

MÖBIUS DISJOINTNESS FOR A CLASS OF EXPONENTIAL FUNCTIONS

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Abstract

A vast class of exponential functions is shown to be deterministic. This class includes functions whose exponents are polynomial-like or ‘piece-wise’ close to polynomials after differentiation. Many of these functions are proved to be disjoint from the Möbius function.

1. Introduction

Let $\mu(n)$ be the Möbius function, that is, $\mu(n)$ is 0 when n is not square free (that is, divisible by a non-trivial square) and is $(-1)^r$ when n is a product of r distinct primes. Many problems in number theory can be reformulated in terms of properties of the Möbius function. For example, the Prime Number Theorem is known to be equivalent to $\sum_{n \leq N} \mu(n) = o(N)$. The Riemann hypothesis holds if and only if $\sum_{n \leq N} \mu(n) = o(N^{\frac{1}{2} + \epsilon})$ for every $\epsilon > 0$.

For a truly random sequence a_n of -1 and 1 , the normalized average $(\sum_{n=1}^N a_n)/N^{\frac{1}{2}}$ obeys Gaussian law in distribution as N tends to infinity, which implies that $\sum_{n \leq N} a_n = o(N^{\frac{1}{2} + \epsilon})$. Under the Riemann hypothesis, the Möbius function shares this property as an indication that certain randomness may exist in the values of the Möbius function. It is widely believed that this randomness predicts significant cancellations in the summation of $\mu(n)\xi(n)$ for any ‘reasonable’ sequence $\xi(n)$. This rather vague principle is known as an instance of the ‘Möbius randomness principle’ (see for example, [20, Section 13.1]). In [36], Sarnak made this principle precise by identifying the notion of ‘reasonable’ and proposed the following conjecture.

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CONJECTURE 1.1 (Sarnak’s Möbius Disjointness Conjecture (SMDC)). *Let $\xi(n)$ be a deterministic sequence. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) \xi(n) = 0.$$

Here, we recall the definition of deterministic sequences. Functions from \mathbb{N} to \mathbb{C} are called *arithmetic functions* or *sequences*. An arithmetic function $f(n)$ is said to be *disjoint* from another one $g(n)$ if $\sum_{n=1}^N f(n) \bar{g}(n) = o(N)$. Let (\mathcal{X}, T) be a topological dynamical system, that is \mathcal{X} is a compact Hausdorff space and $T: \mathcal{X} \rightarrow \mathcal{X}$ a continuous map. We say $n \mapsto F(T^n x_0)$ an *arithmetic function realized in the topological dynamical system (\mathcal{X}, T)* , and (\mathcal{X}, T) a *realization* of this arithmetic function. Functions realized in topological dynamical systems of zero topological entropy are called *deterministic*.

In recent years, a lot of progress has been made on SMDC. See [2–4, 7–9, 13, 14, 16–18, 22–26, 33, 35, 39, 42–44], to list a few. In the following, we shall discuss only the results that are more related to this paper. The goal of this article is to show that a class of exponential functions is deterministic and to verify the conjecture of Sarnak for them.

1.1. Notation

We use $e(f(n))$ to denote the exponential function $\exp(2\pi i f(n))$ when f is a real-valued arithmetic function, where i is the imaginary unit. When we say the exponential function of $f(n)$, we mean $e(f(n))$. Sometimes we call $e(f(n))$ an $f(n)$ *phase*.

We use 1_S to denote the indicator of a predicate S , that is $1_S = 1$ when S is true and $1_S = 0$ when S is false. We also denote $1_A(n) = 1_{n \in A}$ for any subset A of \mathbb{N} . For any finite set C , $|C|$ denotes the cardinality of C .

We use $\{x\}$ and $[x]$ to denote the fractional part and the integer part of a real number x , respectively. We use $\|x\|_{\mathbb{R}/\mathbb{Z}}$ to denote the distance between x and the set \mathbb{Z} , that is, $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min(\{x\}, 1 - \{x\})$. For ease of notation, we drop the subscript and write simply $\|x\|$.

The difference operator Δ is defined on the set of all arithmetic functions, mapping $f(\cdot)$ to $f(\cdot + 1) - f(\cdot)$. The k -th difference operator Δ^k is defined by the composition of Δ with k times.

For two arithmetic functions $f(n)$ and $g(n)$, $f = o(g)$ means $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$; assume that $g(n) \geq 0$ for any $n \in \mathbb{N}$, $f \ll g$ means that there is an absolute constant c such that $|f| \leq cg$; $f = g + O(h)$ means $f - g \ll h$.

1.2. A class of deterministic sequences

It was investigated in [10] and [41] that the set of deterministic sequences is closed under many operations, for example addition, multiplication, inversion, conjugation and translation. This set is also closed under the uniform limit. Moreover, any continuous function of a deterministic sequence is also deterministic.

Motivation. From the above, it is easy to see that the set of deterministic sequences is closed under the difference operator Δ . As we know, Δ has an inverse operator σ (up to an initial value $f(0)$), mapping f to $\sigma(f): n \mapsto \sum_{j < n} f(j)$. Then it is natural to ask whether $\sigma(f)$ is also deterministic when f is deterministic. We may assume f is real in this question because a complex arithmetic

function is deterministic if and only if both its real part and imaginary part are deterministic. However, $\sigma(f)$ could be unbounded (so the corresponding dynamical system is no longer compact) even if f is bounded. It is therefore better to consider what kind of properties f has that can ensure that $e(\sigma(f))$ is deterministic. For this question, we give the following answer, which states that $e(\sigma(f))$ is deterministic when the k -th difference $\Delta^k f(n)$ is ‘piece-wise’ close to polynomials.

THEOREM 1.2 *Let w be a positive integer. Suppose that $p_1(y), \dots, p_w(y)$ are polynomials in $\mathbb{R}[y]$, and $\mathbb{N} = S_1 \cup S_2 \cup \dots \cup S_w$ is a partition of \mathbb{N} with each $1_{S_v}(n)$ deterministic. Let*

$$g(n) = \sum_{v=1}^w 1_{S_v}(n)p_v(n). \tag{1.1}$$

Then for any real-valued arithmetic function $f(n)$ satisfying

$$\lim_{n \rightarrow \infty} \|\Delta^k f(n) - g(n)\| = 0 \tag{1.2}$$

for some $k \in \mathbb{N}$, $e(f(n))$ and $e(\sigma(f)(n))$ are deterministic.

Now, we explain briefly the main idea to prove the above result. The major tool we use is anqie entropy (of arithmetic functions), which was introduced by Ge in [10]. We refer readers to Section 2.1 for knowledge on anqie entropy. To prove Theorem 1.2, we first construct a sequence of arithmetic functions $\{f_N(n)\}_{N=0}^\infty$ with finite ranges that uniformly converges to $f(n)$. By the lower semi-continuity of anqie entropy (see Proposition 2.2), it suffices to show that the anqie entropy of f_N is zero for N large enough. Note that the anqie entropy of f_N (with finite range) is given through the cardinality of different J -blocks occurring in it (see formula (2.1)). So we focus on estimating this cardinality. Our method is to build a one-to-one map from the set of J -blocks occurring in the sequence $f_N(n)$ to the set of pieces of \mathbb{R}^k cut by hyperplanes (for some k depending on J). So the cardinality of the first set is bounded by the cardinality of the latter one. And we prove that the second cardinality has polynomial growth (see Lemma 2.6). We refer readers to Section 3 for more details.

We also consider whether the exponential function of any concatenation of polynomials is deterministic. Before stating the next result, we first introduce the definition of concatenation of arithmetic functions.

DEFINITION 1.3 Let $0 = N_0 < N_1 < \dots$ be a sequence of natural numbers with $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$, and $\{f_i(n)\}_{i=0}^\infty$ be a sequence of arithmetic functions. We say that $f(n)$ is the concatenation of $\{f_i(n)\}_{i=0}^\infty$ with respect to the sequence $\{N_i\}_{i=0}^\infty$ if $f(n) = f_i(n)$ when $N_i \leq n < N_{i+1}$ for $i = 0, 1, \dots$

For the exponential functions of concatenations, we obtain the following result.

THEOREM 1.4 *Let $g(n)$ be defined as in Theorem 1.2. Suppose that $\{f_i(n)\}_{i=0}^\infty$ is a sequence of real-valued arithmetic functions such that*

$$\lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \|\Delta^k f_i(n) - g(n)\| = 0 \tag{1.3}$$

holds for some $k \in \mathbb{N}$. Then for any concatenation $f(n)$ of $\{f_i(n)\}_{i=0}^\infty$, $e(f(n))$ is deterministic.

As an application of Theorem 1.4, we prove that Sarnak's conjecture implies the averages of the products of Möbius and certain exponential functions are small in almost all short intervals (Theorem 3.2).

1.3. Disjointness of Möbius from $e(f(n))$ with the k -th difference of $f(n)$ tending to zero

The disjointness of Möbius from exponential functions is important and has been extensively studied in number theory. For example, the disjointness of $\mu(n)$ from $e(n\alpha)$, for any $\alpha \in \mathbb{R}$, is closely related to the estimate of exponential sums in prime variables, from which one can deduce Vinogradov's three primes theorem. It is known that $\mu(n)$ is disjoint from $e(p(n))$ for $p(n)$ a polynomial (or a sub-polynomial) (see [6], [15, Chapter 6, Theorem 10] and [40, Chapter 3, Theorem 4.6]). Moreover, the upper bound of $\sum_{n=1}^N \mu(n)e(p(n))$ has received much attention (see for example, [1], [21] and [46]) partially due to its close connection with the distribution of the zeros of Dirichlet L-functions.

As we have seen from Theorem 1.2 that $e(f(n))$ is deterministic when $f(n)$ satisfies that there is a $k \in \mathbb{N}$ such that the k -th derivative (or the k -th difference for the discrete case, see condition (1.4) below) tends to zero. Note that SMDC implies that any deterministic sequence is disjoint from the Möbius function. Motivated by this, we are interested in the following problem.

PROBLEM 1. Let $f(n)$ be a real-valued arithmetic function such that

$$\lim_{n \rightarrow \infty} \|\Delta^k f(n)\| = 0 \quad (1.4)$$

for some natural number k . Is $\mu(n)$ disjoint from $e(f(n))$?

Restrictions on the k -th derivative or k -th difference of $f(n)$ often appear in non-trivial estimate of the exponential sums $\sum_{n=1}^N e(f(n))$. For example, Van der Corput's method and the method of exponential pairs (see for example, [11]). The latter one usually requires $\Delta f(n)$ to be approximately cn^{-s} for some $c, s > 0$. This condition is included in (1.4). In this paper, we focus on the sum $\sum_{n=1}^N \mu(n)e(f(n))$ under the restriction (1.4).

In the following we investigate Problem 1 without assuming SMDC. For the case $k=0$, it is obvious that $\mu(n)$ is disjoint from such $e(f(n))$ by the Prime Number Theorem. For the case $k=1$, we have the following result.

PROPOSITION 1.5 *Suppose that $f(n)$ is a real-valued arithmetic function satisfying*

$$\lim_{n \rightarrow \infty} \|\Delta f(n) - c\| = 0$$

for some constant $c \in \mathbb{R}$. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)e(f(n)) = 0.$$

In fact, we have the following general result.

PROPOSITION 1.6 *Suppose that $f(n)$ is a real-valued arithmetic function satisfying that the set $\{e(\Delta f(n)) : n = 0, 1, \dots\}$ has finitely many limit points, and*

$$\lim_{n \rightarrow \infty} \|\Delta^2 f(n) - c\| = 0$$

for some constant $c \in \mathbb{R}$. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)e(f(n)) = 0.$$

For the case $k \geq 2$, we obtain the following result.

THEOREM 1.7 *Let $\tau \in (5/8, 1)$ and $k \geq 2$. Let $f(n)$ be a real-valued arithmetic function satisfying when n large enough,*

$$\|\Delta^k f(n)\| \leq \frac{C}{\exp((\log n)^\tau)} \tag{1.5}$$

for some positive constant C . Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)e(f(n)) = 0.$$

The first ingredient of our proof of the above result is that if the k -th difference of $f(n)$ decays to zero, then $e(f(n))$ can be approximated uniformly by certain concatenations of polynomial phases (see Lemma 4.1). The second one is Matomäki–Radziwiłł–Tao–Teräväinen–Ziegler’s estimate [30] on averages of the correlation of multiplicative functions with polynomial phases in short intervals.

In the following, we provide a sufficient condition, which is weaker than SMDC (see Corollary 3.3), for Problem 1.

PROPOSITION 1.8 *Let k be a given positive integer. Denote \mathcal{D}_k by the set of all polynomials in $\mathbb{R}[y]$ of degrees less than k . Assume that the following estimate holds,*

$$\lim_{h \rightarrow \infty} \limsup_{X \rightarrow \infty} \frac{1}{Xh} \int_X^{2X} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+h} \mu(n)e(p(n)) \right| dx = 0. \tag{1.6}$$

Then for any $f(n)$ satisfying $\lim_{n \rightarrow \infty} \|\Delta^k f(n)\| = 0$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)e(f(n)) = 0.$$

REMARK 1.9 All the results listed in this section also hold if μ is replaced by more general ‘non-pretentious’ 1-bounded (that is, the l^∞ -norm is bounded by 1) multiplicative functions, such as the Liouville function and $\mu(n)\chi(n)$, where χ is a given Dirichlet character.

2. Preliminaries on anqie entropy of arithmetic functions

To use tools from operator algebra to study Sarnak's conjecture, Ge introduced the anqie entropy for arithmetic functions [10]. For a bounded arithmetic function f , its anqie entropy equals the infimum of the topological entropy of all possible realizations of f . For functions with finite ranges, the anqie entropy of such a function f is determined by the number of different J -blocks appearing in the sequence $\{f(n)\}_{n=0}^{\infty}$. Specifically, let $\mathcal{B}_J(f)$ denote the set of all J -blocks occurring in f , that is, $\mathcal{B}_J(f) = \{(f(n), f(n+1), \dots, f(n+J-1)) : n \geq 0\}$, then the anqie entropy of $f(n)$ equals

$$\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J(f)|}{J}, \quad (2.1)$$

where $|\mathcal{B}_J(f)|$ is the cardinality of the set $\mathcal{B}_J(f)$ ([41, Lemma 6.1]).

2.1. Some properties of anqie entropy

The anqie entropy has many nice properties. Here we list some properties which will be used in this paper. The following one is about the algebraic operations. Here and in the sequel, we use $\mathcal{A}(f)$ to denote the anqie entropy of any bounded arithmetic function $f(n)$.

PROPOSITION 2.1 *For any bounded arithmetic functions f, g and continuous function $\phi(z)$ in \mathbb{C} , we have*

$$|\mathcal{A}(f) - \mathcal{A}(g)| \leq \mathcal{A}(f \pm g) \leq \mathcal{A}(f) + \mathcal{A}(g),$$

$$\mathcal{A}(f \cdot g) \leq \mathcal{A}(f) + \mathcal{A}(g), \quad \mathcal{A}(\phi(f)) \leq \mathcal{A}(f).$$

The next one is about the lower semi-continuity of anqie entropy.

PROPOSITION 2.2 *If $\{f_N(n)\}_{N=0}^{\infty}$ is a sequence of bounded arithmetic functions converging to $f(n)$ uniformly with respect to $n \in \mathbb{N}$, then $\liminf_{N \rightarrow \infty} \mathcal{A}(f_N) \geq \mathcal{A}(f)$.*

We refer readers to [10, Section 4] and [41] for details of the above two propositions. Arithmetic functions of zero anqie entropy can be realized in topological dynamical systems of zero topological entropy [10, Section 3]. Based on this fact, we have the following result.

PROPOSITION 2.3 *A sequence $\{f(n)\}_{n=0}^{\infty}$ is deterministic if and only if $\mathcal{A}(f) = 0$.*

2.2. Anqie entropy of arithmetic functions with finite ranges

In this subsection, we show some results on computing anqie entropy of functions with finite ranges, which will be used in the proofs of Theorems 1.2 and 1.4. Let us first recall some basic concepts in symbolic dynamical systems. For a finite set \mathbb{A} , a *block* over \mathbb{A} is a finite sequence of symbols from \mathbb{A} . A *J-block* is a block of length J ($J \geq 1$). For any given (finite or infinite) sequence $x = (x_0, x_1, \dots)$ of symbols from \mathbb{A} , we say that a block w *occurs in* x or x *contains* w if there are natural numbers i, j with

$i \leq j$ such that $(x_i, \dots, x_j) = w$. A concatenation of two blocks $w_1 = (a_1, \dots, a_k)$ and $w_2 = (b_1, \dots, b_l)$ over \mathbb{A} is the block $w_1 w_2 = (a_1, \dots, a_k, b_1, \dots, b_l)$.

Now suppose that $f: \mathbb{N} \rightarrow \mathbb{C}$ has finite range. Let $\mathcal{B}_J(f)$ denote the set of all J -blocks occurring in f , that is,

$$\mathcal{B}_J(f) = \{(f(n), f(n+1), \dots, f(n+J-1)) : n \geq 0\}.$$

A J -block of the form

$$(f(lJ), f(lJ+1), \dots, f(lJ+J-1))$$

for some $l \in \mathbb{N}$ is called a *regular J -block in f* . Denote the set of all regular J -blocks in f by $\mathcal{B}_J^r(f)$. A J -block, which occurs infinitely many times in the sequence $\{f(n)\}_{n=0}^\infty$, is called an *effective J -block in f* . Denote the set of all such blocks in f by $\mathcal{B}_J^e(f)$. A J -block $(a_0, a_1, \dots, a_{J-1})$ is called *regularly effective in f* if there are infinitely many natural numbers l such that

$$(a_0, a_1, \dots, a_{J-1}) = (f(lJ), \dots, f(lJ+J-1)).$$

The set of all regularly effective blocks in f is denoted by $\mathcal{B}_J^{e,r}(f)$.

For a function f taking finitely many values, as we mentioned previously,

$$\mathcal{A}(f) = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J(f)|}{J}. \tag{2.2}$$

In the following, we show that $\mathcal{A}(f)$ also can be computed through the cardinality of $\mathcal{B}_J^r(f)$, $\mathcal{B}_J^e(f)$ or $\mathcal{B}_J^{e,r}(f)$.

Although the proof of the following proposition is not hard, we did not find it in the literature.

PROPOSITION 2.4 *Let $f(n)$ be an arithmetic function with finite range. Then*

$$\mathcal{A}(f) = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J} = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J} = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^e(f)|}{J}. \tag{2.3}$$

Proof. We first show that the first equality in equation (2.3) holds. On the one hand, $\mathcal{B}_J^r(f) \subseteq \mathcal{B}_J(f)$, so by formula (2.2),

$$\mathcal{A}(f) \geq \limsup_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J}.$$

On the other hand, given $J \geq 1$, for any $l \geq 1$ and any (lJ) -block w occurring in f , there is a concatenation of certain $l+1$ successive regular J -blocks in f containing w . Thus $|\mathcal{B}_{lJ}(f)| \leq J |\mathcal{B}_J^r(f)|^{l+1}$.

This implies that

$$\mathcal{A}(f) = \lim_{l \rightarrow \infty} \frac{\log |\mathcal{B}_{lJ}(f)|}{lJ} \leq \frac{\log |\mathcal{B}_J^r(f)|}{J}.$$

We then have

$$\mathcal{A}(f) \leq \liminf_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J}.$$

So $\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J}$ exists and equals $\mathcal{A}(f)$.

Next we show that the second equality in equation (2.3) holds, that is,

$$\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J} = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J}. \quad (2.4)$$

Since $\mathcal{B}_J^{e,r}(f) \subseteq \mathcal{B}_J^r(f)$,

$$\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J} \geq \limsup_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J}. \quad (2.5)$$

So we only need to show that

$$\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^r(f)|}{J} \leq \liminf_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J}. \quad (2.6)$$

In fact, for any given $J \geq 1$, since the set $\mathcal{B}_J^r(f) \setminus \mathcal{B}_J^{e,r}(f)$ is finite, there is an integer $l_J \geq 1$ such that all regular J -blocks in the set

$$\{(f(nJ), \dots, f(nJ + J - 1)) : n \geq l_J\}$$

are regularly effective J -blocks in f . Then for each $l > l_J$, there is at most one regular (lJ) -block in f which is not a concatenation of regularly effective J -blocks in f . This implies $|\mathcal{B}_{lJ}^r(f)| \leq |\mathcal{B}_J^{e,r}(f)|^l + 1$. Therefore

$$\mathcal{A}(f) = \lim_{l \rightarrow \infty} \frac{\log |\mathcal{B}_{lJ}^r(f)|}{lJ} \leq \lim_{l \rightarrow \infty} \frac{\log (|\mathcal{B}_J^{e,r}(f)|^l + 1)}{lJ} = \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J}$$

holds. Letting $J \rightarrow \infty$, we obtain formula (2.6).

At last, note that

$$\mathcal{B}_J^{e,r}(f) \subseteq \mathcal{B}_J^e(f) \subseteq \mathcal{B}_J(f),$$

then

$$\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J} \leq \liminf_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^e(f)|}{J}, \quad \limsup_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^e(f)|}{J} \leq \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J(f)|}{J}.$$

From formula (2.2) and the second equality in (2.3), we have

$$\mathcal{A}(f) = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J(f)|}{J} = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^{e,r}(f)|}{J}.$$

Then $\lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J^e(f)|}{J}$ exists and equals $\mathcal{A}(f)$. So the third equality in equation (2.3) holds. □

To estimate the cardinality of the set of J -blocks occurring in certain sequences, we introduce the following notion.

DEFINITION 2.5 Let $k, m \geq 1$ be integers and $c_1, \dots, c_m \in \mathbb{R}$ be constants. Suppose that for $j = 1, \dots, m$, F_j is a non-zero linear function of x_1, \dots, x_k and H_j is the hyperplane in \mathbb{R}^k given by $F_j(x_1, \dots, x_k) = c_j$. Denote by

$$H_j^+ = \{(x_1, \dots, x_k) \in \mathbb{R}^k : F_j(x_1, \dots, x_k) > c_j\},$$

$$H_j^- = \{(x_1, \dots, x_k) \in \mathbb{R}^k : F_j(x_1, \dots, x_k) < c_j\}.$$

A non-empty subset P of \mathbb{R}^k of the following form

$$P = P_1 \cap P_2 \cap \dots \cap P_m,$$

where each $P_j \in \{H_j^+, H_j^-, H_j\}$, is called a *piece* of \mathbb{R}^k cut by H_1, \dots, H_m .

In the following lemma we give an upper bound for the cardinality of pieces of \mathbb{R}^k cut by hyperplanes.

LEMMA 2.6 Let m, k be integers with $m > k \geq 1$. Suppose that H_1, \dots, H_m are hyperplanes in \mathbb{R}^k . Let $C(H_1, \dots, H_m, k)$ denote the cardinality of pieces of \mathbb{R}^k cut by H_1, \dots, H_m and $W(m, k)$ denote the maximal value of $C(H_1, \dots, H_m, k)$ when H_1, \dots, H_m go through all the possible hyperplanes. Then

$$W(m, k) \leq \sum_{j=0}^k 2^j \binom{m}{j}.$$

In particular, $W(m, k) \leq (k + 1)2^k m^k$.

Proof. Notice that $W(1, k) = 3$ for any $k \geq 1$. If we have

$$W(m, k) \leq W(m-1, k) + 2W(m-1, k-1), \quad (2.7)$$

then one can easily draw the conclusion by induction on m .

Now we show that formula (2.7) holds for $m \geq 2$. Let ψ be the map from the set of all pieces of \mathbb{R}^k cut by H_1, \dots, H_m , denoted by \mathcal{P} , onto the set of all pieces of \mathbb{R}^k cut by H_2, \dots, H_m , denoted by $\tilde{\mathcal{P}}$, given by

$$\psi : P_1 \cap P_2 \cap \dots \cap P_m \mapsto P_2 \cap \dots \cap P_m.$$

Then for any piece $D \in \tilde{\mathcal{P}}$, $|\psi^{-1}(D)| \leq 3$. Now we show that if $|\psi^{-1}(D)| \geq 2$, then $H_1 \cap D$ is non-empty. In fact, the only case we need to consider is when both $H_1^+ \cap D$ and $H_1^- \cap D$ are non-empty. In this case, suppose $p_1 \in H_1^+ \cap D$ and $p_2 \in H_1^- \cap D$, then there is a point $p_0 \in H_1 \cap D$ by the convexity of D . Hence, $|\mathcal{P}| - |\tilde{\mathcal{P}}| = C(H_1, \dots, H_m, k) - C(H_2, \dots, H_m, k)$ does not exceed the cardinality of pieces of the form $H_1 \cap P_2 \cap \dots \cap P_m$ in \mathcal{P} times 2. In the following, we estimate this cardinality.

For each $j = 2, \dots, m$, $H_1 \cap H_j$ is a hyperplane in H_1 , or empty, or equal to H_1 . Denote G_1, \dots, G_n to be the ones which are hyperplanes in H_1 . Then $n \leq m-1$. We claim that if $H_1 \cap P_2 \cap \dots \cap P_m = (H_1 \cap P_2) \cap \dots \cap (H_1 \cap P_m) \neq \emptyset$, then $(H_1 \cap P_2) \cap \dots \cap (H_1 \cap P_m)$ is a piece of H_1 cut by G_1, \dots, G_n . In fact, for $j = 2, \dots, m$, there are at most three cases. When $H_1 \cap H_j = \emptyset$, then $H_1 \cap H_j^+ = H_1$ or $H_1 \cap H_j^- = H_1$ and then $P_j = H_j^+$ or H_j^- , respectively. When $H_1 \cap H_j = H_1$, then $P_j = H_j$ and $H_1 \cap P_j = H_1$. When $H_1 \cap H_j = G_l$ is a hyperplane in H_1 , then $\{H_1 \cap H_j^+, H_1 \cap H_j^-\} = \{G_l^+, G_l^-\}$. Hence $(H_1 \cap P_2) \cap \dots \cap (H_1 \cap P_m)$ is of the form $T_1 \cap \dots \cap T_n$, where each $T_l \in \{G_l, G_l^+, G_l^-\}$.

So the cardinality of pieces of the form $H_1 \cap P_2 \cap \dots \cap P_m$ in \mathcal{P} does not exceed $C(G_1, \dots, G_n, k-1)$ which is at most $W(n, k-1) \leq W(m-1, k-1)$. Therefore, the inequality (2.7) holds and the proof is completed. \square

3. Proofs of Theorems 1.2 and 1.4

Recall that $\|x\| = \inf_{m \in \mathbb{Z}} |x - m| = \min\{\{x\}, 1 - \{x\}\}$. Then $\|\cdot\|$ defines a metric on \mathbb{R}/\mathbb{Z} , and the topology induced by it on \mathbb{R}/\mathbb{Z} is equivalent to the Euclid topology on the unit circle. The following lemma will be used in this section.

LEMMA 3.1 *Let f, g be real-valued arithmetic functions with*

$$\lim_{n \rightarrow \infty} \|\Delta^k f(n) - g(n)\| = 0 \quad (3.1)$$

for some $k \in \mathbb{N}$. Then, for any $\varepsilon > 0$ and positive integer $m \geq 1$, there is some $L \in \mathbb{N}$ such that, whenever $n > L$, the following holds for any j with $0 \leq j \leq m-1$,

$$\|f(n+j) - Y_n(n+j)\| \leq \varepsilon,$$

where $Y_n(n+j)$ is defined to be $f(n+j)$ when $0 \leq j \leq k-1$ and to be the value determined by the following linear equations when $k \leq j \leq m-1$,

$$\Delta^k Y_n(n+j) = g(n+j), \quad j = 0, 1, \dots, m-k-1. \quad (3.2)$$

Proof. When $k=0$, the claim is trivial. In the following, we assume $k \geq 1$. We use induction on m to prove the lemma. For $m \leq k$, since $Y_n(n+j) = f(n+j)$ for $j = 0, 1, \dots, m-1$, choose $L=0$. Then we obtain the claim in the lemma. Assume inductively that the claim holds for some $m_0 \geq k$. In the following we shall prove the claim holds for $m_0 + 1$ case. By condition (3.1) and Proposition A.1,

$$\lim_{n \rightarrow \infty} \left\| \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} f(n+l) - g(n) \right\| = 0.$$

Then for any $\epsilon > 0$, there is an $L_1 > 0$ such that whenever $n > L_1$,

$$\left\| \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} f(n+m_0-k+l) - g(n+m_0-k) \right\| < \epsilon/2,$$

that is,

$$\left\| f(n+m_0) - \left(g(n+m_0-k) - \sum_{l=0}^{k-1} (-1)^{k-l} \binom{k}{l} f(n+m_0-k+l) \right) \right\| < \epsilon/2. \quad (3.3)$$

By the induction hypothesis, there is an $L_0 \in \mathbb{N}$, whenever $n > L_0$,

$$\|f(n+j) - Y_n(n+j)\| < \epsilon/2^{k+1}, \quad j = 0, 1, \dots, m_0 - 1. \quad (3.4)$$

Let $L = \max\{L_0, L_1\}$. Then by equations (3.3) and (3.4), whenever $n > L$,

$$\left\| f(n+m_0) - \left(g(n+m_0-k) - \sum_{l=0}^{k-1} (-1)^{k-l} \binom{k}{l} Y_n(n+m_0-k+l) \right) \right\| < \epsilon. \quad (3.5)$$

Define $Y_n(n+m_0)$ to be the value determined in the following equation,

$$\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} Y_n(n+m_0-k+l) = g(n+m_0-k).$$

Then by equation (3.5),

$$\|f(n+m_0) - Y_n(n+m_0)\| < \epsilon.$$

Combing with equation (3.4), we obtain the claim for $m = m_0 + 1$, completing the induction. \square

THEOREM 1.2 (restated). Let w be a positive integer. Suppose that $p_1(y), \dots, p_w(y)$ are polynomials in $\mathbb{R}[y]$, and $\mathbb{N} = S_1 \cup S_2 \cup \dots \cup S_w$ is a partition of \mathbb{N} with each $1_{S_\nu}(n)$ deterministic. Let

$$g(n) = \sum_{\nu=1}^w 1_{S_\nu}(n) p_\nu(n). \quad (3.6)$$

Then for any real-valued arithmetic function $f(n)$ satisfying

$$\lim_{n \rightarrow \infty} \|\Delta^k f(n) - g(n)\| = 0 \quad (3.7)$$

for some $k \in \mathbb{N}$, $e(f(n))$ and $e(\sigma(f)(n))$ are deterministic. We use the following strategy to prove $\mathcal{A}E(f(n)) = 0$. Firstly, we construct a sequence $\{g_N(n)\}_{N=0}^\infty$ of arithmetic functions with finite ranges to approach $f(n)$ with respect to $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$. Then it suffices to prove $\mathcal{A}E(g_N) = 0$ for N large enough by Proposition 2.2. Secondly, we decompose $\mathcal{B}_{2^m}^{e,r}(g_N)$, the set of all 2^m -regularly effective blocks occurring in g_N , into some subsets $\mathcal{A}_{\nu,m}$ according to the ‘zero entropy partition’ given in $g(n)$. Thirdly, for each ν , we construct a family of hyperplanes $H_{\nu,0}, \dots, H_{\nu,2^m-1}$ in \mathbb{R}^q based on the approximation of $f(n)$ to polynomials after differentiation. Then there is a correspondence between the set of pieces of \mathbb{R}^q cut by $H_{\nu,0}, \dots, H_{\nu,2^m-1}$ and $\mathcal{A}_{\nu,m}$. Precisely, each element in $\mathcal{A}_{\nu,m}$ uniquely determines a piece of \mathbb{R}^q cut by $H_{\nu,0}, \dots, H_{\nu,2^m-1}$. So $|\mathcal{A}_{\nu,m}|$, moreover $|\mathcal{B}_{2^m}^{e,r}(g_N)|$, is bounded by $W(2^m, q)$, which has the polynomial growth rate with respect to 2^m by Lemma 2.6.

Proof of Theorem 1.2. In the following, we first prove that $e(f(n))$ is deterministic. Then a similar argument leads to $e(\sigma(f)(n))$ deterministic since $\Delta^k f(n) = \Delta^{k+1}(\sigma(f))(n)$ by the fact that σ is the inverse of Δ .

Given $N \geq 1$, by Lemma 3.1, for each integer $m \geq 1$, there is a sufficiently large $L_m \in \mathbb{N}$ with $2^m |L_m$, such that whenever $n \geq L_m$, we have

$$\|f(n+j) - Y_n(n+j)\| \leq 1/N, \quad j = 0, \dots, 2^m - 1, \quad (3.8)$$

where $Y_n(n+j)$ is defined to be $f(n+j)$ when $0 \leq j \leq k-1$ and the value determined by the following linear equations when $k \leq j \leq 2^m - 1$,

$$\Delta^k Y_n(n+j) = g(n+j), \quad j = 0, \dots, 2^m - k - 1. \quad (3.9)$$

Moreover, we may further assume that the sequence $\{L_m\}_{m=0}^\infty$ ($L_0 = 0$) chosen above satisfies $L_{m+1} > L_m$ for each $m \geq 1$. Let $d_m = (L_{m+1} - L_m)/2^m$. Then the following is a partition of \mathbb{N} ,

$$\mathbb{N} = \bigcup_{m=0}^\infty \bigcup_{a=0}^{d_m-1} \{L_m + a2^m, L_m + a2^m + 1, \dots, L_m + a2^m + 2^m - 1\}.$$

We define $Y_{L_0+a}(L_0+a) = f(L_0+a)$ for $a = 0, 1, \dots, L_1 - 1$ and define the arithmetic function g_N as follows: for $a = 0, \dots, d_m - 1$ and $j = 0, \dots, 2^m - 1$,

$$g_N(L_m + a2^m + j) = \frac{t}{N} \quad (3.10)$$

when $\{Y_{L_m+a2^m}(L_m+a2^m+j)\} \in [\frac{t}{N}, \frac{t+1}{N})$ for some integer t with $0 \leq t \leq N-1$. By formula (3.8),

$$\|f(L_m+a2^m+j) - g_N(L_m+a2^m+j)\| < 2/N.$$

Then $\sup_{n \in \mathbb{N}} \|f(n) - g_N(n)\| < 2/N$. Hence $\lim_{N \rightarrow \infty} \sup_{n \in \mathbb{N}} |e(f(n)) - e(g_N(n))| = 0$. To prove $e(f(n))$ deterministic (that is $\mathcal{AE}(e(f(n))) = 0$ by Proposition 2.3), it suffices to prove $\mathcal{AE}(g_N(n)) = 0$ for N large enough by Propositions 2.1 and 2.2.

In the remaining part of this proof, we shall prove $\mathcal{AE}(g_N) = 0$ for any given $N \geq 1$. Let η be the function defined as $\eta(n) = \nu$ if $n \in S_\nu$, $\nu = 1, \dots, w$. Then the anqie entropy of η is zero. Note that $\eta(n)$ has finite range. By formula (2.2),

$$\lim_{m \rightarrow \infty} \frac{\log |\mathcal{B}_{2^m}(\eta)|}{2^m} = 0,$$

where $\mathcal{B}_{2^m}(\eta)$ is the set of all 2^m -blocks occurring in η . For any 2^m -block ν in $\mathcal{B}_{2^m}(\eta)$, denote by

$$\mathcal{A}_{\nu,m} = \{(g_N(n2^m), \dots, g_N(n2^m + 2^m - 1)) : n \in \mathbb{N}, n2^m \geq L_m, (\eta(n2^m), \dots, \eta(n2^m + 2^m - 1)) = \nu\}.$$

Recall that $\mathcal{B}_{2^m}^{e,r}(g_N)$ denotes the set of all 2^m -regularly effective blocks occurring in g_N . Then

$$\mathcal{B}_{2^m}^{e,r}(g_N) \subseteq \bigcup_{\nu \in \mathcal{B}_{2^m}(\eta)} \mathcal{A}_{\nu,m}. \tag{3.11}$$

Let $d = \max(\deg(p_1), \dots, \deg(p_w)) + 1$, where we define $\deg(p_i)$ as -1 when $p_i = 0$. In the following, we estimate the cardinality of $\mathcal{A}_{\nu,m}$ for each given m with $2^m > \max(k+1, d)$ and $\nu = (\nu_0, \dots, \nu_{2^m-1})$ in $\mathcal{B}_{2^m}(\eta)$. Denote by $q = k + wd$. If $q = 0$ (that is, $k = 0$ and $d = 0$), then by the definition, $g_N(n) = 0$ when $n \geq L_1$. So it is easy to see that $\mathcal{AE}(g_N) = 0$. Then $\mathcal{AE}(e(f)) = 0$. In the following, we may assume that $q \geq 1$.

We first define linear functions $F_{\nu,0}, \dots, F_{\nu,2^m-1}$ from \mathbb{R}^q to \mathbb{R} . Define $F_{\nu,j}(x_0, x_1, \dots, x_{q-1}) = x_j$ for $j = 0, \dots, k-1$. Assume inductively that we have defined the linear function F_{ν,j_0} for some j_0 with $k-1 \leq j_0 \leq 2^m-2$. Then we define $F_{\nu,j_0+1}(x_0, x_1, \dots, x_{q-1})$ to be the function satisfying

$$\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} F_{\nu,j_0+1-k+l}(x_0, x_1, \dots, x_{q-1}) = P_{\nu,j_0+1-k}(j_0+1-k), \tag{3.12}$$

where

$$P_\nu(j_0+1-k) = \sum_{r=0}^{d-1} x_{k+(v-1)d+r} \prod_{\substack{0 \leq s \leq d-1, \\ s \neq r}} \frac{(j_0+1-k)-s}{r-s}, \quad 1 \leq v \leq w, \tag{3.13}$$

when $d \geq 2$; $P_\nu(j_0+1-k) = 0$ when $d = 0$; $P_\nu(j_0+1-k) = x_{k+(v-1)d}$ when $d = 1$. By equation (3.12), it follows from the inductive assumption that $F_{\nu,j_0+1}(x_0, x_1, \dots, x_{q-1})$ is a linear function of x_0, x_1, \dots, x_{q-1} .

Next, given n with $(\eta(n2^m), \eta(n2^m + 1), \dots, \eta(n2^m + 2^m - 1)) = \nu$ and $n2^m \geq L_m$. Suppose that $L_{m_0} \leq n2^m < L_{m_0+1}$ for some $m_0 \geq m$ and $n2^m = n_02^{m_0} + u$ with $0 \leq u \leq 2^{m_0} - 2^m$. Taking $y_j = \{Y_{n_02^{m_0}}(n2^m + j)\}$ for $0 \leq j \leq k-1$ and $y_{k+(v-1)d+r} = \{p_\nu(n2^m + r)\}$ for $1 \leq v \leq w$, $0 \leq r \leq d-1$. In the following, we show that

$$\{F_{\nu,j}(y_0, y_1, \dots, y_{q-1})\} = \{Y_{n_02^{m_0}}(n2^m + j)\}, j = 0, \dots, 2^m - 1. \quad (3.14)$$

By the definition,

$$F_{\nu,j}(y_0, y_1, \dots, y_{q-1}) = \{Y_{n_02^{m_0}}(n2^m + j)\}, j = 0, \dots, k-1. \quad (3.15)$$

Plugging $(y_0, y_1, \dots, y_{q-1})$ into equation (3.13), we have for $j \geq k$,

$$P_\nu(j-k) = \sum_{r=0}^{d-1} \{p_\nu(n2^m + r)\} \prod_{\substack{0 \leq s \leq d-1, \\ s \neq r}} \frac{(j-k)-s}{r-s}.$$

By Lemma A.4, $\{P_\nu(j-k)\} = \{p_\nu(n2^m + j-k)\}$. Then by equation (3.12),

$$\left\{ \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} F_{\nu,j-k+l}(y_0, y_1, \dots, y_{q-1}) \right\} = \{p_{\nu_{j-k}}(n2^m + j-k)\}, j = k, \dots, 2^m - 1. \quad (3.16)$$

Using the condition (3.9), we have

$$\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} Y_{n_02^{m_0}}(n2^m + j-k+l) = g(n2^m + j-k) = p_{\nu_{j-k}}(n2^m + j-k), j = k, \dots, 2^m - 1.$$

Comparing with equation (3.16) and by (3.15), we conclude that equation (3.14) holds.

Note that $|j-k| \leq 2^m$. Then by equation (3.13), $|P_\nu(j-k)| \leq (d+1)2^{md}$ when $d=0$, or $d \geq 1$, $x_{k+(v-1)d+r} \in [0, 1)$, $r = 0, \dots, d-1$. By equation (3.12) and Proposition A.5, we have

$$|F_{\nu,j}(y_0, y_1, \dots, y_{q-1})| \leq (k+1)j^k(d+1)2^{md} < (k+1)(d+1)2^{mk+md}, j = 0, 1, \dots, 2^m - 1. \quad (3.17)$$

Based on the linear functions $F_{\nu,0}, \dots, F_{\nu,2^m-1}$ constructed above, we define a family of hyperplanes $\mathcal{F} = \{H_{M,t,j} : M, t, j \in \mathbb{Z}, -(k+1)(d+1)2^{m(k+d)} - 1 \leq M \leq (k+1)(d+1)2^{m(k+d)}, 0 \leq t \leq N-1, 0 \leq j \leq 2^m - 1\}$, where

$$H_{M,t,j} = \{(x_0, x_1, \dots, x_{q-1}) \in \mathbb{R}^q : F_{\nu,j}(x_0, x_1, \dots, x_{q-1}) = M + \frac{t}{N}\}.$$

Then $|\mathcal{F}| \leq 2(k+2)(d+1)2^{m(k+d+1)}N$. By Lemma 2.6, there are at most $W(|\mathcal{F}|, q)$ pieces of \mathbb{R}^q cut by the hyperplanes in \mathcal{F} , where

$$W(|\mathcal{F}|, q) \leq (q+1)2^q(2(k+2)(d+1)2^{m(k+d+1)}N)^q. \quad (3.18)$$

Now, we are ready to estimate $|\mathcal{A}_{\nu,m}|$. Let $(g_N(n2^m), \dots, g_N(n2^m + 2^m - 1)) \in \mathcal{A}_{\nu,m}$ with $n2^m \geq L_m$. Then $(\eta(n2^m), \dots, \eta(n2^m + 2^m - 1)) = \nu$. Suppose that $L_{m_0} \leq n2^m < L_{m_0+1}$ for some $m_0 \geq m$ and

$n2^m = n_02^{m_0} + u$ with $0 \leq u \leq 2^{m_0} - 2^m$. Set

$$y_j = \{Y_{n_02^{m_0}}(n2^m + j)\}$$

for $0 \leq j \leq k - 1$ and

$$y_{k+(v-1)d+r} = \{p_v(n2^m + r)\}$$

for $1 \leq v \leq w$, $0 \leq r \leq d - 1$. By formula (3.17), there are integers $M_0, M_1, \dots, M_{2^m-1} \in [-(k+1)(d+1)2^{m(k+d)} - 1, (k+1)(d+1)2^{m(k+d)}]$ and $t_0, t_1, \dots, t_{2^m-1} \in [0, N - 1]$ such that

$$M_j + \frac{t_j}{N} \leq F_{\nu,j}(y_0, y_1, \dots, y_{q-1}) < M_j + \frac{t_j + 1}{N}, \quad j = 0, \dots, 2^m - 1. \tag{3.19}$$

Note that any two pieces of \mathbb{R}^q cut by hyperplanes in \mathcal{F} are disjoint. Let P be the unique piece containing the point $(y_0, y_1, \dots, y_{q-1})$. Then it is not hard to check that formula (3.19) holds for each $(x_0, x_1, \dots, x_{q-1}) \in P$. By equation (3.14), $\{F_{\nu,j}(y_0, y_1, \dots, y_{q-1})\} = \{Y_{n_02^{m_0}}(n2^m + j)\} \in [\frac{t_j}{N}, \frac{t_j+1}{N}]$. Then by formula (3.10), $g_N(n2^m + j) = \frac{t_j}{N}, j = 0, \dots, 2^m - 1$. Moreover, from the above analysis, we conclude that if $n, n' \in \mathbb{N}$ with $(\eta(n2^m), \dots, \eta(n2^m + 2^m - 1)) = (\eta(n'2^m), \dots, \eta(n'2^m + 2^m - 1)) = \nu$, such that the corresponding points $(y_0, y_1, \dots, y_{q-1})$ and $(y'_0, y'_1, \dots, y'_{q-1})$ belong to the same piece of \mathbb{R}^k , then $(g_N(n2^m), \dots, g_N(n2^m + 2^m - 1)) = (g_N(n'2^m), \dots, g_N(n'2^m + 2^m - 1))$. Hence

$$|\mathcal{A}_{\nu,m}| \leq W(|\mathcal{F}|, q).$$

So by equation (3.11) and formula (3.18),

$$|\mathcal{B}_{2^m}^{e,r}(g_N)| \leq \sum_{\nu \in \mathcal{B}_{2^m}(\eta)} |\mathcal{A}_{\nu,m}| \leq (q+1)2^q (2(k+2)(d+1)2^{m(k+d+1)}N)^q |\mathcal{B}_{2^m}(\eta)|. \tag{3.20}$$

Note that N, k, d, q are parameters independent of m . By Proposition 2.4,

$$\mathcal{A}(g_N) = \lim_{m \rightarrow \infty} \frac{\log |\mathcal{B}_{2^m}^{e,r}(g_N)|}{2^m} \leq \lim_{m \rightarrow \infty} \frac{\log |\mathcal{B}_{2^m}(\eta)|}{2^m} = \mathcal{A}(\eta) = 0.$$

So we complete the proof of this theorem. □

Next, we prove Theorem 1.4, which discusses about the anqie entropy of exponential functions of concatenations.

Proof of Theorem 1.4. Given $N \geq 1$. By condition (1.3) and Lemma 3.1, for each integer $m \geq 1$, there is a sufficiently large $L_m \in \mathbb{N}$ with $2^m | L_m$, such that for any $i \in \mathbb{N}$, whenever $n \geq L_m$, we have

$$\|f_i(n+j) - Y_{n,i}(n+j)\| \leq 1/N, j = 0, 1, \dots, 2^m - 1, \quad (3.21)$$

where $Y_{n,i}(n+j)$ equals $f_i(n+j)$ when $0 \leq j \leq k-1$ and is determined by the following linear equations when $k \leq j \leq 2^m - 1$,

$$\Delta^k Y_{n,i}(n+j) = g(n+j), j = 0, 1, \dots, 2^m - k - 1. \quad (3.22)$$

Moreover, we may further assume that the sequence $\{L_m\}_{m=0}^\infty$ ($L_0 = 0$) chosen above satisfies $L_{m+1} > L_m$ for each $m \geq 1$. Let $d_m = (L_{m+1} - L_m)/2^m$. We define $Y_{L_0+a,i}(L_0+a) = f_i(L_0+a)$ for $i \in \mathbb{N}$ and $a = 0, 1, \dots, L_{1,i} - 1$ and a sequence $\{g_{N,i}\}_{i=0}^\infty$ of arithmetic functions with finite ranges as follows: for $a = 0, 1, \dots, d_m - 1$ and $j = 0, 1, \dots, 2^m - 1$, define

$$g_{N,i}(L_m + a2^m + j) = \frac{t}{N}$$

when $\{Y_{L_m+a2^m,i}(L_m + a2^m + j)\} \in [\frac{t}{N}, \frac{t+1}{N})$ for some integer t with $0 \leq t \leq N-1$. By formula (3.21),

$$\|f_i(L_m + a2^m + j) - g_{N,i}(L_m + a2^m + j)\| < 2/N.$$

Then

$$\|f_i(n) - g_{N,i}(n)\| < 2/N, \text{ for any } n, i \in \mathbb{N}. \quad (3.23)$$

Denote by

$$\mathcal{C}_m = \{(g_{N,i}(n2^m), \dots, g_{N,i}(n2^m + 2^m - 1)) : i \in \mathbb{N}, n \in \mathbb{N}, n2^m \geq L_m\}.$$

Recall that

$$g(n) = \sum_{v=1}^w 1_{S_v}(n) p_v(n), \quad (3.24)$$

where each $1_{S_v}(n)$ is deterministic (that is, $\mathcal{A}(1_{S_v}) = 0$ by Proposition 2.3). Define the arithmetic function η by $\eta(n) = v$ if $n \in S_v$, $v = 1, \dots, w$. Then $\mathcal{A}(\eta) = 0$. Recall that $\mathcal{B}_{2^m}(\eta)$ denotes the set of

all 2^m -blocks occurring in η . Given $\nu \in \mathcal{B}_{2^m}(\eta)$, denote by $\tilde{\mathcal{A}}_{\nu,m}$ the set

$$\{(g_{N,i}(n2^m), \dots, g_{N,i}(n2^m + 2^m - 1)) : i \in \mathbb{N}, n \in \mathbb{N}, n2^m \geq L_m, (\eta(n2^m), \dots, \eta(n2^m + 2^m - 1)) = \nu\}.$$

Then

$$\mathcal{C}_m \subseteq \bigcup_{\nu \in \mathcal{B}_{2^m}(\eta)} \tilde{\mathcal{A}}_{\nu,m}. \tag{3.25}$$

Let $d = \max(\deg(p_1), \dots, \deg(p_w)) + 1$ and $q = k + wd$. It follows, from a similar argument to the proof of formula (3.20) in Theorem 1.2, that

$$|\mathcal{C}_m| \leq \sum_{\nu \in \mathcal{B}_{2^m}(\eta)} |\tilde{\mathcal{A}}_{\nu,m}| \leq (q + 1)2^q(2(k + 2)(d + 1)2^{m(k+d+1)}N)^q |\mathcal{B}_{2^m}(\eta)| \tag{3.26}$$

for any m with $2^m > \max(k + 1, d)$. Suppose that $f(n)$ is the concatenation of $\{f_i(n)\}_{i=0}^\infty$ with respect to $\{N_i\}_{i=0}^\infty$. Let $g_N(n)$ be the concatenation of $\{g_{N,i}\}_{i=0}^\infty$ with respect to $\{N_i\}_{i=0}^\infty$, that is,

$$g_N(n) = g_{N,i}(n) \quad \text{if } N_i \leq n < N_{i+1}.$$

By formula (3.23),

$$\|g_N(n) - f(n)\| < 2/N, \text{ for any } n \in \mathbb{N}.$$

This implies that $\lim_{N \rightarrow \infty} \sup_{n \in \mathbb{N}} |e(g_N(n)) - e(f(n))| = 0$. So by Propositions 2.1 and 2.2, to prove $e(f(n))$ deterministic, it suffices to prove $\mathcal{A}(g_N) = 0$ for N large enough.

In the following, to show $\mathcal{A}(g_N) = 0$, we estimate $|\mathcal{B}_{2^m}^{e,r}(g_N)|$ for any given m with $2^m > \max(k + 1, d)$, where $|\mathcal{B}_{2^m}^{e,r}(g_N)|$ is the cardinality of all regularly effective 2^m -blocks occurring in g_N . Let i_m be large enough such that $N_{i+1} - N_i > 2^m$ whenever $i > i_m$. Let $(g_N(n2^m), g_N(n2^m + 1), \dots, g_N(n2^m + 2^m - 1))$ be a 2^m -block in g_N with $n2^m > \max\{L_m, N_{i_m}\}$. It is easy to see that there are two cases about this block: one case is that there is an i_n such that $(g_N(n2^m), g_N(n2^m + 1), \dots, g_N(n2^m + 2^m - 1)) = (g_{N,i_n}(n2^m), g_{N,i_n}(n2^m + 1), \dots, g_{N,i_n}(n2^m + 2^m - 1))$; the other case is that there are two integers $i_n \geq 0$ and $j_n \geq 1$ such that $g_N(n2^m + j) = g_{N,i_n}(n2^m + j)$ when $j = 0, 1, \dots, j_n - 1$ and $g_N(n2^m + j) = g_{N,i_n+1}(n2^m + j)$ when $j = j_n, j_n + 1, \dots, 2^m - 1$. So $|\mathcal{B}_{2^m}^{e,r}(g_N)|$ is less than or equal to $2^m \times |\mathcal{C}_m|^2$. Note that N, k, d, q are parameters independent of m . Then by formula (3.26),

$$\mathcal{A}(g_N) = \lim_{m \rightarrow \infty} \frac{\log |\mathcal{B}_{2^m}^{e,r}(g_N)|}{2^m} \leq \lim_{m \rightarrow \infty} \frac{\log (|\mathcal{B}_{2^m}(\eta)|^2)}{2^m} = 2\mathcal{A}(\eta) = 0.$$

Now, we complete the proof of the theorem. □

As an application of Theorem 1.4, we show that under the assumption of SMDC, for any deterministic sequence $\xi(n)$, $\mu(n)\xi(n)$ does not correlate with $e(f(n))$ in short intervals on average when $f(n)$ satisfies certain conditions. Precisely,

THEOREM 3.2 *Let $g(n)$ be defined as in Theorem 1.2. Suppose that \mathcal{D} is a family of real-valued arithmetic functions such that*

$$\lim_{n \rightarrow \infty} \sup_{f \in \mathcal{D}} \|\Delta^k f(n) - g(n)\| = 0 \quad (3.27)$$

holds for some $k \geq 1$. Then SMDC implies that, for any deterministic sequence $\xi(n)$,

$$\lim_{h \rightarrow \infty} \limsup_{X \rightarrow \infty} \frac{1}{Xh} \int_X^{2X} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} \mu(n) \xi(n) e(f(n)) \right| dx = 0. \quad (3.28)$$

Proof. Let $\{N_i\}_{i=0}^\infty$ be a given sequence of natural numbers with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$ and $\{f_i(n)\}_{i=0}^\infty$ be a sequence in \mathcal{D} and choose $\{\theta_i\}_{i=0}^\infty$ as a sequence of numbers in $[0, 1)$ such that

$$\left| \sum_{N_i \leq n < N_{i+1}} \mu(n) \xi(n) e(f_i(n)) \right| = \left(\sum_{N_i \leq n < N_{i+1}} \mu(n) \xi(n) e(f_i(n)) \right) e(\theta_i).$$

Define $\tilde{f}_i(n) = f_i(n) + \theta_i$. Then $\{\tilde{f}_i(n)\}_{i=0}^\infty$ satisfies condition (3.28) since $k \geq 1$. Let $\tilde{f}(n)$ be the concatenation of $\{\tilde{f}_i(n)\}_{i=0}^\infty$ with respect to the sequence $\{N_i\}_{i=0}^\infty$. Let $f(n) = e(\tilde{f}(n))\xi(n)$. By Proposition 2.1 and Theorem 1.4, $f(n)$ is deterministic. Hence SMDC implies

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n) \xi(n) e(f_i(n)) \right| = 0.$$

Note that the above equation holds for any sequence $\{N_i\}_{i=0}^\infty$ of natural numbers and any sequence $\{f_i(n)\}_{i=0}^\infty$ in \mathcal{D} . By Lemma B.1, this implies that

$$\lim_{h \rightarrow \infty} \limsup_{X \rightarrow \infty} \frac{1}{Xh} \int_X^{2X} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} \mu(n) \xi(n) e(f(n)) \right| dx = 0,$$

as we claimed. \square

If we take $\xi(n) = 1_{n \equiv a \pmod{q}}(n)$ and \mathcal{D} the set of all polynomials of degrees less than a given positive integer in Theorem 3.2, then we have the following corollary.

COROLLARY 3.3 *Let $k \geq 1$ be a given integer. Denote by \mathcal{D}_k the set of all polynomials in $\mathbb{R}[y]$ of degrees less than k . Let $q \geq 1$ and $a \geq 0$ be given integers. Then SMDC implies*

$$\lim_{h \rightarrow \infty} \limsup_{X \rightarrow \infty} \frac{1}{Xh} \int_X^{2X} \sup_{\substack{p(y) \in \mathcal{D}_k \\ n \equiv a \pmod{q}}} \left| \sum_{x \leq n < x+h} \mu(n) e(p(n)) \right| dx = 0. \quad (3.29)$$

As observed in [37], equation (3.29) is implied by the local higher-order Fourier uniformity conjecture, which is deduced from the Chowla conjecture ([5], see also [34]). It is known that Chowla's conjecture implies SMDC. Corollary 3.3 shows that equation (3.29) can be deduced from SMDC.

Here are some results relevant to equation (3.29). For $k = 1$, equation (3.29) has been obtained from the work of Matomäki–Radziwiłł ([28]). For $k \geq 2$, it is open whether (3.29) holds, while recently Matomäki–Radziwiłł–Tao–Teräväinen–Ziegler in [30] established equation (3.29) when $h = X^\theta$ for any fixed $\theta > 0$. Without taking the average on X in equation (3.29), let $h = X^\theta$, the case that $k = 1$ and $\theta > 0.55$ was obtained by Matomäki–Teräväinen in [32]; the case that $k = 2$ and $\theta > 5/8$ was previously established by Zhan in [45] and extended to $\theta > 3/5$ in [32] and the case that $k \geq 2$ and $\theta > 2/3$ was obtained by Matomäki–Shao in [31] (also see [19], [27] for related results on $\Lambda(n)$ instead of $\mu(n)$).

Using a similar idea to the proof of Theorem 1.2, we obtain the following proposition that gives many characteristic functions with zero anqie entropy.

PROPOSITION 3.4 *Let $p_1(y), p_2(y) \in \mathbb{R}[y]$. Suppose that*

$$S = \{n \in \mathbb{N} : \{p_1(n)\} < \{p_2(n)\}\}.$$

Then $1_S(n)$, the characteristic function defined on S , is a deterministic sequence.

Proof. Assume that the degrees of $p_1(n)$ and $p_2(n)$ are both less than k . Then $\Delta^k p_1(n) = \Delta^k p_2(n) = 0$. In the following, for any given integer $J \geq k + 1$, we estimate $|\mathcal{B}_J(1_S)|$, the cardinality of the set of all J -blocks occurring in 1_S .

Firstly, we define linear functions $F_0, F_1, \dots, F_{J-1} : \mathbb{R}^k \rightarrow \mathbb{R}$. Define $F_j(x_0, x_1, \dots, x_{k-1})$ to be x_j when $j = 0, 1, \dots, k - 1$. Assume inductively that we have defined $F_{j_0}(x_0, x_1, \dots, x_{k-1})$ for some $j_0 \geq k - 1$. Then define $F_{j_0+1}(x_0, x_1, \dots, x_{k-1})$ to be the linear function satisfying

$$\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} F_{j_0+1-k+l}(x_0, x_1, \dots, x_{k-1}) = 0, \tag{3.30}$$

equivalently,

$$F_{j_0+1}(x_0, x_1, \dots, x_{k-1}) = - \sum_{l=0}^{k-1} (-1)^{k-l} \binom{k}{l} F_{j_0+1-k+l}(x_0, x_1, \dots, x_{k-1}).$$

By Proposition A.1, it is not hard to see that for $i = 1, 2$ and any $n \in \mathbb{N}$,

$$\{F_j(\{p_i(n)\}, \{p_i(n+1)\}, \dots, \{p_i(n+k-1)\})\} = \{p_i(n+j)\}, j = 0, 1, \dots, J - 1. \tag{3.31}$$

By Proposition A.5,

$$|F_j(\{p_i(n)\}, \{p_i(n+1)\}, \dots, \{p_i(n+k-1)\})| < (k+1)J^k, j = 0, 1, \dots, J - 1. \tag{3.32}$$

Secondly, we define a family of hyperplanes $\mathcal{F} = \{H_{L,j}, H_{M,j}, H_{N,j} : j, L, M, N \in \mathbb{Z}, 0 \leq j \leq J - 1, -2(k+1)J^k - 1 \leq L \leq 2(k+1)J^k, -(k+1)J^k - 1 \leq M, N \leq (k+1)J^k\}$ in \mathbb{R}^{2k} , where

$$H_{L,j} = \{(x_0, x_1, \dots, x_{2k-1}) \in \mathbb{R}^{2k} : F_j(x_0, x_1, \dots, x_{k-1}) - F_j(x_k, x_{k+1}, \dots, x_{2k-1}) = L\},$$

$$H_{M,j} = \{(x_0, x_1, \dots, x_{2k-1}) \in \mathbb{R}^{2k} : F_j(x_0, x_1, \dots, x_{k-1}) = M\}$$

and

$$H_{N,j} = \{(x_0, x_1, \dots, x_{2k-1}) \in \mathbb{R}^{2k} : F_j(x_k, x_{k+1}, \dots, x_{2k-1}) = N\}.$$

Then

$$|\mathcal{F}| \leq 8(k+2)J^{k+1}.$$

By Lemma 2.6, there are at most $W(|\mathcal{F}|, 2k)$ pieces of \mathbb{R}^{2k} cut by the hyperplanes in \mathcal{F} , where

$$W(|\mathcal{F}|, 2k) \leq 8^{2k}(2k+1)2^{2k}(k+2)^{2k}J^{2k(k+1)}.$$

Let \mathcal{P} be the set of all pieces of \mathbb{R}^{2k} cut by the hyperplanes in \mathcal{F} . Recall that $\mathcal{B}_J(1_S)$ denotes the set of all J -blocks occurring in $1_S(n)$. Denote by $|\mathcal{B}_J(1_S)| = C_J$. Suppose that $\mathcal{B}_J(1_S) = \{B_1, \dots, B_{C_J}\}$. Let $n_m = \min\{n \in \mathbb{N} : (1_S(n), 1_S(n+1), \dots, 1_S(n+J-1)) = B_m\}$, for $m = 1, \dots, C_J$. Then $B_m = (1_S(n_m), 1_S(n_m+1), \dots, 1_S(n_m+J-1))$. Define the map

$$\psi : \mathcal{B}_J(1_S) \rightarrow \mathcal{P}$$

by $\psi(B_m) = P_m$, where P_m is the unique piece in \mathcal{P} containing the point $(\{p_1(n_m)\}, \dots, \{p_1(n_m+k-1)\}, \{p_2(n_m)\}, \dots, \{p_2(n_m+k-1)\})$, for $m = 1, \dots, C_J$. Since any two pieces in \mathcal{P} are disjoint, ψ is well-defined.

In the following, we show that ψ is injective. Given $n \in \mathbb{N}$, let

$$(y_0, \dots, y_{k-1}, y_k, \dots, y_{2k-1}) = (\{p_1(n)\}, \dots, \{p_1(n+k-1)\}, \{p_2(n)\}, \dots, \{p_2(n+k-1)\}).$$

Suppose that $(y_0, \dots, y_{k-1}, y_k, \dots, y_{2k-1}) \in P$, a piece of \mathbb{R}^{2k} cut by the hyperplanes in \mathcal{F} . Then by (3.32), there are integers $M_0, \dots, M_{J-1}, N_0, \dots, N_{J-1} \in [-(k+1)J^k - 1, (k+1)J^k]$ and $L_0, \dots, L_{J-1} \in [-2(k+1)J^k - 1, 2(k+1)J^k]$, such that

$$L_j \leq F_j(y_0, \dots, y_{k-1}) - F_j(y_k, \dots, y_{2k-1}) < L_j + 1 \quad (3.33)$$

and

$$M_j \leq F_j(y_0, \dots, y_{k-1}) < M_j + 1, N_j \leq F_j(y_k, \dots, y_{2k-1}) < N_j + 1. \quad (3.34)$$

Moreover, the above inequalities also hold for each point in P . From formulas (3.33) and (3.34), it is not hard to see that for any given j with $0 \leq j \leq J-1$,

$$\{F_j(x_0, x_1, \dots, x_{k-1})\} < \{F_j(x_k, x_{k+1}, \dots, x_{2k-1})\}$$

holds for all $(x_0, x_1, \dots, x_{2k-1}) \in P$ (when $L_j = M_j - N_j - 1$) or

$$\{F_j(x_0, x_1, \dots, x_{k-1})\} \geq \{F_j(x_k, x_{k+1}, \dots, x_{2k-1})\}$$

holds for all $(x_0, x_1, \dots, x_{2k-1}) \in P$ (when $L_j = M_j - N_j$). Then by equation (3.31), we conclude that if $(\{p_1(n)\}, \dots, \{p_1(n+k-1)\}, \{p_2(n)\}, \dots, \{p_2(n+k-1)\})$, $(\{p_1(n')\}, \dots, \{p_1(n'+$

$k - 1\}$, $\{p_2(n')\}, \dots, \{p_2(n' + k - 1)\} \in P$, then $(1_S(n), \dots, 1_S(n + J - 1)) = (1_S(n'), \dots, 1_S(n' + J - 1))$. So ψ is injective. Hence $|\mathcal{B}_J(1_S)| = |\psi(\mathcal{B}_J(1_S))| \leq W(|\mathcal{F}|, 2k) \leq 8^{2k}(2k + 1)2^{2k}(k + 2)^{2k}J^{2k(k+1)}$. Then by formula (2.2),

$$\mathbb{A}(1_S) = \lim_{J \rightarrow \infty} \frac{\log |\mathcal{B}_J(1_S)|}{J} = 0.$$

So $1_S(n)$ is deterministic by Proposition 2.3. □

As an application of the above proposition, we give the following example that satisfies the condition in Theorem 1.2 with $g(n) = \Delta^2 f(n) \neq 0$.

EXAMPLE 3.5 Let $f(n) = \sqrt{3}n\{\sqrt{2}n\}$. Then

$$\Delta^2 f(n) = \begin{cases} 2\sqrt{3}(\sqrt{2} - 1), & n \in S_1, \\ 2\sqrt{3}(\sqrt{2} - 2), & n \in S_2, \\ 2\sqrt{3}(\sqrt{2} - 1) + \sqrt{3}n, & n \in S_3, \\ 2\sqrt{3}(\sqrt{2} - 2) - \sqrt{3}n, & n \in S_4, \end{cases}$$

where $S_1 = \{n \in \mathbb{N} : \{\sqrt{2}(n + 2)\} > \{\sqrt{2}(n + 1)\} > \{\sqrt{2}n\}\}$, $S_2 = \{n \in \mathbb{N} : \{\sqrt{2}(n + 2)\} < \{\sqrt{2}(n + 1)\} < \{\sqrt{2}n\}\}$, $S_3 = \{n \in \mathbb{N} : \{\sqrt{2}(n + 2)\} > \{\sqrt{2}(n + 1)\}, \{\sqrt{2}(n + 1)\} < \{\sqrt{2}n\}\}$ and $S_4 = \{n \in \mathbb{N} : \{\sqrt{2}(n + 2)\} < \{\sqrt{2}(n + 1)\}, \{\sqrt{2}(n + 1)\} > \{\sqrt{2}n\}\}$.

4. The Möbius disjointness of $e(f(n))$ with the k -th difference of $f(n)$ tending to zero

In this section, we shall study the disjointness of the Möbius function from exponential functions of arithmetic functions with the k -th differences tending to a constant (Propositions 1.5, 1.6 and 1.8 and Theorem 1.7). We first show the following property that arithmetic functions with k -th differences tending to zero can be approximated by certain concatenations of polynomials of degrees less than k .

LEMMA 4.1 Suppose that $f(n)$ is a real-valued arithmetic function such that

$$\lim_{n \rightarrow \infty} \|\Delta^k f(n)\| = 0,$$

for some integer $k \geq 1$. Then for any integer $N \geq 1$, there is an increasing sequence $\{N_i\}_{i=0}^\infty$ of natural numbers with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$ and a sequence $\{p_i(y)\}_{i=0}^\infty$ in $\mathbb{R}[y]$ of degrees less than k , such that

$$\|f(n) - g_N(n)\| \leq 1/N, \text{ for any } n \in \mathbb{N}, \tag{4.1}$$

where g_N is the concatenation of $\{p_i(n)\}_{i=0}^\infty$ with respect to $\{N_i\}_{i=0}^\infty$.

Proof. By Lemma 3.1, for each integer $m \geq 1$ and $N \geq 1$, there is a sufficiently large $L_m \in \mathbb{N}$ with $2^m | L_m$ such that, whenever $n \geq L_m$, we have

$$\|f(n+j) - Y_n(n+j)\| \leq 1/N, j = 0, 1, \dots, 2^m - 1, \tag{4.2}$$

where $Y_n(n+j)$ is defined to be $f(n+j)$ when $0 \leq j \leq k-1$ and the value determined by the following linear equations when $k \leq j \leq 2^m - 1$,

$$\Delta^k Y_n(n+j) = 0, j = 0, 1, \dots, 2^m - k - 1.$$

It is not hard to check that

$$Y_n(n+j) = \sum_{l=0}^{k-1} f(n+l) \prod_{t=0, t \neq l}^{k-1} \frac{j-t}{l-t}, j = 0, 1, \dots, 2^m - 1. \tag{4.3}$$

We may further assume that the sequence $\{L_m\}_{m=0}^\infty$ ($L_0 = 0$) chosen above satisfies $L_{m+1} > L_m$ for each m . Let $d_m = (L_{m+1} - L_m)/2^m$. Then the following is a partition of \mathbb{N} .

$$\mathbb{N} = \bigcup_{m=0}^\infty \bigcup_{q=0}^{d_m-1} \{L_m + aq2^m, L_m + aq2^m + 1, \dots, L_m + aq2^m + 2^m - 1\}.$$

Choose the sequence $\{N_i\}_{i=0}^\infty$ with $N_0 < N_1 < N_2 < \dots$ such that

$$\{N_0, N_1, \dots\} = \{L_m + aq2^m : m \in \mathbb{N}, 0 \leq q \leq d_m - 1\}.$$

Define

$$p_i(n) = \sum_{l=0}^{k-1} f(N_i+l) \prod_{t=0, t \neq l}^{k-1} \frac{n - N_i - t}{l - t}. \tag{4.4}$$

It is a polynomial of degree less than k . Let

$$g_N(n) = p_i(n), \text{ when } N_i \leq n \leq N_{i+1} - 1.$$

By formula (4.2) and equation (4.3), we obtain formula (4.1). □

LEMMA 4.2 *Let $\{N_i\}_{i=0}^\infty$ be an increasing sequence of natural numbers with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$. Let $\{p_i(y)\}_{i=0}^\infty$ be a sequence in $\mathbb{R}[y]$ with degrees less than k for some positive integer k . Suppose that $f(n)$ is the concatenation of $\{p_i(n)\}_{i=0}^\infty$ with respect to $\{N_i\}_{i=0}^\infty$. Let*

q be a positive integer and $0 \leq a \leq q - 1$. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{1 \leq n \leq N \\ n \equiv a \pmod{q}}} \mu(n)e(f(n)) = 0 \tag{4.5}$$

if and only if

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \sum_{\substack{N_i \leq n \leq N_{i+1}-1 \\ n \equiv a \pmod{q}}} \mu(n)e(p_i(n)) = 0. \tag{4.6}$$

Proof. It is obvious that (4.5) \Rightarrow (4.6). We now show (4.6) \Rightarrow (4.5). In this process, we need to use a classical result (see [6] for $k = 2$ and [15, Chapter 6, Theorem 10] for $k > 2$) stated as follows,

$$\sup_{\substack{p(y) \in \mathbb{R}[y] \\ \deg(p(y)) < k}} \left| \sum_{\substack{1 \leq n \leq N \\ n \equiv a \pmod{q}}} \mu(n)e(p(n)) \right| \ll N/\log N,$$

where the implied constant at most depends on k and q . By the above inequality and equation (4.6), for any given $\epsilon > 0$, there is a positive integer M such that whenever $m \geq M$ and $N \geq N_M$, we have

$$\sum_{i=0}^{m-1} \sum_{\substack{N_i \leq n \leq N_{i+1}-1 \\ n \equiv a \pmod{q}}} \mu(n)e(p_i(n)) < (\epsilon/2)N_m \tag{4.7}$$

and

$$\sup_{i \in \mathbb{N}} \left| \sum_{\substack{1 \leq n \leq N \\ n \equiv a \pmod{q}}} \mu(n)e(p_i(n)) \right| < (\epsilon/4)N. \tag{4.8}$$

Let $N \geq N_M$. Choose an appropriate $l \geq M$ with $N_l \leq N \leq N_{l+1} - 1$. Then

$$\begin{aligned} \left| \sum_{\substack{n=1 \\ n \equiv a \pmod{q}}}^N \mu(n)e(f(n)) \right| &\leq \left| \sum_{i=0}^{l-1} \sum_{\substack{N_i \leq n < N_{i+1} \\ n \equiv a \pmod{q}}} \mu(n)e(p_i(n)) \right| + \left| \sum_{\substack{N_l \leq n \leq N \\ n \equiv a \pmod{q}}} \mu(n)e(p_l(n)) \right| \\ &< (\epsilon/2)N_l + (\epsilon/2)N < \epsilon N. \end{aligned}$$

Hence we obtain (4.5). □

We now prove Proposition 1.8, which gives a sufficient condition of disjointness between the Möbius function and exponential functions of arithmetic functions with the k -th differences tending to 0.

Proof of Proposition 1.8. By Lemma 4.1, there is a sequence $\{g_M(n)\}_{M=1}^\infty$ such that

$$\lim_{M \rightarrow \infty} \sup_{n \in \mathbb{N}} |e(g_M(n)) - e(f(n))| = 0,$$

where $g_M(n)$ is a certain concatenation of polynomials. Under the assumption of equation (1.6), by Lemmas B.1 and 4.2, we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) e(g_M(n)) = 0.$$

So

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) e(f(n)) = 0,$$

completing the proof. □

Next, we shall prove Propositions 1.5 and 1.6. It is easy to see that Proposition 1.5 directly follows from Proposition 1.6, so in the following, we just give the proof of Proposition 1.6. Before proving it, we need some preparations. The following Dirichlet’s approximation theorem is classical and well-known (see for example, [38, Section 8.2]), which is proved via the fact that if there are $m + 1$ points contained in m regions, then there must be at least two points lying in the same region.

LEMMA 4.3 *Given L real numbers $\theta_1, \dots, \theta_L$ and a positive integer q , then we can find an integer $t \in [1, q^L]$ and integers a_1, \dots, a_L such that $|t\theta_j - a_j| \leq 1/q, j = 1, 2, \dots, L$.*

The following ‘asymptotical periodicity’ of the concatenation of certain linear phases will be used in the proof of Proposition 1.6.

LEMMA 4.4 *Let $\{N_i\}_{i=0}^\infty$ be an increasing sequence of integers with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$. Suppose that $\alpha_0, \alpha_1, \dots$ are real numbers such that the sequence $\{e(\alpha_i)\}_{i=0}^\infty$ has finitely many limit points. Assume that $f(n) = e(n\alpha_i)e(\beta_i)$ when $N_i \leq n < N_{i+1}$ for $i = 0, 1, 2, \dots$, where β_0, β_1, \dots are real numbers. Then there is a δ with $0 < \delta < 1$ and a sequence $\{n_j\}_{j=0}^\infty$ of positive integers with $\lim_{j \rightarrow \infty} n_j = \infty$ such that*

$$\lim_{j \rightarrow \infty} \sum_{l=1}^{n_j^\delta} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |f(n + ln_j) - f(n)|^2 = 0.$$

Proof. Suppose that the limit points of $\{e(\alpha_i)\}_{i=0}^\infty$ are $e(\theta_1), \dots, e(\theta_L)$ for some integer $L \geq 1$. Let $q_j = (j + 6)^2 \pi^2, j \geq 0$. By Lemma 4.3, we can find an integer n_j with $1 \leq n_j \leq q_j^L$ such that $|\theta_s - \frac{a_{s,j}}{n_j}| \leq \frac{1}{n_j q_j}$, where $a_{s,j}$ is some integer for $s = 1, \dots, L$. It is not hard to check that the sequence $\{n_j\}_{j=0}^\infty$ can be chosen to satisfy $\lim_{j \rightarrow \infty} n_j = \infty$. Moreover, there is an i_0 such that when $i \geq i_0$ we can

choose an $s \in \{1, 2, \dots, L\}$ satisfying $\|\alpha_i - \theta_s\| < \frac{1}{n_j q_j}$. Then for $i \geq i_0$, $\|n_j \alpha_i\| < \frac{2}{q_j}$ and $|e(n_j \alpha_i) - 1| = 2|\sin(\pi n_j \alpha_i)| \ll \frac{1}{q_j}$. So, for any given n_j ,

$$\begin{aligned} & \sum_{l=1}^{\frac{1}{2L}} \limsup_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{i=0}^{m-1} \sum_{n=N_i}^{N_{i+1}-1} |f(n+ln_j) - f(n)|^2 + \sum_{n=N_m}^N |f(n+ln_j) - f(n)|^2 \right) \\ &= \sum_{l=1}^{\frac{1}{2L}} \limsup_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{i=0}^{m-1} \sum_{n=N_i}^{N_{i+1}-ln_j} |f(n+ln_j) - f(n)|^2 + \sum_{n=N_m}^{N-ln_j} |f(n+ln_j) - f(n)|^2 \right) \\ &= \sum_{l=1}^{\frac{1}{2L}} \limsup_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{i=0}^{m-1} \sum_{n=N_i}^{N_{i+1}-ln_j} |e(ln_j \alpha_i) - 1|^2 + \sum_{n=N_m}^{N-ln_j} |e(ln_j \alpha_m) - 1|^2 \right) \\ &= \sum_{l=1}^{\frac{1}{2L}} \limsup_{m \rightarrow \infty} \frac{1}{N} \left(\sum_{i=0}^{m-1} \sum_{n=N_i}^{N_{i+1}-ln_j} |e(ln_j \alpha_i) - 1|^2 + \sum_{n=N_m}^{N-ln_j} |e(ln_j \alpha_m) - 1|^2 \right) \\ &\ll \sum_{l=1}^{\frac{1}{2L}} \frac{l^2}{q_j^2} \ll \frac{1}{q_j^2} \rightarrow 0, j \rightarrow \infty. \end{aligned}$$

The claimed result follows by choosing $\delta = \frac{1}{2L}$. □

To prove Proposition 1.6, we also need the following result of the second author [43, Lemma 4.1] on the self-correlation of $\mu(n)e(P(n))$ in short arithmetic progressions.

LEMMA 4.5 *Let $s \geq 1$ and $h \geq 3$ be integers. Suppose that $P(x) \in \mathbb{R}[x]$ is of degree $d \geq 0$.*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left| \frac{1}{h} \sum_{l=1}^h \mu(n+ls)e(P(n+ls)) \right|^2 \ll \frac{s}{\varphi(s)} \frac{\log \log h}{\log h}, \tag{4.9}$$

where φ is the Euler totient function and the implied constant depends on d at most.

Proof of Proposition 1.6. Let $f_0(n) = \frac{\epsilon}{2}n^2 + c_1n + c_0$ for $n \in \mathbb{N}$. Then $\Delta^2 f_0(n) = c$ and $\lim_{n \rightarrow \infty} \|\Delta^2(f - f_0)(n)\| = 0$. For any given $\epsilon > 0$, by Lemma 4.1 and equation (4.4), there is an increasing sequence $\{N_i\}_{i=0}^\infty$ with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$, and a function $g_\epsilon(n)$ defined by $g_\epsilon(n) = n(f(N_i + 1) - f(N_i)) + (N_i + 1)f(N_i) - f(N_i + 1)N_i$ when $N_i \leq n < N_{i+1}$ for $i = 0, 1, \dots$, such that

$$\sup_{n \in \mathbb{N}} |e(f(n)) - e(f_0(n) + g_\epsilon(n))| < \epsilon. \tag{4.10}$$

We claim that

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n) e(f_0(n)) e(g_\epsilon(n)) \right| = 0. \quad (4.11)$$

Using Lemma 4.2,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n) e(f_0(n) + g_\epsilon(n)) = 0.$$

So by formula (4.10), we have for N large enough,

$$\frac{1}{N} \left| \sum_{1 \leq n \leq N} \mu(n) e(f(n)) \right| < 2\epsilon.$$

Letting $\epsilon \rightarrow 0$, we then obtain the statement in this proposition.

We are left to prove claim (4.11). Write $\alpha_i = f(N_i + 1) - f(N_i)$. Choose $\{\beta_i\}_{i=0}^\infty$ as a sequence of real numbers such that

$$\left| \sum_{N_i \leq n < N_{i+1}} \mu(n) e(f_0(n)) e(g_\epsilon(n)) \right| = \sum_{N_i \leq n < N_{i+1}} \mu(n) e(f_0(n)) e(n\alpha_i) e(\beta_i).$$

Define $g(n)$ to be $e(n\alpha_i) e(\beta_i)$ when $N_i \leq n < N_{i+1}$, $i = 0, 1, \dots$. So it suffices to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) e(f_0(n)) g(n) = 0. \quad (4.12)$$

Note that the set $\{e(\alpha_i) : i = 0, 1, \dots\}$ has finitely many limit points. By Lemma 4.4, there is a δ with $0 < \delta < 1$ and a sequence $\{n_j\}_{j=0}^\infty$ of positive integers with $\lim_{j \rightarrow \infty} n_j = \infty$ such that

$$\lim_{j \rightarrow \infty} \frac{1}{n_j^\delta} \sum_{l=1}^{n_j^\delta} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |g(n + l n_j) - g(n)|^2 = 0. \quad (4.13)$$

Note that $\frac{s}{\varphi(s)} \ll \log \log s$. By Lemma 4.5, for any n_j ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left| \frac{1}{n_j^\delta} \sum_{l=1}^{n_j^\delta} \mu(n + l n_j) e(f_0(n + l n_j)) \right|^2 \ll (\log \log n_j) \frac{\log \log n_j^\delta}{\log n_j^\delta}. \quad (4.14)$$

Note that the right side of the above inequality tends to zero as $j \rightarrow \infty$. Hence for any given $\epsilon > 0$, by formulas (4.13) and (4.14), there are positive integers j_0 and M_0 such that whenever $N > M_0$,

$$\frac{1}{n_{j_0}^\delta} \sum_{l=1}^{n_{j_0}^\delta} \frac{1}{N} \sum_{n=1}^N |g(n + l n_{j_0}) - g(n)|^2 < \epsilon^2/9$$

and

$$\frac{1}{N} \sum_{n=1}^N \left| \frac{1}{n_{j_0}^\delta} \sum_{l=1}^{n_{j_0}^\delta} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) \right|^2 < \epsilon^2/9.$$

Then by the Cauchy–Schwarz inequality,

$$\frac{1}{n_{j_0}^\delta} \sum_{l=1}^{n_{j_0}^\delta} \frac{1}{N} \sum_{n=1}^N |g(n + ln_{j_0}) - g(n)| < \epsilon/3 \tag{4.15}$$

and

$$\frac{1}{N} \sum_{n=1}^N \left| \frac{1}{n_{j_0}^\delta} \sum_{l=1}^{n_{j_0}^\delta} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) \right| < \epsilon/3. \tag{4.16}$$

Observe that

$$\frac{1}{N} \sum_{n=1}^N \mu(n) e(f_0(n)) g(n) = \frac{1}{N} \sum_{n=1}^N \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) g(n + ln_{j_0}) + O\left(\frac{ln_{j_0}}{N}\right).$$

Write $h_{j_0} = n_{j_0}^\delta$. Then there is a positive integer M_1 such that whenever $N > M_1$,

$$\left| \frac{1}{N} \sum_{n=1}^N \mu(n) e(f_0(n)) g(n) - \frac{1}{N} \sum_{n=1}^N \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) g(n + ln_{j_0}) \right| < \epsilon/3. \tag{4.17}$$

Let $M_2 = \max\{M_0, M_1\}$. Then by formulas (4.15), (4.16) and (4.17), for $N > M_2$, we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=1}^N \mu(n) e(f_0(n)) g(n) \right| &< \left| \frac{1}{N} \sum_{n=1}^N \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) g(n + ln_{j_0}) \right| + \epsilon/3 \\ &\leq \left| \frac{1}{N} \sum_{n=1}^N \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) (g(n + ln_{j_0}) - g(n)) \right| \\ &\quad + \left| \frac{1}{N} \sum_{n=1}^N \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) g(n) \right| + \epsilon/3 \\ &\leq \frac{1}{N} \sum_{n=1}^N \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} |g(n + ln_{j_0}) - g(n)| \\ &\quad + \frac{1}{N} \sum_{n=1}^N \left| \frac{1}{h_{j_0}} \sum_{l=1}^{h_{j_0}} \mu(n + ln_{j_0}) e(f_0(n + ln_{j_0})) \right| + \epsilon/3 \\ &< \epsilon. \end{aligned}$$

Then we obtain equation (4.12). □

In the remaining part of this section, we shall prove Theorem 1.7. The major ingredient of our proof is Matomäki–Radziwiłł–Tao–Teräväinen–Ziegler’s recent work [30] on averages of the correlation between multiplicative functions and polynomial phases in short intervals. To state this result, we shall use the following distance function of Granville and Soundararajan,

$$\mathbb{D}(g(n), n \mapsto n^{it}; X) := \left(\sum_{p \leq X} \frac{1 - \operatorname{Re}(g(p)p^{-it})}{p} \right)^{\frac{1}{2}}$$

for any multiplicative function $g(n)$ with $|g(n)| \leq 1$ for all $n \in \mathbb{N}$. This distance function was introduced in [12] to measure the pretentiousness between $g(n)$ and $n \mapsto n^{it}$. Throughout this section, define

$$M(g; X, Q) := \inf_{\substack{\chi \bmod q \\ 1 \leq q \leq Q \\ |t| \leq X}} \mathbb{D}(g(n)\chi(n), n \mapsto n^{it}; X)^2 = \inf_{\substack{\chi \bmod q \\ 1 \leq q \leq Q \\ |t| \leq X}} \sum_{p \leq X} \frac{1 - \operatorname{Re}(g(p)\chi(p)p^{-it})}{p}.$$

LEMMA 4.6 [30, Theorems 1.3 and 1.8]. *Let k be a given positive integer, and let $5/8 < \tau < 1$ and $0 < \theta < 1$ be fixed. Denote by \mathcal{D}_k the set of all polynomials in $\mathbb{R}[y]$ of degrees less than k . Let $f: \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function with $|f(n)| \leq 1$ for any $n \in \mathbb{N}$. Suppose that $X \geq 1$, $X^\theta \geq H \geq \exp((\log X)^\tau)$ and $\eta > 0$ are such that*

$$\int_X^{2X} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+H} f(n)e(p(n)) \right| dx \geq \eta HX.$$

Then

$$M(f; AX^k/H^{k-1/2}, Q) \ll_{k,\eta,\theta,\tau} 1$$

for some positive numbers $A, Q \ll_{k,\eta,\theta,\tau} 1$.

The following is a well-known result (see, for example, [29, (1.12)]) about the ‘non-pretentious’ nature of the Möbius function.

PROPOSITION 4.7 *Let $Q > 0$. For $\epsilon > 0$, we have for X large enough,*

$$M(\mu; X, Q) \geq (1/3 - \epsilon) \log \log X + O(1).$$

By the above proposition and Lemma 4.6, we have the following result which states that $\mu(n)$ does not correlate with polynomial phases in short intervals on average.

LEMMA 4.8 *Let k be a given positive integer, and let $5/8 < \tau < 1$ and $0 < \theta < 1$ be fixed. Suppose that $X \geq 1$ and $X^\theta \geq H \geq \exp((\log X)^\tau)$. Denote by \mathcal{D}_k the set of all polynomials in $\mathbb{R}[y]$ of degrees*

less than k . Then for any $\eta > 0$, there is an $X_0 > 0$ (at most depends on k, η, θ, τ) such that whenever $X > X_0$,

$$\int_X^{2X} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+H} \mu(n)e(p(n)) \right| dx < \eta HX.$$

Now applying Lemma 4.8, we show the following result.

THEOREM 4.9 *Let $\tau \in (5/8, 1)$ be given. Let $\{N_i\}_{i=0}^\infty$ be an increasing sequence of natural numbers with $N_0 = 0$ and $N_{i+1} - N_i \geq \exp((\log i)^\tau)$ for i large enough, and let $\{p_i(y)\}_{i=0}^\infty$ be a sequence in $\mathbb{R}[y]$ of degrees less than k for some positive integer k . Then*

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n)e(p_i(n)) \right| = 0. \tag{4.18}$$

Proof. Given ϵ sufficiently small with $(1 - 2\epsilon)\tau > 5/8$. Choose $h_m = \exp((\log N_m)^{(1-2\epsilon)\tau})$. By Lemma 4.8 and a dyadic subdivision, for N_m large enough,

$$\sum_{x=0}^{\lfloor \frac{N_m}{h_m} \rfloor + 1)h_m - 1} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+h_m} \mu(n)e(p(n)) \right| \leq \epsilon N_m h_m, \tag{4.19}$$

where \mathcal{D}_k is the set of polynomials in $\mathbb{R}[y]$ of degrees less than k .

Let $S_j = \{lh_m + j : l = 0, 1, \dots, \lfloor \frac{N_m}{h_m} \rfloor\}$, for $0 \leq j \leq h_m - 1$. Then, by (4.19), there is a j_0 such that

$$\sum_{x \in S_{j_0}} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+h_m} \mu(n)e(p(n)) \right| \leq \epsilon N_m. \tag{4.20}$$

Suppose $S_{j_0} \cap [N_i, N_{i+1}) = \{x_1^{(i)}, x_2^{(i)}, \dots, x_{l_i}^{(i)}\}$, where $x_1^{(i)} < x_2^{(i)} < \dots < x_{l_i}^{(i)}$, $i = 0, 1, \dots, m - 1$. Then

$$\left| \sum_{N_i \leq n < N_{i+1}} \mu(n)e(p_i(n)) \right| \leq \sum_{t=1}^{l_i-1} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x_t^{(i)} \leq n < x_t^{(i)} + h_m} \mu(n)e(p(n)) \right| + 2h_m.$$

So

$$\sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n)e(p_i(n)) \right| \leq \sum_{x \in S_{j_0}} \sup_{p(y) \in \mathcal{D}_k} \left| \sum_{x \leq n < x+h_m} \mu(n)e(p(n)) \right| + 2mh_m.$$

By formula (4.20),

$$\frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n)e(p_i(n)) \right| \leq \frac{\epsilon N_m}{N_m} + \frac{2mh_m}{N_m}.$$

Since $N_{i+1} - N_i \geq \exp((\log i)^\tau)$ for i large enough, we have $m \leq N_m / (\exp((\log N_m)^{(1-\epsilon)\tau}))$ for N_m sufficiently large. Inserting this into the above inequality and letting $m \rightarrow \infty$, we obtain

$$\limsup_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} \mu(n) e(p_i(n)) \right| \leq \epsilon.$$

Letting $\epsilon \rightarrow 0$, we obtain equation (4.18). □

Now we are ready to prove Theorem 1.7, which states the Möbius disjointness of $e(f(n))$ with the k -th difference of $f(n)$ tending to zero as in formula (1.5).

Proof of Theorem 1.7. Firstly, by Proposition A.6, for $j \geq k$, there are integers a_k, \dots, a_j such that for each $n \in \mathbb{N}$,

$$\sum_{s=k}^j a_s \cdot \Delta^k f(n+s-k) = f(n+j) - \sum_{l=0}^{k-1} f(n+l) \prod_{t=0, t \neq l}^{k-1} \frac{j-t}{l-t}, \tag{4.21}$$

where $0 \leq a_s \leq j^{k-1}$ for $s = k, \dots, j$. Note that $\sum_{l=0}^{k-1} f(n+l) \prod_{t=0, t \neq l}^{k-1} \frac{j-t}{l-t} = f(n+j)$ for $j = 0, \dots, k-1$. Then by condition (1.5), for any $j \in \mathbb{N}$,

$$\left\| f(n+j) - \sum_{l=0}^{k-1} f(n+l) \prod_{t=0, t \neq l}^{k-1} \frac{j-t}{l-t} \right\| \leq \frac{Cj^k}{\exp((\log n)^\tau)}.$$

For any positive integer M with $MC \geq 1$, choose $L_0 = 0$ and $L_m = 2^m \lfloor \exp(\log^{\frac{1}{\tau}}(MC2^{mk})) + 1 \rfloor$ for $m = 1, 2, \dots$. Then by the above inequality, we have for $n \geq L_m$ and $j = 0, 1, \dots, 2^m - 1$,

$$\left\| f(n+j) - \sum_{l=0}^{k-1} f(n+l) \prod_{t=0, t \neq l}^{k-1} \frac{j-t}{l-t} \right\| \leq \frac{1}{M}. \tag{4.22}$$

Let $d_m = (L_{m+1} - L_m) / 2^m$. Setting $0 = N_0 < N_1 < N_2 < \dots$ with $\{N_0, N_1, \dots\}$ being the set of $\{L_m + t2^m : m \in \mathbb{N}, 0 \leq t \leq d_m - 1\}$. Assume $L_m = N_{k_m}$. Then $k_{m+1} = k_m + d_m$. Note that $k_0 = 0$. Then for $m \geq 1$,

$$\exp(\log^{\frac{1}{\tau}}(MC2^{mk})) \ll k_m \ll m \exp(\log^{\frac{1}{\tau}}(MC2^{mk})).$$

Choose an appropriate $\epsilon > 0$ with $\tau - \epsilon > 5/8$. Then for m large enough, we have $2^m \geq \exp((\log k_{m+1})^{\tau-\epsilon})$. By the choice of N_i , this leads to $N_{i+1} - N_i > \exp((\log i)^{\tau-\epsilon})$ for i large enough.

Define

$$p_M(n) = \sum_{l=0}^{k-1} f(N_i + l) \prod_{t=0, t \neq l}^{k-1} \frac{n - N_i - t}{l - t}$$

when $N_i \leq n < N_{i+1}$, $i = 0, 1, \dots$. Hence by Theorem 4.9,

$$\lim_{s \rightarrow \infty} \frac{1}{N_s} \sum_{i=0}^{s-1} \sum_{N_i \leq n < N_{i+1}} \mu(n) e(p_M(n)) = 0,$$

and further by Lemma 4.2,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(n) e(p_M(n)) = 0. \quad (4.23)$$

Since $\sup_{n \in \mathbb{N}} \|f(n) - p_M(n)\| \leq \frac{1}{M}$ by equation (4.22), $\lim_{M \rightarrow \infty} \sup_{n \in \mathbb{N}} |e(f(n)) - e(p_M(n))| = 0$. Hence it follows from equation (4.23) that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(n) e(f(n)) = 0$$

as claimed. □

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Appendix A Some properties of the difference operator

Recall that the difference operator Δ is defined as $(\Delta f)(n) = f(n+1) - f(n)$ for any arithmetic function f . In this section, we give some basic properties of the operator Δ^k for $k \in \mathbb{N}$, which are used in this paper. The following one can be easily deduced by induction.

PROPOSITION A.1 For any $k \in \mathbb{N}$ and any arithmetic function f , we have

$$\Delta^k f(n) = \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} f(n+l). \quad (\text{A1})$$

It is known that $\Delta^k f(n) \equiv 0$ if and only if $f(n)$ is a polynomial with degree $k-1$. Moreover, we have the following proposition, which is known as the Lagrange interpolating polynomial.

PROPOSITION A.2 Given integers J, k with $J > k \geq 0$. Suppose that $f(0), \dots, f(J-1)$ satisfy $\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} f(n+l) = 0$ for $n = 0, \dots, J-1-k$. Then

$$f(n) = \sum_{j=0}^{k-1} f(j) \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i}$$

for $n = 0, \dots, J-1$.

Proof. Suppose $g(n) = \sum_{j=0}^{k-1} f(j) \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i}$. Then $g(n)$ is a polynomial of degree at most $k-1$ and satisfies $g(0) = f(0), \dots, g(k-1) = f(k-1)$. By Proposition A.1,

$$\Delta^k g(n) = \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} g(n+l) = 0, \quad n \geq 0.$$

For any given x_0, \dots, x_{k-1} , the solution (x_k, \dots, x_{J-1}) that satisfies $\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} x_{n+l} = 0$ for $n = 0, \dots, J-k-1$ is unique. Then $(f(k), \dots, f(J-1)) = (g(k), \dots, g(J-1))$. Hence $f(n)$ is of the form as claimed in this proposition. \square

The following simple fact is used in this paper.

PROPOSITION A.3 For any $n \in \mathbb{N}$, $k \geq 1$ and $0 \leq j \leq k-1$, $\prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i}$ is an integer.

Proof. For $0 \leq n \leq k-1$ and $n \neq j$, $\prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i} = 0$; for $n = j$, $\prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i} = 1$; for $n \geq k$,

$$\begin{aligned} \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i} &= \prod_{0 \leq i \leq j-1} \frac{n-i}{j-i} \prod_{j+1 \leq i \leq k-1} \frac{n-i}{j-i} \\ &= (-1)^{k-j-1} \frac{n!}{(n-k)! j! (k-1-j)! (n-j)} \\ &= (-1)^{k-j-1} \binom{n-j-1}{n-k} \binom{n}{n-j} \end{aligned}$$

is an integer. □

The next one gives a variant version of Proposition A.2.

LEMMA A.4 *Given integers J, k with $J > k \geq 0$. Suppose that $x_0, \dots, x_{J-1} \in \mathbb{R}$. Then the following two statements are equivalent.*

(i) For $n = 0, \dots, J - 1 - k$,

$$\left\{ \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \{x_{n+l}\} \right\} = 0. \tag{A2}$$

(ii) For $n = 0, \dots, J - 1$,

$$\{x_n\} = \left\{ \sum_{j=0}^{k-1} \{x_j\} \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i} \right\}, \tag{A3}$$

where $\{\cdot\}$ denotes the fractional part function.

Proof. (i) \Rightarrow (ii). We first show that there are integers c_0, c_1, \dots, c_{J-1} such that $x_0 + c_0, x_1 + c_1, \dots, x_{J-1} + c_{J-1}$ satisfy the following linear equation

$$\sum_{l=0}^k (-1)^{k-l} \binom{k}{l} (x_{n+l} + c_{n+l}) = 0, \quad n = 0, \dots, J - 1 - k. \tag{A4}$$

Let $c_0 = \dots = c_{k-1} = 0$. Letting $n = 0$ in equation (A4), by equation (A2) we have that the solution c_k is an integer. Repeating the above process with $n = 1, \dots, J - 1 - k$, we obtain successively solutions c_{k+1}, \dots, c_{J-1} , which are all integers. By equation (A4) and Proposition A.2,

$$x_n + c_n = \sum_{j=0}^{k-1} (x_j + c_j) \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i}, \quad n = 0, \dots, J - 1.$$

By Proposition A.3, we have equation (A3).

(ii) \Rightarrow (i). Assume that x_0, x_1, \dots, x_{J-1} satisfy equation (A3). Let

$$g(n) = \sum_{j=0}^{k-1} \{x_j\} \prod_{\substack{0 \leq i \leq k-1 \\ i \neq j}} \frac{n-i}{j-i}, \quad n = 0, \dots, J - 1.$$

Then $g(n)$ is a polynomial of degree at most $k - 1$. By Proposition A.1,

$$\Delta^k g(n) = \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} g(n+l) = 0, \quad n \geq 0.$$

By (ii), $\{g(0)\} = \{x_0\}$, $\{g(1)\} = \{x_1\}$, \dots , $\{g(J-1)\} = \{x_{J-1}\}$. Then

$$\left\{ \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \{x_{n+l}\} \right\} = \left\{ \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \{g(n+l)\} \right\} = 0, \quad n = 0, \dots, J-1-k.$$

□

The following gives an estimate of $f(n)$ through the initial values and the upper bound of $\Delta^k f(n)$.

PROPOSITION A.5 *Given $n \in \mathbb{N}$ and a real number $c > 0$. Suppose integers $J > k \geq 0$. If $f(n), f(n+1), \dots, f(n+J-1)$ satisfy:*

- (a) $|\Delta^k f(n+j)| \leq c, j = 0, 1, \dots, J-1-k;$
- (b) $f(n+j) \in [0, c], j = 0, 1, \dots, k-1.$

Then we have

$$|f(n+j)| \leq (k+1)j^k c, \quad j = k, k+1, \dots, J-1.$$

Proof. By (b), for $j = 0, 1, \dots, k-2$,

$$|\Delta f(n+j)| \leq c,$$

and by induction,

$$|\Delta^m f(n+j)| \leq 2^{m-1} c, \quad j = 0, 1, \dots, k-m-1, \quad 1 \leq m \leq k-1. \quad (\text{A5})$$

We first claim that, when $1 \leq m \leq k$ and $k-m \leq j \leq J-1-m$, we have

$$|\Delta^m f(n+j)| \leq \sum_{i=0}^{k-m-1} \binom{j-(k-m)+i}{j-(k-m)} 2^{m-1+i} c + \binom{j}{k-m} c, \quad (\text{A6})$$

where $\binom{0}{0} = 1$. In the following we shall prove formula (A6). When $m = k$, the inequality (A6) holds by (a). Assume inductively that formula (A6) holds when $m = m_0 + 1 \leq k$. We shall prove that formula (A6) holds when $m = m_0$ and $k - m_0 \leq j \leq J - 1 - m_0$. For $j = k - m_0$, we have

$$|\Delta^{m_0} f(n+k-m_0)| \leq |\Delta^{m_0+1} f(n+k-m_0-1)| + |\Delta^{m_0} f(n+k-m_0-1)|.$$

Then by the inductive hypothesis on the $m_0 + 1$ case and formula (A5),

$$\begin{aligned} |\Delta^{m_0} f(n+k-m_0)| &\leq 2^{m_0-1}c + \sum_{i=0}^{k-m_0-2} \binom{k-m_0-1-(k-m_0-1)+i}{k-m_0-1-(k-m_0-1)} 2^{m_0+i}c \\ &\quad + \binom{k-m_0-1}{k-m_0-1}c \\ &= \sum_{i=0}^{k-m_0-1} \binom{k-m_0-(k-m_0)+i}{k-m_0-(k-m_0)} 2^{m_0-1+i}c + \binom{k-m_0}{k-m_0}c. \end{aligned}$$

Note that in the above formula the coefficients before $2^{m_0+i}c$ and $2^{m_0-1+i}c$ are all 1. So formula (A6) holds for $m = m_0$ and $j = k - m_0$. Now assume inductively that formula (A6) holds for $m = m_0$ and some $j = j_0 \geq k - m_0$. When $j = j_0 + 1$, we have

$$\begin{aligned} |\Delta^{m_0} f(n+j_0+1)| &\leq |\Delta^{m_0} f(n+j_0)| + |\Delta^{m_0+1} f(n+j_0)| \\ &\leq \sum_{i=0}^{k-m_0-1} \binom{j_0-(k-m_0)+i}{j_0-(k-m_0)} 2^{m_0-1+i}c + \binom{j_0}{k-m_0}c \\ &\quad + \sum_{i=0}^{k-m_0-2} \binom{j_0-(k-m_0-1)+i}{j_0-(k-m_0-1)} 2^{m_0+i}c + \binom{j_0}{k-m_0-1}c \\ &= \sum_{i=0}^{k-m_0-1} \binom{j_0-(k-m_0)+i}{j_0-(k-m_0)} 2^{m_0-1+i}c + \binom{j_0}{k-m_0}c \\ &\quad + \sum_{i=1}^{k-m_0-1} \binom{j_0-(k-m_0)+i}{j_0-(k-m_0-1)} 2^{m_0-1+i}c + \binom{j_0}{k-m_0-1}c \\ &= \sum_{i=1}^{k-m_0-1} \binom{j_0-(k-m_0)+i+1}{j_0-(k-m_0)+1} 2^{m_0-1+i}c + 2^{m_0-1}c + \binom{j_0+1}{k-m_0}c \\ &= \sum_{i=0}^{k-m_0-1} \binom{j_0+1-(k-m_0)+i}{j_0+1-(k-m_0)} 2^{m_0-1+i}c + \binom{j_0+1}{k-m_0}c. \end{aligned}$$

Then formula (A6) holds by the above induction process. In particular, taking $m = 1$ in formula (A6), for $k - 1 \leq j \leq J - 2$,

$$|\Delta f(n+j)| \leq \sum_{i=0}^{k-2} \binom{j-k+i+1}{j-k+1} 2^i c + \binom{j}{k-1} c.$$

Hence for $k \leq j \leq J-1$, by the above inequality,

$$\begin{aligned} |f(n+j)| &\leq |f(n+k-1)| + \sum_{l=k-1}^{j-1} |\Delta f(n+l)| \\ &\leq c + \sum_{l=k-1}^{j-1} \left(\sum_{i=0}^{k-2} \binom{l-k+i+1}{l-k+1} 2^i + \binom{l}{k-1} \right) c \\ &= c + \sum_{i=0}^{k-2} \binom{j-1-k+i+2}{i+1} 2^i c + \binom{j}{k} c. \end{aligned}$$

Since $\binom{j}{k} \leq j^k$ and $\binom{j-k+i+1}{j-k} 2^i \leq j^{i+1}$ when $0 \leq i \leq k-2$,

$$|f(n+j)| \leq (k+1)j^k c, \quad k \leq j \leq J-1.$$

This completes the proof. \square

The following proposition shows that f can be approached by polynomials if the k -th difference of f is small. This proposition will be used in the proof of Theorem 1.7 in Section 5.

PROPOSITION A.6 *Given $j \geq k \geq 1$. There are constants a_k, \dots, a_j such that for each arithmetic function f and each $n \in \mathbb{N}$,*

$$\sum_{l=k}^j a_l \cdot \Delta^k f(n+l-k) = f(n+j) - \sum_{m=0}^{k-1} f(n+m) \prod_{i=0, i \neq m}^{k-1} \frac{j-i}{m-i}. \quad (\text{A7})$$

Proof. Let

$$g(l) = \binom{j-l+k-1}{k-1} = \frac{(j-l+1)(j-l+2)\cdots(j-l+k-1)}{(k-1)!}$$

be a polynomial of l with degree $k-1$. Choose $a_l = g(l)$ when $k \leq l \leq j+k-1$. So $a_l = 0$ when $l = j+1, \dots, j+k-1$. By equation (A1), the left side of equation (A7) is

$$\sum_{l=k}^j a_l \sum_{t=0}^k (-1)^{k-t} \binom{k}{t} f(n+l-k+t). \quad (\text{A8})$$

In the following, we consider the coefficients of $f(n+m)$ in the above formula for $m = 0, \dots, j$. When $m = j$, the coefficient of $f(n+m)$ in (A8) is $a_j = 1$. For the case $j \geq 2k$, when $k \leq m \leq j-k$, the

coefficient of $f(n + m)$ in (A8) is

$$\sum_{t=0}^k a_{m-t+k} (-1)^{k-t} \binom{k}{t} = (-1)^k \sum_{t=0}^k g(m+t) (-1)^{k-t} \binom{k}{t} = (-1)^k \Delta^k g(m) = 0.$$

Notice that $a_l = 0$ for $l = j + 1, \dots, j + k - 1$. When $j - k < m \leq j - 1$, the coefficient of $f(n + m)$ in (A8) is

$$\sum_{t=m+k-j}^k a_{m-t+k} (-1)^{k-t} \binom{k}{t} = \sum_{t=0}^k a_{m-t+k} (-1)^{k-t} \binom{k}{t} = (-1)^k \Delta^k g(m) = 0. \tag{A9}$$

For the case $j \leq 2k - 1$, when $k \leq m \leq j - 1$, by a similar argument to equation (A9), we have that the coefficient of $f(n + m)$ in (A8) is 0. Hence there are constants c_0, \dots, c_{k-1} such that

$$\sum_{l=k}^j a_l \cdot \Delta^k f(n + l - k) = f(n + j) - \sum_{m=0}^{k-1} c_m f(n + m) \tag{A10}$$

holds for each $n \in \mathbb{N}$. To compute c_m , let f_m be the polynomial of degree $k - 1$ such that $f_m(m) = 1$, $f_m(t) = 0$ for $0 \leq t \leq k - 1$ and $t \neq m$. So

$$f_m(t) = \prod_{i=0, i \neq m}^{k-1} \frac{t - i}{m - i}.$$

Note that $\Delta^k f_m = 0$ and equation (A10) holds for any arithmetic function f . Let $f = f_m$ and $n = 0$ in equation (A10), then

$$c_m = f_m(j) = \prod_{i=0, i \neq m}^{k-1} \frac{j - i}{m - i}.$$

Hence equation (A7) holds with $a_l = g(l)$ for $l = k, \dots, j$. □

Appendix B A lemma

In this section, we prove the following lemma which is used in the proof of Theorem 3.2. There are some other methods to prove the following result, while for self-containing, we provide a proof that is adapted to our situation.

LEMMA B.1 *Let $w(n)$ be a bounded arithmetic function and \mathcal{D} be a non-empty family consisting of real-valued arithmetic functions. Then the following two conditions are equivalent.*

- (i) For any increasing sequence $\{N_i\}_{i=0}^\infty$ of natural numbers with $N_0 = 0$ and $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$ and any sequence $\{f_i(n)\}_{i=0}^\infty$ in \mathcal{D} , we have

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| = 0. \quad (\text{B1})$$

- (ii) We have

$$\lim_{h \rightarrow \infty} \limsup_{X \rightarrow \infty} \frac{1}{Xh} \int_X^{2X} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} w(n)e(f(n)) \right| dx = 0. \quad (\text{B2})$$

Proof. We first prove that (i) \Rightarrow (ii). Assume on the contrary that formula (B2) does not hold. Then there is a $\delta \in (0, 1)$ and a sequence $\{h_j\}_{j=0}^\infty$ of positive integers with $\delta h_j > 4$ and $\lim_{j \rightarrow \infty} h_j = \infty$, such that

$$\limsup_{X \rightarrow \infty} \frac{1}{X} \int_X^{2X} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h_j} w(n)e(f(n)) \right| dx > 2\delta h_j.$$

Given h_j , choose X_j large enough with $\delta X_j > 16h_j$ and $X_j > 4X_{j-1}$ satisfying

$$\int_{X_j}^{2X_j} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h_j} w(n)e(f(n)) \right| dx > 2\delta X_j h_j.$$

By the pigeonhole principal, there is a $y_j \in [0, h_j)$, such that

$$\sum_{l_j = \lfloor X_j/h_j \rfloor - 1}^{\lfloor 2X_j/h_j \rfloor + 1} \left(\sup_{f \in \mathcal{D}} \left| \sum_{n=l_j h_j + y_j}^{(l_j+1)h_j + y_j - 1} w(n)e(f(n)) \right| \right) > \frac{3}{2} \delta X_j.$$

Furthermore, for each l_j with $\lfloor X_j/h_j \rfloor - 1 \leq l_j \leq \lfloor 2X_j/h_j \rfloor + 1$, we can find $g_{l_j}(n) \in \mathcal{D}$ such that

$$\sum_{l_j = \lfloor X_j/h_j \rfloor - 1}^{\lfloor 2X_j/h_j \rfloor + 1} \left| \sum_{n=l_j h_j + y_j}^{(l_j+1)h_j + y_j - 1} w(n)e(g_{l_j}(n)) \right| > \delta X_j. \quad (\text{B3})$$

Now we construct $\{N_i\}_{i=0}^\infty$ and $\{f_i(n)\}_{i=0}^\infty$ in the following way. Choose $N_0 = 0$ and $N_i = \lfloor X_{J+1}/h_{J+1} \rfloor h_{J+1} + th_{J+1} + y_{J+1}$ when $i = \sum_{j=1}^J \lfloor X_j/h_j \rfloor - J + 1 + t$, $t = 0, 1, \dots, \lfloor X_{J+1}/h_{J+1} \rfloor - 2$, where $J = 0, 1, \dots$. Then $\lim_{i \rightarrow \infty} (N_{i+1} - N_i) = \infty$. For $J = 0, 1, \dots$, we choose $f_i(n) = g_{\lfloor X_{J+1}/h_{J+1} \rfloor + t}(n)$ when $i = \sum_{j=1}^J \lfloor X_j/h_j \rfloor - J + 1 + t$, $t = 0, 1, \dots, \lfloor X_{J+1}/h_{J+1} \rfloor - 3$ and $f_i(n) = g_{\lfloor X_1/h_1 \rfloor}(n)$ otherwise. Then by equation (B1), there is an m_0 and J_0 with $m_0 = \sum_{j=1}^{J_0-1} \lfloor X_j/h_j \rfloor -$

$J_0 + 2$, such that

$$\frac{1}{N_{m_0 + \lfloor X_{J_0}/h_{J_0} \rfloor - 2}} \left(\sum_{i=0}^{m_0 + \lfloor X_{J_0}/h_{J_0} \rfloor - 3} \left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| \right) < \frac{\delta}{4}. \tag{B4}$$

Note that $N_{m_0} = \lfloor X_{J_0}/h_{J_0} \rfloor h_{J_0} + y_{J_0}$ and $N_{m_0 + \lfloor X_{J_0}/h_{J_0} \rfloor - 2} = 2(\lfloor X_{J_0}/h_{J_0} \rfloor - 1)h_{J_0} + y_{J_0}$. By formula (B3), we have that $\sum_{i=m_0}^{m_0 + \lfloor X_{J_0}/h_{J_0} \rfloor - 3} \left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| > \frac{\delta X_{J_0}}{2}$. Thus the left side of formula (B4) $> \frac{(\delta/2)X_{J_0}}{2X_{J_0} - h_{J_0}} > \frac{\delta}{4}$. This contradicts the right side of formula (B4). Hence equation (B2) holds.

Next, we show that (ii) \Rightarrow (i). Given ϵ with $0 < \epsilon < 1$. Let h be a fixed sufficiently large positive integer. By equation (B2) and a dyadic subdivision, there is an $m_0 \geq 1$ such that whenever $m > m_0$, we have $N_m - N_{m-1} > \frac{2}{\epsilon}h$ and

$$\sum_{x=0}^{\lfloor N_m/h \rfloor + 1} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} w(n)e(f(n)) \right| \leq \epsilon N_m h. \tag{B5}$$

Let $S_j = \{lh + j : l = 0, 1, \dots, \lfloor N_m/h \rfloor\}$, for $0 \leq j \leq h - 1$. Then, by (B5), there is a j_0 such that

$$\sum_{x \in S_{j_0}} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} w(n)e(f(n)) \right| \leq \epsilon N_m. \tag{B6}$$

Suppose that $S_{j_0} \cap [N_i, N_{i+1}) = \{x_1^{(i)}, x_2^{(i)}, \dots, x_{l_i}^{(i)}\}$, where $x_1^{(i)} < x_2^{(i)} < \dots < x_{l_i}^{(i)}$, $i = 0, 1, \dots, m - 1$. Then

$$\left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| \leq \sum_{t=1}^{l_i-1} \sup_{f \in \mathcal{D}} \left| \sum_{x_t^{(i)} \leq n < x_t^{(i)} + h} w(n)e(f(n)) \right| + 2h.$$

Hence

$$\sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| \leq \sum_{x \in S_{j_0}} \sup_{f \in \mathcal{D}} \left| \sum_{x \leq n < x+h} w(n)e(f(n)) \right| + 2mh.$$

By formula (B6),

$$\frac{1}{N_m} \sum_{i=0}^{m-1} \left| \sum_{N_i \leq n < N_{i+1}} w(n)e(f_i(n)) \right| \leq \frac{\epsilon N_m}{N_m} + \frac{2mh}{N_m} < 2\epsilon.$$

So we obtain equation (B1). □

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