ON THE FRØYSHOV INVARIANT AND MONOPOLE LEFSCHETZ NUMBER

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Abstract

Given an involution on a rational homology 3-sphere Y with quotient the 3-sphere, we prove a formula for the Lefschetz number of the map induced by this involution in the reduced monopole Floer homology. This formula is motivated by a variant of Witten's conjecture relating the Donaldson and Seiberg–Witten invariants of 4-manifolds. A key ingredient is a skein-theoretic argument, making use of an exact triangle in monopole Floer homology, that computes the Lefschetz number in terms of the Murasugi signature of the branch set and the sum of Frøyshov invariants associated to spin structures on Y. We discuss various applications of our formula in gauge theory, knot theory, contact geometry, and 4-dimensional topology.

1. Introduction

The monopole Floer homology defined by Kronheimer and Mrowka [**37**] using Seiberg–Witten gauge theory is a powerful invariant of 3–manifolds which has had many important applications in low-dimensional topology. Because of its functoriality [**37**, Theorem 3.4.3], the monopole Floer homology of a 3–manifold is acted upon by its mapping class group. However, the information contained in this action is not easy to extract due to the gauge theoretic nature of the theory. In this paper, we make some first steps towards understanding this action by calculating the Lefschetz numbers of certain involutions making the 3–manifold into a double branched cover over a link in the 3–sphere. Our study is motivated by the calculation of Lefschetz numbers in the instanton Floer homology [**63**] and by a variant [**61**, Conjecture B] of

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Witten's conjecture [**72**] relating the Donaldson and Seiberg–Witten invariants. The following theorem is the main result of the paper, and it comes with many interesting applications.

Theorem A. *Let Y be an oriented rational homology 3–sphere with an involution $\tau : Y \to Y$ making it into a double branched cover of S^3 with branch set a non-empty link L. Denote by $\tau_* : HM^{\mathrm{red}}(Y) \to HM^{\mathrm{red}}(Y)$ the induced map in the reduced monopole Floer homology, and by $\mathrm{Lef}(\tau_*)$ its Lefschetz number. Then*

$$\mathrm{Lef}\,(\tau_*) \;=\; \frac{2^{|L|}}{16}\,\xi(L) \;-\; \sum_{\mathfrak{s}} h(Y,\mathfrak{s}), \tag{1}$$

*where $|L|$ is the number of components of the link L and $\xi(L)$ is its Murasugi signature [**52**], and the last term is the sum over all the spin structures on Y of the Frøyshov invariants $h(Y,\mathfrak{s})$ of the spin manifold $(Y,\mathfrak{s})$. In particular, if the link L is a knot K in S^3,*

$$\mathrm{Lef}(\tau_*) \;=\; \frac{1}{8}\,\mathrm{sign}\,(K) - h(Y), \tag{2}$$

where $\mathrm{sign}(K)$ is the classical knot signature and $h(Y)$ is the Frøyshov invariant for the unique spin structure on Y.

We are using here the rational numbers as the coefficient ring of the monopole Floer homology, and we will continue doing so throughout the paper unless otherwise noted. We expect that a formula similar to (1) will hold for any rational homology sphere Y and a diffeomorphism $\tau : Y \to Y$ of order $n \geqslant 2$. In the special case when the quotient Y/τ is an integral homology sphere Y' and the branch set is a knot $K \subset Y'$, we make this expectation precise and conjecture that

$$\mathrm{Lef}\,(\tau_*) \;=\; n\cdot\lambda(Y') \;+\; \frac{1}{8}\sum_{m=0}^{n-1}\mathrm{sign}^{m/n}(K) \;-\; \sum_{\mathfrak{s}} h(Y,\mathfrak{s}), \tag{3}$$

where $\lambda(Y')$ is the Casson invariant of Y', $\mathrm{sign}^{m/n}(K)$ is the Tristram–Levine signature of K (see [**63**, Section 6]), and the last summation extends to the spin^c structures $\mathfrak{s}$ on Y such that $\tau_*(\mathfrak{s}) = \mathfrak{s}$. The origins of this conjecture will be discussed in Section 1.2.1. In a recent paper [**45**], the authors have proved this conjecture.

Remark 1.1. We will be working throughout with the Frøyshov invariants $h(Y,\mathfrak{s})$, which are defined via monopole homology. However, it is important to note that these are now known to be equivalent to the Heegaard Floer theory correction terms $d(Y,\mathfrak{s})$, for which many more calculations are available. In particular, the work of Kutluhan, Lee, and Taubes [**39**, **40**, **41**, **42**, **43**], or alternatively, the work of Colin, Ghiggini, and Honda [**13**, **14**, **15**] and Taubes [**68**], identifies the monopole homology and the Heegaard Floer homology. Furthermore, by combining the main results of [**59**, **28**, **21**], the absolute $\mathbb{Q}$-gradings in the two theories coincide. Therefore, the relation $d(Y,\mathfrak{s}) = -2h(Y,\mathfrak{s})$ between the Frøyshov invariant and the Heegaard Floer correction term holds for any rational homology spheres. This relation plays a role in our proof of Theorem A (see Proposition 6.4) as well in several of the corollaries.

We conjecture that a version of Theorem A holds for Heegaard Floer homology. This would be established by showing that the isomorphisms cited above between the reduced Floer theories are natural with respect to cobordisms, so that the Lefschetz numbers computed in the two theories are the same.

Remark 1.2. To the best of our knowledge, Theorem A gives the first interpretation of the classical knot signature (and more generally, Murasugi's link signature) in terms of the Seiberg–Witten gauge theory. It can be viewed as a categorification result, with the Lefschetz number substituted for the Euler characteristic. This result should be compared with X.-S. Lin's theorem [**46**] which expresses the signature of a knot in S^3 as (roughly) a signed count of the gauge equivalence classes of certain flat $SU(2)$ connections over the knot exterior.

1.1. An outline of the proof. Since Y is the double branched cover of S^3 with branch set L, we will also use the notation $Y = \Sigma(L)$ and assume that the orientation on Y is pulled back from the standard orientation of S^3. We need to show the vanishing of the link invariant

$$\chi(L) \;=\; \frac{1}{2^{|L|-1}}\left(\mathrm{Lef}\,(\tau_*) + \sum_{\mathfrak{s}} h(\Sigma(L),\mathfrak{s})\right) - \frac{1}{8}\,\xi(L) \tag{4}$$

for all links L with non-zero determinant. This is done by an inductive argument involving a skein relation between $\chi(L)$, $\chi(L_0)$, and $\chi(L_1)$, where L_0, L_1 are resolutions of L at a certain crossing. The skein relation for $\xi(L)$ can be proved directly, and the skein relations for the other two terms are a consequence of an exact triangle relating the monopole Floer homology of $\Sigma(L)$, $\Sigma(L_0)$, and $\Sigma(L_1)$.

While the idea is straightforward, there are several technical obstacles one needs to overcome. First of all, to understand the skein-theoretic behavior of the monopole Lefschetz number (as a rational number), one needs an exact triangle with $\mathbb{Q}$–coefficients; however, the original exact triangle [**38**] has coefficients in $\mathbb{Z}/2$. While one may be able to adapt the proof there by putting suitable plus and minus signs before various terms appearing in the proof, keeping the signs straight is complicated and would require a significant amount of work. Here, we follow an alternative route: we show that, with some extra input from homological algebra, one can deduce a $\mathbb{Q}$–coefficient exact triangle from the corresponding $\mathbb{Z}/2$ exact triangle using the universal coefficient theorem. It is a delicate matter to define the signs involved in the maps of this new exact triangle so that they are compatible with the induced action of τ; we need this compatibility to deduce a vanishing result for the total Lefschetz number.

The second difficulty comes from the fact that the version of monopole Floer homology that appears in the exact triangle is $\widetilde{HM}(Y)$, and it is always infinite dimensional. To discuss the Lefschetz number, one needs to modify $\widetilde{HM}(Y)$ to a finite dimensional vector space by ignoring all generators of sufficiently high degree. However, we lose the exactness of the triangle by such a truncating operation. As a consequence, the skein relation for $\chi(L)$ only holds up to a universal constant C depending on certain combinatorial data including the surgery coefficients. To prove that C always equals zero, we start from the example of two-bridge links. Since such links are known to have vanishing $\chi(L)$, we can use them to show that in some cases, the constant C vanishes and the actual skein relation holds. With the help of this special skein relation, we can produce more examples of links L with $\chi(L) = 0$ and prove the vanishing result for C in a more general situation. Repeating this procedure several times, we eventually produce enough examples to prove that $C = 0$ in all possible cases.

After establishing the skein relation, one might hope to prove $\chi(L) = 0$ by an inductive argument. However, such an argument would need to avoid links with zero determinant because double branched covers of such links, not being rational homology 3-spheres, may have more complicated monopole Floer homology. Unfortunately, it is not clear how to reduce a general non-zero determinant link to the unknot solely by resolving crossing without involving any zero determinant links. To overcome this obstacle, we make use of Mullins's *skein theory for non-zero determinant links* [**51**]. Following his idea, we extend the inductive statement by adding another skein relation relating $\chi(L)$ with $\chi(\bar{L})$, where $\bar{L}$ is obtained from L by a crossing change. The relation is then established by comparing the two exact triangles arising from the triples (L, L_0, L_1) and $(\bar{L}, L_1, L_0)$. Further development of these ideas can be found in Karan [**29**].

1.2. Calculations and applications. Theorem A can be used in several different ways. In some cases (for instance, when Y is an L-space), the monopole Lefschetz number automatically vanishes and we obtain a direct relation between the Frøyshov invariant and the Murasugi signature. In other cases, one can use formula (1) to compute the monopole Lefschetz number. This Lefschetz number contains important information about the action and can be used to explicitly describe the action in some cases, leading to non-trivial conclusions. The applications we present in this paper fall into four different categories: gauge theory, knot theory, contact geometry, and 4-dimensional topology.

1.2.1. An application to gauge theory. Let X be a closed smooth oriented 4-manifold such that

$$H_*(X;\mathbb{Z}) = H_*(S^1\times S^3;\mathbb{Z}) \quad \text{and} \quad H_*(\tilde{X};\mathbb{Q}) = H_*(S^3;\mathbb{Q}), \tag{5}$$

where $\tilde{X}$ is the universal abelian cover of X associated with a choice of generator for $H^1(X;\mathbb{Z}) = \mathbb{Z}$, called a homology orientation on X. Associated with X are two gauge-theoretic invariants whose definition depends on a choice of Riemannian metric on X but which end up being metric independent. The invariant $\lambda_{\mathrm{FO}}(X)$ is roughly one quarter times a signed count of anti-self-dual connections on the trivial $SU(2)$ bundle over X; and the invariant $\lambda_{\mathrm{SW}}(X)$ is roughly a signed count of the Seiberg–Witten monopoles over X plus an index theoretic correction term. The invariant $\lambda_{\mathrm{FO}}(X)$ was originally defined by Furuta and Ohta [**20**] under the more restrictive hypothesis that $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, and the invariant $\lambda_{\mathrm{SW}}(X)$ was defined by Mrowka, Ruberman, and Saveliev [**50**] without any assumption on $\tilde{X}$.

Conjecture B. *For any closed oriented homology oriented smooth 4-manifold X satisfying condition (5), one has*

$$\lambda_{\mathrm{FO}}(X) = -\lambda_{\mathrm{SW}}(X).$$

This conjecture is a slight generalization of [**50**, Conjecture B]. It relates the Donaldson and Seiberg–Witten invariants of certain smooth 4-manifolds and therefore can be thought of as a variant of the Witten conjecture [**72**] for manifolds with vanishing second Betti number. The conjecture has been verified in a number of examples [**61**]. The following theorem, which we prove in this paper, provides further evidence towards it.

Theorem C. *Let $\tau : Y \to Y$ be an involution on a rational homology sphere Y making Y into a double branched cover of S^3 with branch set a knot K, and let X be the mapping torus of τ. Then*

$$\lambda_{\mathrm{FO}}(X) = -\lambda_{\mathrm{SW}}(X) = \frac{1}{8}\,\operatorname{sign}(K).$$

Note that our conjectural formula (3) can be interpreted as a special case of Conjecture B for the mapping torus of a diffeomorphism of order n, by using the splitting formula for $\lambda_{\mathrm{SW}}(X)$ proved in our earlier paper [**44**, Theorem A] and the calculation of $\lambda_{\mathrm{FO}}(X)$ for the finite order mapping tori [**63**, Theorem 1.1].

1.2.2. Strongly non-extendable involutions and Akbulut corks. In [**1**], Akbulut constructed a smooth compact contractible 4-manifold W_1 with boundary an integral homology sphere ∂W_1 and an involution $\tau : \partial W_1 \to \partial W_1$ which can be extended to W_1 as a homeomorphism but not as a diffeomorphism; it was the first example of what is now known as an Akbulut cork. We improve upon this result by constructing the first known example of what we call a 'strongly non-extendable involution'. The precise statement is as follows.

Theorem D. *There exists a smooth involution $\tau : Y \to Y$ on an integral homology 3-sphere Y which has the following two properties:*

(1) *Y bounds a smooth contractible 4-manifold, and*
(2) *τ can not be extended as a diffeomorphism to any $\mathbb{Z}/2$ homology 4-ball that Y bounds.*

One example of a strongly non-extendable involution claimed by Theorem D is the aforementioned involution $\tau : \partial W_1 \to \partial W_1$ of the original Akbulut cork (W_1, τ): we show that τ does not extend to a self-diffeomorphism not just of W_1 but of *any* homology ball that ∂W_1 may bound. We accomplish this by computing the induced action of τ on the monopole Floer homology $\widehat{HM}(\partial W_1; \mathbb{Z})$ with the help of Theorem A and the calculation of Akbulut and Durusoy [**3**].

When the homology 4-ball bounded by Y is contractible, the involution τ always extends to it as a homeomorphism. Using this idea, we give a general construction in Section 8 that results in a large family of new corks. It is worth mentioning that previous examples of corks were usually detected by embedding them in a closed manifold whose smooth structure is changed by the cork twist. (In the terminology of [**7**], they have an *effective* embedding.) On the other hand, the corks we construct do not have an obvious effective embedding and they are detected by monopole Floer homology.

1.2.3. Knot concordance and Khovanov homology thin knots. Recall from [**32**, **33**] that a link L in the 3–sphere is called Khovanov homology thin (over $\mathbb{F}_2$) if its reduced Khovanov homology $\widetilde{Kh}(L; \mathbb{F}_2)$ is supported in a single δ-grading. Such links are rather common: for instance, according to [**48**], all quasi-alternating links, as well as 238 of the 250 prime knots with up to 10 crossings, are Khovanov homology thin. It follows from the spectral sequence of Bloom [**9**] that $HM^{\text{red}}(\Sigma(L)) = 0$ if L is a Khovanov homology thin link. Combined with Theorem A, this leads to the following series of corollaries, the first of which confirms the conjecture of Manolescu and Owens [**47**, Conjecture 1.4].

Corollary E. *For any Khovanov homology thin link L with nonzero determinant, one has the relation*

$$\xi(L) \;=\; 8 \sum_{\mathfrak{s}\in \mathrm{spin}(\Sigma(L))} h(\Sigma(L),\mathfrak{s})$$

between the Murasugi signature of L and the Frøyshov invariants of the double branched cover $\Sigma(L)$.

Corollary F. *For a knot K in S^3, denote by $L(K)$ the Lefschetz number of the map on $HM^{\text{red}}(\Sigma(K))$ induced by the covering translation. Then $L(K)$ is a non-trivial additive concordance invariant which vanishes on Khovanov homology thin knots.*

Corollary G. *Let $\mathcal{C}_{\text{s}}$ be the smooth knot concordance group and $\mathcal{C}_{\text{thin}}$ its subgroup generated by the Khovanov homology thin knots. Then the quotient group $\mathcal{C}_{\text{s}}/\mathcal{C}_{\text{thin}}$ contains a $\mathbb{Z}$–summand.*

1.2.4. The choice of sign in the monopole contact invariant. For any compact contact 3-manifold (Y,ξ), Kronheimer and Mrowka [**36**] (see also [**38**]) defined a contact invariant

$$\psi(Y,\xi) \in \widecheck{HM}(-Y;\mathbb{Z}/2),$$

as well as a monopole homology class

$$\tilde{\psi}(Y,\xi) \in \widecheck{HM}(-Y;\mathbb{Z})/\{\pm 1\}.$$

The notation indicates that when working with integer coefficients, this construction only results in a class in the quotient group $\widecheck{HM}(-Y;\mathbb{Z})/\{\pm 1\}$. In other words, the monopole homology class at hand is only well-defined up to sign. Technically, this happens because the Seiberg–Witten moduli space involved in the construction does not carry a natural orientation. One might hope that the monopole homology class can be defined as an element in $\widecheck{HM}(-Y;\mathbb{Z})$ by further analysis. However, we show that that is not possible: with the help of Theorem A, we construct an involution on the Brieskorn homology sphere $-\Sigma(2,3,7)$ which preserves a certain contact structure but changes the sign of the (non-torsion) contact invariant.

Theorem H. *There exists no canonical choice of sign in the definition of $\tilde{\psi}(Y,\xi)$, or equivalently no canonical lift of $\tilde{\psi}(Y,\xi)$ to $\widetilde{HM}(-Y;\mathbb{Z})$.*

It is worth mentioning that similar contact invariants in Heegaard Floer homology were defined (also with a sign ambiguity) by Ozsváth and Szabó [**55**]. A version of Theorem H in context of Heegaard Floer homology has been proved by Honda, Kazez, and Matić [**26**] using an approach different from ours. As discussed in Remark 1.1, there exists an isomorphism between Heegaard Floer homology and monopole Floer homology which preserves the contact invariant [**13**, **14**, **15**, **68**]. However, since the naturality of this isomorphism has not been established, our result and that of Honda, Kazez, and Matić do not imply each other.

1.3. Organization of the paper. Section 2 sets up the skein theory argument, reducing the proof of Theorem A to the key Proposition 2.2. The proof of that proposition occupies Sections 3–6, which form the bulk of the paper. Section 3 establishes a skein relation for the Murasugi signature $\xi(L)$, and Section 4 sets up a surgery exact triangle with rational coefficients that will be crucial for the remainder of the argument. In Section 5, we show that Proposition 2.2 holds up to certain universal constants C, and organize the rather complicated data necessary to track spin and spin^c structures through the skein theory argument. Section 6 studies the skein exact sequence for a large number of examples, sufficient to show that the constants C vanish, thereby establishing Proposition 2.2 and Theorem A.

The remainder of the paper is devoted to applications. In Section 7 we extend the definition of the Furuta–Ohta invariant λ_{FO} and evaluate it for the mapping torus of an involution on a rational homology sphere with homology sphere quotient. This establishes Theorem C. We calculate the effect of a particular involution on a cork boundary in Section 8 proving the non-extension result Theorem D. Corollaries E, F, and G of Theorem A concerning the knot concordance group are established in Section 9. Finally, Section 10 proves the non-canonical nature of the sign in the contact invariant (Theorem H).

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2. Skein relations and the proof of Theorem A

Let L be an unoriented link in S^3 and $\Sigma(L)$ its double branched cover. A quasi-orientation of L is an orientation on each component of L modulo an overall orientation reversal. The set of quasi-orientations of L will be denoted by $Q(L)$. Turaev [**70**] established a natural bijective correspondence between $Q(L)$ and $\operatorname{spin}(\Sigma(L))$, the set of spin structures on $\Sigma(L)$.

The link L will be called *ramifiable* if $\det(L) \neq 0$ or, equivalently, if $\Sigma(L)$ is a rational homology sphere. Note that all knots are ramifiable. Given a ramifiable link L in S^3, consider the quantity

$$\chi(L) = \frac{1}{2^{|L|-1}} \left(\sum_{\mathfrak{s} \in \operatorname{spin}(\Sigma(L))} h(\Sigma(L), \mathfrak{s}) \; - \; \frac{1}{8} \sum_{\ell \in Q(L)} \sigma(\ell) \; + \; \operatorname{Lef}(\tau_*) \right),$$

where $|L|$ is the number of components of L, $\sigma(\ell)$ is the signature of the link L quasi-oriented by ℓ, $h(\Sigma(L), \mathfrak{s})$ is the Frøyshov invariant of the spin manifold $(\Sigma(L), \mathfrak{s})$, and $\operatorname{Lef}(\tau_*)$ is the Lefschetz number of the map

$$\tau_* : HM^{\mathrm{red}}(\Sigma(L)) \to HM^{\mathrm{red}}(\Sigma(L)) \tag{6}$$

on the reduced monopole Floer homology of $\Sigma(L)$ induced by the covering translation $\tau : \Sigma(L) \to \Sigma(L)$. That the above formula for $\chi(L)$ matches formula (4) can be seen as follows.

Write $L = K_1 \cup \ldots \cup K_m$ as a link of $m = |L|$ components, and choose a quasi-orientation $\ell \in Q(L)$. Recall that the *Murasugi signature* of L is defined as

$$\xi(L) = \sigma(\ell) + \sum_{1 \leqslant i < j \leqslant m} \operatorname{lk}(K_i, K_j).$$

Murasugi [**52**] proved that $\xi(L)$ does not depend on the choice of quasi-orientation ℓ, hence $\xi(L)$ can be defined alternatively as

$$\xi(L) = \frac{1}{2^{m-1}} \sum_{\ell \in Q(L)} \sigma(\ell). \tag{7}$$

The following theorem is then equivalent to Theorem A.

Theorem 2.1. *For any ramifiable link $L \subset S^3$, one has $\chi(L) = 0$.*

Our proof of Theorem 2.1 will rely on skein theory. Given a link L in the 3–sphere, fix its planar projection and consider two resolutions of L at a crossing c as shown in Figure 1; we follow here the convention of [**56**].

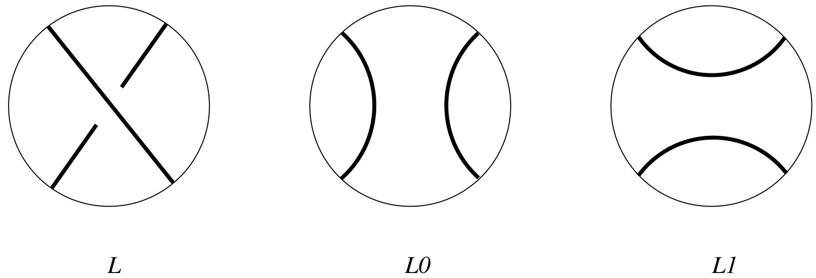


Figure 1.

The links L_0 and L_1 are called the 0-resolution and the 1-resolution of L, respectively, and the triple (L, L_0, L_1) is called a *skein triangle*. Note that a skein triangle possesses a cyclic symmetry: for any link in (L, L_0, L_1), the other two taken in the prescribed cyclic ordering are its 0- and 1-resolutions. This symmetry is best seen when the links are drawn as in Figure 2; see also [**35**, Figure 6]. Denote by $\bar{L}$ the link obtained by changing the crossing c in the link L.

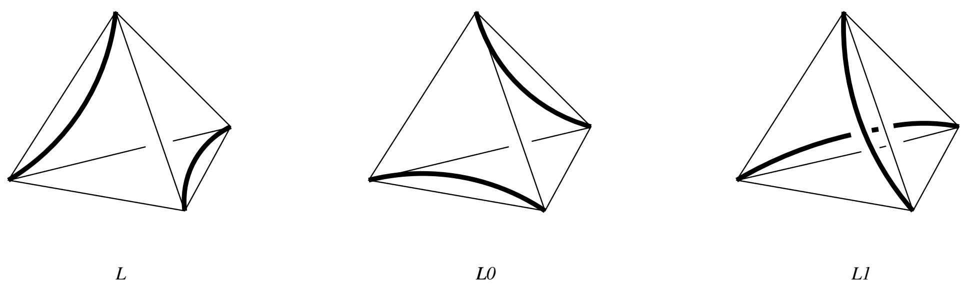


Figure 2.

Proposition 2.2. *Let (L, L_0, L_1) be a skein triangle and assume that L is ramifiable. Then at least two of the three links $\bar{L}$, L_0, L_1 are ramifiable and, in addition,*

(i) if both L_0 and L_1 are ramifiable and $\chi(L_0) = \chi(L_1) = 0$ then $\chi(L) = 0$.

(ii) if one of the links L_0, L_1 is not ramifiable then $\chi(L) = \chi(\bar{L})$.

We will now prove Theorem 2.1 assuming Proposition 2.2; the proof of the proposition will then occupy Section 3 through Section 6.

Proof of Theorem 2.1. The proof is a modification of the proof of [**51**, Theorem 3.3]. We will proceed by induction on the pair $(c(L), |L|)$, where $c(L)$ is the number of crossings in a diagram of L.

The case $(0,1)$ is trivial. The cases $(0,n)$ with $n \geqslant 2$ are vacuous because unlinks with more than one component are not ramifiable. Next, suppose that the statement has been proved for all ramifiable links admitting a diagram with k or fewer crossings. We want to prove it for the case $(k+1,n)$ with $n \geqslant 1$.

First let $n = 1$ then L is a knot admitting a diagram with $k+1$ crossings. By changing $m < k$ crossings we can unknot L, thereby obtaining a sequence of knots

$$L = L^1 \to L^2 \to \cdots \to L^{m+1} = \text{unknot},$$

where L^{a+1} is obtained from L^a by a single crossing change. Denote by L_0^a and L_1^a the two resolutions of L^a. We have $\chi(L^{m+1}) = 0$. To deduce that $\chi(L^a) = 0$ from $\chi(L^{a+1}) = 0$, we will consider the following two cases:

(i) Both L_0^a and L_1^a are ramifiable. Since $c(L_0^a) \leqslant k$ and $c(L_1^a) \leqslant k$, it follows from our induction hypothesis that $\chi(L_0^a) = \chi(L_1^a) = 0$. Proposition 2.2 (i) then implies that $\chi(L^a) = 0$.

(ii) One of the resolutions L_0^a, L_1^a is not ramifiable. Then Proposition 2.2 (ii) implies that $\chi(L^a) = \chi(L^{a+1}) = 0$, and we are finished.

Now let $n \geqslant 2$ so that L is a multi-component link admitting a diagram with $k+1$ crossings. Again, change $m < k$ crossings one by one to obtain a sequence of links

$$L = L^1 \to L^2 \to \cdots \to L^{m+1} = \text{a split link},$$

where L^{a+1} is obtained from L^a by a single crossing change. Since multi-component split links are not ramifiable, we can find $b \leqslant m$ such that $L^1, \cdots, L^b$ are ramifiable and L^{b+1} is not. Proposition 2.2 then implies that both L_0^b and L_1^b are ramifiable. Since $c(L_0^b) \leqslant k$ and $c(L_1^b) \leqslant k$, we conclude that $\chi(L_0^b) = \chi(L_1^b) = 0$ from our induction hypothesis. Proposition 2.2 (i) then implies that $\chi(L^b) = 0$. The deduction that $\chi(L^{a+1}) = 0$ implies $\chi(L^a) = 0$ for all $a < b$ is exactly the same as in the $n = 1$ case. q.e.d.

3. Skein relation for the Murasugi signature

Let (L_0, L_1, L_2) be a skein triangle obtained by resolving a crossing c inside a ball B. For any subscript j viewed as an element of $\mathbb{Z}/3 = \{0,1,2\}$, denote by S_j the standard cobordism surface from L_j to L_{j+1} inside the 4-manifold $[0,1] \times S^3$ obtained by adding a single 1-handle to the product surface outside of $[0,1] \times B$. Denote by W_j the double branched cover of $[0,1] \times S^3$

with branch set S_j. The manifold W_j is an oriented cobordism from $\Sigma(L_j)$ to $\Sigma(L_{j+1})$; it will be described as a surgery cobordism in Section 4. The signature of W_j can be either 0, 1, or -1.

Lemma 3.1. *Let (L_0, L_1, L_2) be a skein triangle such that $|L_2| = |L_0| + 1 = |L_1| + 1$, which is to say that the resolved crossing c is between two different components of L_2. Then*

$$2\xi(L_2) = \xi(L_0) + \xi(L_1) + \operatorname{sign}(W_1) - \operatorname{sign}(W_2). \tag{8}$$

Proof. This can be derived from the Gordon–Litherland [**24**] formula for the Murasugi signature but we will follow a more self-contained approach. It will rely on the disjoint decomposition

$$Q(L_2) = Q_0(L_2) \cup Q_1(L_2),$$

where $Q_0(L_2)$ (resp. $Q_1(L_2)$) consists of the quasi-orientations of L_2 which make L_0 (resp. L_1) into an oriented resolution. For any $\ell \in Q_0(L_2)$, the induced quasi-orientation on L_0 will be denoted by ℓ_0; this establishes a bijective correspondence $Q_0(L_2) = Q(L_0)$. Similarly, for any $\ell \in Q_1(L_2)$, the induced quasi-orientation on L_1 will be denoted by $\ell_1 \in Q(L_1)$; this establishes a bijective correspondence $Q_1(L_2) = Q(L_1)$. We claim that

$$\begin{aligned} \sigma(\ell_0) &= \sigma(\ell) + \operatorname{sign}(W_2) \quad \text{for any} \quad \ell \in Q_0(L_2), \quad \text{and} \\ \sigma(\ell_1) &= \sigma(\ell) - \operatorname{sign}(W_1) \quad \text{for any} \quad \ell \in Q_1(L_2). \end{aligned}$$

These identities, which are essentially due to Murasugi, can be verified as follows. Let $\ell \in Q_0(L_2)$. Since ℓ_0 is an oriented resolution of ℓ, the cobordism surface $S_2 \subset [0,1] \times S^3$ is naturally oriented. Choose an (oriented) Seifert surface for $\ell \subset \partial D^4$ and slightly push its interior into the interior of D^4 to obtain a properly embedded surface $F \subset D^4$. Passing to double branched covers, we obtain

$$\Sigma(D^4 \cup ([0,1] \times S^3), F \cup S_2) \;=\; \Sigma(D^4, F) \;\cup\; \Sigma([0,1] \times S^3, S_2),$$

where $\Sigma(A, B)$ stands for the double branched cover of A with branch set B. Using additivity of the signature and the fact that $\Sigma([0,1] \times S^3, S_2) = W_2$, we obtain

$$\operatorname{sign}(\Sigma(D^4 \cup ([0,1] \times S^3), F \cup S_2)) \;=\; \operatorname{sign}(\Sigma(D^4, F)) + \operatorname{sign}(W_2).$$

Observe that the surface $F \cup S_2$ in $D^4 \cup ([0,1] \times S_2)$ is an embedded surface with boundary ℓ_0 in $\{1\} \times S_2$. It is a classical result (see for instance [**30**]) that

$$\operatorname{sign}(\Sigma(D^4, F)) \;=\; \sigma(\ell) \quad \text{and} \quad \operatorname{sign}(\Sigma(D^4 \cup ([0,1] \times S^3), F \cup S_2)) \;=\; \sigma(\ell_0).$$

Therefore $\sigma(\ell_0) = \sigma(\ell) + \operatorname{sign}(W_2)$ for any $\ell \in Q_0(L_2)$. The proof of the other identity is similar. With these identities in place, the proof of the lemma is completed as follows:

$$\begin{aligned}
2^{|L_2|-1} \cdot \xi(L_2) &= \sum_{\ell \in Q_0(L_2)} \sigma(\ell) + \sum_{\ell \in Q_1(L_2)} \sigma(\ell) \\
&= \sum_{\ell_0 \in Q(L_0)} (\sigma(\ell_0) - \operatorname{sign}(W_2)) + \sum_{\ell_1 \in Q(L_1)} (\sigma(\ell_1) + \operatorname{sign}(W_1)) \\
&= 2^{|L_2|-2} \cdot (\xi(L_0) + \xi(L_1) + \operatorname{sign}(W_1) - \operatorname{sign}(W_2)).
\end{aligned}$$

q.e.d.

4. An exact triangle in monopole Floer homology

In this section, we will establish an exact triangle in the monopole Floer homology with rational coefficients. We will also show that this triangle possesses a certain conjugation symmetry, which will be instrumental in the proof of Proposition 2.2 later in the paper.

4.1. Statement of the exact triangle. Let Y be a compact connected oriented 3-manifold with boundary $\partial Y = T^2$, and let γ_0, γ_1, and γ_2 be oriented simple closed curves on ∂Y such that

$$\#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma_2) = \#(\gamma_2 \cap \gamma_0) = -1,$$

where the algebraic intersection numbers $\#$ are calculated with respect to the boundary orientation on ∂Y. Let $\mathbb{F}_2$ be the field of two elements. It follows from Poincaré duality that the kernel of the map $H_1(\partial Y; \mathbb{F}_2) \to H_1(Y; \mathbb{F}_2)$ is one-dimensional, therefore, we may assume without loss of generality that γ_2 is an $\mathbb{F}_2$ longitude (meaning that $[\gamma_2] = 0 \in H_1(Y; \mathbb{F}_2)$), while γ_0 and γ_1 are not.

For any j viewed as an element of $\mathbb{Z}/3 = \{0,1,2\}$, denote by Y_j the closed manifold obtained from Y by attaching a solid torus to its boundary with meridian γ_j, and by W_j the respective surgery cobordism from Y_j to Y_{j+1}. The cobordism W_j can be obtained by attaching D^4 to the component S^3 in the boundary of the 4-manifold

$$W_j^0 = ([-1,0] \times Y_j) \cup_{\{0\} \times Y} ([0,1] \times Y) \cup_{\{1\} \times Y} ([1,2] \times Y_{j+1}). \tag{9}$$

Lemma 4.1. *The manifolds W_1 and W_2 are spin, and the manifold W_0 is not.*

Proof. Notice that the inclusion $Y \to Y_2$ induces an isomorphism $H^1(Y_2;\mathbb{F}_2) \to H^1(Y;\mathbb{F}_2)$ hence any spin structure on Y can be extended to a spin structure on Y_2. To show that W_1 is spin, start with any spin structure $\mathfrak{s}$ on Y_1 and extend $\mathfrak{s}|_Y$ to a spin structure on Y_2. This gives a spin structure on W_1^0, which extends over D^4 to a spin structure on W_1. A similar argument shows that W_2 is also spin.

To show that W_0 is not spin, we argue as follows. Suppose W_0 has a spin structure $\mathfrak{s}$. By the argument above, $\mathfrak{s}|_{Y_1}$ can be extended to a spin structure on W_1. Glue these two spin structures together to obtain a spin structure on the manifold $W_0 \cup_{Y_1} W_1$. This leads to a contradiction because the manifold

$$W_0 \cup_{Y_1} W_1 = (-W_2) \# \overline{\mathbb{C}\mathrm{P}}^2$$

contains an embedded sphere with self-intersection number -1. q.e.d.

The space of spin^c structures has a natural involution which carries a spin^c structure $\mathfrak{s}$ to its conjugate $\bar{\mathfrak{s}}$. A spin^c structure $\mathfrak{s}$ is called self-conjugate if $c_1(\mathfrak{s}) = \mathfrak{s} - \bar{\mathfrak{s}}$ vanishes. For a fixed self-conjugate spin^c structure $\mathfrak{s}_0$ on Y, we will come up with an exact triangle involving spin^c structures on the manifolds Y_j restricting to the spin^c structure $\mathfrak{s}_0$ on Y. The usual exact triangle, involving all spin^c structures, can be obtained by taking the direct sum of these restricted exact triangles over all possible $\mathfrak{s}_0$.

We will set up some notation first, for use in this and the following section. In the six lines that follow, M can be any of the manifolds Y, Y_j or W_j, $j \in \mathbb{Z}/3$, and we write:

$$\begin{aligned}
\text{tor-spin}^c(M) &= \{\text{equivalence classes of torsion spin}^c \text{ structure on } M\},\\
\text{sc-spin}^c(M) &= \{\text{equivalence classes of self-conjugate spin}^c \text{ structure on } M\},\\
\text{tor-spin}^c(M,\mathfrak{s}_0) &= \{\mathfrak{s} \in \text{tor-spin}^c(M) \mid \mathfrak{s}|_Y = \mathfrak{s}_0\},\\
\text{sc-spin}^c(M,\mathfrak{s}_0) &= \{\mathfrak{s} \in \text{sc-spin}^c(M) \mid \mathfrak{s}|_Y = \mathfrak{s}_0\},\\
\text{spin}^c(M,\mathfrak{s}_0) &= \{\mathfrak{s} \in \text{spin}^c(M) \mid \mathfrak{s}|_Y = \mathfrak{s}_0\}, \text{ and}\\
\text{spin}(M,\mathfrak{s}_0) &= \{\mathfrak{s} \in \text{spin}(M) \mid (\mathfrak{s}|_Y)^c = \mathfrak{s}_0\}, \text{ with the notation } (\mathfrak{s}|_Y)^c \text{ explained below.}
\end{aligned}$$

Remark 4.2. Recall that each spin structure $\mathfrak{s}$ on Y induces a self-conjugate spin^c structure, which we denote by $\mathfrak{s}^c$, and that each self-conjugate spin^c structure on Y is obtained in this fashion. Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two spin structures on Y then $\mathfrak{s}_1^c = \mathfrak{s}_2^c$ if and only if $\mathfrak{s}_1 = \mathfrak{s}_2 + h$ for some h in the image of the coefficient map $H^1(Y;\mathbb{Z}) \to H^1(Y;\mathbb{Z}/2)$. Therefore, each self-conjugate spin^c structure on Y corresponds to $2^{b_1(Y)}$ spin structures. A similar remark applies to the manifolds Y_j and W_j, $j \in \mathbb{Z}/3$.

Our exact triangle will consist of the Floer homology groups[1]

$$\widetilde{HM}(Y_j, [\mathfrak{s}_0]) \;=\; \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y_j,\mathfrak{s}_0)} \widetilde{HM}(Y_j, \mathfrak{s})$$

and the maps between them induced by the cobordisms W_j. To ensure that the composition of any two adjacent maps is zero, we need to assign an appropriate plus or minus sign to each spin^c structure on W_j. We will accomplish this by defining an auxiliary map

$$\mu : \bigcup_{j\in\mathbb{Z}/3} \mathrm{spin}^c(W_j, \mathfrak{s}_0) \longrightarrow \mathbb{F}_2 \tag{10}$$

as follows:

- μ is identically zero on $\mathrm{spin}^c(W_2, \mathfrak{s}_0)$;
- Choose a base point $\mathfrak{s}_1 \in \mathrm{spin}^c(W_0, \mathfrak{s}_0)$ and let $\mu(\mathfrak{s}_1) = 0$. Given an element of $\mathrm{spin}^c(W_0, \mathfrak{s}_0)$, write it in the form $\mathfrak{s}_1 + h$ with $h \in \ker(H^2(W_0;\mathbb{Z}) \to H^2(Y;\mathbb{Z}))$ and let

  $$\mu(\mathfrak{s}_1 + h) = h_{\mathbb{F}_2},$$

  where

  $$h_{\mathbb{F}_2} \in \ker(H^2(W_0;\mathbb{F}_2) \to H^2(Y;\mathbb{F}_2)) = \mathbb{F}_2$$

  is the mod 2 reduction of h;
- The case of $\mathrm{spin}^c(W_1, \mathfrak{s}_0)$ is similar: choose a base point $\mathfrak{s}_2 \in \mathrm{spin}^c(W_1, \mathfrak{s}_0)$ and let

  $$\mu(\mathfrak{s}_2 + h) = h_{\mathbb{F}_2}$$

  for any $h \in \ker(H^2(W_1;\mathbb{Z}) \to H^2(Y;\mathbb{Z}))$.

Proposition 4.3. *$\mu(\mathfrak{s}) = \mu(\bar{\mathfrak{s}})$ for $\mathfrak{s} \in \mathrm{spin}^c(W_1, \mathfrak{s}_0)$ and $\mathfrak{s} \in \mathrm{spin}^c(W_2, \mathfrak{s}_0)$, and $\mu(\mathfrak{s}) = \mu(\bar{\mathfrak{s}}) + 1$ for $\mathfrak{s} \in \mathrm{spin}^c(W_0, \mathfrak{s}_0)$.*

[1] As we mentioned in the introduction, the monopole Floer homology will have the rational numbers as their coefficient ring unless otherwise noted.

Proof. The lemma is trivial for $\mathfrak{s} \in \mathrm{spin}^c(W_2, \mathfrak{s}_0)$. For $\mathfrak{s} \in \mathrm{spin}^c(W_1, \mathfrak{s}_0)$, since $\mathfrak{s} = \bar{\mathfrak{s}} + c_1(\mathfrak{s})$, the difference $\mu(\mathfrak{s}) - \mu(\bar{\mathfrak{s}})$ equals the mod 2 reduction of $c_1(\mathfrak{s})$, which is just the Stiefel–Whitney class $\omega_2(W_1)$. According to Lemma 4.1 the cobordism W_1 is spin, hence $\omega_2(W_1) = 0$ and $\mu(\mathfrak{s}) = \mu(\bar{\mathfrak{s}})$. The proof for $\mathfrak{s} \in \mathrm{spin}^c(W_0, \mathfrak{s}_0)$ is similar. q.e.d.

By functoriality of monopole Floer homology, the cobordism W_j equipped with a spin^c structure $\mathfrak{s} \in \mathrm{spin}^c(W_j, \mathfrak{s}_0)$ induces a map

$$\widetilde{HM}(W_j, \mathfrak{s}) : \widetilde{HM}(Y_j, \mathfrak{s}|_{Y_j}) \to \widetilde{HM}(Y_j, \mathfrak{s}|_{Y_{j+1}}), \quad j \in \mathbb{Z}/3.$$

We will combine these maps into a single map

$$F_{W_j} : \widetilde{HM}(Y_j, [\mathfrak{s}_0]) \to \widetilde{HM}(Y_{j+1}, [\mathfrak{s}_0])$$

defined by the formula

$$F_{W_j} = \sum_{\mathfrak{s} \in \mathrm{spin}^c(W_j, \mathfrak{s}_0)} (-1)^{\mu(\mathfrak{s})} \cdot \widetilde{HM}(W_j, \mathfrak{s}).$$

Note that, up to an overall sign, the maps F_{W_j} are independent of the arbitrary choices of base points in the definition (10).

Proposition 4.4. *We have $F_{W_{j+1}} \circ F_{W_j} = 0$ for all $j \in \mathbb{Z}/3$.*

Proof. Using the composition law for the cobordism induced maps in monopole Floer homology, we obtain

$$F_{W_{j+1}} \circ F_{W_j} = \sum_{\mathfrak{s} \in \mathrm{spin}^c(W_j \cup W_{j+1}, \mathfrak{s}_0)} (-1)^{\mu(\mathfrak{s}|_{W_j}) + \mu(\mathfrak{s}|_{W_{j+1}})} \cdot \widetilde{HM}(W_{j+1} \circ W_j, \mathfrak{s}). \tag{11}$$

The manifold

$$X_j = W_j \cup W_{j+1} = (-W_{j+2}) \# \overline{\mathbb{C}\mathrm{P}}^2$$

has an embedded 2–sphere E_j with self-intersection -1. Therefore, every $\mathfrak{s} \in \mathrm{spin}^c(X_j, \mathfrak{s}_0)$ can be uniquely written as $\mathfrak{s}_1 \# \mathfrak{s}_2$ with $\mathfrak{s}_1 \in \mathrm{spin}^c(-W_{j+2}, \mathfrak{s}_0)$ and $\mathfrak{s}_2 \in \mathrm{spin}^c(\overline{\mathbb{C}\mathrm{P}}^2)$. Let us consider a diffeomorphism of X_j which takes $[E_j] \in H_2(X_j)$ to $-[E_j] \in H_2(X_j)$ and restricts to the identity map on $-W_{j+2}$. Since this diffeomorphism does not change the homology orientation, and since the cobordism map in monopole Floer homology is natural, we obtain the identity

$$\widetilde{HM}(W_{j+1} \circ W_j, \mathfrak{s}_1 \# \mathfrak{s}_2) = \widetilde{HM}(W_{j+1} \circ W_j, \mathfrak{s}_1 \# \bar{\mathfrak{s}}_2).$$

Note that $\bar{\mathfrak{s}}_2$ never equals $\mathfrak{s}_2$ because $\overline{\mathbb{C}\mathrm{P}}^2$ is not spin. As a result, the terms on the right hand side of (11) come in pairs. The proof of the proposition will be complete after we prove the following lemma. q.e.d.

Lemma 4.5. *For any $j \in \mathbb{Z}/3$ and any* spin^c *structures $\mathfrak{s}_1 \in \mathrm{spin}^c(-W_{j+2}, \mathfrak{s}_0)$ and $\mathfrak{s}_2 \in \mathrm{spin}^c(\overline{\mathbb{C}\mathrm{P}}^2)$,*

$$\mu((\mathfrak{s}_1 \# \mathfrak{s}_2)|_{W_j}) + \mu((\mathfrak{s}_1 \# \mathfrak{s}_2)|_{W_{j+1}}) = 1 + \mu((\mathfrak{s}_1 \# \bar{\mathfrak{s}}_2)|_{W_j}) + \mu((\mathfrak{s}_1 \# \bar{\mathfrak{s}}_2)|_{W_{j+1}}) \in \mathbb{F}_2.$$

Proof. Let PD stand for the Poincaré duality isomorphism. Then

$$\mathfrak{s}_1 \# \mathfrak{s}_2 = \mathfrak{s}_1 \# \bar{\mathfrak{s}}_2 + (2k+1) \cdot \mathrm{PD}\,[E_j] \tag{12}$$

for some $k \in \mathbb{Z}$. To prove the lemma, we will compute the mod 2 reductions of $\mathrm{PD}\,[E_j]|_{W_j}$ and $\mathrm{PD}\,[E_j]|_{W_{j+1}}$, which we will denote by $(\mathrm{PD}\,[E_j]|_{W_j})_{\mathbb{F}_2}$ and $(\mathrm{PD}\,[E_j]|_{W_{j+1}})_{\mathbb{F}_2}$, respectively.

Recall that W_j is obtained by attaching a 2–handle H_j to $I \times Y_j$. Since γ_2 (treated as a knot in $\{1\} \times Y_j$) is an $\mathbb{F}_2$ longitude, we can find an immersed, possibly non-orientable surface $\Sigma_2 \subset Y$ with boundary γ_2. Capping Σ_2 off with the surface $\Sigma_1 \subset H_j$ bounded by γ_2, we obtain a closed surface $\Sigma_1 \cup_{\gamma_2} \Sigma_2$ which generates the group

$$\ker(H^2(W_j;\mathbb{F}_2) \to H^2(Y;\mathbb{F}_2))^* = \mathrm{coker}(H_2(Y;\mathbb{F}_2) \to H_2(W_j;\mathbb{F}_2)) = \mathbb{F}_2.$$

As a result, we have

$$(\mathrm{PD}\,[E_j]|_{W_j})_{\mathbb{F}_2} = \#(E_j \cap (\Sigma_1 \cup_{\gamma_2} \Sigma_2)) \pmod 2.$$

Since $\Sigma_1 \cup_{\gamma_2} \Sigma_2$ is contained in W_j, the 2–sphere E_j in the above formula can be replaced by $E_j \cap W_j$, which is a 2–disk $D_j \subset H_j$. The boundary of D_j, denoted by ℓ_{j+1}, is the core of the solid torus $Y_{j+1} \backslash \mathrm{int}(Y)$, therefore,

$$\#(E_j \cap (\Sigma_1 \cup_{\gamma_2} \Sigma_2)) = \#(D_j \cap \Sigma_1),$$

which is the linking number of ℓ_{j+1} and γ_2 inside ∂H_j. After a moment's thought we conclude that

$$\mathrm{lk}(\ell_{j+1}, \gamma_2) = \pm\,\#(\gamma_j \cap \gamma_2)$$

and therefore

$$(\mathrm{PD}\,[E_j]|_{W_j})_{\mathbb{F}_2} = \#(\gamma_j \cap \gamma_2) \pmod 2.$$

A similar argument shows that

$$(\mathrm{PD}\,[E_j]|_{W_{j+1}})_{\mathbb{F}_2} = \#(\gamma_{j+2} \cap \gamma_2) \pmod 2.$$

The rest of the proof is straightforward. We assume that $j = 0$; the other cases are similar. By (12) and the definition of μ, we have

$$\begin{aligned}\mu((\mathfrak{s}_1 \# \mathfrak{s}_2)|_{W_0}) + \mu((\mathfrak{s}_1 \# \mathfrak{s}_2)|_{W_1}) - \mu((\mathfrak{s}_1 \# \bar{\mathfrak{s}}_2)|_{W_0}) - \mu((\mathfrak{s}_1 \# \bar{\mathfrak{s}}_2)|_{W_1}) = \\ (\mathrm{PD}\,[E_0]|_{W_0})_{\mathbb{F}_2} + (\mathrm{PD}\,[E_0]|_{W_1})_{\mathbb{F}_2} = \#(\gamma_0 \cap \gamma_2) + \#(\gamma_2 \cap \gamma_2) = 1 \pmod 2.\end{aligned}$$

q.e.d.

We are now ready to state the main result of this section, the exact triangle in monopole Floer homology with rational coefficients.

Theorem 4.6. *The following sequence of monopole Floer homology groups is exact over the rationals*

$$\cdots \xrightarrow{F_{W_0}} \widetilde{HM}(Y_1, [\mathfrak{s}_0]) \xrightarrow{F_{W_1}} \widetilde{HM}(Y_2, [\mathfrak{s}_0]) \xrightarrow{F_{W_2}} \widetilde{HM}(Y_0, [\mathfrak{s}_0]) \xrightarrow{F_{W_0}} \cdots$$

4.2. Proof of the exact triangle. We already know from Proposition 4.4 that the composite of any two adjacent maps is zero. To complete the proof of exactness, we will combine the universal coefficient theorem with the $\mathbb{F}_2$ coefficient exact triangle proved in [**38**].

Before we go on with the proof, we need to review some basic constructions in monopole Floer homology; see Kronheimer–Mrowka [**37**] for details. For any $j \in \mathbb{Z}/3$, denote by $C^o(Y_j)$ (resp. $C^s(Y_j)$ and $C^u(Y_j)$) the free $\mathbb{Z}$-modules generated by the gauge equivalence classes of irreducible monopoles (resp. boundary stable and boundary unstable monopoles) whose associated spin^c structure belongs to $\mathrm{spin}^c(Y_j, \mathfrak{s}_0)$. By counting points in the zero-dimensional moduli spaces of monopoles on $\mathbb{R} \times Y_j$, we obtain a linear map

$$\partial^o_o(Y_j) : C^o(Y_j) \to C^o(Y_j)$$

and its companions $\partial^o_s(Y_j),\ \partial^u_o(Y_j),\ \partial^u_s(Y_j),\ \bar{\partial}^s_u(Y_j),\ \bar{\partial}^u_s(Y_j),\ \bar{\partial}^s_s(Y_j),\ \bar{\partial}^u_u(Y_j)$. Note that the last four maps count only reducible monopoles. Set

$$\check{C}(Y_j) = C^o(Y_j) \oplus C^s(Y_j)$$

and define the map

$$\check{\partial}(Y_j) : \check{C}(Y_j) \to \check{C}(Y_j)$$

by the matrix

$$\begin{bmatrix} \partial^o_o(Y_j) & -\partial^u_o(Y_j)\bar{\partial}^s_u(Y_j) \\ \partial^o_s(Y_j) & \bar{\partial}^s_s(Y_j) - \partial^u_s(Y_j)\bar{\partial}^s_u(Y_j) \end{bmatrix}.$$

One can check that $\check{\partial}(Y_j) \circ \check{\partial}(Y_j) = 0$. The homology of the chain complex $(\check{C}(Y_j), \check{\partial}(Y_j))$ is the monopole Floer homology $\widecheck{HM}(Y_j, [\mathfrak{s}_0]; \mathbb{Z})$. To obtain homology with rational coefficients, we set $\check{C}(Y_j)_{\mathbb{Q}} = \check{C}(Y_j) \otimes_{\mathbb{Z}} \mathbb{Q}$ and use the linear map $\check{\partial}(Y_j)_{\mathbb{Q}} = \check{\partial}(Y_j) \otimes \mathrm{id}$ as the boundary operator. (We will henceforth use similar notations without further explanation). Consider the manifold with cylindrical ends

$$W_j^* = ((-\infty, 0] \times Y_j) \cup_{Y_j} W_j \cup_{Y_{j+1}} ([0, +\infty) \times Y_{j+1}).$$

(In what follows, the superscript $*$ will indicate attaching a product end to the boundary of the manifold at hand). For any $\mathfrak{s} \in \mathrm{spin}^c(W_j, \mathfrak{s}_0)$, the count of monopoles on W_j^* defines the map

$$m^o_o(W_j, \mathfrak{s}) : C^o(Y_j) \to C^o(Y_j)$$

and its companion maps $m^o_s(W_j, \mathfrak{s})$, $m^u_o(W_j, \mathfrak{s})$, $m^u_s(W_j, \mathfrak{s})$, $\bar{m}^s_u(W_j, \mathfrak{s})$, $\bar{m}^s_s(W_j, \mathfrak{s})$, $\bar{m}^u_s(W_j, \mathfrak{s})$, and $\bar{m}^u_u(W_j, \mathfrak{s})$. Define the map

$$\check{m}(W_j, \mathfrak{s}) : \check{C}(Y_j) \to \check{C}(Y_{j+1})$$

by the matrix

$$\begin{bmatrix} m^o_o(W_j, \mathfrak{s}) & -m^u_o(W_j, \mathfrak{s})\bar{\partial}^s_u(Y_j) - \partial^u_o(Y_{j+1})\bar{m}^s_u(W_j, \mathfrak{s}) \\ m^o_s(W_j, \mathfrak{s}) & \bar{m}^s_s(W_j, \mathfrak{s}) - m^u_s(W_j, \mathfrak{s})\bar{\partial}^s_u(Y_j) - \partial^u_s(Y_{j+1})\bar{m}^s_u(W_j, \mathfrak{s}) \end{bmatrix}$$

and sum over all the spin^c structures with proper signs to obtain the map

$$\check{m}(W_j) := \sum_{\mathfrak{s} \in \mathrm{spin}^c(W_j, \mathfrak{s}_0)} (-1)^{\mu(\mathfrak{s})} \cdot \check{m}(W_j, \mathfrak{s}) : \check{C}(Y_j) \to \check{C}(Y_{j+1}).$$

This is the chain map that induces the map F_{W_j} in the exact triangle.

As our next step, we will construct an explicit null-homotopy of the composite $\check{m}(W_{j+1}) \circ \check{m}(W_j)$. To this end, recall that the composite cobordism

$$X_j = W_j \cup_{Y_{j+1}} W_{j+1}$$

from Y_j to Y_{j+2} contains an embedded 2–sphere E_j with self-intersection number -1. Denote by S_j the boundary of a normal neighborhood of E_j. Let

$$Q_j = \{\, g_T \mid T \in \mathbb{R}\,\}$$

be the family of metrics on X_j constructed as follows. Start with an arbitrary metric g_0 on X_j which is a product metric near ∂X_j, Y_{j+1}, and S_j, and which has the property that the metric it induces on S_j is close enough to the round metric to have positive scalar curvature. For any $T \in \mathbb{R}$, the metric g_T is then obtained from g_0 by inserting the cylinder $[T,-T] \times S_j$ into a normal neighborhood of S_j if $T < 0$, and by inserting the cylinder $[-T,T] \times Y_{j+1}$ into a normal neighborhood of Y_{j+1} if $T > 0$.

Given a spin^c structure $\mathfrak{s}$ on X_j, we again count monopoles on the manifold X_j^* with cylindrical ends over the whole family Q_j to define the map

$$H^o_o(X_j,\mathfrak{s}) : C^o(Y_j) \to C^o(Y_{j+2})$$

as well as its companion maps $H^o_s(X_j,\mathfrak{s})$, $H^u_o(X_j,\mathfrak{s})$, $H^u_s(X_j,\mathfrak{s})$, $\bar{H}^s_u(X_j,\mathfrak{s})$, $\bar{H}^s_s(X_j,\mathfrak{s})$, $\bar{H}^u_s(X_j,\mathfrak{s})$, and $\bar{H}^u_u(X_j,\mathfrak{s})$. Using these maps, we define the map

$$\check{H}(X_j,\mathfrak{s}) : \check{C}(Y_j) \to \check{C}(Y_{j+2})$$

by the matrix

$$\left[\begin{array}{c|l} H^o_o(X_j,\mathfrak{s}) & \begin{array}{l} -H^u_o(X_j,\mathfrak{s})\bar{\partial}^s_u(Y_j) \\ -m^u_o(W_{j+1},\mathfrak{s})\bar{m}^s_u(W_j,\mathfrak{s}|_{W_j}) - \partial^u_o(Y_{j+2})\bar{H}^s_u(X_j,\mathfrak{s}|_{W_j}) \end{array} \\ \hline H^o_s(X_j,\mathfrak{s}) & \begin{array}{l} \bar{H}^s_s(X_j,\mathfrak{s}) - H^u_s(X_j,\mathfrak{s})\bar{\partial}^s_u(Y_j) \\ -m^u_s(W_{j+1},\mathfrak{s}|_{W_{j+1}})\bar{m}^s_u(W_j,\mathfrak{s}|_{W_j}) - \partial^u_s(Y_{j+2})\bar{H}^s_u(X_j,\mathfrak{s}) \end{array} \end{array}\right]$$

Note that an $\mathbb{F}_2$ version of this map can be found in [**38**, page 491], and the correct sign assignments for its integral version in [**37**, (26.12)]. By summing up over the spin^c structures, we obtain the map

$$\check{H}(X_j) = \sum_{\mathfrak{s}\in \mathrm{spin}^c(X_j,\mathfrak{s}_0)} (-1)^{\mu(\mathfrak{s}|_{W_j})+\mu(\mathfrak{s}|_{W_{j+1}})}\,\check{H}(X_j,\mathfrak{s}) : \check{C}(Y_j) \to \check{C}(Y_{j+2}).$$

Proposition 4.7. *(1) One has the equality*

$$\check{\partial}(Y_{j+2}) \circ \check{H}(X_j) + \check{H}(X_j) \circ \check{\partial}(Y_j) = \check{m}(W_{j+1}) \circ \check{m}(W_j). \tag{13}$$

(2) The map $\Psi_j : \check{C}(Y_j) \to \check{C}(Y_j)$ *defined as*

$$\check{H}(X_{j+1}) \circ \check{m}(W_j) - \check{m}(W_{j+2}) \circ \check{H}(X_j)$$

is an anti-chain map. Moreover, the map

$$(\Psi_j)_{\mathbb{Q}} : \check{C}(Y_j)_{\mathbb{Q}} \to \check{C}(Y_j)_{\mathbb{Q}}$$

induces an isomorphism in homology.

Proof. (1) We can upgrade the proof of the mod 2 version [**38**, Proposition 5.2] of this result to the integers as follows: Let B_j be the (closed) normal neighborhood of E_j and Z_j the closure of $X_j \backslash B_j$. The family Q_j of metrics on X_j can be completed by adding the disjoint union $Z_j^* \cup B_j^*$ at $T = -\infty$ and the disjoint union $W_j^* \cup W_{j+1}^*$ at $T = +\infty$. Denote this new family by $\bar{Q}_j$. Given monopoles $\mathfrak{a}$ on Y_j and $\mathfrak{b}$ on Y_{j+2} and a spin^c structure $\mathfrak{s}$ on X_j, consider the parametrized moduli space

$$\mathcal{M}(\mathfrak{a}, (X_j^*, \mathfrak{s}), \mathfrak{b})_{\bar{Q}}$$

on the manifold $(X_j^*, \mathfrak{s})$ and construct its compactification $\mathcal{M}^+(\mathfrak{a}, (X_j^*, \mathfrak{s}), \mathfrak{b})_{\bar{Q}}$ by adding in broken trajectories. When this moduli space is one-dimensional, the number of its boundary points, counted with sign, must be zero. This gives us a boundary identity. By adding these boundary identities over all possible $\mathfrak{s}$ with sign $(-1)^{\mu(\mathfrak{s}|_{W_j}) + \mu(\mathfrak{s}|_{W_{j+1}})}$, we obtain various summed up boundary identities for different $(\mathfrak{a}, \mathfrak{b})$.

We claim that the points in $\mathcal{M}^+(\mathfrak{a}, (X_j^*, \mathfrak{s}), \mathfrak{b})_{g-\infty}$ do not contribute to these identities: As explained in the proof of [**38**, Proposition 5.2], these points always come in pairs of the form (γ, γ') and (γ, γ''), where γ is a (possibly broken) solution over Z_j^* and γ' and γ'' are reducible solutions over B_j^*. Moreover, γ' and γ'' correspond to conjugate spin^c structures over B_j^*. Since $b_2^+(B_j) = b_1(B_j) = 0$, all reducible monopoles over B_j^* are positive. Therefore, (γ, γ') and (γ, γ'') contribute to their respective boundary identities with the same sign. By Lemma 4.5, when we take the sum with the weights $(-1)^{\mu(\mathfrak{s}|_{W_j}) + \mu(\mathfrak{s}|_{W_{j+1}})}$ these contributions cancel.

The rest of the proof proceeds exactly as in [**38**, Proposition 5.2]. In [**37**, Lemma 26.2.3], several similar boundary identities are obtained by considering one-dimensional moduli spaces of monopoles for a family of metrics parametrized by $[0, 1]$. As a consequence of our claim, the summed up boundary identities we have here can be obtained from the identities there by removing terms corresponding to $T = 0$. For example, we have

$$\begin{aligned}0 = \sum_{\mathfrak{s} \in \mathrm{spin}^c(X_j, \mathfrak{s}_0)} &(-1)^{\mu(\mathfrak{s}|_{W_{j+1}}) + \mu(\mathfrak{s}|_{W_j})} (-H^o_o(X_j, \mathfrak{s}) \partial^o_o(Y_j) - \partial^o_o(Y_{j+2}) H^o_o(X_j, \mathfrak{s}) \\ &+ H^u_o(X_j, \mathfrak{s}) \bar{\partial}^s_u(Y_j) \partial^o_s(Y_j) + \partial^u_o(Y_{j+2}) \bar{H}^s_u(X_j, \mathfrak{s}) \partial^o_s(Y_j) + \partial^u_o(Y_{j+2}) \bar{\partial}^s_u(Y_{j+2}) H^o_s(X_j, \mathfrak{s}) \\ &+ m^o_o(W_{j+1}, \mathfrak{s}|_{W_{j+1}}) m^o_o(W_j, \mathfrak{s}|_{W_j}) + m^u_o(W_{j+1}, \mathfrak{s}|_{W_{j+1}}) \overline{m}^s_u(W_j, \mathfrak{s}|_{W_j}) \partial^o_s(Y_j) \\ &- m^u_o(W_{j+1}, \mathfrak{s}|_{W_{j+1}}) \overline{\partial}^s_u(Y_{j+1}) m^o_s(W_j, \mathfrak{s}|_{W_j}) - \partial^u_o(Y_{j+2}) \overline{m}^s_u(W_{j+1}, \mathfrak{s}|_{W_{j+1}}) m^o_s(W_j, \mathfrak{s}|_{W_j}))\end{aligned}$$

Using these identities, formula (13) can be proved by an elementary (but cumbersome) calculation.

(2) The fact that Ψ_j is an anti-chain map follows easily from (1). According to [**38**, Lemma 5.11], the map

$$\Psi_j \otimes \mathrm{id} : \check{C}(Y_j) \otimes \mathbb{F}_2 \to \check{C}(Y_j) \otimes \mathbb{F}_2$$

induces an isomorphism in homology. By the universal coefficient theorem, the map $\Psi_{\mathbb{Q}}$ also induces an isomorphism in homology. q.e.d.

The proof of Theorem 4.6 is now completed by the following 'triangle detection lemma'. The mod 2 version of this lemma appears as Lemma 4.2 in [**56**]. The proof of the version at hand is essentially the same.[2]

Lemma 4.8. *For any $j \in \mathbb{Z}/3$, let (C_j, ∂_j) be a chain complex over the rationals. Suppose that there are chain maps $f_j : C_j \to C_{j+1}$ satisfying the following two conditions:*

- *the composite $f_{j+1} \circ f_j$ is null-homotopic by a chain homotopy $H_j : C_j \to C_{j+2}$ with*

$$\partial H_j + H_j \partial = f_{j+1} \circ f_j, \quad \textit{and}$$

- *the map*

$$\psi_j = H_{j+1} \circ f_j - f_{j+2} \circ H_j : C_j \to C_j,$$

which is an anti-chain map by the first condition, induces an isomorphism in homology.

Then the sequence

$$\cdots \longrightarrow H_*(C_j) \xrightarrow{(f_j)_*} H_*(C_{j+1}) \xrightarrow{(f_{j+1})_*} H_*(C_{j+2}) \longrightarrow \cdots$$

is exact.

[2] A version of this lemma over the integers can be found as Lemma 7.1 in [**35**]. Our sign conventions here are slightly different.

5. Skein relations up to constants

Let (L_0, L_1, L_2) be a skein triangle obtained by resolving a crossing c of the link $L = L_2$ as shown in Figure 1.

Definition 5.1. The skein triangle (L_0, L_1, L_2) will be called *admissible* if

1) $|L_2| = |L_0| + 1 = |L_1| + 1$, which is equivalent to saying that the resolved crossing c is between two different components of L_2, and
2) at least one of the links L_0, L_1, and L_2 is ramifiable.

In Section 2 we defined a link $\bar{L}_2$ by changing the crossing c. Using the cyclic symmetry as in Figure 2, we can find a link projection of L_0 such that L_1 and L_2 are the two resolutions of L_0 at a crossing c. Then we define $\bar{L}_0$ as the crossing change of L_0 at c. The link $\bar{L}_1$ is defined similarly.

Lemma 5.2. *If (L_0, L_1, L_2) is an admissible skein triangle then at most one of the six links L_0, L_1, L_2, $\bar{L}_0$, $\bar{L}_1$, $\bar{L}_2$ is not ramifiable. In particular, we have three more admissible skein triangles, $(\bar{L}_1, L_0, L_2)$, $(L_1, \bar{L}_0, L_2)$, and $(L_1, L_0, \bar{L}_2)$.*

Proof. By our definition of $\bar{L}$, all of the triples $(\bar{L}_1, L_0, L_2)$, $(L_1, \bar{L}_0, L_2)$, and $(L_1, L_0, \bar{L}_2)$ are skein triangles. Now suppose that two of the links L_0, L_1, L_2, $\bar{L}_0$, $\bar{L}_1$, and $\bar{L}_2$ are not ramifiable. By [**51**, Claim 3.2], these two links have to be $\bar{L}_j$ and $\bar{L}_{j+1}$ for some $j \in \mathbb{Z}/3$. Recall that after putting suitable signs, the determinants of the three links in a skein triangle add up to zero. Therefore, from the skein triangles $(L_{j-1}, L_{j+1}, \bar{L}_j)$ and $(L_j, L_{j-1}, \bar{L}_{j+1})$ we deduce that $\det(L_0) = \det(L_1) = \det(L_2)$. Since (L_0, L_1, L_2) is a skein triangle, this implies that $\det(L_0) = \det(L_1) = \det(L_2) = 0$. This contradicts Condition (2) of Definition 5.1. q.e.d.

Let (L_0, L_1, L_2) be an admissible skein triangle and $B \subset S^3$ a small ball containing the resolved crossing c. Denote by Y the double branched cover of $S^3 \backslash B$ with branch set $(S^3 \backslash B) \cap L_2$ then Y is a manifold with torus boundary ∂Y.

Definition 5.3. A *boundary framing* is a pair of oriented simple closed curves (m, l) on ∂Y such that

(1) $\#(m \cap l) = -1$,
(2) $[l] = 0 \in H_1(Y; \mathbb{Q})$, and
(3) either m or l represents the zero element in $H_1(Y; \mathbb{F}_2)$.

One can easily check that a boundary framing always exists. Once a boundary framing (m, l) is fixed, we will define the following numbers:

- The divisibility of the longitude

$$t(Y) = \min \{ a \in \mathbb{Z} \mid a > 0 \text{ and } a \cdot [l] = 0 \in H_1(Y; \mathbb{Z}) \};$$

- Set $s(Y) = 0$ if l represents the zero element in $H_1(Y; \mathbb{F}_2)$ and set $s(Y) = 1$ otherwise;
- The double branched cover $Y_j = \Sigma(L_j)$, $j \in \mathbb{Z}/3$, is obtained from Y by attaching a solid torus along ∂Y, matching the meridian with a simple closed curve γ_j on ∂Y. We will orient the curves γ_j by the following two conditions:
  (a) $\#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma_2) = \#(\gamma_2 \cap \gamma_0) = -1$, where the algebraic intersection numbers $\#$ are calculated with respect to the boundary orientation on ∂Y (see Ozsváth–Szabó [**56**, Section 2]), and
  (b) $\#(\gamma_2 \cap m) > 0$ when $s(Y) = 0$ and $\#(\gamma_2 \cap l) > 0$ when $s(Y) = 1$ (this makes sense because γ_2 represents zero in $H_1(Y; \mathbb{F}_2)$ by Definition 5.1 (1)).

  Having oriented the curves γ_j this way, we define the integers (p_j, q_j) by the equality $[\gamma_j] = p_j \cdot [m] + q_j \cdot [l]$, which holds in $H_1(\partial Y; \mathbb{Z})$.

Definition 5.4. Given a boundary framing (m, l), we define the *resolution data* for the admissible skein triangle (L_0, L_1, L_2) as the six-tuple

$$(t(Y), s(Y), (p_0, q_0), (p_1, q_1))$$

(we should note that $(p_2, q_2) = (-p_0 - p_1, -q_0 - q_1)$ since $[\gamma_0] + [\gamma_1] + [\gamma_2] = 0 \in H_1(\partial Y; \mathbb{Z})$).

The main goal of this section is to establish the following 'skein relations up to universal constants'. We will show later in Section 6 that these universal constants actually vanish.

Theorem 5.5. *Let (L_0, L_1, L_2) be an admissible skein triangle, and fix a boundary framing (m, l) on the boundary ∂Y of the manifold Y as above. Then*

1) if all of the links L_0, L_1, L_2 are ramifiable,

$$2\chi(L_2) = \chi(L_0) + \chi(L_1) + C(t(Y), s(Y), (p_0, q_0), (p_1, q_1)); \tag{14}$$

2) if L_j is not ramifiable for some $j \in \mathbb{Z}/3$ then $L_{j\pm1}$ and $\bar{L}_{j\pm1}$ are all ramifiable and, in addition,

$$\chi(\bar{L}_{j-1}) = \chi(L_{j-1}) + C_j^-(t(Y), s(Y), (p_0, q_0), (p_1, q_1)) \text{ and} \tag{15}$$

$$\chi(\bar{L}_{j+1}) = \chi(L_{j+1}) + C_j^+(t(Y), s(Y), (p_0, q_0), (p_1, q_1)), \tag{16}$$

where $C(t(Y), s(Y), (p_0, q_0), (p_1, q_1))$ and $C_j^{\pm}(t(Y), s(Y), (p_0, q_0), (p_1, q_1))$ are certain universal constants depending only on the resolution data.

5.1. The action of covering translations. We will use τ_M to denote covering translations on various double branched covers M such as $M = Y_j$ or $M = W_j$.

Lemma 5.6. *Let (L_0, L_1, L_2) be an admissible skein triangle. Then exactly one of the following two options is realized:*

1) if all L_j are ramifiable, there exists a unique $n \in \mathbb{Z}/3$ such that $|p_n| > |p_{n\pm1}|$ and, in addition,
- $b_1(Y_j) = b_1(W_j) = 0$ *for all* $j \in \mathbb{Z}/3$,
- $b_2^+(W_{n+1}) = 1$, $b_2^+(W_{n-1}) = b_2^+(W_n) = 0$, *and*
- $b_2^-(W_{n+1}) = 0$, $b_2^-(W_{n-1}) = b_2^-(W_n) = 1$.

2) if L_n is not ramifiable for some $n \in \mathbb{Z}/3$ then
- $b_1(Y_n) = 1$ *and* $b_1(Y_{n-1}) = b_1(Y_{n+1}) = 0$,
- $b_1(W_j) = b_2^+(W_j) = 0$ *for all* $j \in \mathbb{Z}/3$, *and*
- $b_2^-(W_{n+1}) = 1$ *and* $b_2^-(W_{n-1}) = b_2^-(W_n) = 0$.

Proof. The claims about b_1 follow easily from the Mayer–Vietoris sequence. As for the $b_2^{\pm}$ claims, note that the inequality $|p_n| > |p_{n\pm1}|$ is equivalent to p_n having an opposite sign to both p_{n-1} and p_{n+1}. The result then follows from the explicit calculation of the cup-product structure on $H^2(W_j)$ in Lemma 5.12. q.e.d.

Lemma 5.7. *The covering translations act as follows:*

1) $\tau_Y^*(a) = -a$ *for any* $a \in H^2(Y; \mathbb{Z})$;

2) $\tau_{Y_j}^*(b) = -b$ *for any* $b \in H^2(Y_j; \mathbb{Z})$;

3) $\tau_{W_j}^*(c) = -c$ *for any* $c \in H^2(W_j; \mathbb{Z})$;

4) $\tau_{Y_j}^*(d) = -d$ *for any* $d \in H_1(Y_j; \mathbb{Z})$;

5) $\tau_{W_j}^*(e) = -e$ *for any* $e \in H_1(W_j; \mathbb{Z})$;

6) $\tau_Y^*(\mathfrak{s}) = \bar{\mathfrak{s}}$ *for any* $\mathfrak{s} \in \mathrm{spin}^c(Y)$*;*

7) $\tau_{Y_j}^*(\mathfrak{s}) = \bar{\mathfrak{s}}$ *for any* $\mathfrak{s} \in \mathrm{spin}^c(Y_j)$*;*

8) $\tau_{X_j}^*(\mathfrak{s}) = \bar{\mathfrak{s}}$ *for any* $\mathfrak{s} \in \mathrm{spin}^c(X_j)$*.*

Proof. Since $H_2(Y;\mathbb{Z}) = 0$, the universal coefficient theorem provides a natural identification of $H^2(Y;\mathbb{Z})$ with the torsion part of $H_1(Y;\mathbb{Z})$. Therefore, to prove (1), we only need to show that τ acts on $H_1(Y;\mathbb{Z})$ as -1. To see this, let $\tilde{\alpha}$ be a loop in Y which does not intersect the branch locus. The image of $\tilde{\alpha}$ under the covering map $Y \to S^3\backslash B$ will be called α. Since $S^3\backslash B$ is contractible, there is a continuous map $f : D^2 \to S^3\backslash B$ with $f(\partial D^2) = \alpha$. Lift f to a map $\tilde{f} : F \to Y$, where F is a double branched cover of D^2. If $\tilde{f}(\partial F)$ equals $\tilde{\alpha}$ or $\tau(\tilde{\alpha})$, then $\tilde{\alpha}$ is null-homologous. Otherwise, we have $\tilde{f}(\partial F) = \tilde{\alpha} + \tau(\tilde{\alpha})$, which implies $[\tilde{\alpha}] = -[\tau(\tilde{\alpha})]$ and proves (1). Claim (4) can be proved similarly, while (2) is just the Poincaré dual of (4), and (5) follows from (4) and the fact that $H_1(\partial W_j;\mathbb{Z}) \to H_1(W_j;\mathbb{Z})$ is onto.

We will next prove (3) under the assumption that $b_1(Y_j) = 0$ (otherwise, $b_1(Y_{j+1}) = 0$, and the argument is similar). Using the Mayer–Vietoris sequence for the decomposition of W_j into $I \times Y_j$ and the 2-handle, we obtain an exact sequence

$$0 \longrightarrow H^1(S^1 \times D^2;\mathbb{Z}) \xrightarrow{\ \partial\ } H^2(W_j;\mathbb{Z}) \xrightarrow{\ i^*\ } H^2(I \times Y_j;\mathbb{Z}) \longrightarrow \cdots$$

The maps induced by τ on the cohomology groups in this sequence are compatible with ∂ and i^*. For any element $\alpha \in H^2(W_j;\mathbb{Z})$, it follows from Claim (2) that $i^*(\tau_{W_j}^*(\alpha) + \alpha) = \tau_{Y_j}^*(i^*(\alpha)) + i^*(\alpha) = 0$. Therefore, there exists $\beta \in H^1(S^1 \times D^2;\mathbb{Z})$ such that $\partial\beta = \tau_{W_j}^*(\alpha) + \alpha$. Notice that

$$\partial(-\beta) = \partial(\tau_{S^1\times D^2}^*(\beta)) = \tau_{W_j}^*(\tau_{W_j}^*(\alpha) + \alpha) = \tau_{W_j}^*(\alpha) + \alpha = \partial\beta.$$

Since ∂ is injective, β must vanish and therefore $\tau_{W_j}^*(\alpha) + \alpha = 0$ as claimed in (3).

Let us now turn our attention to the action of τ^* on spin^c structures, starting with (7). Turaev [**70**, Section 2.2] established a natural one-to-one correspondence between spin structures on Y_j and quasi-orientations on L_j. Using this correspondence, one can easily see that all the spin structures on Y_j are invariant under τ_{Y_j}. This proves (7) for self-conjugate spin^c structures. Since any spin^c structure can be written as $\mathfrak{s} + h$, with $\mathfrak{s}$ self-conjugate and $h \in H^2(Y;\mathbb{Z})$, claim (7) follows from (2). To prove (6), consider $\mathfrak{s}_0 = \mathfrak{s}_1|_Y$ with $\mathfrak{s}_1$ a spin^c structure on Y_1. By (7) we have $\tau_Y^*(\mathfrak{s}_0) = \bar{\mathfrak{s}}_0$. Then we express a general spin^c structure as $\mathfrak{s}_0 + h$ with $h \in H^2(Y;\mathbb{Z})$

and use (1). We are left with (8). Take any $\mathfrak{s} \in \mathrm{spin}^c(W_j)$. It follows from (7) that

$$\tau_{W_j}^*(\mathfrak{s})|_{Y_j} = \tau_{Y_j}^*(\mathfrak{s}|_{Y_j}) = \bar{\mathfrak{s}}|_{Y_j}.$$

Therefore, $\tau_{W_j}^*(\mathfrak{s}) = \bar{\mathfrak{s}} + h$ for some $h \in \ker(H^2(W_j;\mathbb{Z}) \to H^2(Y_j;\mathbb{Z}))$. Using (3), we conclude that $c_1(\tau_{W_j}^*\mathfrak{s}) = -c_1(\mathfrak{s}) = c_1(\bar{\mathfrak{s}})$ and therefore $2h = 0$. However, it follows from the Mayer–Vietoris exact sequence that $\ker(H^2(W_j;\mathbb{Z}) \to H^2(Y_j;\mathbb{Z}))$ is torsion free. Therefore, $h = 0$, and claim (8) is proved. q.e.d.

5.2. Spin^c systems and their equivalence. In this section we introduce the concept of a spin^c system on a skein triangle and relate the spin^c systems corresponding to admissible skein triangles with the same resolution data.

Definition 5.8. Let (L_0, L_1, L_2) be a skein triangle and $\mathfrak{s}_0$ a fixed self-conjugate spin^c structure on Y. A spin^c *system* $\mathcal{S}((L_0, L_1, L_2), \mathfrak{s}_0)$ is the set

$$\bigcup_{j \in \mathbb{Z}/3} (\mathrm{spin}^c(W_j, \mathfrak{s}_0) \cup \mathrm{spin}^c(Y_j, \mathfrak{s}_0))$$

endowed with the following additional structure:

1) the restriction map $r_{j,j+1} : \mathrm{spin}^c(W_j, \mathfrak{s}_0) \longrightarrow \mathrm{spin}^c(Y_j, \mathfrak{s}_0) \times \mathrm{spin}^c(Y_{j+1}, \mathfrak{s}_0)$ for every $j \in \mathbb{Z}/3$,
2) for every $\mathfrak{s} \in \mathrm{spin}^c(W_j, \mathfrak{s}_0)$ with $r_{j,j+1}(\mathfrak{s})$ torsion, the Chern number $c_1(\mathfrak{s})^2 \in \mathbb{Q}$ (see (19)), and
3) the involution $\tau_{W_j}^*$ on $\mathrm{spin}^c(W_j, \mathfrak{s}_0)$ and the involution $\tau_{Y_j}^*$ on $\mathrm{spin}^c(Y_j, \mathfrak{s}_0)$. By Lemma 5.7, these act by conjugation on the set of spin^c structures.

Definition 5.9. Two spin^c systems $\mathcal{S}((L_0, L_1, L_2), \mathfrak{s}_0)$ and $\mathcal{S}((L_0', L_1', L_2'), \mathfrak{s}_0')$ are called *equivalent* if $\mathrm{sign}(W_j) = \mathrm{sign}(W_j')$ for all $j \in \mathbb{Z}/3$ and there exist bijections

$$\hat{\theta}_j : \mathrm{spin}^c(W_j, \mathfrak{s}_0) \to \mathrm{spin}^c(W_j', \mathfrak{s}_0') \quad \text{and} \quad \theta_j : \mathrm{spin}^c(Y_j, \mathfrak{s}_0) \to \mathrm{spin}^c(Y_j', \mathfrak{s}_0')$$

which are compatible with the additional structures (1), (2), and (3) in the obvious way. We use $\sim$ to denote this equivalence relation.

Theorem 5.10. *Let (L_0, L_1, L_2) and (L_0', L_1', L_2') be admissible skein triangles. Suppose that, for a suitable choice of boundary framings, the resolution data of (L_0, L_1, L_2) matches that of (L_0', L_1', L_2'). Then there exist disjoint decompositions*

$$\text{sc-spin}^c(Y) = A_0 \cup A_1 \quad \textit{and} \quad \text{sc-spin}^c(Y') = A_0' \cup A_1'$$

with the following properties:

- $|A_0| = |A_1|$ *and* $|A_0'| = |A_1'|$ *(the vertical bars stand for the cardinality of a set), and*
- *for* $i = 0, 1$ *and any* $\mathfrak{s}_0 \in A_i$ *and* $\mathfrak{s}_0' \in A_i'$*, we have*

$$\mathcal{S}((L_0, L_1, L_2), \mathfrak{s}_0) \sim \mathcal{S}((L_0', L_1', L_2'), \mathfrak{s}_0').$$

The proof of Theorem 5.10 will take up the rest of this subsection. The idea of the proof is straightforward: we give an explicit description of the topology and the spin^c structures on Y_j and W_j in terms of the resolution data.

We begin by studying the algebraic topology of cobordisms W_j. Let us consider the decompositions

$$W_j = (I \times Y) \cup_{I \times \partial Y} D^4 \quad \text{and} \quad Y_j = Y \cup_{\partial Y} (S^1 \times D^2).$$

The gluing map in the first decomposition (which matches the decomposition (9)) identifies ∂Y with the standard torus in $S^3 = \partial D^4$ by sending γ_j and γ_{j+1} to the standard meridian m and longitude l, respectively. The corresponding Mayer–Vietoris exact sequences are of the form

$$\dots \longrightarrow H^1(Y) \longrightarrow H^1(\partial Y) \xrightarrow{\partial_{W_j}} H^2(W_j) \xrightarrow{i_{W_j}} H^2(Y) \longrightarrow \dots \tag{17}$$

and

$$\dots \longrightarrow H^1(Y) \oplus H^1(S^1 \times D^2) \longrightarrow H^1(\partial Y) \xrightarrow{\partial_{Y_j}} H^2(Y_j) \xrightarrow{i_{Y_j}} H^2(Y) \longrightarrow \dots \tag{18}$$

Let $\hat{\gamma}_j$, $\hat{m}$, and $\hat{l} \in H^1(\partial Y)$ be the Poincaré duals of $[\gamma_j]$, $[m]$, and $[l] \in H_1(\partial Y)$, respectively. The following lemma is a direct consequence of (17) and (18).

Lemma 5.11. *There are isomorphisms*

$$\begin{aligned} \ker i_{W_j} &= \mathbb{Z} \oplus \mathbb{Z}/\langle (0, t(Y)) \rangle \quad \textit{with generators} \quad \partial_{W_j}(\hat{m}),\ \partial_{W_j}(\hat{l}), \quad \textit{and} \\ \ker i_{Y_j} &= \mathbb{Z} \oplus \mathbb{Z}/\langle (0, t(Y)), (p_j, q_j) \rangle \quad \textit{with generators} \quad \partial_{Y_j}(\hat{m}),\ \partial_{Y_j}(\hat{l}), \end{aligned}$$

under which the map $\ker i_{W_j} \to \ker i_{Y_j} \oplus \ker i_{Y_{j+1}}$ *induced by the boundary inclusions is the natural projection map.*

Recall that, for a cohomology class $\alpha \in H^2(W_j)$ with $\alpha|_{\partial W_j}$ torsion, the number $\alpha^2 \in \mathbb{Q}$ is defined as follows. Let $u_j : H^2(W_j, \partial W_j) \to H^2(W_j)$ be the map induced by the inclusion.

Then there exists a non-zero integer k and a cohomology class $\beta \in H^2(W_j, \partial W_j)$ such that $k\alpha = u_j(\beta)$. We define

$$\alpha^2 \;=\; \frac{1}{k^2}\,\langle\, \beta \smile \beta,\, [W_j, \partial W_j]\,\rangle. \tag{19}$$

Lemma 5.12. *(1) Suppose that $p_j \neq 0$ and $p_{j+1} \neq 0$. Then*

$$(\partial_{W_j}(a\hat{m} + b\hat{l}))^2 \;=\; \frac{a^2}{p_j \cdot p_{j+1}}. \tag{20}$$

(2) Suppose that either $p_j = 0$ or $p_{j+1} = 0$. Then $\alpha^2 = 0$ for any $\alpha \in H^2(W_j)$ with $\alpha|_{\partial W_j}$ torsion.

Proof. Associated with the decomposition $W_j = (I \times Y) \cup_{I \times \partial Y} D^4$ is the Mayer–Vietoris exact sequence in homology,

$$\ldots \longrightarrow H_2(Y) \xrightarrow{\;0\;} H_2(W_j) \xrightarrow{\;\partial\;} H_1(\partial Y) \longrightarrow H_1(Y) \longrightarrow \ldots$$

from which we conclude that $H_2(W_j)$ is a copy of $\mathbb{Z}$ generated by the homology class $[\Sigma]$ of a surface Σ with $\partial[\Sigma] = t(Y)[l]$. The surface Σ splits as $F_1 \cup F_2$, where F_1 is an embedded surface in $I \times Y$ bounded by $t(Y)$ copies of l, and F_2 is the Seifert Surface for the right handed $(t(Y) \cdot p_j, t(Y) \cdot p_{j+1})$ torus link in ∂D^4. From this description, we see that the homological self-intersection number of Σ equals the linking number between two parallel copies of the $(t(Y) \cdot p_j, t(Y) \cdot p_{j+1})$ torus link, which equals $t(Y)^2 \cdot p_j \cdot p_{j+1}$. Therefore,

$$\langle\, \mathrm{PD}\,[\Sigma] \smile \mathrm{PD}\,[\Sigma],\, [W_j, \partial W_j]\,\rangle = \langle\, u_j(\mathrm{PD}\,[\Sigma]),\, [\Sigma]\,\rangle = t(Y)^2 \cdot p_j \cdot p_{j+1}.$$

Comparing this to

$$\langle\, \partial_{W_j}\hat{m},\, [\Sigma]\,\rangle = \langle \hat{m},\, \partial[\Sigma]\,\rangle = \langle \hat{m},\, t(Y)[l]\,\rangle = t(Y)$$

we obtain

$$u_j(\mathrm{PD}\,[\Sigma]) = \partial_{W_j}(\, t_Y \cdot p_j \cdot p_{j+1}\, \hat{m} + k\hat{l}\,)$$

for some integer k, whose value is of no importance to us because $\partial_{W_j}(\hat{l})$ is torsion. From this we deduce that

$$(\partial_{W_j}(a\hat{m} + b\hat{l}))^2 = \frac{a^2}{(t(Y) \cdot p_j \cdot p_{j+1})^2} \cdot \langle\, \mathrm{PD}\,[\Sigma] \smile \mathrm{PD}\,[\Sigma],\, [W_j, \partial W_j]\,\rangle = \frac{a^2}{p_j \cdot p_{j+1}}.$$

This completes the proof of (1). To prove (2), observe that the map $H^2(W_j;\mathbb{Q}) \to H^2(Y_{j+1};\mathbb{Q})$ is injective when $p_j = 0$, and that the map $H^2(W_j;\mathbb{Q}) \to H^2(Y_j;\mathbb{Q})$ is injective when $p_{j+1} = 0$. Therefore, any element α with $\alpha|_{\partial W}$ torsion is a torsion itself. q.e.d.

Let $X_j = W_j \cup W_{j+1}$ be the composite cobordism from Y_j to Y_{j+2}. As we mentioned in the proof of the exact triangle in Section 4.2, there exists an embedded 2–sphere $E_j \subset X_j$ with homological self-intersection -1. It is obtained by gluing a disk $D_1 \subset W_j$ to a disk $D_2 \subset W_{j+1}$ along $-\partial D_1 = \partial D_2 = l_{j+1}$, where l_{j+1} is the core of the solid torus $Y_{j+1} \backslash \operatorname{int} Y$. Orient E_j so that l_{j+1} is homotopic to γ_{j+2} in $Y_{j+1} \backslash \operatorname{int} Y$. Also recall a decomposition $X_j = (-W_{j+2}) \# \overline{\mathbb{C}\mathrm{P}}^2$, which induces an isomorphism

$$\rho_{j,j+1} : H^2(W_{j+2}) \oplus H^2(\overline{\mathbb{C}\mathrm{P}}^2) \longrightarrow H^2(X_j). \tag{21}$$

Lemma 5.13. *For any integers a, b, c, denote by ξ the image of $(\partial_{W_{j+2}}(a\hat{m} + b\hat{l}), c \cdot \mathrm{PD}\,[E_j])$ under the map $\rho_{j,j+1}$. Then*

$$\xi|_{W_j} = \partial_{W_j}(a\hat{m} + b\hat{l} + c\hat{\gamma}_j) \quad \text{and} \quad \xi|_{W_{j+1}} = \partial_{W_{j+1}}(a\hat{m} + b\hat{l} + c\hat{\gamma}_{j+2}).$$

Proof. It is a direct consequence of the naturality of the boundary map in the Mayer–Vietoris exact sequence that

$$\rho_{j,j+1}(\partial_{W_{j+2}}(a\hat{m} + b\hat{l}), 0)|_{W_n} = \partial_{W_n}(a\hat{m} + b\hat{l}) \quad \text{for} \quad n = j, j+1.$$

We still need to show that

$$\rho_{j,j+1}(0, \mathrm{PD}\,[E_j])|_{W_j} = \partial_{W_j}(\hat{\gamma}_j) \quad \text{and} \quad \rho_{j,j+1}(0, \mathrm{PD}\,[E_j])|_{W_{j+1}} = \partial_{W_{j+1}}(\hat{\gamma}_{j+2}).$$

The Poincare dual of $\rho_{j,j+1}(0, \mathrm{PD}\,[E_j])$ is realized by the sphere E_j. Therefore, the restriction of $\rho_{j,j+1}(0, \mathrm{PD}\,[E_j])$ to W_j equals the Poincare dual of $[E_j \cap W_j] \in H_2(W_j, \partial W_j)$. Using the fact that $E_j \cap W_j$ is a disk contained in the two-handle $D^4 \subset W_j$ with the boundary l_{j+1}, one can easily verify that $[E_j \cap W_j]$ equals the Poincare dual of $\partial_{W_j}(\hat{\gamma}_j)$. This finishes the proof of the first formula. The proof of the second formula is similar. q.e.d.

We will next study the spin and spin^c structures on the manifolds Y_j and W_j. First, define a map

$$\lambda : \mathrm{spin}(Y) \to H^1(\partial Y; \mathbb{F}_2)$$

as follows (compare with Turaev [**71**]). Fix a diffeomorphism $\varphi : \partial Y \to \mathbb{R}^2/\mathbb{Z}^2$ and let x_1, x_2 be the standard coordinates on $\mathbb{R}^2$. Pull back the vector fields $\partial/\partial x_1$, $\partial/\partial x_2$ via φ to obtain vector fields $\vec{v}_1$, $\vec{v}_2$ on ∂Y. Any loop γ in ∂Y gives rise to the loop

$$\tilde{\gamma}(x) = (\vec{n}(x), \vec{v}_1(x), \vec{v}_2(x))$$

in the frame bundle of Y, where $\vec{n}(x)$ is the outward normal vector at $x \in \partial Y$. We define $\lambda(\mathfrak{s})$ to be the unique cohomology class in $H^1(\partial Y, \mathbb{F}_2)$ with the property that $\langle \lambda(\mathfrak{s}), [\gamma] \rangle = 0$ if and only if $\tilde{\gamma}$ can be lifted to a loop in the spin bundle for $\mathfrak{s}$.

Lemma 5.14. *The map* $\lambda : \operatorname{spin}(Y) \to H^1(\partial Y; \mathbb{F}_2)$ *has the following properties:*

(1) λ *does not depend on the choice of diffeomorphism* φ*,*

(2) $\lambda(\mathfrak{s} + \omega) = \mathfrak{s} + \omega|_{\partial Y}$ *for any* $\mathfrak{s} \in \operatorname{spin}(Y)$ *and* $\omega \in H^1(Y; \mathbb{F}_2)$*, and*

(3) *for any* $j \in \mathbb{Z}/3$*, a spin structure* $\mathfrak{s}$ *can be extended to* Y_j *if and only if* $\langle \lambda(\mathfrak{s}), \gamma_j \rangle = 1 \in \mathbb{F}_2$*. The extension is unique if it exists.*

Proof. This is immediate from the definition of the map λ. q.e.d.

Lemma 5.15. *Any* $\mathfrak{s} \in \operatorname{spin}(Y)$ *extends to a spin structure on* Y_2*, and to a spin structure on one of the manifolds* Y_0 *and* Y_1 *but not the other. For any* $\mathfrak{s}_0 \in \operatorname{sc-spin^c}(Y)$*, we have the following identity for the counts of self-conjugate* spin^c *structures:*

$$2^{b_1(Y_0)} \cdot |\operatorname{sc-spin^c}(Y_0, \mathfrak{s}_0)| + 2^{b_1(Y_1)} \cdot |\operatorname{sc-spin^c}(Y_1, \mathfrak{s}_0)| = 2^{b_1(Y_2)} \cdot |\operatorname{sc-spin^c}(Y_2, \mathfrak{s}_0)|. \tag{22}$$

Proof. Since the map $H^1(Y_2; \mathbb{F}_2) \to H^1(Y; \mathbb{F}_2)$ is an isomorphism, any spin structure $\mathfrak{s}$ on Y can be extended to a spin structure on Y_2. This implies that

$$\langle \lambda(\mathfrak{s}), [\gamma_0] \rangle + \langle \lambda(\mathfrak{s}), [\gamma_1] \rangle = \langle \lambda(\mathfrak{s}), [\gamma_2] \rangle = 1. \tag{23}$$

It now follows from Lemma 5.14 (3) that $\mathfrak{s}$ can be extended a spin structure on exactly one of the manifolds Y_0 and Y_1. This finishes the proof of the first statement.

According to Remark 4.2, a self-conjugate spin^c structure on Y_j corresponds to $2^{b_1(Y_j)}$ spin structures. Therefore,

$$2^{b_1(Y_j)} \cdot |\operatorname{sc-spin^c}(Y_j, \mathfrak{s}_0)| = |\operatorname{spin}(Y_j, \mathfrak{s}_0)|,$$

and (22) is equivalent to

$$|\operatorname{spin}(Y_2, \mathfrak{s}_0)| = |\operatorname{spin}(Y_0, \mathfrak{s}_0)| + |\operatorname{spin}(Y_1, \mathfrak{s}_0)|,$$

which follows easily from the first statement. q.e.d.

Lemma 5.16. *A self-conjugate* $\mathrm{spin^c}$ *structure* $\mathfrak{s}_0 \in \text{sc-spin}^c(Y)$ *has the following extension properties to the cobordisms* W_j*:*

1) $\text{sc-spin}^c(W_0, \mathfrak{s}_0) = \varnothing$*;*

2) If $t(Y)$ *is odd, then* $\text{sc-spin}^c(W_1, \mathfrak{s}_0) \neq \varnothing$ *and* $\text{sc-spin}^c(W_2, \mathfrak{s}_0) \neq \varnothing$*;*

3) If $t(Y)$ *is even, there is a disjoint decomposition*

$$\text{sc-spin}^c(Y) = A_0 \cup A_1 \tag{24}$$

such that $|A_0| = |A_1|$ *and, in addition,*

$$\begin{aligned} &\mathfrak{s}_0 \in A_0 \quad \textit{if and only if} \quad \text{sc-spin}^c(W_1, \mathfrak{s}_0) = \varnothing \;\; \textit{and} \;\; \text{sc-spin}^c(W_2, \mathfrak{s}_0) \neq \varnothing, \\ &\mathfrak{s}_0 \in A_1 \quad \textit{if and only if} \quad \text{sc-spin}^c(W_1, \mathfrak{s}_0) \neq \varnothing \;\; \textit{and} \;\; \text{sc-spin}^c(W_2, \mathfrak{s}_0) = \varnothing. \end{aligned} \tag{25}$$

Proof. Denote by $\mathfrak{s}_0^1$ and $\mathfrak{s}_0^2$ the two spin structures on Y corresponding to the self-conjugate $\mathrm{spin^c}$ structure $\mathfrak{s}_0$ on Y. Then

$$\text{sc-spin}^c(W_j, \mathfrak{s}_0) \neq \varnothing \quad \text{if and only if} \quad \mathrm{spin}(W_j, \mathfrak{s}_0^1) \neq \varnothing \;\; \text{or} \;\; \mathrm{spin}(W_j, \mathfrak{s}_0^2) \neq \varnothing. \tag{26}$$

Since the cobordism W_j is obtained by attaching D^4 to the manifold W_j^0 along S^3, see (9), we conclude that, for both $k = 1$ and $k = 2$,

$$\begin{aligned} \mathrm{spin}(W_j, \mathfrak{s}_0^k) \neq \varnothing \quad &\text{if and only if} \quad \mathrm{spin}(Y_j, \mathfrak{s}_0^k) \neq \varnothing \;\; \text{and} \;\; \mathrm{spin}(Y_{j+1}, \mathfrak{s}_0^k) \neq \varnothing \\ &\text{if and only if} \quad \langle \lambda(\mathfrak{s}_0^k), [\lambda_j] \rangle = \langle \lambda(\mathfrak{s}_0^k), [\lambda_{j+1}] \rangle = 1. \end{aligned}$$

With this understood, (1) follows from Lemma 5.15. Since $\mathfrak{s}_0^1$ and $\mathfrak{s}_0^2$ correspond to the same $\mathrm{spin^c}$ structure, we can write $\mathfrak{s}_0^1 = \mathfrak{s}_0^2 + (\omega_{\mathbb{F}_2})|_{\partial Y}$, where $\omega_{\mathbb{F}_2}$ is the mod 2 reduction of the generator $\omega \in H^1(Y; \mathbb{Z})$. This implies that

$$\lambda(\mathfrak{s}_0^1) = \lambda(\mathfrak{s}_0^2) + (\omega_{\mathbb{F}_2})|_{\partial Y}.$$

It is not difficult to see that $(\omega_{\mathbb{F}_2})|_{\partial Y} \neq 0$ if and only if $t(Y)$ is odd. Therefore, if $t(Y)$ is odd, $\lambda(\mathfrak{s}_0^1) \neq \lambda(\mathfrak{s}_0^2)$. By Lemma 5.15, one of $\mathfrak{s}_0^k$ $(k = 1, 2)$ can be extended over Y_0 (and hence W_2), while the other one can be extended over Y_1 (and hence W_1). Claim (2) now follows from (26). If $t(Y)$ is even, $\lambda(\mathfrak{s}_0^1) = \lambda(\mathfrak{s}_0^2)$. Define the sets

$$A_j = \{\mathfrak{s}_0 \mid \langle \lambda(\mathfrak{s}_0^k), [\gamma_j] \rangle = 0 \;\; \text{for} \;\; k = 1, 2\}, \quad j = 0, 1,$$

then (24) and (25) follow directly from (23), and the equality $|A_0| = |A_1|$ can be verified as follows:

$$|A_0| = \frac{1}{2}\,|\operatorname{spin}(Y_0)| = \frac{1}{4}\,|\operatorname{spin}(Y)| = \frac{1}{2}\,|\operatorname{spin}(Y_1)| = |A_1|.$$

q.e.d.

Remark 5.17. The disjoint decomposition $\text{sc-spin}^c(Y) = A_0 \cup A_1$ of (24) with the additional properties (25) holds only for even $t(Y)$. We will extend it to the case of odd $t(Y)$ by choosing an arbitrary disjoint decomposition such that $|A_0| = |A_1|$.

In our next step toward the proof of Theorem 5.10, we will study the set of 'relative characteristic vectors' defined as

$$\operatorname{Char}(W_j, \mathfrak{s}_0) = \{c_1(\mathfrak{s}) \mid \mathfrak{s} \in \operatorname{spin}^c(W_j, \mathfrak{s}_0)\}.$$

To state the following set of results about this set, we need to recall the maps $i_{W_j} : H^2(W_j) \to H^2(Y)$ and $\partial_{W_j} : H^1(\partial Y) \to H^2(W_j)$ from the Mayer–Vietoris exact sequence (17).

Lemma 5.18. *The set* $\operatorname{Char}(W_j, \mathfrak{s}_0)$ *is a coset of* $2\ker i_{W_j}$ *inside of* $\ker i_{W_j}$ *which can be described precisely as follows:*

1) Suppose $t(Y)$ *is odd. Then, for any* $\mathfrak{s}_0 \in \text{sc-spin}^c(Y)$*,*
  - *a)* $\operatorname{Char}(W_j, \mathfrak{s}_0) = 2\ker i_{W_j}$ *for* $j = 1, 2$*,*
  - *b)* $\operatorname{Char}(W_0, \mathfrak{s}_0) = \partial_{W_0}(\hat{m}) + 2\ker i_{W_0}$*.*

2) Suppose $t(Y)$ *is even and* $\mathfrak{s}_0 \in A_0$*. Then*
  - *a)* $\operatorname{Char}(W_2, \mathfrak{s}_0) = 2\ker i_{W_2}$*,*
  - *b)* $\operatorname{Char}(W_1, \mathfrak{s}_0) = \partial_{W_1}(\hat{\gamma}_2) + 2\ker i_{W_1}$*,*
  - *c)* $\operatorname{Char}(W_0, \mathfrak{s}_0) = \partial_{W_0}(\hat{\gamma}_0) + 2\ker i_{W_0}$*.*

3) Suppose $t(Y)$ *is even and* $\mathfrak{s}_0 \in A_1$*. Then*
  - *a)* $\operatorname{Char}(W_1, \mathfrak{s}_0) = 2\ker i_{W_2}$*,*
  - *b)* $\operatorname{Char}(W_2, \mathfrak{s}_0) = \partial_{W_2}(\hat{\gamma}_2) + 2\ker i_{W_2}$*,*
  - *c)* $\operatorname{Char}(W_0, \mathfrak{s}_0) = \partial_{W_0}(\hat{\gamma}_1) + 2\ker i_{W_0}$*.*

Proof. We will only prove case (2) because cases (1) and (3) are similar. In case (2), we have even $t(Y)$ and $\mathfrak{s}_0 \in A_0$. Since $\text{sc-spin}^c(W_2, \mathfrak{s}_0) \neq \varnothing$, the coset $\operatorname{Char}(W_2, \mathfrak{s}_0)$ must contain zero. This proves (a). To prove (b), note that the image of $\operatorname{Char}(W_1, \mathfrak{s}_0)$ under the restriction map $\ker i_{W_1} \to \ker i_{Y_1}$ does not contain zero because $\text{sc-spin}^c(Y_1, \mathfrak{s}_0) = \varnothing$, while the image of

$\mathrm{Char}(W_1, \mathfrak{s}_0)$ under the map $\ker i_{W_1} \to \ker i_{Y_2}$ contains zero because $\text{sc-spin}^c(Y_2, \mathfrak{s}_0) \neq \varnothing$. It is now not difficult to check that $\partial_{W_2}(\hat{\gamma}_2) + 2 \ker i_{W_2}$ is the only one coset (of the four) satisfying these requirements. This proves (b). Case (c) is similar. q.e.d.

Let (L_0, L_1, L_2) and (L_0', L_1', L_2') be two admissible skein triangles with the same resolution data, and let us fix decompositions

$$\text{sc-spin}^c(Y) = A_0 \cup A_1 \quad \text{and} \quad \text{sc-spin}^c(Y') = A_0' \cup A_1'$$

as in Lemma 5.16 and Remark 5.17. Combining all of the above lemmas, we obtain the following result.

Proposition 5.19. *Let $\mathfrak{s}_0 \in A_n$ and $\mathfrak{s}_0' \in A_n'$ with $n = 0$ or $n = 1$. Then there exist isomorphisms $\hat{\xi}_j : \ker i_{W_j} \to \ker i_{W_j'}$ and $\xi_j : \ker i_{Y_j} \to \ker i_{Y_j'}$ with the following properties:*

1) $\hat{\xi}_j(\alpha)|_{Y_j'} = \xi_j(\alpha|_{Y_j})$ and $\hat{\xi}_{j+1}(\alpha)|_{Y_{j+1}'} = \xi_{j+1}(\alpha|_{Y_{j+1}})$;

2) For any $\alpha \in \ker i_{W_j}$ with $\alpha|_{\partial W_j}$ torsion, we have $\alpha^2 = (\hat{\xi}_j(\alpha))^2$;

3) $\hat{\xi}_j(\mathrm{Char}(W_j, \mathfrak{s}_0)) = \mathrm{Char}(W_j', \mathfrak{s}_0')$,

4) Let $\rho_{j,j+1}$ be the isomorphism (21) and $\rho_{j,j+1}'$ the corresponding isomorphism for the skein triangle (L_0', L_1', L_2'). Then, for any $\beta \in \ker i_{W_{j+2}}$ and any integer k we have

$$\hat{\xi}_j(\rho_{j,j+1}(\beta, k \cdot \mathrm{PD}\,[E_j])|_{W_j}) = \rho_{j,j+1}'(\hat{\xi}_{j+2}(\beta), k \cdot \mathrm{PD}\,[E_j'])|_{W_j'} \quad \text{and}$$
$$\hat{\xi}_{j+1}(\rho_{j,j+1}(\beta, k \cdot \mathrm{PD}\,[E_j])|_{W_{j+1}}) = \rho_{j,j+1}'(\hat{\xi}_{j+2}(\beta), k \cdot \mathrm{PD}\,[E_j'])|_{W_{j+1}'}.$$

With all the necessary preparations now in place, we are finally ready to prove the main result of this subsection, Theorem 5.10.

Proof of Theorem 5.10. Let A_0, A_1, A_0', and A_1' be as above, and $\mathfrak{s}_0 \in A_n$ and $\mathfrak{s}_0' \in A_n'$ for $n = 0$ or $n = 1$. We first pick any $\mathfrak{s}_{W_0} \in \mathrm{spin}^c(W_0, \mathfrak{s}_0)$. It follows from Proposition 5.19 (3) that $\hat{\xi}_0(c_1(\mathfrak{s}_{W_0})) \in \mathrm{Char}(W_0', \mathfrak{s}_0')$ and there exists $\mathfrak{s}_{W_0'}$ such that $c_1(\mathfrak{s}_{W_0'}) = \hat{\xi}_0(c_1(\mathfrak{s}_{W_0}))$. Denote by $\tilde{\mathfrak{s}}$ the spin^c structure on $\overline{\mathbb{CP}}^2$ with $c_1(\tilde{\mathfrak{s}}) = \mathrm{PD}\,[E_j] = \mathrm{PD}\,[E_j']$ and let

$$\mathfrak{s}_{W_j} = (\mathfrak{s}_{W_0} \# \tilde{\mathfrak{s}})|_{W_j} \quad \text{and} \quad \mathfrak{s}_{W_j'} = (\mathfrak{s}_{W_0'} \# \tilde{\mathfrak{s}})|_{W_j'} \quad \text{for } j = 1, 2, \quad \text{and}$$

$$\mathfrak{s}_{Y_j} = \mathfrak{s}_{W_j}|_{Y_j} \quad \text{and} \quad \mathfrak{s}_{Y_j'} = \mathfrak{s}_{W_j'}|_{Y_j'} \quad \text{for } j = 0, 1, 2.$$

Using Proposition 5.19 (4) one can show that

$$\hat{\xi}_j(c_1(\mathfrak{s}_{W_j})) = c_1(\mathfrak{s}_{W_j'}) \quad \text{and} \quad \xi_j(c_1(\mathfrak{s}_{Y_j})) = c_1(\mathfrak{s}_{Y_j'}).$$

We now define the map $\hat{\theta}_j : \mathrm{spin}^c(W_j, \mathfrak{s}_0) \to \mathrm{spin}^c(W'_j, \mathfrak{s}'_0)$ by the formula

$$\hat{\theta}_j(\mathfrak{s}_{W_j} + h) = \mathfrak{s}_{W'_j} + \hat{\xi}_j(h) \quad \text{for} \quad h \in \ker i_{W_j}$$

and the map $\theta_j : \mathrm{spin}^c(Y_j, \mathfrak{s}_0) \to \mathrm{spin}^c(Y'_j, \mathfrak{s}'_0)$ by the formula

$$\theta_j(\mathfrak{s}_{Y_j} + h) = \mathfrak{s}_{Y'_j} + \xi_j(h) \quad \text{for} \quad h \in \ker i_{Y_j}.$$

It is not difficult to verify that $\hat{\theta}_j$ and θ_j are compatible with the additional structures in Definition 5.8 and that they provide the desired equivalence $\mathcal{S}((L_0, L_1, L_2), \mathfrak{s}_0) \sim \mathcal{S}((L'_0, L'_1, L'_2), \mathfrak{s}'_0)$. q.e.d.

5.3. Truncated Floer homology. Recall that, according to Theorem 4.6, we have the following Floer exact triangle over the rationals

$$\dots \xrightarrow{F_{W_0}} \widetilde{HM}(Y_1, [\mathfrak{s}_0]) \xrightarrow{F_{W_1}} \widetilde{HM}(Y_2, [\mathfrak{s}_0]) \xrightarrow{F_{W_2}} \widetilde{HM}(Y_0, [\mathfrak{s}_0]) \xrightarrow{F_{W_0}} \dots \tag{27}$$

Let us introduce the constants

$$c_0 = b_2^+(W_2) + b_1(Y_0), \quad c_1 = b_2^+(W_1) + b_1(Y_2), \quad \text{and} \quad c_2 = 0,$$

and use them to define 'twisted' versions of the maps $\check{\tau}_*^j$ induced by the covering translations on $\widetilde{HM}(Y_j, [\mathfrak{s}_0])$ by the formula

$$f_j = (-1)^{c_j} \cdot \check{\tau}_*^j : \widetilde{HM}(Y_j, [\mathfrak{s}_0]) \to \widetilde{HM}(Y_j, [\mathfrak{s}_0]).$$

Lemma 5.20. *The maps f_j are compatible with the maps F_{W_j} in the sense that*

$$f_{j+1} \circ F_{W_j} = F_{W_j} \circ f_j \quad \textit{for} \quad j \in \mathbb{Z}/3. \tag{28}$$

Proof. Since the covering translation τ_{W_j} on W_j extends the covering translations τ_{Y_j} and $\tau_{Y_{j+1}}$ on its boundary components, the functoriality of the monopole Floer homology and Lemma 5.7 imply that

$$\begin{aligned} \check{\tau}_*^{j+1} \circ \widetilde{HM}(W_j, \mathfrak{s}) &= (-1)^{b_2^+(W_j) + b_1(Y_{j+1})} \cdot \widetilde{HM}(W_j, \tau_{W_j}^* \mathfrak{s}) \circ \check{\tau}_*^j \\ &= (-1)^{b_2^+(W_j) + b_1(Y_{j+1})} \cdot \widetilde{HM}(W_j, \bar{\mathfrak{s}}) \circ \check{\tau}_*^j, \end{aligned} \tag{29}$$

for any $\mathfrak{s} \in \mathrm{spin}^c(W_j, [\mathfrak{s}_0])$. To explain the extra factor $(-1)^{b_2^+(W_j) + b_1(Y_{j+1})}$, we recall that the homology orientation, that is, an orientation of the vector space

$$\bigwedge^{\max} \left(H^1(W_j; \mathbb{R}) \oplus I^+(W_j; \mathbb{R}) \oplus H^1(Y_{j+1}; \mathbb{R}) \right),$$

is involved in the definition of the map $\widetilde{HM}(W)$; see [**37**, Definition 3.4.1]. Here, $I^+(W_j)$ stands for a maximum positive subspace for the intersection form on $\operatorname{im}(H^2(W_j,\partial W_j) \to H^2(W_j))$. By Lemma 5.7, the covering translation τ_{W_j} acts as the negative identity on the space

$$H^1(W_j;\mathbb{R}) \oplus I^+(W_j;\mathbb{R}) \oplus H^1(Y_{j+1};\mathbb{R}),$$

thereby changing the homology orientation by the factor of

$$(-1)^{b_2^+(W_j)+b_1(W_j)+b_1(Y_{j+1})}$$

Recall that the map F_{W_j} was defined in Section 4 by the formula

$$F_{W_j} \;=\; \sum_{\mathfrak{s}\in\operatorname{spin}^c(W_j,\mathfrak{s}_0)} (-1)^{\mu(\mathfrak{s})}\cdot \widetilde{HM}(W_j,\mathfrak{s}),$$

where $\mu(\mathfrak{s})$ is the $\mathbb{F}_2$-valued function defined in (10). Therefore, in order to deduce (28) from (29), we just need to check the relation

$$b_2^+(W_j) + b_1(Y_{j+1}) + c_{j+1} + c_j \;=\; \mu(\mathfrak{s}) - \mu(\bar{\mathfrak{s}}) \pmod 2$$

for any $\mathfrak{s} \in \operatorname{spin}^c(W_j,\mathfrak{s}_0)$. For $j = 1$ and $j = 2$, this is immediate from Proposition 4.3. For $j = 0$, this follows from Proposition 4.3 and the identity

$$b_1(Y_0) + b_1(Y_1) + b_1(Y_2) + b_2^+(W_0) + b_2^+(W_1) + b_2^+(W_2) = 1, \tag{30}$$

which is a consequence of Lemma 5.6. q.e.d.

Since $\widetilde{HM}(Y_j,[\mathfrak{s}_0])$ usually has infinite rank as a $\mathbb{Z}$–module, we will truncate it before discussing Lefschetz numbers.

Definition 5.21. For any rational number q and $j \in \mathbb{Z}/3$, define the *truncated monopole Floer homology* as

$$\widetilde{HM}_{\leqslant q}(Y_j,[\mathfrak{s}_0]) = \Big(\bigoplus_{\substack{\mathfrak{s}|_Y=\mathfrak{s}_0\\ c_1(\mathfrak{s})\text{ torsion}}}\;\bigoplus_{a\leqslant q}\;\widetilde{HM}_a(Y_j,\mathfrak{s})\Big)\oplus\Big(\bigoplus_{\substack{\mathfrak{s}|_Y=\mathfrak{s}_0\\ c_1(\mathfrak{s})\text{ non-torsion}}}\widetilde{HM}(Y_j,\mathfrak{s})\Big).$$

Also define

$$\widetilde{HM}_{>q}(Y_j,[\mathfrak{s}_0]) \;=\; \widetilde{HM}(Y_j,[\mathfrak{s}_0])\Big/\widetilde{HM}_{\leqslant q}(Y_j,[\mathfrak{s}_0]).$$

We wish to find truncations of $\widetilde{HM}(Y_j, [\mathfrak{s}_0])$ for all $j \in \mathbb{Z}/3$ which are preserved by the maps F_{W_j}. To this end, recall the map

$$\rho : \text{tor-spin}^c(Y_j) \to [0, 2)$$

defined by the formula

$$\rho(\mathfrak{s}) \equiv \frac{1}{4}\left(c_1(\hat{\mathfrak{s}})^2 - \mathrm{sign}(X)\right) \pmod 2$$

for any choice of smooth compact spin^c manifold $(X, \hat{\mathfrak{s}})$ with the spin^c boundary $(Y_j, \mathfrak{s})$ (see [**6**]).

Definition 5.22. Let $\tilde{\mathfrak{s}} \in \text{sc-spin}^c(Y_2, \mathfrak{s}_0)$ and choose an even integer $N > 0$ large enough so that, for each $j \in \mathbb{Z}/3$, the following two conditions are satisfied:

1) the natural map $\widetilde{HM}_{\leqslant q}(Y_j, [\mathfrak{s}_0]) \to HM^{\mathrm{red}}(Y_j, [\mathfrak{s}_0])$ is surjective, and
2) for any $\mathfrak{s} \in \mathrm{spin}^c(Y_j, \mathfrak{s}_0)$, there exists a finite set

$$\{a_1, a_2, ..., a_n\} \subset \widetilde{HM}(Y_j, \mathfrak{s})$$

representing a set of generators for $HM_{\mathrm{red}}(Y_j, \mathfrak{s})$ as a quotient $\mathbb{Q}[U]$-module, such that

$$F_{W_j}(a_i) \subset \widetilde{HM}_{\leqslant N}(Y_{j+1}, [\mathfrak{s}_0]).$$

(That this can be achieved follows from [**37**, Lemma 25.3.1]).

The truncated triangle with parameter $(N, \tilde{\mathfrak{s}})$ is then defined as the 3-periodic chain complex

$$\cdots \xrightarrow{F_{W_0}(N,\tilde{\mathfrak{s}})} \widetilde{HM}_{\leqslant N+\rho(\tilde{\mathfrak{s}})+o(1)}(Y_1, [\mathfrak{s}_0]) \xrightarrow{F_{W_1}(N,\tilde{\mathfrak{s}})} \widetilde{HM}_{\leqslant N+\rho(\tilde{\mathfrak{s}})+o(2)}(Y_2, [\mathfrak{s}_0])$$
$$\xrightarrow{F_{W_2}(N,\tilde{\mathfrak{s}})} \widetilde{HM}_{\leqslant N+\rho(\tilde{\mathfrak{s}})+o(0)}(Y_0, [\mathfrak{s}_0]) \xrightarrow{F_{W_0}(N,\tilde{\mathfrak{s}})} \cdots$$

where $F_{W_j}(N, \tilde{\mathfrak{s}})$ is the restriction of F_{W_j} and $o(j)$ is defined as follows:

1) if $b_1(Y_n) = 1$ for some $n \in \mathbb{Z}/3$ then $o(n) = o(n-1) = o(n+1) = 0$;
2) if $b_2^+(W_n) = 1$ for some $n \in \mathbb{Z}/3$ then $o(n+1) = 0$, $o(n) = 1/2$, and $o(n-1) = 1/4$.

(Note that by Lemma 5.6, exactly one of these two cases occurs). We denote this truncated triangle by $\mathfrak{C}^{\leqslant}(N, \tilde{\mathfrak{s}})$. It is a subcomplex of the exact triangle (27), and we denote by $\mathfrak{C}^{>}(N, \tilde{\mathfrak{s}})$ the quotient complex.

Lemma 5.23. *The image* $F_{W_j}(\widetilde{HM}_{\leqslant N+\rho(\tilde{\mathfrak{s}})+o(j)}(Y_j,[\mathfrak{s}_0]))$ *is contained in*

$$\widetilde{HM}_{\leqslant N+\rho(\tilde{\mathfrak{s}})+o(j+1)}(Y_{j+1},[\mathfrak{s}_0]),$$

and therefore the map $F_{W_j}(N,\tilde{\mathfrak{s}})$ *is well defined.*

Proof. We will only give the proof in the case when $b_1(Y_n) = 1$ for some $n \in \mathbb{Z}/3$ since the other case is similar.

We will start with the map $F_{W_{n+1}}$. Since $b_1(Y_n) = 1$, we have $\gamma_n = l$, which implies that $|p_{n\pm 1}| = 1$ and $p_{n-1} + p_{n+1} = -p_n = 0$. Using (20) and Lemma 5.6, we obtain

$$\frac{1}{4}\left(c_1^2(\mathfrak{s}) - 2\chi(W_{n+1}) - 3\sigma(W_{n+1})\right) \leqslant \frac{1}{4}\left(-1-2+3\right) = 0$$

for any $\mathfrak{s} \in \operatorname{spin}^c(W_{n+1},[\mathfrak{s}_0])$. Therefore, $F_{W_{n+1}}$ decreases the absolute grading, and the statement follows from the fact that $o(n+1) = o(n-1)$.

Let us now consider the map $F_{W_{n-1}}$. For a given $\mathfrak{s} \in \operatorname{spin}^c(W_{n-1},\mathfrak{s}_0)$, there are two possibilities:

- $\mathfrak{s}|_{Y_n}$ is non-torsion. There is nothing to prove in this case because no truncation is done on $\widetilde{HM}(Y_n,\mathfrak{s}|_{Y_n})$.
- $\mathfrak{s}|_{Y_n}$ is torsion. Since $b_1(Y_{n-1}) = 0$, the restriction $\mathfrak{s}|_{\partial W_n}$ is torsion and the map $\widetilde{HM}(W_n,\mathfrak{s})$ has $\mathbb{Q}$–degree

$$\frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(W_n) - 3\sigma(W_n)\right) = \frac{1}{4}\left(0-2-0\right) = -\frac{1}{2}.$$

  The statement follows from the fact that $o(n) > o(n-1) - 1/2$.

Finally, consider the map F_{W_n}. For a given $\mathfrak{s} \in \operatorname{spin}^c(W_n,\mathfrak{s}_0)$, there are again two possibilities:

- $\mathfrak{s}|_{Y_n}$ is non-torsion. Then $\widetilde{HM}(Y_n,\mathfrak{s}|_{Y_n}) = HM^{\mathrm{red}}(Y_n,\mathfrak{s}|_{Y_n})$ and the statement follows from Part (2) of Definition 5.22.
- $\mathfrak{s}|_{Y_n}$ is torsion. As in the corresponding case for $F_{W_{n-1}}$, the map $\widetilde{HM}(W_n,\mathfrak{s})$ has $\mathbb{Q}$–degree $-1/2$, and the statement follows from the fact that $o(n+1) > o(n) - 1/2$.

q.e.d.

Note that, in general, neither $\mathfrak{C}^{\leqslant}(N,\tilde{\mathfrak{s}})$ nor $\mathfrak{C}^{>}(N,\tilde{\mathfrak{s}})$ is exact. We denote their homology groups by $\{H_j^{\leqslant}(N,\tilde{\mathfrak{s}})\}$ and $\{H_j^{>}(N,\tilde{\mathfrak{s}})\}$, respectively. The absolute $\mathbb{Z}/2$ grading on $\widetilde{HM}(Y_j,[\mathfrak{s}_0])$ induces an absolute $\mathbb{Z}/2$ grading on these homology groups. The maps f_j give rise to involutions on both $\mathfrak{C}^{\leqslant}(N,\tilde{\mathfrak{s}})$ and $\mathfrak{C}^{>}(N,\tilde{\mathfrak{s}})$. We denote the corresponding chain maps by $f_j^{\leqslant}(N,\tilde{\mathfrak{s}})$ and $f_j^{>}(N,\tilde{\mathfrak{s}})$.

We also denote the induced maps on $H_j^{\leqslant}(N,\tilde{\mathfrak{s}})$ and $H_j^{>}(N,\tilde{\mathfrak{s}})$ by, respectively, $f_j^{\leqslant}(N,\tilde{\mathfrak{s}})_*$ and $f_j^{>}(N,\tilde{\mathfrak{s}})_*$. With a slight abuse of language, we will call all of these involutions *covering involutions.*

Lemma 5.24. *For any $j \in \mathbb{Z}/3$, there is an isomorphism*

$$\xi_j : H_j^{\leqslant}(N,\tilde{\mathfrak{s}}) \longrightarrow H_{j+1}^{>}(N,\tilde{\mathfrak{s}})$$

compatible with the covering involution. The map ξ_j shifts the absolute $\mathbb{Z}/2$ grading by the same amount as the map F_{W_j}.

Proof. Treat the exact triangle (27) as a chain complex with trivial homology. Call this chain complex $\mathfrak{C}$. Then we have a short exact sequence

$$0 \longrightarrow \mathfrak{C}_*^{\leqslant}(N,\tilde{\mathfrak{s}}) \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{C}_*^{>}(N,\tilde{\mathfrak{s}}) \longrightarrow 0,$$

and the statement follows from the long exact sequence it generates in homology. q.e.d.

Next, we will show that the chain complex $\mathfrak{C}^{>}(N,\tilde{\mathfrak{s}})$ only depends on the equivalence class of the spin^c system $\mathcal{S}((L_0,L_1,L_2),\mathfrak{s}_0)$. To make this statement precise, consider another admissible skein triple (L_0',L_1',L_2'), and let $\mathfrak{s}_0'$ be a self-conjugate spin^c structure on $Y' = \Sigma(S^3\backslash B')$, where B' is a small ball containing the resolved crossing. We suppose that there exists an equivalence

$$\mathcal{S}((L_0,L_1,L_2),\mathfrak{s}_0) \sim \mathcal{S}((L_0',L_1',L_2'),\mathfrak{s}_0')$$

provided by the maps $\{\theta_j\}$ and $\{\hat{\theta}_j\}$ as in Definition 5.9. We write $\tilde{\mathfrak{s}}' = \theta_j(\tilde{\mathfrak{s}})$ and choose N' large enough as to satisfy the conditions of Definition 5.22. All of the above constructions can be repeated with (L_0',L_1',L_2') in place of (L_0,L_1,L_2) and $(N',\tilde{\mathfrak{s}}')$ in place of $(N,\tilde{\mathfrak{s}})$.

Lemma 5.25. *There is an isomorphism between the chain complexes $\mathfrak{C}^{>}(N,\tilde{\mathfrak{s}})$ and $\mathfrak{C}^{>}(N',\tilde{\mathfrak{s}}')$. This isomorphism preserves the covering involution, absolute $\mathbb{Z}/2$ grading and the relative $\mathbb{Q}$-grading.*

Proof. The chain complex $\mathfrak{C}^{>}(N,\tilde{\mathfrak{s}})$ can be explicitly described in terms of the spin^c structures and their Chern classes. For example, when $b_1(Y_j) = 0$, each $\mathfrak{s} \in \text{tor-spin}^c(Y_j,\mathfrak{s}_0)$ contributes a summand $\mathcal{T}(\mathfrak{s}) = \mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U]$ to $\widetilde{HM}_j^{>}(N,\tilde{s})$, supported in even $\mathbb{Z}/2$ grading. The smallest $\mathbb{Q}$–degree in this summand is given by

$$\min\{a \mid a \in 2\mathbb{Z} + \rho(s),\ a > N + \rho(\tilde{s}) + o(j)\}.$$

The covering involution interchanges $\mathcal{T}(\mathfrak{s})$ and $\mathcal{T}(\bar{\mathfrak{s}})$. We have a similar description for the chain maps in $\mathfrak{C}^{>}(N, \tilde{\mathfrak{s}})$. The lemma can now be checked using this description and the corresponding description for $\mathfrak{C}^{>}(N', \tilde{\mathfrak{s}}')$. q.e.d.

Corollary 5.26. *For each $j \in \mathbb{Z}/3$, there exists an isomorphism $H_j^{\leqslant}(N', \tilde{\mathfrak{s}}') \to H_j^{\leqslant}(N, \tilde{\mathfrak{s}})$ that is compatible with the covering involution and preserves the absolute $\mathbb{Z}/2$ grading.*

Proof. This follows by combining Lemma 5.25 with Lemma 5.24. q.e.d.

Lemma 5.27. *The following identity holds for the Lefschetz numbers on the truncated monopole Floer homology:*

$$\begin{aligned}
&\mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_0,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) + \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_1,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) - \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_2,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) \\
&\quad = (-1)^{b_2^+(W_2)+b_1(Y_0)} \mathrm{Lef}(f_0^{\leqslant}(N, \tilde{\mathfrak{s}})) + (-1)^{b_2^+(W_1)+b_1(Y_2)} \mathrm{Lef}(f_1^{\leqslant}(N, \tilde{\mathfrak{s}})) - \mathrm{Lef}(f_2^{\leqslant}(N, \tilde{\mathfrak{s}})) \\
&\quad = (-1)^{b_2^+(W_2)+b_1(Y_0)} \mathrm{Lef}(f_0^{\leqslant}(N, \tilde{\mathfrak{s}})_*) + (-1)^{b_1^+(W_1)+b_1(Y_2)} \mathrm{Lef}(f_1^{\leqslant}(N, \tilde{\mathfrak{s}})_*) - \mathrm{Lef}(f_2^{\leqslant}(N, \tilde{\mathfrak{s}})_*).
\end{aligned}$$

A similar equality holds for (L_0', L_1', L_2').

Proof. The first equality should be clear from the definition of f_j. The second equality is based on the following observation: by [**38**, Proposition 2.5], the map $F^{\leqslant}_{W_j}(N, \tilde{\mathfrak{s}})$ preserves the absolute $\mathbb{Z}/2$ grading if and only if

$$\frac{1}{2}(\chi(W_j) + \sigma(W_j) - b_1(Y_j) + b_1(Y_{j+1})) = 0 \pmod 2.$$

Using Lemma 5.6, it is not difficult to check that this is equivalent to the condition

$$b_2^+(W_j) + b_1(Y_{j+1}) = 0 \pmod 2.$$

This is exactly when the sign before $\mathrm{Lef}(f_j^{\leqslant}(N, \tilde{\mathfrak{s}}))$ differs from the sign before $\mathrm{Lef}(f_{j+1}^{\leqslant}(N, \tilde{\mathfrak{s}}))$ (see (30). As a result, this kind of alternating sum of the Lefschetz numbers for the chain map equals the corresponding sum for the induced map on homology. q.e.d.

Corollary 5.28. *We have the following equality of Lefschetz numbers*

$$\begin{aligned}
&\mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_0,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) + \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_1,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) - \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_2,[\mathfrak{s}_0]}(N, \tilde{\mathfrak{s}})) \\
&\qquad = \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_0',[\mathfrak{s}_0']}(N', \tilde{\mathfrak{s}}')) + \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_1',[\mathfrak{s}_0']}(N', \tilde{\mathfrak{s}}')) - \mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_2',[\mathfrak{s}_0']}(N', \tilde{\mathfrak{s}}')).
\end{aligned}$$

Proof. This follows from Corollary 5.26 and Lemma 5.27. q.e.d.

As our next step, we will study relations between the Lefschetz numbers $\mathrm{Lef}(\check{\tau}^{\leqslant}_{Y_j,[\mathfrak{s}_0]}(N,\tilde{\mathfrak{s}}))$ and the corresponding Lefschetz numbers on the reduced Floer homology.

Definition 5.29. For any $\mathfrak{s} \in \text{sc-spin}^c(Y_j)$, define the *normalized Lefschetz number* $\mathrm{Lef}^\circ(Y_j,\mathfrak{s})$ of the map

$$\tau^{\mathrm{red}}_{Y_j,\mathfrak{s}} : HM^{\mathrm{red}}(Y_j,\mathfrak{s}) \to HM^{\mathrm{red}}(Y_j,\mathfrak{s})$$

as follows:

- if $b_1(Y_j) = 0$, we let
$$\mathrm{Lef}^\circ(Y_j,\mathfrak{s}) = \mathrm{Lef}(\tau^{\mathrm{red}}_{Y_j,\mathfrak{s}}) + h(Y_j,\mathfrak{s}).$$
- if $b_1(Y_j) = 1$, recall that (as in Heegaard Floer theory [**54**, §4.2]) there are two Frøyshov invariants $h_0(Y_j,\mathfrak{s})$ and $h_1(Y_j,\mathfrak{s})$ (see the proof of Lemma 5.30 below). We let
$$\mathrm{Lef}^\circ(Y_j,\mathfrak{s}) = \mathrm{Lef}(\tau^{\mathrm{red}}_{Y_j,\mathfrak{s}}) + h_0(Y_j,\mathfrak{s}) + h_1(Y_j,\mathfrak{s}).$$

Lemma 5.30. *For any $\mathfrak{s} \in \text{sc-spin}^c(Y_j)$ and any rational number q, consider the map*

$$\check{\tau}^{\leqslant q}_{Y_j,\mathfrak{s}} : \bigoplus_{a\leqslant q} \widetilde{HM}_a(Y_j,\mathfrak{s}) \longrightarrow \bigoplus_{a\leqslant q} \widetilde{HM}_a(Y_j,\mathfrak{s}).$$

For all sufficiently large q, its Lefschetz number satisfies the equality

$$\mathrm{Lef}(\check{\tau}^{\leqslant q}_{Y_j,\mathfrak{s}}) - 2^{b_1(Y_j)-1}q \;=\; \mathrm{Lef}^\circ(Y_j,\mathfrak{s}) + C(b_1(Y_j), q-\rho(\mathfrak{s})), \tag{31}$$

where $C(b_1(Y_j), q-\rho(\mathfrak{s}))$ is constant depending only on $b_1(Y_j)$ and the mod 2 reduction of $q-\rho(\mathfrak{s})$ in $\mathbb{Q}/2\mathbb{Z}$.

Proof. Let us assume that $b_1(Y_j) = 1$; the case of $b_1(Y_j) = 0$ is similar (and easier). We have the following (non-canonical) decomposition for $\widetilde{HM}(Y,\mathfrak{s})$:

$$(\mathbb{Q}[U,U^{-1}]/\mathbb{Q}[U])_{-2h_0(Y,\mathfrak{s})} \oplus (\mathbb{Q}[U,U^{-1}]/\mathbb{Q}[U])_{-2h_1(Y,\mathfrak{s})} \oplus HM^{\mathrm{red}}(Y_j,\mathfrak{s}),$$

with the lower indices indicating the absolute grading of the bottom of the U–tail. Regarding the absolute $\mathbb{Z}/2$ grading, the first summand is supported in the even grading while the second summand is supported in the odd grading. With respect to this decomposition, the map $\check{\tau}_{Y_j,\mathfrak{s}}$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & \tau^{\mathrm{red}}_{Y_j,\mathfrak{s}} \end{bmatrix}$$

(the action on the second summand is -1 because $\tau_{Y_j}^*$ acts as negative identity on $H^1(Y_j;\mathbb{R})$; see [**37**, Theorem 35.1.1]). Therefore, if q is large enough so that $HM^{\mathrm{red}}(Y_j,\mathfrak{s})$ is supported in degree less than q, we obtain

$$\operatorname{Lef}(\check{\tau}^{\leqslant q}_{Y_j,\mathfrak{s}}) - q - \operatorname{Lef}^{\circ}(Y_j,\mathfrak{s}) = \left|\{k \in \mathbb{Z}^{\geqslant 0} \mid -2h_0(Y,\mathfrak{s}) + 2k \leqslant q\}\right| - q/2 + \left|\{k \in \mathbb{Z}^{\geqslant 0} \mid -2h_1(Y,\mathfrak{s}) + 2k \leqslant q\}\right| - q/2.$$

Clearly, this number only depends on the mod 2 reduction of $q+2h_0(Y,\mathfrak{s})$ and $q+2h_1(Y,\mathfrak{s})$. To complete the proof, we observe that $\rho(\mathfrak{s}) + 2h_0(Y,\mathfrak{s}) \in 2\mathbb{Z}$ and $\rho(\mathfrak{s}) + 2h_1(Y,\mathfrak{s}) \in 2\mathbb{Z}+1$, which follows directly from the definition of the absolute grading in monopole Floer homology. q.e.d.

By setting $q = N + o(j) + \rho(\tilde{\mathfrak{s}})$ and taking the sum of the equalities (31) over all spin$^{\mathrm{c}}$-structures $\mathfrak{s} \in \text{sc-spin}^c(Y_j,\mathfrak{s}_0)$, we obtain the equality

$$\operatorname{Lef}(\check{\tau}^{\leqslant}_{Y_j,[\mathfrak{s}_0]}(N,\tilde{\mathfrak{s}})) - \sum_{\mathfrak{s}\in \text{sc-spin}^c(Y_j,\mathfrak{s}_0)} \operatorname{Lef}^{\circ}(Y_j,\mathfrak{s}) = 2^{b_1(Y_j)-1} \cdot |\text{sc-spin}^c(Y_j,\mathfrak{s}_0)| \cdot (N+\rho(\tilde{\mathfrak{s}})) + C, \tag{32}$$

where C is a constant depending on $b_1(Y_j)$, $o(j)$, $|\text{sc-spin}^{\mathrm{c}}(Y_j,\mathfrak{s}_0)|$, and the mod 2 reduction of $\rho(\mathfrak{s}) - \rho(\tilde{\mathfrak{s}})$ for $\mathfrak{s} \in \text{sc-spin}^{\mathrm{c}}(Y_j,\mathfrak{s}_0)$.

Corollary 5.31. *If $\mathcal{S}((L_0,L_1,L_2),\mathfrak{s}_0) \sim \mathcal{S}((L_0',L_1',L_2'),\mathfrak{s}_0')$ then*

$$\sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_0,\mathfrak{s}_0)} \operatorname{Lef}^{\circ}(Y_0,\mathfrak{s}) + \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_1,\mathfrak{s}_0)} \operatorname{Lef}^{\circ}(Y_1,\mathfrak{s}) - \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_2,\mathfrak{s}_0)} \operatorname{Lef}^{\circ}(Y_2,\mathfrak{s}) = \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_0',\mathfrak{s}_0')} \operatorname{Lef}^{\circ}(Y_0',\mathfrak{s}) + \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_1',\mathfrak{s}_0')} \operatorname{Lef}^{\circ}(Y_1',\mathfrak{s}) - \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_2',\mathfrak{s}_0')} \operatorname{Lef}^{\circ}(Y_2',\mathfrak{s})$$

In addition,

$$\frac{1}{2^{|L_2|-2}}\Big(\sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_0)} \operatorname{Lef}^{\circ}(Y_0,\mathfrak{s}) + \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_1)} \operatorname{Lef}^{\circ}(Y_1,\mathfrak{s}) - \sum_{\mathfrak{s}\in \text{sc-spin}^{\mathrm{c}}(Y_2)} \operatorname{Lef}^{\circ}(Y_2,\mathfrak{s})\Big) = C(t(Y),s(Y),(p_0,q_0),(p_1,q_1)). \tag{33}$$

Proof. Since $b_1(Y_j) = b_1(Y_j')$, $o(j) = o'(j)$, $|\,\text{sc-spin}^{\mathrm{c}}(Y_j,\mathfrak{s}_0)| = |\,\text{sc-spin}^{\mathrm{c}}(Y_j',\mathfrak{s}_0')|$, and

$$\rho(\mathfrak{s}) - \rho(\tilde{\mathfrak{s}}) \;=\; \rho(\theta(\mathfrak{s})) - \rho(\tilde{\mathfrak{s}}') \pmod 2,$$

the constant C in (32) equals the corresponding constant for Y'. Now we add the equalities (32) for Y_0, Y_1 and subtract the one for Y_2. By comparing the result with the corresponding result for Y_j' and applying (22) and Corollary 5.28, we finish the proof of the first claim. The second claim follows easily from (1) and Theorem 5.10. q.e.d.

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.5. For any ramifiable link L_j, it follows from formula (7) for the Murasugi signature that

$$\frac{1}{2^{|L_j|-1}} \sum_{\mathfrak{s}\in \text{sc-spin}^c(Y_j)} \operatorname{Lef}^\circ(Y_j,\mathfrak{s}) \;-\; \frac{1}{8}\,\xi(L_j) \;=\; \chi(L_j). \tag{34}$$

Therefore, (14) follows from (33) and Lemma 3.1. This proves statement (1) of Theorem 5.5.

The first assertion of statement (2) follows from Lemma 5.2. To prove the second assertion, suppose that L_2 is not ramifiable. Then γ_2 represents zero elements in both $H_1(Y;\mathbb{F}_2)$ and $H_1(Y;\mathbb{Q})$. Let $(t(Y), s(Y), (p_0,q_0), (p_1,q_1))$ be the resolution data for (L_0,L_1,L_2). Then $s(Y) = 0$ and $(p_2,q_2) = (0,1)$, which implies that $(p_1,q_1) = (-p_0,-q_0-1)$. It follows from Lemma 5.2 that $(\bar{L}_1, L_0, L_2)$ forms an admissible skein triangle, and one can check that its resolution data is $(t(Y), s(Y), (p_0,q_0-1), (-p_0,-q_0))$. The equality (33) now reads

$$\frac{1}{2^{|L_2|-2}} \Big(\sum_{\mathfrak{s}\in \text{sc-spin}^c(Y_0)} \operatorname{Lef}^\circ(Y_0,\mathfrak{s}) + \sum_{\mathfrak{s}\in \text{sc-spin}^c(\bar{Y}_1)} \operatorname{Lef}^\circ(\overline{Y}_1,\mathfrak{s}) - \sum_{\mathfrak{s}\in \text{sc-spin}^c(Y_2)} \operatorname{Lef}^\circ(Y_2,\mathfrak{s}) \Big) \\ = C(t(Y), s(Y), (p_0,q_0-1), (-p_0,-q_0)), \tag{35}$$

where $\overline{Y}_1$ stands for the double branched cover of S^3 with branch set $\bar{L}_1$. Subtracting (33) from (35), we obtain

$$\frac{1}{2^{|L_1|-1}} \Big(\sum_{\mathfrak{s}\in \text{sc-spin}^c(\bar{Y}_1)} \operatorname{Lef}^\circ(\bar{Y}_1,\mathfrak{s}) - \sum_{\mathfrak{s}\in \text{sc-spin}^c(Y_1)} \operatorname{Lef}^\circ(Y_1,\mathfrak{s}) \Big) \\ = C(t(Y), s(Y), (p_0,q_0-1), (-p_0,-q_0)) - C(t(Y), s(Y), (p_0,q_0), (p_1,q_1)).$$

Combining this with Lemma 3.1 and (34), we finish the proof of (15) in the case of $n = 2$. The proofs of (16) for $n = 2$, and of both (15) and (16) for $n = 0$ and $n = 1$ are similar. q.e.d.

6. Vanishing of the universal constants

In this section, we will prove that the constants $C(t(Y), s(Y), (p_0,q_0), (p_1,q_1))$ and $C_j^{\pm}(t(Y), s(Y), (p_0,q_0), (p_1,q_1))$ in Theorem 5.5 vanish for all resolution data. The cyclic symmetry of skein triangles will then imply Proposition 2.2 and therefore finish the proof of Theorem 2.1.

Definition 6.1. A six-tuple $(t, s, (p_0, q_0), (p_1, q_1))$, where t is a positive integer, $s \in \mathbb{Z}/2$, and $p_j, q_j \in \mathbb{Z}$, $j = 0, 1$, is called *admissible* is the following four conditions are satisfied:

1) $p_0 q_1 - p_1 q_0 = 1$,
2) $s = 0$ if t is odd,
3) $p_2 = -p_0 - p_1$ is even when $s = 0$, and $q_2 = -q_0 - q_1$ is even when $s = 1$.

Lemma 6.2. *The resolution data* $(t(Y), s(Y), (p_0, q_0), (p_1, q_1))$ *associated with an admissible skein triangle* (L_0, L_1, L_2) *as in Definition 5.4 is an admissible six-tuple.*

Proof. This can be easily verified: (1) corresponds to the requirement that $\#(\gamma_0 \cap \gamma_1) = -1$, (2) follows from the fact that l is an $\mathbb{F}_2$ longitude when $t(Y)$ is odd, and (3) is true because γ_2 is an $\mathbb{F}_2$ longitude by part (1) of Definition 5.1. q.e.d.

Theorem 6.3. *Every admissible six-tuple* $(t, s, (p_0, q_0), (p_1, q_1))$ *is the resolution data of an admissible skein triangle. Furthermore,*

1) if $(p_j, q_j) \neq (0, 1)$ *for all* $j \in \mathbb{Z}/3$ *then* $C(t, s, (p_0, q_0), (p_1, q_1)) = 0$*, and*
2) if $(p_j, q_j) = (0, 1)$ *for some* $j \in \mathbb{Z}/3$ *then* $C^{\pm}_j(t, s, (p_0, q_0), (p_1, q_1)) = 0$.

Our proof of Theorem 6.3 is inspired by the proof of [**57**, Theorem 7.5]. The idea is roughly as follows: starting with two-bridge links, which are alternating and hence have vanishing χ, we will generate sufficiently many examples of Montesinos links with vanishing χ to cover all possible admissible six-tuples. We will then apply Theorem 5.5 to conclude that the constants in question are all zero.

Proposition 6.4. *Theorem 6.3 holds for all admissible six-tuples with* $t = 1$.

Proof. Observe that since $t = 1$ is odd, we automatically have $s = 0$ by Definition 6.1, so the admissible six-tuples at hand are of the form $(1, 0, (p_0, q_0), (p_1, q_1))$ with $p_0 q_1 - p_1 q_0 = 1$ and even $p_2 = -p_0 - p_1$. Let us consider an unknot in S^3 with $Y = S^1 \times D^2$ and the standard boundary framing (m, l) on ∂Y, which has $t = 1$ and $s = 0$. For every $j \in \mathbb{Z}/3$, the manifold Y_j obtained by the p_j/q_j surgery on the unknot is a lens space of the form $Y_j = \Sigma(L_j)$, where L_j is a two-bridge link. The links $\bar{L}_j$ are also two-bridge, and hence alternating. It then follows from [**47**, Theorem 1.2] and [**17**, Lemma 3.4], combined with the relation cited in Remark 1.1 between the Heegaard Floer and monopole correction terms that $\chi(L_j) = \chi(\bar{L}_j) = 0$. The result will now follow from Theorem 5.5 as soon as we show that (L_0, L_1, L_2) is an admissible skein triangle. But the latter is a special case of the more general result proved below in Lemma 6.6.

q.e.d.

To continue, we will introduce some notation. Choose three distinct circle fibers in $S^2 \times S^1$ and remove their disjoint open tubular neighborhoods. The resulting manifold will be called N. The tori T_j, $j = 1, 2, 3$, on the boundary of N have natural framings (x_j, h), where h is the circle fiber and the curves x_1, x_2, and x_3 co-bound a section of the product circle bundle.

Denote by $\mathbb{Q}^+ = \mathbb{Q} \cup \{\infty\}$ the extended set of rational numbers, with the convention that $\infty = 1/0$. Given three numbers $a_j/b_j \in \mathbb{Q}^+$ with co-prime (a_j, b_j), $j = 1, 2, 3$, denote by $Y(a_1/b_1, a_2/b_2, a_3/b_3)$ the closed manifold obtained by attaching to N three solid tori along their boundaries so that their meridians match the curves $a_j x_j + b_j h$. A direct calculation shows that the first homology group of $Y(a_1/b_1, a_2/b_2, a_3/b_3)$ is finite if and only if $b_1 a_2 a_3 + a_1 b_2 a_3 + a_1 a_2 b_3 \neq 0$, in which case

$$|H_1(Y(a_1/b_1, a_2/b_2, a_3/b_3); \mathbb{Z})| = |b_1 a_2 a_3 + a_1 b_2 a_3 + a_1 a_2 b_3|. \tag{36}$$

A surgery description of $Y(a_1/b_1, a_2/b_2, a_3/b_3)$ is shown in Figure 3. Denote by $Y(a_1/b_1, a_2/b_2, \bullet)$ the manifold with a single boundary component obtained by attaching to N just the first two solid tori.

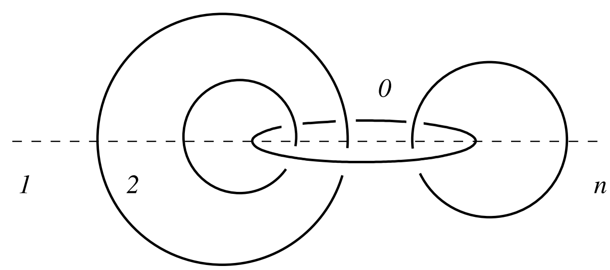


Figure 3. The manifold $Y(a_1/b_1, a_2/b_2, a_3/b_3)$

The 180° rotation with respect to the dotted line in Figure 3 makes $Y(a_1/b_1, a_2/b_2, a_3/b_3)$ into a double branched cover over S^3 with branch set the Montesinos link $K(a_1/b_1, a_2/b_2, a_3/b_3)$ pictured in Figure 4.

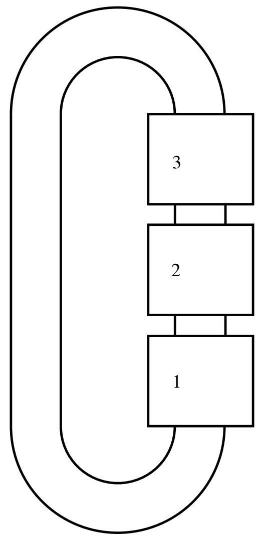


Figure 4. The Montesinos link $K(a_1/b_1, a_2/b_2, a_3/b_3)$

Each of the boxes marked a_j/b_j in the figure stands for the rational tangle $T(a_j/b_j)$ obtained from a continued fraction decomposition

$$a_j/b_j = [t_1, \ldots, t_{k_j}] = t_1 - \cfrac{1}{t_2 - \cfrac{1}{\cdots - \cfrac{1}{t_{k_j}}}} \tag{37}$$

by applying consecutive twists to neighboring endpoints starting from two unknotted and unlinked arcs. Our conventions for rational tangles should be clear from the examples in Figure 5.

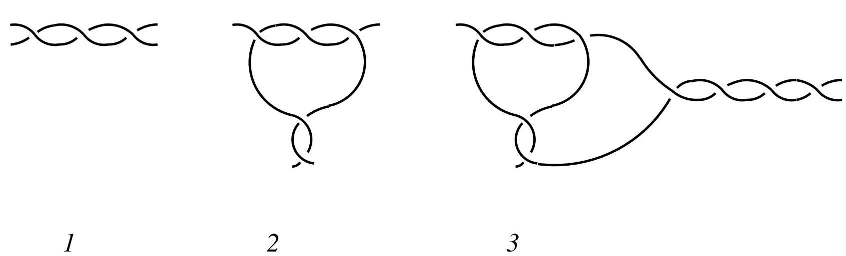


Figure 5. Examples of rational tangles

We will study skein triangles formed by these Montesinos links. Given p_0/q_0 and $p_1/q_1 \in \mathbb{Q}^+$ with co-prime (p_0, q_0) and (p_1, q_1), define the *distance* between them by the formula

$$\Delta(p_0/q_0, p_1/q_1) = |p_0 q_1 - p_1 q_0|. \tag{38}$$

We will say that three points in $\mathbb{Q}^+$ form a *triangle* if the distance between any two of them is equal to 1. Two triangles T_1 and T_2 are called *adjacent* if the intersection $T_1 \cap T_2$ consists of exactly two points.

Lemma 6.5. *For any $r, s, t \in \mathbb{Q}^+$ with $\Delta(r, s) = 1$, there exists a chain of triangles S_0, $S_1, \ldots, S_n$ such that $r, s \in S_0$, $t \in S_n$, and S_i is adjacent to S_{i+1} for all $i = 0, \ldots, n$.*

Proof. The modular group $PSL(2,\mathbb{Z})$ acts on the set $\mathbb{Q}^+$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \frac{ap+bq}{cp+dq}.$$

This action preserves the distance (38) and hence sends triangles to triangles. It follows from the general properties of the modular group (and can also be checked directly) that, for any pair $r, s \in \mathbb{Q}^+$ of distance 1, there exists $A \in PSL(2,\mathbb{Z})$ such that $A \cdot r = 0$ and $A \cdot s = \infty$. Therefore, we may assume without loss of generality that $r = 0$ and $s = \infty$.

Observe that there are exactly two choices for the triangle S_0 with vertices 0 and ∞: in one of these triangles, the third vertex is 1, and in the other -1. Therefore, we will find a chain of triangles S_0, $S_1, \ldots, S_n$ connecting 0 and ∞ to $t = p/q$ as soon as we find a chain of triangles $S_1, \ldots, S_n$ connecting 0 and ± 1 to $t = p/q$. First suppose that the triangle S_0 has vertices 0, 1, and ∞. The matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

sends 0, 1, and p/q into ∞, 0, and $(p-q)/p$, respectively, and turns the problem at hand into the problem of finding a chain of triangles $S_1, \ldots, S_n$ connecting 0 and ∞ to $(p-q)/p$. This is, of course, the original problem with $t = p/q$ replaced by $t = (p-q)/p$. Similarly, the matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

sends 0, 1, and p/q into 0, ∞, and $p/(q-p)$, respectively, thereby replacing $t = p/q$ by $t = p/(q-p)$. If the triangle S_0 has vertices 0, -1 , and ∞, the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

can be used to replace $t = p/q$ with $t = (p+q)/p$ and $t = p/(p+q)$, respectively. In summary, $t = p/q$ can be replaced with any one of the four fractions $(p \pm q)/q$ and $p/(q \pm p)$. One can find a sequence of such replacements making any $t = p/q$ into $t = 1$, for which there is an obvious solution. q.e.d.

Lemma 6.6. *For any $p,\ q \in \mathbb{Q}^+$ and any adjacent triangles $\{r, s, t\}$ and $\{r, s, t'\}$, one can find a planar projection of the link $K(p, q, t)$ and a crossing c such that*

- *the two resolutions of $K(p, q, t)$ at the crossing c are $K(p, q, r)$ and $K(p, q, s)$, and*
- *the link $K(p, q, t)$ with the crossing c changed is $K(p, q, t')$.*

In particular, each of the sets

$$\{K(p, q, r), K(p, q, s), K(p, q, t)\} \quad \text{and} \quad \{K(p, q, r), K(p, q, s), K(p, q, t')\}$$

forms an admissible skein triangle, possibly after a permutation. For both skein triangles, the manifold Y with torus boundary is just $Y(p, q, \bullet)$.

Proof. Let B^3 be a 3-ball in S^3 which contains the third rational tangle in all of the Montesinos links at hand. Identify its boundary ∂B^3 with the quotient $(\mathbb{R}^2/\mathbb{Z}^2)/\pm 1$ of the torus $\mathbb{R}^2/\mathbb{Z}^2$ by the hyperelliptic involution. The standard action of $SL(2, \mathbb{Z})$ on the plane $\mathbb{R}^2$ induces an action of $PSL(2, \mathbb{Z})$ on ∂B^3 which permutes the points $(0, 0)$, $(0, 1/2)$, $(1/2, 0)$, and $(1/2, 1/2)$. Every $A \in PSL(2, \mathbb{Z})$ gives a homeomorphism $A : \partial B^3 \to \partial B^3$ which extends to a homeomorphism $f_A : B^3 \to B^3$ by the coning construction, $f_A(t \cdot x) = t \cdot A(x)$. The homeomorphism f_A sends a rational tangle $T(\ell)$ to a rational tangle $T(A \cdot \ell)$, which can be seen by factoring A into a product of Dehn twists

$$A = \begin{pmatrix} q & s \\ p & r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_k & 0 \end{pmatrix}$$

using a continued fraction $p/q = [t_1, \ldots, t_k]$ as in (37). Now, given $r, s \in \mathbb{Q}^+$, there exists $A \in PSL(2, \mathbb{Z})$ such that $A \cdot r = 0$, $A \cdot s = \infty$, and then necessarily $\{A \cdot t, A \cdot t'\} = \{\pm 1\}$. By the coning construction, f_A extends to the exterior of B^3, resulting in a homeomorphism of S^3. This homeomorphism turns the original tangle decompositions into tangle decompositions of the form

$$K(p,q,r) = T' \cup T(0), \quad K(p,q,s) = T' \cup T(\infty), \quad \text{and}$$
$$\{K(p,q,t),\, K(p,q,t')\} = \{\,T' \cup T(1),\, T' \cup T(-1)\},$$

where T' is a certain tangle in the exterior of B^3. The conclusion of the lemma is now clear. q.e.d.

Proposition 6.7. *Suppose the link $K(r_1, r_2, r_3)$ is ramifiable and $1/r_j$ is an integer or infinity for some j. Then $\chi(K(r_1, r_2, r_3)) = 0$.*

Proof. In this case, $K(r_1, r_2, r_3)$ is a two-bridge link and, in particular, it is alternating. The result now follows from [**47**] and [**17**] as in the proof of Proposition 6.4. q.e.d.

Proposition 6.8. *For any $p, q, r \in \mathbb{Q}^+$, suppose that $K(p,q,r)$ is ramifiable and Theorem 6.3 holds for all admissible six-tuples with $t = t(Y(p,q,\bullet))$. Then $\chi(K(p,q,r)) = 0$.*

Proof. Use formula (36) to find a positive integer k such that both $K(p,q,1/k)$ and $K(p,q,$ $1/(k+1))$ are ramifiable. It follows from Proposition 6.7 that

$$\chi(K(p,q,1/k) = \chi(K(p,q,1/(k+1)) = 0.$$

Since $\Delta(1/k, 1/(k+1)) = 1$, Lemma 6.5 supplies us with a chain of triangles $S_0, S_1, \ldots, S_n$ such that $1/k, 1/(k+1) \in S_0$, $r \in S_n$, and S_i is adjacent to S_{i+1} for all $i = 0, \ldots, n$. We claim that for any $m = 0, \ldots, n$ and $s \in S_m$ such that $K(p,q,s)$ is ramifiable,

$$\chi(K(p,q,s)) = 0.$$

We will proceed by induction on m. First, suppose that $m = 0$. If $s = 1/k$ or $1/(k+1)$, the claim follows from Proposition 6.7; otherwise, it follows from Theorem 5.5 (1). Next, suppose that the claim holds for m and prove it for $m+1$. Write $S_{m+1} = \{s, u, v\}$ and suppose that $K(p,q,s)$ is ramifiable. If $s \in S_m$ then the claim follows from the induction hypothesis. Otherwise, write $S_m = \{u, v, w\}$ and consider two possibilities. One possibility is that both $K(p,q,u)$ and $K(p,q,v)$ are ramifiable. Then $\chi(K(p,q,u)) = \chi(K(p,q,v)) = 0$ by the induction hypothesis and the vanishing of $\chi(K(p,q,s))$ follows from Theorem 5.5 (1). The other possibility is that one of $K(p,q,u)$ and $K(p,q,v)$ is not ramifiable. Then by Theorem 5.5 (2), the link $K(p,q,w)$ is ramifiable and $\chi(K(p,q,s)) = \chi(K(p,q,w)) = 0$ by the induction hypothesis. q.e.d.

The following lemma will be helpful in computing $t(Y(n, a/b, \bullet))$. It uses the notation $\operatorname{div}_2(m) = \max\{c \in \mathbb{N} \mid 2^c \text{ divides } m\}$.

Lemma 6.9. *For any integer n and co-prime integers a and b with $b_1(Y(n, a/b, \bullet)) = 1$, the integer $t(Y(n, a/b, \bullet))$ is a divisor of n. In particular, $t(Y(n, a/b, \bullet)) \leqslant |n|$, with $t(Y(n, a/b, \bullet)) = |n|$ if and only there exists an integer k such that $a = kn$ and n divides $k + b$. Furthermore, if $\operatorname{div}_2(n) \neq \operatorname{div}_2(a)$ then the integer $t(Y(n, a/b, \bullet))$ is odd.*

Proof. We will use the notation $Y = Y(n, a/b, \bullet)$. The homology group $H_1(Y;\mathbb{Z})$ is generated by the homology classes $[x_1]$, $[x_2]$, $[x_3]$, and $[h]$ subject to the relations

$$n \cdot [x_1] + [h] = 0, \quad a \cdot [x_2] + b \cdot [h] = 0, \quad [x_1] + [x_2] + [x_3] = 0.$$

One can easily see that the kernel of the map $H_1(\partial Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ is generated by the homology class

$$\frac{a + bn}{\gcd(n, a)} \cdot [h] - \frac{na}{\gcd(n, a)} \cdot [x_3],$$

and therefore

$$t(Y) = \gcd\left(\frac{a + bn}{\gcd(n, a)}, \frac{na}{\gcd(n, a)}\right).$$

To prove the first statement of the lemma, write $n = \gcd(n, a) \cdot n'$ and $a = \gcd(n, a) \cdot a'$ with the relatively prime n' and a'. Then

$$t(Y) = \gcd(a' + bn', \gcd(n, a) \cdot n'a').$$

Note that any prime p that divides the product $n'a'$ must divide either n' or a' but not both. In either case, p cannot divide $a' + bn'$ because a' and b are relatively prime. Therefore, all the common divisors of $a' + bn'$ and $\gcd(n, a) \cdot n'a'$ must also be divisors of $\gcd(n, a)$, which implies that $t(Y)$ is a divisor of $\gcd(n, a)$ and hence of n.

If $t(Y) = |n|$, the integer n must divide $\gcd(n, a)$ implying that $n = \gcd(n, a)$ and $a = kn$ for some integer k. Since $n' = 1$ and $a' = k$, the fact that n divides $a' + bn'$ is equivalent to saying that n divides $k + b$.

Finally, suppose $\operatorname{div}_2(n) \neq \operatorname{div}_2(a)$, then $n'a'$ must be even. If n' is even then a' is odd hence $a' + bn'$ must be odd. On the other hand, if a' is even then both n' and b are odd hence $a' + bn'$ must still be odd. In both cases, $t(Y)$ is odd because it divides $a' + bn'$. q.e.d.

Proposition 6.10. *Theorem 6.3 holds for all admissible six-tuples with t odd.*

Proof. Observe that since t is odd, s must be zero by Definition 6.1. We will proceed by induction on t. The case $t = 1$ was proved in Proposition 6.4. Suppose the statement holds for all odd $t = 1, \ldots, 2n-1$ and consider links of the form $K(2n+1, -(2n+1), a/b)$. An easy calculation with formula (36) shows that such a link is ramifiable if and only if $a/b \neq \infty$.

It follows from Lemma 6.9 that $t(Y(2n+1, \bullet, a/b))$ and $t(Y(\bullet, -(2n+1), a/b))$ are divisors of $2n+1$. We claim that they can not be both $2n+1$: otherwise, by Lemma 6.9 again, there would exist an integer k such that $a = (2n+1)k$ and $2n+1$ divides both $b+k$ and $b-k$, which would contradict the assumption that a and b are co-prime. Together with Proposition 6.8 and the induction hypothesis, this claim implies that

$$\chi(K(2n+1, -(2n+1), a/b)) = 0$$

for any $a/b \neq \infty$. Since $t(Y(2n+1, -(2n+1), \bullet)) = 2n+1$ by Lemma 6.9, all the constants $C(2n+1, 0, (p_0, q_0), (p_1, q_1))$ and $C_j^{\pm}(2n+1, 0, (p_0, q_0), (p_1, q_1))$ must vanish by Theorem 5.5. This completes the inductive step and hence the proof of the proposition. q.e.d.

Proposition 6.11. *Suppose that* $K(n, a_2/b_2, a_3/b_3)$ *is ramifiable and* $\operatorname{div}_2(n) \neq \operatorname{div}_2(a_2)$. *Then* $\chi(K(n, a_2/b_2, a_3/b_3)) = 0$.

Proof. When $\operatorname{div}_2(n) \neq \operatorname{div}_2(a_2)$, the integer $t(K(n, a_2/b_2, \bullet))$ must be odd by Lemma 6.9. The result now follows from Proposition 6.10 and Proposition 6.8. q.e.d.

The following simple lemma will be instrumental in completing the proof of Theorem 6.3.

Lemma 6.12. *For any admissible six-tuple* $(t, s, (p_0, q_0), (p_1, q_1))$,

$$C(t, s, (p_0, q_0), (p_1, q_1)) = C(t, s, (p_0, q_0 + kp_0), (p_1, q_1 + kp_1)) \text{ and}$$
$$C_j^{\pm}(t, s, (p_0, q_0), (p_1, q_1)) = C_j^{\pm}(t, s, (p_0, q_0 + kp_0), (p_1, q_1 + kp_1)),$$

assuming the constants are defined. Here, k can be any integer when $s = 0$, and k can be any even integer when $s = 1$.

Proof. The right hand sides of the equalities in Theorem 5.5 do not depend on the choice of framing. Therefore, we can replace the framing (m, l) by $(m - kl, l)$ without changing the corresponding constants (for this to be true, k needs to be even when $s = 1$ so that $m - kl$ is still an $\mathbb{F}_2$ longitude). This proves the statement of the lemma. q.e.d.

Proposition 6.13. *Theorem 6.3 (1) holds for any t.*

Proof. The case of odd t was dealt with in Proposition 6.10 hence we will focus on the case of $t = 2n$ with positive n. We will consider two separate cases, those of $s = 0$ and $s = 1$.

Let us first suppose that $s = 1$. It follows from the homological calculation in the proof of Lemma 6.9 that the $\mathbb{Q}$–longitude l of the manifold $Y(2n, -2n, \bullet)$ is x_3 with divisibility $2n$, while its $\mathbb{F}_2$ longitude m can be chosen to be h. We wish to show that $C(2n, 1, (p_0, q_0), (p_1, q_1)) = 0$ for any admissible six-tuple $(2n, 1, (p_0, q_0), (p_1, q_1))$ with non-zero p_0, p_1, and p_2 (recall that $q_2 = -q_0 - q_1$ and $p_2 = -p_0 - p_1$). By Lemma 6.12, it suffices to show that $C(2n, 1, (p_0, q_0 + kp_0), (p_1, q_1 + kp_1)) = 0$ for some even integer k.

Since $m = h$ and $l = x_3$, the constant $C(2n, 1, (p_0, q_0 + kp_0), (p_1, q_1 + kq_1))$ arises in the admissible skein triangle comprising the links

$$L_j = K(2n, -2n, (q_j + kp_j)/p_j), \quad j = 0, 1, 2.$$

Since q_2 is even by Definition 6.1 (3), the integers q_0, q_1, and p_2 must be odd. Therefore,

$$\operatorname{div}_2(q_j + kp_j) = 0 \neq \operatorname{div}_2(2n)$$

for $j = 0, 1$ and any even k. Using Proposition 6.11, we conclude that $\chi(L_0) = \chi(L_1) = 0$. To show that $\chi(L_2) = 0$, we just need to find an even integer k such that

$$\operatorname{div}_2(q_2 + kp_2) \neq \operatorname{div}_2(2n).$$

This can be done as follows: since p_2 and $2^{\operatorname{div}_2(2n)+1}$ are co-prime, there exists an (obviously even) k such that $2^{\operatorname{div}_2(2n)+1}$ divides $q_2 + kp_2$, which implies that $\operatorname{div}_2(q_2 + kp_2) > \operatorname{div}_2(2n)$. Now that we know that $\chi(L_j) = 0$ for $j = 0, 1, 2$, we use Theorem 5.5 and Lemma 6.12 to conclude that $C(2n, 1, (p_0, q_0), (p_1, q_1)) = 0$.

Let us now suppose that $s = 0$. Our argument will be similar to that in the $s = 1$ case but with the manifold $Y(4n, 4n/(2n - 1), \bullet)$. The $\mathbb{Q}$–longitude l of this manifold is $-2x_3 + h$ with divisibility $2n$, and it also happens to be its $\mathbb{F}_2$ longitude. We set $m = x_3$. As before, for any admissible six-tuple $(2n, 0, (p_0, q_0), (p_1, q_1))$ with non-zero p_0, p_1, and p_2, we want to show that

$$C(2n, 0, (p_0, q_0 + kp_0), (p_1, q_1 + kp_1)) = 0$$

for some integer k. This constant arises in the admissible skein exact triangle with the links

$$L_j = K(4n, 4n/(2n - 1), (p_j - 2q_j - 2kp_j)/(q_j + kp_j)), \quad j = 0, 1, 2.$$

Since $(2n, 0, (p_0, q_0), (p_1, q_1))$ is an admissible six-tuple, p_2 is even and q_2, p_0, and p_1 are odd. Therefore,

$$\mathrm{div}_2(p_j - 2q_j - 2kp_j) = 0 \neq \mathrm{div}_2(4n)$$

for $j = 0, 1$ and any k. Using Proposition 6.11, we conclude that $\chi(L_0) = \chi(L_1) = 0$. Now, we wish to find an integer k such that

$$\mathrm{div}_2(p_2 - 2q_2 - 2kp_2) \neq \mathrm{div}_2(4n).$$

If $p_2 = 0 \pmod 4$ this is true for any k because $\mathrm{div}_2(p_2 - 2q_2 - 2kp_2) = 1$. Let us now assume that $p_2 = 4\ell + 2$. Then $p_2 - 2q_2 - 2kp_2 = 4(\ell + (1 - q_2)/2 - k(2\ell + 1))$. Since $2\ell + 1$ is odd, we can choose k so that $2^{\mathrm{div}_2(4n)}$ divides $(\ell + (1 - q_2)/2 - k(2\ell + 1))$ and therefore

$$\mathrm{div}_2(p_2 - 2q_2 - 2kp_2) > \mathrm{div}_2(4n).$$

In either case, Proposition 6.11 implies that $\chi(L_2) = 0$ for a properly chosen k. Theorem 5.5 now completes the proof. q.e.d.

Lemma 6.14. *Suppose that n is even and the link $K(n, a_2/b_2, a_3/b_3)$ is ramifiable. Then*

$$\chi(K(n, a_2/b_2, a_3/b_3)) = 0.$$

Proof. If either a_2 or a_3 is odd, this follows from Proposition 6.11, hence we will focus on the case of even a_2 and a_3. Since a_3 and b_3 are co-prime, there exist integers c_3 and d_3 such that $a_3d_3 - c_3b_3 = 1$. By replacing (c_3, d_3) by $(c_3 + ka_3, d_3 + kb_3)$ if necessary and using (36), we may assume that the links

$$L = K(n, a_2/b_2, c_3/d_3) \quad \text{and} \quad L' = K(n, a_2/b_2, (a_3 + c_3)/(b_3 + d_3))$$

are both ramifiable. Since c_3 is odd and a_3 is even, we have $\chi(L) = \chi(L') = 0$ by Proposition 6.11. After a cyclic permutation if necessary, the triple $(K(n, a_2/b_2, b_3/a_3), L, L')$ forms an admissible skein triangle consisting of three ramifiable links. By Proposition 6.13,

$$\chi(K(n, a_2/b_2, a_3/b_3)) = 0.$$

q.e.d.

Proposition 6.15. *Theorem 6.3 (2) holds for any t.*

Proof. The case that t is odd has been dealt with in Proposition 6.10 so we will assume that $t = 2m$. By Lemma 6.14, all ramifiable links of the form $K(2m, -2m, p/q)$ and $K(4m, 4m/(2m-1), p/q)$ have vanishing χ. These links cover all possible admissible skein triangles with $t = 2m$. Therefore, we can use Theorem 5.5 to conclude that $C^{\pm}_n(2m, s, (p_0, q_0), (p_1, q_1))$ equals zero, as long as it is defined. q.e.d.

Proposition 6.13 together with Proposition 6.15 finishes the proof of Theorem 6.3.

7. The Seiberg–Witten and Furuta–Ohta invariants of mapping tori

Let Y be the double branched cover of a knot K in an integral homology sphere Y'. The manifold Y is a rational homology sphere, which comes equipped with the covering translation $\tau : Y \to Y$. The mapping torus of τ is the smooth 4-manifold $X = ([0,1]\times Y)\,/\,(0,x) \sim (1,\tau(x))$ with the product orientation. We will show in Section 7.1 that X has the integral homology of $S^1 \times S^3$ and that it has a well defined invariant $\lambda_{\mathrm{FO}}(X)$ of the type introduced by Furuta–Ohta [**20**]. The following theorem is the main result of this section.

Theorem 7.1. *Let $\lambda(Y')$ be the Casson invariant of Y', and* $\operatorname{sign}(K)$ *the signature of the knot K. Then*

$$\lambda_{\mathrm{FO}}(X) \;=\; 2\cdot\lambda(Y') \;+\; \frac{1}{8}\,\operatorname{sign}(K).$$

Proof of Theorem C. Applying Theorem 7.1 to the homology sphere $Y' = S^3$, we obtain

$$\lambda_{\mathrm{FO}}(X) = \frac{1}{8}\,\operatorname{sign}(K).$$

On the other hand, using the splitting theorem [**44**, Theorem A] together with Theorem A of this paper, we have

$$\lambda_{\mathrm{SW}}(X) = -\operatorname{Lef}(\tau_*) - h(Y,\mathfrak{s}) = -\frac{1}{8}\,\operatorname{sign}(K)$$

for the unique spin structure on Y. This completes the proof. q.e.d.

Theorem 7.1 was proved in [**16**] and [**63**] under the assumption that Y is an integral homology sphere. Our proof here will rely on the extension of those techniques to the general case at hand.

7.1. Preliminaries. We begin in this section with some topological preliminaries, including an extension of the Furuta–Ohta invariant $\lambda_{\mathrm{FO}}(X)$ to a wider class of manifolds than that in the original paper [**20**].

The Furuta–Ohta invariant was originally defined in [**20**] for smooth 4-manifolds X satisfying two conditions, $H_*(X;\mathbb{Z}) = H_*(S^1 \times S^3;\mathbb{Z})$ and $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, where $\tilde{X}$ is the universal abelian cover of X. To fix the signs, one needs to fix an orientation on X as well as a homology orientation, i.e. a choice of generator of $H^1(X;\mathbb{Z})$. The mapping tori we consider in this section provide examples of manifolds X which satisfy the first condition but not the second (which can only be guaranteed if we use rational coefficients). Therefore, we need an extension of the Furuta–Ohta work to define $\lambda_{\mathrm{FO}}(X)$ in this case.

Let X be an arbitrary smooth oriented 4-manifold such that $H_*(X;\mathbb{Z}) = H_*(S^1 \times S^3;\mathbb{Z})$ and $H_*(\tilde{X};\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. Let $\mathcal{M}^*(X)$ be the moduli space of irreducible ASD connections in a trivial $SU(2)$–bundle $E \to X$. All such connections are necessarily flat hence we can identify $\mathcal{M}^*(X)$ with the irreducible part of the $SU(2)$–character variety of $\pi_1(X)$.

Lemma 7.2. *The moduli space $\mathcal{M}^*(X)$ is compact.*

Proof. The spectral sequence argument of Furuta–Ohta [**20**, Section 4.1] works in our situation with little change to show that the $SU(2)$ character variety of $\pi_1 X$ has the right Zariski dimension at the reducible representations, and hence the set of reducibles is a single isolated component of the character variety, which is obviously compact. q.e.d.

Given the compactness of $\mathcal{M}^*(X)$, the definition of the Furuta–Ohta invariant proceeds exactly as in [**20**] and [**63**] giving a well-defined invariant

$$\lambda_{\mathrm{FO}}(X) \;=\; \frac{1}{4}\,\#\mathcal{M}^*(X) \in \mathbb{Q}, \tag{39}$$

where $\#\mathcal{M}^*(X)$ stands for the count of points in the (possibly perturbed) moduli space $\mathcal{M}^*(X)$ with the signs determined by a choice of orientation and homology orientation on X.

Remark 7.3. The original definition of λ_{FO} in [**20**] had a denominator of $1/2$, which was replaced by the $1/4$ in equation (39) in [**63**] to match the conjectured mod 2 equality with the Rohlin invariant [**20**, Conjecture 4.5]. It is not obvious from the definition that the original λ_{FO} should even be an integer, although this turns out to be true [**60**, Section 5]. On the other hand, Theorem 7.1 makes it clear that the generalized λ_{FO} invariant defined herein is not an integer, since the signature of a knot can be an arbitrary even integer. We conjecture that with the normalization used in this paper, $\lambda_{\mathrm{FO}}(X)$ reduces mod 2 to the Rohlin invariant of X, defined as an element of $\mathbb{Q}/2\mathbb{Z}$. This conjecture was confirmed in [**63**] for the mapping tori of finite order diffeomorphisms of integral homology spheres, and now the formula of Theorem 7.1 reduced mod 2 implies that the conjecture is also true for all of the mapping tori X in Theorem 7.1.

Remark 7.4. A closer examination of the argument in [**20**, Section 4.1] shows that the following hypotheses would allow for a well-defined λ_{FO} invariant: X has the integral homology of $S^1 \times S^3$ and, for every non-trivial $U(1)$ representation α, the cohomology $H^1(X; \mathbb{C}_\alpha)$ vanishes. Examples of such manifolds X may be obtained by surgery on a knot in S^4 whose Alexander polynomial has no roots on the unit circle. For instance, the spin of the figure-eight knot in the 3-sphere has this property, as do the Cappell–Shaneson knots [**10**]. The latter knots are fibered with fiber T^3, and hence it is not difficult to count the irreducible $SU(2)$ representations of $\pi_1(X)$. For example, one of the Cappell–Shaneson knots gives rise to a 3–torus fibration X over the circle with the monodromy

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and hence has fundamental group with presentation

$$\pi_1(X) \;=\; \langle t, x, y, z \mid [x, y] = 1, [y, z] = 1, [x, z] = 1, txt^{-1} = y, tyt^{-1} = yz, tzt^{-1} = x \rangle.$$

A direct calculation shows that, up to conjugation, $\pi_1(X)$ admits a unique irreducible $SU(2)$ representation given by

$$t = j, \quad x = e^{2\pi i/3}, \quad y = z = e^{-2\pi i/3},$$

and that this representation gives a non-degenerate point in the instanton moduli space on X. Therefore, the generalized λ_{FO} invariant of X equals $\pm 1/4$. On the other hand, the spin structure on the torus fiber induced from its embedding in X is the group-invariant one [**11**]. Since the Rohlin invariant of this spin structure equals 1, the generalized $\lambda_{\mathrm{FO}}(X)$ does not reduce mod 2 to the Rohlin invariant of X.

For the rest of Section 7, we will assume that X is the mapping torus of $\tau : Y \to Y$, an involution which exhibits Y as the double branched cover of an integral homology sphere Y' with branch set a knot K.

Lemma 7.5. *The manifold X has the integral homology of $S^1 \times S^3$.*

Proof. Let $\Delta_K(t)$ be the Alexander polynomial of the knot K normalized so that $\Delta_K(1) = 1$ and $\Delta_K(t^{-1}) = \Delta_K(t)$. Then $H_1(Y)$ is a finite group of order $|\Delta_K(-1)|$ on which τ_* acts as minus identity, see Lemma 5.7 or [**31**, Theorem 5.5.1]. Since $|\Delta_K(-1)|$ is odd, the fixed point set of $\tau_* : H_1(Y) \to H_1(Y)$ must be zero. Now, the natural projection $X \to S^1$ gives rise to a locally trivial bundle with fiber Y. The E^2 page of its Leray–Serre spectral sequence is

$$E^2_{pq} = H_p(S^1, \mathcal{H}_q(Y)),$$

where $\mathcal{H}_q(Y)$ is the local coefficient system associated with the fiber bundle. The groups E^2_{pq} vanish for all $p \geqslant 2$ hence the spectral sequence collapses at its E_2 page. This implies that

$$H_1(X) = H_1(S^1, \mathcal{H}_0(Y)) \oplus H_0(S^1, \mathcal{H}_1(Y)) = \mathbb{Z} \oplus H_0(S^1, \mathcal{H}_1(Y)).$$

The generator of $\pi_1(S^1)$ acts on $H_1(Y)$ as $\tau_* : H_1(Y) \to H_1(Y)$, therefore, $H_0(S^1, \mathcal{H}_1(Y)) = \operatorname{Fix}(\tau_*) = 0$ and hence $H_1(X) = \mathbb{Z}$. Similarly,

$$H_2(X) = H_1(S^1, \mathcal{H}_1(Y)) \oplus H_0(S^1, \mathcal{H}_2(Y)) = 0$$

because $\operatorname{Fix}(\tau_*) = 0$ and $H_2(Y) = H^1(Y) = 0$. This completes the proof. q.e.d.

Since the $\tilde{X} = \mathbb{R} \times Y$, where Y is a rational homology sphere, both conditions $H_*(X;\mathbb{Z}) = H_*(S^1 \times S^3;\mathbb{Z})$ and $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ are satisfied, and the invariant $\lambda_{\mathrm{FO}}(X)$ is well defined by the formula (39). To prove that $\lambda_{\mathrm{FO}}(X)$ is given by the formula of Theorem 7.1, we need to analyze the moduli spaces $\mathcal{M}^*(X)$ that go into its definition.

7.2. Equivariant theory. We will first describe $\mathcal{M}^*(X)$ in terms of $\mathcal{R}(Y)$, the $SU(2)$ character variety of $\pi_1(Y)$. To this end, consider the splitting

$$\mathcal{R}(Y) \;=\; \{\theta\} \sqcup \mathcal{R}_{\mathrm{ab}}(Y) \sqcup \mathcal{R}_{\mathrm{irr}}(Y),$$

whose three components consist of the trivial representation and the conjugacy classes of abelian (that is, non-trivial reducible) and irreducible representations, respectively. Note that θ is the only central representation $\pi_1(Y) \to SU(2)$ because Y is a $\mathbb{Z}/2$ homology sphere ; see the proof of Lemma 7.5. This decomposition is preserved by the map $\tau^* : \mathcal{R}(Y) \to \mathcal{R}(Y)$.

Lemma 7.6. *The involution τ^* acts as the identity on $\mathcal{R}_{\mathrm{ab}}(Y)$.*

Proof. Up to conjugation, any abelian representation $\pi_1(Y) \to SU(2)$ can be factored through a representation $\alpha : H_1(Y) \to U(1)$, where $U(1)$ stands for the group of unit complex numbers in $SU(2)$. Since the involution τ_* acts as minus identity on $H_1(Y)$, we have $\tau^*\alpha = \alpha^{-1}$, which is obviously a conjugate of α. Moreover, any unit quaternion u which conjugates α^{-1} to α must belong to $j \cdot U(1)$ because α is not a central representation. q.e.d.

Let $\mathcal{R}^\tau(Y)$ be the fixed point set of the involution τ^* acting on $\mathcal{R}(Y)\backslash\{\theta\} = \mathcal{R}_{\text{ab}}(Y) \sqcup \mathcal{R}_{\text{irr}}(Y)$. It follows from the above lemma that

$$\mathcal{R}^\tau(Y) \;=\; \mathcal{R}_{\text{ab}}(Y) \,\sqcup\, \mathcal{R}^\tau_{\text{irr}}(Y).$$

Proposition 7.7. *Let $i : Y \to X$ be the inclusion map given by the formula $i(x) = [0, x]$. Then the induced map*

$$i^* : \mathcal{M}^*(X) \to \mathcal{R}^\tau(Y) \tag{40}$$

is well defined, and is a one-to-one correspondence over $\mathcal{R}_{\text{ab}}(Y)$ and a two-to-one correspondence over $\mathcal{R}^\tau_{\text{irr}}(Y)$.

Proof. The natural projection $X \to S^1$ is a locally trivial bundle whose homotopy exact sequence

$$0 \longrightarrow \pi_1(Y) \longrightarrow \pi_1(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

splits, making $\pi_1(X)$ into a semi-direct product of $\pi_1(Y)$ and $\mathbb{Z}$. Let t be a generator of $\mathbb{Z}$ then every representation $A : \pi_1(X) \to SU(2)$ determines and is uniquely determined by the pair (α, u) where $u = A(t)$ and $\alpha = i^*A : \pi_1(Y) \to SU(2)$ is a representation such that $\tau^*\alpha = u\alpha u^{-1}$. In particular, the conjugacy class of α is fixed by τ^*.

If $\alpha = \theta$ then A must be reducible, hence α is not in the image of i^*. If α is non-trivial abelian, we can conjugate it to a representation whose image is in the group of unit complex numbers in $SU(2)$. Then α is of the form $\alpha = i^*A$ with $u = A(t)$ in the circle $j \cdot U(1)$, as in the proof of Lemma 7.6. In particular, A is irreducible and $u^2 = -1$. Since any two quaternions in $j \cdot U(1)$ are conjugate to each other by a unit complex number, the map i^* is a one-to-one correspondence over $\mathcal{R}_{\text{ab}}(Y)$. Finally, let α be an irreducible representation with the character in $\mathcal{R}^\tau(Y)$. Then there is a unit quaternion u such that $\tau^*\alpha = u\alpha u^{-1}$, and therefore α is in the image of i^*. Moreover, there are exactly two different choices of u such that $\tau^*\alpha = u\alpha u^{-1}$ because if $u_1\alpha u_1^{-1} = u_2\alpha u_2^{-1}$ then $u_1 = \pm\, u_2$ since α is irreducible. The irreducibility of α also implies that $u^2 = \pm 1$. In this case, the map i^* is a two-to-one correspondence. q.e.d.

Remark 7.8. It follows from the above proof that the characters in $\mathcal{M}^*(X)$ that are mapped by i^* to $\mathcal{R}_{\mathrm{ab}}(Y)$ are binary dihedral, while those mapped to $\mathcal{R}^\tau_{\mathrm{irr}}(Y)$ are not.

The Zariski tangent space to $\mathcal{R}^\tau(Y)$ at a point $[\alpha] \in \mathcal{R}^\tau(Y)$ is the fixed point set of the map $\tau^* : T_{[\alpha]}\mathcal{R}(Y) \to T_{[\alpha]}\mathcal{R}(Y)$. Using an identification $T_{[\alpha]}\mathcal{R}(Y) = H^1(Y, \operatorname{ad}\alpha)$ and the fact that $\tau^*\alpha = u\alpha u^{-1}$, this set can be described in cohomological terms as the fixed point set of the map

$$\operatorname{Ad} u \circ \tau^* : H^1(Y, \operatorname{ad}\alpha) \to H^1(Y, \operatorname{ad}\alpha).$$

We will call $\mathcal{R}^\tau(Y)$ *non-degenerate* if the equivariant cohomology groups

$$H^1_\tau(Y, \operatorname{ad}\alpha) = \operatorname{Fix}\left(\operatorname{Ad} u \circ \tau^* : H^1(Y, \operatorname{ad}\alpha) \to H^1(Y, \operatorname{ad}\alpha)\right)$$

vanish for all $[\alpha] \in \mathcal{R}^\tau(Y)$. The moduli space $\mathcal{M}^*(X)$ is called *non-degenerate* if $\operatorname{coker}(d^*_A \oplus d^+_A) = 0$ for all $[A] \in \mathcal{M}^*(X)$. Since $\operatorname{ind}(d^* \oplus d^+_A) = \dim \mathcal{M}^*(X) = 0$, this is equivalent to $\ker(d^*_A \oplus d^+_A) = 0$ and, since A is flat and irreducible, to simply $H^1(X; \operatorname{ad} A) = 0$.

Proposition 7.9. *The moduli space $\mathcal{M}^*(X)$ is non-degenerate if and only if $\mathcal{R}^\tau(Y)$ is non-degenerate.*

Proof. The group $H^1(X, \operatorname{ad} A)$ can be computed with the help of the Leray–Serre spectral sequence of the fibration $X \to S^1$ with fiber Y. The E_2–page of this spectral sequence is

$$E_2^{pq} = H^p(S^1, \mathcal{H}^q(Y, \operatorname{ad}\alpha)),$$

where $\alpha = i^*A$ and $\mathcal{H}^q(Y, \operatorname{ad}\alpha)$ is the local coefficient system associated with the fibration. The groups E_2^{pq} vanish for all $p \geqslant 2$, so the spectral collapses at the E_2–page, and

$$H^1(X, \operatorname{ad} A) = H^1(S^1, \mathcal{H}^0(Y, \operatorname{ad}\alpha)) \oplus H^0(S^1, \mathcal{H}^1(Y, \operatorname{ad}\alpha)). \tag{41}$$

The generator of $\pi_1(S^1)$ acts on the cohomology groups $H^*(Y, \operatorname{ad}\alpha)$ as

$$\operatorname{Ad} u \circ \tau^* : H^*(Y, \operatorname{ad}\alpha) \to H^*(Y, \operatorname{ad}\alpha),$$

where u is such that $\tau^*\alpha = u\alpha u^{-1}$. If α is irreducible, $H^0(Y, \operatorname{ad}\alpha) = 0$ and the first summand in (41) vanishes. If α is non-trivial abelian, we may assume without loss of generality that it takes values in the group $U(1)$ of unit complex numbers. Then $\tau^*\alpha = u\alpha u^{-1}$ for some $u \in j \cdot U(1)$ and $H^0(Y, \operatorname{ad}\alpha) = i \cdot \mathbb{R}$ as a subspace of $\mathfrak{su}(2)$, with $\tau^* = \mathrm{id}$. One can easily check that $\operatorname{Ad} u$ acts as minus identity on $i \cdot \mathbb{R}$ hence the first summand in (41) again vanishes. The second summand in (41) is the fixed point set of τ^* acting on $H^1(Y, \operatorname{ad}\alpha)$, which is the equivariant cohomology $H^1_\tau(Y, \operatorname{ad}\alpha)$. Thus we conclude that $H^1(X, \operatorname{ad} A) = H^1_\tau(Y, \operatorname{ad}\alpha)$, which completes the proof. q.e.d.

Let us assume that $\mathcal{R}^\tau(Y)$ is non-degenerate. For any $[\alpha] \in \mathcal{R}^\tau(Y)$, its orientation will be given by

$$(-1)^{\operatorname{sf}^\tau(\theta,\alpha)}$$

where $\operatorname{sf}^\tau(\theta, \alpha)$ is the mod 2 equivariant spectral flow defined in [**63**, Section 3.4] for irreducible α. That definition extends word for word to abelian α after one resolves the technical issue of the existence of a constant lift, which we will do next.

Let P be an $SU(2)$ bundle over Y with a fixed trivialization and α an abelian flat connection in P; we are abusing notations by using the same symbol for the connection and its holonomy. It follows from Lemma 7.6 that τ admits a lift $\tilde\tau : P \to P$ such that $\tilde\tau^*\alpha = \alpha$. Since α is abelian, this lift is defined uniquely up to the stabilizer of α, which is a copy of $U(1)$ in $SU(2)$. The lift $\tilde\tau$ can be written in the base-fiber coordinates as $\tilde\tau(x, y) = (\tau(x), \rho(x) \cdot y)$ for some function $\rho : Y \to SU(2)$. We call it *constant* if there exists $u \in SU(2)$ such that $\rho(x) = u$ for all $x \in SU(2)$.

Lemma 7.10. *By changing α within its gauge equivalence class, one may assume that $\tilde\tau$ is a constant lift with $u^2 = -1$.*

Proof. The equation $\tilde\tau^*\alpha = \alpha$ implies that $(\tilde\tau^2)^*\alpha = \alpha$ hence the gauge transformation $\tilde\tau^2$ belongs to the stabilizer of the connection α. If $x \in \operatorname{Fix}(\tau)$ then $\tilde\tau^2(x, y) = (x, \rho(x)^2 \cdot y)$ hence $\rho(x)^2$ is a unit complex number independent of x. This implies that $\rho(x)$ itself is a unit complex number unless $\rho(x)^2 = -1$. It is this last case that must be realized because, at the level of holonomy representations, $\tau^*\alpha = \alpha^{-1}$ is conjugate to α by an element $u \in SU(2)$ with $u^2 = -1$; see the proof of Lemma 7.6. Since $\rho(x)^2 = -1$ describes a single conjugacy class $\operatorname{tr}\rho(x) = 0$ in $SU(2)$, we may assume that $\rho(x) = u$ for all $x \in \operatorname{Fix}(\tau)$.

To finish the proof, we will follow the argument of [**63**, Section 2.2]. Let $u : P \to P$ be the constant lift $u(x, y) = (\tau(x), u \cdot y)$ and consider the $SO(3)$ orbifold bundles $P/\tilde\tau$ and P/u over the integral homology sphere Y'. All such bundles are classified by the holonomy around the singular set in Y'. Since this holonomy equals $\operatorname{ad}(u)$ in both cases, the bundles $P/\tilde\tau$ and P/u must be isomorphic, with any isomorphism pulling back to a gauge transformation $g : P \to P$ that relates the lifts $\tilde\tau$ and u. q.e.d.

Proposition 7.11. *Assuming that the moduli space $\mathcal{R}^\tau(Y)$ is non-degenerate, the map* (40) *is orientation preserving.*

Proof. The proof from [**63**, Section 3] extends to the current situation with no change. q.e.d.

7.3. Orbifold theory. Under the continued non-degeneracy assumption, we will now describe $\mathcal{R}^\tau(Y)$ in terms of orbifold representations. Let us consider the orbifold fundamental group $\pi_1^V(Y', K) = \pi_1(Y' - N(K))/\langle\mu^2\rangle$, where μ is a meridian of K. This group can be included into the split orbifold exact sequence

$$1 \longrightarrow \pi_1 Y \xrightarrow{\ \pi_*\ } \pi_1^V(Y', K) \xrightarrow{\ j\ } \mathbb{Z}/2 \longrightarrow 1.$$

Denote by $\mathcal{R}^V(Y', K; SO(3))$ the character variety of irreducible $SO(3)$ representations of the group $\pi_1^V(Y', K)$, and also introduce the character variety $\mathcal{R}^\tau(Y; SO(3))$ of irreducible representations $\pi_1 Y \to SO(3)$.

Proposition 7.12. *The pull back of representations via the map π_* in the orbifold exact sequence gives rise to a one-to-one correspondence*

$$\pi^* : \mathcal{R}^V(Y', K; SO(3)) \longrightarrow \mathcal{R}^\tau(Y; SO(3)).$$

Proof. One can easily see that a representation $\alpha' : \pi_1^V(Y', K) \to SO(3)$ pulls back to a trivial representation $\theta : \pi_1 Y \to SO(3)$ if and only if α' is reducible. The same argument as in [**16**, Proposition 3.3] shows that all pull-back representations belong to $\mathcal{R}^\tau(Y, SO(3))$. The inverse map for π^* is constructed as follows: given $[\alpha] \in \mathcal{R}^\tau(Y, SO(3))$ choose $v \in SO(3)$ such that $\tau^*\alpha = v\alpha v^{-1}$, and define a representation α' of $\pi_1^V(Y', K) = \pi_1 Y \rtimes \mathbb{Z}/2$ by the formula

$$\alpha'(g \cdot \mu^k) \;=\; \alpha(g) \cdot v^k. \tag{42}$$

If α is irreducible, the element v is unique hence formula (42) gives an inverse map. If α is non-trivial abelian, lift it to a $U(1)$ representation using the fact that Y is a $\mathbb{Z}/2$ homology sphere. The proof of Lemma 7.6 then tells us that $v = \operatorname{Ad} u$ for some $u \in j \cdot U(1)$. Since any two elements of $j \cdot U(1)$ are conjugate to each other by a unit complex number, formula (42) again gives an inverse map. q.e.d.

The representations $\pi_1^V(Y', K) \to SO(3)$ need not lift to $SU(2)$ representations. However, they lift to projective representations $\pi_1^V(Y', K) \to SU(2)$ sending μ^2 to ± 1. The character variety of such projective representations will be denoted by $\mathcal{R}^V(Y', K)$, and it will be oriented using the orbifold spectral flow.

Proposition 7.13. *The correspondence of Proposition 7.12 gives rise to an orientation preserving correspondence $\mathcal{R}^V(Y', K) \to \mathcal{R}^\tau(Y)$ which is one-to-one over $\mathcal{R}_{\mathrm{ab}}(Y)$ and two-to-one over $\mathcal{R}^\tau_{\mathrm{irr}}(Y)$.*

Proof. Let us consider the adjoint representation $\mathrm{Ad} : SU(2) \to SO(3)$ and the induced maps

$$\mathcal{R}^\tau(Y) \to \mathcal{R}^\tau(Y; SO(3)) \quad \text{and} \quad \mathcal{R}^V(Y', K) \to \mathcal{R}^V(Y', K; SO(3)).$$

The first map is a one-to-one correspondence because Y is a $\mathbb{Z}/2$ homology sphere. The second map is the quotient map by the action of $\mathbb{Z}/2$ sending the image of the meridian μ to its negative. The fixed points of this action are precisely the binary dihedral projective representations $\alpha' : \pi_1^V(Y', K) \to SU(2)$. Now, the proof will be finished as soon as we show that an irreducible projective representation $\alpha' : \pi_1^V(Y', K) \to SU(2)$ is binary dihedral if and only if its pull back representation $\pi^*\alpha' : \pi_1(Y) \to SU(2)$ is abelian.

If $\pi^*\alpha'$ is abelian, its image belongs to $U(1) \subset SU(2)$ and the image of α' to its $\mathbb{Z}/2$ extension. This extension is the binary dihedral group $U(1) \cup j \cdot U(1)$. Conversely, it follows from the orbifold exact sequence that $\pi_1 Y$ is the commutator subgroup of $\pi_1^V(Y', K)$ therefore, if α' is binary dihedral, the image of $\pi^*\alpha'$ must belong to the commutator subgroup of $U(1) \cup j \cdot U(1)$, which is of course the group $U(1)$.

Since the orbifold spectral flow matches the equivariant spectral flow used to orient $\mathcal{R}^\tau(Y)$, the above correspondence is orientation preserving. q.e.d.

7.4. Perturbations. In this section, we will remove the assumption that $\mathcal{R}^\tau(Y)$ is non-degenerate which we used until now. To accomplish that, we will switch from the language of representations to the language of connections. Let P a trivialized $SU(2)$ bundle over Y. Any endomorphism $\tilde{\tau} : P \to P$ which lifts the involution τ induces an action on the space of connections $\mathcal{A}(Y)$ by pull back. Since any two such lifts are related by a gauge transformation, this action defines a well defined action on the configuration space $\mathcal{B}(Y) = \mathcal{A}(Y)/\mathcal{G}(Y)$. The fixed point set of this action will be denoted by $\mathcal{B}^\tau(Y)$.

The irreducible part of $\mathcal{B}^\tau(Y)$ was studied in [**63**] hence we will only deal with reducible connections. In fact, we will further restrict ourselves to constant lifts u with $u^2 = -1$ because any flat abelian connection α admits such a lift; see Lemma 7.10.

Let $\mathcal{A}^u(Y) \subset \mathcal{A}(Y)$ consist of all non-trivial connections A such that $u^*A = A$, and $\mathcal{G}^u(Y) \subset \mathcal{G}(Y)$ of all gauge transformations g such that $gu = ug$. The quotient space $\mathcal{A}^u(Y)/\mathcal{G}^u(Y)$ will be denoted by $\mathcal{B}^u(Y)$. The following lemma is a key to making the arguments of [**63**] work in the case of abelian connections.

Lemma 7.14. *The group $\mathcal{G}^u(Y)$ acts on $\mathcal{A}^u(Y)$ with the stabilizer $\{\pm 1\}$. Moreover, the natural map $\mathcal{B}^u(Y) \to \mathcal{B}^\tau(Y)$ is a two-to-one correspondence to its image on the irreducible part of $\mathcal{B}^u(Y)$, and a one-to-one correspondence on the reducible part.*

Proof. For the sake of simplicity, we will assume that reducible connections have their holonomy in the subgroup $U(1)$ of unit complex numbers in $SU(2)$, and that $u \in j \cdot U(1)$. Let us suppose that $g^*A = A$ for a connection $A \in \mathcal{A}^u(Y)$ and a gauge transformation $g \in \mathcal{G}^u(Y)$. If A is irreducible, we automatically have $g = \pm 1$. If A is non-trivial abelian, then g is a complex number, and the condition $ug = gu$ implies that $g = \pm 1$.

To prove the second statement, consider a connection A such that $u^*A = A$ and consider its gauge equivalence class in $\mathcal{B}^\tau(Y)$. It consists of all connections g^*A such that $u^*g^*A = g^*A$. Since $A = u^*A$, we immediately conclude that $u^*g^*A = g^*u^*A$ so that ug and gu differ by an element in the stabilizer of A. If A is irreducible, its stabilizer consists of ± 1 hence $ug = \pm gu$. The group of gauge transformations satisfying this condition contains $\mathcal{G}^u(Y)$ as a subgroup of index two, which leads to the desired two-to-one correspondence. If A is non-trivial abelian, its stabilizer consists of unit complex numbers. Therefore, we can write $ug = c^2 gu$ with $c \in U(1)$ or, equivalently, $ucg = cgu$. This provides us with a gauge transformation $cg \in \mathcal{G}^u(Y)$ such that $(cg)^*A = A$, yielding the one-to-one correspondence on the reducible part. q.e.d.

With this lemma in place, the proof of Proposition 7.7 can be re-stated in gauge-theoretic terms as in [**63**, Proposition 3.1]. The treatment of perturbations in our case is then essentially identical to that in [**16**] and [**63**], one important observation being that the orbifold representations α' that pull back to abelian representations of $\pi_1(Y)$ are in fact irreducible. This fact is used in the proof of [**16**, Lemma 3.8], which supplies us with sufficiently many admissible perturbations.

7.5. Proof of Theorem 7.1. The outcome of Section 7.2 and Section 7.3 is that, perhaps after perturbing as in Section 7.4, we have two orientation preserving correspondences,

$$\mathcal{M}^*(X) \longrightarrow \mathcal{R}^\tau(Y) \longleftarrow \mathcal{R}^V(Y', K),$$

both of which are one-to-one over $\mathcal{R}_{\mathrm{ab}}(Y)$ and two-to-one over $\mathcal{R}^{\tau}_{\mathrm{irr}}(Y)$ (we omit perturbations in our notations). These correspondences give rise to an orientation preserving one-to-one correspondence between $\mathcal{M}^*(X)$ and $\mathcal{R}^V(Y',K)$. The proof of Theorem A will be complete after we express the signed count of points in $\mathcal{R}^V(Y',K)$ in terms of the Casson invariant of Y' and the knot signature of K.

The character variety $\mathcal{R}^V(Y',K)$ of projective representations α' splits into two components corresponding to whether the square of $\alpha'(\mu)$ equals $+1$ or -1. Let E be the exterior of the knot K then this splitting corresponds to the splitting

$$\mathcal{R}^V(Y',K) \;=\; \mathcal{S}_0(E,SU(2)) \,\cup\, \mathcal{S}_{1/2}(E,SU(2)) \,\cup\, \mathcal{S}_1(E,SU(2))$$

of [**16**, Proposition 3.4], where $\mathcal{S}_a(E,SU(2))$ comprises the conjugacy classes of representations $\gamma : \pi_1 X \to SU(2)$ such that $\operatorname{tr}\gamma(\mu) = 2\cos(2\pi a)$. According to Herald [**25**], the signed count of points in $\mathcal{S}_0(E,SU(2)) \,\cup\, \mathcal{S}_1(E,SU(2))$ equals $4\cdot\lambda(Y')$, while the signed count of points in $\mathcal{S}_{1/2}(E,SU(2))$ equals $4\cdot\lambda(Y') + 1/2\ \operatorname{sign}(K)$. Adding up the two counts and dividing by four we obtain the desired formula

$$\lambda_{\mathrm{FO}}(X) \;=\; 2\cdot\lambda(Y') \;+\; \frac{1}{8}\,\operatorname{sign}(K).$$

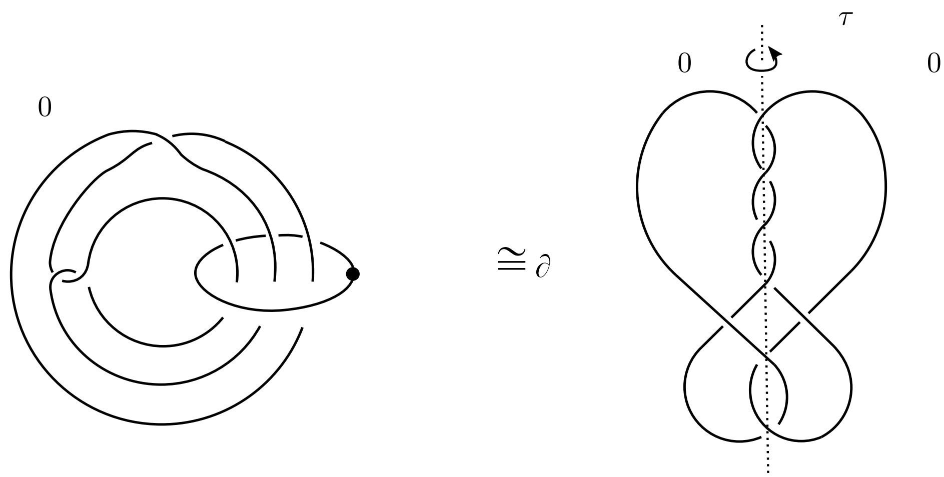
8. Strongly non-extendable involutions and Akbulut corks

A cork is a pair (W,τ) which consists of a smooth compact contractible 4–manifold W and an involution τ on its boundary that does not extend to a self-diffeomorphism of W. Sometimes the definition of a cork includes the hypothesis that W have a Stein structure (see for instance [**2**, Definition 10.3]) but we do not require this.

8.1. Strongly non-extendable involutions. Figure 6 (a) shows the cork constructed by Akbulut [**1**], and Figure 6 (b) shows the involution τ on its boundary. This cork will be called W_1, and its boundary Y_1.

Theorem 8.1. *The involution $\tau : Y_1 \to Y_1$ does not extend to a diffeomorphism of any smooth $\mathbb{Z}/2$ homology 4-ball bounded by Y_1.*

The proof of Theorem 8.1 makes use of a gluing theorem for Seiberg–Witten invariants, which we briefly summarize. Let $(X,\mathfrak{s})$ be a smooth closed oriented 4-manifold with a spin$^{\mathrm{c}}$ structure $\mathfrak{s}$ and $b_2^+(X) > 1$. Suppose that X is decomposed as $X = X_1 \cup X_2$ with $b_2^+(X_2) > 1$. Let Y be



(a) (b)

Figure 6. Akbulut cork W_1 and the involution on $Y_1 = \partial W_1$

the oriented boundary of X_1 and consider two cobordisms, M_1 from S^3 to Y and M_2 from Y to S^3, obtained by removing open 4-balls from X_1 and X_2, respectively. Let $\mathfrak{s}_i$ be the induced spin^c structures on M_i, $i = 1, 2$, and $\mathfrak{s}_0$ the induced spin^c structure on Y. Then we have two maps in monopole homology,

$$\widehat{HM}_*(M_1, \mathfrak{s}_1) : \widehat{HM}_*(S^3) \to \widehat{HM}_*(Y, \mathfrak{s}_0) \quad \text{and}$$
$$\overrightarrow{HM}^*(M_2, \mathfrak{s}_2) : \widetilde{HM}^*(S^3) \to \widehat{HM}^*(Y, \mathfrak{s}_0).$$

Denote by $\check{1}$ and $\hat{1}$ the canonical generators of $\widetilde{HM}^*(S^3)$ and $\widehat{HM}_*(S^3)$. The gluing theorem expresses the Seiberg–Witten invariant of $(X, \mathfrak{s})$ as follows.

Proposition 8.2. *Suppose that Y is a rational homology sphere. Then*

$$SW(X, \mathfrak{s}) = \langle\, \widehat{HM}_*(M_1, \mathfrak{s}_1)(\hat{1}), \overrightarrow{HM}^*(M_2, \mathfrak{s}_2)(\check{1})\,\rangle. \tag{43}$$

Formula (43) is a slight strengthening of the formula that appears just before [**37**, Definition 3.6.3], in that (43) holds for each spin^c structure separately, rather than for the sum over the spin^c structures on X, as would be the case for $b_1(Y) > 0$. Our strengthened formula follows from the remark on [**37**, page 569] following the proof of Proposition 27.4.1. (Separating the spin^c structures can also be achieved using local coefficients as in [**37**, Section 3.7–3.8], but we do not need this in our situation.)

The following simple algebraic lemma is presumably well-known.

Lemma 8.3. *Let A be a 2×2 matrix with $A^2 = I$ and $\operatorname{tr}(A) = -2$. Then $A = -I$, where I stands for the identity matrix.*

Proof. By the Cayley–Hamilton theorem, we have that $A^2 - \operatorname{tr}(A)\cdot A + \det(A)\cdot I = 0$, where $\det(A) = \pm 1$. If $\det(A) = -1$, we obtain $\operatorname{tr}(A)\cdot A = 0$, which contradicts the invertibility of A. Hence $\det(A) = 1$, which implies that $A = -I$. q.e.d.

Proof of Theorem 8.1. We will omit the spin^c structure $\mathfrak{s}_0$ from our notations. We claim first that the action of τ_* on $HM^{\mathrm{red}}(Y_1)$ is minus the identity. To prove this, we will combine our Theorem A with a Heegaard Floer homology calculation by Akbulut and Durusoy [**3**]. They work with a picture that is the mirror image of Figure 6 (a) and show that $HF^+(-Y_1) \cong \mathcal{T}_{(0)} \oplus \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(0)}$, where the first summand is a tower $\mathbb{Z}[U, U^{-1}]/U\cdot\mathbb{Z}[U]$ with the lowest degree in grading 0. It follows that $HF^{\mathrm{red}}(-Y_1) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(0)}$ and $HF^{\mathrm{red}}(Y_1) \cong \mathbb{Z}\oplus\mathbb{Z}$, with both summands of odd grading (with respect to the absolute $\mathbb{Z}/2$ grading). The parity can be checked using the formula

$$\lambda(Y) = \chi(HF^{\mathrm{red}}(Y)) - 1/2\cdot d(Y)$$

of [**54**, Theorem 1.3], where $\lambda(Y)$ is the Casson invariant of Y. Since $\lambda(Y_1) = -2$, see for instance [**65**], and $d(Y_1) = 0$, both summands in $HF^{\mathrm{red}}(Y_1)$ must have odd grading. We translate this computation into the monopole homology, keeping in mind the isomorphisms

$$\widehat{HM}_a(Y) \;\cong\; (\widetilde{HM}_{-1-a}(-Y))^* \;\cong\; (HF^+_{-1-a}(-Y))^*. \tag{44}$$

The grading shift for the first 'duality' isomorphism is [**37**, Proposition 28.3.4], while the second equality of the absolute $\mathbb{Q}$-gradings is deduced from [**59**, **27**, **21**].

Now, the involution τ makes Y_1 into a double branched cover of the 3-sphere with branch set a knot $K_1 \subset S^3$. As described in [**60**] and drawn in Figure 7, the knot K_1 is obtained from the left-handed $(5,6)$-torus knot on six strings by adding one full left-handed twist on two adjacent strings. In particular, the signature of K_1 is 16. Using Theorem A, we compute

$$\operatorname{tr}(\tau_*) = -\operatorname{Lef}(\tau_*) = -\frac{1}{8}\operatorname{sign}(K_1) = -2$$

and, using Lemma 8.3, conclude that

$$\tau_* = -I : HF_{\mathrm{red}}(Y_1) \to HF_{\mathrm{red}}(Y_1). \tag{45}$$

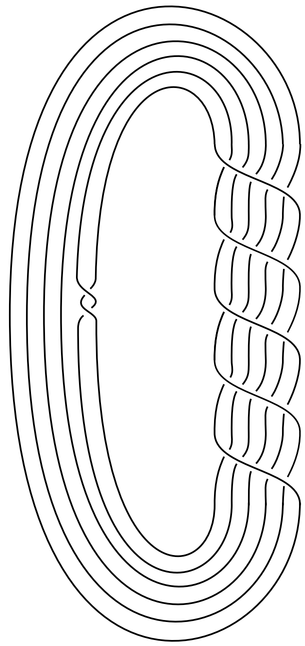


Figure 7.

In order to compute the action of τ_* on $\widehat{HM}(Y_1)$, consider the short exact sequence in monopole homology

$$0 \longrightarrow HM^{\mathrm{red}}(Y_1) \longrightarrow \widehat{HM}_{-1}(Y_1) \xrightarrow{f} \overline{HM}_{-2}(Y_1) \longrightarrow 0.$$

Since $HM^{\mathrm{red}}(Y_1) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\overline{HM}_{-2}(Y_1) \cong \mathbb{Z}$, the group $\widehat{HM}_{-1}(Y_1)$ must be free of rank 3. We define a splitting $\overline{HM}_{-2}(Y_1) \to \widehat{HM}_{-1}(Y_1)$ of this short exact sequence by sending the canonical generator $1 \in \overline{HM}_{-2}(Y_1)$ to the element $e_W = \widehat{HM}_*(M_1, \mathfrak{s}_1)(\hat{1}) \in \widehat{HM}_{-1}(Y_1)$ as above, where M_1 is obtained from W_1 by removing an open 4-ball. Using the fact that $\overline{HM}_*(M_1, \mathfrak{s}_1)$ maps the canonical generator of $\overline{HM}_{-2}(S^3)$ to that of $\overline{HM}_{-2}(Y_1)$, we see that $f(e_W) = 1$.

For any choice of free generators $\{e_0, e_1\}$ of $HM^{\mathrm{red}}(Y_1)$ we have a set of free generators $\{e_0, e_1, e_W\}$ of $\widehat{HM}_{-1}(Y_1)$. The action of τ_* on $\widehat{HM}_{-1}(Y_1)$ is then given by a matrix of the form

$$(46) \qquad \begin{pmatrix} -1 & 0 & p \\ 0 & -1 & q \\ 0 & 0 & 1 \end{pmatrix}$$

with some unknown integers p and q. In what follows, we will extract some information about p and q from the fact that a cork twist on W_1 changes the Seiberg–Witten invariant of a certain closed 4-manifold.

There is an embedding [**22**, Figure 9.5] (see also [**1**]) of W_1 into the blown up $K3$–surface, $X = K3 \# \overline{\mathbb{C}\mathrm{P}}^2$, such that the the cork twist results in the manifold

$$X^\tau = W_1 \cup_\tau (X - \mathrm{int}(W_1))$$

with the trivial Seiberg–Witten invariant. On the other hand, the blowup formula for Seiberg–Witten invariants [**19**] implies that the Seiberg–Witten invariant of X equals 1 for the spin^c structure $\mathfrak{s}$ whose first Chern class is the generator of $H^2(\overline{\mathbb{C}\mathrm{P}}^2)$. Since Y_1 is an integral homology sphere, there is an obvious correspondence, $\mathfrak{s} \leftrightarrow \mathfrak{s}^\tau$, between spin^c structures on X and X^τ. Using the gluing formula (43) with $X_1 = W_1$ and $X_2 = X - \mathrm{int}(W_1)$, we obtain

$$\begin{aligned} \mathrm{SW}(X, \mathfrak{s}) &= \langle \widehat{HM}_*(M_1, \mathfrak{s}_1)(\hat{1}), \overrightarrow{HM}^*(M_2, \mathfrak{s}_2)(\check{1}) \rangle = 1 \quad \text{and} \\ \mathrm{SW}(X^\tau, \mathfrak{s}^\tau) &= \langle \tau_*(\widehat{HM}_*(M_1, \mathfrak{s}_1)(\hat{1})), \overrightarrow{HM}^*(M_2, \mathfrak{s}_2)(\check{1}) \rangle = 0. \end{aligned}$$

If we write $\overrightarrow{HM}^*(M_2, \mathfrak{s}_2)(\check{1}) = a e_0^* + b e_1^* + c e_W^*$ with respect to the dual basis of $\widehat{HM}^{-1}(Y_1)$, the above formulas reduce to

$$\mathrm{SW}(X, \mathfrak{s}) = c = 1 \quad \text{and} \quad \mathrm{SW}(X^\tau, \mathfrak{s}^\tau) = ap + bq + c = 0,$$

implying that $ap + bq + 1 = 0$ and, in particular, that the integers p and q are co-prime. Therefore, by a change of basis $\{e_0, e_1\}$, we can turn the matrix (46) of the involution τ_* into

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, suppose that Y_1 bounds another smooth $\mathbb{Z}/2$-homology 4-ball W'. W' has a unique spin structure $\mathfrak{s}_{W'}$, which must be preserved by any diffeomorphism. By studying the the spin manifold $(W', \mathfrak{s}_{W'})$, one defines the element $e_{W'} \in \widehat{HM}_{-1}(Y_1)$ by the same procedure as e_W. As before, $f(e_{W'}) = 1$. Suppose that τ extends to a diffeomorphism on W'. Then, by naturality of monopole Floer homology, one must have

$$\tau^*(e_{W'}) = e_{W'}.$$

But since the kernel of $A - I$ is generated by the vector $(0,1,2)$, we have $e_{W'} = (0, c, 2c)$ for some integer c. In particular, $f(e_{W'}) = 2c$ is an even integer, which contradicts $f(e_{W'}) = 1$. q.e.d.

Remark 8.4. We can prove the same non-extension result for other involutions on homology spheres, even those that are not the boundaries of contractible manifolds. For example, an extension [**53**] of Taubes' result [**67**] (plus the fact [**18**] that $\Sigma(2,3,7)$ bounds a spin manifold with intersection form $E_8 \oplus H$) implies that the homology sphere $\Sigma(2,3,7) \# -\Sigma(2,3,7)$ does not bound a smooth contractible manifold. On the other hand, we can construct an involution on this manifold as follows. View $\Sigma(2,3,7)$ as the link of a singularity,

$$\Sigma(2,3,7) \;=\; \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^7 = 0,\; |x|^2 + |y|^2 + |z|^2 = 1\},$$

and consider the involutions τ_0 and τ_1 acting on $\Sigma(2,3,7)$ by the formula

$$\tau_0(x,y,z) = (-x,y,z) \quad \text{and} \quad \tau_1(x,y,z) = (\bar{x},\bar{y},\bar{z}). \tag{47}$$

Let τ_i^* denote the map on $HM^{\mathrm{red}}(\Sigma(2,3,7);\mathbb{Q}) = \mathbb{Q}$ induced by τ_i, $i = 0,1$. The involution τ_0 is isotopic to the identity hence τ_0^* is the identity; the action of τ_1^* is computed in Section 10 below as negative one. Suppose $\tau = \tau_0 \# \tau_1$ extends as a diffeomorphism on some $\mathbb{Z}/2$ homology ball with boundary $\Sigma(2,3,7) \# -\Sigma(2,3,7)$. Adding a 3-handle results in a $\mathbb{Z}/2$ homology cobordism W from $\Sigma(2,3,7)$ to itself that admits a self-diffeomorphism restricting to τ_0 and τ_1 on its two boundary components. By functoriality of monopole Floer homology, W induces trivial map on $HM^{\mathrm{red}}(\Sigma(2,3,7))$. This contradicts the splitting formula for λ_{SW} [**44**, Theorem A] and the fact that it reduces mod 2 to the Rohlin invariant [**50**, Theorem A].

8.2. Constructing corks. Starting with the cork W_1, one can construct a number of other corks by the method we describe in this subsection. Recall that the involution $\tau : \partial W_1 \to \partial W_1$ makes ∂W_1 into a double branched cover $\Sigma(K_1)$ of the 3–sphere with branch set the knot $K_1 \subset S^3$ shown in Figure 7. Let K be an arbitrary knot in S^3 smoothly concordant to K_1. The double branched cover of $I \times S^3$ with branch set the concordance is a $\mathbb{Z}/2$ homology cobordism

U_K from $\partial W_1 = \Sigma(K_1)$ to $\Sigma(K)$. The manifold

$$W_K = W_1 \cup_{\partial W_1} U_K$$

is a smooth $\mathbb{Z}/2$ homology 4-ball with the natural involution $\tau_K : \partial W_K \to \partial W_K$ on its boundary given by the covering translation.

Corollary 8.5. *The involution τ_K can be extended to W_K as a homeomorphism but not as a diffeomorphism. Moreover, if $\pi_1(U_K)$ is normally generated by the image of $\pi_1(\partial W_1)$ then the manifold W_K is contractible and therefore (W_K, τ_K) is a cork.*

Proof. The involution τ_K extends as a homeomorphism because τ does. To prove that τ_K does not extend as a diffeomorphism, consider the $\mathbb{Z}/2$ homology ball

$$W \;=\; W_K \cup_{\Sigma(K)} (-U_K)$$

with boundary Y_1, where $-U_K$ denotes U_K with reversed orientation. Suppose τ_K extends as a diffeomorphism on W_K. By gluing this diffeomorphism with the covering translation on $-U_K$, we obtain a diffeomorphism on W that extends the involution τ on its boundary. This contradicts Theorem 8.1. q.e.d.

Examples of knots K which are concordant to K_1 and, at the same time, satisfy the condition of Corollary 8.5 can be constructed using the technique of infection [**12**]. Choose a knot η in the complement of K_1 that is unknotted in S^3 and has even linking number with K_1. Let J be any slice knot in S^3. Denote by $\nu(\eta)$ and $\nu(J)$ open tubular neighborhoods of the two knots. Then

$$(S^3 - \nu(\eta)) \cup (S^3 - \nu(J))$$

is diffeomorphic to S^3, provided we glue the meridian of J to the longitude of η, and vice versa. Under this diffeomorphism, the knot K_1 becomes a new knot, $K(J, \eta)$.

One can similarly 'infect' the product concordance from K_1 to itself by removing $I \times \nu(\eta)$ from $I \times S^3$ and gluing in the exterior of a concordance $C \subset I \times S^3$ from the unknot to J; see Gordon [**23**]. This gives a concordance $C(J, \eta)$ from K_1 to $K(J, \eta)$. Writing $U_{K(J,\eta)}$ for the double branched cover of $I \times S^3$ with branch set $C(J, \eta)$, we claim that $U_{K(J,\eta)}$ is a $\mathbb{Z}/2$ homology cobordism from $\partial W_1 = \Sigma(K_1)$ to $\Sigma(K(J, \eta))$ whose fundamental group is normally generated by $\pi_1(\Sigma(K_1))$.

To see this, note that by the assumption on the linking number, the preimage of the cylinder $I \times \eta$ in $I \times \Sigma(K_1)$ consists of two cylinders, $I \times \eta_1$ and $I \times \eta_2$. Therefore,

$$U_{K(J,\eta)} = ((I \times \Sigma(K_1)) - (I \times \nu(\eta_1)) - (I \times \nu(\eta_2))) \cup \big((I \times S^3) - \nu(C)\big) \cup \big((I \times S^3) - \nu(C)\big)\,.$$

In this identification, the longitude for each copy of C is glued to the corresponding meridian of η_1 or η_2. Since the bottom of C is an unknot, this means that the group $\pi_1(U_{K(J,\eta)})$, computed via van Kampen's theorem, is normally generated by $\pi_1(\Sigma(K_1))$ and two copies of $\pi_1((I \times S^3) - \nu(C))$. But the meridians of the two copies of C, which normally generate $\pi_1((I \times S^3) - \nu(C))$, are the longitudes of $I \times \eta_1$ and $I \times \eta_2$. Since these are in $\pi_1(\Sigma(K_1))$, it follows that $\pi_1(\Sigma(K_1))$ normally generates $\pi_1(U_{K(J,\eta)})$.

One can also construct concordances to which Corollary 8.5 would apply by replacing a tangle in K_1 with one that is concordant to it; see Kirby–Lickorish [**34**] and Bleiler [**8**]. As we mentioned in the introduction, the corks are usually detected with the help of an effective embedding. A good example illustrating this point would be the corks constructed in [**5**] using a similar trick with invertible homology cobordisms. However, this is not how the corks in Corollary 8.5 are detected: there does not seem to exist an obvious effective embedding that would help detect them.

8.3. A re-gluing formula. The above calculation of the induced action of τ on monopole Floer homology allows us to determine the effect of cutting and gluing along the homology sphere Y_1 via τ in a more general situation.

Theorem 8.6. *Let Y_1 be the manifold with involution τ shown in Figure 6 (b), and X a smooth closed oriented 4-manifold with $b_2^+(X) > 1$ decomposed as $X = X_1 \cup X_2$ with $b_2^+(X_2) \geqslant 1$ and $\partial X_2 = Y_1$. Let X^τ be the manifold obtained by cutting X open along Y_1 and regluing using τ. Then*

$$SW(X, s) = (-1)^{b_1(X_1) + b_2^+(X_1)} SW(X^\tau, s^\tau).$$

Proof of Theorem 8.6. We wish to apply the gluing formula of Section 8.1 to $Y = -Y_1$ (note that the orientation convention for Y_1 in the above theorem is opposite of that in Section 8.1). The key to doing that are the following two observations:

(1) Write $M_i = X_i - \operatorname{int}(B^4)$ then the absolute $\mathbb{Z}/2$ grading of $\widehat{HM}^*(M_1, s_1)(\hat{1})$ is equal to $b_1(X_1) + b_2^+(X_1) + 1 \pmod 2$.

(2) The isomorphisms (44) and the formula (45) imply that τ^* acts as identity on $\widehat{HM}_{\text{odd}}$ $(-Y_1)$ and minus identity on $\widehat{HM}_{\text{even}}(-Y_1) = HM_{\text{red}}(-Y_1)$.

Now, if $b_2^+(X_2) > 1$, the result follows from Proposition 8.2. If $b_2^+(X_2) = 1$ then both manifolds X_1 and X_2 in the splitting $X = X_1 \cup X_2$ have positive b_2^+ and the result follows from the pairing formula [**37**, Equation 3.22]. q.e.d.

Corollary 8.7. *In the situation of Theorem 8.6, twisting the manifold X along Y_1 via the involution τ can only kill the Seiberg–Witten invariant of X when the piece bounded by Y_1 is negative definite.*

In particular, if $X_1 = -W_1$, the cork twist cannot change the Seiberg–Witten invariant of X. This is perhaps more readily seen via the blow-up formula for the Seiberg–Witten invariants, using the fact (which is implicit in [**4**]) that the cork twist extends over $W_1 \,\#\, \mathbb{C}\mathrm{P}^2$.

9. Knot concordance and Khovanov homology thin knots

In this section, we prove the results in Section 1.2.3 from the introduction. We start with the following lemma, which is presumably well known.

Lemma 9.1. *Let L be a ramifiable link in the 3-sphere that is Khovanov homology thin over $\mathbb{F}_2$. Then $\Sigma(L)$ is a monopole L-space over the rationals, that is,*

$$HM^{\text{red}}(\Sigma(L);\mathbb{Q}) \;=\; 0.$$

Proof. Let us fix an orientation on the link L. According to Bloom [**9**], there is a spectral sequence whose E_2 page is $\widetilde{Kh}\,(L;\mathbb{F}_2)$ and which converges to $\widetilde{HM}(-\Sigma(L);\mathbb{F}_2)$ (we refer to [**9**, Section 8] for the definition of this tilde-version of monopole Floer homology). In particular, this implies that

$$\dim_{\mathbb{F}_2}(\widetilde{Kh}\,(L;\mathbb{F}_2)) \;\geqslant\; \dim_{\mathbb{F}_2}(\widetilde{HM}(-\Sigma(L);\mathbb{F}_2)). \tag{48}$$

Recall from [**32**] that the reduced Khovanov cohomology categorifies the Jones polynomial J_L. Together with the Khovanov homology thin condition, this implies that

$$\dim_{\mathbb{F}_2}(\widetilde{Kh}\,(L;\mathbb{F}_2)) = |J_L(-1)| = |H_1(\Sigma(L);\mathbb{Z})|. \tag{49}$$

Combining (48) and (49) with the universal coefficient theorem, we obtain

$$|H_1(\Sigma(L);\mathbb{Z})| \;\geqslant\; \dim_{\mathbb{F}_2}(\widetilde{HM}(-\Sigma(L);\mathbb{F}_2)) \;\geqslant\; \dim_{\mathbb{Q}}(\widetilde{HM}(-\Sigma(L);\mathbb{Q})). \tag{50}$$

By the definition of $\widetilde{HM}$, one has

$$\dim_{\mathbb{Q}}(\widetilde{HM}(-\Sigma(L),\mathfrak{s};\mathbb{Q})) \geqslant 1$$

for any spin^c structure $\mathfrak{s}$, with equality holding if and only if $HM^{\text{red}}(-\Sigma(L),\mathfrak{s};\mathbb{Q}) = 0$. Therefore, (50) implies that $-\Sigma(L)$ is a monopole L–space over the rationals. By the duality in the reduced monopole homology, $\Sigma(L)$ is a monopole L–space over the rationals as well. q.e.d.

Proof of Corollary E. This is a direct consequence of Lemma 9.1 and Theorem A. q.e.d.

Proof of Corollary F. In the case of a knot K, the Murasugi signature of K equals its usual signature, and the double branch cover $\Sigma(K)$ has a unique spin structure. Therefore, Theorem A reduces to the statement that

$$L(K) \;=\; \frac{1}{8}\,\sigma(K) \;-\; h(\Sigma(K)),$$

where $\sigma(K)$ and $h(\Sigma(K))$ are additive concordance invariants. This makes $L(K)$ into an additive concordance invariant. This invariant is non-trivial: for example, if K is the right handed $(3,7)$-torus knot, $\sigma(K) = -8$ and $h(\Sigma(K)) = h(\Sigma(2,3,7)) = 0$, hence $L(K) = -1$. q.e.d.

Proof of Corollary G. This is immediate from Corollary F. q.e.d.

10. Monopole Contact invariant

In this section we will prove Theorem H. Consider the Brieskorn homology sphere $Y = \Sigma(2,3,7)$, along with the involution $\tau_1(x,y,z) = (\bar{x},\bar{y},\bar{z})$ described in Remark 8.4. We give Y the canonical orientation as a link of singularity. By combining the calculation of Heegaard Floer homology in [**58**] with the identification between Heegaard Floer and monopole Floer homology, we obtain

$$\widetilde{HM}(Y;\mathbb{Z}) = HF^+(Y;\mathbb{Z}) = \mathcal{T}_{(0)} \oplus \mathbb{Z}_{(-1)} \quad \text{and} \quad HM^{\text{red}}(Y) = \widetilde{HM}_{(-1)}(Y;\mathbb{Z}) = \mathbb{Z}_{(-1)}.$$

To determine the induced action of τ on $HM^{\text{red}}(Y)$ we will use the fact that τ makes Y into a double branched cover of the 3-sphere with branch set the $(2,-3,-7)$ pretzel knot K, pictured as the Montesinos knot $K(2,-3,-7)$ in Figure 4. Either by a direct calculation, or by using the formula of [**66**, Section 7] for the knot signature in terms of the $\bar{\mu}$–invariant, we obtain

$$\sigma(K) = 8\,\bar{\mu}(\Sigma(2,3,7)) = 8.$$

Theorem A then tells us that

$$\mathrm{Lef}(\tau_*) \;=\; \frac{1}{8}\,\sigma(K) - h(Y) \;=\; 1$$

and therefore $\tau_* : HM^{\mathrm{red}}(Y) \to HM^{\mathrm{red}}(Y)$ is negative identity.

According to [**69**] (see also [**49**, Theorem 1.6]), the manifold $-Y$ admits a unique (up to isotopy) tight contact structure ξ, which is Stein fillable and has Gompf invariant $\theta = 2$. By applying the involution τ, we obtain another contact structure $\tau^*(\xi)$. Since $\tau^*(\xi)$ is also tight, it must be isotopic to ξ.

Now, suppose there is a canonical choice of the contact element $\tilde{\psi}(-Y,\xi) \in \widetilde{HM}(Y;\mathbb{Z})$. This element is non-zero and it is supported in degree $-(\theta+2)/4 = -1$. Since τ_* acts as negative identity on $HM^{\mathrm{red}}(Y) = \widetilde{HM}_{(-1)}(Y;\mathbb{Z})$, we have

$$\tilde{\psi}(-Y,\xi) \neq -\tilde{\psi}(-Y,\xi) = \tau_*(\tilde{\psi}(-Y,\xi)) = \tilde{\psi}(-Y,\tau_*(\xi)).$$

However, this contradicts the naturality of the contact invariant because $\tau_*(\xi)$ and ξ are isotopic.

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