SU(2)-CYCLIC SURGERIES ON KNOTS

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Abstract. A surgery on a knot in S^3 is called $SU(2)$-cyclic if it gives a manifold whose fundamental group has no noncyclic $SU(2)$ representations. Using holonomy perturbations on the Chern-Simons functional, we prove that two $SU(2)$-cyclic surgery coefficients $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ should satisfy $|p_1q_2 - p_2q_1| \leq |p_1| + |p_2|$. This is an analog of Culler-Gordon-Luecke-Shalen's cyclic surgery theorem.

1. Introduction

Definition 1.1. *A closed orientable 3-manifold M is called $SU(2)$-cyclic (or $SO(3)$-cyclic) if there exists no homomorphism $\rho : \pi_1(M) \to SU(2)$ (or $SO(3)$) with noncyclic image.*

Suppose $K \subset S^3$ is a knot. For $r \in \mathbb{Q}$, we denote the manifold obtained by doing r-surgery on K by $K(r)$.

Definition 1.2. *A surgery on K with coefficient r is called $SU(2)$-cyclic (or $SO(3)$-cyclic) if $K(r)$ is $SU(2)$-cyclic (or $SO(3)$-cyclic).*

Remark 1.3. *Notice that $H_1(K(r); \mathbb{Z})$ is a cyclic group. We see that an $SU(2)$ representation of $\pi_1(K(r))$ has noncyclic image if and only if it is irreducible.*

We have the following exact sequence:

$$0 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 0$$

It's easy to see that an $SO(3)$-cyclic surgery is always an $SU(2)$-cyclic surgery. Using some basic obstruction theory, we get:

Lemma 1.4. *If $r = \frac{p}{q}$ is an $SU(2)$-cyclic surgery with p odd, then r is an $SO(3)$-cyclic surgery.*

In [4], Kronheimer and Mrowka proved the following theorem:

Theorem 1.5 (Kronheimer, Mrowka 2003 [4])**.** *Any r-surgery on a nontrivial knot with surgery coefficient $|r| \leq 2$ is not $SU(2)$-cyclic.*

In particular, this theorem gave a proof for the Property-P Conjecture:

Corollary 1.6 (Kronheimer, Mrowka 2003 [4])**.** *A nontrivial surgery on a nontrivial knot does not give simply connected 3-manifold.*

Obviously, lens spaces are all $SU(2)$-cyclic and $SO(3)$-cyclic. Thus all cyclic surgeries (the surgeries which give lens spaces) are $SO(3)$-cyclic. Therefore, we have:

Example 1.7. *The* $pq - \frac{1}{r}$ $(r \in \mathbb{Z})$ *surgeries on the* (p,q)*-torus knot are cyclic and hence* $SO(3)$*-cyclic.*

Dunfield [3] gives the following example:

Example 1.8. *The* $18, \frac{37}{2}, 19$ *surgeries on the* $(-2,3,7)$*-pretzel knot are* $SO(3)$*-cyclic. The* $18, 19$ *surgeries give lens spaces, while* $K(\frac{37}{2})$ *is a graph manifold obtained by gluing the left-handed trefoil knot complement and the right-handed trefoil complement.*

Another related theorem is Culler-Gordon-Luecke-Shalen's cyclic surgery theorem (we only state the case for knot surgery):

Theorem 1.9 (Culler-Gordon-Luecke-Shalen [7])**.** *Suppose that* K *is not a torus knot and* r, s *are both cyclic surgeries, then* $\triangle(r,s) \leq 1$*.*

Here for two rational numbers $r = \frac{p_1}{q_1}$ and $s = \frac{p_2}{q_2}$, the distance $\triangle(r,s)$ is defined to be $|p_1q_2 - p_2q_1|$.

Since $\frac{1}{0}$-surgery is always cyclic, this theorem implies that when K is not a torus knot, r-surgery can be cyclic only if $r \in \mathbb{Z}$. Moreover, there are at most two such integers, and if there are two then they must be successive.

Although Example 1.8 shows that Theorem 1.9 is not true for $SU(2)$-cyclic or $SO(3)$-cyclic surgeries, we have the following analogous result, which is the main theorem of this paper.

Theorem 1.10. *Consider a nontrivial knot* $K \subset S^3$ *and two surgeries with coefficients* $r_1 = p_1/q_2$ *and* $r_2 = p_2/q_2$*. We have the following:*

- *If* r_1, r_2 *are both* $SU(2)$*-cyclic, then* $\triangle(r_1, r_2) \leq |p_1| + |p_2|$*.*
- *If* r_1, r_2 *are both* $SO(3)$*-cyclic, then* $2\triangle(r_1, r_2) \leq |p_1| + |p_2|$*.*

Combining this theorem with Lemma 1.4, we get the following corollaries.

Corollary 1.11. *Suppose* r_1, r_2 *are both* $SU(2)$*-cyclic. If* p_1 *is odd, then* $2\triangle(r_1, r_2) \leq 2|p_1| + |p_2|$*. If* p_1, p_2 *are both odd, then* $2\triangle(r_1, r_2) \leq |p_1| + |p_2|$*.*

Corollary 1.12. *If* r_1, r_2 *on* K *are both* $SO(3)$*-cyclic surgeries, then* $r_1r_2 > 0$*. If* r_1, r_2 *on* K *are both* $SU(2)$*-cyclic surgeries, then* $r_1r_2 > 0$ *unless* r_1 *and* r_2 *are both even integers.*

Corollary 1.13. *For a nontrivial surgery on a nontrivial amphichiral knot* K *with coefficient* r*, we have the following:*

- *It can never be* $SO(3)$*-cyclic.*
- *If it is* $SU(2)$*-cyclic, then* r *is an even integer and some* $\frac{r}{2}$*-th root of unity is a root of* Δ_K *(the Alexander polynomial of* K*).*

Remark 1.14. *Actually, we haven't found any examples of* $SU(2)$*-cyclic surgeries on an amphichiral knot. It would be interesting to know whether there exists such a surgery.*

We know that $\Delta_K(1) = \pm 1$ for any knot K while $\Phi_p(1) = p$ for any prime number p (Φ_p is the p-th cyclotomic polynomial). Therefor the Alexander polynomial Δ_K never has the p-th root of unity as its root. We get:

Example 1.15. *If p is prime, then the $2p$-surgery on an non-trivial amphichiral knot is not $SU(2)$-cyclic.*

Remark 1.16. *In [4], Kronheimer and Mrowka asked whether there exists $SU(2)$-cyclic surgery with coefficient 3 or 4. We see that there exist no such surgeries for nontrivial amphichiral knots.*

Using the criterion in Corollary 1.13, we checked the amphichiral knots with crossing number ≤ 10 and get:

Example 1.17. *All the nontrivial amphichiral knots with crossing number ≤ 10 except perhaps 8_{18} and 10_{99} in Rolfsen's knot table admit no $SU(2)$-cyclic surgeries. For 8_{18} and 10_{99}, we have no examples of $SU(2)$-cyclic surgeries.*

Corollary 1.18. *Given a nontrivial knot K and an integer q, there exist at most finitely many $p \in \mathbb{Z}$ such that $(p, q) = 1$ and the $\frac{p}{q}$-surgery on K is $SO(3)$-cyclic. For the $SU(2)$ case, the only possible exception is when $q = 1$ and infinitely many even p.*

In particular, any nontrivial knot admits only finitely many integer $SO(3)$-cyclic surgeries and only finite many odd $SU(2)$-cyclic surgeries.

The paper is organized as follows: in section 2, we prove a result about the boundary holonomy of the representations after reviewing some preliminaries and basic constructions related to holonomy perturbations. In section 3, we prove the main theorem and the corollaries.

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2. A result on the boundary holonomy of knot complement

Let ρ be an $SU(2)$ representation of $\pi_1(S^3 - N(K))$ ($N(K)$ denotes the open tubular neighborhood of K). We denote by $m, l \in \pi_1(S^3 - N(K))$ the meridian and the longitude of K respectively. Since m, l commute with each other, after a conjugation in $SU(2)$, we can assume that:

$$\rho(m) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ \rho(l) = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix}.$$

We say that the points $\pm(\theta, \eta)$ are the boundary holonomies of ρ. We denote the torus $(\mathbb{R}/2\pi\mathbb{Z}) \oplus (\mathbb{R}/2\pi\mathbb{Z})$ by $\mathbb{T}$. The main result of this section is the following proposition:

Proposition 2.1. *Let K be a nontrivial knot. Suppose γ is a closed curve on $\mathbb{T}$ parameterized as:*

$$(x - g_1 \circ g_2(x), g_2(x) + \pi), \ x \in [-\pi, \pi]$$

where g_1, g_2 are any smooth periodic odd functions of period 2π. Then the image of γ contains the boundary holonomy of some $SU(2)$ representation of $\pi_1(S^3 - N(K))$.

The proof of Theorem 1.10 begins with assuming $\frac{p}{q}$ and $\frac{r}{s}$ are two $SU(2)$-cyclic surgery slopes. This implies that there is a certain pair of curves on the torus T with slopes $\frac{p}{q}$ and $\frac{r}{s}$ whose union S cannot contain the boundary holonomy of any irreducible $SU(2)$ representation. After small modifications near the lines $\{\eta = 2\pi\mathbb{Z}\}$, we may further assume S does not contain the boundary holonomy of any $SU(2)$ representation. Since the set of all such boundary holonomies is a compact set, there is a small neighborhood of S which avoid all such boundary holonomies. The geometric and combinatorial arguments in section 3 then shows that if p, q, r, s do not satisfy the condition in the theorem, then arbitrary small neighborhood of S contains a curve which can be parameterized as in the above proposition, giving a contradiction.

Before giving the proof of Proposition 2.1, we review some background materials that will be useful in our discussion. Although most of them can be found in [4], [8] and [12], we still include some details here for completeness.

Consider the closed manifold $K(0)$. We have $b_1(K(0)) = 1$. Let E be the rank 2 unitary bundle over $K(0)$ with $c_1(E)$ the Poincaré dual of the meridian m and let $\mathfrak{g}_E$ be the bundle whose sections are traceless, skew-hermitian endomorphisms of E. We denote by $\mathcal{A}$ the affine space of $SO(3)$ connections of $\mathfrak{g}_E$. After fixing a reference connection $A_0 \in \mathcal{A}$, we can define a functional $CS : \mathcal{A} \to \mathbb{R}$, which is call the Chern-Simon functional. Although the explicit formula of this functional will not be used in our discussion, we still give it here for completeness:

$$CS(A) = \frac{1}{4}\int_{K(0)} Tr(2\omega \wedge F_{A_0} + \omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega).$$

Here $\omega \in i\Omega^1(\mathfrak{g}_E)$ equals $A - A_0$ and F_{A_0} is the curvature of A_0.

The critical points of the Chern-Simons functional are the flat connections. Floer introduced the holonomy perturbations as follows. Take a function $\phi : SU(2) \to \mathbb{R}$ which is invariant under conjugation. Then it is uniquely determined by the even, $2\pi-$periodic function:

$$f(x) := \phi \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \tag{1}$$

Let Σ be a compact 2-manifold with boundary. Consider an embedding $\Sigma \times S^1$ in $K(0)$ such that $\mathfrak{g}_E$ is trivial over it. Fix a trivialization of $\mathfrak{g}_E$ over $\Sigma \times S^1$ and take a 2-form μ which is supported in the interior of Σ with integral 1. Using the trivialization, we can lift A to a connection $\bar{A}$ on the trivialized $SU(2)$ bundle $\widetilde{P}$ over $\Sigma \times S^1$. We consider the functional:

$$\Phi : \mathcal{A} \to \mathbb{R}$$

$$\Phi(A) := \int_{p\in\Sigma} \phi(\mathrm{Hol}_{\{p\}\times S^1}(\bar{A}))\mu(p) \tag{2}$$

Here $\mathrm{Hol}_{\{p\}\times S^1}$ is the holonomy along $\{p\} \times S^1$.

We decompose $K(0)$ into three parts: $(S^3 - N(K)) \underset{\{0\}\times l\times m}{\cup} ([0,1] \times l \times m) \underset{\{1\}\times l\times m}{\cup} (D^2 \times m)$. We have meridians and longitudes on both side of the thicken torus. Denote them by m_0, l_0, m_1, l_1 respectively. Note that we should be careful that m_0 is the meridian of the knot complement but m_1 the longitude of the attached solid torus. Also, l_0 is the longitude of the knot complement but l_1 is the meridian of the attached solid torus.

For our purpose, we will consider two types of perturbations:

- Set $\Sigma \cong D^2$ and $i_1(D^2 \times S^1) = (D^2 \times m) \subset K(0)$. That means we use the holonomy along m to do the perturbation. We denote this perturbation by Φ_1
- Set $\Sigma \cong m \times [0,1]$ (Σ is an annulus) and $i_2(\Sigma \times S^1) = (m \times [0,1]) \times l \subset K(0)$. That means we embed a thickened torus and use the holonomy along l to do the perturbation. We denote this perturbation by Φ_2.

We choose a trivialization of $\mathfrak{g}_E$ over $(D^2 \times m) \cup (m \times [0,1] \times l)$ and use it to lift the connection A to a $SU(2)$-connection $\overline{A}$ on $\widetilde{P}$. Now use formula (2) and consider the perturbed Chern-Simons functional $\widehat{CS} = CS + \Phi_1 + \Phi_2 : \mathcal{A} \to \mathbb{R}$.

The following theorem, which plays a central role in our discussion, is proved in [6]:

Theorem 2.2 (Kronheimer, Mrowka [6]). *If K is a nontrivial knot, then for any holonomy perturbation, the perturbed Chern-Simons functional $\widehat{CS}$ over $K(0)$ always has at least one critical point.*

Remark 2.3. *We mention that the proof of this theorem is highly nontrivial. It combines Gabai's result about taut foliation in* [2]*, Eliashberg-Thurston's theorem about symplectic filling in* [14] *and* [15]*, Taubes's result about the Seiberg-Witten invariants of the symplectic four-manifold in* [10]*, Feehan and Leness's work about Witten's conjecture in* [9] *and Kronheimer-Mrowka's work about the refinement of Eliashberg-Thurston's theorem in* [6]*. However, in our discussion, we will use this theorem directly without going into any part of its proof.*

The critical points of $\widehat{CS}$ is completely determined in the following lemma:

Lemma 2.4. *If $A \in \mathcal{A}$ is a critical point of $\widehat{CS}$, then:*

- *A is flat on $S^3 - N(K) \subset K(0)$.*
- *We can choose a suitable trivialization of the $\mathfrak{g}_E|_{(D^2 \times m) \cup (m \times [0,1] \times l)}$ such that the lifted connection $\bar{A}$ satisfies:*
  $Hol_{m_0}(\overline{A}) = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix}, Hol_{m_1}(\overline{A}) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}, Hol_{l_0}(\overline{A}) = \begin{pmatrix} e^{i\eta_0} & 0 \\ 0 & e^{-i\eta_0} \end{pmatrix}$
  and $Hol_{l_1}(\overline{A}) = \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & e^{-i\eta_1} \end{pmatrix}$ *with* $\eta_0 = \eta_1 = -f_2'(\theta_1)$ *and* $\theta_0 - \theta_1 = -f_1'(\eta_0)$.

Remark 2.5. *Recall that we chose $\phi_i : SU(2) \to \mathbb{R}$ to define the perturbation Φ_i ($i = 1, 2$), which gives us $f_i : \mathbb{R} \to \mathbb{R}$ by formula (1).*

Proof of Lemma 2.4. By Lemma 4 in [8] and Lemma 2.2 in [12], A is flat on $S^3 - N(K) \subset K(0)$ and near $(m \times l \times \{0\}) \cup (m \times l \times \{1\})$. Moreover, we can choose a suitable trivialization of $\widetilde{P}$ such that $Hol_{m_0}(\overline{A}) = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix}, Hol_{m_1}(\overline{A}) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}, Hol_{l_0}(\overline{A}) = \begin{pmatrix} e^{i\eta_0} & 0 \\ 0 & e^{-i\eta_0} \end{pmatrix}, Hol_{l_1}(\overline{A}) = \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & e^{-i\eta_1} \end{pmatrix}$ and $\theta_0 - \theta_1 = -f_1'(\eta_0) + 2\mathbb{Z}\pi$. Also, we can choose another trivialization of $\widetilde{P}$ such that $Hol_{m_1}(\overline{A}) = \begin{pmatrix} e^{i\theta_1'} & 0 \\ 0 & e^{-i\theta_1'} \end{pmatrix}, Hol_{l_1}(\overline{A}) = \begin{pmatrix} e^{i\eta_1'} & 0 \\ 0 & e^{-i\eta_1'} \end{pmatrix}$

and $\eta_1' = -f_2'(\theta_1') + 2\mathbb{Z}\pi$. Since different trivialzations give the same holonomy modulo conjugation. We have $(\theta_1', \eta_1') = \pm(\theta_1, \eta_1)$. Since f_2' is an odd function, we have $\eta_1 = -f_2'(\theta_1) + 2\mathbb{Z}\pi$. $\square$

Now suppose A is a critical point. Because $\mathfrak{g}_E$ is trivial over $\pi_1(S^3 - N(K))$, we fix a trivialization of $\mathfrak{g}_E|_{S^3-N(K)}$. Using this trivialization, we lift the connection A to a $SU(2)$-connection $\widetilde{A}$ over $S^3 - N(K)$. By taking the holonomy of $\widetilde{A}$, we get a representation $\rho : \pi_1(S^3 - N(K)) \to SU(2)$.

Definition 2.6. *We define the subset R_K of $\mathbb{T}$ as:*

$$\{(\theta, \eta) | (\theta, \eta) \textit{ is the boundary holonomy of some representation } \rho\}$$

By the well-know relation between flat connections and representations of the fundamental group, this set can also be defined as:

$$\{(\theta, \eta) | \exists \textit{ flat connection } \widetilde{A} \textit{ over } S^3 - N(K) \textit{ s.t. } Hol_m(\widetilde{A}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, Hol_l(\widetilde{A}) = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix}\}$$

We summarize the properties of R_K in the following lemma. Some of them are proved in [4]. But since we change the statement a little, we give the proof here for completeness.

Lemma 2.7. *R_K has the following properties:*

- *1) Any point in R_K off the line $\{\eta = 2\pi\mathbb{Z}\}$ gives some irreducible representation.*
- *2) R_K is a closed subset of $\mathbb{T}$.*
- *3) R_K is invariant under the translation $(\theta, \eta) \to (\theta + \pi, \eta)$.*
- *4) $R_K \cap \{\theta = k\pi\} = (k\pi, 2k'\pi), (k, k' \in \mathbb{Z})$.*
- *5) $\exists \epsilon > 0$ such that $\forall k \in \mathbb{Z}, R_K \cap \{\theta \in [k\pi - \epsilon, k\pi + \epsilon]\} \cap \{\eta \neq 2\mathbb{Z}\pi\} = \emptyset$.*

Proof. 1) Any point in R_K gives a representation $\rho : \pi_1(S^3 - N(K)) \to SU(2)$. If ρ is reducible, then ρ factors through $H_1(S^3 - N(K); \mathbb{Z})$, which implies that $\rho(l) = 1 \in SU(2)$ and $\eta \in 2\pi\mathbb{Z}$.

2) R_K is closed because $\pi_1(S^3 - K)$ is finitely generated and $SU(2)$ is compact.

3) We have a map $\rho_0 : \pi_1(S^3 - K) \to H_1(S^3 - K; \mathbb{Z}) \to \mathbb{Z}_2 \subset SU(2)$ with $\rho_0(m) = -1 \in SU(2)$ and $\rho_0(l) = 1 \in SU(2)$. For any homomorphism $\rho : \pi_1(S^3 - N(K)) \to SU(2)$, we can multiply it by ρ_0 to get another representation ρ' such that $\rho'(l) = \rho(l)$ and $\rho'(m) = -\rho(m)$. By the definition of R_K, this implies 3).

4) Suppose ρ is given by a point with $\theta = 0$. Then $\rho(m) = 1 \in SU(2)$ and ρ factors through $\pi_1(S^3)$, which is a trivial. We get $\rho(l) = 1$ and $\eta = 2k'\pi$. For the case $\theta = \pi$, we use 3).

5) Look at a small neighborhood U of $(0, 0) \in R_K$ in $\mathbb{T}$. The point $(0, 0)$ is given by the restriction of the trivial representation ρ_1. The deformations of ρ_1 are governed by $H^1(\pi_1(S^3 - K); \mathbb{R}^3) \cong \mathbb{R}^3$. But every vector in this $\mathbb{R}^3$ can be realized by the some reducible representation. We see that in a small neighborhood of ρ_1, all the representations are reducible. Thus $U \cap R_K \cap \{\eta \neq 2\mathbb{Z}\pi\} = \emptyset$ if U is small enough. Use 4) and the compactness of R_K, we prove 5) for the case k is even. Then we use 3) to prove the case of odd k. $\square$

Lemma 2.8. *If A is a critical point of the perturbed Chern-Simons functional, then $(\theta_0, \eta_0 + \pi) \in R_K$, where θ_0 and η_0 are defined in Lemma 2.4.*

Proof. By Lemma 2.4, A is flat on $S^3 - N(K)$. After choosing a trivialization of $\mathfrak{g}_E|_{S^3-N(K)}$, we can lift A to a flat $SU(2)$ connection $\tilde{A}$, whose holonomy gives a point of R_K. Recall that to define θ_0, η_0 in Lemma 2.4, we also choose a trivialization of $\mathfrak{g}_E|_{(D^2\times m)\cup(m\times[0,1]\times l)}$ and lift A to a connection $\bar{A}$. Because the bundle $\mathfrak{g}_E$ is nontrivial, these two trivializations do not agree with each other on the common boundary $\{0\} \times l \times m$. Actually, they differ each other by a map $h : \{0\} \times l \times m \to SO(3)$ with $h_*(l_0) = 1 \in \mathbb{Z}_2 \cong \pi_1(SO(3))$ and $h_*(m_0) = 0 \in \pi_1(SO(3))$. Therefore, we see that $(\mathrm{Hol}_{m_0}(\widetilde{A}), \mathrm{Hol}_{l_0}(\widetilde{A})) \in SU(2) \times SU(2)$ is conjugate with $(\mathrm{Hol}_{m_0}(\overline{A}), -\mathrm{Hol}_{l_0}(\overline{A}))$. After a change of the trivialization of $\mathfrak{g}_E|_{S^3-N(K)}$, we can assume that $(\mathrm{Hol}_{m_0}(\overline{A}), -\mathrm{Hol}_{l_0}(\overline{A})) = (\mathrm{Hol}_{m_0}(\widetilde{A}), \mathrm{Hol}_{l_0}(\widetilde{A}))$. By the second description of R_K, we have $(\theta_0, \eta_0 + \pi) \in R_K$. □

Now we start the proof of Proposition 2.1:

Proof. We can find even, 2π-periodic functions f_i $(i = 1, 2)$ such that $f_1'(x) = g_1(x)$ and $f_2'(x) = -g_2(x)$ and use them to define the holonomy perturbations Φ_1, Φ_2. By Theorem 2.2, the perturbed Chern-Simons functional $\widehat{CS}$ has at least one critical point. Let θ_i, η_i $(i = 0, 1)$ be the numbers in Lemma 2.4 corresponding to this critical point. Then we have $\eta_0 = g_2(\theta_1)$ and $\theta_0 = \theta_1 - g_1 \circ g_2(\theta_1)$. Therefore, the image of the loop γ contains the point $(\theta_0, \eta_0 + \pi)$, which is the boundary holonomy of some representation ρ by Lemma 2.8. □

3. Proof of the Main Theorem and its Corollaries

3.1. **Proof of the main theorem.** Now suppose $K \subset S^3$ is a nontrivial knot. Denote the set $R_K \cap \{\eta \neq 2\mathbb{Z}\pi\}$ by R_K^*. For $r = \frac{p}{q}$, we define the following sets:

$$S(r) := \{(\theta, \eta) | (p\theta + q\eta) \in 2\mathbb{Z}\pi \text{ or } (p\theta + p\pi + q\eta) \in 2\mathbb{Z}\pi\}$$

$$\widehat{S}(r) := \{(\theta, \eta) | (p\theta + q\eta) \in \mathbb{Z}\pi\}$$

Notice that when p is odd, we have $\widehat{S}(r) = S(r)$.

Lemma 3.1. *If r is an $SU(2)$-cyclic surgery, then $R_K^* \cap S(r) = \emptyset$. If r is an $SO(3)$-cyclic surgery, then $R_K^* \cap \widehat{S}(r) = \emptyset$.*

Proof. If $(\theta, \eta) \in R_K^*$ satisfies $p\theta + q\eta \in 2\mathbb{Z}\pi$, then it gives a representation $\rho : \pi_1(S^3 - N(K)) \to SU(2)$ with $\rho(pm + ql) = 1 \in SU(2)$. Thus ρ factors through $\pi_1(K(r))$. By (1) of Lemma 2.7, ρ is noncyclic. This is a contradiction with our assumption that r is a $SU(2)$-cyclic surgery. We see that $R_K^* \cap \{(\theta, \eta) | (p\theta + q\eta) \in 2\mathbb{Z}\pi\} = \emptyset$. By (3) of Lemma2.7, we also have $R_K^* \cap \{(\theta, \eta) | (p\theta + p\pi + q\eta) \in 2\mathbb{Z}\pi\} = \emptyset$. We have proved the first assertion. The second assertion can be proved similarly. □

Since we are considering the subsets of $\mathbb{T}$, it will be convenient to fix a region $W = \{(\theta, \eta) | \theta \in (-\infty, \infty), \eta \in [0, 2\pi]\} \subset \mathbb{R}^2$ and define W^* to be $\{(\theta, \eta) | \theta \in (-\infty, \infty), \eta \in (0, 2\pi)\}$. We can work in W and W^* and then project to $\mathbb{T}$.

For two different numbers $r_1 = \frac{p_1}{q_1}, r_2 = \frac{p_2}{q_2}$. We define another two numbers:

$$d_1(r_1,r_2) = \begin{cases} \frac{2\pi|p_1|}{\Delta(r_1,r_2)} & \text{if } p_2 \text{ is even} \\ \frac{\pi|p_1|}{\Delta(r_1,r_2)} & \text{if } p_2 \text{ is odd} \end{cases} ; d_2(r_1,r_2) = \begin{cases} \frac{2\pi|p_2|}{\Delta(r_1,r_2)} & \text{if } p_1 \text{ is even} \\ \frac{\pi|p_2|}{\Delta(r_1,r_2)} & \text{if } p_1 \text{ is odd} \end{cases}$$

The intersections $S(r_i)\cap W^*$ are just line segments with slope $-r_i$ and $S(r_1)\cap S(r_2)\cap W^*$ consists of isolated points. We say two intersection points in $S(r_1)\cap S(r_2)\cap W^*$ are adjacent in $S(r_i)$ $(i = 1, 2)$ if they lie in the same component of $S(r_i) \cap W^*$ and there is no other intersection point between them. We define two intersection points in $\widehat{S}(r_1) \cap \widehat{S}(r_2) \cap W^*$ to be adjacent in $\widehat{S}(r_i)$ in a similar way.

The following lemma is easy to prove:

Lemma 3.2. *(1) If two intersection points $(\theta, \eta), (\theta', \eta') \in S(r_1)\cap S(r_2)\cap W^*$ are adjacent in $S(r_i)$, then $|\eta - \eta'| = d_i(r_1, r_2)$ $(i = 1, 2)$.*

(2) If two intersection points $(\theta, \eta), (\theta', \eta') \in \widehat{S}(r_1) \cap \widehat{S}(r_2) \cap W^*$ are adjacent in $\widehat{S}(r_i)$, then $|\eta - \eta'| = \frac{\pi|p_i|}{\Delta(r_1,r_2)}$ $(i = 1, 2)$.

(3) For $(\theta, \eta) \in S(r_1) \cap S(r_2) \cap W^*$, if $\eta > d_i(r_1, r_2)$, then we can find $(\theta', \eta') \in S(r_1) \cap S(r_2) \cap W^*$ such that they are adjacent in $S(r_i)$ and $\eta' < \eta$. If $\eta < 2\pi - d_i(r_1, r_2)$, then we can find $(\theta', \eta') \in S(r_1) \cap S(r_2) \cap W^*$ such that they are adjacent in $S(r_i)$ and $\eta' > \eta$.

(4) For $(\theta, \eta) \in \widehat{S}(r_1) \cap \widehat{S}(r_2) \cap W^*$, if $\eta > \frac{\pi|p_i|}{\Delta(r_1,r_2)}$, then we can find $(\theta', \eta') \in \widehat{S}(r_1) \cap \widehat{S}(r_2) \cap W^*$ such that they are adjacent in $\widehat{S}(r_i)$ and $\eta' < \eta$. If $\eta < 2\pi - \frac{\pi|p_i|}{\Delta(r_1,r_2)}$, then we can find $(\theta', \eta') \in \widehat{S}(r_1) \cap \widehat{S}(r_2) \cap W^*$ such that they are adjacent in $\widehat{S}(r_i)$ and $\eta' > \eta$.

Now we can start the proof of our main theorem:

Proof of Theorem 1.10. Let r_1, r_2 be two $SU(2)$-cyclic surgeries. Since the theorem is trivial when $r_1 = r_2$, we always assume that $r_1 \neq r_2$. By Theorem 1.5, we have $|r_i| > 2$. Moreover, when r_1 or r_2 equals $\frac{1}{0}$, the identities in the theorem and corollaries can be easily deduced from Theorem 1.5. Thus we can assume $p_i \neq 0$ and $q_i \neq 0$. Suppose $d_1(r_1, r_2) + d_2(r_1, r_2) < 2\pi$. By Lemma 3.1 and (4) of Lemma 2.7, we have $R^*_K \cap (S(r_1) \bigcup S(r_2) \bigcup \{\theta = k\pi\}) = \emptyset$. We will construct a piecewise linear path $L : [-1, 1] \to W$ such that $Im(L) \subset S(r_1) \bigcup S(r_2) \bigcup \{\theta = k\pi\}$. There are two cases:

(1) Suppose $r_1 < -2 < 2 < r_2$. Let $L(0) = (0, \pi)$. Then as t increases, L first goes up along the line $\theta = 0$ to $(0, 2\pi)$. Since $(0, 2\pi) \in S(r_2)$, L can go down along $S(r_2)$ to the lowest intersection point $(\theta_1, \eta_1) \in S(r_1)\cap S(r_2)\cap W^*$ in this component. By (3) of Lemma 3.2, we have $\eta_1 \leq d_2(r_1, r_2)$, which also implies $\eta_1 < 2\pi - d_1(r_2, r_2)$ By our assumption.

Again by Lemma 3.2, (θ_1, η_1) is not the highest intersection point in the component of $S(r_1) \cap W^*$ containing it. Thus L can go along $S(r_1)$ to the highest intersection point. Notice that this point is still in W^*. After that, L again goes along $S(r_2)$ to the lowest intersection point. Repeat this procedure until L hits the line $\theta = \pi$. Then L goes along $\theta = \pi$ to the point (π, π). We have defined $L(t)$ for $t \in [0, 1]$. Reflecting along $(0, \pi)$, we can define $L(t)$ for $t \in [-1, 0]$.

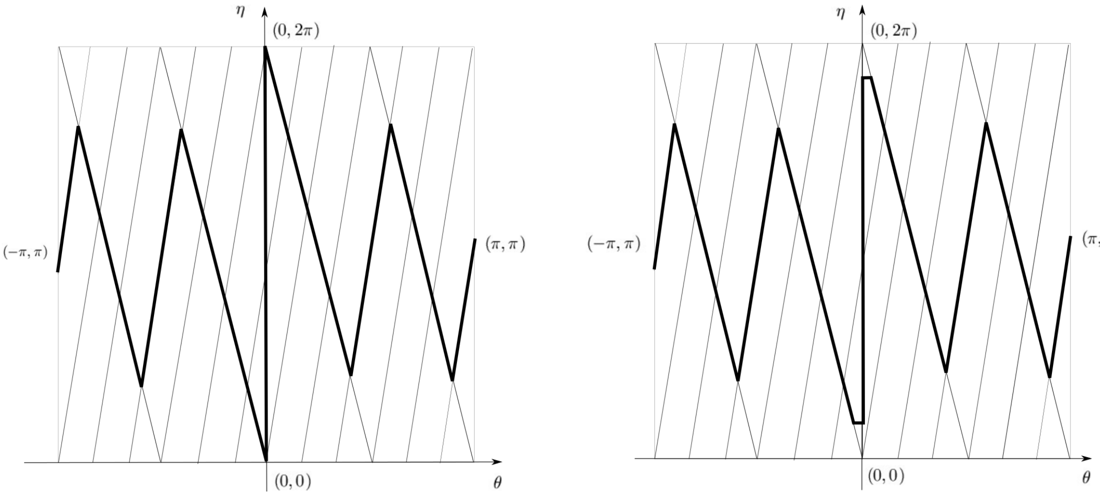

FIGURE 1. The path L (left) and the path $\widehat{L}$ (right) when $r_1 = -3, r_2 = 4$

(2) Suppose r_1, r_2 are of the same sign. We consider the case $2 < r_1 < r_2$ and the other case is similar. Set $L(0) = (0, \pi)$ and let L goes along $\theta = 0$ to $(0, 2\pi)$. Then L moves down alone $S(r_1)$ to the lowest intersection point in W^*. After that L moves along $S(r_2)$ to the highest intersection point. The difference from case (1) is that we repeat this procedure until L intersects the line $l \subset S(r_1)$ which passes through $(\pi, 0)$. It is easy to see that this happens before L hits $\theta = \pi$. Then L goes along l to $(\pi, 0)$ and then goes along the line $\theta = \pi$ to (π, π). By reflecting along the point $(0, \pi)$, we define $L(t)$ for any $t \in [-1, 1]$.

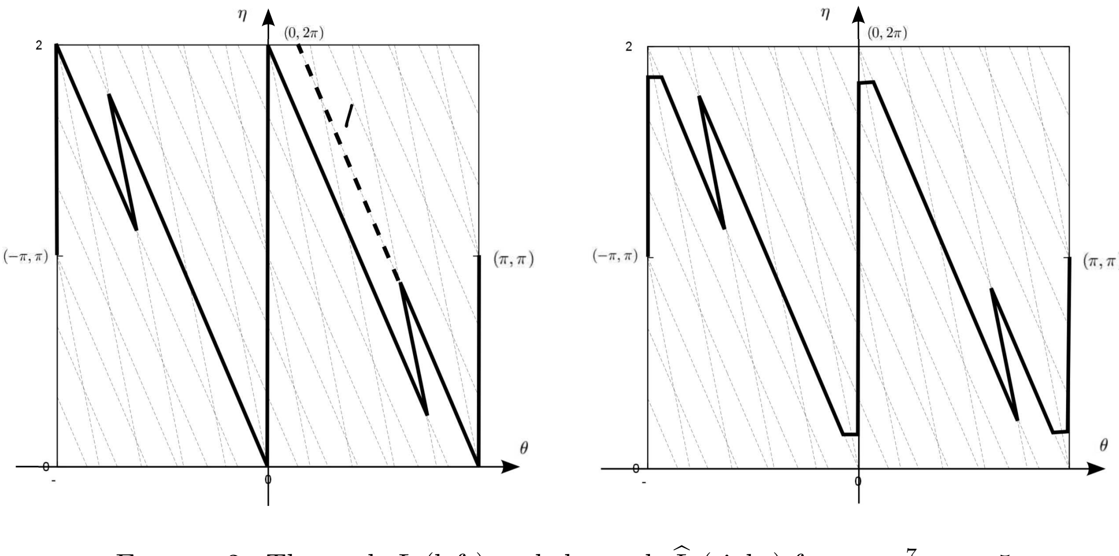

FIGURE 2. The path L (left) and the path $\widehat{L}$ (right) for $r_1 = \frac{7}{3}, r_2 = 5$

In both cases, the image of L, which we denote by $Im(L)$, is contained in $S(r_1) \cup S(r_2) \cup \{\theta = k\pi\}$. Thus $R_K^* \cap Im(L) = \emptyset$. Notice that $Im(L)$ intersects the line $\eta = 0$ and $\eta = 2\pi$ at $(0,0), (0,2\pi)$ in case (1) and at $(0,0), (0,2\pi), (\pi,0), (-\pi,2\pi)$ in case (2). We need to do small modification around these points. Take the point $(0,2\pi)$ for example. We choose a small neighborhood U of $(0,2\pi)$ and remove $Im(L) \cap U$. Then we replace it with a short horizontal line segment $\eta = 2\pi - \varepsilon$. After doing this modification, we get a map $\widehat{L} : [-1,1] \to W^*$, which still satisfies $Im(\widehat{L}) \cap R_K = \emptyset$ by 5) of Lemma 2.7. Moreover, we can require that $Im(\widehat{L})$ is symmetric under the reflection about $(0,\pi)$. Suppose $\widehat{L}(t) = (\theta(t), \eta(t))$. By the compactness of R_K, there exists a small neighborhood N of $Im(\widehat{L})$ such that $N \cap R_K = \emptyset$.

In case (1), the path $\widehat{L}$ "goes forward", which means that $\theta(t) \geq \theta(t')$ if $t \geq t'$. Since $(0,\pi), (\pm\pi,\pi) \in Im(\widehat{L})$ and $Im(\widehat{L})$ is symmetric under the reflection of $(0,\pi)$, there exists a smooth odd function g_2 with period 2π such that the loop $\gamma \subset \mathbb{T}$ defined as $\{(\theta,\eta) | \eta = g_2(\theta) + \pi\}$ is contained in N. Therefore, this loop does not intersect R_K. In other words, the image of γ does not contain the boundary holonomy of any $SU(2)$ representation of $\pi_1(S^3 - N(K))$. Setting the function g_2 as above and $g_1 \equiv 0$, we get a contradiction from Proposition 2.1.

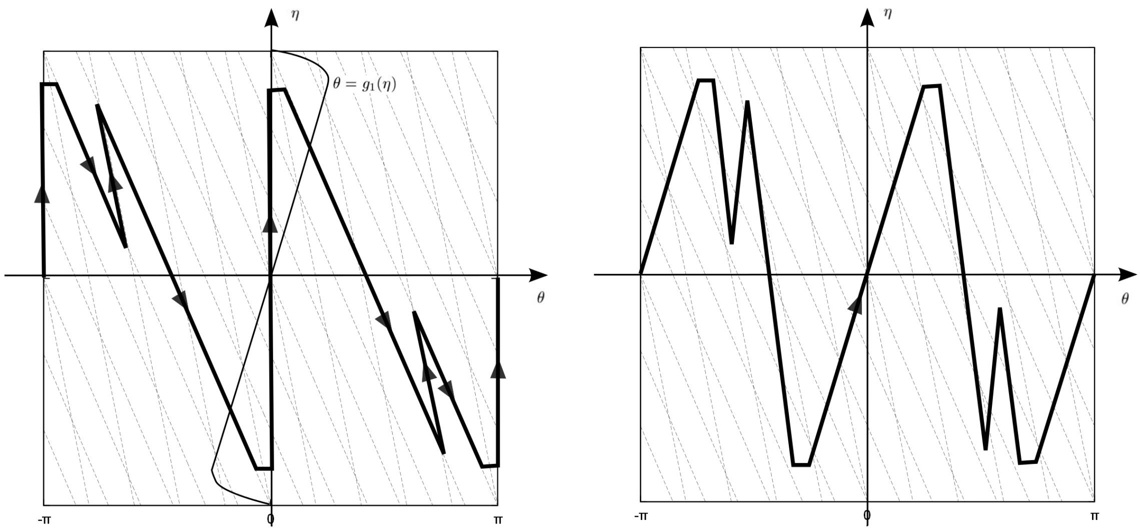

FIGURE 3. The path $\{(\theta, \eta - \pi) | (\theta,\eta) \in Im(\hat{L})\}$ (left) and the path $\{(\theta + g_1(\eta - \pi), \eta - \pi) | (\theta,\eta) \in Im(\hat{L})\}$ (right) for $r_1 = \frac{7}{3}, r_2 = 5$

In case (2), the path $\widehat{L}$ does not always go forward and our argument needs to be modified. Take the case $2 < r_1 < r_2$ for example (see Figure 3). By the construction of $\widehat{L}$, there exists a small $\epsilon > 0$ such that $Im(\widehat{L})$ is contained in the region $\epsilon < \eta < 2\pi - \epsilon$. Choose a number $r_0 \in (r_1, r_2)$. There exists an odd, 2π-periodic function g_1 such that $g_1(\eta) = \frac{\eta}{r_0}, \forall \eta \in [\epsilon, 2\pi - \epsilon]$.

Notice that the image of $\widehat{L}$ only consists of the following 4 types of segments:

- i) horizontal line that goes forward,
- ii) going down line with slope $-r_1$,
- iii) going up line with slope $-r_2$,
- iv) going up line with slope $+\infty$.

We see that the path $\{(\theta + g_1(\eta-\pi), \eta-\pi) | (\theta,\eta) \in Im(\hat{L})\}$ goes forward. Therefore, its neighborhood $\{(\theta + g_1(\eta-\pi), \eta-\pi) | (\theta,\eta) \in N\}$ contains the the graph of some odd, 2π-periodic function g_2. In other words, the image of the loop γ defined as:

$$\{(\theta,\eta) | g_2(\theta + g_1(\eta-\pi)) = \eta - \pi\}$$

is contained in N. Setting $x = \theta + g_1(\eta-\pi)$, we obtain a parametrization of γ as:

$$(x - g_1 \circ g_2(x), g_2(x) + \pi).$$

Notice that R_K does not intersect the image of γ since it is contained in N. This is a contradiction with Proposition 2.1 again.

We finish the proof of the $SU(2)$-cyclic case. The $SO(3)$-cyclic case can be proved similarly by considering $\widehat{S}(r_i)$ instead of $S(r_i)$. □

Actually, we have proved that if r_1, r_2 are both $SU(2)$-cyclic, then $d_1(r_1,r_2)+d_2(r_1,r_2) \geq 2\pi$. When p_i is odd, this gives the conclusions of Corollary 1.11. Corollary 1.12 and Corollary 1.18 are easy to prove using the main theorem.

3.2. **Relation with the Alexander polynomial.** In this subsection, we will give some relations between the $SU(2)$-cyclic surgeries and the Alexander polynomial and prove Corollary 1.13.

Suppose $d_1(r_1,r_2) + d_2(r_1,r_2) = 2\pi$ (for example $r_1 = -r_2 = 2k$) and r_1, r_2 are both $SU(2)$-cyclic. Let's try to repeat the argument as before. We focus on the case $r_2 < 0 < r_1$ and the other cases are similar. Consider $S(r_i) \subset \mathbb{T}$ $(i = 1, 2)$, then $R_K^* \cap S(r_i) = \emptyset$. We now construct $L : [-1, 1] \to W$. Set $L(0) = (0, \pi)$ and L goes upwards along $\theta = 0$ to $(0, 2\pi)$. Then L goes down along $S(r_2)$ the the lowest intersection point $(\theta_1, \eta_1) \in S(r_1) \cap S(r_2) \cap W$. After that, L goes up along $S(r_1)$ to the highest intersection point $(\theta_2, \eta_2) \in S(r_1) \cap S(r_2) \cap W$. Notice unlike the previous case, here we consider W instead of W^*. The reason is that it is now possible that the lowest intersection point in $(\theta_2, \eta_2) \in S(r_1) \cap S(r_2) \cap W^*$ is also the highest one (see Figure 4). We repeat this procedure and get $L : [-1, 1] \to W$. As before, we need to modify L to $\widehat{L}$ whose image is contained in W^*. The trouble appears: L may contain some points like $(\theta_0, 0)$ or $(\theta_0, 2\pi)$ with $\theta_0 \neq 0$ or $\pm\pi$. In general, we don't have the result like 5) of Lemma 2.8 which allows us to modify L near these points without intersecting R_K.

Consider the case of $(\theta_0, 0)$ (the other case is similar). Suppose that we can choose a small neighborhood U of $(\theta_0, 0)$ such that $R_K^* \cap U = \emptyset$. We just replace $Im(L) \cap U$ by some short, horizontal line $l \subset U \cap W^*$. If we can do this for every point in $Im(L) \cap (W \backslash W^*)$, we can construct $\widehat{L}$ and get the contradiction as before. If we can't do this for some point $(\theta_0, 0) \in S(r_1)$, then there exist a sequence $(\theta_n, \eta_n) \in R_K^*$ converging to $(\theta_0, 0)$ as $n \to \infty$. Each (θ_n, η_n) gives an irreducible representation $\rho_n : \pi_1(S^3 - N(K)) \to SU(2)$. It is easy to see that these representations are also irreducible as $SL(2, \mathbb{C})$ representations. By the compactness of $SU(2)$ representation variety, after passing to subsequence, ρ_n

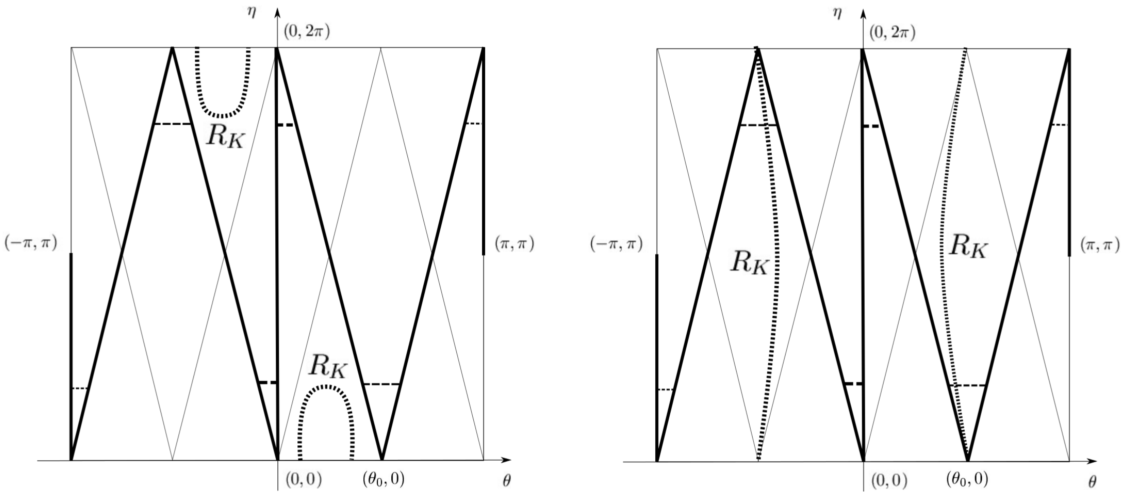

FIGURE 4. When $r_1 = 4, r_2 = -4$, we can modify L near $(\theta_0, 0)$ in the left picture but we can't modify L in the right picture.

converge to some ρ_∞ with boundary holonomy $(\theta_0, 0) \in S(r_1)$. Recall that we have a representation $\pi_1(S^3 - K) \to \pm 1 \to SU(2)$ such that m is mapped to -1. After multiplying ρ_∞ by this representation if necessary, we get a representation of ρ'_∞ of $\pi_1(S^3 - N(K))$ such that $\rho'_\infty(p_1 m + q_1 l) = 1$. Since r_1 is an $SU(2)$-cyclic surgery, this representation must be cyclic. In particular, this implies that ρ_∞ is cyclic. Thus we get a sequence of irreducible $SL(2,\mathbb{C})$ representations converging to a reducible $SL(2,\mathbb{C})$ representation ρ_∞ with $\rho_\infty(m) = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix}$. Now we apply the following proposition in [11]:

Proposition 3.3 ([11]). *Let M be the complement of a knot K in a homology 3-sphere. Suppose that ρ is a reducible representation of $\pi_1(M)$ such that the character of ρ lies on a component of $\chi(M)$ (the character variety of M) which also contains the character of an irreducible representation. Then $\rho(m)$ has an eigenvalue whose square is a root of Δ_K (the Alexander polynomial of K).*

Using this result, we see that $e^{2i\theta_0}$ is a root of Δ_K. Since $(\theta_0, 0) \in S(r_1)$, we see that Δ_K has a root which is a p_1-th root of unity for odd p_1 and $\frac{p_1}{2}$-th root of unity for even p_1.

By considering the intersection point $(\theta_0, 2\pi) \in Im(L) \cap \{\eta = 2\pi\}$, we can get the same conclusion for p_2. In particular, we get the following:

Proposition 3.4. *Suppose that $r_1 = \frac{p_1}{q_1}, r_2 = \frac{p_2}{q_2}$ are two $SU(2)$-cyclic surgeries with $d_1(r_1, r_2) + d_2(r_1, r_2) = 2\pi$ and p_1, p_2 even, then the Alexander polynomial of K has a root which is either a $\frac{p_1}{2}$-th or a $\frac{p_2}{2}$-th root of unity.*

Notice that if K is amphichiral, then the r-surgery is $SU(2)$-cyclic implies that the $-r$-surgery is also $SU(2)$-cyclic. By Corollary 1.12, we see that r is an even integer and

$d_1(r,-r)+d_2(r,-r)=2\pi$. Therefore, Corollary 1.13 is a straightforward consequence of the proposition above.

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