^{\widehat{\otimes}3}$. Via q_1 , we can naturally consider

$$\mathfrak{S}^{\widehat{\otimes}3} := \mathfrak{S}[[y - u, w - u, \{s_j - T_j, r_j - T_j\}_{j=1, \dots, d}]]$$

as an \mathfrak{S} -subalgebra of $B^{\widehat{\otimes}3}$, where $s_j := q_2(T_j), r_j := q_3(T_j), y := q_2(u)$ and $w := q_3(u)$. Let

$$\begin{aligned} J^{(2)} &= (E, y - u, \{s_j - T_j\}_{j=1, \dots, d}) \subset \mathfrak{S}^{\widehat{\otimes}2} \quad \text{and} \\ J^{(3)} &= (E, y - u, w - u, \{s_j - T_j, r_j - T_j\}_{j=1, \dots, d}) \subset \mathfrak{S}^{\widehat{\otimes}3}. \end{aligned}$$

For $i = 2, 3$, $\mathfrak{S}^{\widehat{\otimes}i}$ is naturally a (p, E) -completed δ - \mathfrak{S} -algebra and $\mathfrak{S}^{(i)} \cong \mathfrak{S}^{\widehat{\otimes}i} \left\{ \frac{J^{(i)}}{E} \right\}_\delta^\wedge$.

Lemma 3.5. *The maps $p_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ (resp. $q_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(3)}$) for $i = 1, 2$ (resp. $i = 1, 2, 3$) are classically faithfully flat.*

Proof. We only consider $p_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$. The proof for $q_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(3)}$ is similar. Note that $\mathfrak{S}^{(2)}$ is classically (p, E) -complete by [9, Lem. 3.7 (1)], and $p_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is (p, E) -completely flat. In particular, the induced map $\mathfrak{S}/(p, E)^n \rightarrow \mathfrak{S}^{(2)}/(p, E)^n$ is flat for each $n \geq 1$. Since \mathfrak{S} is noetherian, $p_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is classically flat by [42, Tag 0912]. Note that p_i is a section of the diagonal map $\mathfrak{S}^{(2)} \rightarrow \mathfrak{S}$. So if N is any non-zero \mathfrak{S} -module, then $\mathfrak{S}^{(2)} \otimes_{p_i, \mathfrak{S}} N \neq 0$. Thus, p_i is classically faithfully flat. \square

Corollary 3.6. *$\mathfrak{S}^{(2)}$ is p -torsion free and E -torsion free. Furthermore,*

$$\mathfrak{S}^{(2)}[p^{-1}] \cap \mathfrak{S}^{(2)}[E^{-1}] = \mathfrak{S}^{(2)},$$

and $\mathfrak{S}^{(2)}[E^{-1}]$ is p -adically separated.

Proof. Since \mathfrak{S} is torsion free and $p_1: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is classically flat, $\mathfrak{S}^{(2)}$ is p -torsion free and E -torsion free. We deduce by Lemma 3.1 and $\mathfrak{S}[p^{-1}] \cap \mathfrak{S}[E^{-1}] = \mathfrak{S}$ that

$$\mathfrak{S}^{(2)}[p^{-1}] \cap \mathfrak{S}^{(2)}[E^{-1}] = (\mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{S}[p^{-1}]) \cap (\mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{S}[E^{-1}]) = \mathfrak{S}^{(2)}.$$

Since $\mathfrak{S}^{(2)}$ is p -adically complete, this also implies that $\mathfrak{S}^{(2)}[E^{-1}]$ is p -adically separated. \square

Example 3.7 (The \mathbf{A}_{inf} -prism). Let (ξ) be the kernel of $\theta: \mathbf{A}_{\text{inf}}(\overline{R}) \rightarrow \overline{R}^\wedge$. Then $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)) \in R_\Delta$, with the structure map $R \rightarrow \mathbf{A}_{\text{inf}}(\overline{R})/(\xi)$ given by the natural inclusion $R \rightarrow \overline{R}^\wedge$. Note that the map $f_{\pi^\flat, T_i^\flat}: \mathfrak{S} \rightarrow \mathbf{A}_{\text{inf}}(\overline{R})$ given by $u \mapsto [\pi^\flat]$ and $T_i \mapsto [T_i^\flat]$ induces a map of prisms $(\mathfrak{S}, (E)) \rightarrow (\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$ over R . Moreover, each $\sigma \in \mathcal{G}_R$ induces a map of prisms $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)) \rightarrow (\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$ satisfying $\sigma \circ f_{\pi^\flat, T_i^\flat} = f_{\sigma(\pi^\flat), \sigma(T_i^\flat)}$.

Example 3.8 (The $\mathbf{OA}_{\text{cris}}$ -prism and its Frobenius twists). Consider the surjective map $\theta_{R_0}: \mathbf{OA}_{\text{cris}}(\overline{R}) \rightarrow \overline{R}^\wedge$. The map

$$\varphi: \mathbf{OA}_{\text{cris}}(\overline{R})/(p) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})/(p)$$

factors through

$$\mathbf{OA}_{\text{cris}}(\overline{R})/(p) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})/((p) + \ker(\theta_{R_0})) \cong \overline{R}/(p) \xrightarrow{h} \mathbf{OA}_{\text{cris}}(\overline{R})/(p).$$

The pair $(\mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ defines a prism in R_{Δ} , where the structure map $R \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})/(p)$ is given by the composite of $R \rightarrow \overline{R}/(p)$ and h (defined in above factorization). Consider the composite $\mathbf{A}_{\text{inf}}(\overline{R}) \xrightarrow{\varphi} \mathbf{A}_{\text{inf}}(\overline{R}) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})$, where the second map is the natural inclusion. This induces a map of prisms $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)) \xrightarrow{\varphi} (\mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ over R , which is compatible with Frobenii and \mathcal{G}_R -actions.

On the other hand, consider the prism $(R_0, (p))$ in R_{Δ} , where the structure map $R \rightarrow R_0/(p)$ is given by the natural projection $R \rightarrow R/(\pi) \cong R_0/(p)$. For any integer $j \geq 1$, write $(\phi_j \mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ for the prism in R_{Δ} whose underlying δ -pair is $(\mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ and the structure map $R \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})/(p)$ is given the structure map of $(\mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ composed with $\varphi^j: \mathbf{OA}_{\text{cris}}(\overline{R}) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})$. For a sufficiently large j , there exists a map of prisms $(R_0, (p)) \rightarrow (\phi_j \mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ given by Dwork's trick. Indeed, let $e = [K : K_0]$ be the ramification index, and choose an integer l such that $p^l \geq e$. Consider $\varphi^{l+1}: \mathbf{OA}_{\text{cris}}(\overline{R}) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})$. Taking modulo the ideal $(p) \subset \mathbf{OA}_{\text{cris}}(\overline{R})$, this induces a map $\overline{R}/(p) \rightarrow \mathbf{OA}_{\text{cris}}(\overline{R})/(p)$ as above, which further factors through $\overline{R}/(\pi)$. Thus, the ring map $R_0 \xrightarrow{\varphi^{l+1}} \mathbf{OA}_{\text{cris}}(\overline{R})$ induces a map $(R_0, (p)) \rightarrow (\phi_l \mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ of prisms over R .

Example 3.9 (The Breuil prism). Let S denote the p -adically completed PD-envelope of \mathfrak{S} with respect to (E) , equipped with the Frobenius extending φ on \mathfrak{S} . Note that $c := \frac{\varphi(E)}{p}$ is a unit in S . So $(S, (p)) \in R_{\Delta}$ with the structure map given by $R \cong \mathfrak{S}/(E) \xrightarrow{\varphi} S/(p)$, and we have a map of prisms $\varphi: (\mathfrak{S}, (E)) \rightarrow (S, (p))$.

For $i = 2, 3$, let $S^{(i)} := D_{\mathfrak{S}^{\widehat{\otimes} i}}(J^{(i)})^{\wedge}$ be the p -adically completed PD-envelope of $\mathfrak{S}^{\widehat{\otimes} i}$ with respect to the ideal $J^{(i)}$. We set

$$z_0 := \frac{y - u}{E} \quad \text{and} \quad z_j := \frac{s_j - T_j}{E} \quad (j = 1, \dots, d).$$

Let $A_{\text{max}}^{(2)}$ be the p -adic completion of \mathfrak{S} -subalgebra of $(\mathfrak{S}[p^{-1}])[z_0, z_1, \dots, z_d]$ generated by $\frac{E}{p}$ and $\{\gamma_n(z_j)\}_{n \geq 1, j=0, \dots, d}$. By [17, §2.2], we have a ring endomorphism $\varphi: A_{\text{max}}^{(2)} \rightarrow A_{\text{max}}^{(2)}$ extending $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ and satisfying

$$\varphi(z_0) = \frac{y^p - u^p}{\varphi(E)} \quad \text{and} \quad \varphi(z_j) = \frac{s_j^p - T_j^p}{\varphi(E)} \quad (j = 1, \dots, d).$$

Since $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ is injective, $\varphi: A_{\text{max}}^{(2)} \rightarrow A_{\text{max}}^{(2)}$ is injective. By *ibid.*, we have a natural ring map $\mathfrak{S}^{(2)} \rightarrow A_{\text{max}}^{(2)}$ which is injective and compatible with φ .

Let $S_2 := \mathfrak{S}^{\widehat{\otimes} 2}[\gamma_n(E), \gamma_n(y - u), \{\gamma_n(s_j - T_j)\}_{n \geq 1, j=0, \dots, d}] \subset \mathfrak{S}^{\widehat{\otimes} 2}[p^{-1}]$. Note that $S_2 \subset A_{\text{max}}^{(2)}$ since $\gamma_n(y - u) = z_0^n \gamma_n(E) \in A_{\text{max}}^{(2)}$ and similarly for $\gamma_n(s_j - T_j)$. Since E , $y - u$, and $\{s_j - T_j\}_{j=1, \dots, d}$ form a regular sequence in $\mathfrak{S}^{\widehat{\otimes} 2}$, S_2 is the PD-envelope of $\mathfrak{S}^{\widehat{\otimes} 2}$ for $J^{(2)}$ by [9, Cor. 2.38]. Then $S^{(2)}$ is the p -adic completion of S_2 . As a subring

of $(R_0[p^{-1}])[[u, y - u, s_1 - T_1, \dots, s_d - T_d]]$, we have

$$S^{(2)} = \left\{ \sum a_{i_0, \dots, i_{d+1}} \gamma_{i_0}(E) \gamma_{i_1}(y - u) \gamma_{i_2}(s_1 - T_1) \cdots \gamma_{i_{d+1}}(s_d - T_d) \mid \right. \\ \left. a_{i_0, \dots, i_{d+1}} \in \mathfrak{S}^{\widehat{\otimes} 2}, a_{i_0, \dots, i_{d+1}} \rightarrow 0 \text{ (as } i_0 + \cdots + i_{d+1} \rightarrow \infty) \right\}$$

where the sum goes over the multi-index (i_0, \dots, i_{d+1}) of non-negative integers and $a_{i_0, \dots, i_{d+1}} \rightarrow 0$ means in the p -adic topology. Note that $S^{(2)}$ is a δ -ring by [9, Cor. 2.38]. Similarly, $S^{(3)}$ is a δ -ring.

Lemma 3.10. *For $i = 2, 3$, we have an embedding $\mathfrak{S}^{(i)} \xrightarrow{\varphi} S^{(i)}$.*

Proof. We prove the statement for $\varphi: \mathfrak{S}^{(2)} \rightarrow S^{(2)}$, and the proof for $\varphi: \mathfrak{S}^{(3)} \rightarrow S^{(3)}$ is analogous. It suffices to show $\varphi(\delta^n(z_j)) \in S^{(2)}$ for $j = 0, \dots, d$. Since $S^{(2)}$ is a δ -ring, we have

$$\varphi(z_0) = c^{-1} \frac{\varphi(y - u)}{p} = c^{-1} \left(\frac{(y - u)^p}{p} + \delta(y - u) \right) \in S^{(2)},$$

and similarly $\varphi(z_j) \in S^{(2)}$ for $j = 1, \dots, d$. Again since $S^{(2)}$ is a δ -ring, we have $\varphi(\delta^n(z_j)) = \delta^n(\varphi(z_j)) \in S^{(2)}$ for any $n \geq 0$. \square

3.2. Completed prismatic F -crystals in the small affine case. In this subsection, we introduce completed prismatic crystals and completed prismatic F -crystals on the absolute prismatic site.

We first introduce the notion of finitely generated completed prismatic crystals. Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K (or a CDVR of mixed characteristic $(0, p)$).

Definition 3.11. A *finitely generated completed crystal of \mathcal{O}_Δ -modules* on \mathfrak{X}_Δ is a sheaf \mathcal{F} of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ such that

- (1) for each $(A, I) \in \mathfrak{X}_\Delta$, the evaluation $\mathcal{F}_A := \mathcal{F}(A, I)$ of \mathcal{F} on (A, I) is a finitely generated and classically (p, I) -complete A -module;
- (2) for any morphism $(A, I) \rightarrow (B, IB)$ of bounded prisms over \mathfrak{X} , the canonical linearized transition map

$$B \widehat{\otimes}_A \mathcal{F}_A \rightarrow \mathcal{F}_B$$

is an isomorphism, where $B \widehat{\otimes}_A \mathcal{F}_A$ denotes the (p, I) -adically completed tensor product $\varprojlim_n (B \otimes_A \mathcal{F}_A) / (p, I)^n (B \otimes_A \mathcal{F}_A)$.

We also call such a sheaf a *finitely generated completed prismatic crystal* on \mathfrak{X} , or a *completed prismatic crystal* on \mathfrak{X} for short.

Similarly, a *finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules* on \mathfrak{X}_Δ is a sheaf \mathcal{F}_n of $\mathcal{O}_{\Delta, n}$ -modules on \mathfrak{X}_Δ such that

- (1) for each $(A, I) \in \mathfrak{X}_\Delta$, the evaluation $\mathcal{F}_{n, A} := \mathcal{F}_n(A, I)$ of \mathcal{F}_n on (A, I) is a finitely generated $A/(p, I)^n$ -module;
- (2) for any morphism $(A, I) \rightarrow (B, IB)$ of bounded prisms over \mathfrak{X} , the canonical linearized transition map $B \otimes_A \mathcal{F}_{n, A} \rightarrow \mathcal{F}_{n, B}$ is an isomorphism.

Remark 3.12. Let \mathcal{F} be a finitely generated completed prismatic crystal on \mathfrak{X} and let $(A, I) \rightarrow (B, IB)$ be a map of bounded prisms over \mathfrak{X} . Since \mathcal{F}_A is a finitely generated A -module and B is classically (p, IB) -complete, the natural map

$$(3.1) \quad B \otimes_A \mathcal{F}_A \rightarrow B \widehat{\otimes}_A \mathcal{F}_A \xrightarrow{\cong} \mathcal{F}_B$$

is surjective. Since (p, IB) is a finitely generated ideal of B , the map (3.1) induces an isomorphism $B/(p, IB)^n \otimes_A \mathcal{F}_A \xrightarrow{\cong} \mathcal{F}_B/(p, IB)^n \mathcal{F}_B$ by [49, Thm. 1.2 (2)]. Moreover, the map (3.1) is an isomorphism if A and B are both noetherian or if A is noetherian and the map $A \rightarrow B$ is classically flat. The latter case follows from [42, Tag 0912]. It is also an isomorphism if \mathcal{F}_A is a finite projective A -module.

Lemma 3.13. *Let \mathcal{F} be a finitely generated completed crystal of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ . Then for each $n \geq 1$, the association $(A, I) \mapsto \mathcal{F}_A/(p, I)^n \mathcal{F}_A$ represents the quotient sheaf $\mathcal{F}/(p, \mathcal{I}_\Delta)^n \mathcal{F}$ and defines a finitely generated crystal \mathcal{F}_n of $\mathcal{O}_{\Delta, n}$ -modules on \mathfrak{X}_Δ . Moreover, we have isomorphisms of \mathcal{O}_Δ -modules $\mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_{\Delta, n+1}} \mathcal{F}_{n+1} \xrightarrow{\cong} \mathcal{F}_n$ and $\mathcal{F} \cong \varprojlim_n \mathcal{F}_n \cong \text{Rlim } \mathcal{F}_n$.*

Conversely, let $(\mathcal{F}_n)_n$ be an inverse system of sheaves of \mathcal{O}_Δ -modules such that \mathcal{F}_n is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules and such that the projection $\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$ induces an isomorphism $\mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_{\Delta, n+1}} \mathcal{F}_{n+1} \xrightarrow{\cong} \mathcal{F}_n$ for each n . Then $\mathcal{F} := \varprojlim_n \mathcal{F}_n$ is a finitely generated completed crystal of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ , and we have isomorphisms of \mathcal{O}_Δ -modules $\mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_\Delta} \mathcal{F} \cong \mathcal{F}_n$ and $\mathcal{F} \cong \text{Rlim } \mathcal{F}_n$.

Proof. Let \mathcal{F} be a finitely generated completed crystal of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ . Let $(A, I) \rightarrow (B, IB)$ be a (p, I) -completely faithfully flat map of bounded prisms over \mathfrak{X} . Set $B' = (B \otimes_A B)_{(p, I)}^\wedge$. By Lemma 3.3, B' is (p, I) -completely faithfully flat over A and $(B', IB') \in \mathfrak{X}_\Delta$ is the self-fiber product of (B, IB) over (A, I) . Let $p_1, p_2: (B, IB) \rightarrow (B', IB')$ be the two maps of bounded prisms over \mathfrak{X} . Since $B/(p, I)^n B$ is classically faithfully flat over $A/(p, I)^n$ and since $B/(p, I)^n B \otimes_{A/(p, I)^n} B/(p, I)^n B \cong B'/(p, I)^n B'$, we have an exact sequence

$$0 \rightarrow \mathcal{F}_{n, A} \rightarrow B/(p, I)^n B \otimes_{A/(p, I)^n} \mathcal{F}_{n, A} \xrightarrow{p_1 \otimes 1 - p_2 \otimes 1} B'/(p, I)^n B' \otimes_{A/(p, I)^n} \mathcal{F}_{n, A}.$$

On the other hand, since \mathcal{F} is a completed prismatic crystal, we have the isomorphisms $B \widehat{\otimes}_A \mathcal{F}_A \cong \mathcal{F}_B$ and $B' \widehat{\otimes}_A \mathcal{F}_A \cong \mathcal{F}_{B'}$. It follows that the above exact sequence is identified with

$$0 \rightarrow \mathcal{F}_{n, A} \rightarrow \mathcal{F}_{n, B} \xrightarrow{p_1^* - p_2^*} \mathcal{F}_{n, B'}.$$

This implies that \mathcal{F}_n is a sheaf on \mathfrak{X}_Δ , representing the quotient sheaf $\mathcal{F}/(p, \mathcal{I}_\Delta)^n \mathcal{F}$. Since \mathcal{F}_A is a finitely generated classically (p, I) -complete A -module, \mathcal{F}_n is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules. Moreover, we have isomorphisms of \mathcal{O}_Δ -modules $\mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_{\Delta, n+1}} \mathcal{F}_{n+1} \xrightarrow{\cong} \mathcal{F}_n$ and $\mathcal{F} \xrightarrow{\cong} \varprojlim_n \mathcal{F}_n$. Finally, since the absolute prismatic topos is replete and $\mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$ is surjective for every n , we obtain $\varprojlim_n \mathcal{F}_n \cong \text{Rlim } \mathcal{F}_n$ by [6, Prop. 3.1.10].

Conversely, let $(\mathcal{F}_n)_n$ be an inverse system of sheaves of \mathcal{O}_Δ -modules satisfying the properties as in the lemma. An argument similar to the previous paragraph

shows that the association $(A, I) \mapsto A/(p, I)^n \otimes_{A/(p, I)^{n+1}} \mathcal{F}_{n+1, A}$ represents the sheaf $\mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_{\Delta, n+1}} \mathcal{F}_{n+1}$. Set $\mathcal{F} := \varprojlim_n \mathcal{F}_n$ and take any $(A, I) \in \mathfrak{X}_{\Delta}$. Then we have $\mathcal{F}(A, I) = (\varprojlim_n \mathcal{F}_n)(A, I) = \varprojlim_n \mathcal{F}_{n, A}$ and $A/(p, I)^n \otimes_{A/(p, I)^{n+1}} \mathcal{F}_{n+1, A} \cong \mathcal{F}_{n, A}$. It follows from [49, Thm 2.8] that $\mathcal{F}(A, I)$ is a finitely generated and classically (p, I) -complete A -module with $\mathcal{F}(A, I)/(p, I)^n \mathcal{F}(A, I) \cong \mathcal{F}_{A, n}$. Moreover, for a morphism $(A, I) \rightarrow (B, IB)$ of bounded prisms over \mathfrak{X} , we have $B \otimes_A \mathcal{F}_{n, A} \cong \mathcal{F}_{n, B}$. It follows that the canonical map $B \widehat{\otimes}_A \mathcal{F}(A, I) \rightarrow \mathcal{F}(B, IB)$ is an isomorphism. Hence \mathcal{F} is a finitely generated completed crystal of \mathcal{O}_{Δ} -modules. Now the remaining assertions follow easily. \square

To define completed prismatic F -crystals in the affine case, let us first introduce the following terminologies.

Definition 3.14. Let R be a base ring and keep the notation as in § 2.1.

- (1) We say that a finite \mathfrak{S} -module N is *projective away from* (p, E) if N is torsion free, $N[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$, and $N[E^{-1}]_p^\wedge$ is projective over $\mathfrak{S}[E^{-1}]_p^\wedge = \mathcal{O}_{\mathcal{E}}$.
- (2) We say that a finite \mathfrak{S} -module N is *saturated* if N is torsion free and

$$N = N[p^{-1}] \cap N[E^{-1}].$$

- (3) Let r be a non-negative integer and let N be an \mathfrak{S} -module equipped with a φ -semi-linear endomorphism $\varphi_N: N \rightarrow N$. We say that the pair (N, φ_N) has *E -height $\leq r$* if

$$1 \otimes \varphi_N: \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} N \rightarrow N$$

is injective and its cokernel is killed by $E(u)^r$. We say that (N, φ_N) has finite *E -height* if it has E -height $\leq r$ for some r .

We will also use these terminologies for a finite module over an \mathfrak{S} -algebra.

Remark 3.15.

- (1) Let N be a torsion free finite \mathfrak{S} -module. Then N is saturated if and only if the natural map

$$N/pN \rightarrow N[E^{-1}]/pN[E^{-1}]$$

is injective. Since $N[E^{-1}]/pN[E^{-1}] \cong N[E^{-1}]_p^\wedge/pN[E^{-1}]_p^\wedge \cong N[u^{-1}]/pN[u^{-1}]$, we deduce that N is saturated if and only if $N = N[p^{-1}] \cap N[E^{-1}]_p^\wedge$, or equivalently, $N = N[p^{-1}] \cap N[u^{-1}]$.

- (2) Assume that either R is small over \mathcal{O}_K or $R = \mathcal{O}_L$. We will show that if (N, φ_N) is a torsion free finite \mathfrak{S} -module with Frobenius of finite E -height, then $N[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ (Proposition 4.13).

We now introduce the notion of completed prismatic F -crystals on R , which will be our main object of study. We make Assumption 2.9: R is small over \mathcal{O}_K or $R = \mathcal{O}_L$.

Definition 3.16. A *completed F -crystal of \mathcal{O}_{Δ} -modules* on R_{Δ} is a pair $(\mathcal{F}, \varphi_{\mathcal{F}})$, where \mathcal{F} is a finitely generated completed crystal of \mathcal{O}_{Δ} -modules on R_{Δ} and

$$1 \otimes \varphi_{\mathcal{F}}: \varphi^* \mathcal{F} \rightarrow \mathcal{F}$$

is a morphism of \mathcal{O}_{Δ} -modules such that

- (1) $\mathcal{F}_{\mathfrak{S}} := \mathcal{F}(\mathfrak{S}, E)$ is projective away from (p, E) and saturated;
- (2) the pair $(\mathcal{F}_{\mathfrak{S}}, \varphi_{\mathcal{F}_{\mathfrak{S}}})$ has finite E -height.

We also call such an object a *completed prismatic F -crystal* on R . The morphisms between completed F -crystals of \mathcal{O}_{Δ} -modules are \mathcal{O}_{Δ} -module maps compatible with Frobenii.

We write $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ for the category of completed F -crystals of \mathcal{O}_{Δ} -modules on R_{Δ} . Let $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_{\Delta})$ denote the full subcategory of $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ consisting of objects $(\mathcal{F}, \varphi_{\mathcal{F}})$ where \mathcal{F} is a locally free \mathcal{O}_{Δ} -module. For a fixed non-negative integer r , we let $\mathrm{CR}_{[0, r]}^{\wedge, \varphi}(R_{\Delta})$ and $\mathrm{Vect}_{[0, r]}^{\varphi}(R_{\Delta})$ denote the full subcategories consisting of objects for which $(\mathcal{F}_{\mathfrak{S}}, \varphi_{\mathcal{F}_{\mathfrak{S}}})$ has E -height $\leq r$.

Remark 3.17. When R is small over \mathcal{O}_K , the above definition agrees with Definition 1.1 by Remark 3.15(2)/Proposition 4.13. In § 3.6, we will define completed prismatic F -crystals on a smooth p -adic formal scheme by gluing.

Remark 3.18. When $R = \mathcal{O}_L := R_{(\pi)}^{\wedge}$ (i.e., a CDVR with residue field having a finite p -basis and a uniformizer finite over $W(k)$), any finite \mathfrak{S}_L -module which is projective away from (p, E) and saturated is free over \mathfrak{S}_L , since \mathfrak{S}_L is a regular local ring of dimension 2. Thus, by Proposition 3.25 below, the category $\mathrm{CR}^{\wedge, \varphi}((\mathcal{O}_L)_{\Delta})$ is equal to the category $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}((\mathcal{O}_L)_{\Delta})$. Furthermore, when $R = \mathcal{O}_K$ (i.e., a CDVR with perfect residue field), our category $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}((\mathcal{O}_K)_{\Delta})$ coincides with the full subcategory of $\mathrm{Vect}^{\varphi}(\mathrm{Spf}(\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ defined in [7, Def. 4.1] consisting of *effective* prismatic F -crystals of vector bundles.

Let us explain that the definition of completed prismatic F -crystals is independent of the choice of a Breuil–Kisin prism, namely, a uniformizer $\pi \in \mathcal{O}_K$ and a W -subalgebra $R_0 \subset R$ (Corollary 3.21). For this, we need the following two lemmas.

Lemma 3.19. *Let \mathcal{F} be a finitely generated completed prismatic crystal on R equipped with a morphism $1 \otimes \varphi_{\mathcal{F}}: \varphi^* \mathcal{F} \rightarrow \mathcal{F}$ of \mathcal{O}_{Δ} -modules. Fix a uniformizer $\pi \in \mathcal{O}_K$ with minimal polynomial $E(u)$ and associated Breuil–Kisin prism $(\mathfrak{S}, (E))$, and let $(\mathfrak{S}, (E)) \rightarrow (B, EB)$ be a classically flat map of bounded prisms over R . Then the following properties hold:*

- (1) if $\mathcal{F}_{\mathfrak{S}}$ is projective away from (p, E) and saturated as an \mathfrak{S} -module, then \mathcal{F}_B is projective away from (p, E) and saturated as a B -module;
- (2) for a non-negative integer r , if the pair $(\mathcal{F}_{\mathfrak{S}}, \varphi_{\mathcal{F}_{\mathfrak{S}}})$ has E -height $\leq r$, then $(\mathcal{F}_B, \varphi_{\mathcal{F}_B})$ has E -height $\leq r$.

Moreover, the converse also holds if $\mathfrak{S} \rightarrow B$ is classically faithfully flat.

Proof. Note $B \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \cong \mathcal{F}_B$ by Remark 3.12.

(1) Suppose $\mathcal{F}_{\mathfrak{S}}$ is projective away from (p, E) and saturated as an \mathfrak{S} -module. Then \mathcal{F}_B is p -torsion free, and $\mathcal{F}_B[p^{-1}]$ is projective over $B[p^{-1}]$. It follows that $\mathcal{F}_B \subset \mathcal{F}_B[p^{-1}]$ is torsion free. Since \mathfrak{S} is noetherian and $\mathcal{F}_{\mathfrak{S}}$ is finitely generated, we see that the induced map $\mathfrak{S}[E^{-1}]_p^{\wedge} \rightarrow B[E^{-1}]_p^{\wedge}$ is classically flat and $B[E^{-1}]_p^{\wedge} \otimes_{\mathfrak{S}[E^{-1}]_p^{\wedge}} \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge} \cong \mathcal{F}_B[E^{-1}]_p^{\wedge}$ by [42, Tag 0912]. We deduce that $\mathcal{F}_B[E^{-1}]_p^{\wedge}$ is projective over $B[E^{-1}]_p^{\wedge}$. Thus \mathcal{F}_B is projective away from (p, E) . Since $\mathcal{F}_{\mathfrak{S}}$ is saturated, Lemma 3.1

implies that

$$\mathcal{F}_B = B \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} = B \otimes_{\mathfrak{S}} (\mathcal{F}_{\mathfrak{S}}[p^{-1}] \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]) = \mathcal{F}_B[p^{-1}] \cap \mathcal{F}_B[E^{-1}].$$

This means that \mathcal{F}_B is saturated.

(2) The assertion follows from $\text{Coker}(1 \otimes \varphi_{\mathcal{F}_{\mathfrak{S}}}) \otimes_{\mathfrak{S}} B = \text{Coker}(1 \otimes \varphi_{\mathcal{F}_B})$.

Finally, if the map $\mathfrak{S} \rightarrow B$ is classically faithfully flat, then so is $\mathfrak{S}[E^{-1}]_p^\wedge \rightarrow B[E^{-1}]_p^\wedge$. Hence the converse direction follows similarly. \square

Suppose R is small over \mathcal{O}_K . Let $\pi' \in \mathcal{O}_K$ be another of uniformizer of \mathcal{O}_K , $E'(y) \in W[y]$ the Eisenstein polynomial for π' , and R'_0 a $W\langle\langle(T'_1)^{\pm 1}, \dots, (T'_d)^{\pm 1}\rangle\rangle$ -algebra with $R'_0 \otimes_W \mathcal{O}_K = R$ as in Remark 2.2. Set $\mathfrak{S}' := R'_0[[y]]$ equipped with Frobenius given by $\varphi(T'_i) = (T'_i)^p$ and $\varphi(y) = y^p$. Then we have a Breuil–Kisin prism $(\mathfrak{S}', (E')) \in R_{\Delta}$ with the structure map $R \xrightarrow{\cong} \mathfrak{S}'/(E')$.

Lemma 3.20.

- (1) *The absolute product of $(\mathfrak{S}, (E))$ and $(\mathfrak{S}', (E'))$ exists in R_{Δ} . Write $(\mathfrak{S}_{\pi, \pi'}^{(2)}, I)$ for the absolute product. We also have $I = E\mathfrak{S}_{\pi, \pi'}^{(2)} = E'\mathfrak{S}_{\pi, \pi'}^{(2)}$.*
- (2) *The maps $\mathfrak{S} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ and $\mathfrak{S}' \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ are classically faithfully flat.*

Proof. (1) Consider the p -adically complete tensor-product $\widehat{\mathfrak{S}}_{\mathbf{Z}_p} \widehat{\mathfrak{S}}'$, and let

$$d: \widehat{\mathfrak{S}}_{\mathbf{Z}_p} \widehat{\mathfrak{S}}' \rightarrow R$$

be the composite of the natural projections $\widehat{\mathfrak{S}}_{\mathbf{Z}_p} \widehat{\mathfrak{S}}' \rightarrow \mathfrak{S}/(E) \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathfrak{S}}'/(E') \cong R \widehat{\otimes}_{\mathbf{Z}_p} R$ and $R \widehat{\otimes}_{\mathbf{Z}_p} R \rightarrow R$. Let J be the kernel of d . We claim that the absolute product of $(\mathfrak{S}, (E))$ and $(\mathfrak{S}', (E'))$ in R_{Δ} is given by

$$\mathfrak{S}_{\pi, \pi'}^{(2)} = \widehat{\mathfrak{S}}_{\mathbf{Z}_p} \widehat{\mathfrak{S}}' \left\{ \frac{J}{E} \right\}_{\delta}^{\wedge},$$

where $\{\cdot\}_{\delta}^{\wedge}$ means adjoining elements in the category of derived (p, E) -complete simplicial δ - \mathfrak{S} -algebras. Indeed, by [8, Cor. 3.14], $(\mathfrak{S}_{\pi, \pi'}^{(2)}, (E))$ is a (p, E) -completely flat prism over $(\mathfrak{S}, (E))$. We have a natural map of prisms $(\mathfrak{S}', (E')) \rightarrow (\mathfrak{S}_{\pi, \pi'}^{(2)}, (E))$, and thus $(E')\mathfrak{S}_{\pi, \pi'}^{(2)} = (E)\mathfrak{S}_{\pi, \pi'}^{(2)}$ by [9, Lem. 3.5]. So the construction is symmetric, and $(\mathfrak{S}_{\pi, \pi'}^{(2)}, (E))$ is also a (p, E') -completely flat prism over $(\mathfrak{S}', (E'))$. By [9, Lem. 3.7 (2)], $(\mathfrak{S}_{\pi, \pi'}^{(2)}, (E))$ is bounded. The universal property can be checked similarly as in Example 3.4.

(2) The classical flatness follows by a similar argument as in the proof of Lemma 3.5: note that $\mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically (p, E) -complete by [9, Lem. 3.7 (1)]. Since $\mathfrak{S} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is (p, E) -completely flat and \mathfrak{S} is noetherian, $\mathfrak{S} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically flat by [42, Tag 0912].

Consider the composite $R_0 \rightarrow \mathfrak{S} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$, where the first map is given by the natural inclusion $R_0 \hookrightarrow R_0[[u]] = \mathfrak{S}$ (which is classically faithfully flat). Since $\mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically $(p, E(u))$ -complete, it is u -complete and u lies in the radical of $\mathfrak{S}_{\pi, \pi'}^{(2)}$. Thus,

to prove $\mathfrak{S} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically faithfully flat, it suffices to show that $R_0 \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically faithfully flat. Let $\mathfrak{P} \subset R$ be a maximal ideal, and let $\mathfrak{m} = R_0 \cap \mathfrak{P}$ and $\mathfrak{m}' = R'_0 \cap \mathfrak{P}$ be the corresponding maximal ideals of R_0 and R'_0 respectively. Let $(R_0)_{\mathfrak{m}}^{\wedge}$ denote the \mathfrak{m} -adic completion of the localization $(R_0)_{\mathfrak{m}}$. It is shown in the proof of Proposition 4.13 below that $(R_0)_{\mathfrak{m}}^{\wedge}$ is equipped with the Frobenius induced from R_0 , and that $(R_0)_{\mathfrak{m}}^{\wedge} \cong W(k_1)[[t_1, \dots, t_d]]$ where $k_1 := R/\mathfrak{P}$ is a finite extension of k . Similarly, we have $(R'_0)_{\mathfrak{m}'}^{\wedge} \cong W(k_1)[[t'_1, \dots, t'_d]]$.

Let $A_{\mathfrak{P}}^{(2)}$ be the absolute product of $((R_0)_{\mathfrak{m}}^{\wedge}[u], (E))$ and $((R'_0)_{\mathfrak{m}'}^{\wedge}[y], (E'))$ constructed as in (1) with R_0 (resp. R'_0) replaced by $(R_0)_{\mathfrak{m}}^{\wedge}$ (resp. $(R'_0)_{\mathfrak{m}'}^{\wedge}$). Note that the map

$$(3.2) \quad f_{\mathfrak{m}}: W(k_1)[[t_1, \dots, t_d]] \cong (R_0)_{\mathfrak{m}}^{\wedge} \rightarrow A_{\mathfrak{P}}^{(2)}$$

is classically flat similarly as above. Consider the induced map

$$W(k_1)[[t_1, \dots, t_d]]/(t_1, \dots, t_d) \cong W(k_1) \rightarrow A_{\mathfrak{P}}^{(2)}/(t_1, \dots, t_d)A_{\mathfrak{P}}^{(2)}.$$

From the explicit construction of the absolute product $A_{\mathfrak{P}}^{(2)}$ in (1), we deduce that $1 \notin (t_1, \dots, t_d)A_{\mathfrak{P}}^{(2)}$, and so $A_{\mathfrak{P}}^{(2)}/(t_1, \dots, t_d)A_{\mathfrak{P}}^{(2)}$ is not the zero ring. Furthermore, since $A_{\mathfrak{P}}^{(2)}$ is classically p -complete, p lies in the radical of $A_{\mathfrak{P}}^{(2)}/(t_1, \dots, t_d)A_{\mathfrak{P}}^{(2)}$. Thus, $A_{\mathfrak{P}}^{(2)}$ has a maximal ideal which lies over the maximal ideal (p, t_1, \dots, t_d) of $(R_0)_{\mathfrak{m}}^{\wedge}$, and the map $f_{\mathfrak{m}}$ in (3.2) is classically faithfully flat.

Now, consider the map $(R_0)_{\mathfrak{m}}^{\wedge} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)} \otimes_{R_0} (R_0)_{\mathfrak{m}}^{\wedge}$ induced from $R_0 \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$. We claim that $(R_0)_{\mathfrak{m}}^{\wedge} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)} \otimes_{R_0} (R_0)_{\mathfrak{m}}^{\wedge}$ is classically faithfully flat. The classical flatness is clear. Note that by [8, Cor. 3.14], the construction of the absolute product in (1) commutes with (p, E) -completely flat base change. Thus, the map $f_{\mathfrak{m}}: (R_0)_{\mathfrak{m}}^{\wedge} \rightarrow A_{\mathfrak{P}}^{(2)}$ in (3.2) naturally factors through $(R_0)_{\mathfrak{m}}^{\wedge} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)} \otimes_{R_0} (R_0)_{\mathfrak{m}}^{\wedge}$. Since $f_{\mathfrak{m}}$ is classically faithfully flat, so is the flat map $(R_0)_{\mathfrak{m}}^{\wedge} \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)} \otimes_{R_0} (R_0)_{\mathfrak{m}}^{\wedge}$. Now since the claim holds for any maximal ideal $\mathfrak{m} \subset R_0$, $R_0 \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is classically faithfully flat.

By symmetry, $\mathfrak{S}' \rightarrow \mathfrak{S}_{\pi, \pi'}^{(2)}$ is also classically faithfully flat. \square

Corollary 3.21. *Definition 3.16 of completed prismatic F -crystals is independent of the choice of a uniformizer $\pi \in \mathcal{O}_K$ and a W -subalgebra R_0 of R .*

Proof. This follows from Lemmas 3.19 and 3.20. \square

Remark 3.22 (Restriction of completed prismatic F -crystals). Suppose R is small over \mathcal{O}_K , and let $R \rightarrow R'$ be a p -adically completed étale map. Let $R'_0 \subset R'$ such that $R'_0 \otimes_W \mathcal{O}_K \cong R'$ with a p -adically completed étale map $R_0 \rightarrow R'_0$ as in Remark 2.2. Note that the Frobenius on R_0 extends uniquely to a Frobenius on R'_0 . For $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$, consider its restriction $\mathcal{F}|_{(R')_{\Delta}}$ to $(R')_{\Delta}$. Since $\mathfrak{S} = R_0[[u]] \rightarrow R'_0[[u]]$ is classically flat, we deduce from Remark 3.12 and Lemma 3.19 that $\mathcal{F}|_{(R')_{\Delta}}(R'_0[[u]], (E)) = R'_0[[u]] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$ and $\mathcal{F}|_{R'} \in \text{CR}^{\wedge, \varphi}((R')_{\Delta})$. We similarly have the restriction of completed prismatic F -crystals for the maps $\bar{R} \rightarrow \mathcal{O}_L$ and $R \rightarrow \mathcal{O}_{K_g}$ as in Notation 2.8.

We now study some properties of completed prismatic F -crystals on R . Let $\mathfrak{S}_L := \mathcal{O}_{L_0}[[u]]$ equipped with Frobenius given by $\varphi(u) = u^p$. Note that $(\mathfrak{S}_L, (E)) \in R_{\Delta}$ with $R \rightarrow \mathfrak{S}_L/(E) = \mathcal{O}_L = R_{(\pi)}^{\wedge}$ and that the natural map $\mathfrak{S} \rightarrow \mathfrak{S}_L$ induces a map of prisms $(\mathfrak{S}, (E)) \rightarrow (\mathfrak{S}_L, (E))$ over R . Let $\mathcal{O}_{\mathcal{E}, L}$ denote the p -adic completion of $\mathfrak{S}_L[u^{-1}]$.

Lemma 3.23. *Let $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$. Then the following properties hold.*

- (1) *We have $\mathcal{F}_{\mathfrak{S}_L} \cong \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$. Furthermore, $\mathcal{F}_{\mathfrak{S}_L}$ is finite free over \mathfrak{S}_L .*
- (2) *We have $\mathcal{F}_{\mathfrak{S}} = \mathcal{F}_{\mathfrak{S}_L} \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge}$ as submodules of $\mathcal{O}_{\mathcal{E}, L} \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$.*
- (3) *The natural map*

$$\mathfrak{S}^{(2)} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

is injective for $i = 1, 2$.

- (4) *For any map of bounded prisms $(\mathfrak{S}, (E)) \rightarrow (A, EA)$ over R , the natural map*

$$A[p^{-1}] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathcal{F}_A[p^{-1}]$$

is a φ -compatible isomorphism of $A[p^{-1}]$ -modules.

Proof. (1) Since \mathfrak{S}_L is noetherian and $\mathfrak{S} \rightarrow \mathfrak{S}_L$ is classically flat, we deduce by a similar argument as in Remark 3.22 that $\mathcal{F}_{\mathfrak{S}_L} \cong \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$, $\mathcal{F}_{\mathfrak{S}_L}$ is torsion free, and

$$\mathcal{F}_{\mathfrak{S}_L}[p^{-1}] \cap \mathcal{F}_{\mathfrak{S}_L}[E^{-1}] = \mathcal{F}_{\mathfrak{S}_L}.$$

So $\{p, E\}$ forms a regular sequence for $\mathcal{F}_{\mathfrak{S}_L}$. Since \mathfrak{S}_L is a regular local ring of dimension 2, $\mathcal{F}_{\mathfrak{S}_L}$ is finite free over \mathfrak{S}_L .

(2) It suffices to show $\mathcal{F}_{\mathfrak{S}_L} \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge} \subset \mathcal{F}_{\mathfrak{S}}$. Since $\mathcal{F}_{\mathfrak{S}}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ and $\mathfrak{S}_L \cap \mathcal{O}_{\mathcal{E}} = \mathfrak{S}$, we have by Lemma 3.1 that

$$\begin{aligned} \mathcal{F}_{\mathfrak{S}_L}[p^{-1}] \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge}[p^{-1}] &= (\mathfrak{S}_L[p^{-1}] \otimes_{\mathfrak{S}[p^{-1}]} \mathcal{F}_{\mathfrak{S}}[p^{-1}]) \cap (\mathcal{O}_{\mathcal{E}}[p^{-1}] \otimes_{\mathfrak{S}[p^{-1}]} \mathcal{F}_{\mathfrak{S}}[p^{-1}]) \\ &= \mathcal{F}_{\mathfrak{S}}[p^{-1}]. \end{aligned}$$

Thus,

$$\mathcal{F}_{\mathfrak{S}_L} \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge} \subset \mathcal{F}_{\mathfrak{S}}[p^{-1}] \cap \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^{\wedge} = \mathcal{F}_{\mathfrak{S}}.$$

- (3) The natural map

$$\mathfrak{S}^{(2)} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

is injective since $\mathcal{F}_{\mathfrak{S}} \rightarrow \mathcal{F}_{\mathfrak{S}}[p^{-1}]$ is injective and $p_i: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is classically flat by Lemma 3.5. Furthermore, since $\mathcal{F}_{\mathfrak{S}}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ and $\mathfrak{S}^{(2)}[p^{-1}] \rightarrow \mathfrak{S}^{(2)}[1/E]_p^{\wedge}[p^{-1}]$ is injective by Corollary 3.6, the natural map

$$\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_i, \mathfrak{S}[p^{-1}]} \mathcal{F}_{\mathfrak{S}}[p^{-1}] \rightarrow \mathfrak{S}^{(2)}[1/E]_p^{\wedge}[p^{-1}] \otimes_{p_i, \mathfrak{S}[p^{-1}]} \mathcal{F}_{\mathfrak{S}}[p^{-1}]$$

is injective. So the composite map

$$\mathfrak{S}^{(2)} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathfrak{S}^{(2)}[1/E]_p^{\wedge}[p^{-1}] \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

is injective. The composite factors through the map

$$\mathfrak{S}^{(2)} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathfrak{S}^{(2)}[1/E]_p^{\wedge} \otimes_{p_i, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}},$$

which is therefore injective.

(4) By the definition of a finitely generated completed prismatic crystal, the map $A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow \mathcal{F}_A$ is an isomorphism. Since $\mathcal{F}_{\mathfrak{S}}$ is finitely generated over \mathfrak{S} and A is classically (p, E) -complete, the natural map

$$A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

is surjective. Note that $A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$ is equipped with a Frobenius endomorphism induced from that on $A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$, since $\varphi((p, E)^n) \subset (p, E)^n$ for each $n \geq 1$. So we have a φ -compatible surjection

$$(A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}] \twoheadrightarrow (A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}],$$

and it suffices to show that this map is also injective.

Since $\mathcal{F}_{\mathfrak{S}}[p^{-1}]$ is finite projective over $\mathfrak{S}[p^{-1}]$, there exists an $\mathfrak{S}[p^{-1}]$ -module Q such that $\mathcal{F}_{\mathfrak{S}}[p^{-1}] \oplus Q$ is finite free over $\mathfrak{S}[p^{-1}]$. We have an \mathfrak{S} -submodule $N \subset \mathcal{F}_{\mathfrak{S}}[p^{-1}] \oplus Q$ with $N[p^{-1}] = \mathcal{F}_{\mathfrak{S}}[p^{-1}] \oplus Q$ such that N is free over \mathfrak{S} and that the inclusion $\mathcal{F}_{\mathfrak{S}} \hookrightarrow N[p^{-1}]$ factors through $\mathcal{F}_{\mathfrak{S}} \hookrightarrow N \subset N[p^{-1}]$.

Consider the induced map $A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow A \otimes_{\mathfrak{S}} N$. Note that $A \otimes_{\mathfrak{S}} N$ is (p, E) -complete since N is finite free over \mathfrak{S} . Thus, this map factors through

$$A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \rightarrow A \otimes_{\mathfrak{S}} N.$$

On the other hand, since $\mathcal{F}_{\mathfrak{S}}[p^{-1}]$ is a direct summand of $N[p^{-1}]$, the map $(A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}] \rightarrow (A \otimes_{\mathfrak{S}} N)[p^{-1}]$ is injective. Since it factors through

$$(A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}] \rightarrow (A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}] \rightarrow (A \otimes_{\mathfrak{S}} N)[p^{-1}],$$

the map $(A \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}] \rightarrow (A \widehat{\otimes}_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})[p^{-1}]$ in question is also injective. \square

3.3. Completed prismatic F -crystals in terms of descent data. Keep Assumption 2.9. We can explicitly describe the category $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ in terms of certain descent data as follows.

Definition 3.24. Let $\mathrm{DD}_{\mathfrak{S}}$ denote the category consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f)$ where

- (1) \mathfrak{M} is a finite \mathfrak{S} -module that is projective away from (p, E) and saturated;
- (2) $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semi-linear endomorphism such that $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ has finite E -height;
- (3) $f: \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$ is an isomorphism of $\mathfrak{S}^{(2)}$ -modules that is compatible with Frobenii and satisfies the cocycle condition over $\mathfrak{S}^{(3)}$.

The morphisms of $\mathrm{DD}_{\mathfrak{S}}$ are \mathfrak{S} -linear maps compatible with all structures.

For a fixed non-negative integer r , let $\mathrm{DD}_{\mathfrak{S}, [0, r]}$ denote the full subcategory consisting of objects for which $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ has E -height $\leq r$.

We call an object of $\mathrm{DD}_{\mathfrak{S}}$ an *integral Kisin descent datum*. One can also consider a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f)$ where $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is as above and $f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$ is an isomorphism of $\mathfrak{S}^{(2)}[p^{-1}]$ -modules that is compatible with Frobenii and satisfies the cocycle condition over $\mathfrak{S}^{(3)}[p^{-1}]$. Such an object is called a *rational Kisin descent datum*.

Proposition 3.25. *The association $\mathcal{F} \mapsto \mathcal{F}_{\mathfrak{S}} = \mathcal{F}(\mathfrak{S}, (E))$ gives rise to a functor $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \mathrm{DD}_{\mathfrak{S}}$ and induces equivalences of categories*

$$\mathrm{CR}^{\wedge, \varphi}(R_{\Delta}) \cong \mathrm{DD}_{\mathfrak{S}} \quad \text{and} \quad \mathrm{CR}_{[0, r]}^{\wedge, \varphi}(R_{\Delta}) \cong \mathrm{DD}_{\mathfrak{S}, [0, r]}.$$

Furthermore, under this equivalence, $(\mathfrak{M}, \varphi, f)$ corresponds to an object in $\mathrm{Vect}_{\mathrm{eff}}^{\varphi}(R_{\Delta})$ if and only if \mathfrak{M} is finite projective over \mathfrak{S} .

Proof. Let $\mathcal{F} \in \mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$. By Lemma 3.5 and Remark 3.12, we have an isomorphism of $\mathfrak{S}^{(2)}$ -modules

$$f: \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \xrightarrow{\cong} \mathcal{F}_{\mathfrak{S}^{(2)}} \leftarrow \cong \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

satisfying the cocycle condition over $\mathfrak{S}^{(3)}$. Thus, any completed crystal in $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ naturally gives an object in $\mathrm{DD}_{\mathfrak{S}}$ via $\mathcal{F} \mapsto \mathcal{F}_{\mathfrak{S}}$, which gives a functor from $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ to $\mathrm{DD}_{\mathfrak{S}}$.

Conversely, let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f) \in \mathrm{DD}_{\mathfrak{S}}$. Take any prism $(A, I) \in R_{\Delta}$. By [17, Lem. 4.1.8], there exists a prism $(B, IB) \in R_{\Delta}$ which covers (A, I) and admits a map $(\mathfrak{S}, (E)) \rightarrow (B, IB)$ over R . By Lemma 3.3, the pushout of the diagram $(B, IB) \leftarrow (A, I) \rightarrow (B, IB)$ of maps of bounded prism over R is represented by $(B \otimes_A B)_{(p, I)}^{\wedge}$, and $(B \otimes_A B \otimes_A B)_{(p, I)}^{\wedge}$ satisfies a similar property for the self-triple cofiber product. By the universal property of $\mathfrak{S}^{(2)}$ and $\mathfrak{S}^{(3)}$, we have maps $\mathfrak{S}^{(2)} \rightarrow (B \otimes_A B)_{(p, I)}^{\wedge}$ and $\mathfrak{S}^{(3)} \rightarrow (B \otimes_A B \otimes_A B)_{(p, I)}^{\wedge}$.

Consider the B -module $B \otimes_{\mathfrak{S}} \mathfrak{M}$. The base change of the descent datum $f: \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$ along $\mathfrak{S}^{(2)} \rightarrow (B \otimes_A B)_{(p, I)}^{\wedge}$ gives a descent datum of $B \otimes_{\mathfrak{S}} \mathfrak{M}$, namely, a $(B \otimes_A B)_{(p, I)}^{\wedge}$ -linear isomorphism

$$f_B: (B \otimes_A B)_{(p, I)}^{\wedge} \otimes_{p_1, B} (B \otimes_{\mathfrak{S}} \mathfrak{M}) \xrightarrow{\cong} (B \otimes_A B)_{(p, I)}^{\wedge} \otimes_{p_2, B} (B \otimes_{\mathfrak{S}} \mathfrak{M})$$

satisfying the cocycle condition over $(B \otimes_A B \otimes_A B)_{(p, I)}^{\wedge}$. By reducing modulo $(p, I)^n$, f_B induces a compatible system of isomorphisms

$$f_{B, n}: (B \otimes_A B)/(p, I)^n \otimes_{p_1, B} (B \otimes_{\mathfrak{S}} \mathfrak{M}) \xrightarrow{\cong} (B \otimes_A B)/(p, I)^n \otimes_{p_2, B} (B \otimes_{\mathfrak{S}} \mathfrak{M})$$

satisfying the cocycle condition over $(B \otimes_A B \otimes_A B)/(p, I)^n$ for each $n \geq 1$.

Since $A \rightarrow B$ is (p, I) -completely faithfully flat, each $f_{B, n}$ defines a finitely generated $A/(p, I)^n$ -module $\mathcal{F}_{n, A}$ by the usual faithfully flat descent. We claim that $\mathcal{F}_{n, A}$ is independent of the choice of the cover $(A, I) \rightarrow (B, IB)$ and that the association $(A, I) \mapsto \mathcal{F}_{n, A}$ defines a sheaf \mathcal{F}_n of \mathcal{O}_{Δ} -modules on R_{Δ} . To see the former, take another prism $(B', IB') \in R_{\Delta}$ which covers (A, I) and admits a map $(\mathfrak{S}, (E)) \rightarrow (B', IB')$ over R . Let $\mathcal{F}'_{n, A}$ denote the finitely generated $A/(p, I)^n$ -module given by the descent of $((B' \otimes_{\mathfrak{S}} \mathfrak{M})/(p, I)^n, f_{B', n})$. By Lemma 3.3, the pushout of the diagram $(B, IB) \leftarrow (A, I) \rightarrow (B', IB')$ of maps of bounded prism is represented by $(B \otimes_A B')_{(p, I)}^{\wedge}$. Since the maps $B \rightarrow (B \otimes_A B')_{(p, I)}^{\wedge}$ and $B' \rightarrow (B \otimes_A B')_{(p, I)}^{\wedge}$ are (p, I) -completely faithfully flat, we can canonically identify both $\mathcal{F}_{n, A}$ and $\mathcal{F}'_{n, A}$ with the descent of $((B \otimes_A B')_{(p, I)}^{\wedge} \otimes_{\mathfrak{S}} \mathfrak{M})/(p, I)^n, f_{(B \otimes_A B')_{(p, I)}^{\wedge}, n})$. To see that the association $(A, I) \mapsto \mathcal{F}_{n, A}$ defines a sheaf \mathcal{F}_n of \mathcal{O}_{Δ} -modules on R_{Δ} , take a (p, I) -completely

faithfully flat map of prisms $(A, I) \rightarrow (A', IA')$ over R . Then the pushout of the diagram $(A', IA') \leftarrow (A, I) \rightarrow (A', IA')$ is represented by $(A' \otimes_A A')^{\wedge}_{(p, I)}$. Hence we need to show the exactness of the sequence

$$(3.3) \quad 0 \rightarrow \mathcal{F}_{n, A} \rightarrow \mathcal{F}_{n, A'} \rightarrow \mathcal{F}_{n, (A' \otimes_A A')^{\wedge}_{(p, I)}}.$$

On the other hand, we see that $(A' \otimes_A B)^{\wedge}_{(p, I)}$ (resp. $(A' \otimes_A A' \otimes_A B)^{\wedge}_{(p, I)}$) together with the ideal generated by I gives a bounded prism over R that admits a map from $(\mathfrak{S}, (E))$ over R and covers (A', IA') (resp. $((A' \otimes_A A')^{\wedge}_{(p, I)}, I(A' \otimes_A A')^{\wedge}_{(p, I)})$). By construction, we have a left exact sequence

$$0 \rightarrow (B \otimes_{\mathfrak{S}} \mathfrak{M})/(p, I)^n \rightarrow (A' \otimes B \otimes_{\mathfrak{S}} \mathfrak{M})/(p, I)^n \rightarrow (A' \otimes_A A' \otimes_A B \otimes_{\mathfrak{S}} \mathfrak{M})/(p, I)^n.$$

Since this left exact sequence is the base change of the sequence (3.3) along the classically faithfully flat map $A/(p, I)^n \rightarrow B/(p, I)^n$, we conclude that the sequence (3.3) is left exact. This completes the verification of the claim.

The sheaf \mathcal{F}_n is equipped with an induced Frobenius, since $\varphi((p, I)^n) \subset (p, I)^n$. Furthermore, \mathcal{F}_n is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules. This follows from a similar argument as in the above paragraph and the verification is left to the reader. We also remark that $\{\mathcal{F}_n\}_{n \geq 1}$ forms an inverse system of sheaves of \mathcal{O}_{Δ} -modules such that $\mathcal{O}_{\Delta, n+1} \otimes_{\mathcal{O}_{\Delta, n}} \mathcal{F}_{n+1} \cong \mathcal{F}_n$. Hence $\mathcal{F} := \varprojlim_n \mathcal{F}_n$ is a completed prismatic crystal on R equipped with Frobenius by Lemma 3.13. By construction, we see $\mathcal{F}(\mathfrak{S}, (E)) = \mathfrak{M}$. As a result, $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$. This proves the essential surjectivity.

The fully faithfulness also follows directly from a similar argument as above. Obviously, this equivalence also induces $\text{CR}^{\wedge, \varphi}_{[0, r]}(R_{\Delta}) \cong \text{DD}_{\mathfrak{S}, [0, r]}$. The last assertion follows from [42, Tag 0D4B]. \square

3.4. Étale realization and the main theorem in the small affine case. We now formulate our main theorem. For this, we first attach to a completed prismatic F -crystal \mathcal{F} on R a finite free \mathbf{Z}_p -representation $T(\mathcal{F})$ of \mathcal{G}_R . This will be based on the results in [7, §3] (see also [34]). Keep Assumption 2.9: R is small over \mathcal{O}_K or $R = \mathcal{O}_L$.

Recall that $\text{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1}$ denotes the category of crystals of vector bundles \mathcal{V} on $(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)$ together with isomorphisms $\varphi_{\mathcal{V}}: \varphi^* \mathcal{V} \cong \mathcal{V}$ [7, Def. 3.2], and that there is a natural equivalence of categories

$$\text{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1} \cong \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R)$$

given by $(\mathcal{V}, \varphi_{\mathcal{V}}) \mapsto \mathcal{V}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))^{\varphi_{\mathcal{V}}=1}$ (see [7, Cor. 3.8], [34, Thm. 3.2]).

Proposition 3.26.

- (1) The natural functor $\text{Vect}_{\text{eff}}^{\varphi}(R_{\Delta}) \rightarrow \text{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1}$ given by $\mathcal{F} \mapsto \mathcal{F}_{\text{ét}} := \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p \otimes_{\mathcal{O}_{\Delta}} \mathcal{F}$ extends to a faithful functor
- $$\text{CR}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \text{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1}; \quad \mathcal{F} \mapsto \mathcal{F}_{\text{ét}}.$$

- (2) Define a contravariant functor $T: \text{CR}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R)$ by

$$T(\mathcal{F}) := ((\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)))^{\varphi_{\mathcal{F}_{\text{ét}}}=1})^{\vee}.$$

Then it satisfies the following properties:

(a) there is a \mathcal{G}_R -equivariant identification

$$T(\mathcal{F})[p^{-1}]^\vee = (W(\overline{R}^\flat[(\pi^b)^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})})^{\varphi=1},$$

where the \mathcal{G}_R -action on the right hand side is the tensor product of those on $W(\overline{R}^\flat[(\pi^b)^{-1}])$ and on $\mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})} = \mathcal{F}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$;

(b) $\mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^\wedge$ is the étale φ -module associated with $T(\mathcal{F})^\vee|_{\mathcal{G}_{\overline{R}^\infty}}$ via Proposition 2.16.

(c) If R is small over \mathcal{O}_K and if $R \rightarrow R'$ is a p -adically completed étale map together with a compatible W -map $R_0 \rightarrow R'_0$, then T is compatible with the restrictions $\text{CR}^{\wedge, \varphi}(R_\Delta) \rightarrow \text{CR}^{\wedge, \varphi}(R'_\Delta)$ (see Remark 3.22) and $\text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_R) \rightarrow \text{Rep}_{\mathbf{Z}_p}^{\text{pr}}(\mathcal{G}_{R'})$. We also have the analogous compatibility for the base changes along $R \rightarrow \mathcal{O}_L$ and $R \rightarrow \mathcal{O}_{K_g}$.

We call the functor T the *étale realization functor*.

Proof. (1) Let $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_\Delta)$. By Proposition 3.25, we have an $\mathfrak{S}^{(2)}$ -linear isomorphism

$$f: \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$$

satisfying the cocycle condition over $\mathfrak{S}^{(3)}$. Let $\mathcal{M} = \mathcal{F}_{\mathfrak{S}}[E^{-1}]_p^\wedge$, which is an étale φ -module finite projective over $\mathcal{O}_{\mathcal{E}}$. By extending the scalar, f induces a descent datum

$$\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_1, \mathcal{O}_{\mathcal{E}}} \mathcal{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_2, \mathcal{O}_{\mathcal{E}}} \mathcal{M}.$$

By [17, Lem. 4.1.8] and [48, Prop. 3.2], this descent datum gives a Laurent F -crystal $\mathcal{F}_{\text{ét}} \in \text{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1}$. The faithfulness of $\mathcal{F} \mapsto \mathcal{F}_{\text{ét}}$ follows from construction and Lemma 3.23 (3). It is straightforward to see that if $\mathcal{F} \in \text{Vect}_{\text{ét}}^\varphi(R_\Delta)$, then this construction yields $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge \otimes_{\mathcal{O}_\Delta} \mathcal{F}$.

(2) By the paragraph before the proposition, $((\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)))^{\varphi_{\mathcal{F}_{\text{ét}}}=1})^\vee$ is a finite free \mathbf{Z}_p -representation of \mathcal{G}_R for $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_\Delta)$, and T is well-defined.

First we verify (a). By construction in (1), we have

$$\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)) \cong \mathbf{A}_{\text{inf}}(\overline{R})[E^{-1}]_p^\wedge \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M} \cong W(\overline{R}^\flat[(\pi^b)^{-1}]) \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}.$$

On the other hand, it follows from Lemma 3.23 (4) and Example 3.7 that

$$\mathcal{F}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))[p^{-1}] \cong \mathbf{A}_{\text{inf}}(\overline{R})[p^{-1}] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}.$$

Thus, we deduce

$$\begin{aligned} T(\mathcal{F})[p^{-1}]^\vee &= (\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi)))^{\varphi_{\mathcal{F}_{\text{ét}}}=1}[p^{-1}] = (\mathcal{F}_{\text{ét}}(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))[p^{-1}])^{\varphi_{\mathcal{F}_{\text{ét}}}=1} \\ &\cong (W(\overline{R}^\flat[(\pi^b)^{-1}])[p^{-1}] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}})^{\varphi=1} \\ &\cong (W(\overline{R}^\flat[(\pi^b)^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})})^{\varphi=1}. \end{aligned}$$

Since $\mathcal{F}_{\text{ét}}$ is a crystal, the \mathcal{G}_R -action on the prism $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$ induces the \mathcal{G}_R -action on $(W(\overline{R}^\flat[(\pi^b)^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})})^{\varphi=1}$. It is easy to check that the above isomorphism $T(\mathcal{F})[p^{-1}]^\vee \cong (W(\overline{R}^\flat[(\pi^b)^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})})^{\varphi=1}$ is \mathcal{G}_R -equivariant.

Next we prove (b). Since $T(\mathcal{F})^\vee \cong (W(\overline{R}^b[(\pi^b)^{-1}]) \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1}$, it suffices to show that the natural injective map

$$(\widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1} \rightarrow (W(\overline{R}^b[(\pi^b)^{-1}]) \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1}$$

is also surjective. Consider the φ -equivariant base change $b_g: R_0 \rightarrow W(k_g)$ as in Notation 2.8, which extends \mathcal{O}_K -linearly to $R \rightarrow \mathcal{O}_{K_g}$. Write $\widehat{\mathcal{O}}_{\varepsilon, g}^{\text{ur}}$ for the ring constructed as in § 2.3 for the base ring \mathcal{O}_{K_g} . We have a natural commutative diagram

$$\begin{array}{ccc} (\widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1} & \longrightarrow & (W(\overline{R}^b[(\pi^b)^{-1}]) \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1} \\ \downarrow & & \downarrow \\ (\widehat{\mathcal{O}}_{\varepsilon, g}^{\text{ur}} \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1} & \longrightarrow & (W(\overline{K}_g^b) \otimes_{\mathcal{O}_\varepsilon} \mathcal{M})^{\varphi=1}. \end{array}$$

All the maps in the diagram are injective by construction. The left vertical map is bijective by the functoriality as explained in the end of § 2.3. The bottom horizontal map is bijective by [21, Lem. 2.1.4]. Thus, the other two maps are also bijective.

We now prove (c). Let $R \rightarrow R'$ be a p -complete étale map. From the above construction, we have an induced map of \mathbf{Z}_p -modules

$$T(\mathcal{F})^\vee \rightarrow T(\mathcal{F}|_{R'})^\vee$$

which is injective and compatible with $\mathcal{G}_{R'}$ -actions. By part (b) and the functoriality of étale φ -modules as in the end of § 2.3, this map $T(\mathcal{F})^\vee \rightarrow T(\mathcal{F}|_{R'})^\vee$ is an isomorphism. The statements for $R \rightarrow \mathcal{O}_L$ and $R \rightarrow \mathcal{O}_{K_g}$ follow from a similar argument. \square

Example 3.27. Recall the Breuil–Kisin twist $\mathcal{O}_\Delta\{1\} \in \text{Vect}^\varphi(R_\Delta)$ from [7, Ex. 4.5]. It is an invertible \mathcal{O}_Δ -module with $\varphi^*\mathcal{O}_\Delta\{1\} \cong \mathcal{I}_\Delta^{-1}\mathcal{O}_\Delta\{1\}$ and is given informally by $\mathcal{O}_\Delta\{1\} := \bigotimes_{i \geq 0} (\varphi^i)^*\mathcal{I}_\Delta$. For $n \in \mathbb{Z}$, set $\mathcal{O}_\Delta\{n\} := (\mathcal{O}_\Delta\{1\})^{\otimes n}$. This is an invertible \mathcal{O}_Δ -module such that $\varphi^*\mathcal{O}_\Delta\{n\} \cong \mathcal{I}_\Delta^{-n}\mathcal{O}_\Delta\{n\}$. By [7, Ex. 4.9], we have $T(\mathcal{O}_\Delta\{n\}) = \mathbf{Z}_p(-n)$.¹

For $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_\Delta)$, consider a sheaf of \mathcal{O}_Δ -modules $\mathcal{F}\{n\} := \mathcal{F} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta\{n\}$. Suppose that the image of the induced map $\varphi^*(\mathcal{F}\{n\})_{\mathfrak{S}} \rightarrow (\mathcal{F}\{n\})_{\mathfrak{S}}[E^{-1}]$ lies in $(\mathcal{F}\{n\})_{\mathfrak{S}}$. It follows directly from Proposition 3.25 that $\mathcal{F}\{n\} \in \text{CR}^{\wedge, \varphi}(R_\Delta)$. We claim that $T(\mathcal{F}\{n\}) \cong T(\mathcal{F}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-n)$. To see this, note that we have a natural \mathcal{G}_R -equivariant map $T(\mathcal{F})^\vee \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(n) = T(\mathcal{F})^\vee \otimes_{\mathbf{Z}_p} T(\mathcal{O}_\Delta\{n\})^\vee \rightarrow T(\mathcal{F}\{n\})^\vee$. Since the equivalence in Proposition 2.16 is compatible with tensor products and duals, it follows from the proof of Proposition 3.26 (2b) that this map is bijective, which shows the claim.

Now we state our main theorem. Let $\text{Rep}_{\mathbf{Z}_p, \geq 0}^{\text{cris}}(\mathcal{G}_R)$ (resp. $\text{Rep}_{\mathbf{Z}_p, [0, r]}^{\text{cris}}(\mathcal{G}_R)$) denote the category of crystalline \mathbf{Z}_p -representations of \mathcal{G}_R with non-negative Hodge–Tate weights (resp. with Hodge–Tate weights in $[0, r]$). Note that $\mathbf{Z}_p(1)$ has Hodge–Tate weight one by our convention.

¹Recall that our étale realization functor T is the dual of that of [7].

Theorem 3.28. *We keep Assumption 2.9.*

- (1) *The étale realization T as in Proposition 3.26 gives a fully faithful functor from $\mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$ to $\mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$. Moreover, T restricts to $\mathrm{CR}_{[0, r]}^{\wedge, \varphi}(R_{\Delta}) \rightarrow \mathrm{Rep}_{\mathbf{Z}_p, [0, r]}^{\mathrm{cris}}(\mathcal{G}_R)$.*
- (2) *The functor T gives an equivalence $\mathrm{CR}_{[0, r]}^{\wedge, \varphi}(R_{\Delta}) \cong \mathrm{Rep}_{\mathbf{Z}_p, [0, r]}^{\mathrm{cris}}(\mathcal{G}_R)$.*

Remark 3.29. Note that for every crystalline \mathbf{Z}_p -representation T_0 of \mathcal{G}_R , there exists $n \in \mathbf{Z}$ such that $T_0 \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(n) \in \mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$, and that the étale realization functor T is compatible with Breuil–Kisin twists by Example 3.27. Hence as in [27, §1.2], one can extend the definition of completed prismatic F -crystals in a way that the resulting category is equivalent to $\mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{cris}}(\mathcal{G}_R)$, the category of \mathbf{Z}_p -crystalline representations of \mathcal{G}_R . We leave it to the reader to make a precise formulation.

We prove the first part here. The second part will be proved in the next section.

Proof of Theorem 3.28 (1). Let $\mathcal{F} \in \mathrm{CR}_{[0, r]}^{\wedge, \varphi}(R_{\Delta})$. Consider the map $R \rightarrow \mathcal{O}_{K_g}$ as in Notation 2.8. By Remarks 3.22 and 3.18, we have $\mathcal{F}|_{\mathcal{O}_{K_g}} \in \mathrm{CR}_{[0, r]}^{\wedge, \varphi}((\mathcal{O}_{K_g})_{\Delta}) = \mathrm{Vect}_{[0, r]}^{\varphi}((\mathcal{O}_{K_g})_{\Delta})$. Thus, by [7, Thm. 1.2] (see also [17, Thm. 4.1.10]), we have $T(\mathcal{F}|_{\mathcal{O}_{K_g}}) \in \mathrm{Rep}_{\mathbf{Z}_p, [0, r]}^{\mathrm{cris}}(G_{K_g})$ where $G_{K_g} := \mathcal{G}_{\mathcal{O}_{K_g}}$. Note that by Proposition 3.26 (2c), $T(\mathcal{F}|_{\mathcal{O}_{K_g}})$ is equal to $T(\mathcal{F})|_{G_{K_g}}$.

Let $V(\mathcal{F}) = T(\mathcal{F})[p^{-1}]$ denote the corresponding \mathbf{Q}_p -representation of \mathcal{G}_R . By Proposition 3.26 (2a), we see

$$V(\mathcal{F})^{\vee} \cong (W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\mathrm{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})})^{\varphi=1}.$$

By Lemma 3.23 (4), we have $\mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}] \cong \mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}] \otimes_{\mathfrak{S}[p^{-1}]} \mathcal{F}_{\mathfrak{S}}[p^{-1}]$, which is finite projective over $\mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}]$. Since $\mathcal{F}_{\mathrm{ét}}(\mathbf{A}_{\mathrm{inf}}(\overline{R}), (\xi))$ is an étale φ -module finite projective over $W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])$, we obtain

$$(3.4) \quad W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \otimes_{\mathbf{Q}_p} V(\mathcal{F})^{\vee} \cong W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}]} \mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}].$$

Equation (3.4) induces

$$W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \otimes_{\mathbf{Q}_p} V(\mathcal{F})^{\vee} = W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \otimes_{\mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}]} \mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}]$$

by the base change $W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}]) \rightarrow W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])$. By [5, Lem. 4.26], this restricts to the equality

$$(3.5) \quad W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}][\mu^{-1}] \otimes_{\mathbf{Q}_p} V(\mathcal{F})^{\vee} = W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}][\mu^{-1}] \otimes_{\mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}]} \mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}]$$

where $\mu := [\varepsilon] - 1$. We have

$$W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}] \cap W(\mathcal{O}_{K_g}^{\flat}[(\pi^{\flat})^{-1}])[p^{-1}][\mu^{-1}] = \mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}][\mu^{-1}]$$

by Lemma 3.31 below. Thus, using Lemma 3.1, we deduce from Equations (3.4) and (3.5) that

$$(3.6) \quad \mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}][\mu^{-1}] \otimes_{\mathbf{Q}_p} V(\mathcal{F})^{\vee} = \mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}][\mu^{-1}]$$

since $\mathcal{F}_{\mathbf{A}_{\mathrm{inf}}(\overline{R})}[p^{-1}]$ is projective over $\mathbf{A}_{\mathrm{inf}}(\overline{R})[p^{-1}]$.

Consider the map of prisms $(\mathfrak{S}, (E)) \rightarrow (R_0, (p))$ over R given by $u \mapsto 0$. Let $D(\mathcal{F}) := \mathcal{F}_{R_0}[p^{-1}]$. We have $D(\mathcal{F}) \cong R_0[p^{-1}] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$ by Lemma 3.23 (4), and $1 \otimes \varphi: \varphi^* D(\mathcal{F}) \rightarrow D(\mathcal{F})$ is an isomorphism. Choose a positive integer l with $p^l \geq e$ as in Example 3.8 so that we have the map of prisms $(R_0, (p)) \xrightarrow{\varphi^{l+1}} (\phi_l \mathbf{OA}_{\text{cris}}(\overline{R}), (p))$ over R . Thus,

$$\begin{aligned} \mathbf{OA}_{\text{cris}}(\overline{R})[p^{-1}] \otimes_{\varphi^{l+1}, R_0} D(\mathcal{F}) &\cong \mathcal{F}(\phi_l \mathbf{OA}_{\text{cris}}(\overline{R}), (p))[p^{-1}] \\ &\cong \mathbf{OA}_{\text{cris}}(\overline{R})[p^{-1}] \otimes_{\varphi^l, \mathbf{OA}_{\text{cris}}(\overline{R})} \mathcal{F}(\mathbf{OA}_{\text{cris}}(\overline{R}), (p)) \end{aligned}$$

by Lemma 3.23 (4), and we obtain

$$(3.7) \quad \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\varphi^{l+1}, R_0} D(\mathcal{F}) \cong \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\varphi^l, \mathbf{OA}_{\text{cris}}(\overline{R})} \mathcal{F}(\mathbf{OA}_{\text{cris}}(\overline{R}), (p)).$$

On the other hand, again by Lemma 3.23 (4),

$$\mathcal{F}(\phi_l \mathbf{OA}_{\text{cris}}(\overline{R}), (p))[p^{-1}] \cong \mathbf{OA}_{\text{cris}}(\overline{R})[p^{-1}] \otimes_{\varphi^{l+1}, \mathbf{A}_{\text{inf}}(\overline{R})} \mathcal{F}_{\mathbf{A}_{\text{inf}}(\overline{R})}[p^{-1}].$$

So Equation (3.6) gives

$$(3.8) \quad \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V(\mathcal{F})^\vee \cong \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\varphi^l, \mathbf{OA}_{\text{cris}}(\overline{R})} \mathcal{F}(\mathbf{OA}_{\text{cris}}(\overline{R}), (p)).$$

By Equations (3.7), (3.8), and $\mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\varphi^{l+1}, R_0} D(\mathcal{F}) \cong \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{R_0} D(\mathcal{F})$ obtained by $(l+1)$ -times iterations of the isomorphism $1 \otimes \varphi$, we deduce the isomorphism

$$(3.9) \quad \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{R_0} D(\mathcal{F}) \cong \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V(\mathcal{F})^\vee$$

that is compatible with \mathcal{G}_R -actions and φ . Since $(\mathbf{OB}_{\text{cris}}(\overline{R}))^{\mathcal{G}_R} = R_0[p^{-1}]$, we deduce from this isomorphism that $\mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V(\mathcal{F})^\vee$ is spanned by its \mathcal{G}_R -invariants as an $\mathbf{OB}_{\text{cris}}(\overline{R})$ -module. It follows that $\alpha_{\text{cris}}(V(\mathcal{F})^\vee)$ is surjective and thus $V(\mathcal{F})^\vee$ is crystalline. So $V(\mathcal{F})$ is crystalline. Note that $V(\mathcal{F})$ has Hodge–Tate weights in $[0, r]$, since it has Hodge–Tate weights in $[0, r]$ considered as a representation of G_{K_g} .

The faithfulness follows from the construction of the étale realization T in Proposition 3.26. For the fullness, let $\mathcal{F}_1, \mathcal{F}_2 \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$, and suppose we have a map $h: T(\mathcal{F}_1) \rightarrow T(\mathcal{F}_2)$ of representations of \mathcal{G}_R . By Proposition 3.26 (2b), $h|_{\mathcal{G}_{\overline{R}_\infty}}$ induces a map

$$(\mathcal{F}_2)_{\mathfrak{S}}[E^{-1}]_p^\wedge \rightarrow (\mathcal{F}_1)_{\mathfrak{S}}[E^{-1}]_p^\wedge$$

of étale φ -modules over \mathcal{O}_ε . On the other hand, by Lemma 3.23 (1) and [20, Prop. 4.2.7], $h|_{\mathcal{G}_{\overline{L}_\infty}}$ induces a φ -equivariant map

$$(\mathcal{F}_2)_{\mathfrak{S}_L} \rightarrow (\mathcal{F}_1)_{\mathfrak{S}_L}$$

of \mathfrak{S}_L -modules. These two maps are compatible after the base changes to $\mathcal{O}_{\varepsilon, L}$, and thus we obtain an induced φ -equivariant map of \mathfrak{S} -modules

$$(\mathcal{F}_2)_{\mathfrak{S}}[E^{-1}]_p^\wedge \cap (\mathcal{F}_2)_{\mathfrak{S}_L} \rightarrow (\mathcal{F}_1)_{\mathfrak{S}}[E^{-1}]_p^\wedge \cap (\mathcal{F}_1)_{\mathfrak{S}_L},$$

i.e., a map $f: (\mathcal{F}_2)_{\mathfrak{S}} \rightarrow (\mathcal{F}_1)_{\mathfrak{S}}$ by Lemma 3.23 (2).

By the construction of the étale realization, f is compatible with the descent data

$$\mathfrak{S}^{(2)}[1/E]_p^\wedge \otimes_{\mathcal{O}_{p_i, \varepsilon}} (\mathcal{F}_i)_{\mathfrak{S}}[E^{-1}]_p^\wedge \xrightarrow{\cong} \mathfrak{S}^{(2)}[1/E]_p^\wedge \otimes_{p_i, \mathcal{O}_\varepsilon} (\mathcal{F}_i)_{\mathfrak{S}}[E^{-1}]_p^\wedge$$

for $i = 1, 2$. So by Lemma 3.23 (3), the map $f: (\mathcal{F}_2)_{\mathfrak{S}} \rightarrow (\mathcal{F}_1)_{\mathfrak{S}}$ is compatible with the descent data

$$\mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} (\mathcal{F}_i)_{\mathfrak{S}} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} (\mathcal{F}_i)_{\mathfrak{S}}$$

for $i = 1, 2$. Thus, the fullness follows from Proposition 3.25. \square

Remark 3.30. For $\mathcal{F} \in \mathrm{CR}^{\wedge, \varphi}(R_{\Delta})$, the isomorphism (3.9) in the above proof shows that there is an isomorphism $\mathcal{F}(R_0, (p))[p^{-1}] \cong D_{\mathrm{cris}}^{\vee}(T(\mathcal{F})[p^{-1}])$ as φ -modules over $R_0[p^{-1}]$. Since φ is an isomorphism on $D_{\mathrm{cris}}^{\vee}(T(\mathcal{F})[p^{-1}])$, Lemma 3.23 (4) for the map of prisms $(\mathfrak{S}, E) \xrightarrow{u \rightarrow 0} (R_0, (p)) \xrightarrow{\varphi} (R_0, (p))$ gives a φ -equivalent $R_0[p^{-1}]$ -linear isomorphism

$$(R_0 \otimes_{\varphi, R_0} \mathcal{F}_{\mathfrak{S}}/u\mathcal{F}_{\mathfrak{S}})[p^{-1}] \cong D_{\mathrm{cris}}^{\vee}(T(\mathcal{F})[p^{-1}]).$$

In Remark 4.35, we will explain how to obtain the connection on $D_{\mathrm{cris}}^{\vee}(T(\mathcal{F})[p^{-1}])$ and the filtration on $R \otimes_{R_0} D_{\mathrm{cris}}^{\vee}(T(\mathcal{F})[p^{-1}])$ from \mathcal{F} under the above isomorphism.

We used the following lemma in the proof of Theorem 3.28 (1).

Lemma 3.31. *We have*

$$W(\overline{R}^{\flat}[(\pi^{\flat})^{-1}]) \cap W(\mathcal{O}_{\overline{K}_g}^{\flat}) = \mathbf{A}_{\mathrm{inf}}(\overline{R})$$

as subrings of $W(\overline{K}_g^{\flat})$.

Proof. Recall $\mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\overline{R}^{\flat})$. For any $x \in \overline{R}^{\flat}[(\pi^{\flat})^{-1}]$, if its π^{\flat} -adic valuation as an element in \overline{K}_g^{\flat} is ≥ 0 , then $x \in \overline{R}^{\flat}$. Thus, $\overline{R}^{\flat}[(\pi^{\flat})^{-1}] \cap \mathcal{O}_{\overline{K}_g}^{\flat} = \overline{R}^{\flat}$, which implies the statement. \square

3.5. Height one case. This subsection discusses the case where crystalline representations have Hodge–Tate weights in $[0, 1]$, and studies the relation to p -divisible groups. We keep Assumption 2.9: R is small over \mathcal{O}_K or $R = \mathcal{O}_L$. We first recall a main result in [2] on classifying p -divisible groups over R via prismatic F -crystals on R .

For a p -complete R -algebra with bounded p^{∞} -torsion, let R_{QSYN} denote the big quasi-syntomic site of R (cf. [2, §3.3]). By [2, Cor. 3.3.10], the functor $R_{\Delta} \rightarrow R_{\mathrm{QSYN}}$ sending (A, I) to $R \rightarrow A/I$ is cocontinuous, so it defines a morphism of topoi

$$u: \mathrm{Shv}(R_{\Delta}) \rightarrow \mathrm{Shv}(R_{\mathrm{QSYN}}).$$

For a p -divisible group H over R , we consider the sheaf $\mathfrak{M}_{\Delta}(H) := \mathcal{E}xt_{R_{\Delta}}^1(u^{-1}H, \mathcal{O}_{\Delta})$ on R_{Δ} . Let $\varphi_{\mathfrak{M}_{\Delta}(H)}$ be the endomorphism of $\mathfrak{M}_{\Delta}(H)$ induced from φ on \mathcal{O}_{Δ} . The following is proved in [2].

Theorem 3.32 (Anschütz–Le Bras). *The assignment*

$$H \mapsto (\mathfrak{M}_{\Delta}(H), \varphi_{\mathfrak{M}_{\Delta}(H)})$$

defines a functor from the category $\mathrm{BT}(R)$ of p -divisible groups over R to $\mathrm{Vect}_{[0,1]}^{\varphi}(R_{\Delta})$. Furthermore, this is an equivalence of categories.

Proof. Let $H \in \text{BT}(R)$. By [2, Thm. 4.6.6, Lem. 4.2.4], we see that $(\mathfrak{M}_\Delta(H), \varphi_{\mathfrak{M}_\Delta(H)})$ is an object in $\text{Vect}_{[0,1]}^\varphi(R_\Delta)$. So the assignment defines a functor from $\text{BT}(R)$ to $\text{Vect}_{[0,1]}^\varphi(R_\Delta)$. By [2, Thm. 4.6.9, Prop. 5.2.3], this is an equivalence. \square

Remark 3.33. The above theorem holds for any p -complete regular ring as in [2]. We make Assumption 2.9 since we consider the étale realization below.

For $H \in \text{BT}(R)$, we write $T_p(H)$ for its Tate module. Note that we have a natural \mathcal{G}_R -equivariant isomorphism

$$T_p(H) \cong \text{Hom}_{\text{BT}(\overline{R}^\wedge)}((\mathbf{Q}_p/\mathbf{Z}_p)_{\overline{R}^\wedge}, H_{\overline{R}^\wedge}).$$

Proposition 3.34. *There exists a natural \mathcal{G}_R -equivariant isomorphism*

$$T_p(H) \cong T(\mathfrak{M}_\Delta(H)),$$

where $T(\mathfrak{M}_\Delta(H))$ is the étale realization of $\mathfrak{M}_\Delta(H) \in \text{Vect}_{[0,1]}^\varphi(R_\Delta) \subset \text{CR}^{\wedge, \varphi}(R_\Delta)$ as in Proposition 3.26.

Proof. Since \overline{R}^\wedge is an integral perfectoid ring, the prism $(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$ is the final object of $(\overline{R}^\wedge)_\Delta$. Let $M = \mathfrak{M}_\Delta(H)(\mathbf{A}_{\text{inf}}(\overline{R}), (\xi))$. By [36, Prop. 1.39] (where the covariant version of $\mathfrak{M}_\Delta(\cdot)$ is used), we have a \mathcal{G}_R -equivariant isomorphism

$$T_p(H) \cong (M^\vee)^{\varphi=1}.$$

We claim that the natural injective map

$$(M^\vee)^{\varphi=1} \rightarrow (M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\overline{R}^\flat[(\pi^b)^{-1}]))^{\varphi=1}$$

is also surjective. Let $x \in (M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\overline{R}^\flat[(\pi^b)^{-1}]))^{\varphi=1}$, and consider the base change $R \rightarrow \mathcal{O}_{K_g}$ as before. By [5, Lem. 4.26], we have $x \in M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\mathcal{O}_{K_g}^\flat)[\mu^{-1}]$. Since M^\vee is projective over $\mathbf{A}_{\text{inf}}(\overline{R})$, we deduce from Lemmas 3.1 and 3.31 that

$$x \in (M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\overline{R}^\flat[(\pi^b)^{-1}])) \cap (M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\mathcal{O}_{K_g}^\flat)[\mu^{-1}]) = M^\vee[\mu^{-1}].$$

Thus, $x \in (M^\vee[\mu^{-1}])^{\varphi=1} = (M^\vee)^{\varphi=1}$ by [36, Prop. 1.39], which proves the claim. Since $(M^\vee \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\overline{R}^\flat[(\pi^b)^{-1}]))^{\varphi=1} \cong ((M \otimes_{\mathbf{A}_{\text{inf}}(\overline{R})} W(\overline{R}^\flat[(\pi^b)^{-1}]))^{\varphi=1})^\vee$, it follows from the definition of the étale realization functor that $T_p(H) \cong T(\mathfrak{M}_\Delta(H))$. \square

Based on an example in [47, § 5.4], we now present an example of a crystalline representation with Hodge–Tate weights in $[0, 1]$ that does not come from a p -divisible group. By Theorems 3.28, 3.32, and Proposition 3.34, such an example implies that the inclusion $\text{Vect}_{\text{eff}}^\varphi(R_\Delta) \subset \text{CR}^{\wedge, \varphi}(R_\Delta)$ is strict in general.

Example 3.35. Let $R_0 = W(k)\langle T^{\pm 1} \rangle$ and $R = R_0 \otimes_{W(k)} \mathcal{O}_K$. Suppose $p \geq 3$ and the ramification index $[K : K_0]$ is p . Let \mathfrak{M}_1 be a free $\mathfrak{S}/p\mathfrak{S}$ -module with a basis $\{e_1, e_2, e_3\}$, equipped with Frobenius given by

$$\varphi = \begin{pmatrix} (1+T)^{p-1} - u & 0 & u \\ 0 & u^{p-1} & (1+T)^p u^{p-1} - (1+T) \\ 1 & 0 & 0 \end{pmatrix}$$

and with the trivial connection $\nabla(e_i) = 0$ for $i = 1, 2, 3$. Note that for

$$\psi = \begin{pmatrix} 0 & 0 & u^p \\ (1+T) - (1+T)^p u^{p-1} & u & (u - (1+T)^{p-1})((1+T) - (1+T)^p u^{p-1}) \\ u^{p-1} & 0 & u^{p-1}(u - (1+T)^{p-1}) \end{pmatrix},$$

we have $\varphi\psi = \psi\varphi = u^p I_3$. Thus $\mathfrak{M}_1 \in (\text{Mod FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ in the sense of [25, Def. 9.2]. By [25, Thm. 9.8], \mathfrak{M}_1 is associated with a finite flat group scheme H_1 over R .

Let \mathfrak{M}_2 be a free $\mathfrak{S}/p\mathfrak{S}$ -module with a basis $\{f\}$, equipped with Frobenius given by $\varphi(f) = f$ and with the trivial connection $\nabla(f) = 0$. Then $\mathfrak{M}_2 \in (\text{Mod FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ and it is associated with a finite flat group scheme H_2 over R .

Let $h: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ be a map of torsion Kisin modules given by $\begin{pmatrix} 1+T & u & (1+T)u \end{pmatrix}$. Since h is not surjective, the associated map $H_2 \rightarrow H_1$ of finite flat group schemes is not a monomorphism. On the other hand, the induced maps $H_2[p^{-1}] \rightarrow H_1[p^{-1}]$ and $H_2 \times_R \mathcal{O}_L \rightarrow H_1 \times_R \mathcal{O}_L$ are monomorphisms of finite flat group schemes over $R[p^{-1}]$ and \mathcal{O}_L , respectively.

By [3, Thm. 3.1.1], there exists $a \in R$ with $a \notin (\pi, 1+T) \subset R$ such that $(H_1)_{R'}$ can be embedded into some p -divisible group H over R' , the p -adic completion of $R[a^{-1}]$. Consider the p -divisible group H' over $R'[p^{-1}]$ given by

$$H' := H_{R'[p^{-1}]} / (H_2)_{R'[p^{-1}]},$$

and let V be the associated representation of $\mathcal{G}_{R'}$. Since the map $(H_2)_{R'} \times_{R'} \mathcal{O}_L \rightarrow (H_1)_{R'} \times_{R'} \mathcal{O}_L$ is a monomorphism, the p -divisible group $H' \times_{R'[p^{-1}]} L$ over L extends to a p -divisible group over \mathcal{O}_L . So $V|_{G_L}$ is crystalline with Hodge–Tate weights in $[0, 1]$, where $G_L := \mathcal{G}_{\mathcal{O}_L} = \text{Gal}(\overline{L}/L)$. In particular, V is de Rham by [30, Thm. 1.5]. Recall that Tsuji’s purity theorem [46, Thm. 5.4.8] states that if $W \in \text{Rep}_{\mathbf{Q}_p}(\mathcal{G}_{R'})$ is de Rham and if $W|_{G_L}$ is crystalline, then W is crystalline. Hence we conclude that V is a crystalline representation of $\mathcal{G}_{R'}$ with Hodge–Tate weights in $[0, 1]$.

On the other hand, we claim that H' cannot be extended to a p -divisible group over R' . Suppose otherwise, i.e., suppose that H' extends to a p -divisible group $H'_{R'}$ over R' . Let R_1 be the $(\pi, 1+T)$ -adic completion of the localization $R'_{(\pi, 1+T)}$. Let $\mathfrak{m} \in \text{Spec } R_1$ be the closed point, and let $U := \text{Spec } R_1 - \mathfrak{m}$ be the open subscheme of $\text{Spec } R_1$. Note that by the construction of $h: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ above, the induced map $(H_2)_{R_1} \rightarrow (H_1)_{R_1}$ is not a monomorphism whereas the restriction $(H_2)_U \rightarrow (H_1)_U$ to U is a monomorphism. In particular, by [44, Thm. 4], we have $H'_{R'} \times_{R'} U = (H \times_{R'} U) / (H_2)_U$ as p -divisible groups over U . The isogeny $H \times_{R'} U \rightarrow H'_{R'} \times_{R'} U$ extends to an isogeny $i: H \times_{R'} R_1 \rightarrow H'_{R'} \times_{R'} R_1$. The kernel of i is a finite flat group scheme over R_1 whose restriction to U is $(H_2)_U$. Since R_1 is a regular local ring of dimension 2, the kernel of i is then equal to $(H_2)_{R_1}$, and $(H_2)_{R_1}$ embeds into $H \times_{R'} R_1$. This contradicts that $(H_2)_{R_1} \rightarrow (H_1)_{R_1}$ is not a monomorphism, so H cannot be extended to a p -divisible group over R' .

In the above example, the ramification index $e := [K : K_0]$ needs to be large. In fact, when $e < p - 1$, such an example does not exist.

Theorem 3.36 ([33, Thm. 1.2]). *Suppose $e < p - 1$ (so that $p \geq 3$). Assume moreover that R_0/pR_0 is a UFD, and that R_0 is complete with respect to some ideal*

$J \subset R_0$ containing p such that R_0/J is finitely generated over some field. Then for any $T \in \text{Rep}_{\mathbf{Z}_p, [0,1]}^{\text{cris}}(\mathcal{G}_R)$, there exists a p -divisible group G over R such that $T_p(G) \cong T$ as representations of \mathcal{G}_R .

In particular, we have $\text{Vect}_{[0,1]}^\varphi(R_\Delta) = \text{CR}_{[0,1]}^{\wedge, \varphi}(R_\Delta)$ under the assumptions of the above theorem.

Remark 3.37. In fact, we have a little stronger result: $\text{Vect}_{[0,1]}^\varphi(R_\Delta) = \text{CR}_{[0,1]}^{\wedge, \varphi}(R_\Delta)$ if $e < p-1$ and R_0 is small over \mathcal{O}_K . To see this, we use arguments in our proof of Theorem 3.28 (2) (the essential surjectivity) in § 4. More precisely, for $T \in \text{Rep}_{\mathbf{Z}_p, [0,1]}^{\text{cris}}(\mathcal{G}_R)$, the associated completed prismatic F -crystal $\mathcal{F} \in \text{CR}_{[0,1]}^{\wedge, \varphi}(R_\Delta)$ satisfies $\mathcal{F}_\mathfrak{S} = \mathfrak{M}$ where \mathfrak{M} is given by the construction in § 4.4. By Remark 4.23, $\mathcal{F}_\mathfrak{S}$ is projective over \mathfrak{S} when $e < p-1$. Hence Proposition 3.25 implies $\text{Vect}_{[0,1]}^\varphi(R_\Delta) = \text{CR}_{[0,1]}^{\wedge, \varphi}(R_\Delta)$ when $e < p-1$ (even without the additional assumptions on R_0 in Theorem 3.36). By [33, Rem. 4.6], we can similarly deduce $\text{Vect}_{[0,r]}^\varphi(R_\Delta) = \text{CR}_{[0,r]}^{\wedge, \varphi}(R_\Delta)$ when $er < p-1$. In particular, when $r = 0$, we have $\text{Vect}_{[0,0]}^\varphi(R_\Delta) = \text{CR}_{[0,0]}^{\wedge, \varphi}(R_\Delta)$ for any e .

3.6. Completed prismatic F -crystals on a smooth p -adic formal scheme.

This subsection globalizes the construction and the main theorem in § 3.4. Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K . To define the category $\text{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta)$ by gluing, we need to show the descent property of completed prismatic F -crystals with respect to Zariski open coverings.

Lemma 3.38. *Let $\mathfrak{X} = \bigcup_{\lambda \in \Lambda} \text{Spf } R_\lambda$ be an affine open covering of \mathfrak{X} . For a sheaf \mathcal{F} of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ , it is a finitely generated completed prismatic crystal on \mathfrak{X} if and only if $\mathcal{F}|_{\text{Spf } R_\lambda}$ is a finitely generated completed prismatic crystal on $\text{Spf } R_\lambda$ for every λ .*

Proof. The necessity is obvious. To show the sufficiency, assume that $\mathcal{F}|_{\text{Spf } R_\lambda}$ is a finitely generated completed prismatic crystal on $\text{Spf } R_\lambda$ for every λ . Consider the quotient sheaf $\mathcal{F}_n := \mathcal{F}/(p, \mathcal{I}_\Delta)^n \mathcal{F} = \mathcal{O}_{\Delta, n} \otimes_{\mathcal{O}_\Delta} \mathcal{F}$ on \mathfrak{X}_Δ for each $n \in \mathbf{N}$. Then $(\mathcal{F}_n)_n$ forms an inverse system of \mathcal{O}_Δ -modules and the natural morphism $\mathcal{F} \rightarrow \varprojlim_n \mathcal{F}_n$ is an isomorphism since it is so on $(\text{Spf } R_\lambda)_\Delta$ for each λ . By Lemma 3.13, it is enough to show that \mathcal{F}_n is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules for every $n \in \mathbf{N}$.

Take any $(A, I) \in \mathfrak{X}_\Delta$. Then there exist a finite affine open covering $\text{Spf } A/I = \bigcup_{j=1}^l \text{Spf } R_j$ and an element $\lambda_j \in \Lambda$ for each $j = 1, \dots, l$ such that the map $\text{Spf } R_j \rightarrow \text{Spf } A/I \rightarrow \mathfrak{X}$ factors through $\text{Spf } R_{\lambda_j} \subset \mathfrak{X}$. Since $A/I \rightarrow R_j$ is p -completely étale map, it lifts uniquely to a (p, I) -completely étale map $A \rightarrow A_j$ of δ -rings (cf. [9, Construction 4.4]) and defines $(A_j, IA_j) \in (\text{Spf } R_{\lambda_j})_\Delta \subset \mathfrak{X}_\Delta$. Set $B := \prod_{j=1}^l A_j$. Then B admits a natural δ -structure and $(B, IB) \in \mathfrak{X}_\Delta$. Moreover, $(A, I) \rightarrow (B, IB)$ is (p, I) -completely faithfully flat. Let (B', IB') be the object of \mathfrak{X}_Δ corresponding to the pushout of the diagram $(B, IB) \leftarrow (A, IA) \rightarrow (B, IB)$ of maps of bounded prisms over \mathfrak{X} . Note $B'/(p, I)^n B' = B/(p, I)^n B \otimes_{A/(p, I)^n} B/(p, I)^n B$ by Lemma 3.3. Let p_1 and p_2 denote the two structure maps $B \rightarrow B'$.

Since \mathcal{F}_n is a sheaf on \mathfrak{X}_Δ , we have an exact sequence

$$0 \rightarrow \mathcal{F}_n(A, I) \rightarrow \mathcal{F}_n(B, IB) \xrightarrow{p_1^* - p_2^*} \mathcal{F}_n(B', IB').$$

By definition of B , we also have an identification $\mathcal{F}_n(B, IB) = \prod_{j=1}^l \mathcal{F}_n(A_j, IA_j)$. By assumption, $\mathcal{F}_n|_{(\mathrm{Spf} R_\lambda)_\Delta}$ is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules. Hence we have a $B'/(p, I)^n B'$ -linear isomorphism

$$\eta: B' \otimes_{p_1, B} \mathcal{F}_n(B, IB) \cong \mathcal{F}_n(B', IB') \cong B' \otimes_{p_2, B} \mathcal{F}_n(B, IB)$$

satisfying the cocycle condition over $B/(p, I)^n B \otimes_A B/(p, I)^n B \otimes_A B/(p, I)^n B$. Since $A/(p, I)^n \rightarrow B/(p, I)^n B$ is classically faithfully flat, it follows from the faithfully flat descent that $\mathcal{F}_n(A, I)$ is a finitely generated $A/(p, I)^n$ -module and $B \otimes_A \mathcal{F}_n(A, I) \cong \mathcal{F}_n(B, IB)$.

Let $(A, I) \rightarrow (\tilde{A}, I\tilde{A})$ be a map of bounded prisms over \mathfrak{X} . Set $\tilde{B} := \tilde{A} \widehat{\otimes}_A B$. Then \tilde{B} admits a natural δ -structure and $(\tilde{B}, I\tilde{B}) \in \mathfrak{X}_\Delta$. Moreover, $(\tilde{A}, I\tilde{A}) \rightarrow (\tilde{B}, I\tilde{B})$ is (p, I) -completely faithfully flat. By the same argument as above, $\mathcal{F}_n(\tilde{A}, I\tilde{A})$ is an $\tilde{A}/(p, I)^n \tilde{A}$ -module with $\tilde{B} \otimes_{\tilde{A}} \mathcal{F}_n(\tilde{A}, I\tilde{A}) \cong \mathcal{F}_n(\tilde{B}, I\tilde{B})$. Since $\mathcal{F}_n|_{(\mathrm{Spf} R_\lambda)_\Delta}$ is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules, we also have $\tilde{B} \otimes_B \mathcal{F}_n(B, IB) \cong \mathcal{F}_n(\tilde{B}, I\tilde{B})$. Hence the natural map $\tilde{A} \otimes_A \mathcal{F}_n(A, I) \rightarrow \mathcal{F}_n(\tilde{A}, I\tilde{A})$ is an isomorphism since it is so after tensored with $\tilde{B}/(p, I)\tilde{B}$ over $\tilde{A}/(p, I)^n \tilde{A}$. Therefore \mathcal{F}_n is a finitely generated crystal of $\mathcal{O}_{\Delta, n}$ -modules on \mathfrak{X}_Δ . \square

Remark 3.39. An analogue of Lemma 3.38 holds for an étale covering of \mathfrak{X} in place of an affine open covering. The verification is left to the reader.

Recall that for an integral domain R that is small over \mathcal{O}_K , we defined the category $\mathrm{CR}^{\wedge, \varphi}(R_\Delta)$ of completed prismatic F -crystals on R in Definition 3.16.

Lemma 3.40. *Assume that $\mathfrak{X} = \mathrm{Spf} R$ is affine that is connected and small over \mathcal{O}_K and let \mathcal{F} be a sheaf of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ together with $1 \otimes \varphi_{\mathcal{F}}: \varphi^* \mathcal{F} \rightarrow \mathcal{F}$. Then $(\mathcal{F}, \varphi_{\mathcal{F}}) \in \mathrm{CR}^{\wedge, \varphi}(R_\Delta)$ if and only if there exists an affine open covering $\mathfrak{X} = \bigcup_{\lambda \in \Lambda} \mathrm{Spf} R_\lambda$ such that for each λ , R_λ is connected and small over \mathcal{O}_K , and $(\mathcal{F}|_{(\mathrm{Spf} R_\lambda)_\Delta}, \varphi|_{(\mathrm{Spf} R_\lambda)_\Delta}) \in \mathrm{CR}^{\wedge, \varphi}(R_\lambda)$.*

Proof. The necessity is straightforward. For the sufficiency, choose a p -complete étale map $R_0 \rightarrow R_{\lambda, 0}$ that induces $R \rightarrow R_\lambda$ after the base change along $W \rightarrow \mathcal{O}_K$. Set $R'_0 := \prod_{\lambda \in \Lambda} R_{\lambda, 0}$ and extend the Frobenius on R_0 to R'_0 . Let $\mathfrak{S}' := R'_0[[u]]$ and equip it with Frobenius φ extending the one on R'_0 by $\varphi(u) = u^p$. Via $\mathfrak{S}'/(E) \cong \prod_{\lambda \in \Lambda} R_\lambda \leftarrow R$, we regard $(\mathfrak{S}', (E))$ as an object of R_Δ . Since $(\mathfrak{S}, (E)) \rightarrow (\mathfrak{S}', (E))$ is a classically faithfully flat map of bounded prisms over R , the sufficiency follows from Lemmas 3.19 and 3.38. \square

Definition 3.41. Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K . A *completed F -crystal of \mathcal{O}_Δ -modules* on \mathfrak{X}_Δ is a pair $(\mathcal{F}, \varphi_{\mathcal{F}})$, where \mathcal{F} is a finitely generated completed crystal of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ and

$$1 \otimes \varphi_{\mathcal{F}}: \varphi^* \mathcal{F} \rightarrow \mathcal{F}$$

is a morphism of \mathcal{O}_Δ -modules satisfying the following property: there exists an affine open covering $\mathfrak{X} = \bigcup_{\lambda \in \Lambda} \mathrm{Spf} R_\lambda$ such that each R_λ is connected and small over \mathcal{O}_K in the sense of Definition 2.1 and such that $(\mathcal{F}|_{(\mathrm{Spf} R_\lambda)_\Delta}, \varphi_{\mathcal{F}}|_{(\mathrm{Spf} R_\lambda)_\Delta}) \in \mathrm{CR}^{\wedge, \varphi}(R_\lambda)$. When

$\mathfrak{X} = \mathrm{Spf} R$ is affine that is connected and small over \mathcal{O}_K , this definition coincides with Definition 3.16 by Lemma 3.40.

We also call such an object a *completed prismatic F -crystal* on \mathfrak{X} . The morphisms between completed F -crystals of \mathcal{O}_Δ -modules are \mathcal{O}_Δ -module maps compatible with Frobenii $\varphi_{\mathcal{F}}$.

We write $\mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta)$ for the category of completed F -crystals of \mathcal{O}_Δ -modules on \mathfrak{X}_Δ . Let $\mathrm{Vect}_{\mathrm{eff}}^\varphi(\mathfrak{X}_\Delta)$ denote the full subcategory of $\mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta)$ consisting of objects $(\mathcal{F}, \varphi_{\mathcal{F}})$ where \mathcal{F} is a locally free \mathcal{O}_Δ -module. For a fixed non-negative integer r , we let $\mathrm{CR}_{[0, r]}^{\wedge, \varphi}(\mathfrak{X}_\Delta)$ and $\mathrm{Vect}_{[0, r]}^\varphi(\mathfrak{X}_\Delta)$ denote the full subcategories consisting of objects for which, locally, $(\mathcal{F}_\mathfrak{E}, \varphi_{\mathcal{F}_\mathfrak{E}})$ has E -height $\leq r$.

Let \mathfrak{X}_η denote the adic generic fiber of \mathfrak{X} . Recall that $\mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1}$ denotes the category of crystals of vector bundles \mathcal{V} on $(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)$ together with isomorphisms $\varphi_{\mathcal{V}}: \varphi^*\mathcal{V} \cong \mathcal{V}$ [7, Def. 3.2] and that there is a natural equivalence of categories

$$\mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1} \cong \mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta),$$

where $\mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta)$ denotes the category of étale \mathbf{Z}_p -local systems on \mathfrak{X}_η (see [7, Cor. 3.8]).

Proposition 3.42. *The natural functor $\mathrm{Vect}_{\mathrm{eff}}^\varphi(\mathfrak{X}_\Delta) \rightarrow \mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1}$ given by $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{ét}} := \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge \otimes_{\mathcal{O}_\Delta} \mathcal{F}$ extends to a faithful functor*

$$\mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta) \rightarrow \mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1} : \mathcal{F} \mapsto \mathcal{F}_{\mathrm{ét}}.$$

Proof. Take an affine open covering $\mathfrak{X} = \bigcup_\lambda \mathrm{Spf} R_\lambda$ such that R_λ is connected and small over \mathcal{O}_K for each λ . Then for each λ , $\mathcal{F}|_{(\mathrm{Spf} R_\lambda)_\Delta}$ is naturally an object of $\mathrm{CR}^{\wedge, \varphi}((R_\lambda)_\Delta)$. Hence by Proposition 3.26 (1), we obtain an object $(\mathcal{F}|_{(\mathrm{Spf} R_\lambda)_\Delta})_{\mathrm{ét}}$ of $\mathrm{Vect}((\mathrm{Spf} R_\lambda)_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1}$ together with an identification

$$(\mathcal{F}|_{(\mathrm{Spf} R_\lambda)_\Delta})_{\mathrm{ét}}|_{(\mathrm{Spf} R_\lambda \times_{\mathfrak{X}} \mathrm{Spf} R_{\lambda'})_\Delta} \cong (\mathcal{F}|_{(\mathrm{Spf} R_{\lambda'})_\Delta})_{\mathrm{ét}}|_{(\mathrm{Spf} R_\lambda \times_{\mathfrak{X}} \mathrm{Spf} R_{\lambda'})_\Delta}$$

satisfying the cocycle condition over $(\mathrm{Spf} R_\lambda \times_{\mathfrak{X}} \mathrm{Spf} R_{\lambda'} \times_{\mathfrak{X}} \mathrm{Spf} R_{\lambda''})_\Delta$. Hence they glue to an object $\mathcal{F}_{\mathrm{ét}}$ of $\mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1}$. It is immediate to see that $\mathcal{F}_{\mathrm{ét}}$ is independent of the choice of the affine open covering and this gives the desired faithful functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{ét}}$. \square

Definition 3.43. Define a contravariant functor $T: \mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta) \rightarrow \mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta)$ to be the composite

$$\mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta) \rightarrow \mathrm{Vect}(\mathfrak{X}_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge)^{\varphi=1} \cong \mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta) \xrightarrow{(\)^\vee} \mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta),$$

where the last functor sends \mathbb{L} to its dual \mathbf{Z}_p -local system \mathbb{L}^\vee . We call the functor T the *étale realization functor*. Note that we use the contravariant convention as opposed to the covariant convention in [7].

Notation 3.44. Let $\mathrm{Loc}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathfrak{X}_\eta)$ denote the full subcategory of $\mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta)$ consisting of \mathbf{Z}_p -local systems \mathbb{L} on \mathfrak{X}_η such that $\mathbb{L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a crystalline local system on \mathfrak{X}_η with non-negative Hodge–Tate weights. See Appendix A for the definition of crystalline local systems on \mathfrak{X}_η .

Theorem 3.45. *Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K and let \mathfrak{X}_η denote its adic generic fiber. The étale realization functor T induces the equivalence of categories*

$$T: \mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta) \xrightarrow{\cong} \mathrm{Loc}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathfrak{X}_\eta).$$

Proof. By Theorem 3.28 (1), we see that T factors through $\mathrm{Loc}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathfrak{X}_\eta) \subset \mathrm{Loc}_{\mathbf{Z}_p}(\mathfrak{X}_\eta)$ and T is fully faithful since both properties are of local nature. Once the full faithfulness is established, it follows from Proposition 3.26 (2c), Theorem 3.28 (2), and gluing that $T: \mathrm{CR}^{\wedge, \varphi}(\mathfrak{X}_\Delta) \rightarrow \mathrm{Loc}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathfrak{X}_\eta)$ is also essentially surjective. \square

4. QUASI-KISIN MODULES ASSOCIATED WITH CRYSTALLINE REPRESENTATIONS

The goal of this section is to prove the second part of Theorem 3.28 (the essential surjectivity of the étale realization functor). Recall $\mathfrak{S} = R_0[[u]]$ as in Notation 2.6. Given a \mathbf{Z}_p -lattice T of a crystalline representation of \mathcal{G}_R , we will construct a certain \mathfrak{S} -module equipped with a Frobenius and a connection, which we call a *quasi-Kisin module* associated with T .

In § 4.1, we introduce quasi-Kisin modules (Definition 4.1) and attach a rational Kisin descent datum to a quasi-Kisin module (Construction 4.3 and Propositions 4.6, 4.9). The proof crucially uses explicit computations of elements in $A_{\max}^{(2)}$ (Lemmas 4.4 and 4.8). Section 4.2 shows, under Assumption 2.9, that if $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a finitely generated torsion free φ -module of finite E -height over \mathfrak{S} , then $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ (Proposition 4.13). In § 4.3, we consider the special case where $R = \mathcal{O}_L$ and establish some preliminary results. In § 4.4-4.5, we construct a quasi-Kisin module associated with $T \in \mathrm{Rep}_{\mathbf{Z}_p, \geq 0}^{\mathrm{cris}}(\mathcal{G}_R)$. Finally, § 4.6 completes the proof of Theorem 3.28 by spreading the rational Kisin descent datum to an integral Kisin descent datum via the theory of étale φ -modules.

Since some of the arguments work for a general base ring R , which may be of some interest, we let R be a base ring over \mathcal{O}_K as in Set-up 2.3 unless otherwise noted.

4.1. Quasi-Kisin modules and associated rational Kisin descent data. Recall that S denotes the p -adically completed divided power envelope of \mathfrak{S} with respect to $(E(u))$, equipped with the Frobenius extending that on \mathfrak{S} . Let $\mathrm{Fil}^i S$ be the PD-filtration of S . Namely, $\mathrm{Fil}^i S$ is the p -adically completed ideal of S generated by the divided powers $\gamma_j(E(u))$ ($j \geq i$), where $\gamma_j(x) := \frac{x^j}{j!}$. Let $N_u: S \rightarrow S$ be the R_0 -linear derivation given by $N_u(u) = -u$, and let $\partial_u: S \rightarrow S$ be the R_0 -linear derivation given by $\partial_u(u) = 1$. Note that $-u\partial_u = N_u$. We also have a natural integrable connection $\nabla = \nabla_S: S \rightarrow S \otimes_{R_0} \widehat{\Omega}_{R_0}$ given by the universal derivation on R_0 , which commutes with N_u .

Definition 4.1. Let r be a non-negative integer. A *quasi-Kisin module* over \mathfrak{S} of E -height $\leq r$ is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ where

- (1) \mathfrak{M} is a finitely generated \mathfrak{S} -module that is projective away from (p, E) and saturated;
- (2) $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semi-linear endomorphism such that $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ has E -height $\leq r$;

- (3) if we set $M := R_0 \otimes_{\varphi, R_0} \mathfrak{M}/u\mathfrak{M}$ equipped with the induced tensor-product Frobenius, then

$$\nabla_{\mathfrak{M}}: M[p^{-1}] \rightarrow M[p^{-1}] \otimes_{R_0} \widehat{\Omega}_{R_0}$$

is a topologically quasi-nilpotent integrable connection commuting with Frobenius and satisfies the S -Griffiths transversality (see below).

Let us explain the definition of the S -Griffiths transversality. Set $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and define a decreasing filtration $F^i \mathcal{M}[p^{-1}]$ by

$$F^i \mathcal{M}[p^{-1}] := \{x \in \mathcal{M}[p^{-1}] \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in (\text{Fil}^i S[p^{-1}]) \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

By Lemma 4.2 below, we have $\mathcal{M}[p^{-1}] \cong S[p^{-1}] \otimes_{R_0[p^{-1}]} M[p^{-1}]$, which admits a connection

$$\nabla_{\mathcal{M}[p^{-1}]}: \mathcal{M}[p^{-1}] \rightarrow \mathcal{M}[p^{-1}] \otimes_{R_0} \widehat{\Omega}_{R_0}$$

given by $\nabla_{\mathcal{M}[p^{-1}]} = \nabla_{S[p^{-1}]} \otimes 1 + 1 \otimes \nabla_{\mathfrak{M}}$ so that φ is horizontal. Let $\partial_u: \mathcal{M}[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$ be the derivation given by $\partial_{u, S[p^{-1}]} \otimes 1$. We say that the connection $\nabla_{\mathfrak{M}}$ or $\nabla_{\mathcal{M}[p^{-1}]}$ satisfies the S -Griffiths transversality if, for every i ,

$$\partial_u(F^{i+1} \mathcal{M}[p^{-1}]) \subset F^i \mathcal{M}[p^{-1}] \text{ and } \nabla_{\mathcal{M}[p^{-1}]}(F^{i+1} \mathcal{M}[p^{-1}]) \subset (F^i \mathcal{M}[p^{-1}]) \otimes_{R_0} \widehat{\Omega}_{R_0}.$$

Lemma 4.2. *Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a φ -module finite torsion free over \mathfrak{S} of E -height $\leq r$ such that $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$. Let $M := R_0 \otimes_{\varphi, R_0} \mathfrak{M}/u\mathfrak{M}$ and $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ equipped with the induced Frobenius. Consider the projection $q: \mathcal{M} \rightarrow M$ induced by the φ -compatible projection $S \rightarrow R_0$, $u \mapsto 0$. Then q admits a unique φ -compatible section $s: M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$. Furthermore, $1 \otimes s: S[p^{-1}] \otimes_{R_0[p^{-1}]} M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$ is an isomorphism.*

Proof. Since \mathfrak{M} has E -height $\leq r$, the map

$$(1 \otimes \varphi)[p^{-1}]: \varphi^* M[p^{-1}] := (R_0 \otimes_{\varphi, R_0} M)[p^{-1}] \rightarrow M[p^{-1}]$$

is an isomorphism, and the preimage of M is contained in $p^{-r}(\varphi^* M)$. It then follows from the standard argument as in the proof of [25, Lem. 3.14] that there exists a unique φ -compatible section $s: M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$. Furthermore, the map $1 \otimes s: S[p^{-1}] \otimes_{R_0[p^{-1}]} M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$ is a map of projective $S[p^{-1}]$ -modules of the same rank. So by a similar argument as in the proof of [35, Lem. 4.17], $1 \otimes s$ is an isomorphism. \square

Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ be a quasi-Kisin module of E -height $\leq r$. We associate with $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ a rational Kisin descent datum, namely, an isomorphism of $\mathfrak{S}^{(2)}[p^{-1}]$ -modules

$$f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$$

satisfying the cocycle condition over $\mathfrak{S}^{(3)}$ and compatible with Frobenius.

First we will construct an isomorphism of $S^{(2)}[p^{-1}]$ -modules

$$f_S: S^{(2)}[p^{-1}] \otimes_{p_1, S} \mathcal{M} \xrightarrow{\cong} S^{(2)}[p^{-1}] \otimes_{p_2, S} \mathcal{M}$$

satisfying the cocycle condition over $S^{(3)}[p^{-1}]$ and compatible with Frobenius and filtration. For each $i = 1, \dots, d$, let $\partial_{T_i}: M[p^{-1}] \rightarrow M[p^{-1}]$ be the derivation given

by $\nabla: M[p^{-1}] \rightarrow M[p^{-1}] \otimes_{R_0} \widehat{\Omega}_{R_0} \cong \bigoplus_{i=1}^d M[p^{-1}] \cdot dT_i$ composed with the projection to the i -th factor.

Construction 4.3. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ be a quasi-Kisin module of E -height $\leq r$. Identify $\mathcal{M}[p^{-1}]$ with $\mathcal{D} := S[p^{-1}] \otimes_{R_0} M$ as in Lemma 4.2. Let $\partial_u: \mathcal{D} \rightarrow \mathcal{D}$ be the derivation given by $\partial_{u,S} \otimes 1$, and for $i = 1, \dots, d$, let $\partial_{T_i}: \mathcal{D} \rightarrow \mathcal{D}$ be the derivation given by $\partial_{T_i,S} \otimes 1 + 1 \otimes \partial_{T_i,M}$. We define $f_S: S^{(2)} \otimes_{p_1,S} \mathcal{D} \rightarrow S^{(2)} \otimes_{p_2,S} \mathcal{D}$ by

$$f_S(x) = \sum \partial_u^{j_0} \partial_{T_1}^{j_1} \cdots \partial_{T_d}^{j_d}(x) \cdot \gamma_{j_0}(p_2(u) - p_1(u)) \prod_{i=1}^d \gamma_{j_i}(p_2(T_i) - p_1(T_i)),$$

where the sum goes over the multi-index (j_0, \dots, j_d) of non-negative integers. Note that ∂_u and ∂_{T_i} 's are topologically quasi-nilpotent, so the above sum converges. It follows from a standard computation that this defines a φ -compatible isomorphism of $S^{(2)}[p^{-1}]$ -modules $f_S: S^{(2)} \otimes_{p_1,S} \mathcal{D} \xrightarrow{\cong} S^{(2)} \otimes_{p_2,S} \mathcal{D}$ satisfying the cocycle condition over $S^{(3)}[p^{-1}]$.

By the identification $\mathcal{M}[p^{-1}] = \mathcal{D}$, we obtain a descent datum $f_S: S^{(2)}[p^{-1}] \otimes_{p_1,S} \mathcal{M} \xrightarrow{\cong} S^{(2)}[p^{-1}] \otimes_{p_2,S} \mathcal{M}$. Since $\nabla_{\mathcal{M}[p^{-1}]}$ satisfies the S -Griffiths transversality, we see that f_S is compatible with filtrations (see below for the filtration on $S^{(2)}$).

Before proceeding, let us discuss filtrations on subrings of $A_{\max}^{(2)}[p^{-1}]$ defined after Example 3.9. For any subring $B \subset A_{\max}^{(2)}[p^{-1}]$ that is stable under $\varphi_{A_{\max}^{(2)}[p^{-1}]}$, define

$$\mathrm{Fil}^m B := B \cap E^m A_{\max}^{(2)}[p^{-1}].$$

In particular, we have $\mathrm{Fil}^m \mathfrak{S} = E^m \mathfrak{S}$ and $\mathrm{Fil}^m \mathfrak{S}^{(2)} = E^m \mathfrak{S}^{(2)}$ by [17, Cor. 2.2.8]. Note that $\mathrm{Fil}^m S^{(2)}$ is compatible with the PD-filtration on $S^{(2)}$, i.e.,

$$\mathrm{Fil}^m S^{(2)} = \left\{ \sum_{i_0 + \dots + i_{d+1} \geq m} a_{i_0, \dots, i_{d+1}} \gamma_{i_0}(E) \gamma_{i_1}(y - u) \gamma_{i_2}(s_1 - T_1) \cdots \gamma_{i_{d+1}}(s_d - T_d) \mid a_{i_0, \dots, i_{d+1}} \in \mathfrak{S}^{\widehat{\otimes} 2}, a_{i_0, \dots, i_{d+1}} \rightarrow 0 \text{ (as } i_0 + \dots + i_{d+1} \rightarrow \infty) \right\}.$$

Lemma 4.4. *Assume $p \geq 3$ and let r be a fixed non-negative integer. There exists an integer $h_0 > r$ such that if $m \geq h_0$ and $x \in S^{(2)}[E^{-1}]$ with $E^r x \in \mathrm{Fil}^m S^{(2)}$, then $\varphi(x) = a + b$ for some $a \in \mathfrak{S}^{(2)}$ and $b \in \mathrm{Fil}^{m+1} S^{(2)}$ (as elements in $A_{\max}^{(2)}$).*

Proof. By the explicit description of $\mathrm{Fil}^m S^{(2)}$, since $y - u = Ez_0$, $s_j - T_j = Ez_j$ and $z_j \in \mathfrak{S}^{(2)}$, we can write $E^r x = \sum_{i \geq m} c_i \gamma_i(E)$ for some $c_i \in \mathfrak{S}^{(2)}$ with $c_i \rightarrow 0$ p -adically as $i \rightarrow \infty$. So

$$\varphi(x) = \sum_{i \geq m} \varphi(c_i) \varphi\left(\frac{\gamma_i(E)}{E^r}\right).$$

It suffices to show that there exists $h_0 > r$ such that if $m \geq h_0$ then $\varphi\left(\frac{E^{m-r}}{m!}\right) = a_m + b_m$ for some $a_m \in \mathfrak{S}$ and $b_m \in \mathrm{Fil}^{m+1} S$. For this, note that $\varphi(E) = E^p + pt$ for some $t \in \mathfrak{S}$. So

$$\varphi(E)^{m-r} = (E^p + pt)^{m-r} = \sum_{i=0}^{m-r} \binom{m-r}{i} E^{p(m-r-i)} (pt)^i.$$

Let $v_p(\cdot)$ be the p -adic valuation with $v_p(p) = 1$. Since $v_p(m!) < \frac{m}{p-1}$, we have

$$a_m := \frac{1}{m!} \sum_{i \geq \frac{m}{p-1}}^{m-r} \binom{m-r}{i} E^{p(m-r-i)}(pt)^i \in \mathfrak{S}.$$

Consider $b_m := \frac{1}{m!} \sum_{0 \leq i < \frac{m}{p-1}} \binom{m-r}{i} E^{p(m-r-i)}(pt)^i$. If $p(m-r - \frac{m}{p-1}) \geq m+1$, i.e., $m \geq \frac{(p-1)(pr+1)}{p^2-3p+1}$ (since $p > 2$), then $b_m \in \text{Fil}^{m+1} S$. Hence, we can set $h_0 = \lceil \frac{(p-1)(pr+1)}{p^2-3p+1} \rceil$. \square

We now return to the discussion on the quasi-Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$. Set $\mathfrak{M}^* := \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. For $j = 1, 2$, let $\mathfrak{M}_j^{(2)} := \mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M}$, $\mathfrak{M}_j^{*,(2)} := \mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M}^*$, and $\mathfrak{M}_{\max, j}^{*,(2)} := A_{\max}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}^*$. If B is a subring of $A_{\max}^{(2)}[p^{-1}]$ stable under $\varphi_{A_{\max}^{(2)}[p^{-1}]}$ and if $p_j: \mathfrak{S} \rightarrow A_{\max}^{(2)}[p^{-1}]$ factors through B , then define

$$\text{Fil}^i(B \otimes_{p_j, \mathfrak{S}} \mathfrak{M}^*) := \{x \in B \otimes_{p_j, \mathfrak{S}} \mathfrak{M}^* \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in \text{Fil}^i B \otimes_{p_j, \mathfrak{S}} \mathfrak{M}\}.$$

Note that

$$\begin{aligned} \text{Fil}^i \mathfrak{M}^* &= \{x \in \mathfrak{M}^* \mid (1 \otimes \varphi)(x) \in E^i \mathfrak{M}\} \quad \text{and} \\ \text{Fil}^i \mathfrak{M}_j^{*,(2)} &= \{x \in \mathfrak{M}_j^{*,(2)} \mid (1 \otimes \varphi)(x) \in E^i \mathfrak{M}_j^{(2)}\}. \end{aligned}$$

Since \mathfrak{M} has E -height $\leq r$, $1 \otimes \varphi: \text{Fil}^r \mathfrak{M}^* \rightarrow E^r \mathfrak{M}$ is an isomorphism. Let $\varphi_r: \text{Fil}^r \mathfrak{M}^* \rightarrow \mathfrak{M}^*$ be the φ -semi-linear map given by the composite

$$\varphi_r: \text{Fil}^r \mathfrak{M}^* \xrightarrow{1 \otimes \varphi} E^r \mathfrak{M} \cong E^r \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\frac{\varphi \otimes 1}{\varphi(E^r)}} \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathfrak{M}^*$$

Note that $\varphi_r(\text{Fil}^r \mathfrak{M}^*)$ generates \mathfrak{M}^* as an \mathfrak{S} -module. Similarly, we define the φ -semi-linear map $\varphi_r: \text{Fil}^r \mathfrak{M}_j^{*,(2)} \rightarrow \mathfrak{M}_j^{*,(2)}$.

Lemma 4.5. *We have $(\text{Fil}^i \mathfrak{M}_{\max, j}^{*,(2)}) \cap \mathfrak{M}_j^{*,(2)} = \text{Fil}^i \mathfrak{M}_j^{*,(2)}$.*

Proof. By assumption, (p, u) forms a regular sequence for \mathfrak{M} as an \mathfrak{S} -module. So (p, E) is a regular sequence for \mathfrak{M} , and $\mathfrak{M}/E\mathfrak{M}$ is p -torsion free. Since $p_j: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is classically flat by Lemma 3.5, $\mathfrak{M}_j^{(2)}/E\mathfrak{M}_j^{(2)}$ is p -torsion free. In particular, $E^i \mathfrak{M}_j^{(2)}[p^{-1}] \cap \mathfrak{M}_j^{(2)} = E^i \mathfrak{M}_j^{(2)}$.

It suffices to show

$$(E^i A_{\max}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}) \cap \mathfrak{M}_j^{(2)} = E^i \mathfrak{M}_j^{(2)}$$

as submodules of $A_{\max}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}$, which makes sense since $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ by assumption. Since $E^i A_{\max}^{(2)}[p^{-1}] \cap \mathfrak{S}^{(2)}[p^{-1}] = E^i \mathfrak{S}^{(2)}[p^{-1}]$, Lemma 3.1 implies that

$$\begin{aligned} (E^i A_{\max}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}[p^{-1}]) \cap \mathfrak{M}_j^{(2)}[p^{-1}] &= E^i \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}[p^{-1}]} \mathfrak{M}[p^{-1}] \\ &= E^i \mathfrak{M}_j^{(2)}[p^{-1}]. \end{aligned}$$

Since $E^i \mathfrak{M}_j^{(2)}[p^{-1}] \cap \mathfrak{M}_j^{(2)} = E^i \mathfrak{M}_j^{(2)}$ by above, the assertion follows. \square

We can now show that f_S defines a rational Kisin descent datum when $p \geq 3$. The same result also holds for $p = 2$ (Proposition 4.9) with a similar but longer proof, and we postpone the latter case.

Proposition 4.6. *Assume $p \geq 3$. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ be a quasi-Kisin module of E -height $\leq r$. There exists a unique rational Kisin descent datum*

$$f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$$

such that $\text{id}_{S^{(2)}} \otimes_{\varphi, \mathfrak{S}^{(2)}} f = f_S$, where f_S is defined as in Construction 4.3.

Proof. For $j = 1, 2$, we write $\mathcal{M}_j^{(2)}$ for the image of $S^{(2)} \otimes_{p_j, S} \mathcal{M}$ in $S^{(2)}[p^{-1}] \otimes_{p_j, S} \mathcal{M}$ under the natural map. We will show that there exists a unique $\mathfrak{S}^{(2)}[p^{-1}]$ -linear map

$$f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$$

such that $\text{id}_{S^{(2)}} \otimes_{\varphi, \mathfrak{S}^{(2)}} f = f_S$. Let $h_0 > r$ be a constant given as in Lemma 4.4. Note that by the explicit description of $\text{Fil}^m S^{(2)}$, for any $x \in S^{(2)}$, we have $p^{h_0}x = y + z$ for some $y \in \mathfrak{S}^{\otimes 2}$ and $z \in \text{Fil}^{h_0} S^{(2)}$. Thus, we can take a sufficiently large integer $n \geq 0$ such that $f'_S := p^n f_S$ satisfies $f'_S(\mathcal{M}_1^{(2)}) \subset \mathcal{M}_2^{(2)}$ and

$$f'_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)} + \text{Fil}^{h_0} S^{(2)} \cdot \mathcal{M}_2^{(2)}$$

as submodules of $A_{\max}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}^*$. We claim that

$$f'_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)} + \text{Fil}^m S^{(2)} \cdot \mathcal{M}_2^{(2)}$$

for any $m \geq h_0$. We induct on m . Suppose that the claim holds for $m (\geq h_0)$. Let $w \in \text{Fil}^r \mathfrak{M}^*$. We can write

$$f'_S(w) = z + \sum_i a_i w_i$$

for some $z \in \mathfrak{M}_2^{*,(2)}$, $a_i \in \text{Fil}^m S^{(2)}$, $w_i \in \mathfrak{M}^*$ (with finitely many indices i). Note that $z \in (\text{Fil}^r \mathfrak{M}_{\max, 2}^{*,(2)}) \cap \mathfrak{M}_2^{*,(2)} = \text{Fil}^r \mathfrak{M}_2^{*,(2)}$ by Lemma 4.5. Let $a'_i = \frac{a_i}{E^r} \in S^{(2)}[E^{-1}]$. Then $f'_S(w) = z + \sum_i a'_i \cdot E^r w_i$ with $E^r w_i \in \text{Fil}^r \mathfrak{M}^*$.

We have

$$f'_S(\varphi_r(w)) = \varphi_r(z) + \sum_i \varphi(a'_i) \varphi_r(E^r w_i).$$

Since $E^r a'_i \in \text{Fil}^m S^{(2)}$, we have $\varphi(a'_i) = b_i + c_i$ for some $b_i \in \mathfrak{S}^{(2)}$ and $c_i \in \text{Fil}^{m+1} S^{(2)}$ by Lemma 4.4. Thus, $f'_S(\varphi_r(w)) \in \mathfrak{M}_2^{*,(2)} + \text{Fil}^{m+1} S^{(2)} \cdot \mathcal{M}_2^{(2)}$. Since $\varphi_r(\text{Fil}^r \mathfrak{M}^*)$ generates \mathfrak{M}^* as \mathfrak{S} -modules, the claim follows.

Since $\mathfrak{M}[p^{-1}]$ is finite projective over $\mathfrak{S}[p^{-1}]$ by assumption and the filtration $\{\text{Fil}^m S^{(2)}[p^{-1}]\}$ is separated, we deduce that $f'_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)}[p^{-1}]$. By increasing n if necessary, we may further assume $f'_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)}$. Then $f'_S(\mathfrak{M}_1^{*,(2)}) \subset \mathfrak{M}_2^{*,(2)}$, and

$$f'_S(\text{Fil}^r \mathfrak{M}_1^{*,(2)}) \subset (\text{Fil}^r \mathfrak{M}_{\max, 2}^{*,(2)}) \cap \mathfrak{M}_2^{*,(2)} = \text{Fil}^r \mathfrak{M}_2^{*,(2)}$$

by Lemma 4.5. Consider the composite of the isomorphisms

$$\text{Fil}^r \mathfrak{M}^* \xrightarrow{1 \otimes \varphi} E(u)^r \mathfrak{M} \cong \mathfrak{M}.$$

Since $p_j: \mathfrak{S} \rightarrow \mathfrak{S}^{(2)}$ is classically faithfully flat by Lemma 3.5, we obtain the isomorphism $\mathrm{Fil}^r \mathfrak{M}_j^{*,(2)} \cong \mathfrak{M}_j^{(2)}$ of $\mathfrak{S}^{(2)}$ -modules for $j = 1, 2$. Via these isomorphisms, $f'_S: \mathrm{Fil}^r \mathfrak{M}_1^{*,(2)} \rightarrow \mathrm{Fil}^r \mathfrak{M}_2^{*,(2)}$ induces a map $f': \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$ of $\mathfrak{S}^{(2)}$ -modules. If we set $f := p^{-n} f'$, then we have $\mathrm{id}_{\mathfrak{S}^{(2)}} \otimes_{\varphi, \mathfrak{S}^{(2)}} f = f_S$. The uniqueness is obvious.

By applying the same argument to f_S^{-1} , we conclude that f is an isomorphism. Hence f is a rational Kisin descent datum. \square

We now explain how to obtain a rational Kisin descent datum from f_S when $p = 2$. We consider two auxiliary subrings \tilde{S}, \hat{S} of $A_{\max}^{(2)}$, defined by

$$\begin{aligned} \tilde{S} &:= \mathfrak{S}^{(2)} \left[\left[\frac{E^2}{2} \right] \right] = \left\{ \sum_{i \geq 0} a_i \left(\frac{E^2}{2} \right)^i \mid a_i \in \mathfrak{S}^{(2)} \right\} \quad \text{and} \\ \hat{S} &:= \mathfrak{S}^{(2)} \left[\left[\frac{E^4}{2} \right] \right] = \left\{ \sum_{i \geq 0} a_i \left(\frac{E^4}{2} \right)^i \mid a_i \in \mathfrak{S}^{(2)} \right\}. \end{aligned}$$

Since $\varphi(E) = E^2 + 2\delta(E)$ and $\mathfrak{S}^{(2)}$ is 2-adically complete, both \tilde{S} and \hat{S} are stable under the ring endomorphism φ on $A_{\max}^{(2)}$. The following is shown in [17].

Lemma 4.7 (cf. [17, Lem. 2.2.10]). *Suppose $p = 2$. The following properties hold.*

- (1) $\varphi(A_{\max}^{(2)}) \subset \tilde{S}$ and $\varphi(\tilde{S}) \subset \hat{S}$.
- (2) For every positive integer h , we have

$$\mathrm{Fil}^h \tilde{S} = \left\{ \sum_{i \geq h} a_i \frac{E^i}{2^{\lfloor \frac{i}{2} \rfloor}} \mid a_i \in \mathfrak{S}^{(2)} \right\} \quad \text{and} \quad \mathrm{Fil}^h \hat{S} = \left\{ \sum_{i \geq h} a_i \frac{E^i}{2^{\lfloor \frac{i}{4} \rfloor}} \mid a_i \in \mathfrak{S}^{(2)} \right\}.$$

Lemma 4.8. *Assume $p = 2$, and let r be a fixed non-negative integer. There exists an integer $h_0 > r$ such that if $m \geq h_0$ and $x \in \hat{S}[E^{-1}]$ with $E^r x \in \mathrm{Fil}^m \hat{S}$, then $\varphi(x) = a + b$ for some $a \in \mathfrak{S}^{(2)}$ and $b \in \mathrm{Fil}^{m+1} \hat{S}$ (as elements in $A_{\max}^{(2)}$).*

Proof. By Lemma 4.7 (2), we can write

$$E^r x = \sum_{i \geq m} c_i \frac{E^i}{2^{\lfloor \frac{i}{4} \rfloor}}$$

for some $c_i \in \mathfrak{S}^{(2)}$. So

$$\varphi(x) = \sum_{i \geq m} \varphi(c_i) \frac{\varphi(E^{i-r})}{2^{\lfloor \frac{i}{4} \rfloor}}.$$

It suffices to show that there exists $h_0 > r$ such that if $m \geq h_0$ then $\frac{\varphi(E^{m-r})}{2^{\lfloor \frac{m}{4} \rfloor}} = a_m + b_m$ for some $a_m \in (2, u)^{m-r-\lfloor \frac{m}{4} \rfloor} \mathfrak{S}$ and $b_m \in \mathrm{Fil}^{m+1} \hat{S}$. For this, note that

$$\varphi(E^{m-r}) = (E^2 + 2\delta(E))^{m-r} = \sum_{i=0}^{m-r} \binom{m-r}{i} E^{2(m-r-i)} (2\delta(E))^i.$$

We have

$$a_m := \frac{1}{2^{\lfloor \frac{m}{4} \rfloor}} \sum_{i \geq \lfloor \frac{m}{4} \rfloor}^{m-r} \binom{m-r}{i} E^{2(m-r-i)} (2\delta(E))^i \in (2, u)^{m-r-\lfloor \frac{m}{4} \rfloor} \mathfrak{S}.$$

Set $b_m := \frac{1}{2^{\lfloor \frac{m}{4} \rfloor}} \sum_{0 \leq i \leq \lfloor \frac{m}{4} \rfloor - 1} \binom{m-r}{i} E^{2(m-r-i)} (2\delta(E))^i$. If $2(m-r-\lfloor \frac{m}{4} \rfloor + 1) \geq m+1$,

then $b_m \in \text{Fil}^{m+1} \widehat{S}$. Since

$$2\left(m-r-\left\lfloor \frac{m}{4} \right\rfloor + 1\right) \geq 2\left(m-r-\frac{m}{4} + 1\right) = \frac{3}{2}m - 2r + 2,$$

we can set $h_0 = 4r + 1$. \square

Using Lemmas 4.7 and 4.8, we now construct a rational Kisin datum when $p = 2$.

Proposition 4.9. *Assume $p = 2$. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathfrak{M}})$ be a quasi-Kisin module of E -height $\leq r$. There exists a unique rational Kisin descent datum*

$$f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$$

such that $\text{id}_{S^{(2)}} \otimes_{\varphi, \mathfrak{S}^{(2)}} f = f_S$, where f_S is defined as in Construction 4.3.

Proof. For $j = 1, 2$, write $\mathcal{M}_j^{(2)}$ for the image of $S^{(2)} \otimes_{p_j, S} \mathcal{M}$ in $S^{(2)}[p^{-1}] \otimes_{p_j, S} \mathcal{M}$ under the natural map. We first claim that $f_S(\mathfrak{M}^*) \subset \widetilde{S}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$. For this, take a sufficiently large integer $n \geq 0$ such that $f'_S := p^n f_S$ satisfies $f'_S(\mathcal{M}_1^{(2)}) \subset \mathcal{M}_2^{(2)}$ and

$$f'_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)} + \text{Fil}^r S^{(2)} \cdot \mathcal{M}_2^{(2)}$$

as submodules of $A_{\max}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}^*$. Let $w \in \text{Fil}^r \mathfrak{M}^*$. We can write

$$f'_S(w) = z + \sum_i a_i w_i$$

for some $z \in \mathfrak{M}_2^{*,(2)}$, $a_i \in \text{Fil}^r S^{(2)}$, $w_i \in \mathfrak{M}^*$ (with finitely many indices i). Note that $z \in (\text{Fil}^r \mathfrak{M}_{\max, 2}^{*,(2)}) \cap \mathfrak{M}_2^{*,(2)} = \text{Fil}^r \mathfrak{M}_2^{*,(2)}$ by Lemma 4.5. Since $a_i \in \text{Fil}^r S^{(2)}$, it follows from the explicit description of $\text{Fil}^r S^{(2)}$ that $a'_i := p^r \frac{a_i}{E^r}$ lies in $A_{\max}^{(2)}$. We have $f'_S(p^r w) = p^r z + \sum_i a'_i \cdot E^r w_i$ with $E^r w_i \in \text{Fil}^r \mathfrak{M}^*$ as elements in $A_{\max}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}^*$, and so

$$f'_S(\varphi_r(p^r w)) = \varphi_r(p^r z) + \sum_i \varphi(a'_i) \varphi_r(E^r w_i).$$

Note that $\varphi(a'_i) \in \widetilde{S}$ by Lemma 4.7 (1). Thus, we deduce $f_S(\varphi_r(w)) \in \widetilde{S}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$. Since $\varphi_r(\text{Fil}^r \mathfrak{M}^*)$ generates \mathfrak{M}^* as \mathfrak{S} -modules, the claim follows.

Let $a \in \text{Fil}^r \widetilde{S}$. Since $\lfloor \frac{i-r}{2} \rfloor - (\lfloor \frac{i}{2} \rfloor - r) \geq 0$, it follows from Lemma 4.7 (2) that $p^r \frac{a}{E^r} \in \widetilde{S}$. Furthermore, $\varphi(\widetilde{S}) \subset \widehat{S}$ by Lemma 4.7 (1). Thus, starting with $f_S(\mathfrak{M}^*) \subset \widetilde{S}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}$, we can repeat a similar argument to further obtain

$$f_S(\mathfrak{M}^*) \subset \widehat{S}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}.$$

As in the proof of Proposition 4.6 with Lemma 4.8 in place of Lemma 4.4, we deduce $f_S(\mathfrak{M}^*) \subset \mathfrak{M}_2^{*,(2)}[p^{-1}]$. The rest of the proof proceeds exactly as in the proof of Proposition 4.6. \square

We end this subsection with a simple lemma.

Lemma 4.10. *Let \mathfrak{M} be a finitely generated \mathfrak{S} -module which is projective away from (p, E) and saturated. Then the natural map*

$$\mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow (\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}) \cap (\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathfrak{S}} \mathfrak{M})$$

is an isomorphism.

Proof. Note first that the maps

$$\mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \quad \text{and} \quad \mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathfrak{S}} \mathfrak{M}$$

are injective by the same argument as in the proof of Lemma 3.23 (3).

We need to show that the injective map

$$\mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \hookrightarrow (\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}) \cap (\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathfrak{S}} \mathfrak{M})$$

is also surjective. Suppose not. Set $\mathfrak{L} := (\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M}) \cap (\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathfrak{S}} \mathfrak{M})$ for simplicity. For any \mathbf{Z}_p -module Q , write Q/p for Q/pQ . Then the induced map

$$\mathfrak{S}^{(2)}/p \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{L}/p$$

is not injective since $\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M} = \mathfrak{L}[p^{-1}]$. On the other hand, by the saturation assumption, we have $\mathfrak{M}[p^{-1}] \cap \mathfrak{M}[E^{-1}] = \mathfrak{M}$. So by Lemmas 3.1 and 3.5,

$$(\mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M}[p^{-1}]) \cap (\mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M}[E^{-1}]) = \mathfrak{S}^{(2)} \otimes_{p_j, \mathfrak{S}} \mathfrak{M}.$$

This implies that the map

$$\mathfrak{S}^{(2)}/p \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)}[E^{-1}]_p^\wedge / p \otimes_{p_j, \mathfrak{S}} \mathfrak{M} = \mathfrak{S}^{(2)}[E^{-1}]/p \otimes_{p_j, \mathfrak{S}} \mathfrak{M}$$

is injective. This factors through the map $\mathfrak{S}^{(2)}/p \otimes_{p_j, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{L}/p$, which therefore is injective. This gives a contradiction, and the surjectivity follows. \square

4.2. Projectivity of $\mathfrak{M}[p^{-1}]$ under Assumption 2.9. In this subsection, assume that either R is small over \mathcal{O}_K or $R = \mathcal{O}_L$ (Assumption 2.9). We will show that if $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a finitely generated torsion free φ -module of finite E -height over \mathfrak{S} , then $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$ (Proposition 4.13). For this, we need two preliminary results.

Lemma 4.11. *Let k_1 be a perfect field of characteristic p , and let A be a power-series ring $W(k_1)[[s_1, \dots, s_a]]$. Suppose that A is equipped with a Frobenius endomorphism φ extending the Witt vector Frobenius on $W(k_1)$. Then there exist $t_1, \dots, t_a \in A$ such that $A = W(k_1)[[t_1, \dots, t_a]]$ and $\varphi(t_i)$ has zero constant term for each i .*

Proof. Write $\varphi(s_i) = s_i^p + p(f_i(s_1, \dots, s_a)) + pb_i$ where $f_i(s_1, \dots, s_a) \in A$ satisfying $f_i(0, \dots, 0) = 0$ and $b_i \in W(k_1)$. Write $v_p(\cdot)$ for the p -adic valuation on $W(k_1)$ with $v_p(p) = 1$. Suppose $b_i \neq 0$ for some i , and define $I = \{j \mid v_p(b_j) = \min_{1 \leq i \leq a} \{v_p(b_i)\}\}$.

Let $i_0 \in I$, and let $c_{i_0} \in W(k_1)$ such that $\varphi(c_{i_0}) = b_{i_0}$. We claim that if we replace s_{i_0} by $s_{i_0} - pc_{i_0}$, then $\varphi(s_i) = s_i^p + p(f'_i(s_1, \dots, s_a)) + pb'_i$ satisfying $v_p(b'_i) \geq$

$\min\{v_p(b_i), v_p(b_{i_0}) + 1\}$ for each $i = 1, \dots, a$, and $v_p(b'_{i_0}) \geq v_p(b_{i_0}) + 1$ and $v_p(b'_i) = v_p(b_i)$ if $i_0 \neq i \in I$. Here, f'_i and b'_i denote the corresponding polynomial and the constant replacing f_i and b_i , respectively. To check the claim, note that

$$\begin{aligned} \varphi(s_{i_0} - pc_{i_0}) &= s_{i_0}^p + p(f_{i_0}(s_1, \dots, s_a)) + pb_{i_0} - \varphi(pc_{i_0}) \\ &= (s_{i_0} - pc_{i_0} + pc_{i_0})^p + pf_{i_0}(s_1, \dots, s_{i_0} - pc_{i_0} + pc_{i_0}, \dots, s_a). \end{aligned}$$

Since $v_p(c_{i_0}) = v_p(b_{i_0})$, we have $v_p(b'_{i_0}) \geq v_p(b_{i_0}) + 1$. For $i \neq i_0$, we have

$$\varphi(s_i) = s_i^p + p(f_i(s_1, \dots, s_{i_0} - pc_{i_0} + pc_{i_0}, \dots, s_a)) + pb_i.$$

So $v_p(b'_i) \geq \min\{v_p(c_{i_0}) + 1, v_p(b_i)\}$. Furthermore, if $i \in I$ (with $i \neq i_0$), then $v_p(b'_i) = v_p(b_i)$. This proves the claim.

Thus, if $\#I \geq 2$, then after replacing s_{i_0} by $s_{i_0} - pc_{i_0}$, $\#I$ decreases by 1. If $\#I = 1$, then after replacing s_{i_0} by $s_{i_0} - pc_{i_0}$, we have

$$\min_{1 \leq i \leq a} \{v_p(b'_i)\} \geq 1 + \min_{1 \leq i \leq a} \{v_p(b_i)\}.$$

By repeating the above process, we deduce that there exist $c_1, \dots, c_a \in W(k_1)$ such that for $t_i = s_i - pc_i$, $\varphi(t_i)$ has zero constant term for each i . It is clear that $A = W(k_1)[[t_1, \dots, t_a]]$. \square

Lemma 4.12. *Let k_1 be a perfect field of characteristic p , and let A be a power-series ring $W(k_1)[[t_1, \dots, t_a]]$. Suppose that A is equipped with a Frobenius endomorphism φ extending the Witt vector Frobenius on $W(k_1)$ such that $\varphi(t_i) \in A$ has zero constant term for each i . Let $\mathfrak{S}_A := A[[u]]$ equipped with Frobenius extending that on A by $\varphi(u) = u^p$. Let \mathfrak{N} be a finite \mathfrak{S}_A -module equipped with a φ -semi-linear endomorphism $\varphi: \mathfrak{N} \rightarrow \mathfrak{N}$ such that the induced map $1 \otimes \varphi: (\mathfrak{S}_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{N})[E(u)^{-1}] \rightarrow \mathfrak{N}[E(u)^{-1}]$ is an isomorphism. Then $\mathfrak{N}[p^{-1}]$ is projective over $\mathfrak{S}_A[p^{-1}]$.*

Proof. We induct on a . The base case $a = 0$ (i.e., $A = W(k_1)$) is proved in [5, Prop. 4.3]. Suppose $a \geq 1$. Let J be the non-zero Fitting ideal of \mathfrak{N} over \mathfrak{S}_A with the smallest index. It suffices to show that $J\mathfrak{S}_A[p^{-1}] = \mathfrak{S}_A[p^{-1}]$. Assume the contrary. Since Fitting ideals are compatible under base change, we have

$$(4.1) \quad J\mathfrak{S}_A[E(u)^{-1}] = \varphi(J)\mathfrak{S}_A[E(u)^{-1}]$$

as ideals of $\mathfrak{S}_A[E(u)^{-1}]$, and so

$$(4.2) \quad (\mathfrak{S}_A/J)[E(u)^{-1}] = (\mathfrak{S}_A/\varphi(J))[E(u)^{-1}].$$

Write $K_1 = W(k_1)[p^{-1}]$. Let B be the rigid analytic open unit ball in coordinates (t_1, \dots, t_a, u) . Hence the set of $\overline{K_1}$ -valued points of B is given by

$$\{(t_1, \dots, t_a, u) \in \overline{K_1}^{a+1} \mid 0 \leq |t_i|, |u| < 1\},$$

where we use the p -adic norm such that $|p| = p^{-1}$. We have a natural map $\mathfrak{S}_A[p^{-1}] \rightarrow \mathcal{O}_B$ whose image is dense. Note that by [15, Lem. 7.1.9], we have a functorial bijection between the set of maximal ideals of $\mathfrak{S}_A[p^{-1}]$ and the points of B . Moreover, the Frobenius φ on \mathfrak{S}_A induces an endomorphism on B , for which we still write φ .

For any real number c with $0 < c < 1$, set

$$\begin{aligned} M_c &:= \{(x_1, \dots, x_{a+1}) \in \mathbf{R}^{a+1} \mid 0 \leq x_i \leq c\} \quad \text{and} \\ V_c &:= \{(x_1, \dots, x_{a+1}) \in \mathbf{R}^{a+1} \mid 0 \leq x_i < 1 \text{ for } 1 \leq i \leq a, x_{a+1} = c\}. \end{aligned}$$

Consider the $\overline{K_1}$ -valued points of $\text{Spec}(\mathfrak{S}_A[p^{-1}]/J)$, and let $Z = \{(|t_1|, \dots, |t_a|, |u|)\}$ be the set of corresponding $(a+1)$ -tuple norms. Define

$$Z' = \{(|t_1|, \dots, |t_a|, |u|) \mid (|\varphi(t_i)|, |u|^p) \in Z\}.$$

By Equation (4.2), we have $Z - V_{|\pi|} = Z' - V_{|\pi|}$. For $i = 1, \dots, a$, let $y_i, t_i \in \overline{K_1}$ with $0 \leq |y_i| < 1$ and $0 \leq |t_i| < 1$ such that $\varphi(t_i) = y_i$. Note that by the assumption on $\varphi(t_i)$'s,

$$|y_i| \leq \max\{|t_i|^p, p^{-1}|t_i|\}, \dots, p^{-1}|t_a|\}$$

for each i . So we have

$$(4.3) \quad \max_{1 \leq i \leq a} \{|y_i|\} \leq \max_{1 \leq i \leq a} \{|t_i|^p, p^{-1}|t_i|\}.$$

First we show that Z contains a point with $|u| < |\pi|$. Suppose otherwise. Since $Z - V_{|\pi|} = Z' - V_{|\pi|}$, we deduce that if $Z \cap V_c \neq \emptyset$, then $c = |\pi|^{p^{-n}}$ for some integer $n \geq 0$. By [23, §4, Eq. (5), (6)], the rigid analytic K_1 -space $(\text{Spf}(\mathfrak{S}_A/J))^{\text{rig}}$ has finitely many connected components. So there exists a finite set of non-negative integers $\{n_1, \dots, n_m\}$ such that $Z \cap V_c \neq \emptyset$ if and only if $c = |\pi|^{p^{-n_i}}$ for some i . Without loss of generality, let n_1 be maximal among $\{n_1, \dots, n_m\}$. Since $Z - V_{|\pi|} = Z' - V_{|\pi|}$, we have $Z \cap V_{|\pi|^{p^{-(n_1+1)}}} \neq \emptyset$, which is a contradiction. Thus, Z contains a point with $|u| < |\pi|$.

Next we show $(0, \dots, 0) \in Z$, i.e., $J\mathfrak{S}_A[p^{-1}] \subset (t_1, \dots, t_a, u)\mathfrak{S}_A[p^{-1}]$. Suppose otherwise. Then there exists $f(t_1, \dots, t_a, u) \in J$ whose constant term is non-zero, and let b be the norm of the constant term. Since Z contains a point with $|u| < |\pi|$, we deduce from $Z - V_{|\pi|} = Z' - V_{|\pi|}$ and the inequality (4.3) that $Z \cap M_\epsilon \neq \emptyset$ for any sufficiently small $\epsilon > 0$. But $|f(t_1, \dots, t_a, u)| = b > 0$ if $(|t_1|, \dots, |t_a|, |u|) \in M_\epsilon$ for any sufficiently small $\epsilon > 0$, which is a contradiction. Thus, $(0, \dots, 0) \in Z$.

On the other hand, we claim $J\mathfrak{S}_A[p^{-1}] \not\subset I\mathfrak{S}_A[p^{-1}]$ where $I = (t_1, \dots, t_a) \subset \mathfrak{S}_A$. Suppose otherwise. Take $n \geq 0$ such that $J' := p^n J$ satisfies $J' \subset I\mathfrak{S}_A$. We show by induction that $J' \subset (p, I)^m \cap I$ (as ideals of \mathfrak{S}_A) for each $m \geq 0$. The base case $m = 0$ is clear. Suppose $J' \subset (p, I)^m \cap I$. By Equation (4.1) and the assumption on $\varphi(t_i)$'s, we have

$$E(u)^s J' \subset \varphi((p, I)^m \cap I) \subset (p, I)^{m+1} \cap I$$

for some integer $s \geq 0$. So it suffices to show that if $f \in \mathfrak{S}_A$ satisfies $E(u)f \in (p, I)^{m+1} \cap I$, then $f \in (p, I)^{m+1} \cap I$. For this, choose a set of generators $g_1, \dots, g_b \in A = W(k_1)[[t_1, \dots, t_a]]$ of $(p, I)^{m+1} \cap I$. We have $E(u)f = \sum_{i=1}^b g_i h_i$ for some $h_i \in \mathfrak{S}_A$. Note that we can write

$$h_i = \sum_{j=0}^{e-1} c_{ij} u^j + E(u)h'_i$$

for some $c_{ij} \in W(k_1)[[t_1, \dots, t_a]]$ and $h'_i \in \mathfrak{S}_A$. So

$$E(u)f = \sum_{j=0}^{e-1} \left(\sum_{i=1}^b c_{ij} g_i \right) u^j + E(u) \sum_{i=1}^b g_i h'_i.$$

Setting $u = \pi$ in the above equation, we get $\sum_{j=0}^{e-1} (\sum_{i=1}^b c_{ij} g_i) \pi^j = 0$ as an element in $\mathcal{O}_{K'}[[t_1, \dots, t_a]]$ where $K' := W(k_1) \otimes_{W(k)} K$. This implies $\sum_{i=1}^b c_{ij} g_i = 0$ for each $j = 0, \dots, e-1$, and thus $f = \sum_{i=1}^b g_i h'_i \in (p, I)^{m+1} \cap I$. Hence $J' \subset (p, I)^m \cap I$ for each $m \geq 0$. Since \mathfrak{S}_A is (p, I) -adically separated, we have $J' = 0$ and thus $J = 0$, which is a contradiction. This proves the claim.

Finally, consider the φ -equivariant projection $\mathfrak{S}_A \rightarrow \mathfrak{S}_{A_0} := \mathfrak{S}_A / I\mathfrak{S}_A \cong W(k_1)[[u]]$. Let $J_0 \subset \mathfrak{S}_{A_0}$ be the image of J . Since $J\mathfrak{S}_A[p^{-1}] \not\subset I\mathfrak{S}[p^{-1}]$, we have $J_0 \neq (0)$. Moreover, $J_0\mathfrak{S}_{A_0}[p^{-1}] \neq \mathfrak{S}_{A_0}[p^{-1}]$ since $(0, \dots, 0) \in Z$. On the other hand, Equation (4.2) gives via $\mathfrak{S}_A \rightarrow \mathfrak{S}_{A_0}$

$$(\mathfrak{S}_{A_0}/J_0)[E(u)^{-1}] = (\mathfrak{S}_{A_0}/\varphi(J_0))[E(u)^{-1}],$$

which gives a contradiction by inductive hypothesis. Hence, $J\mathfrak{S}_A[p^{-1}] = \mathfrak{S}_A[p^{-1}]$. \square

Let us return to the discussion on the projectivity of $\mathfrak{M}[p^{-1}]$.

Proposition 4.13. *Suppose that R satisfies Assumption 2.9: R is small over \mathcal{O}_K or $R = \mathcal{O}_L$. If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a finitely generated torsion free φ -module of finite E -height over \mathfrak{S} , then $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$*

Proof. The case where $R = \mathcal{O}_L$ follows from [5, Prop. 4.3] for \mathcal{O}_{K_g} (cf. Notation 2.8) and the classically faithful flatness of $\mathfrak{S}_L[p^{-1}] \rightarrow \mathfrak{S}_g[p^{-1}] := \mathcal{O}_{K_{0,g}}[[u]][p^{-1}]$. Consider the case where R_0 is the p -adic completion of an étale extension of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$. Note that the Krull dimension of R_0 is the same as that of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$. Let $\mathfrak{m} \subset R_0$ be any maximal ideal, and let $(R_0)_{\mathfrak{m}}^{\wedge}$ denote the \mathfrak{m} -adic completion of the localization $(R_0)_{\mathfrak{m}}$. Since $\mathfrak{m} \cap W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ is a maximal ideal of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$, the residue field $k_{\mathfrak{m}} := (R_0)_{\mathfrak{m}}^{\wedge} / \mathfrak{m}(R_0)_{\mathfrak{m}}^{\wedge}$ is a finite extension of k . Note that since $p \in \mathfrak{m}$ and $\varphi(\mathfrak{m}) \subset \mathfrak{m}$, $(R_0)_{\mathfrak{m}}^{\wedge}$ is equipped with the Frobenius induced from R_0 . Let $f: W(k) \rightarrow (R_0)_{\mathfrak{m}}^{\wedge}$ be the composite $W(k) \rightarrow W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle \rightarrow (R_0)_{\mathfrak{m}}^{\wedge}$, which is compatible with φ . Since $W(k) \rightarrow W(k_{\mathfrak{m}})$ is étale, f factors uniquely through $W(k) \rightarrow W(k_{\mathfrak{m}}) \rightarrow (R_0)_{\mathfrak{m}}^{\wedge}$. By unicity, $W(k_{\mathfrak{m}}) \rightarrow (R_0)_{\mathfrak{m}}^{\wedge}$ is compatible with φ . Since $p \notin \mathfrak{m}^2$, $\{p\}$ can be extended to a minimal set generating \mathfrak{m} , and the map $W(k_{\mathfrak{m}}) \rightarrow (R_0)_{\mathfrak{m}}^{\wedge}$ extends to an isomorphism

$$W(k_{\mathfrak{m}})[[t_1, \dots, t_d]] \xrightarrow{\cong} (R_0)_{\mathfrak{m}}^{\wedge}.$$

Furthermore, by Lemma 4.11, t_1, \dots, t_d can be chosen such that $\varphi(t_i)$ has zero constant term for each i (where φ on $W(k_{\mathfrak{m}})[[t_1, \dots, t_d]]$ is given by the above isomorphism).

Now, let $\mathfrak{P} \subset \mathfrak{S}[p^{-1}]$ be any maximal ideal. Then the prime ideal $\mathfrak{q} = \mathfrak{S} \cap \mathfrak{P}$ is maximal among the prime ideals of \mathfrak{S} not containing p . Thus, $\mathfrak{n} := \sqrt{\mathfrak{q} + p\mathfrak{S}}$ is a maximal ideal of \mathfrak{S} . Let $\mathfrak{S}_{\mathfrak{n}}^{\wedge}$ be the \mathfrak{n} -adic completion of the localization $\mathfrak{S}_{\mathfrak{n}}$. By the above discussion, $\mathfrak{S}_{\mathfrak{n}}^{\wedge} \cong W(k_{\mathfrak{n}})[[t_1, \dots, t_d]][[u]]$ for some finite extension $k_{\mathfrak{n}}$ of k and

t_1, \dots, t_d such that $W(k_n) \hookrightarrow \mathfrak{S}_n^\wedge$ is compatible with φ and $\varphi(t_i)$ has zero constant term for each i .

Let $\mathfrak{M}_n := \mathfrak{S}_n^\wedge \otimes_{\mathfrak{S}} \mathfrak{M}$ equipped with the induced tensor-product Frobenius. By Lemma 4.12, $\mathfrak{M}_n[p^{-1}]$ is projective over $\mathfrak{S}_n^\wedge[p^{-1}]$. Let $\mathfrak{P}_n \subset \mathfrak{S}_n^\wedge[p^{-1}]$ be a maximal ideal lying over $\mathfrak{P} \subset \mathfrak{S}[p^{-1}]$. Note that the natural map on localizations

$$(\mathfrak{S}[p^{-1}])_{\mathfrak{P}} \rightarrow (\mathfrak{S}_n^\wedge[p^{-1}])_{\mathfrak{P}_n}$$

is classically faithfully flat. Since $(\mathfrak{S}_n^\wedge[p^{-1}])_{\mathfrak{P}_n} \otimes_{\mathfrak{S}_n^\wedge[p^{-1}]} \mathfrak{M}_n[p^{-1}]$ is finite projective over $(\mathfrak{S}_n^\wedge[p^{-1}])_{\mathfrak{P}_n}$, we deduce that $(\mathfrak{S}[p^{-1}])_{\mathfrak{P}} \otimes_{\mathfrak{S}[p^{-1}]} \mathfrak{M}[p^{-1}]$ is projective over $(\mathfrak{S}[p^{-1}])_{\mathfrak{P}}$. This holds for any maximal ideal $\mathfrak{P} \subset \mathfrak{S}[p^{-1}]$, so $\mathfrak{M}[p^{-1}]$ is projective over $\mathfrak{S}[p^{-1}]$. \square

4.3. Crystalline representations and Breuil–Kisin modules in the CDVR case. We follow Notation 2.8. In particular, recall that \mathcal{O}_L denotes the p -adic completion of $R_{(\pi)}$. Then L is a complete discrete valuation field whose residue field has a finite p -basis given by $\{T_1, \dots, T_d\}$. We first consider crystalline representations of $\mathcal{G}_{\mathcal{O}_L} = \text{Gal}(\overline{L}/L)$, and study certain properties of the associated Breuil–Kisin modules. By the abuse of notation, we also write G_L and $G_{\tilde{L}_\infty}$ for the Galois groups $\mathcal{G}_{\mathcal{O}_L}$ and $\mathcal{G}_{\tilde{\mathcal{O}}_{L,\infty}}$, respectively (see (2.2) for the definition of $\tilde{\mathcal{O}}_{L,\infty}$).

Fix a non-negative integer r . Let V be a crystalline \mathbf{Q}_p -representation of G_L with Hodge–Tate weights in $[0, r]$. By [13, Prop. 4.17], there exists an \mathfrak{S}_L -module \mathfrak{M}_L satisfying the following properties:

- \mathfrak{M}_L is finite free over \mathfrak{S}_L ;
- \mathfrak{M}_L is equipped with a φ -semi-linear endomorphism $\varphi_{\mathfrak{M}_L}: \mathfrak{M}_L \rightarrow \mathfrak{M}_L$ with E -height $\leq r$;
- Set

$$M_L := \mathcal{O}_{L_0} \otimes_{\varphi, \mathcal{O}_{L_0}} \mathfrak{M}_L / u\mathfrak{M}_L$$

and equip it with the induced tensor-product Frobenius. We have a natural isomorphism of L_0 -modules $M_L[p^{-1}] \cong D_{\text{cris}}^\vee(V)$ compatible with Frobenii. Via this isomorphism, $M_L[p^{-1}]$ admits a topologically quasi-nilpotent connection $\nabla_{\mathfrak{M}_L}$.

We call the triple $(\mathfrak{M}_L, \varphi_{\mathfrak{M}_L}, \nabla_{\mathfrak{M}_L})$ the *Breuil–Kisin module associated with V* . Note that [13] considers $\mathfrak{M}_L / u\mathfrak{M}_L$ instead of the Frobenius pullback M_L . However, we have a natural isomorphism of L_0 -modules $M_L[p^{-1}] \xrightarrow{\cong} (\mathfrak{M}_L / u\mathfrak{M}_L)[p^{-1}]$ compatible with Frobenii. Following [25], we use M_L since it is more suitable when we consider the filtration.

Let S_L be the p -adically completed divided power envelope of \mathfrak{S}_L with respect to $(E(u))$. The Frobenius on \mathfrak{S}_L extends uniquely to S_L . For each integer $i \geq 0$, let $\text{Fil}^i S_L$ be the PD-filtration of S_L as before. Let $N_u: S_L \rightarrow S_L$ be the \mathcal{O}_{L_0} -linear derivation given by $N_u(u) = -u$. We also have a natural integrable connection $\nabla: S_L \rightarrow S_L \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}}$ given by the universal derivation on \mathcal{O}_{L_0} , which commutes with N_u .

Set

$$\mathcal{M}_L := S_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$$

equipped with the induced Frobenius. If we let $q: S_L \rightarrow \mathcal{O}_{L_0}$ denote the φ -compatible projection given by $u \mapsto 0$, it induces the projection $q: \mathcal{M}_L \rightarrow M_L$.

We define two filtrations on $\mathcal{M}_L[p^{-1}]$ and study their compatibility. Let

$$\mathcal{D}_L := S_L[p^{-1}] \otimes_{L_0} D_{\text{cris}}^\vee(V).$$

By the above isomorphism $M_L[p^{-1}] \cong D_{\text{cris}}^\vee(V)$ and Lemma 4.2, we have a φ -equivariant identification $\mathcal{M}_L[p^{-1}] = \mathcal{D}_L$. Let $N_u: \mathcal{D}_L \rightarrow \mathcal{D}_L$ be the L_0 -linear derivation given by $N_{u, S_L} \otimes 1$, and let $\nabla: \mathcal{D}_L \rightarrow \mathcal{D}_L \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}}$ be the connection given by $\nabla_{S_L} \otimes 1 + 1 \otimes \nabla_{D_{\text{cris}}^\vee(V)}$. Define a decreasing filtration on \mathcal{D}_L by $S_L[p^{-1}]$ -submodules $\text{Fil}^i \mathcal{D}_L$, inductively as follows: $\text{Fil}^0 \mathcal{D}_L = \mathcal{D}_L$ and

$$\text{Fil}^{i+1} \mathcal{D}_L = \{x \in \mathcal{D}_L \mid N_u(x) \in \text{Fil}^i \mathcal{D}_L, q_\pi(x) \in \text{Fil}^{i+1}(L \otimes_{L_0} D_{\text{cris}}^\vee(V))\},$$

where $q_\pi: \mathcal{D}_L \rightarrow L \otimes_{L_0} D_{\text{cris}}^\vee(V)$ is the map induced by $S_L[p^{-1}] \rightarrow L$, $u \mapsto \pi$. The following is proved in [35]. Note that [35, §4.1] assumes $p > 2$ and $r \leq p - 2$, but the results we will cite in this subsection hold without these assumptions.

Lemma 4.14 ([35, Lem. 4.2]). *The connection ∇ on \mathcal{D}_L satisfies the Griffiths transversality:*

$$\nabla(\text{Fil}^{i+1} \mathcal{D}_L) \subset \text{Fil}^i \mathcal{D}_L \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}}.$$

For the second filtration, let

$$F^i \mathcal{M}_L[p^{-1}] := \{x \in \mathcal{M}_L[p^{-1}] \mid (1 \otimes \varphi_{\mathfrak{M}_L})(x) \in (\text{Fil}^i S_L[p^{-1}]) \otimes_{\mathfrak{S}_L} \mathfrak{M}_L\}.$$

We will see that these two filtrations coincide under the identification $\mathcal{M}_L[p^{-1}] = \mathcal{D}_L$ and thus $\nabla_{\mathfrak{M}_L}$ satisfies the S_L -Griffiths transversality. For this, consider the base change along $\mathcal{O}_{L_0} \rightarrow W(k_g)$ as in Notation 2.8. Note that $W(k_g)$ is a complete discrete valuation ring with *perfect* residue field. Let S_g be the p -adically completed divided power envelope of $\mathfrak{S}_g := W(k_g)[[u]]$ with respect to $(E(u))$. It is equipped with φ , PD-filtration, and N_u similarly as above. Let

$$\mathfrak{M}_g = \mathfrak{S}_g \otimes_{\mathfrak{S}_L} \mathfrak{M}_L, \quad \mathcal{M}_g = S_g \otimes_{\varphi, \mathfrak{S}_g} \mathfrak{M}_g, \quad \text{and} \quad \mathcal{D}_g = S_g[p^{-1}] \otimes_{W(k_g)} D_{\text{cris}}^\vee(V|_{G_{K_g}}).$$

We can identify $\mathcal{M}_g[p^{-1}] = \mathcal{D}_g$ compatibly with φ , and define two filtrations $\text{Fil}^i \mathcal{D}_g$ and $F^i \mathcal{M}_g[p^{-1}]$ similarly as above. By the proof of [32, Cor. 3.2.3], we have

$$\text{Fil}^i \mathcal{D}_g = F^i \mathcal{M}_g[p^{-1}].$$

Note also $K_{0,g} \otimes_{L_0} D_{\text{cris}}^\vee(V) \xrightarrow{\cong} D_{\text{cris}}^\vee(V|_{G_{K_g}})$ by [38, 4B].

Lemma 4.15. *Under the φ -equivariant identification $\mathcal{M}_L[p^{-1}] = \mathcal{D}_L$, we have*

$$F^i \mathcal{D}_L = \text{Fil}^i \mathcal{D}_L.$$

In particular, the triple $(\mathfrak{M}_L, \varphi_{\mathfrak{M}_L}, \nabla_{\mathfrak{M}_L})$ is a quasi-Kisin module of E -height $\leq r$ over \mathfrak{S}_L .

Proof. We consider \mathcal{D}_L as a $S_L[p^{-1}]$ -submodule of \mathcal{D}_g via $S_g \otimes_{S_L} \mathcal{D}_L = \mathcal{D}_g$. Recall that π is a uniformizer of \mathcal{O}_L , and let $e = [L : L_0]$. Note that any $x \in S_L$ can be written as $x = \sum_{i \geq 0} \frac{E(u)^i}{i!} (\sum_{j=0}^{e-1} a_{ij} u^j)$ for some $a_{ij} \in \mathcal{O}_{L_0}$ (with $a_{ij} \rightarrow 0$ p -adically as $i \rightarrow \infty$). Furthermore, a_{ij} 's can be seen to be uniquely determined by inductively

setting $u = \pi$. The analogous statement holds for the elements in S_g , and thus we have $S_L \cap \mathrm{Fil}^i S_g = \mathrm{Fil}^i S_L$.

Let $x \in \mathrm{Fil}^i \mathcal{D}_L$. Since $\mathrm{Fil}^i \mathcal{D}_L \subset \mathrm{Fil}^i \mathcal{D}_g = \mathrm{F}^i \mathcal{D}_g$, we have

$$(1 \otimes \varphi)(x) \in (\mathrm{Fil}^i S_g[p^{-1}]) \otimes_{\mathfrak{S}_L} \mathfrak{M}_L.$$

Since $S_L[p^{-1}] \cap \mathrm{Fil}^i S_g[p^{-1}] = \mathrm{Fil}^i S_L[p^{-1}]$ and $\mathfrak{M}_L[p^{-1}]$ is projective over $\mathfrak{S}_L[p^{-1}]$, we deduce $(1 \otimes \varphi)(x) \in (\mathrm{Fil}^i S_L[p^{-1}]) \otimes_{\mathfrak{S}_L} \mathfrak{M}_L$ by Lemma 3.1. Thus, $\mathrm{Fil}^i \mathcal{D}_L \subset \mathrm{F}^i \mathcal{D}_L$.

Conversely, let $x \in \mathrm{F}^i \mathcal{D}_L$. Note that

$$(L \otimes_{L_0} D_{\mathrm{cris}}^\vee(V)) \cap \mathrm{Fil}^i(K_g \otimes_{W(k_g)} D_{\mathrm{cris}}^\vee(V|_{G_{K_g}})) = \mathrm{Fil}^i(L \otimes_{L_0} D_{\mathrm{cris}}^\vee(V)).$$

Hence we deduce by induction on i that $\mathcal{D}_L \cap \mathrm{Fil}^i \mathcal{D}_g = \mathrm{Fil}^i \mathcal{D}_L$. Since $\mathrm{F}^i \mathcal{D}_L \subset \mathrm{F}^i \mathcal{D}_g = \mathrm{Fil}^i \mathcal{D}_g$, we have $x \in \mathcal{D}_L \cap \mathrm{Fil}^i \mathcal{D}_g = \mathrm{Fil}^i \mathcal{D}_L$. Hence, $\mathrm{F}^i \mathcal{D}_L \subset \mathrm{Fil}^i \mathcal{D}_L$. The second assertion follows from the first and Lemma 4.14: $N_u(\mathrm{Fil}^{i+1} \mathcal{D}_L) \subset \mathrm{Fil}^i \mathcal{D}_L$ by definition, and it is straightforward to check $\partial_u(\mathrm{Fil}^{i+1} \mathcal{D}_L) \subset \mathrm{Fil}^i \mathcal{D}_L$ by induction. \square

Next we will explain how to recover V from \mathcal{D}_L as a representation of G_L . Note that the embedding $\mathfrak{S}_L \rightarrow W(\mathcal{O}_{\overline{L}}^b)$ given in § 2.3 extends to $S_L \rightarrow \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})$, which is compatible with φ , filtrations, and $G_{\tilde{L}_\infty}$ -actions. We have a natural $\mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})$ -semi-linear $G_{\tilde{L}_\infty}$ -action on $\mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L$ given by the $G_{\tilde{L}_\infty}$ -action on $\mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}]$ and the trivial $G_{\tilde{L}_\infty}$ -action on \mathcal{D}_L . As in [35, §4], we can extend this to a G_L -action using differential operators as follows. For each $i = 1, \dots, d$, let $N_{T_i}: \mathcal{D}_L \rightarrow \mathcal{D}_L$ be the derivation given by $\nabla: \mathcal{D}_L \rightarrow \mathcal{D}_L \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}} \cong \bigoplus_{i=1}^d \mathcal{D}_L \cdot d \log T_i$ composed with the projection to the i -th factor. Note that $N_{T_i} = T_i \partial_{T_i}$, where $\partial_{T_i}: \mathcal{D}_L \rightarrow \mathcal{D}_L$ is the derivation as in § 4.1. For any $\sigma \in G_L$, write

$$\underline{\varepsilon}(\sigma) := \frac{\sigma([\pi^b])}{[\pi^b]} \quad \text{and} \quad \underline{\mu}_i(\sigma) := \frac{\sigma([T_i^b])}{[T_i^b]} \quad (i = 1, \dots, d).$$

Note that $\log(\underline{\varepsilon}(\sigma))$ and $\log(\underline{\mu}_i(\sigma))$ lie in $\mathrm{Fil}^1 \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})$. For any element $a \otimes x \in \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L$, define

$$(4.4) \quad \sigma(a \otimes x) = \sum \sigma(a) \gamma_{i_0}(-\log(\underline{\varepsilon}(\sigma))) \gamma_{i_1}(\log(\underline{\mu}_1(\sigma))) \cdots \gamma_{i_d}(\log(\underline{\mu}_d(\sigma))) \cdot N_u^{i_0} N_{T_1}^{i_1} \cdots N_{T_d}^{i_d}(x)$$

where the sum goes over the multi-index (i_0, i_1, \dots, i_d) of non-negative integers. Since $\nabla_{\mathcal{D}_L}$ is topologically quasi-nilpotent and since $\gamma_j(-\log(\underline{\varepsilon}(\sigma)))$, $\gamma_j(\log(\underline{\mu}_i(\sigma))) \rightarrow 0$ p -adically as $j \rightarrow \infty$, the above sum converges. It is standard to check that this gives a well-defined $\mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})$ -semi-linear G_L -action compatible with φ . Furthermore, this G_L -action preserves the filtration since $\log(\underline{\varepsilon}(\sigma))$, $\log(\underline{\mu}_i(\sigma)) \in \mathrm{Fil}^1 \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})$ and since N_u and ∇ satisfy the Griffiths transversality by definition and Lemma 4.14.

Let

$$V(\mathcal{D}_L) := \mathrm{Hom}_{S_L, \mathrm{Fil}, \varphi}(\mathcal{D}_L, \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}]).$$

Using the identification

$$\mathrm{Hom}_{S_L, \mathrm{Fil}, \varphi}(\mathcal{D}_L, \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}]) = \mathrm{Hom}_{S_L, \mathrm{Fil}, \varphi}(\mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L, \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}]),$$

we define the G_L -action on $V(\mathcal{D}_L)$ by setting $\sigma(f)(x) = \sigma(f(\sigma^{-1}(x)))$ for any $x \in \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L$.

Proposition 4.16 (cf. [35, §4]). *There is a natural G_L -equivariant isomorphism*

$$V(\mathcal{D}_L) \cong V.$$

Proof. This is proved in [35, §4], and we sketch the proof here. We first study how above constructions are related to étale φ -modules. If we let $\mathcal{M}_L = \mathcal{O}_{\mathcal{E},L} \otimes_{\mathfrak{S}_L} \mathfrak{M}_L$ with the induced φ , then \mathcal{M}_L is an étale φ -module over $\mathcal{O}_{\mathcal{E},L}$. Consider the $G_{\tilde{L}_\infty}$ -equivariant map

$$\mathrm{Hom}_{\mathfrak{S}_L, \varphi}(\mathfrak{M}_L, \widehat{\mathfrak{S}}_L^{\mathrm{ur}}) \rightarrow T^\vee(\mathcal{M}_L) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},L}, \varphi}(\mathcal{M}, \widehat{\mathcal{O}}_{\mathcal{E},L}^{\mathrm{ur}})$$

induced by the embedding $\widehat{\mathfrak{S}}_L^{\mathrm{ur}} \rightarrow \widehat{\mathcal{O}}_{\mathcal{E},L}^{\mathrm{ur}}$. By [35, Lem. 4.6], this map is an isomorphism.

The embedding $\varphi: \widehat{\mathfrak{S}}_L^{\mathrm{ur}} \rightarrow \mathbf{A}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ induces a natural $G_{\tilde{L}_\infty}$ -equivariant injective map

$$\mathrm{Hom}_{\mathfrak{S}_L, \varphi}(\mathfrak{M}_L, \widehat{\mathfrak{S}}_L^{\mathrm{ur}})[p^{-1}] \rightarrow V(\mathcal{D}_L)$$

by Lemma 4.2. On the other hand, any $f \in V(\mathcal{D}_L)$ induces a φ -equivariant map $f': D_{\mathrm{cris}}^\vee(V) \rightarrow \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ via the map $D_{\mathrm{cris}}^\vee(V) \rightarrow \mathcal{D}_L$. We see that f' is also compatible with filtration, since $\mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \cong \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{K}_g})$ and the induced map $D_{\mathrm{cris}}^\vee(V|_{G_{K_g}}) \rightarrow \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{K}_g})$ is compatible with filtration by the proof of [10, Lem. 8.1.2] and [32, §3.4]. So we obtain a natural injective map

$$V(\mathcal{D}_L) \rightarrow \mathrm{Hom}_{\mathrm{Fil}, \varphi}(D_{\mathrm{cris}}^\vee(V), \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})).$$

Since $\mathrm{Hom}_{\mathfrak{S}_L, \varphi}(\mathfrak{M}_L, \widehat{\mathfrak{S}}_L^{\mathrm{ur}})[p^{-1}]$ and V are \mathbf{Q}_p -vector spaces of the same dimension, it suffices to show that $\mathrm{Hom}_{\mathrm{Fil}, \varphi}(D_{\mathrm{cris}}^\vee(V), \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}))$ admits a G_L -action compatibly with $V(\mathcal{D}_L)$ and there exists a natural isomorphism $\mathrm{Hom}_{\mathrm{Fil}, \varphi}(D_{\mathrm{cris}}^\vee(V), \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})) \cong V$ as G_L -representations.

Write $D = D_{\mathrm{cris}}^\vee(V)$ for simplicity. By Lemma 2.11, we have a $\mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ -linear isomorphism

$$\mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})\{X_1, \dots, X_d\} \cong \mathbf{OB}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$$

given by $X_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat]$. Consider the projection

$$\mathrm{pr}: \mathbf{OB}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \rightarrow \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$$

given by $X_i = T_i - [T_i^\flat] \mapsto 0$, which is compatible with filtrations and φ . This induces the projection

$$\mathrm{pr}: \mathbf{OB}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \otimes_{\iota_1, L_0} D \rightarrow \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \otimes_{\iota_2, L_0} D$$

compatible with φ and filtrations (after tensoring with L over L_0). Here, $\iota_1: L_0 \rightarrow \mathbf{OB}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ is the natural map given by $T_i \mapsto T_i \otimes 1$, and $\iota_2: L_0 \rightarrow \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ is the map given by $T_i \mapsto [T_i^\flat]$.

We define a $\mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}})$ -linear section s to pr as follows. For $x \in D$, let

$$s(x) = \sum (-1)^{i_1 + \dots + i_d} \gamma_{i_1}(\log(\frac{T_1}{[T_1^\flat]})) \cdots \gamma_{i_d}(\log(\frac{T_d}{[T_d^\flat]})) N_{T_1}^{i_1} \cdots N_{T_d}^{i_d}(x)$$

where the sum goes over the multi-index (i_1, \dots, i_d) of non-negative integers. It is shown in [35, §4.1] that s is a well-defined section inducing an isomorphism

$$s: \mathbf{B}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \otimes_{\iota_2, L_0} D \xrightarrow{\cong} (\mathbf{OB}_{\mathrm{cris}}(\mathcal{O}_{\tilde{L}}) \otimes_{\iota_1, L_0} D)^{\nabla=0}$$

of $\mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}})$ -modules, compatibly with filtrations and φ . Furthermore, if we define G_L -action on $\mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D$ by

$$\sigma(a \otimes x) = \sum \sigma(a) \gamma_{i_1}(\log(\underline{\mu}_1(\sigma))) \cdots \gamma_{i_d}(\log(\underline{\mu}_d(\sigma))) \cdot N_{T_1}^{i_1} \cdots N_{T_d}^{i_d}(x)$$

for $\sigma \in G_L$ and $a \otimes x \in \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D$, then by *loc. cit.*, the map

$$V(\mathcal{D}_L) \rightarrow \text{Hom}_{\text{Fil}, \varphi}(D, \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}})) = \text{Hom}_{\text{Fil}, \varphi}(\mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D, \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}))$$

is G_L -equivariant, and s induces a G_L -equivariant isomorphism

$$\text{Hom}_{\text{Fil}, \varphi}(D, \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}})) \cong \text{Hom}_{\text{Fil}, \varphi, \nabla}(D, \mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}})) = V.$$

This shows that $V(\mathcal{D}_L) \cong V$ as representations of G_L . \square

Since $\text{Hom}_{\mathfrak{S}_L, \varphi}(\mathfrak{M}_L, \widehat{\mathfrak{S}}_L^{\text{ur}}[p^{-1}]) \cong V$ as $G_{\overline{L}_\infty}$ -representations, we have a natural map $\mathfrak{M}_L \rightarrow \widehat{\mathfrak{S}}_L^{\text{ur}} \otimes_{\mathbf{Z}_p} V^\vee$. Via the embedding $\widehat{\mathfrak{S}}_L^{\text{ur}} \xrightarrow{\varphi} \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \hookrightarrow \mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}})$, this induces a map $\mathfrak{M}_L[p^{-1}] \rightarrow V^\vee \otimes_{\mathbf{Q}_p} \mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}})$. Composing this with the section $M_L[p^{-1}] \rightarrow \mathfrak{M}_L[p^{-1}]$ in Lemma 4.2, we obtain a φ -compatible map

$$M_L[p^{-1}] \rightarrow \mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\mathbf{Q}_p} V^\vee.$$

Write $D = D_{\text{cris}}^\vee(V)$ as before. If we compose the above map with

$$\mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\mathbf{Q}_p} V^\vee \xrightarrow{\alpha_{\text{cris}}^{-1}} \mathbf{OB}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_1, L_0} D \xrightarrow{\text{pr}} \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D,$$

then we obtain a φ -compatible map

$$M_L[p^{-1}] \rightarrow \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D.$$

We will use the following proposition in § 4.5.

Proposition 4.17. *The image of the above map $M_L[p^{-1}] \rightarrow \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D$ lies in*

$$D = L_0 \otimes_{L_0} D \subset \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D.$$

Furthermore, the induced map $M_L[p^{-1}] \rightarrow D$ is an isomorphism of L_0 -modules.

Proof. The construction of G_L -equivariant isomorphisms

$$V(\mathcal{D}_L) \xrightarrow{\cong} \text{Hom}_{\text{Fil}, \varphi}(D, \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}})) \xrightarrow{\cong} V$$

and diagram chasing implies that the above map injects into $D \subset \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, L_0} D$. Since $M_L[p^{-1}]$ and D are L_0 -vector spaces of the same dimension, the induced map $M_L[p^{-1}] \rightarrow D$ is an isomorphism. \square

4.4. Construction of the quasi-Kisin module I: definition of \mathfrak{M} . We now work over a general base ring: consider R and $\mathfrak{S} = R_0[[u]]$ as in Set-up 2.3 and Notation 2.6.

Let V be a crystalline \mathbf{Q}_p -representation of \mathcal{G}_R with Hodge–Tate weights in $[0, r]$, and let T be a \mathbf{Z}_p -lattice of V stable under \mathcal{G}_R -action. Let

$$\mathcal{M} := \mathcal{M}^\vee(T) = \text{Hom}_{\mathcal{G}_{\overline{R}_\infty}}(T, \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}})$$

be the associated étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module. In the following, we will construct a quasi-Kisin module over \mathfrak{S} of E -height $\leq r$ associated with T .

Consider the base change along the map $R \rightarrow \mathcal{O}_L$ as in Notation 2.8. If we consider T as a representation of $G_L := \mathcal{G}_{\mathcal{O}_L}$ via $G_L \rightarrow \mathcal{G}_R$, then T is a \mathbf{Z}_p -lattice in a crystalline G_L -representation with Hodge–Tate weights in $[0, r]$. Note that $\mathcal{M}_L := \mathcal{O}_{\mathcal{E},L} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{M} \cong \mathrm{Hom}_{G_{\tilde{L}_\infty}}(T, \widehat{\mathcal{O}}_{\mathcal{E},L}^{\mathrm{ur}})$ as étale $(\varphi, \mathcal{O}_{\mathcal{E},L})$ -modules.

Lemma 4.18. *There exists a unique \mathfrak{S}_L -submodule \mathfrak{M}_L of \mathcal{M}_L stable under Frobenius such that the following properties hold.*

- \mathfrak{M}_L with the induced Frobenius is a quasi-Kisin module over \mathfrak{S}_L of E -height $\leq r$. Furthermore, \mathfrak{M}_L is free over \mathfrak{S}_L ;
- $\mathcal{O}_{\mathcal{E},L} \otimes_{\mathfrak{S}_L} \mathfrak{M}_L = \mathcal{M}_L$;
- if we let $M_L = \mathcal{O}_{L_0} \otimes_{\varphi, \mathcal{O}_{L_0}} \mathfrak{M}_L / u\mathfrak{M}_L$, then $M_L[p^{-1}] \cong D_{\mathrm{cris}}^\vee(V|_{G_L})$ compatibly with Frobenius and connection.

Proof. By [13, Cor. 4.18] and Lemma 4.15, there exists a quasi-Kisin module \mathfrak{N} over \mathfrak{S}_L of E -height $\leq r$ such that \mathfrak{N} is free over \mathfrak{S}_L and $\mathcal{O}_{L_0}[p^{-1}] \otimes_{\varphi, \mathcal{O}_{L_0}} \mathfrak{N} / u\mathfrak{N} \cong D_{\mathrm{cris}}^\vee(V|_{G_L})$ compatibly with Frobenius and connections. By [20, Lem. 4.2.9], $\mathfrak{M}_L := \mathfrak{N}[p^{-1}] \cap \mathcal{M}_L$ satisfies the required properties. The uniqueness also follows from the cited lemma. \square

Construction 4.19. Let T be a crystalline \mathbf{Z}_p -representation of \mathcal{G}_R with Hodge–Tate weights in $[0, r]$ and keep the notation as above. We set

$$\mathfrak{M} := \mathfrak{M}(T) := \mathfrak{M}_L \cap \mathcal{M} \subset \mathcal{M}_L.$$

This is an \mathfrak{S} -module. Moreover, since \mathfrak{M}_L and \mathcal{M} are p -adically complete and torsion free, so is \mathfrak{M} .

We will show that \mathfrak{M} is a quasi-Kisin module over \mathfrak{S} of E -height $\leq r$ satisfying $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$ and $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathfrak{M}_L$.

Proposition 4.20. *The \mathfrak{S} -module \mathfrak{M} is finitely generated. Furthermore, we have $\mathfrak{M}[u^{-1}] \cap \mathfrak{M}[p^{-1}] = \mathfrak{M}$.*

Proof. Note that the cokernels of the maps $\mathfrak{S}_L \rightarrow \mathcal{O}_{\mathcal{E},L}$ and $\mathcal{O}_{\mathcal{E}} \rightarrow \mathcal{O}_{\mathcal{E},L}$ are p -torsion free. So the maps $\mathfrak{M}_L/p\mathfrak{M}_L \rightarrow \mathcal{M}_L/p\mathcal{M}_L$ and $\mathcal{M}/p\mathcal{M} \rightarrow \mathcal{M}_L/p\mathcal{M}_L$ are injective. By the proof of [33, Lem. 4.1], the intersection $\mathfrak{M}_L/p\mathfrak{M}_L \cap \mathcal{M}/p\mathcal{M}$ inside $\mathcal{M}_L/p\mathcal{M}_L$ is finite over \mathfrak{S} . To show that \mathfrak{M} is finite over \mathfrak{S} , it suffices to prove that the natural map $\mathfrak{M}/p\mathfrak{M} \rightarrow \mathfrak{M}_L/p\mathfrak{M}_L \cap \mathcal{M}/p\mathcal{M}$ is injective since \mathfrak{S} is noetherian and \mathfrak{M} is p -adically complete.

We have

$$p\mathfrak{M}_L \cap \mathfrak{M} = p\mathfrak{M}_L \cap \mathcal{M} \subset p\mathcal{M}_L \cap \mathcal{M}.$$

Since $p\mathcal{O}_{\mathcal{E},L} \cap \mathcal{O}_{\mathcal{E}} = p\mathcal{O}_{\mathcal{E}}$ and \mathcal{M} is classically flat over $\mathcal{O}_{\mathcal{E}}$, we have $p\mathcal{M}_L \cap \mathcal{M} \subset p\mathcal{M}$. Thus,

$$p\mathfrak{M}_L \cap \mathfrak{M} \subset p\mathfrak{M}_L \cap p\mathcal{M} = p\mathfrak{M},$$

where the last equality follows from the p -torsion freeness of \mathcal{M}_L . Thus the map $\mathfrak{M}/p\mathfrak{M} \rightarrow \mathfrak{M}_L/p\mathfrak{M}_L$ is injective. Since this map factors as $\mathfrak{M}/p\mathfrak{M} \rightarrow \mathfrak{M}_L/p\mathfrak{M}_L \cap \mathcal{M}/p\mathcal{M} \rightarrow \mathfrak{M}_L/p\mathfrak{M}_L$, we deduce the desired injectivity.

For the second part, since \mathfrak{M} is torsion free, we have $\mathfrak{M} \subset \mathfrak{M}[u^{-1}] \cap \mathfrak{M}[p^{-1}]$. On the other hand,

$$\mathfrak{M}[u^{-1}] \cap \mathfrak{M}[p^{-1}] \subset \mathfrak{M}_L[u^{-1}] \cap \mathfrak{M}_L[p^{-1}] = \mathfrak{M}_L,$$

and thus

$$\mathfrak{M}[u^{-1}] \cap \mathfrak{M}[p^{-1}] \subset \mathfrak{M}[u^{-1}] \cap \mathfrak{M}_L \subset \mathcal{M} \cap \mathfrak{M}_L = \mathfrak{M}.$$

□

Since the Frobenius endomorphisms on \mathcal{M} and \mathfrak{M}_L are compatible with that on $\mathcal{M}_{\mathcal{O}_L}$, we have an induced Frobenius $\varphi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}$.

Proposition 4.21. *The \mathfrak{S} -module \mathfrak{M} with Frobenius has E -height $\leq r$.*

Proof. Since the composite of maps $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{M} \xrightarrow{1 \otimes \varphi} \mathcal{M}$ is injective, $1 \otimes \varphi: \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$ is injective. Consider the natural map $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$. Since $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$ are p -adically complete by Lemma 2.7 and since the induced map $\mathfrak{S}/(p) \otimes_{\varphi, \mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathcal{O}_{\mathcal{E}}/(p) \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism, the map $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. Thus, the map

$$\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{M} \rightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{M}$$

is an isomorphism. On the other hand, since R_0/pR_0 has a finite p -basis which is also a p -basis of $\mathcal{O}_{L_0}/p\mathcal{O}_{L_0}$, the natural map $\mathfrak{S}/(p) \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}_L \rightarrow \mathfrak{S}_L/(p) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{S}_L$ is an isomorphism. Since $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}_L$ and $\mathfrak{S}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{S}_L$ are p -adically complete, the map $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}_L \rightarrow \mathfrak{S}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{S}_L$ is an isomorphism. Hence

$$\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_L \rightarrow \mathfrak{S}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$$

is an isomorphism.

Now, let $x \in \mathfrak{M}$. There exists a unique $y_1 \in \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{M} = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{M}$ such that $(1 \otimes \varphi)(y_1) = E(u)^r x$. On the other hand, there exists a unique $y_2 \in \mathfrak{S}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_L$ such that $(1 \otimes \varphi)(y_2) = E(u)^r x$. Hence, we have

$$y_1 = y_2 \in (\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{M}) \cap (\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_L) = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$$

by Lemma 3.1 since $\varphi: \mathfrak{S} \rightarrow \mathfrak{S}$ is classically flat. □

Next we will show that \mathfrak{M} satisfies $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$ and $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathfrak{M}_L$. For this, we consider another description of \mathfrak{M} as an inverse limit of p -power torsion \mathfrak{S} -modules as follows. Let

$$\mathfrak{M}_{L,n} := \mathfrak{M}_L/p^n \mathfrak{M}_L, \quad \mathcal{M}_n := \mathcal{M}/p^n \mathcal{M}, \quad \text{and} \quad \mathcal{M}_{L,n} := \mathcal{M}_L/p^n \mathcal{M}_L.$$

Then $\mathfrak{M}_{L,n}$ and \mathcal{M}_n are submodules of $\mathcal{M}_{L,n}$, and we set

$$\mathfrak{M}_{(n)} := \mathfrak{M}_{L,n} \cap \mathcal{M}_n \subset \mathcal{M}_{L,n}.$$

For any positive integers $i > j$, let $q_{i,j}$ denote the natural projection $\mathcal{M}_i \rightarrow \mathcal{M}_j$ given by modulo p^j , as well as its restriction $q_{i,j}: \mathfrak{M}_{(i)} \rightarrow \mathfrak{M}_{(j)}$. We have the commutative

diagram

$$\begin{array}{ccccc} \ker(q_{i,j}) & \hookrightarrow & \mathfrak{M}_{(i)} & \xrightarrow{q_{i,j}} & \mathfrak{M}_{(j)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{i-j} & \hookrightarrow & \mathcal{M}_i & \xrightarrow{q_{i,j}} & \mathcal{M}_j. \end{array}$$

Lemma 4.22. *We have a natural isomorphism*

$$\mathfrak{M} \cong \varprojlim_n \mathfrak{M}_{(n)}.$$

Proof. Recall that \mathfrak{M} is p -adically complete. By a similar argument as in the proof of Proposition 4.20, the natural map $\mathfrak{M}/p^n\mathfrak{M} \rightarrow \mathfrak{M}_{(n)}$ is injective for each $n \geq 1$. So the induced map

$$f: \mathfrak{M} \xrightarrow{\cong} \varprojlim_n \mathfrak{M}/p^n\mathfrak{M} \rightarrow \varprojlim_n \mathfrak{M}_{(n)}$$

is injective. On the other hand, let $x = (x_n)_{n \geq 1} \in \varprojlim_n \mathfrak{M}_{(n)}$. Note that x_n lies in both $\mathfrak{M}_{L,n}$ and \mathcal{M}_n as an element in $\mathcal{M}_{L,n}$. Thus,

$$x \in \left(\varprojlim_n \mathfrak{M}_{L,n} \right) \cap \left(\varprojlim_n \mathcal{M}_n \right) \subset \varprojlim_n \mathcal{M}_{L,n},$$

i.e., $x \in \mathfrak{M}_L \cap \mathcal{M} \subset \mathcal{M}_L$. This implies that x lies in the image of f , and thus f is surjective. \square

Remark 4.23. Suppose $r = 1$ and $e < p - 1$. By the above lemma and [33, Prop. 4.3, 4.5], \mathfrak{M} is projective over \mathfrak{S} . For general $r \geq 0$, as noted in [33, Rem 4.6], the \mathfrak{S} -module \mathfrak{M} is projective when $er < p - 1$. In particular, when $r = 0$, \mathfrak{M} is projective for any e .

Proposition 4.24. *The following properties hold for $\mathfrak{M}_{(n)}$:*

- (1) $\mathfrak{M}_{(n)}$ is a finitely generated \mathfrak{S} -module;
- (2) $\mathfrak{M}_{(n)}[u^{-1}] \cong \mathcal{M}_n$ and $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}_{(n)} \cong \mathfrak{M}_{L,n}$;
- (3) $\mathfrak{M}_{(n)}$ has E -height $\leq r$.

Proof. Since the composite of maps $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{(n)} \rightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{M}_n \xrightarrow{1 \otimes \varphi} \mathcal{M}_n$ is injective, $1 \otimes \varphi: \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_{(n)} \rightarrow \mathfrak{M}_{(n)}$ is injective. Thus, all statements follow from the same argument as in the proofs of [33, Lem. 4.1, 4.2] (where the case $r = 1$ is studied). \square

Consider the set \mathcal{A}_n consisting of \mathfrak{S} -submodules \mathfrak{N} of $\mathfrak{M}_{(n)}$ that are stable under φ , have E -height $\leq r$, and satisfy $\mathfrak{N}[u^{-1}] = \mathcal{M}_n$. Let

$$\mathfrak{M}_{(n)}^{\circ} := \bigcap_{\mathfrak{N} \in \mathcal{A}_n} \mathfrak{N} \subset \mathfrak{M}_{(n)}.$$

Lemma 4.25. *The following properties hold for $\mathfrak{M}_{(n)}^{\circ}$:*

- (1) $\mathfrak{M}_{(n)}^{\circ} \in \mathcal{A}_n$;
- (2) $\mathfrak{M}_{(n)}^{\circ} \subset q_{n+1,n}(\mathfrak{M}_{(n+1)}^{\circ})$.

Proof. (1) Let $e := [K : K_0]$ be the ramification index. We first show that for each fixed n , there exists an integer $s = s(n)$ such that $u^s \mathfrak{M}_{(n)} \subset \mathfrak{N} \subset \mathfrak{M}_{(n)}$ for all $\mathfrak{N} \in \mathcal{A}_n$. Choose an integer $a = a(n) \geq r$ such that $E(u)^a \equiv u^{ea} \pmod{p^n}$. Let $\mathfrak{N} \in \mathcal{A}_n$ and $\mathcal{L} := \mathfrak{M}_{(n)}/\mathfrak{N}$. Without loss of generality, assume $\mathcal{L} \neq 0$. Note that $\mathfrak{M}_{(n)}$ and \mathfrak{N} have E -height $\leq r$ and thus E -height $\leq a$. Hence we have unique \mathfrak{S} -linear maps $\psi_{\mathfrak{M}_{(n)}} : \mathfrak{M}_{(n)} \rightarrow \varphi^* \mathfrak{M}_{(n)}$ and $\psi_{\mathfrak{N}} : \mathfrak{N} \rightarrow \varphi^* \mathfrak{N}$ such that $\psi_{\mathfrak{N}} \circ (1 \otimes \varphi_{\mathfrak{N}}) = u^{ea} \text{Id}_{\varphi^* \mathfrak{N}}$ and $\psi_{\mathfrak{M}_{(n)}} \circ (1 \otimes \varphi_{\mathfrak{M}_{(n)}}) = u^{ea} \text{Id}_{\varphi^* \mathfrak{M}_{(n)}}$.

The exact sequence $0 \rightarrow \mathfrak{N} \rightarrow \mathfrak{M}_{(n)} \rightarrow \mathcal{L} \rightarrow 0$ induces the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \varphi^* \mathfrak{N} & \longrightarrow & \varphi^* \mathfrak{M}_{(n)} & \longrightarrow & \varphi^* \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow 1 \otimes \varphi_{\mathfrak{N}} & & \downarrow 1 \otimes \varphi_{\mathfrak{M}_{(n)}} & & \downarrow 1 \otimes \varphi_{\mathcal{L}} & & \\
 0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M}_{(n)} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow \psi_{\mathfrak{N}} & & \downarrow \psi_{\mathfrak{M}_{(n)}} & & \downarrow \psi_{\mathcal{L}} & & \\
 0 & \longrightarrow & \varphi^* \mathfrak{N} & \longrightarrow & \varphi^* \mathfrak{M}_{(n)} & \longrightarrow & \varphi^* \mathcal{L} & \longrightarrow & 0.
 \end{array}$$

Here, $1 \otimes \varphi_{\mathcal{L}}$ and $\psi_{\mathcal{L}}$ are the maps induced by $1 \otimes \varphi_{\mathfrak{M}_{(n)}}$ and $\psi_{\mathfrak{M}_{(n)}}$ respectively. We have $\psi_{\mathcal{L}} \circ (1 \otimes \varphi_{\mathcal{L}}) = u^{ea} \text{Id}_{\varphi^* \mathcal{L}}$.

We will show that $u^s \mathcal{L} = 0$ for $s = \lceil \frac{ea+p}{p-1} \rceil$. Since $\mathfrak{N}[u^{-1}] = \mathcal{M}_n = \mathfrak{M}_{(n)}[u^{-1}]$, \mathcal{L} is killed by some u -power. Take an integer $l \geq 1$ such that $u^l \mathcal{L} = 0$ and $u^{l-1} \mathcal{L} \neq 0$. Pick $x \in \mathcal{L}$ so that $u^{l-1} x \neq 0$. Set $y = 1 \otimes x \in \varphi^* \mathcal{L}$. Then $u^{pl} y = 1 \otimes u^l x = 0$ but $u^{p(l-1)} y = 1 \otimes (u^{l-1} x) \neq 0$, since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is classically faithfully flat by Lemma 2.7. Let $z = (1 \otimes \varphi_{\mathcal{L}})(y) \in \mathcal{L}$. Since $u^l \mathcal{L} = 0$, we have $u^l z = 0$ and thus

$$0 = \psi_{\mathcal{L}}(u^l z) = u^l (\psi_{\mathcal{L}} \circ (1 \otimes \varphi_{\mathcal{L}}))(y) = u^{ea+l} y.$$

So $ea + l > p(l-1)$, i.e., $l < \frac{ea+p}{p-1}$. Hence $u^s \mathcal{L} = 0$ for $s = \lceil \frac{ea+p}{p-1} \rceil$. This implies that $u^s \mathfrak{M}_{(n)} \subset \mathfrak{M}_{(n)}^\circ \subset \mathfrak{M}_{(n)}$, and $\mathfrak{M}_{(n)}^\circ[u^{-1}] = \mathcal{M}_n$.

It remains to show that $\mathfrak{M}_{(n)}^\circ$ has E -height $\leq r$. Let $x \in \mathfrak{M}_{(n)}^\circ$. We need to show there exists $y \in \varphi^* \mathfrak{M}_{(n)}^\circ$ such that $(1 \otimes \varphi)(y) = E(u)^r x$. For each $\mathfrak{N} \in \mathcal{A}_n$, we have $x \in \mathfrak{N}$, and there exists $y \in \varphi^* \mathfrak{N}$, which is unique as an element of $\varphi^* \mathcal{M}_n$, such that $(1 \otimes \varphi)(y) = E(u)^r x$. Since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is finite free by Lemma 2.7, we deduce

$$y \in \bigcap_{\mathfrak{N} \in \mathcal{A}_n} (\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}) = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \left(\bigcap_{\mathfrak{N} \in \mathcal{A}_n} \mathfrak{N} \right) = \varphi^* \mathfrak{M}_{(n)}^\circ.$$

(2) Since $\mathfrak{M}_{(n+1)}^\circ[u^{-1}] = \mathcal{M}_{n+1}$ and $q_{n+1,n}(\mathcal{M}_{n+1}) = \mathcal{M}_n$, we have $q_{n+1,n}(\mathfrak{M}_{(n+1)}^\circ)[u^{-1}] = \mathcal{M}_n$. So it suffices to show that $q_{n+1,n}(\mathfrak{M}_{(n+1)}^\circ)$ has E -height $\leq r$. Let $\mathfrak{K} = \text{Ker}(q_{n+1,n})$.

We have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \varphi^* \mathfrak{K} & \longrightarrow & \varphi^* \mathfrak{M}_{(n+1)}^\circ & \longrightarrow & \varphi^* q_{n+1,n}(\mathfrak{M}_{(n+1)}^\circ) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow 1 \otimes \varphi & & \downarrow 1 \otimes \varphi & & \\
 0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \mathfrak{M}_{(n+1)}^\circ & \xrightarrow{q_{n+1,n}} & q_{n+1,n}(\mathfrak{M}_{(n+1)}^\circ) & \longrightarrow & 0
 \end{array}$$

Since $q_{n+1,n}(\mathfrak{M}_{(n+1)}^\circ) \subset \mathcal{M}_n$, the rightmost vertical map is injective. From the first part, $1 \otimes \varphi$ in the middle column has cokernel killed by $E(u)^r$. Hence the cokernel of $1 \otimes \varphi$ in the rightmost column is killed by $E(u)^r$. \square

Proposition 4.26. *The natural maps*

$$\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow \mathcal{M} \quad \text{and} \quad \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}_L.$$

are isomorphisms.

Proof. We first prove $\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$. Since $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_\mathcal{E}$ and \mathcal{M} are p -adically complete and \mathcal{M} is p -torsion free, it suffices to show that the induced map

$$f: \mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}/p\mathfrak{M} \cong (\mathfrak{M}/p\mathfrak{M})[u^{-1}] \rightarrow \mathcal{M}_1$$

is an isomorphism. It is shown in the proof of Proposition 4.20 that $\mathfrak{M}/p\mathfrak{M} \rightarrow \mathfrak{M}_{(1)}$ is injective. Hence f is injective. By Lemma 4.25, we have $\mathfrak{M}/p\mathfrak{M} \supset \mathfrak{M}_{(1)}^\circ$ and thus f is surjective.

For the second isomorphism, note that $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}/p\mathfrak{M} \rightarrow \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}_{(1)} \cong \mathfrak{M}_{L,1}$ is injective since $\mathfrak{S} \rightarrow \mathfrak{S}_L$ is classically flat. Since $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$ is p -adically complete and \mathfrak{M}_L is p -torsion free, $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}_L$ is injective. In particular, $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$ is a finite torsion free \mathfrak{S}_L -module. Furthermore, since $\mathfrak{S} \rightarrow \mathfrak{S}_L$ is classically flat, we have

$$(\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M})[u^{-1}] \cap (\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M})[p^{-1}] = \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_L$$

by Lemma 3.1 and Proposition 4.20. Thus, $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$ is a finite free \mathfrak{S}_L -module.

Since \mathfrak{M} has E -height $\leq r$ by Proposition 4.21, $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$ with the induced Frobenius has E -height $\leq r$. We have

$$\begin{aligned} \mathcal{O}_{\mathcal{E},L} \otimes_{\mathfrak{S}_L} (\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}) &\cong \mathcal{O}_{\mathcal{E},L} \otimes_{\mathcal{O}_\mathcal{E}} (\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}) \\ &\cong \mathcal{O}_{\mathcal{E},L} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{M} \cong \mathcal{O}_{\mathcal{E},L} \otimes_{\mathfrak{S}_L} \mathfrak{M}_L. \end{aligned}$$

Hence, we deduce $\mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M} \cong \mathfrak{M}_L$ by [20, Prop. 4.2.5, 4.2.7]. \square

4.5. Construction of the quasi-Kisin module II: definition of ∇ . We further suppose that either R is small over \mathcal{O}_K or $R = \mathcal{O}_L$ (Assumption 2.9).

Let $M = R_0 \otimes_{\varphi, R_0} \mathfrak{M}/u\mathfrak{M}$. The R_0 -module M is equipped with the induced tensor-product Frobenius. We will construct a natural φ -equivariant isomorphism $M[p^{-1}] \cong D_{\text{cris}}^\vee(V)$, via which we define ∇ on $M[p^{-1}]$. Consider the φ -equivariant map $R_0 \rightarrow W(k_g)$ as in Notation 2.8, which naturally factors as $R_0 \rightarrow \mathcal{O}_{L_0} \rightarrow W(k_g)$.

Lemma 4.27. *The natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant map*

$$\text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\text{ur}}) \rightarrow T^\vee(\mathcal{M}) = \text{Hom}_{\mathcal{O}_\mathcal{E}, \varphi}(\mathcal{M}, \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}) \cong T$$

is an isomorphism.

Proof. This follows from Lemma 3.31 and a similar argument as in the proof of [35, Lem. 4.6]. \square

Proposition 4.28. *Suppose that R satisfies Assumption 2.9. There exists a natural φ -compatible isomorphism $M[p^{-1}] \xrightarrow{\cong} D_{\text{cris}}^\vee(V)$.*

Proof. Let $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ equipped with the induced Frobenius. Consider the φ -compatible projection $S \twoheadrightarrow R_0$ given by $u \mapsto 0$. This induces the projection $q: \mathcal{M} \twoheadrightarrow M$. By Lemma 4.2, Propositions 4.13 and 4.21, the projection q admits a unique φ -compatible section $s: M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$, and $1 \otimes s: S[p^{-1}] \otimes_{R_0[p^{-1}]} M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$ is an isomorphism.

We first construct a φ -equivariant map $M[p^{-1}] \rightarrow D_{\text{cris}}^{\vee}(V)$ similarly as in § 4.3. By Lemma 4.27, we have a natural map $\mathfrak{M} \rightarrow \widehat{\mathfrak{S}}^{\text{ur}} \otimes_{\mathbf{Q}_p} V^{\vee}$. This extends to a map $\mathcal{M} \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V^{\vee}$ via the embedding $\widehat{\mathfrak{S}}^{\text{ur}} \xrightarrow{\varphi} \mathbf{B}_{\text{cris}}(\overline{R}) \hookrightarrow \mathbf{OB}_{\text{cris}}(\overline{R})$. So by composing with the section $s: M[p^{-1}] \rightarrow \mathcal{M}[p^{-1}]$ given in Lemma 4.2, we get a φ -compatible map

$$(4.5) \quad M[p^{-1}] \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V^{\vee}.$$

By Lemma 2.11, we have the projection $\text{pr}: \mathbf{OB}_{\text{cris}}(\overline{R}) \rightarrow \mathbf{B}_{\text{cris}}(\overline{R})$ given by $T_i \otimes 1 - 1 \otimes [T_i^{\flat}] \mapsto 0$. This induces the projection

$$\text{pr}: \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\iota_1, R_0} D_{\text{cris}}^{\vee}(V) \rightarrow \mathbf{B}_{\text{cris}}(\overline{R}) \otimes_{\iota_2, R_0} D_{\text{cris}}^{\vee}(V),$$

where $\iota_1: R_0 \rightarrow \mathbf{OB}_{\text{cris}}(\overline{R})$ is the natural map given by $T_i \mapsto T_i \otimes 1$ and $\iota_2: R_0 \rightarrow \mathbf{B}_{\text{cris}}(\overline{R})$ is the embedding given by $T_i \mapsto [T_i^{\flat}]$.

If we compose the map (4.5) with

$$\mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V^{\vee} \xrightarrow{\alpha_{\text{cris}}^{-1}} \mathbf{OB}_{\text{cris}}(\overline{R}) \otimes_{\iota_1, R_0} D_{\text{cris}}^{\vee}(V) \xrightarrow{\text{pr}} \mathbf{B}_{\text{cris}}(\overline{R}) \otimes_{\iota_2, R_0} D_{\text{cris}}^{\vee}(V),$$

then we obtain a φ -equivariant map

$$f: M[p^{-1}] \rightarrow \mathbf{B}_{\text{cris}}(\overline{R}) \otimes_{\iota_2, R_0} D_{\text{cris}}^{\vee}(V).$$

By Proposition 4.17, we have

$$f(M[p^{-1}]) \subset D_{\text{cris}}^{\vee}(V|_{G_L}) = L_0 \otimes_{R_0} D_{\text{cris}}^{\vee}(V) \subset \mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}}) \otimes_{\iota_2, R_0} D_{\text{cris}}^{\vee}(V).$$

We claim that

$$\mathbf{B}_{\text{cris}}(\overline{R}) \cap L_0 = R_0[p^{-1}]$$

as subrings of $\mathbf{B}_{\text{cris}}(\mathcal{O}_{\overline{L}})$. Since $\mathcal{O}_{L_0} \subset \mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}})$, it suffices to show

$$\mathbf{A}_{\text{cris}}(\overline{R}) \cap \mathcal{O}_{L_0} = R_0.$$

We clearly have $R_0 \subset \mathbf{A}_{\text{cris}}(\overline{R}) \cap \mathcal{O}_{L_0}$. Let $x \in \mathbf{A}_{\text{cris}}(\overline{R}) \cap \mathcal{O}_{L_0}$. Then

$$\theta(x) \in \overline{R}^{\wedge} \cap \mathcal{O}_{L_0} = R_0 \subset \mathcal{O}_{\overline{L}}^{\wedge},$$

which implies $x \in R_0$. This shows the claim.

Hence, we have by Lemma 3.1 that

$$f(M[p^{-1}]) \subset (\mathbf{B}_{\text{cris}}(\overline{R}) \otimes_{R_0[p^{-1}]} D_{\text{cris}}^{\vee}(V)) \cap (L_0 \otimes_{R_0[p^{-1}]} D_{\text{cris}}^{\vee}(V)) = D_{\text{cris}}^{\vee}(V),$$

since $D_{\text{cris}}^{\vee}(V)$ is projective over $R_0[p^{-1}]$. This gives a natural φ -equivariant map $f: M[p^{-1}] \rightarrow D_{\text{cris}}^{\vee}(V)$.

It remains to show that f is an isomorphism. Note that by Proposition 4.17, it suffices to consider the case where R_0 is the p -adic completion of an étale extension

of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$. By Propositions 4.13 and 4.26, $M[p^{-1}]$ is projective over $R_0[p^{-1}]$ of rank equal to $\text{rank } D_{\text{cris}}^{\vee}(V)$. Moreover, by Proposition 4.17, the map

$$L_0 \otimes_{R_0} M[p^{-1}] \rightarrow D_{\text{cris}}^{\vee}(V|_{G_L}) = L_0 \otimes_{R_0} D_{\text{cris}}^{\vee}(V)$$

induced by f is an isomorphism. In particular, $f: M[p^{-1}] \rightarrow D_{\text{cris}}^{\vee}(V)$ is injective. Let $I \subset R_0[p^{-1}]$ be the invertible ideal given by the determinant of f . Since $1 \otimes \varphi: \varphi^* M[p^{-1}] \rightarrow M[p^{-1}]$ and $1 \otimes \varphi: \varphi^* D_{\text{cris}}^{\vee}(V) \rightarrow D_{\text{cris}}^{\vee}(V)$ are isomorphisms, we have

$$\varphi(I)R_0[p^{-1}] = IR_0[p^{-1}].$$

So $I = R_0[p^{-1}]$ by Proposition 4.30 (which is based on Lemma 4.29) below, and f is an isomorphism. \square

Lemma 4.29. *Let k_1 be a perfect field of characteristic p , and let $A = W(k_1)[[t_1, \dots, t_d]]$ be a power-series ring. Suppose that A is equipped with a Frobenius endomorphism φ extending the Witt vector Frobenius on $W(k_1)$. Let $I \subset A[p^{-1}]$ be an invertible ideal such that $IA[p^{-1}] = \varphi(I)A[p^{-1}]$. Then $I = A[p^{-1}]$.*

Proof. Suppose $I \neq A[p^{-1}]$. Since A is a UFD, so is $A[p^{-1}]$. Hence I is principal, say, generated by x . Since p is a prime element of A , we may choose x so that $x \in A \setminus pA$ and x is not a unit in A . Write $\varphi(x) = x^p + py$ for some $y \in A$. Since $p \nmid x$, we deduce from $\varphi(I)A[p^{-1}] = IA[p^{-1}]$ that $y = xz$ for some $z \in A$. Thus,

$$\varphi(x) = x(x^{p-1} + pz).$$

$x^{p-1} + pz$ is not a unit in A since it lies in the ideal (x, p) which is contained in the maximal ideal of A . Note that $p \nmid (x^{p-1} + pz)$ as elements in A . Thus, $x^{p-1} + pz$ is not a unit in $A[p^{-1}]$, which contradicts $\varphi(I)A[p^{-1}] = IA[p^{-1}]$. Hence, $I = A[p^{-1}]$. \square

Proposition 4.30. *Suppose R_0 is the p -adic completion of an étale extension of $W(k)\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$. Let $I \subset R_0[p^{-1}]$ be an invertible ideal such that $\varphi(I)R_0[p^{-1}] = IR_0[p^{-1}]$. Then $I = R_0[p^{-1}]$.*

Proof. Let $\mathfrak{P} \subset R_0[p^{-1}]$ be any maximal ideal. Then the prime ideal $\mathfrak{q} = R_0 \cap \mathfrak{P}$ is maximal among the prime ideals of R_0 not containing p , and $\mathfrak{n} := \sqrt{\mathfrak{q} + pR_0}$ is a maximal ideal of R_0 . Let $(R_0)_{\mathfrak{n}}^{\wedge}$ be the \mathfrak{n} -adic completion of the localization $(R_0)_{\mathfrak{n}}$. As in the proof of Proposition 4.13, $(R_0)_{\mathfrak{n}}^{\wedge} \cong W(k_{\mathfrak{n}})[[t_1, \dots, t_d]]$ for some finite extension $k_{\mathfrak{n}}$ of k and t_1, \dots, t_d such that $W(k_{\mathfrak{n}}) \hookrightarrow (R_0)_{\mathfrak{n}}^{\wedge}$ is compatible with φ .

Since the natural map $R_0[p^{-1}] \rightarrow (R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}]$ is classically flat, we have $I(R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}] = (R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}]$ by Lemma 4.29. Let $\mathfrak{P}_{\mathfrak{n}} \subset (R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}]$ be a maximal ideal lying over $\mathfrak{P} \subset R_0[p^{-1}]$. Note that the natural map on localizations

$$(R_0[p^{-1}])_{\mathfrak{P}} \rightarrow ((R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}])_{\mathfrak{P}_{\mathfrak{n}}}$$

is classically faithfully flat. Since $I((R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}])_{\mathfrak{P}_{\mathfrak{n}}} = ((R_0)_{\mathfrak{n}}^{\wedge}[p^{-1}])_{\mathfrak{P}_{\mathfrak{n}}}$, we deduce that $I(R_0[p^{-1}])_{\mathfrak{P}} = (R_0[p^{-1}])_{\mathfrak{P}}$. This holds for any maximal ideal $\mathfrak{P} \subset R_0[p^{-1}]$, so $I = R_0[p^{-1}]$. \square

By Proposition 4.28, the connection on $D_{\text{cris}}^{\vee}(V)$ defines a connection

$$\nabla_{\mathfrak{M}}: M[p^{-1}] \rightarrow M[p^{-1}] \otimes_{R_0} \widehat{\Omega}_{R_0}.$$

Finally, we will show that $\nabla_{\mathfrak{M}}$ satisfies the S -Griffiths transversality. For this, we study the compatibility between two filtrations as in § 4.3. By Lemma 4.2 and Proposition 4.28, we have a natural φ -equivariant isomorphism

$$\mathcal{M}[p^{-1}] = S[p^{-1}] \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong S[p^{-1}] \otimes_{R_0} D_{\text{cris}}^{\vee}(V).$$

Let $\mathcal{D} := S[p^{-1}] \otimes_{R_0} D_{\text{cris}}^{\vee}(V)$, and identify $\mathcal{D} = \mathcal{M}[p^{-1}]$ via the above isomorphism. Let $N_u: \mathcal{D} \rightarrow \mathcal{D}$ be the R_0 -linear derivation given by $N_u = N_{u,S} \otimes 1$, and let $\nabla: \mathcal{D} \rightarrow \mathcal{D} \otimes_{R_0} \widehat{\Omega}_{R_0}$ be the connection given by $\nabla_S \otimes 1 + 1 \otimes \nabla_{D_{\text{cris}}^{\vee}(V)}$. We consider two filtrations on $\mathcal{D} = \mathcal{M}[p^{-1}]$. For the first filtration, set $\text{Fil}^0 \mathcal{D} = \mathcal{D}$, and inductively define for $i \geq 1$

$$\text{Fil}^i \mathcal{D} := \{x \in \mathcal{D} \mid N_u(x) \in \text{Fil}^{i-1} \mathcal{D}, q_{\pi}(x) \in \text{Fil}^i(R \otimes_{R_0} D_{\text{cris}}^{\vee}(V))\},$$

where $q_{\pi}: \mathcal{D} \rightarrow R \otimes_{R_0} D_{\text{cris}}^{\vee}(V)$ is the map given by $u \mapsto \pi$. For the second filtration, let

$$\text{F}^i \mathcal{M}[p^{-1}] := \{x \in S[p^{-1}] \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in (\text{Fil}^i S[p^{-1}]) \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

Lemma 4.31. *We have*

$$\text{F}^i \mathcal{M}[p^{-1}] = \text{Fil}^i \mathcal{D}.$$

Furthermore,

$$\nabla(\text{Fil}^i \mathcal{D}) \subset \text{Fil}^{i-1} \mathcal{D} \otimes_{R_0} \widehat{\Omega}_{R_0}.$$

In particular, ∇ satisfies the S -Griffiths transversality.

Proof. By Proposition 4.26, the first part follows from a similar argument as in the proof of Lemma 4.15 using the base change along $R_0 \rightarrow W(k_g)$. The second part on the Griffiths transversality follows from a similar argument as in the proof of [35, Lem. 4.2]. Note that $N_u(\text{Fil}^i \mathcal{D}) \subset \text{Fil}^{i-1} \mathcal{D}$ by definition, and it is straightforward to check $\partial_u(\text{Fil}^i \mathcal{D}) \subset \text{Fil}^{i-1} \mathcal{D}$ by induction. \square

Combining this with Propositions 4.13 and 4.21, we conclude the following.

Proposition 4.32. *Suppose that R satisfies Assumption 2.9. With the above structures, \mathfrak{M} is a quasi-Kisin module over \mathfrak{S} of E -height $\leq r$.*

4.6. Proof of the second part of Theorem 3.28. Throughout this subsection, we suppose that R satisfies Assumption 2.9.

Let V be a finite free \mathbf{Q}_p -representation of \mathcal{G}_R , which is crystalline with Hodge-Tate weights in $[0, r]$. Let $T \subset V$ be a \mathbf{Z}_p -lattice stable under the \mathcal{G}_R -action, and let \mathfrak{M} be the quasi-Kisin module over \mathfrak{S} of E -height $\leq r$ associated with T as in Construction 4.19 and Proposition 4.32.

By Propositions 4.6 and 4.9, we have a rational Kisin descent datum

$$f: \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}.$$

On the other hand, since T is a finite free \mathbf{Z}_p -representation of \mathcal{G}_R , [7, Cor. 3.8] (see also [34, Thm 3.2]) gives an isomorphism of $\mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge}$ -modules

$$f_1: \mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge} \otimes_{p_1, \mathcal{O}_{\mathfrak{E}}} \mathcal{M} \xrightarrow{\cong} \mathfrak{S}^{(2)}[E^{-1}]_p^{\wedge} \otimes_{p_2, \mathcal{O}_{\mathfrak{E}}} \mathcal{M}$$

satisfying the cocycle condition over $\mathfrak{S}^{(3)}[E^{-1}]_p^\wedge$. Here, by Proposition 4.26, $\mathcal{M} = \mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the étale φ -module finite projective over $\mathcal{O}_\mathcal{E}$ associated with T (contravariantly). Note $\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathfrak{S}} \mathfrak{M} = \mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_j, \mathcal{O}_\mathcal{E}} \mathcal{M}$ since $\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} = \mathcal{M}$.

Proposition 4.33. *Under the identification*

$$\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge[p^{-1}] \otimes_{p_j, \mathfrak{S}} \mathfrak{M} = \mathfrak{S}^{(2)}[E^{-1}]_p^\wedge[p^{-1}] \otimes_{p_j, \mathcal{O}_\mathcal{E}} \mathcal{M},$$

the maps $\text{id}_{\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge[p^{-1}]} \otimes_{\mathfrak{S}^{(2)}[p^{-1}]} f$ and $\text{id}_{\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge[p^{-1}]} \otimes_{\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge} f_1$ coincide.

To show Proposition 4.33, let us consider the base change along $R_0 \rightarrow \mathcal{O}_{L_0}$ as before. Recall that $(\mathfrak{S}_L, (E)) = (\mathcal{O}_{L_0}[[u]], (E))$ is a prism in R_Δ with the structure map $R \rightarrow \mathcal{O}_L = \mathfrak{S}_L/(E)$. Let $(\mathfrak{S}_L^{(2)}, (E))$ be the self-product of $(\mathfrak{S}_L, (E))$ in $(\mathcal{O}_L)_\Delta$. Considering $(\mathfrak{S}_L^{(2)}, (E))$ as a prism in R_Δ , the maps f and f_1 induce the descent data

$$f_L: \mathfrak{S}_L^{(2)}[p^{-1}] \otimes_{p_1, \mathfrak{S}_L} \mathfrak{M}_L \xrightarrow{\cong} \mathfrak{S}_L^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}_L} \mathfrak{M}_L$$

and

$$f_{1,L}: \mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge \otimes_{p_1, \mathfrak{S}_L} \mathfrak{M}_L \xrightarrow{\cong} \mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge \otimes_{p_2, \mathfrak{S}_L} \mathfrak{M}_L,$$

respectively. Here, $\mathfrak{M}_L = \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$ by Proposition 4.26. Since the map $\mathfrak{S}^{(2)} \rightarrow \mathfrak{S}_L^{(2)}$ is injective, Proposition 4.33 follows if we show that f_L and $f_{1,L}$ coincide over $\mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge[p^{-1}]$. For this, we need the following proposition.

Proposition 4.34. *There exists an \mathfrak{S}_L -submodule $\mathfrak{N}_L \subset \mathfrak{M}_L$ with $\mathfrak{N}_L[p^{-1}] = \mathfrak{M}_L[p^{-1}]$ such that f_L induces an isomorphism of $\mathfrak{S}_L^{(2)}$ -modules*

$$f_L: \mathfrak{S}_L^{(2)} \otimes_{p_1, \mathfrak{S}_L} \mathfrak{N}_L \xrightarrow{\cong} \mathfrak{S}_L^{(2)} \otimes_{p_2, \mathfrak{S}_L} \mathfrak{N}_L.$$

Furthermore, \mathfrak{N}_L can be chosen to be finite free over \mathfrak{S}_L of E -height $\leq r$ and stable under $\varphi_{\mathfrak{M}_L}$.

Proof. Since $p_i: \mathfrak{S}_L \rightarrow \mathfrak{S}_L^{(2)}$ is classically faithfully flat by Lemma 3.5, the first part follows directly from the proof of [16, Prop. 3.6.5]. We recall some points here. Note that for any \mathfrak{S}_L -submodule $\mathfrak{N}_L \subset \mathfrak{M}_L$, the induced map $p_i^* \mathfrak{N}_L \rightarrow p_i^* \mathfrak{M}_L$ for $i = 1, 2$ is injective, where $p_i^* \mathfrak{N}_L$ denotes $\mathfrak{S}_L^{(2)} \otimes_{p_i, \mathfrak{S}_L} \mathfrak{N}_L$. Take an integer $n \geq 0$ such that $p^n f_L$ maps $p_1^* \mathfrak{M}_L$ into $p_2^* \mathfrak{M}_L$. It suffices to find an \mathfrak{S}_L -submodule $\mathfrak{N}_L \subset \mathfrak{M}_L$ such that $p^n \mathfrak{M}_L \subset \mathfrak{N}_L$ and f_L maps $p_1^* \mathfrak{N}_L$ to $p_2^* \mathfrak{N}_L$. The map f_L induces a map

$$f_L: p_1^*(\mathfrak{M}_L/p^n \mathfrak{M}_L) \rightarrow p_2^*(p^{-n} \mathfrak{M}_L/\mathfrak{M}_L).$$

This induces a morphism

$$\beta: \mathfrak{M}_L/p^n \mathfrak{M}_L \rightarrow \Phi(p^{-n} \mathfrak{M}_L/\mathfrak{M}_L)$$

where $\Phi := (p_1)_* p_2^*$. Let \mathfrak{N}_L be the kernel of the composite

$$\mathfrak{M}_L \rightarrow \mathfrak{M}_L/p^n \mathfrak{M}_L \xrightarrow{\beta} \Phi(p^{-n} \mathfrak{M}_L/\mathfrak{M}_L).$$

Then $f_L(p_1^* \mathfrak{N}_L) \subset p_2^* \mathfrak{M}_L$, and by the proof of [16, Prop. 3.6.5] (cf. also the proof of [37, Thm. 1.9]), we further have $f_L(p_1^* \mathfrak{N}_L) \subset p_2^* \mathfrak{N}_L$.

Since f_L is compatible with Frobenius, β as above is compatible with φ . Thus, \mathfrak{N}_L constructed as above is stable under φ . Consider the exact sequence

$$0 \rightarrow \mathfrak{N}_L \rightarrow \mathfrak{M}_L \rightarrow \beta(\mathfrak{M}_L) \rightarrow 0$$

where $\beta(\mathfrak{M}_L) \subset \Phi(p^{-n}\mathfrak{M}_L/\mathfrak{M}_L)$ denotes the image of \mathfrak{M}_L under the above composite. Under the classically faithfully flat base change along $\mathfrak{S}_L \rightarrow \mathfrak{S}_g$, this induces an exact sequence

$$0 \rightarrow \mathfrak{S}_g \otimes_{\mathfrak{S}_L} \mathfrak{N}_L \rightarrow \mathfrak{S}_g \otimes_{\mathfrak{S}_L} \mathfrak{M}_L \rightarrow \mathfrak{S}_g \otimes_{\mathfrak{S}_L} \beta(\mathfrak{M}_L) \rightarrow 0.$$

Note that $\mathfrak{S}_g \otimes_{\mathfrak{S}_L} \mathfrak{M}_L$ is a Kisin module of E -height $\leq r$ that is finite free over \mathfrak{S}_g . Furthermore, $\beta(\mathfrak{M}_L)$ is u -torsion free since $p^{-n}\mathfrak{M}_L/\mathfrak{M}_L$ is finite free over $\mathfrak{S}_L/p^n\mathfrak{S}_L$, and so $\mathfrak{S}_g \otimes_{\mathfrak{S}_L} \beta(\mathfrak{M}_L)$ is u -torsion free. Thus, $\mathfrak{S}_g \otimes_{\mathfrak{S}_L} \beta(\mathfrak{M}_L)$ is a torsion Kisin module over \mathfrak{S}_g of E -height $\leq r$ by [31, Prop. 2.3.2]. Then $\mathfrak{S}_g \otimes_{\mathfrak{S}_L} \mathfrak{N}_L$ is a Kisin module of E -height $\leq r$ finite free over \mathfrak{S}_g by [31, Cor. 2.3.8]. Since $\mathfrak{S}_L \rightarrow \mathfrak{S}_g$ is classically faithfully flat, \mathfrak{N}_L is finite free over \mathfrak{S}_L and has E -height $\leq r$. \square

Proof of Proposition 4.33. By the above proposition, $\mathcal{N}_L := \mathcal{O}_{\varepsilon,L} \otimes_{\mathfrak{S}_L} \mathfrak{N}_L$ is an étale φ -module finite free over $\mathcal{O}_{\varepsilon,L}$, and f_L induces an isomorphism of $\mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge$ -modules

$$f_L : \mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge \otimes_{p_1, \mathcal{O}_{\varepsilon,L}} \mathcal{N}_L \xrightarrow{\cong} \mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge \otimes_{p_2, \mathcal{O}_{\varepsilon,L}} \mathcal{N}_L$$

satisfying the cocycle condition over $\mathfrak{S}_L^{(3)}[E^{-1}]_p^\wedge$. As in the proof of Proposition 3.26, this corresponds to a finite free \mathbf{Z}_p -representation T' of G_L . Furthermore, by [7, Cor. 3.7, Ex. 3.4], T' is determined by the G_L -action on $W(\mathcal{O}_{\overline{L}}^b[(\pi^b)^{-1}]) \otimes_{\mathcal{O}_{\varepsilon,L}} \mathcal{N}_L$. On the other hand, note that $(\mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}}), (p)) \in R_\Delta$ similarly as in Example 3.8, and the composite $S \rightarrow S_L \rightarrow \mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}})$ gives a map of prisms $(S, (p)) \rightarrow (\mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}}), (p))$ over R . Thus, by the construction of the descent datum f and definition of f_S in Construction 4.3, the G_L -action on $\mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{N}_L \cong \mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L$ (with $\mathcal{D}_L = S_L[p^{-1}] \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$) is given by

$$f_S(a \otimes x) = \sum \sigma(a) \partial_u^{j_0} \partial_{T_1}^{j_1} \cdots \partial_{T_d}^{j_d}(x) \cdot \gamma_{j_0}(\sigma([\pi^b]) - [\pi^b]) \prod_{i=1}^d \gamma_{j_i}(\sigma([T_i^b]) - [T_i^b]),$$

for $\sigma \in G_L$ and $a \otimes x \in \mathbf{A}_{\text{cris}}(\mathcal{O}_{\overline{L}})[p^{-1}] \otimes_{S_L} \mathcal{D}_L$ (where the sum goes over the multi-index (j_0, \dots, j_d) of non-negative integers). By [29, §8.1], this is the same as the G_L -action given by Equation (4.4), and it is proved in § 4.3 that this gives a \mathbf{Q}_p -representation of G_L isomorphic to $T[p^{-1}] = V$. Thus, $T[p^{-1}] \cong T'[p^{-1}]$ as representations of G_L . This proves the claim that f_L and $f_{1,L}$ coincide over $\mathfrak{S}_L^{(2)}[E^{-1}]_p^\wedge[p^{-1}]$ and thus Proposition 4.33. \square

By Proposition 4.33, we see that the descent data f and f_1 induce a map

$$f : \mathfrak{M} \rightarrow (\mathfrak{S}^{(2)}[p^{-1}] \otimes_{p_2, \mathfrak{S}} \mathfrak{M}) \cap (\mathfrak{S}^{(2)}[E^{-1}]_p^\wedge \otimes_{p_2, \mathfrak{S}} \mathfrak{M}).$$

End of the proof of Theorem 3.28 (2). By Lemma 4.10, we see that f and f_1 induce a map

$$f_{\text{int}} : \mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathfrak{M}.$$

By applying a similar argument to f^{-1} and f_1^{-1} , we deduce that f_{int} is an isomorphism. Namely, f_{int} is a descent datum. Since f is compatible with φ , so is f_{int} . By

Proposition 3.25, the triple $(\mathfrak{M}, \varphi, f_{\text{int}})$ gives rise to a completed prismatic F -crystal \mathcal{F} on R with $\mathfrak{M} = \mathcal{F}_{\mathfrak{S}}$. It is straightforward to see $T(\mathcal{F}) = T$. Hence the functor T in Theorem 3.28 is essentially surjective (when R satisfies Assumption 2.9). \square

Remark 4.35. We continue the discussion in Remark 3.30. For $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$, let $V = T(\mathcal{F})[p^{-1}] \in \text{Rep}_{\mathbf{Z}_p, \geq 0}^{\text{cris}}(\mathcal{G}_R)$. Consider the φ -equivalent $R_0[p^{-1}]$ -linear isomorphism

$$h: (R_0 \otimes_{\varphi, R_0} \mathcal{F}_{\mathfrak{S}}/u\mathcal{F}_{\mathfrak{S}})[p^{-1}] \cong D_{\text{cris}}^{\vee}(V)$$

in Remark 3.30. Note that $\mathcal{F}_S[p^{-1}] \cong S[p^{-1}] \otimes_{\varphi, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}$ by Lemma 3.23 (4) for the map of prisms $(\mathfrak{S}, E) \rightarrow (S, (p))$. So h induces a φ -compatible isomorphism $\mathcal{F}_S[p^{-1}] \cong S[p^{-1}] \otimes_{R_0} D_{\text{cris}}^{\vee}(V)$ by Lemma 4.2.

Since $\mathcal{F} \in \text{CR}^{\wedge, \varphi}(R_{\Delta})$, we have an isomorphism of $\mathfrak{S}^{(2)}$ -modules

$$\mathfrak{S}^{(2)} \otimes_{p_1, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \xrightarrow{\cong} \mathfrak{S}^{(2)} \otimes_{p_2, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}}.$$

Under the map $\varphi: \mathfrak{S}^{(2)} \rightarrow S^{(2)}$, this induces an isomorphism of $S^{(2)}[p^{-1}]$ -modules

$$(4.6) \quad f_S: S^{(2)}[p^{-1}] \otimes_{p_1, S} \mathcal{F}_S \xrightarrow{\cong} S^{(2)}[p^{-1}] \otimes_{p_2, S} \mathcal{F}_S.$$

Note that f_S reduces to the identity after the base change along $S^{(2)} \rightarrow S$ and it satisfies the cocycle condition over $S^{(3)}$. Let $\nu: R_0 \otimes_W R_0 \rightarrow R_0$ be the natural projection, and let $R_0^{(2)}$ be the p -adically completed divided power envelope of $R_0 \otimes_W R_0$ with respect to $\text{Ker}(\nu)$. We also write $\nu: R_0^{(2)} \rightarrow R_0$ for the induced map. Consider the map $S^{(2)} \rightarrow R_0^{(2)}$ given by $u, y \mapsto 0$. From the isomorphism (4.6), we obtain an isomorphism of $R_0^{(2)}[p^{-1}]$ -modules

$$f_{R_0}: R_0^{(2)}[p^{-1}] \otimes_{p_1, R_0} D_{\text{cris}}^{\vee}(V) \xrightarrow{\cong} R_0^{(2)}[p^{-1}] \otimes_{p_2, R_0} D_{\text{cris}}^{\vee}(V)$$

such that f_{R_0} reduces to the identity after the base change along ν and it satisfies a similar cocycle condition. Since $\widehat{\Omega}_{R_0} \cong \text{Ker}(\nu)/(\text{Ker}(\nu))^{[2]}$ where $(\text{Ker}(\nu))^{[2]}$ denotes the divided square of $\text{Ker}(\nu)$, the isomorphism f_{R_0} gives an integrable connection $\nabla: D_{\text{cris}}^{\vee}(V) \rightarrow D_{\text{cris}}^{\vee}(V) \otimes_{R_0} \widehat{\Omega}_{R_0}$. On the other hand, we have the natural integrable connection on $D_{\text{cris}}^{\vee}(V)$ induced by that on $\mathbf{OB}_{\text{cris}}(\overline{R})$ (cf. § 2.2). In the proof of Theorem 3.28 (2) on essential surjectivity, the isomorphism (4.6) is obtained by Construction 4.3 using the natural connection on $D_{\text{cris}}^{\vee}(V)$ as in § 4.5. Thus, ∇ given above agrees with the natural connection on $D_{\text{cris}}^{\vee}(V)$.

Define the filtration on $\mathcal{F}_S[p^{-1}]$ by

$$\text{F}^i \mathcal{F}_S[p^{-1}] := \{x \in S[p^{-1}] \otimes_{\varphi, \mathfrak{S}} \mathcal{F}_{\mathfrak{S}} \mid (1 \otimes \varphi)(x) \in \text{Fil}^i S[p^{-1}] \otimes_{\mathfrak{S}} \mathcal{F}_{\mathfrak{S}}\}.$$

Let $\text{Fil}^i(R \otimes_{R_0} D_{\text{cris}}^{\vee}(V))$ be the quotient filtration given by $\text{F}^i \mathcal{F}_S[p^{-1}]$ under the map $q_{\pi}: S \rightarrow R, u \mapsto \pi$. By the proof of Theorem 3.28 (2), Lemma 4.31 and a similar argument as in the proof of [10, Prop. 6.2.2.3], this quotient filtration agrees with the natural filtration on $R \otimes_{R_0} D_{\text{cris}}^{\vee}(V)$ as in § 2.2. In this way, we can directly obtain the filtered (φ, ∇) -module $(D_{\text{cris}}^{\vee}(T(\mathcal{F})[p^{-1}]), \nabla, \text{Fil}^i(R \otimes_{R_0} D_{\text{cris}}^{\vee}(T(\mathcal{F})[p^{-1}]))$ from \mathcal{F} .

Corollary 4.36. *The étale realization functor gives an equivalence of categories from $\text{Vect}_{[0, r]}^{\varphi}((\mathcal{O}_L)_{\Delta})$ to $\text{Rep}_{\mathbf{Z}_p, [0, r]}^{\text{cris}}(G_L)$.*

Proof. By Remark 3.18, the category $\mathrm{CR}_{[0,r]}^{\wedge, \varphi}((\mathcal{O}_L)_{\Delta})$ is equal to $\mathrm{Vect}_{[0,r]}^{\varphi}((\mathcal{O}_L)_{\Delta})$. So the statement follows from Theorem 3.28. \square

Remark 4.37. As a corollary, we can deduce that the construction of Brinon–Trihan in [13] is independent of the choice of a uniformizer and the Kummer tower.

APPENDIX A. CRYSTALLINE LOCAL SYSTEMS

Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K and let X denote its adic generic fiber. In this appendix, we define the notion of crystalline local systems on X , which is used in § 3.6. The definition of crystalline local systems goes back to the work [18, V f)] of Faltings. Tan and Tong [43] also define crystalline local systems in the unramified case $\mathcal{O}_K = W$ and prove that their definition agrees with the one given by Faltings. Since we also work on the ramified case, we give a minimal foundation that generalizes part of the work of Tan and Tong.

For our purpose, we work in two steps: when there exists a smooth p -adic formal scheme \mathfrak{X}_0 over W such that $\mathfrak{X} \cong \mathfrak{X}_0 \otimes_W \mathcal{O}_K$, we define the pro-étale sheaf $\mathcal{O}_{\mathbb{B}_{\mathrm{cris}}}$ on X and use it to define crystalline local systems. Note that this assumption is satisfied Zariski locally, e.g., by considering a Zariski open covering consisting of small affines. In the general case, we define crystalline local systems via gluing.

Let $X_{\mathrm{pro\acute{e}t}}$ denote the pro-étale site defined in [40, §3] and [41]. It admits the morphism of site $\nu: X_{\mathrm{pro\acute{e}t}} \rightarrow X_{\acute{e}t}$.

Definition A.1. We introduce sheaves on $X_{\mathrm{pro\acute{e}t}}$.

(1) ([40, Def. 4.1, 5.10]). Set

$$\mathcal{O}_X^+ := \nu^{-1} \mathcal{O}_{X_{\acute{e}t}}^+, \quad \widehat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+ / p^n, \quad \widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}_X^+[p^{-1}], \quad \text{and} \quad \widehat{\mathcal{O}}_{X^b}^+ := \varprojlim_{\Phi: x \rightarrow x^p} \widehat{\mathcal{O}}_X^+ / p.$$

(2) ([40, Def. 6.1]). Set $\mathbb{A}_{\mathrm{inf}} := W(\widehat{\mathcal{O}}_{X^b}^+)$ and $\mathbb{B}_{\mathrm{inf}} := \mathbb{A}_{\mathrm{inf}}[p^{-1}]$. We have ring morphisms $\theta: \mathbb{A}_{\mathrm{inf}} \rightarrow \widehat{\mathcal{O}}_X^+$ and $\theta: \mathbb{B}_{\mathrm{inf}} \rightarrow \widehat{\mathcal{O}}_X$.

(3) ([43, Def. 2.1]). Let $\mathbb{A}_{\mathrm{cris}}^0$ be the PD-envelope of $\mathbb{A}_{\mathrm{inf}}$ with respect to the ideal sheaf $\mathrm{Ker} \theta$, and set $\mathbb{A}_{\mathrm{cris}} := \varprojlim \mathbb{A}_{\mathrm{cris}}^0 / p^n$. Note that the series $t := \log[\varepsilon]$ converges and is a nonzero-divisor in $\mathbb{A}_{\mathrm{cris}}|_{X_{\overline{K}}}$. See [43, (2A.6), Cor. 2.24].

Now assume that \mathfrak{X} admits a W -model, namely, there exists a smooth p -adic formal scheme \mathfrak{X}_0 over W such that $\mathfrak{X} \cong \mathfrak{X}_0 \otimes_W \mathcal{O}_K$. Let X_0 denote the adic generic fiber of \mathfrak{X}_0 . Hence we have a canonical identification $X \cong X_0 \times_{\mathrm{Spa}(W[p^{-1}], W)} \mathrm{Spa}(K, \mathcal{O}_K)$. In [43, § 2B], Tan and Tong defined the structural crystalline period sheaves $\mathcal{O}_{\mathbb{A}_{\mathrm{cris}}, X_0}$ and $\mathcal{O}_{\mathbb{B}_{\mathrm{cris}}, X_0}$ on $(X_0)_{\mathrm{pro\acute{e}t}}$. We define structural crystalline sheaves on $X_{\mathrm{pro\acute{e}t}}$ in a similar way.

Definition A.2 (cf. [43, § 2B]). Consider the morphisms of sites

$$w: X_{\mathrm{pro\acute{e}t}} \rightarrow X_{\acute{e}t} \rightarrow \mathfrak{X}_{\acute{e}t} \rightarrow (\mathfrak{X}_0)_{\acute{e}t}.$$

Define sheaves $\mathcal{O}_X^{\mathrm{ur}+}$ and $\mathcal{O}_X^{\mathrm{ur}}$ on $X_{\mathrm{pro\acute{e}t}}$ by

$$\mathcal{O}_X^{\mathrm{ur}+} := \mathcal{O}_X^{\mathrm{ur}/\mathfrak{X}_0^+} := w^{-1} \mathcal{O}_{(\mathfrak{X}_0)_{\acute{e}t}} \quad \text{and} \quad \mathcal{O}_X^{\mathrm{ur}} := \mathcal{O}_X^{\mathrm{ur}/\mathfrak{X}_0} := w^{-1} \mathcal{O}_{(\mathfrak{X}_0)_{\acute{e}t}}[p^{-1}].$$

Set $\mathcal{O}\mathbb{A}_{\text{inf}} := \mathcal{O}_X^{\text{ur}+} \otimes_{\mathbf{Z}} \mathbb{A}_{\text{inf}}$. By extending the scalars, we have an $\mathcal{O}_X^{\text{ur}+}$ -algebra morphism $\theta_X: \mathcal{O}\mathbb{A}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X^+$. Define $\mathcal{O}\mathbb{A}_{\text{cris}}$ to be the p -adic completion of the PD-envelope $\mathcal{O}\mathbb{A}_{\text{cris}}^0$ of $\mathcal{O}\mathbb{A}_{\text{inf}}$ with respect to the ideal sheaf $\text{Ker } \theta_X$. Note that $\mathcal{O}\mathbb{A}_{\text{cris}}$ is an \mathbb{A}_{cris} -algebra. Set

$$\mathcal{O}\mathbb{B}_{\text{cris}}^+ := \mathcal{O}\mathbb{A}_{\text{cris}}[p^{-1}] \quad \text{and} \quad \mathcal{O}\mathbb{B}_{\text{cris}} := \mathcal{O}\mathbb{B}_{\text{cris}}^+[t^{-1}].$$

Here the sheaf $\mathcal{O}\mathbb{B}_{\text{cris}}^+|_{X_{\overline{K}}}[t^{-1}]$ on $X_{\text{proét}/X_{\overline{K}}}$ naturally descends to a sheaf on $X_{\text{proét}}$ and $\mathcal{O}\mathbb{B}_{\text{cris}}^+[t^{-1}]$ denotes the corresponding sheaf. These sheaves are equipped with a decreasing filtration and a connection that satisfy the Griffiths transversality, which we omit to explain. See the remark below.

Remark A.3. The definitions of our sheaves $\mathcal{O}\mathbb{A}_{\text{inf}}$ and $\mathcal{O}\mathbb{A}_{\text{cris}}^0$ are slightly different from the ones given by Tan and Tong: we use $\otimes_{\mathbf{Z}}$ instead of \otimes_W to define $\mathcal{O}\mathbb{A}_{\text{inf}}$. However, our $\mathcal{O}\mathbb{A}_{\text{cris}}$ still coincides with theirs in the unramified case $\mathcal{O}_K = W$ since k is perfect. Hence it follows that $\mathcal{O}\mathbb{A}_{\text{cris}} \cong \mathcal{O}\mathbb{A}_{\text{cris}, X_0}|_{X_{\text{proét}}}$ and $\mathcal{O}\mathbb{B}_{\text{cris}} \cong \mathcal{O}\mathbb{B}_{\text{cris}, X_0}|_{X_{\text{proét}}}$. In particular, one can define the additional structures on $\mathcal{O}\mathbb{A}_{\text{cris}}$ and $\mathcal{O}\mathbb{B}_{\text{cris}}$ directly from [43].

Proposition A.4 (cf. [43, Cor. 2.19]). *Let $\mathfrak{U}_0 = \text{Spf } R_0 \in (\mathfrak{X}_0)_{\text{ét}}$ be affine such that R_0 is connected and small over W . With the notation as in § 2.1, set $R = R_0 \otimes_W \mathcal{O}_K$ and $U = \text{Spa}(R[p^{-1}], R)$, and let $\overline{U} \in X_{\text{proét}}$ denote the affinoid perfectoid corresponding to the pro-étale cover $(\overline{R}[p^{-1}], \overline{R})$ of $(R[p^{-1}], R)$. Then there is a natural isomorphism of $R_0 \otimes_W \mathbf{B}_{\text{cris}}(\overline{R})$ -modules*

$$\mathbf{OB}_{\text{cris}}(\overline{R}) \xrightarrow{\cong} \mathcal{O}\mathbb{B}_{\text{cris}}(\overline{U})$$

that is strictly compatible with filtrations. Moreover, for every $i > 0$ and $j \in \mathbf{Z}$, we have

$$H^i(\overline{U}, \mathcal{O}\mathbb{B}_{\text{cris}}) = H^i(\overline{U}, \text{Fil}^j \mathcal{O}\mathbb{B}_{\text{cris}}) = 0.$$

Proof. Note that we have natural identifications $\overline{R} = \overline{R}_0$ and $\mathbf{OB}_{\text{cris}}(\overline{R}) = \mathbf{OB}_{\text{cris}}(\overline{R}_0)$. Now the proposition is nothing but [43, Cor. 2.19] for \mathfrak{U}_0 and $\overline{U} \in (X_0)_{\text{proét}}$. Note that *loc. cit.* only claims that the map $\mathbf{OB}_{\text{cris}}(\overline{R}) \rightarrow \mathcal{O}\mathbb{B}_{\text{cris}}(\overline{U})$ is an isomorphism of $R_0 \otimes_W \mathbf{B}_{\text{cris}}(\overline{\mathcal{O}_K})$ -modules, but its proof together with [43, Cor. 2.8] shows that the map is indeed an isomorphism of $R_0 \otimes_W \mathbf{B}_{\text{cris}}(\overline{R})$ -modules. \square

Remark A.5. The modules $\mathbf{OB}_{\text{cris}}(\overline{R})$ and $\mathcal{O}\mathbb{B}_{\text{cris}}(\overline{U})$ admit an action of \mathcal{G}_R (or even \mathcal{G}_{R_0}). The above isomorphism is compatible with the Galois actions. It is also compatible with the restriction along any étale morphism $\text{Spf } R'_0 \rightarrow \text{Spf } R_0$.

Remark A.6. By the same argument, we also have a description of $\mathcal{O}\mathbb{A}_{\text{cris}}(\overline{U})$ similar to [43, Lem. 2.18].

We now explain the crystalline formalism. Let $\text{Loc}_{\mathbf{Z}_p}(X)$ (resp. $\text{ILoc}_{\mathbf{Z}_p}(X)$) denote the category of étale \mathbf{Z}_p -local systems (resp. étale isogeny \mathbf{Z}_p -local systems) on X . See [24, § 1.4, 8.4] for the precise formulation. By [40, Prop. 8.2], $\text{Loc}_{\mathbf{Z}_p}(X)$ is equivalent to the category of $\widehat{\mathbf{Z}}_p$ -local systems on $X_{\text{proét}}$. We also note that if $\mathfrak{X} = \text{Spf } R$

is connected and affine with $X = \mathrm{Spa}(R[p^{-1}], R)$, then there are equivalences of categories

$$\mathrm{Loc}_{\mathbf{Z}_p}(X) \cong \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{pr}}(\mathcal{G}_R) \quad \text{and} \quad \mathrm{ILoc}_{\mathbf{Z}_p}(X) \cong \mathrm{Rep}_{\mathbf{Q}_p}(\mathcal{G}_R).$$

Definition A.7 (cf. [43, Def. 3.12]). Keep the assumption on the existence of \mathfrak{X}_0 . For an étale isogeny \mathbf{Z}_p -local system \mathbb{L} on X with corresponding $\widehat{\mathbf{Q}}_p$ -local system $\widehat{\mathbb{L}}$ on $X_{\mathrm{pro\acute{e}t}}$, we set

$$D_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathcal{O}_{\mathrm{Bcris}} \otimes_{\widehat{\mathbf{Q}}_p} \widehat{\mathbb{L}}) \quad \text{and} \quad \mathrm{Fil}^i D_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathrm{Fil}^i \mathcal{O}_{\mathrm{Bcris}} \otimes_{\widehat{\mathbf{Q}}_p} \widehat{\mathbb{L}}).$$

Note that these are sheaves of $\mathcal{O}_{(\mathfrak{X}_0)_{\acute{e}t}}[p^{-1}]$ -modules.

We say that \mathbb{L} is *crystalline* (with respect to \mathfrak{X}_0) if

- (1) the $\mathcal{O}_{(\mathfrak{X}_0)_{\acute{e}t}}[p^{-1}]$ -modules $D_{\mathrm{cris}}(\mathbb{L})$ and $\mathrm{Fil}^i D_{\mathrm{cris}}(\mathbb{L})$ ($i \in \mathbf{Z}$) are all coherent, and
- (2) the adjunction morphism

$$(A.1) \quad \mathcal{O}_{\mathrm{Bcris}} \otimes_{\mathcal{O}_X[p^{-1}]} w^{-1} D_{\mathrm{cris}}(\mathbb{L}) \rightarrow \mathcal{O}_{\mathrm{Bcris}} \otimes_{\widehat{\mathbf{Q}}_p} \widehat{\mathbb{L}}$$

is an isomorphism of $\mathcal{O}_{\mathrm{Bcris}}$ -modules.

Remark A.8. In the unramified case, Tan and Tong [43, Def. 3.10] define crystalline local systems using the notion of association with a convergent filtered F -isocrystal, and they prove that their definition is equivalent to conditions (1) and (2) above in [43, Prop. 3.13].

Lemma A.9 (cf. [43, Lem. 3.14]). *Assume that \mathfrak{X} admits a W -model \mathfrak{X}_0 and let $\mathbb{L} \in \mathrm{ILoc}_{\mathbf{Z}_p}(X)$. For each small and connected affine formal scheme $\mathfrak{U}_0 = \mathrm{Spf} R_0$ that is étale over \mathfrak{X}_0 , set $R := R_0 \otimes_W \mathcal{O}_K$ and $U := \mathrm{Spa}(R[p^{-1}], R)$, and let V_U denote the \mathbf{Q}_p -representation of \mathcal{G}_R corresponding to $\mathbb{L}|_U$. Then there exist natural isomorphisms of $R_0[p^{-1}]$ -modules*

$$D_{\mathrm{cris}}(\mathbb{L})(\mathfrak{U}_0) \xrightarrow{\cong} D_{\mathrm{cris}}(V_U) \quad \text{and} \quad (\mathrm{Fil}^i D_{\mathrm{cris}}(\mathbb{L}))(\mathfrak{U}_0) \xrightarrow{\cong} \mathrm{Fil}^i D_{\mathrm{cris}}(V_U) \quad (i \in \mathbf{Z}).$$

Moreover, if we write $\overline{U} \in X_{\mathrm{pro\acute{e}t}}$ for the affinoid perfectoid attached to $(\overline{R}[p^{-1}], \overline{R})$, then the evaluation of the adjunction morphism (A.1) at \overline{U} coincides with

$$\alpha_{\mathrm{cris}}(V_U): \mathbf{OB}_{\mathrm{cris}}(\overline{R}) \otimes_{R_0[p^{-1}]} D_{\mathrm{cris}}(V_U) \rightarrow \mathbf{OB}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{Q}_p} V_U$$

under the identification $D_{\mathrm{cris}}(\mathbb{L})(\mathfrak{U}_0) \cong D_{\mathrm{cris}}(V_U)$.

Proof. The proof in [43, Lem. 3.14] also works in the current setting if one uses Proposition A.4 in place of [43, Cor. 2.19]. The second assertion follows from the construction. \square

Proposition A.10. *Assume that \mathfrak{X} admits a W -model \mathfrak{X}_0 and let $\mathbb{L} \in \mathrm{ILoc}_{\mathbf{Z}_p}(X)$. Then \mathbb{L} is crystalline with respect to \mathfrak{X}_0 in the sense of Definition A.7 if and only if there exists an étale covering $\{\mathfrak{U}_{\lambda,0} \rightarrow \mathfrak{X}_0\}$ of small and connected affine $\mathfrak{U}_{\lambda,0} = \mathrm{Spf} R_{\lambda,0}$ such that the \mathbf{Q}_p -representation V_λ of \mathcal{G}_{R_λ} corresponding to $\mathbb{L}|_{\mathrm{Spa}(R_\lambda[p^{-1}], R_\lambda)}$ is crystalline in the sense of Definition 2.12, where $R_\lambda := R_{\lambda,0} \otimes_W \mathcal{O}_K$. In particular, the notion of crystalline local systems on X does not depend on the choice of a W -model of \mathfrak{X} .*

Proof. The necessity follows from Lemma A.9. For the sufficiency, observe that both of conditions (1) and (2) in Definition A.7 can be verified locally on $(\mathfrak{X}_0)_{\text{ét}}$. So we may assume $\mathfrak{X}_0 = \text{Spf } R_{\lambda,0}$ for some λ . To simplify the notation, write R_0 for $R_{\lambda,0}$ and V for V_λ .

First we verify condition (1). Since the proof is similar, we only show that $D_{\text{cris}}(\mathbb{L})$ is a coherent $\mathcal{O}_{(\mathfrak{X}_0)_{\text{ét}}}[p^{-1}]$ -module. Take any connected and affine $\mathfrak{U}_0 = \text{Spf } R'_0 \in (\mathfrak{X}_0)_{\text{ét}}$. We need to show that the natural morphism

$$(A.2) \quad R'_0[p^{-1}] \otimes_{R_0[p^{-1}]} D_{\text{cris}}(\mathbb{L})(\mathfrak{X}_0) \rightarrow D_{\text{cris}}(\mathbb{L})(\mathfrak{U}_0)$$

is an isomorphism. Set $R' := R'_0 \otimes_W \mathcal{O}_K$. Then R' is connected and small over \mathcal{O}_K . By Lemma A.9, we have identifications $D_{\text{cris}}(\mathbb{L})(\mathfrak{X}_0) \xrightarrow{\cong} D_{\text{cris}}(V)$ and $D_{\text{cris}}(\mathbb{L})(\mathfrak{U}_0) \xrightarrow{\cong} D_{\text{cris}}(V|_{\mathcal{G}_{R'}})$. Since V is crystalline, the map (A.2) is an isomorphism by Lemma 2.13. Now that we have verified condition (1), condition (2) follows from the proof of [43, Cor. 3.15 and 3.16] with Remark A.6, Proposition A.4, and Lemma A.9 in place of Lemma 2.18, Corollary 2.19, and Lemma 3.14 of [43]. This completes the proof of the sufficiency. The last assertion follows from [12, Prop. 8.3.5]. \square

With these preparations, we define the notion of crystalline local systems via gluing.

Definition A.11. Let \mathfrak{X} be a smooth p -adic formal scheme over \mathcal{O}_K and let X denote its adic generic fiber. An étale isogeny \mathbf{Z}_p -local system \mathbb{L} on X is said to be *crystalline* if there exists an open covering $\mathfrak{X} = \bigcup_\lambda \mathfrak{U}_\lambda$ such that each \mathfrak{U}_λ admits a W -model and such that for each λ , $\mathbb{L}|_{U_\lambda}$ is crystalline in the sense of Definition A.7 where U_λ denotes the adic generic fiber of \mathfrak{U}_λ . By Proposition A.10, this definition coincides with Definition A.7 when \mathfrak{X} itself admits a W -model.

An étale \mathbf{Z}_p -local system \mathbb{L} on X is said to be *crystalline* if the associated isogeny \mathbf{Z}_p -local system is crystalline.

Remark A.12. One could define crystalline local systems by introducing a period sheaf $\mathcal{O}_{\mathbb{B}_{\max,K}}$ on $X_{\text{proét}}$ that generalizes the period ring $A_{\max}(R)[p^{-1}, t^{-1}]$ appearing in the proof of [12, Prop. 8.3.5]. This period sheaf is defined without fixing a W -model of \mathfrak{X} and thus one could bypass the gluing approach.

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