PIN(2)-EQUIVARIANT KO-THEORY AND INTERSECTION FORMS OF SPIN FOUR-MANIFOLDS

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Abstract. Using Seiberg-Witten Floer spectrum and Pin(2)-equivariant KO-theory, we prove new Furuta-type inequalities on the intersection forms of spin cobordisms between homology 3-spheres. As an application, we give explicit constrains on the intersection forms of spin 4-manifolds bounded by Brieskorn spheres $\pm\Sigma(2,3,6k\pm 1)$. Along the way, we also give an alternative proof of Furuta's improvement of 10/8-theorem for closed spin-4 manifolds.

1. Introduction

A natural question in 4-dimensional topology is: which nontrivial symmetric bilinear form can be realized as the intersection form of a closed, smooth, spin 4-manifold X. Such form should be even and unimodular. Therefore, it is indefinite by Donaldson's diagonalizability theorem [6, 7]. After changing the orientation of X if necessary, we can assume that the signature $\sigma(X)$ is non-positive. Then the intersection form can be decomposed as $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ with $p\geq 0, q>0$. Matsumoto's 11/8 conjecture [16] states that $b_2(X)\geq \frac{11}{8}|\sigma(X)|$, which can be rephrased as $q\geq \frac{3p}{2}$. An important result is the following 10/8 theorem of Furuta.

Theorem 1.1 (Furuta [12])**.** *Suppose X is an oriented closed spin four-manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p\geq 0, q>0$. Then we have $q\geq p+1$.*

Furuta's proof made use of the finite dimensional approximation of the Seiberg-Witten equations on closed 4-manifolds and Pin(2)-equivariant K-theory. By doing destabilization and appealing to a result by Stolz [32], Minami [21] and Schmidt [27] independently proved the following improvement:

Theorem 1.2 (Minami [21], Schmidt [27])**.** *Let X be a smooth, oriented, closed spin four-manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p\geq 0, q>0$. Then we have:*

$$q\geq\begin{cases} p+1, & p\equiv 0,2 \ mod\ 8\\ p+2, & p\equiv 4\ mod\ 8\\ p+3, & p\equiv 6\ mod\ 8.\end{cases}\tag{1}$$

Remark 1.3. *p is always an even integer by Rokhlin's theorem* [25]*.*

An interesting observation is that Schmidt's calculation in [27] about the Adams operations actually implies an alternative proof of the following further improvement, which was first proved by Furuta-Kametani [13]. We will give the proof in Section 3.

Theorem 1.4 (Furuta-Kametani [13]). *Let X be a smooth, oriented, closed spin four-manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q > 0$. Then $q \geq p+3$ when $p \equiv 0$ mod 8.*

Another direction is to consider the intersection form of a spin 4-manifold with given boundary. Suppose X is not closed but has boundary components, which are homology three-spheres. The intersection form of X is still even and unimodular but can be definite now. For the definite case, various constrains are found in [8, 9, 10, 24, 15, 19].

For the indefinite case, Furuta-Li [14] and Manolescu [18] proved the following theorem independently[1].

Theorem 1.5 (Furuta-Li [14], Manolescu [18]). *To each oriented homology 3-sphere Y, we can associate an invariant $\kappa(Y) \in \mathbb{Z}$ with the following properties:*

(i) Suppose W is a smooth, spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then:

$$\kappa(Y_1) + q \geq \kappa(Y_0) + p - 1.$$

(ii) Suppose W is a smooth, oriented spin manifold with a single boundary Y, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $q > 0$. Then:

$$\kappa(Y) + q \geq p + 1.$$

Both Furuta-Li and Manolescu proved this theorem by considering the Pin(2)-equivariant K-theory on the Seiberg-Witten Floer spectrum. Some new bounds can be obtained from this theorem. For example, the Brieskorn sphere $+\Sigma(2,3,12n+1)$ does not bound a spin 4-manifold with intersection form $p(-E_8) \oplus p\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 0$.

The main purpose of this paper is to extend Theorem 1.2 to the case of spin cobordisms and get more constrains on the intersection form of a spin 4-manifold with boundary. Here is the first result:

Theorem 1.6. *For any $k \in \mathbb{Z}/8$, we can associate an invariant $\kappa o_k(Y)$ to each oriented homology sphere Y, with the following properties:*

- *(1) $2\kappa o_k(Y)$ is an integer whose mod 2 reduction is the Rokhlin invariant $\mu(Y)$.*
- *(2) Suppose W is an oriented smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q \geq 0$. Let $p = 4l + m$ for $l \in \mathbb{Z}$ and $m = 0, 1, 2, 3$. Then for any $k \in \mathbb{Z}/8$, we have the following inequalities:*

  (i) If $(\mu(Y_0), m) = (0,0), (0,3), (1,0), (1,1)$, then:

$$\kappa o_k(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^q_{k+q}. \tag{2}$$

  (ii) If $(\mu(Y_0), m) = (0,1), (0,2), (1,2), (1,3)$, then:

$$\kappa o_{k+4}(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^{4+q}_{k+q}. \tag{3}$$

  Here $\beta^j_k = \sum_{i=0}^{j-1} \alpha_{k-i}$ where $\alpha_i = 1$ for $i \equiv 1, 2, 3, 5$ mod 8 and $\alpha_i = 0$ for $i \equiv 0, 4, 6, 7$ mod 8 (β^0_k is defined to be 0). The constants $h(\mu(Y_0), m)$ are listed below:

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[1] We give Manolescu's statement here. Furuta-Li's statement is slightly different.

	$m=0$	$m=1$	$m=2$	$m=3$
$\mu(Y_0)=0$	0	5/2	3	3/2
$\mu(Y_0)=1$	0	1/2	3	7/2

.

Remark 1.7. *When m is even, $\mu(Y_0)=\mu(Y_1)$ and $h(\mu(Y_0),m)$ is an integer. When m is odd, $\mu(Y_0)\neq\mu(Y_1)$ and $h(\mu(Y_0),m)$ is a half-integer.*

Setting $p=q=0$ in (2) of Theorem 1.6, we get:

Corollary 1.8. *If two homology spheres Y_0, Y_1 are homology cobordant to each other, then $\kappa o_k(Y_0)=\kappa o_k(Y_1)$ for any $k\in\mathbb{Z}/8$.*

The definition of κo_k is similar to that of κ [14, 18]. Roughly, $\kappa o_k(Y)$ is defined as follows. Pick a metric g on Y. By doing finite dimensional approximation to the Seiberg-Witten equations on (Y,g), we get a topological space I_ν with an action by $G=Pin(2)$. After changing I_ν by suitable suspension or desuspension, we consider the following construction: The inclusion of the S^1-fixed point set $I_\nu^{S^1}$ induces a map between the equivariant KO-groups $i^*:\widetilde{KO}_G(I_\nu)\to\widetilde{KO}_G(I_\nu^{S^1})$. We choose a specific reduction $\varphi:\widetilde{KO}_G(I_\nu^{S^1})\to\mathbb{Z}$. It can be proved that the image of $\varphi\circ i^*$ is an ideal generated by $2^a\in\mathbb{Z}$. We define a as $\kappa o_k(Y)$. Different $k\in\mathbb{Z}/8$ correspond to different suspensions.

In Section 8, we calculate some examples using the results in [18] about the Seiberg-Witten Floer spectrum of $\pm\Sigma(2,3,r)$.

Theorem 1.9. *(a) We have $\kappa o_i(S^3)=0$ for any $i\in\mathbb{Z}/8$.*

(b) For a positive integer r with gcd$(r,6)=1$, let $\Sigma(2,3,r)$ be the Brieskorn spheres oriented as boundaries of negative plumbings and let $-\Sigma(2,3,r)$ be the same Brieskorn spheres with the orientations reversed. Then $\kappa o_i(\pm\Sigma(2,3,r))$ are listed below:

	κo_0	κo_1	κo_2	κo_3	κo_4	κo_5	κo_6	κo_7
$\Sigma(2,3,12n-1)$	1	1	1	0	0	0	0	0
$-\Sigma(2,3,12n-1)$	0	0	-1	-1	0	0	0	0
$\Sigma(2,3,12n-5)$	1/2	1/2	1/2	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
$-\Sigma(2,3,12n-5)$	3/2	3/2	1/2	$-1/2$	$-1/2$	$-1/2$	$-1/2$	1/2
$\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$-\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$\Sigma(2,3,12n+5)$	3/2	3/2	1/2	$-1/2$	$-1/2$	$-1/2$	1/2	3/2
$-\Sigma(2,3,12n+5)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$

.

Remark 1.10. *We see that $\kappa o_k(-Y)\neq-\kappa o_k(Y)$ in general, while $\kappa o_k(Y\#(-Y))$ is always 0 by Corollary 1.8. Therefore, κo_k is not additive under connected sum.*

If we apply (2) of Theorem 1.6 to the case $Y_0=Y_1=S^3$, the result is weaker than Theorem 1.2. As the case in K-theory (See [18]), we can remedy this by considering the special property of $Y_0\cong S^3$, which is called the Floer KO_G-split condition.

Theorem 1.11. *Let W be an oriented, smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $p\geq 0, q>0$. Suppose Y_0 is Floer KO_G-split. Let $p=4l+m$ for $l\in\mathbb{Z}$ and $m=0,1,2,3$. Then we have the following inequalities:*

(1)If $(\mu(Y_0), m) = (0,0), (0,3), (1,0), (1,1)$*, then:*

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_{4+q}(Y_1) + \beta^q_{4+q}. \tag{4}$$

(2)If $(\mu(Y_0), m) = (0,1), (0,2), (1,2), (1,3)$*, then:*

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_q(Y_1) + \beta^{4+q}_q. \tag{5}$$

Here β^*_* *and* $h(\mu(Y_0), m)$ *are the constants defined in Theorem 1.6.*

In particular, S^3 is Floer KO_G-split. Applying $Y_0 = S^3$ to the previous theorem, we get the following useful corollary:

Corollary 1.12. *Let* W *be an oriented smooth spin 4-manifold whose boundary is a homology sphere* Y*. Suppose the intersection form of* W *is* $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *with* $p \geq 0, q > 0$*. Then we have the following inequalities:*

- *If* $p = 4l$*, then* $2l < \kappa o_{4+q}(Y) + \beta^q_{4+q}$*.*
- *If* $p = 4l+1$*, then* $2l + \frac{5}{2} < \kappa o_q(Y) + \beta^{4+q}_q$*.*
- *If* $p = 4l+2$*, then* $2l + 3 < \kappa o_q(Y) + \beta^{4+q}_q$*.*
- *If* $p = 4l+3$*, then* $2l + \frac{3}{2} < \kappa o_{4+q}(Y) + \beta^q_{4+q}$*.*

Remark 1.13. *If we set* $Y = S^3$ *in Corollary 1.12, we will recover Theorem 1.2. However, Corollary 1.12 is not enough to prove Theorem 1.4. In order to get the relative version of Theorem 1.4, we have to apply similar constructions on the fixed point set of the Adams operation. This will not be done in the present paper.*

Combining the results in Theorem 1.9 with Corollary 1.12, we get some new explicit bounds on the intersection forms of spin four-manifolds bounded by $\pm\Sigma(2,3,r)$. We give two of them here and refer to Section 8.2 for a complete list.

Example 1.14. *We have the following conclusions:*

- $-\Sigma(2,3,12n-1)$ *does not bound a spin four-manifold with intersection form* $p(-E_8) \oplus (p+1) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *for* $p > 0$*.*
- $-\Sigma(2,3,12n-5)$ *does not bound a spin four-manifold with intersection form* $p(-E_8) \oplus p \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *for* $p > 1$*.*

The paper is organized as follows: In Section 2, we discuss some background material about Pin(2)-equivariant KO-theory. In Section 3, we prove Theorem 1.4 after recalling some basic facts and properties of the Adams operations. In Section 4, we review the basic properties of the Seiberg-Witten Floer spectrum. The numerical invariant κo_k is defined in Section 5 and Theorem 1.6 is proved in Section 6. In Section 7, we introduce the Floer KO_G-split condition and prove Theorem 1.11. In Section 8, we prove Theorem 1.9 and use Corollary 1.12 and Theorem 1.4 to obtain new constrains on the intersection form of a spin four-manifold with given boundary.

Acknowledgement Many of the constructions are parallel to those in [14, 18] and are credited throughout. I wish to thank Ciprian Manolescu for suggesting the problem that leads to the results in this paper, and for his encouragement and enthusiasm. I am also grateful to the referee for comments on a previous version of this paper.

2. Equivariant KO-theory

2.1. **General Theory.** In this subsection, we review some general facts about equivariant KO-theory, mostly from [28] and [5]. See [2], [3] for basic facts about ordinary K-theory and KO-theory.

Let G be a compact topological group and X be a compact G-space. We denote the Grothendieck group of real G-bundles over X by $KO_G(X)$.

Fact 2.1. *$KO_G(pt) = RO(G)$. Here $RO(G)$ denotes the real representation ring of G. For a general X, $KO_G(X)$ is a $RO(G)$-algebra (with unit).*

Remark 2.2. *In this paper, we will not distinguish a representation of G with its representation space.*

Fact 2.3. *A continuous G-map $f : X \to Y$ induces a map $f^* : KO_G(Y) \to KO_G(X)$.*

Fact 2.4. *For each subgroup $H \subseteq G$, by restricting the G action to H, which makes a G-bundle into an H-bundle, we get a functorial restriction map $r : KO_G(X) \to KO_H(X)$.*

Fact 2.5. *If G acts freely on X, then the pull back map $KO(X/G) \to KO_G(X)$ is a ring isomorphism.*

Fact 2.6. *For a real irreducible representation space V of G, $End_G(V)$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathbb{Z}Ir_{\mathbb{R}}$, $\mathbb{Z}Ir_{\mathbb{C}}$ and $\mathbb{Z}Ir_{\mathbb{H}}$ denote the free abelian groups generated by irreducible representations of respective types and let $KSp(X)$ be the the Grothendieck group of quaternionic vector bundles over X. Then if G acts trivially on X, we have:*

$$KO_G(X) = (KO(X) \otimes \mathbb{Z}Ir_{\mathbb{R}}) \oplus (K(X) \otimes \mathbb{Z}Ir_{\mathbb{C}}) \oplus (KSp(X) \otimes \mathbb{Z}Ir_{\mathbb{H}}). \quad (6)$$

Now suppose X has a distinguished base point p which is fixed by G. Then we define $\widetilde{KO}_G(X)$ (the reduced KO-group) to be the kernel of the map $KO_G(X) \to KO_G(p)$. For based space X with trivial action, we also have:

$$\widetilde{KO}_G(X) = (\widetilde{KO}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{R}}) \oplus (\widetilde{K}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{KSp}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}). \quad (7)$$

The following fact is proved as Corollary 3.1.6 in [2]. ([2] only proved the complex K-theory case but the proof works without modification in the real case.)

Fact 2.7. *Suppose X is a finite, based G-CW complex and the G-action is free away from the base point. Then any element in $\widetilde{KO}_G(X) \cong \widetilde{KO}(X/G)$ is nilpotent.*

Recall that the augmentation ideal $\mathfrak{a} \subset RO(G)$ is the kernel of the forgetful map $RO(G) \cong KO_G(pt) \to KO(pt) \cong \mathbb{Z}$. Any element in $\mathfrak{a}$ defines an element in $\widetilde{KO}_G(X)$. By the above fact, we get:

Fact 2.8. *Suppose X is a finite, based G-CW complex and the G-action is free away from the base point. Then any element in the augmentation ideal acts on $\widetilde{KO}^*_G(X)$ nilpotently.*

Fact 2.9. *For pointed spaces X, Y, there is a natural product map $\widetilde{KO}_G(X) \otimes \widetilde{KO}_G(Y) \to \widetilde{KO}_G(X \wedge Y)$.*

Fact 2.10. *For pointed spaces X, Y, we have $\widetilde{KO}_G(X \vee Y) \cong \widetilde{KO}_G(X) \oplus \widetilde{KO}_G(Y)$*

Let V be a real representation space of G. Denote the reduced suspension $V^+ \wedge X$ by $\Sigma^V X$. The following equivariant version of real Bott periodicity theorem was proved in [5].

Fact 2.11. *Suppose the dimension n of V is divisible by 8 and V is a spin representation (which means the group action $G \to SO(n) \subset End(V)$ factors through $Spin(n)$). Then we have the Bott isomorphism $\varphi_V : \widetilde{KO}_G(X) \cong \widetilde{KO}_G(\Sigma^V X)$, given by the multiplication of the Bott Class $b_V \in \widetilde{KO}_G(V^+)$ under the natural map $\widetilde{KO}_G(V^+) \otimes \widetilde{KO}_G(X) \to \widetilde{KO}_G(\Sigma^V X)$. Bott isomorphism is funtorial under the pointed map $X \to X'$.*

Fact 2.12. *Bott classes behave well under the restriction map, which means that $i^* b_V = b_{i^*(V)}$. Here i^* is the restriction map (see Fact 2.4) and $i^*(V)$ is the the restriction of the representation to the subgroup.*

2.2. **Pin(2)-equivariant KO-theory.** In this section, we will review some important facts about Pin(2)-equivariant KO-theory. The detailed discussions can be found in [27]. From now on, we assume $G \cong Pin(2)$ unless otherwise noted. Recall that the group Pin(2) can be defined as $S^1 \oplus jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}$. We have:

$$RO(Pin(2)) \cong \mathbb{Z}[D, K, H]/(D^2 - 1, DK - K, DH - H, H^2 - 4(1 + D + K)).$$

The representation space of D is $\mathbb{R}$ where the identity component $S^1 \subset Pin(2)$ acts trivially and $j \in Pin(2)$ act as multiplication by -1.

The representation space of K is $\mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R}$ where $z \in S^1 \subset Pin(2)$ acts as multiplication by z^2 (in $\mathbb{C}$) and j acts as reflection along the diagonal.

The representation space of H is $\mathbb{H}$ where the action is given by the left multiplication of $Pin(2) \subset \mathbb{H}$.

We will also write $\mathbb{R}$ as the trivial one dimensional representation of G.

Following the notation of [27], we denote $\widetilde{KO}_G((kD + lH)^+)$ by $KO_G(kD + lH)$ (we choose ∞ as the base point). Then for $k, l, m, n \in \mathbb{Z}_{\geq 0}$ we have the multiplication map:

$$KO_G(kD + lH) \otimes KO_G(mD + nH) \to KO_G((k + m)D + (l + n)H). \tag{8}$$

In order to define this map, we need to fix the identification between $(kD \oplus lH) \oplus (mD \oplus nH)$ and $(k + m)D \oplus (l + n)H$ by sending $(x_1 \oplus y_1) \oplus (x_2 \oplus y_2)$ to $(x_1, x_2) \oplus (y_1, y_2)$. By considering the G-equivariant homotopy, it is not hard to see that the multiplication map is commutative when k or l is even. (We will prove that the multiplication map is actually commutative for any k, l, after we give the structure of $KO_G(kD + lH)$ in Theorem 2.13.)

It is easy to prove (see [27]) that $8D$, $H+4D$ and $2H$ are spin representations. Therefore, we can choose Bott classes $b_{8D} \in KO_G(8D)$, $b_{2H} \in KO_G(2H)$ and $b_{H+4D} \in KO_G(H + 4D)$. Multiplication by these classes induces isomorphism $KO_G(kD + lH) \cong KO_G((k + 8)D + lH) \cong KO_G((k + 4)D + (l + 1)H) \cong KO_G(kD + (l + 2)H)$. Since the Bott classes are in the center, it doesn't matter whether we multiply on the left or on the right. Moreover, we can choose the Bott classes to be compatible with each other, which means that $b_{8D} b_{2H} = b^2_{H+4D}$. We will fix the choice of these Bott classes throughout this paper.

For $k, l \in \mathbb{Z}$, the $RO(G)$-module $KO_G(kD + lH)$ is defined to be $KO_G((k + 8a)D + (l + 2b)H)$ for any $a, b \in \mathbb{Z}$ which make $k + 8a \geq 0$ and $l + 2b \geq 0$. Since the Bott

Classes are chosen to be compatible, the groups defined by different choices of a, b are canonically identified to each other. Again because the Bott classes are in the center, the multiplication map (8) can now be extended to all $k, l, m, n \in \mathbb{Z}$.

Consider the inclusion $i : 7D^+ \to 8D^+$. There is a unique element $\gamma(D) \in KO_G(-D)$ which satisfies $\gamma(D)b_{8D} = i^*(b_{8D})$. The map $KO_G((k+1)D + lH) \xrightarrow{\cdot\gamma(D)} KO_G(kD + lH)$ is just the map induced by the inclusion $kD \oplus lH \to (k+1)D \oplus lH$ for $k, l \geq 0$. Similarly, we can define $\gamma(H) \in KO_G(-H)$ and $\gamma(H + 4D) = \gamma(H)\gamma(D)^4$. Since left multiplication and right multiplication by $\gamma(D)$ or $\gamma(H)$ just correspond to different inclusions of subspaces, which are homotopic to each other, we see that $\gamma(D)$ and $\gamma(H)$ are both in the center.

By Bott periodicity, we only have to compute $KO_G(lD)$ for $l = -2, -1, 0, ..., 5$. This was done in [27] and we list the result here:

Theorem 2.13 (Schmidt [27])**.** *As $\mathbb{Z}$-modules we have the following isomorphisms:*

- *1) $KO_G(pt) \cong RO(Pin(2)) \cong \mathbb{Z}[D, A, B]/(D^2 - 1, DA - A, DB - B, B^2 - 4(A - 2B))$, where $A = K - (1 + D)$ and $B = H - 2(1 + D)$.* [2]
- *2) $KO_G(-lD) \cong \mathbb{Z} \oplus \oplus_{n\geq 1}\mathbb{Z}/2$ for $l = 1, 2$ generated by $\gamma(D)^{|l|}$ and $\gamma(D)^{|l|}A^n$.*
- *3) $KO_G(D) \cong \mathbb{Z}$, generated by $\eta(D)$.*
- *4) $KO_G(lD) \cong \mathbb{Z} \oplus \oplus_{m\geq 0}\mathbb{Z}/2$ for $l = 2, 3$. The generators are $\eta(D)^2$ and $\gamma(D)^2A^m c$ for $l = 2$; $\gamma(D)\lambda(D)$ and $\gamma(D)A^m c$ for $l = 3$.*
- *5) $KO_G(4D)$ is freely generated by $\lambda(D), D\lambda(D), A^n\lambda(D)$ and $A^m c$ for $m \geq 0$ and $n \geq 1$.*
- *6) $KO_G(5D) \cong \mathbb{Z}$, generated by $\eta(D)\lambda(D)$.*

Corollary 2.14. *The multiplication map (8) is commutative.*

Proof. We just need to check $\gamma(D)$, $\eta(D)$, $\lambda(D)$, c commute with each other. This is easy since $\lambda(D)$ and c are in $KO_G(kD)$ for even k, while $\gamma(D)$ is in the center by our discussion before. □

For our purpose, we don't need to know the explicit constructions of $\eta(D), \lambda(D)$ and c. We just need to know the following properties of them.

$\eta(D)$ is the Hurewicz image of an element $\tilde{\eta}(D) \in \pi^0_G(D)$ (G-equivariant stable cohomotopy group of D^+). If we forget about the G-action, $\tilde{\eta}(D)$ is just the Hopf map in $\pi_1^{\mathrm{st}}(\mathrm{pt})$.

For $\lambda(D)$ and $c \in KO_G(4D)$, by Bott periodicity and formula (7), we have isomorphisms:

$$KO_G(4D) \cong KO_G(8D + 4) \cong KO_G(4)$$
$$\cong (\widetilde{KO}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{R}}) \oplus (\widetilde{K}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{KSp}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}).$$

(Here $4 \in RO(G)$ denotes the trivial 4-dimensional real representation.)

We can choose suitable Bott classes such that under these isomorphisms, $\lambda(D)$ corresponds to $([V_H] - 4\mathbb{R}) \otimes 1 \in \widetilde{KO}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{R}}$ and c corresponds to $([V_{\mathbb{H}}] - \mathbb{H}) \otimes H \in \widetilde{KSp}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}$. Here $V_{\mathbb{H}}$ is the quaternion Hopf bundle over $S^4 \cong \mathbb{H}P^2$. $\mathbb{H}$ and $\mathbb{R}$ denote the trivial bundles and $1, H$ are elements in $RO(G)$.

[2] There is a typo in [27], where the relation between A and B is $B^2 - 2(A - 2B)$.

Let $\lambda(H)$ and $c(H)$ be the image of $\lambda(D)$ and c under the Bott isomorphism $KO_G(4D) \cong KO_G(8D+H) \cong KO_G(H)$. Then $KO_G(H)$ is generated by $\lambda(H)$ and $c(H)$ as $RO(G)$-algebra.

Remark 2.15. *Notice that the element $[V_H] \otimes H \in KSpS^4 \otimes \mathbb{Z}Ir_{\mathbb{H}}$ is represented by the bundle $V_H \otimes_{\mathbb{H}} H$. Hence it is a real bundle of dimension* 4 *(not* 16*).*

For further discussions, we need to know the multiplicative structures of $KO_G(lD)$, which are also given in [27]. We list some of them that are useful for us:

Theorem 2.16 (Schmidt [27])**.** *The following relations hold:*

- *1)* $H\lambda(D) = 4c$, $Hc = (A+2+2D)\lambda(D)$, $Dc = c$.
- *2)* $(D+1)\gamma(D) = 2A\gamma(D) = B\gamma(D) = 0$.
- *3)* $(D+1)\eta(D) = A\eta(D) = B\eta(D) = 0$.
- *4)* $\gamma(D)\eta(D) = 1-D$, $\gamma(D)\lambda(D) = \eta(D)^3$.
- *5)* $\gamma(D)^8 b_{8D} = 8(1-D)$, $\gamma(H)^2 b_{2H} = K - 2H + D + 5$.
- *6)* $\gamma(H+4D)b_{H+4D} = 4(1-D)$.
- *7)* $\eta(D)\lambda(D) = \gamma(D)^3 b_{8D}$, $\eta(D)c = 0$.
- *8)* $\gamma(H)\lambda(H) = 4 - H$ *and* $\gamma(H)c(H) = H - 1 - D - K$.

3. The Adams operations

3.1. **Basic properties.** In this subsection, we give a quick review about the basic properties of the Adams operations. See [2] and [31] for more detailed discussions. Some of the calculations can be found in [27] but we give them here for completeness. For simplicity and concreteness, we only deal with $\psi^k : KO_G(X) \to KO_G(X)$ for an actual G-space X and we don't do localizations (like [27]).

Let $KO_G(X)[[t]]$ be the formal power series with coefficients in $KO_G(X)$. For a bundle E over X, we define $\lambda_t(E) \in KO_G(X)[[t]]$ to be $\sum\limits_{i=0} t^i[\lambda^i(E)]$. Here $\lambda^i(E)$ is the i-th exterior power of E. We let $\psi^0(E) = \text{rank}(E)$ and define $\psi_t(E) = \sum\limits_{i=0} t^i\psi^i(E) \in KO_G(X)[[t]]$ by

$$\psi_t(E) = \psi^0(E) - t\frac{d}{dt}(\log\lambda_{-t}(x)). \tag{9}$$

It turns out that for any $k \in \mathbb{Z}_{\geq 0}$, ψ^k extends to a well defined operation on $KO_G(X)$, which satisfies the following nice properties:

- (1) ψ^k is functorial with respect to continuous maps $f : X \to X'$.
- (2) ψ^k maps $\widetilde{KO}_G(X)$ to $\widetilde{KO}_G(X)$.
- (3) For all $x, y \in KO_G(X)$, $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$ and $\psi^k(xy) = \psi^k(x)\psi^k(y)$.
- (4) If x is a line bundle, then $\psi^k(x) = x^k$.

The effect of the Adams operations on the Bott classes can be described by the Bott cannibalistic class. Given a spin G-bundle E over X with rank $n \equiv 0 \bmod 8$, the Bott cannibalistic class $\theta_k^{\text{or}}(E) \in RO(G)$ is defined by the equation:

$$\psi^k(b_E) = \theta_k^{\text{or}}(E) \cdot b_E \text{ for } k > 1. \tag{10}$$

When k is odd, this can be explicitly written as (see [31]):[3]

$$\theta_k^{\mathrm{or}}(E) = k^{n/2} \prod_{u \in J} \lambda_{-u}(E)(1-u)^{-n}. \tag{11}$$

Here J is a set of k-th unit roots $u \neq 1$ such that J contains exactly one element from each pair $\{u, u^{-1}\}$. Notice that we can define $\theta_k^{\mathrm{or}}(E)$ for any real bundle E of even dimension using formula (11). It can be shown that:

$$\theta_k^{\mathrm{or}}(E+F) = \theta_k^{\mathrm{or}}(E)\theta_k^{\mathrm{or}}(F).$$

Now let's specialize to the case $k = 3$. By formula (9), it is easy to check that $\psi^3(x) = x^3 - 3\lambda^2(x)x + 3\lambda^3(x)$. We want to calculate the action of ψ^3 on $RO(G)$. Since the G-action on H preserves the orientation, we have $\lambda^3(H) = \lambda^1(H) = H$. Using complexification, it is easy to show $\lambda^2(H) = K + D + 3$. Also, we have $\lambda^2(K) = D$. Therefore, we get[4]:

$$\psi^3(D) = D,\ \psi^3(H) = HK - H,\ \psi^3(K) = K^3 - 3K,$$
$$\psi^3(A) = A^3 + 6A^2 + 9A, \quad \psi^3(B) = AB + B + 4A.$$

Also, applying formula (11), we get:

$$\theta_3^{\mathrm{or}}(2) = 3,\ \theta_3^{\mathrm{or}}(2D) = 1 + 2D,\ \theta_3^{\mathrm{or}}(H) = A + B + 4D + 5.$$

3.2. **Proof of Theorem 1.4.** The central part of the proof is the following proposition:

Proposition 3.1. *For any integers $r, a, b \geq 0$ and $l > 0$, there does not exist G-equivariant map*

$$f : (r\mathbb{R} + aD + (4l+b)H)^+ \to (r\mathbb{R} + (a+8l+2)D + bH)^+$$

which induces homotopy equivalence on the G-fixed point set.

Proof. Suppose there exists such a map f. After suspension by copies of $\mathbb{R}, D$ and H, we can assume $a = 8l' + 6$, $r = 8d$ and $b = 2k$. Let $V_1 = 8d\mathbb{R} + 2kH + 8(l + l' + 1)D$ and $V_2 = 8d\mathbb{R} + (4l + 2k)H + (8l' + 8)D$. Let b_{V_1} and b_{V_2} be the Bott classes of V_1 and V_2, respectively. Consider the element $x = f^*(b_{V_1})$. By the Bott isomorphism and (2) of Theorem 2.13, we can write x as $b_{V_2}\gamma(D)^2\alpha$ for some $\alpha \in RO(G)$. Moreover, we can assume $\alpha = p + Ah(A)$ for some integer p and some polynomial $h(A)$ whose coefficients are either 0 or 1.

Claim: p is even and $h = 0$.

This is essentially a special case of Proposition 5.21 in [27] for $\mathcal{KO}(4l, 8l+2)$.[5]

By formula (10), we have: $\psi^3(b_{V_1}) = \theta_3^{\mathrm{or}}(V_1) \cdot b_{V_1}$, which implies:

$$\psi^3(x) = f^*(\psi^3(b_{V_1})) = \theta_3^{\mathrm{or}}(V_1) \cdot x. \tag{12}$$

Notice that $x = i^*(b_{V_2} \cdot \alpha)$ where $i : (8d\mathbb{R} + (4l+2k)H + (8l'+6)D)^+ \to V_2^+$ is the standard inclusion. By formula (10), we have:

$$\psi^3(x) = i^*(\psi^3(b_{V_2} \cdot \alpha)) = \theta_3^{\mathrm{or}}(V_2) b_{V_2} \psi^3(\alpha) \cdot \gamma(D)^2. \tag{13}$$

[3] There is a typo in 3.10.4 [31].

[4] There is a typo in [27], where $\psi^3(H) = HK - K$.

[5] There is an error in [27] for $\mathcal{KO}(c, d)$ when $4c - d \equiv -3 \bmod 8$, but we will not consider this case here.

Comparing equation (12) and equation (13), we get:

$$(\theta_3^{\rm or}(V_2)\psi^3(\alpha) - \theta_3^{\rm or}(V_1)\alpha)\gamma(D)^2 = 0 \tag{14}$$

We can calculate:

$$\theta_3^{\rm or}(V_1) = 3^{4d}(1+2D)^{4l+4l'+4}(A+B+4D+5)^{2k},$$

$$\theta_3^{\rm or}(V_2) = 3^{4d}(1+2D)^{4l'+4}(A+B+4D+5)^{2k+4l}.$$

Notice that $2A\gamma(D) = B\gamma(D) = (1+D)\gamma(D) = 0$, we can simplify equation (14) as:

$$3^{4d}((A+1)^{2k}\alpha - (A+1)^{4l+2k}\psi^3(\alpha))\cdot\gamma(D)^2 = 0. \tag{15}$$

Since $\alpha = p + Ah(A)$, we have $\psi^3(\alpha) = p + (A^3+6A^2+9A)h(A^3+6A^2+9A)$. Using the relation $2A\gamma(D) = 0$, we can further simplify equation (15) and get:

$$3^{4d}\cdot g(A)\cdot\gamma(D)^2 = 0 \tag{16}$$

Here $g(A) = (A+1)^{2k}(p+Ah(A)) - (A+1)^{2k+4l}(p+(A^3+A)h(A^3+A))$.

By (2) of Theorem 2.13, we see that if we expand $g(A)$ as a polynomial in A, the degree-0 coefficient should be 0 and all other coefficients should be even. By our assumption, the coefficients of h are either 0 or 1. Checking the leading coefficient of $g(A)$, it is easy to see that $h = 0$ and $g(A) = p((A+1)^{2k} - (A+1)^{2k+4l})$. This implies that p is even. The claim is proved.

Now consider the commutative diagram:

$$\begin{array}{ccc} \widetilde{KO}_G(V_1^+) & \xrightarrow{f^*} & \widetilde{KO}_G((8d\mathbb{R}+(8l'+6)D+(4l+2k)H)^+) \\ \downarrow{\scriptstyle \cdot\gamma(H)^{2k}\gamma(D)^{8l+8l'+8}} & & \downarrow{\scriptstyle \cdot\gamma(H)^{4l+2k}\gamma(D)^{8l'+6}} \\ \widetilde{KO}_G((8d\mathbb{R})^+) & \xrightarrow{\cong} & \widetilde{KO}_G((8d\mathbb{R})^+). \end{array} \tag{17}$$

The vertical maps are given by the inclusions of subspaces. The bottom map is an i-somorphism because f induces a homotopy equivalence on the G-fixed point set. Any automorphism on $\widetilde{KO}_G((8d\mathbb{R})^+)$ is given by the multiplication of a unit $\tilde{u} \in RO(G)$. Therefore, we obtain :

$$\tilde{u}\cdot b_{V_1}\cdot\gamma(H)^{2k}\gamma(D)^{8l+8l'+8} = x\cdot\gamma(H)^{4l+2k}\gamma(D)^{8l'+6} = b_{V_2}\cdot\gamma(D)^{8l'+8}\gamma(H)^{4l+2k}\cdot p \tag{18}$$

Applying the relations in Theorem 2.16, we simplify this as :

$$(K-2H+D+5)^{2l+k}(8(1-D))^{l'+1}\cdot p = (K-2H+D+5)^k(8(1-D))^{l+l'+1}\cdot\tilde{u}. \tag{19}$$

Now consider the ring homomorphism $\varphi_0 : RO(G) \to \mathbb{Z}$ defined by $\varphi_0(D) = -1, \varphi_0(A) = \varphi_0(B) = 0$. Notice that $\varphi_0(\tilde{u}) = \pm 1$ since $\tilde{u}$ is a unit. We get $p = \pm 1$, which is a contradiction. This finishes the proof of Proposition 3.1. □

Now suppose X is a closed, oriented, smooth spin four-manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p = 8l > 0$ and $q < p+3$. After doing surgery on loops and connect sum copies of $S^2\times S^2$, we can assume $b_1(W) = 0$ and $q = 8l+2$. As shown in [12], by doing finite dimensional approximation of the Seiberg-Witten equations on W, we get an G-equivariant map:

$$f : (aD + (4l + b)H)^+ \to ((a + 8l + 2)D + bH)^+ \text{ for some } a, b > 0.$$

Moreover, f induces a homotopy equivalence on the G-fixed point set. This is a contradiction to Proposition 3.1. Therefore, Theorem 1.4 is proved.

4. Pin(2)-equivariant Seiberg-Witten Floer theory

In [17], [18] and [19], Manolescu constructed a Pin(2)-equivariant spectrum class $S(Y, \mathfrak{s})$ for each rational homology sphere Y with a spin structure $\mathfrak{s}$. We will not repeat the constructions here but just collect some useful properties. See [17], [18] and [19] for the explicit constructions.

Definition 4.1. *Let $s \in \mathbb{Z}_{\geq 0}$. A space of type SWF (at level s) is a pointed, finite G-CW complex X with the following properties:*

- *(a) The S^1-fixed point set X^{S^1} is G-homotopy equivalent to the sphere $(sD)^+$. We define $lev(X)$ to be s.*
- *(b) The action of G is free on the complement $X - X^{S^1}$.*

Definition 4.2. *Let X, X' be two spaces of type SWF at level k and k' respectively. A pointed G-map $f : X \to X'$ is called admissible if f preserves the base point and satisfies one of the following two conditions:*

- *(1) $k < k'$ and the induced map on the G-fixed point set $f^G : X^G \to X'^G$ is a homotopy equivalence.*
- *(2) $k = k'$ and the induced map on the S^1-fixed point set $f^{S^1} : X^{S^1} \to X'^{S^1}$ is a homotopy equivalence.*

Now consider the set of triples (X, a, b) where X is a space of type SWF and $a \in \mathbb{Z}, b \in \mathbb{Q}$.

Definition 4.3. *We say that (X, a, b) is stable equivalent to (X', a', b') if $b - b' \in \mathbb{Z}$ and for some $M, N, r > 0$, there exists a G-homotopy equivalence:*

$$\Sigma^{r\mathbb{R}}\Sigma^{(M-a)D}\Sigma^{(N-b)H}X \cong \Sigma^{r\mathbb{R}}\Sigma^{(M-a')D}\Sigma^{(N-b')H}X'.$$

(Here $\mathbb{R}$ denotes the trivial representation of G.)

Remark 4.4. *In* [18], *Manolescu worked with stable even equivalence, which requires X to be a space of type SWF at even level.*

This triple can be thought of the "formal de-suspension" of X with a copies of D and b copies of H. We denote $\mathfrak{C}$ to be the set of stable equivalence classes of triples (X, a, b). Informally, we call an element in $\mathfrak{C}$ a spectrum class.

Definition 4.5. *For a spectrum class $S = [(X, a, b)] \in \mathfrak{C}$, we let*

$$lev(S) = lev(X) - a.$$

Remark 4.6. *By considering the S^1-fixed point set, we see that two spaces of type SWF at different levels are not G-homotopic to each other. Using this fact, it is easy to prove that $lev(S)$ is a well defined quality.*

For $r \in \mathbb{Z}$ and $s \in \mathbb{Q}$, we can define the formal suspension $\Sigma^{rD+sH} : \mathfrak{C} \to \mathfrak{C}$ by sending $[(X, a, b)]$ to $[(X, a-r, b-s)]$. It's easy to check that this is a well defined operation on the set $\mathfrak{C}$.

Now suppose Y is an oriented rational homology three-sphere with a metric g and a spin structure $\mathfrak{s}$. Let $\mathbb{S}$ be the associated spinor bundle. We consider the global Coulomb splice:

$$V = i\ker d^* \oplus \Gamma(\mathbb{S}) \subset i\Omega^1(Y) \oplus \Gamma(\mathbb{S}).$$

Using the quaternionic structure on $\mathbb{S}$, we can define a natural action of G on V: $e^{i\theta} \in G$ takes (α, ϕ) to $(e^{i\theta}\alpha, \phi)$ and $j \in G$ takes (α, ϕ) to $(-\alpha, j\phi)$.

Now we consider the self-adjoint first order elliptic operator $l : V \to V$ defined by $l(\alpha, \phi) = (*d\alpha, \not{D}\phi)$ where $\not{D}$ is the Dirac operator [6]. For any $\tau < \nu$, let V^τ_ν be the subspace spanned by the eigenvectors of l with eigenvalues in the interval $(\tau, \nu]$. Then V^τ_ν is a finite dimensional G-representation space which is isomorphic to $kD \oplus lH$. We denote k by $\dim_{\mathbb{R}} V(D)^\nu_\tau$ and l by $\dim_{\mathbb{H}} V(H)^\nu_\tau$.

We pick $-\nu << 0 << \nu$. By considering the equivariant Conley index of the gradient flow of $CSD|_{V^\nu_{-\nu}}$ (see [17] and [18]), we get a G-space I_ν of type SWF at level $\dim_{\mathbb{R}} V(D)^0_{-\nu}$.

Next, we need to recall the definition of $n(Y, \mathfrak{s}, g)$. Choose a compact smooth spin four-manifold N with $\partial N = Y$. Let $\text{ind}_{\mathbb{C}} \not{D}(N)$ be the index of Dirac operator on N (with Atiyah-Patodi-Singer boundary conditions). We can define:

$$n(Y, \mathfrak{s}, g) := \text{ind}_{\mathbb{C}} \not{D}(N) + \frac{\sigma(N)}{8}. \tag{20}$$

Remark 4.7. *It can be proved that this definition does not depend on the choice of N. For a rational homology sphere Y, we have $n(Y, \mathfrak{s}, g) \in \frac{1}{8}\mathbb{Z}$. When Y is an integral homology sphere, $n(Y, \mathfrak{s}, g)$ is an integer and has the same parity as the Rokhlin invariant $\mu(Y)$.*

We can consider the following element in $\mathfrak{C}$:[7]

$$S(Y, \mathfrak{s}) := [(I_\nu, \dim_{\mathbb{R}} V(D)^0_{-\nu}, \dim_{\mathbb{H}} V(H)^0_{-\nu} + \frac{1}{2} n(Y, \mathfrak{s}, g))]. \tag{21}$$

Notice that the level of $S(Y, \mathfrak{s})$ is always 0.

Theorem 4.8 (Manolescu [17],[18])**.** *The element $S(Y, \mathfrak{s}) \in \mathfrak{C}$ is independent of the metric g, the cut-off ν and the other choices in the construction. Thus $S(Y, \mathfrak{s})$ is an invariant of the pair $(Y, \mathfrak{s})$.*

Remark 4.9. *In this paper, since we only use the numerical invariants, we don't need to make $\mathfrak{C}$ a category and $S(Y, \mathfrak{s})$ a functor. Therefore, we don't define $S(Y, \mathfrak{s})$ as natural spectrum invariant. See Section 3.4 of* [19] *for a discussion about naturality.*

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[6] Since Y is a rational homology sphere, there is a unique flat spin-connection on $\mathbb{S}$, we choose it as the base connection and use it to define $\not{D}$.

[7] Our convention is different from [17] and [18] , where the second component in the triple denotes the complex dimension of the G-representation.

Suppose W is a smooth spin cobordism between rational homology three spheres Y_0 and Y_1 with $b_1(W) = 0$. Further, we assume W is equipped with a matric g and a spin structure $\mathfrak{t}$ such that $g|_{Y_i} = g_i$ and $\mathfrak{t}|_{Y_i} = \mathfrak{s}_i$.

The following theorem is important for our constructions:

Theorem 4.10 (Manolescu [17],[18]). *By doing finite dimensional approximation for the Seiberg-Witten equations on W, we obtain an admissible map:*

$$f : \Sigma^{a_0 D}\Sigma^{b_0 H}(I_0)_\nu \to \Sigma^{a_1 D}\Sigma^{b_1 H}(I_1)_\nu. \tag{22}$$

Here, $(I_0)_\nu$ and $(I_1)_\nu$ are the Conley indices for the approximated Seiberg-Witten flow. Let V_i denotes the Coulomb slice on Y_i, for $i = 0, 1$. The differences in the suspension indices are:

$$a_0 - a_1 = dim_{\mathbb{R}} V_1(D)^0_{-\nu} - dim_{\mathbb{R}} V_0(D)^0_{-\nu} - b_2^+(W) \tag{23}$$

and

$$b_0 - b_1 = dim_{\mathbb{H}} V_1(H)^0_{-\nu} - dim_{\mathbb{H}} V_0(H)^0_{-\nu} + \frac{1}{2}n(Y_1, \mathfrak{s}_1, g_1) - \frac{1}{2}n(Y_0, \mathfrak{s}_0, g_0) - \frac{\sigma(W)}{16}. \tag{24}$$

5. Numerical Invariants

Let Y be a rational homology sphere and $\mathfrak{s}$ be a spin structure on Y. In the previous section, we defined an invariant $S(Y, s) \in \mathfrak{C}$. In this section, we will extract a set of numerical invariants $\kappa o_i(Y, s)$ from $S(Y, s)$, for $i \in \mathbb{Z}/8$.

Definition 5.1. *For $l = -2, -1, 0, ..., 5$, we define the group homomorphisms $\varphi_l : KO(lD) \to \mathbb{Z}$ as following (see Theorem 2.13):*

- *1) For $l = 0$, $\varphi_l(D) = -1$ and $\varphi_l(A) = \varphi_l(B) = 0$, then extend φ_l by the multiplicative structure on $RO(G)$.*
- *2) For $l = -1, -2$, $\varphi_l(\gamma(D)^{|l|}) = 1$ and $\varphi_l(\gamma(D)^{|l|}A^n) = 0$ for $n \geq 1$.*
- *3) For $l = 1$, $\varphi_l(\eta(D)) = 1$.*
- *4) For $l = 2$, $\varphi_l(\eta(D)^2) = 1$ and $\varphi_l(\gamma(D)^2 A^m c) = 0$.*
- *5) For $l = 3$, $\varphi_l(\gamma(D)\lambda(D)) = 1$ and $\varphi_l(\gamma(D)A^m c) = 0$.*
- *6) For $l = 4$, $\varphi_l(\lambda(D)) = 1$, $\varphi_l(D\lambda(D)) = -1$, and $\varphi_l(A^n\lambda(D)) = \varphi_l(A^m c) = 0$.*
- *7) For $l = 5$, $\varphi_l(\eta(D)\lambda(D)) = 1$.*

For the other $l \in \mathbb{Z}$, we use the Bott isomorphism to identify $KO(lD)$ with $KO((l-8k)D)$ for $-2 \leq l - 8k \leq 5$ and apply the above definition.

Lemma 5.2. *For any $a \in KO_G(pt)$ and $b \in KO_G(kD)$, we have $\varphi_0(a)\varphi_k(b) = \varphi_k(a \cdot b)$.*

Proof. This is a straightforward calculation using Theorem 2.13 and Theorem 2.16. □

Remark 5.3. *φ_0 is just taking the trace of $j \in Pin(2)$. While the other φ_l are defined such that the torsion elements are killed and Lemma 5.2 holds.*

We consider the map $\tau : D^+ \to D^+$ which maps x to $-x$. By suspension with copies of D, we get an admissible involution $\tau : (kD)^+ \to (kD)^+$ for $k > 0$.

The following lemma is a straightforward corollary of the equivariant Hopf theorem (see [30]).

Lemma 5.4. *When $0 \leq k < l$, any admissible map $f : (kD)^+ \to (lD)^+$ is G-homotopic to the standard inclusion. For $0 \leq k = l$, any admissible map $f : (kD)^+ \to (kD)^+$ is either homotopic to τ or to the identity map, depending on $deg(f)$.*

τ induces the involution $\tau^* : KO_G(kD) \to KO_G(kD)$. For $k, l > 0$ and any $a \in KO_G(kD), b \in KO_G(lD)$, the following equality is easy to check by Lemma 5.4:

$$\tau^*(a) \cdot b = a \cdot \tau^*(b) = \tau^*(a \cdot b) \text{ and } \tau^*(a) \cdot \tau^*(b) = a \cdot b. \tag{25}$$

Using this fact, we can define $\tau^* : KO_G(kD) \to KO_G(kD)$ for any $k \in \mathbb{Z}$ by identifying $KO_G(kD)$ with $KO_G(k'D)$ for any $0 < k' \equiv k \bmod 8$ using Bott periodicity. Moreover, formula (25) now holds for all $k, l \in \mathbb{Z}$.

Now consider the element $u \in RO(G)$ defined by $\tau^*(b_{8D}) = u \cdot b_{8D}$. Then for $l \in \mathbb{Z}$ and any element $\alpha \in KO_G(lD)$, we have $\tau^*(\alpha) \cdot b_{8D} = \alpha \cdot \tau^*(b_{8D}) = (u\alpha) \cdot b_{8D}$, which implies $\tau^*(\alpha) = u\alpha$.

Lemma 5.5. *We have the following properties about τ^* and u:*

- *(1) τ^* acts as identity on $KO_G(lD)$ for $l \neq 0, 4$ mod 8.*
- *(2) u is a unit with $\varphi_0(u) = 1$.*
- *(3) $\varphi_l \circ \tau^* = \varphi_l$ for any $l \in \mathbb{Z}$.*

Proof. (1) We have $\gamma(D)b_{8D} = i^*(b_{8D})$ where i^* is the inclusion $(7D)^+ \to (8D)^+$. Therefore, we get $\tau^*(\gamma(D)b_{8D}) = (\tau \circ i)^*(b_{8D})$. By Lemma 5.4, $\tau \circ i$ is G-homotopic to i, thus $\tau^*(\gamma(D)b_{8D}) = i^*(b_{8D}) = \gamma(D)b_{8D}$, which implies that $\tau^*(\gamma(D)) = \gamma(D)$.

Since τ^* induces an involution on $KO_G(D) \cong \mathbb{Z}$, we have $\tau^*(\eta(D)) = \pm\eta(D)$. But since $\tau^*(\eta(D)) \cdot \gamma(D) = \eta(D) \cdot \tau^*(\gamma(D)) = \eta(D)\gamma(D) = 1 - D \neq -\eta(D)\gamma(D)$, we get $\tau^*(\eta(D)) = \eta(D)$.

By formula (25), $\tau^*(a) = a$ implies $\tau^*(ab) = ab$ for any a, b. Therefore we see that τ^* acts as the identity map on $KO_G(kD)$ for $k \neq 0, 4$ mod 8.

(2) $u^2 = 1$ because $\tau^2 = \text{id}$. Since $u \cdot (1 - D) = \tau^*(1 - D) = \tau^*(\gamma(D) \cdot \eta(D)) = \gamma(D) \cdot \eta(D) = 1 - D$, we see that $(u - 1)(1 - D) = 0$. We get $\varphi_0(u) = 1$ by Lemma 5.2.

(3) is straightforward from (2) and Lemma 5.2. □

Now suppose X is a space of type SWF at level l. A choice of G-homotopy equivalence $X^{S^1} \cong (lD)^+$ gives us an inclusion map $i : (lD)^+ \to X$, which we call a trivialization. A trivialization induces the map $i^* : \widetilde{KO}_G(X) \to KO_G(lD)$. Consider the map $\varphi_l \circ i^* : \widetilde{KO}_G(X) \to \mathbb{Z}$.

Proposition 5.6. *The submodule $Im(i^*)$ and the map $\varphi_l \circ i^*$ are both independent of the choice of the trivialization. Moreover, we have $Im(\varphi_l \circ i^*) = (2^k)$ for some $k \in \mathbb{Z}_{\geq 0}$.*

Proof. By Lemma 5.4, there are two possible trivializations i and $i \circ \tau$. We have $\text{Im}(i \circ \tau)^* = \tau^*(\text{Im} i^*) = u \cdot \text{Im}(i^*)$. Since u is a unit, the multiplication by u does not change the submodule $\text{Im}(i^*)$. Moreover, we have $\varphi_l \circ (i \circ \tau)^* = \varphi_l \circ \tau^* \circ i^* = \varphi_l \circ i^*$ by (3) of Lemma 5.5.

For the second statement, we consider the exact sequence:

$$... \to \widetilde{KO}_G(X) \xrightarrow{i^*} KO_G(lD) \xrightarrow{\delta} \widetilde{KO}^1_G(X/X^{S^1}) \to ...$$

Since the G action is free away from the basepoint and $(1-D) \in RO(G)$ is in the augmentation ideal, $(1-D)$ acts on $\widetilde{KO}^1_G(X/X^{S^1})$ nilpotently by Fact 2.8. Therefore, we can find $m \gg 0$ such that $(1-D)^m KO_G(lD) \subset \ker(\delta) = \mathrm{Im}(i^*)$. It follows that $2^m \in \mathrm{Im}(\varphi_l \circ i^*)$ and $\mathrm{Im}(\varphi_l \circ i^*) = (2^k)$ for some $0 \leq k \leq m$. $\square$

Proposition 5.6 justify the following definition:

Definition 5.7. *For a G-space X of type SWF at level l, we define $\mathcal{J}(X)$ to be the image of i^* for any trivialization i and let $\kappa o(X)$ be the integer k such that $\varphi_l(\mathcal{J}(X)) = (2^k)$.*

Let's study the property of $\mathcal{J}(X)$ and $\kappa o(X)$. First recall that we defined the constants $\beta^0_k = 0$ and $\beta^j_k = \sum_{i=0}^{j-1} \alpha_{k-i}$ for $j \geq 1$, where $\alpha_i = 1$ for $i \equiv 1,2,3,5 \bmod 8$ and $\alpha_i = 0$ for $i \equiv 0,4,6,7 \bmod 8$. It's easy to see that $\beta^k_j = \beta^k_{j'}$ for $j \equiv j' (\mathrm{mod} 8)$. The integers β^k_j are important because of the following proposition:

Proposition 5.8. *For integers $0 \leq j \leq k$ and an admissible map $i : ((k-j)D)^+ \to (kD)^+$, we have the following commutative diagram, where the map $m^j_k : \mathbb{Z} \to \mathbb{Z}$ is the multiplication of $2^{\beta^j_k}$.*

$$\begin{array}{ccc} KO_G(kD) & \xrightarrow{i^*} & KO_G((k-j)D) \\ \downarrow \varphi_k & & \downarrow \varphi_{k-j} \\ \mathbb{Z} & \xrightarrow{m^j_k} & \mathbb{Z} \end{array} \tag{26}$$

Proof. The case $j = 0$ follows from Lemma 5.5. When $j > 0$, by Lemma 5.4, the map i is G-homotopic to the standard inclusion. Because of the associativity of i^* and m^k_l, we only need to prove the case $j = 1$. In this case, the map i^* is just the multiplication by $\gamma(D)$ and m^1_k is the multiplication by 2^{α_k}. Since both φ_k and i^* are compatible with Bott isomorphism, we only need to check the case $k = 1, 2, ..., 8$. This can be proved by straightforward calculations using Definition 5.1, Theorem 2.16 and Theorem 2.13. $\square$

The following proposition studies the behavior of $\mathcal{J}(X)$ and $\kappa o(X)$ under the Bott isomorphism:

Proposition 5.9. *Let X be a space of type SWF at level k. We have the following:*

- *(1) $\mathcal{J}(X) \cdot b_{8D} = \mathcal{J}(\Sigma^{8D} X)$ and $\kappa o(\Sigma^{8D} X) = \kappa o(X)$.*
- *(2) $\mathcal{J}(X) \cdot (K - 2H + D + 5) = \mathcal{J}(\Sigma^{2H} X)$ and $\kappa o(\Sigma^{2H} X) = \kappa o(X) + 2$.*
- *(3) $\kappa o(\Sigma^{H+4D} X) = \kappa o(X) + 3 - \beta^4_{k+4}$.*

Proof. (1) Since $(\Sigma^{8D} X)^{S^1} = \Sigma^{8D}(X^{S^1})$, statement (1) follows from the functoriality of the Bott isomorphism.

(2) We have the commutative diagram induced by the inclusions of subspaces:

$$\begin{array}{ccc} \widetilde{KO}_G(\Sigma^{2H} X) & \longrightarrow & \widetilde{KO}_G(X) \\ \downarrow & & \downarrow \\ \widetilde{KO}_G((\Sigma^{2H} X)^{S^1}) & \xrightarrow{\cong} & \widetilde{KO}_G(X^{S^1}). \end{array} \tag{27}$$

Since $(\Sigma^{2H}X)^{S^1} = \Sigma^{2H}(X^{S^1})$, the map in the bottom row is the identity. If we identify $\widetilde{KO}_G(\Sigma^{2H}X)$ with $\widetilde{KO}_G(X)$ using the Bott isomorphism, then the top horizontal map is the multiplication by $\gamma(H)^2 b_{2H} = K - 2H + D + 5$ (by Theorem 2.16). This implies $\mathcal{J}(\Sigma^{2H}X) = (K - 2H + D + 5)\mathcal{J}(X)$. We also have $\kappa o(\Sigma^{2H}X) = \kappa o(X) + 2$ since $\varphi_0(K - 2H + D + 5) = 4$.

(3) Again, by inclusions of subspaces, we have:

$$\begin{array}{ccc} \widetilde{KO}_G(\Sigma^{H+4D}X) & \longrightarrow & \widetilde{KO}_G(X) \\ \downarrow & & \downarrow \\ KO_G((\Sigma^{H+4D}X)^{S^1}) & \xrightarrow{\cdot\gamma(D)^4} & KO_G(X^{S^1}). \end{array}$$

Since $(\Sigma^{H+4D}X)^{S^1} \cong \Sigma^{4D}(X^{S^1})$, the bottom horizontal map is the multiplication by $\gamma(D)^4$. If we identify $\widetilde{KO}_G(\Sigma^{H+4D}X)$ with $\widetilde{KO}_G(X)$ using the Bott isomorphism, the top horizontal map is the multiplication by $\gamma(H + 4D)b_{H+4D} = 4(1 - D)$ (by Theorem 2.16). Therefore, under appropriate trivializations, we see that the maps $i_1^* : \widetilde{KO}_G(X) \cong \widetilde{KO}_G(\Sigma^{H+4D}X) \to KO_G((k+4)D)$ and $i_2^* : \widetilde{KO}_G(X) \to KO_G(kD)$ are related by $\gamma(D)^4 \cdot i_1^*(x) = 4(1 - D) \cdot i_2^*(x)$. Since $\varphi_0(4(1 - D)) = 8$, statement (3) follows from Proposition 5.8 (for $j = 4$) and Lemma 5.2. □

We have the following proposition, which is the analogue of Lemma 3.8 in [18].

Proposition 5.10. *Let X_1 and X_2 be spaces of type SWF. Suppose there is a based G-equivariant homotopy equivalence f from $\Sigma^{r\mathbb{R}}X_1$ to $\Sigma^{r\mathbb{R}}X_2$, for some $r \geq 0$. Then we have $\mathcal{J}(X_1) = \mathcal{J}(X_2)$ and $\kappa o(X_1) = \kappa o(X_2)$.*

Proof. The proof in [18] works with some modifications. Suppose X_1, X_2 are both at level k. By (1) of Proposition 5.9, we can replace X_i by $\Sigma^{8D}X_i$ and assume $k > 1$. Also, we can suspend some more copies of $\mathbb{R}$ and assume that $8|r$. Choose trivilizations i_1, i_2 of X_1 and X_2, respectively. They give homotopy equivalences $(r\mathbb{R} + kD)^+ \cong (\Sigma^{r\mathbb{R}}X_1)^{S^1}$ and $(r\mathbb{R} + kD)^+ \cong (\Sigma^{r\mathbb{R}}X_2)^{S^1}$. Composing them with $f^{S^1} : (\Sigma^{r\mathbb{R}}X_1)^{S^1} \to (\Sigma^{r\mathbb{R}}X_2)^{S^1}$, we get the equivariant homotopy equivalence $h : (r\mathbb{R} + kD)^+ \to (r\mathbb{R} + kD)^+$. Since $k > 1$, by equivariant Hopf theorem, h is based homotopic to $\tau_1 \wedge \tau_2$. The map $\tau_1 : (r\mathbb{R})^+ \to (r\mathbb{R})^+$ is either identity or a map with degree -1. Therefore, $\tau_1^*(b_{r\mathbb{R}}) = a \cdot b_{r\mathbb{R}}$ where $b_{r\mathbb{R}}$ is the Bott class and $a \in RO(G)$ is a unit. Also, $\tau_2 : (kD)^+ \to (kD)^+$ is either identity or the map τ we defined before. Therefore, $\tau_2^*(x)$ is either x or ux (see Lemma 5.5). We have shown that the map $h^* : \widetilde{KO}_G((r\mathbb{R} + kD)^+) \to \widetilde{KO}_G((r\mathbb{R} + kD)^+)$ is just multiplication by some unit in $RO(G)$, which does not change any submodule.

Now consider the following commutative diagram:

$$\begin{array}{ccccccc} \widetilde{KO}_G(X_2) & \xrightarrow{\cong} & \widetilde{KO}_G(\Sigma^{r\mathbb{R}}X_2) & \xrightarrow{f^*} & \widetilde{KO}_G(\Sigma^{r\mathbb{R}}X_1) & \xrightarrow{\cong} & \widetilde{KO}_G(X_1) \\ \downarrow{\scriptstyle i_2^*} & & \downarrow{\scriptstyle (\Sigma^{r\mathbb{R}}i_2)^*} & & \downarrow{\scriptstyle (\Sigma^{r\mathbb{R}}i_1)^*} & & \downarrow{\scriptstyle i_1^*} \\ KO_G(kD) & \xrightarrow{\cong} & \widetilde{KO}_G((r\mathbb{R} + kD)^+) & \xrightarrow{h^*} & \widetilde{KO}_G((r\mathbb{R} + kD)^+) & \xrightarrow{\cong} & KO_G(kD). \end{array}$$

In each row, the first map is a Bott isomorphism and the third map is the inverse to a Bott isomorphism. We see that $b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*) = h^*(b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*)) = b_{r\mathbb{R}} \cdot \mathrm{Im}(i_1^*)$. Therefore, we have $\mathrm{Im}(i_1^*) = \mathrm{Im}(i_2^*)$, which implies $\kappa o(X_1) = \kappa o(X_2)$. □

Definition 5.11. *For a spectrum class $S = [(X, a, b)] \in \mathfrak{C}$, we let*

$$\kappa o(S) = \kappa o(\Sigma^{(8M-a)D}\Sigma^{(2N-b')H}X) - 2N - s \tag{28}$$

for any $M, N, b' \in \mathbb{Z}$ and $s \in [0, 1)$ making $8M - a \geq 0, 2N - b' \geq 0$ and $b = b' + s$.

Proposition 5.12. *$\kappa o(S)$ is well defined.*

Proof. By (1) and (2) of Proposition 5.9, it's easy to prove that the righthand side of formula (5.11) is independent of the choice of M, N. By choosing $M, N \gg 0$, we see that changing the representative of S from (X, a, b) to $(\Sigma^D X, a+1, b)$ or $(\Sigma^H X, a, b+1)$ does not change the value of $\kappa o(S)$. By Definition 4.3 and Proposition 5.10, we proved that $\kappa o(S)$ does not change when we change the representative of the spectrum class. □

By definition of the suspension of a spectrum class and Proposition 5.9, it is easy to prove:

Proposition 5.13. *For any spectrum class $S \in \mathfrak{C}$ at level k, we have:*
- $\kappa o(\Sigma^{8D}S) = \kappa o(S)$.
- $\kappa o(\Sigma^{2H}S) = \kappa o(S) + 2$.
- $\kappa o(\Sigma^{H+4D}S) = \kappa o(S) + 3 - \beta^4_{k+4}$.

With these discussions, we can now define the invariants for three manifolds.

Definition 5.14. *For an oriented rational homology sphere Y and a spin structure $\mathfrak{s}$ on Y, we define $\kappa o_i(Y, \mathfrak{s}) = \kappa o(\Sigma^{iD}S(Y, \mathfrak{s}))$ for any $i \in \mathbb{Z}_{\geq 0}$. Then $\kappa o_i(Y, \mathfrak{s}) = \kappa o_{i+8}(Y, \mathfrak{s})$, which allow us to define $\kappa o_i(Y, \mathfrak{s})$ for $i \in \mathbb{Z}/8$.*

6. Proof of Theorem 1.6

In this section, we will prove Theorem 1.6.

Let X_0, X_1 be be two spaces of type SWF at level k_0 and k_1, respectively. Suppose there is an admissible map $f : X_0 \to X_1$ (which implies $k_0 \leq k_1$). By Lemma 5.8, we can choose suitable trivializations such that the following diagram commutes.

$$\begin{array}{ccc}
\widetilde{KO}_G(X_1) & \xrightarrow{f^*} & \widetilde{KO}_G(X_0) \\
\downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_0^*} \\
KO_G(k_1 D) & \xrightarrow{(f^{S^1})^*} & KO_G(k_0 D) \\
\downarrow{\scriptstyle \varphi_{k_1}} & & \downarrow{\scriptstyle \varphi_{k_0}} \\
\mathbb{Z} & \xrightarrow{m_{k_1}^{k_1-k_0}} & \mathbb{Z}
\end{array}$$

Therefore, we get $m_{k_1}^{k_1-k_0}(\mathrm{Im}(\varphi_{k_1} \circ i_1^*)) \subset \mathrm{Im}(\varphi_{k_0} \circ i_0^*)$. This implies that $(2^{\kappa o(X_1) + \beta_{k_1}^{k_1-k_0}}) \subset (2^{\kappa o(X_0)}) \subset \mathbb{Z}$. Therefore, we get the following proposition:

Proposition 6.1. *Let X_0, X_1 be two spaces of type SWF at level k_0 and k_1, respectively. Suppose there is an admissible map $f : X_0 \to X_1$. Then we have:*

$$\kappa o(X_0) \leq \kappa o(X_1) + \beta_{k_1}^{k_1-k_0}. \tag{29}$$

Next we generalize the above inequality to the spectrum classes:

Definition 6.2. *Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes. We call S_0 dominates S_1 if we can find representatives $S_i = [(X_i, a, b)]$ for $i = 1, 2$ and an admissible map f from X_0 to X_1.*

Proposition 6.3. *Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes at level k_0 and k_1 respectively. Suppose S_0 dominates S_1, then we have:*

$$\kappa o(S_0) \leq \kappa o(S_1) + \beta_{k_1}^{k_1-k_0}. \tag{30}$$

Proof. Since an admissible map $f : X_0 \to X_1$ gives an admissible map $\Sigma^{aH+bD} f : \Sigma^{aH+bD} X_0 \to \Sigma^{aH+bD} X_1$ for any $a, b \in \mathbb{Z}_{\geq 0}$. This proposition is a straightforward corollary of Proposition 6.1 and Definition 5.11. □

By considering the natural inclusion $X \to \Sigma^D X$, it is easy to see that S always dominates $\Sigma^D S$. Therefore ,we get the following corollary, which will be useful in Section 8.

Corollary 6.4. *For any spectrum class $S \in \mathfrak{C}$ at level k. We have:*

$$\kappa o(S) \leq \kappa o(\Sigma^D S) + \alpha_{k+1}.$$

Now let Y_0, Y_1 be two rational homology three-spheres and $\mathfrak{s}_i$ be spin structures on them respectively. Suppose $(W, \mathfrak{s})$ is a smooth oriented spin cobordism from $(Y_0, \mathfrak{s}_0)$ to $(Y_1, \mathfrak{s}_1)$. After doing surgery along loops in W, we can assume $b_1(W) = 0$ without loss of generality. Then by Theorem 4.10, we see that $\Sigma^{-\frac{\sigma(W)}{16}H} S(Y_0, \mathfrak{s}_0)$ dominates $\Sigma^{b_2^+(W)D} S(Y_1, \mathfrak{s}_1)$. We can do suspensions and prove $\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0))$ dominates $\Sigma^{(b_2^+(W)+k)D} S(Y_1, \mathfrak{s}_1)$ for any $k \in \mathbb{Z}$. Applying Proposition 6.3, we get:

Theorem 6.5. *Suppose $(W, \mathfrak{s})$ is a smooth, oriented spin cobordism from $(Y_0, \mathfrak{s}_0)$ to $(Y_1, \mathfrak{s}_1)$. Then for any $k \in \mathbb{Z}$, we have the inequality:*

$$\kappa o_{k+b_2^+(W)}(Y_1, \mathfrak{s}_1) + \beta_{k+b_2^+(W)}^{b_2^+(W)} \geq \kappa o(\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0))). \tag{31}$$

In general, $\kappa o(\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0)))$ can be expressed by $\kappa o_k(Y_0, \mathfrak{s}_0)$ or $\kappa o_{k+4}(Y_0, \mathfrak{s}_0)$, but the explicit formula is messy. For simplicity, we now focus on the integral homology sphere case.

Remark 6.6. *Suppose Y is an oriented integral homology three-sphere. There is a unique spin structure $\mathfrak{s}$ on Y and we simply write $S(Y, \mathfrak{s})$ and $\kappa o_i(Y, \mathfrak{s})$ as $S(Y)$ and $\kappa o_i(Y)$, respectively.*

Suppose both Y_i are integral homology spheres, then the intersection form of W is a unimodular, even form. Let's assume that the intersection from can be decomposed as:

$$p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \text{ for } p, q \geq 0.$$

In this case, we have $\frac{\sigma(W)}{16} = -\frac{p}{2}$ and $b_2^+(W) = q$. Recall that the spectrum class invariant $S(Y_0)$ is defined by $[(I_\nu, \dim_{\mathbb{R}} V(D)^0_{-\nu}, \dim_{\mathbb{H}} V(H)^0_{-\nu} + \frac{1}{2}n(Y_0, \mathfrak{s}, g))]$. The third component of this triple may be an integer or a half integer, depending on the Rokhlin invariant $\mu(Y_0)$.

Proposition 6.7. *Let Y_0 be an integral homology three sphere and $p \in \mathbb{Z}_{\geq 0}$. Then we have the following relations.*

(1) Suppose $\mu(Y_0) = 0 \in \mathbb{Z}_2$.

- *For $p = 4l$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l$.*
- *For $p = 4l+1$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + \frac{5}{2} + 2l - \beta^4_k$.*
- *For $p = 4l+2$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + 3 + 2l - \beta^4_k$.*
- *For $p = 4l+3$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l + \frac{3}{2}$.*

(2) Suppose $\mu(Y_0) = 1 \in \mathbb{Z}_2$.

- *For $p = 4l$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l$.*
- *For $p = 4l+1$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l + \frac{1}{2}$.*
- *For $p = 4l+2$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + 3 + 2l - \beta^4_k$.*
- *For $p = 4l+3$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + \frac{7}{2} + 2l - \beta^4_k$.*

Proof. Let's denote $(I_\nu, \dim_{\mathbb{R}} V(D)^0_{-\nu}, \dim_{\mathbb{H}} V(H)^0_{-\nu} + \frac{1}{2}n(Y_0, \mathfrak{s}, g))$ by (X, a, b).

For $\mu(Y_0) = 0$ and $p = 4l$, we have $b \in \mathbb{Z}$. Take $M, N \gg 0$ and let $N' = N + l$. Then by Definition 5.11, we have:

$$\begin{aligned} \kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0)) &= \kappa o(\Sigma^{(8M+k-a)D} \Sigma^{(2N+2l-b)H} X) - 2N \\ &= \kappa o(\Sigma^{(8M+k-a)D} \Sigma^{(2N'-b)H} X) - 2N' + 2l = \kappa o_k(Y) + 2l. \end{aligned} \tag{32}$$

For $p = 4l+1$, take $M, N \gg 0$ and let $N' = N + l$. Then we have:

$$\begin{aligned} \kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0)) &= \kappa o(\Sigma^{(8M+k-a)D} \Sigma^{(2N+2l+1-b)H} X) - 2N - \frac{1}{2} \\ &= \kappa o(\Sigma^{H}(\Sigma^{kD}(X, a, b))) + 2l - \frac{1}{2} = \kappa o_{k+4}(Y_0) + \frac{5}{2} + 2l - \beta^4_k. \end{aligned} \tag{33}$$

The other cases can be proved similarly. □

Now combining the above proposition and Theorem 6.5, we proved Theorem 1.6.

7. KO_G-Split condition

Now consider the space $X = (8kD + (2l+1)H)^+$ for $k, l \in \mathbb{Z}_{\geq 0}$. We have the map induced by the inclusion:

$$i^* : \widetilde{KO}_G(X) \to KO_G(8kD).$$

By Theorem 2.13, we see that $KO_G(8kD + (2l+1)H)$ is generated by $(b_{2H})^l(b_{8D})^k\lambda(H)$ and $(b_{2H})^l(b_{8D})^k c(H)$ as $RO(G)$-module and the map i^* is multiplication by $\gamma(H)^{2l+1}$. Using Proposition 2.16, we get:

$$\begin{aligned} i^*((b_{2H})^l(b_{8D})^k\lambda(H)) = (2 + A - 2D - 2B)^l(2 - 2D - B) \cdot (b_{8D})^k, \\ i^*((b_{2H})^l(b_{8D})^k c(H)) = (A - 2B)^l(B - A) \cdot (b_{8D})^k. \end{aligned} \tag{34}$$

The above discussion motivates the following definition:

Definition 7.1. *Let X be a space of type SWF at level $8k$. X is called even KO_G-split if $\mathcal{J}(X)$ is the submodule generated by $(2 + A - 2D - 2B)^l(2 - 2D - B) \cdot (b_{8D})^k$ and $(A - 2B)^l(B - A) \cdot (b_{8D})^k$ for some $l \in \mathbb{Z}_{\geq 0}$.*

Next, we consider the space $X = ((8k + 4)D + 2lH)^+$. The map:

$$i^* : \widetilde{KO}_G(X) \to KO_G((8k + 4)D)$$

is just multiplication of $\gamma(H)^{2l}$. We know $\widetilde{KO}_G(X) = KO_G((8k + 4)D) \cdot (b_{2H})^l$ by the Bott isomorphism. Since $\gamma(H)^{2l}(b_{2H})^l = (K - 2H + D + 5)^l = (A + 2D + 6 - 2H)^l$ (see Theorem 2.16), we have $\mathrm{Im}(i^*) = (A+2D+6-2H)^l \cdot KO_G((8k+4)D) \subset KO_G((8k+4)D)$. This motivates the following definition:

Definition 7.2. *Let X be a space of type SWF at level $8k+4$. X is called odd KO_G-split if $\mathcal{J}(X) = (A + 2D + 6 - 2H)^l \cdot KO_G((8k + 4)D)$ for some $l \in \mathbb{Z}_{\geq 0}$.*

KO_G-split spaces are special because of the following proposition (compare Proposition 6.1).

Proposition 7.3. *Let X_0, X_1 be two spaces of type SWF at level k_0, k_1 respectively and f be an admissible map from X_0 to X_1. Suppose $k_0 < k_1$ and X_0 is odd or even KO_G-split (which implies that $k_0 \equiv 0$ or 4 mod 8). Then we have:*

$$\kappa o(X_0) < \kappa o(X_1) + \beta_{k_1}^{k_1 - k_0}. \tag{35}$$

Before proving this proposition, we need to make a digression into the general properties of $KO_G(4D)$ and $RO(G)$.

Lemma 7.4. *The following properties holds:*

- *(1) Any element in $RO(G)$ can be uniquely written as $bD + f(A) + Bg(A)$ for some polynomials f, g and integer b.*
- *(2) Any element in $RO(G)$ can be uniquely written as $bD + f(A) + Hg(A)$ for some polynomials f, g and integer b.*
- *(3) Any element in $KO_G(4D)$ can be uniquely written as $bD\lambda(D) + f(A)\lambda(D) + g(A)c$ for some polynomials f, g and integer b.*
- *(4) The map $RO(G) \to KO_G(4D)$ defined by multiplication of $\lambda(D)$ is injective.*
- *(5) An element $\omega = bD\lambda(D) + f(A)\lambda(D) + g(A)c$ belongs to $RO(G)\lambda(D)$ if and only if $4|g(A)$. Moreover, if $(A + 2D + 6 - 2H)^l\omega \in RO(G) \cdot \lambda(D)$ for some l, then $\omega \in RO(G) \cdot \lambda(D)$.*
- *(6) Suppose $(A - 2B)^l h(A, B) = 0 \in RO(G)$ for some two-variable polynomial h in A, B. Then we have $h(A, B) = 0$ in $RO(G)$.*

- *(7) Suppose $f(D) = h(A, B)$ for some 2-variable polynomial h without degree-0 term and some polynomial f. Then $h(A, B) = 0$.*

Proof. (1),(2),(3),(4) can be proved by straightforward calculation using Theorem 2.13. The first statement of (5) is the corollary of (2),(3) and the relation $H\lambda(D) = 4c$. Let's prove the second statement of (5). We have $Hc = (1+D+K)\lambda(D)$ and $(2D+6)c = 8c = 2H\lambda(D)$. Therefore, $(A + 2D + 6 - 2H)^l\omega \in RO(G)\lambda(D)$ implies $A^l\omega \in RO(G)\lambda(D)$. It follows that $4|A^lg(A)$, which implies $4|g(A)$ and $\omega \in RO(G)\lambda(D)$.

For (6), we can assume that $h(A, B) = f(A) + Bg(A)$ for some polynomials f, g. Consider the map $\psi : RO(G) \to \mathbb{Q}[x]$ defined by $\psi(D) = 1, \psi(B) = x$ and $\psi(A) = \frac{x^2}{4} + 2x$. Then $0 = \psi((A - 2B)^l(f(A) + Bg(A))) = (\frac{x^2}{4})^l(f(\frac{x^2}{4} + 2x) + xg(\frac{x^2}{4} + 2x))$, which implies $0 = f(\frac{x^2}{4} + 2x) + xg(\frac{x^2}{4} + 2x)$. Considering the leading term in x, we see that $f(x) = g(x) = 0$.

For (7), we can simplify $h(A, B)$ as $Ag_1(A) + Bg_2(A)$ for some polynomials g_1, g_2 by the relation $B^2 - 4(A - 2B) = 0$. Then the conclusion follows from (1). □

Lemma 7.5. *Suppose $a(1 - D)\lambda(D) \in (A + 2D + 6 - 2H)^l KO_G(4D)$ for some $a \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$. Then we have $2^{2l+1}|\varphi_4(a(1 - D)\lambda(D))$.*

Proof. Since $\varphi_4(a(1 - D)\lambda(D)) = 2a$, the conclusion is trivial when $l = 0$. Now suppose $l > 0$. Let $a(1 - D)\lambda(D) = (A + 2D + 6 - 2H)^l \cdot \omega$ for some $\omega \in KO_G(4D)$. By (5) of Lemma 7.4, we see that $\omega \in RO(G)\lambda(D)$. Write ω as $(bD + f(A) + Bg(A))\lambda(D)$. By (4) of Lemma 7.4, we get $a(1 - D) = (A - 2B - 2D + 2)^l(bD + f(A) + Bg(A))$. Using the relation $(1 - D)A = (1 - D)B = 0$, we can simplify this equality as $a(1 - D) - (f(0) + bD)(2 - 2D)^l = (A - 2B)^l(b + f(A) + Bg(A))$. By (7) of Lemma 7.4, we get that $(A - 2B)^l(b + f(A) + Bg(A)) = 0 \in RO(G)$. By (6) of Lemma 7.4, we have $b+f(A)+Bg(A) = 0$. This implies that $\omega = b(D-1)\lambda(D)$ and $\varphi_4(a(1-D)\lambda(D)) = -2^{2l+1}b$ for some $b \in \mathbb{Z}$. □

Lemma 7.6. *Suppose $a(1-D)$ is in the ideal of $RO(G)$ generated by $(2+A-2D-2B)^l(2-2D - B)$ and $(A - 2B)^l(B - A)$ for some $l \in \mathbb{Z}_{\geq 0}$. Then we have $2^{2l+3}|\varphi_0(a(1 - D))$.*

Proof. We assume $l > 0$ first. By (1) of Lemma 7.4 and the relation $A(1-D) = B(1-D) = 0$, we have can express $a(1 - D)$ as:

$$\begin{aligned} &(2 - 2D - B)(2 - 2D + A - 2B)^l(b(1 - D) + f_1(A) + Bg_1(A)) \\ &\qquad + (A - 2B)^l(B - A)(f_2(A) + Bg_2(A)) \end{aligned} \tag{36}$$

for some integer b and polynomials f_1, f_2, g_1, g_2.

As in the proof of Lemma 7.5, we can simplify this formula and use (7) of Lemma 7.4 to get:

$$-B(A-2B)^l(f_1(A)+Bg_1(A))+(A-2B)^l(B-A)(f_2(A)+Bg_2(A)) = 0 \in RO(G). \tag{37}$$

We have $-B(f_1(A) + Bg_1(A)) + (B - A)(f_2(A) + Bg_2(A)) = 0$ by (6) of Lemma 7.4. Simplifying this, we obtain:

$$-4Ag_1(A)-Af_2(A)+4Ag_2(A)+B(-f_1(A)+f_2(A)+8g_1(A)-Ag_2(A)-8g_2(A)) = 0. \tag{38}$$

This implies $-4Ag_1(A)-Af_2(A)+4Ag_2(A)=0$ and $-f_1(A)+8g_1(A)+f_2(A)-Ag_2(A)-8g_2(A)=0$. Considering the degree-1 term of the first identity, we get $4|f_2(0)$. Also, we have $8|-f_1(0)+f_2(0)$ by checking the degree-0 term of the second identity. Therefore, we have $4|f_1(0)$, which implies $\varphi_0(a(1-D))=2^{2l+2}(2b+f_1(0))$ can be divided by 2^{2l+3}.

The case $l=0$ is similar. We also get the identity (38). □

Proof of Proposition 7.3: Consider the commutative diagram:

$$
\begin{array}{ccc}
\widetilde{KO}_G(X_1) & \xrightarrow{f^*} & \widetilde{KO}_G(X_0) \\
\downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_0^*} \\
KO_G(k_1D) & \xrightarrow{(f^{S^1})^*} & KO_G(k_0D) \\
\downarrow{\scriptstyle \varphi_{k_1}} & & \downarrow{\scriptstyle \varphi_{k_0}} \\
\mathbb{Z} & \xrightarrow{m_{k_1}^{k_1-k_0}} & \mathbb{Z}.
\end{array}
$$

(1) Suppose X_0 is odd KO_G-split. Then $k_0=8k+4$ for some integer k and $KO_G(k_0D)=KO_G(4D)\cdot(b_{8D})^k$ by the Bott isomorphism. $\mathrm{Im}(i_0^*)=(A+2D+6-2H)^l\cdot KO_G(4D)\cdot(b_{8D})^k$ for some $l\in\mathbb{Z}_{\geq 0}$. A simple calculation shows that $\kappa o(X_0)=2l$. Suppose $\kappa o(X_1)=r$. Then we can find an element $z\in\widetilde{KO}_G(X_1)$ such that $\varphi_{k_1}i_1^*(z)=2^r$. Therefore, $\varphi_{k_0}(\omega)=2^{r+\beta_{k_1}^{k_1-k_0}}$ where $\omega=(f^{S^1})^*(i_1^*(z))$. Since $k_1>k_0$, the map $(f^{S^1})^*$ factors through $KO_G((k_0+1)D)\to KO_G(k_0D)$. Therefore, we see that $\omega=\gamma(D)\cdot(a\eta(D)\lambda(D))\cdot(b_{8D})^k=a(1-D)\lambda(D)\cdot(b_{8D})^k$ for some $a\in\mathbb{Z}$. Because of the commutative diagram, we have $\omega\in\mathrm{Im}(i_0^*)$. By Lemma 7.5, we get $2^{2l+1}|\varphi_{k_0}(\omega)$. This implies $2l+1\leq r+\beta_{k_1}^{k_1-k_0}$.

(2) Suppose X_0 is even KO_G-split with $k_0=8k$. Notice that $\kappa o(X)=2l+2$ if $\mathcal{J}(X)$ is the submodule generated by $(2+A-2D-2B)^l(2-2D-B)(b_{8D})^k$ and $(A-2B)^l(B-A)(b_{8D})^k$. Using Lemma 7.6, the proof is almost the same with the previous case. □

By Proposition 5.9, we see that $\Sigma^{2H}X$ and $\Sigma^{8D}X$ are even (odd) KO_G-split if X is even (odd) KO_G-split. Therefore, Proposition 5.10 justifies the following definition:

Definition 7.7. *A spectrum class $S=[(X,a,b+r)]$ with $a,b\in\mathbb{Z},r\in[0,1)$ is called even (odd) KO_G-split if for integers $M,N\gg 0$, $\Sigma^{(8M-a)D}\Sigma^{(2N-b)H}X$ is even (odd) KO_G-split.*

Example 7.8. *For any $a,b\in\mathbb{Z}$ and $r\in[0,1)$, $[(S^0,8a,2b+1+r)]$ is even KO_G-split and $[(S^0,8a+4,2b+r)]$ is odd KO_G-split.*

The following proposition is easy to prove using Proposition 7.3

Proposition 7.9. *Let $S_0,S_1\in\mathfrak{C}$ be two spectrum classes at level k_0,k_1 respectively, with $k_0<k_1$. Suppose S_0 is even or odd KO_G-split and S_0 dominates S_1, then we have:*

$$\kappa o(S_0)<\kappa o(S_1)+\beta_{k_1}^{k_1-k_0}. \tag{39}$$

Now let Y be a homology sphere. Recall that we have a spectrum class invariant $S(Y)$ at level 0.

Definition 7.10. *Y is called Floer KO_G-split if $\Sigma^H S(Y)$ is even KO_G-split and $\Sigma^{4D} S(Y)$ is odd KO_G-split.*

Remark 7.11. *For simple examples like $Y = \pm\Sigma(2,3,12n+1)$ or $\pm\Sigma(2,3,12n+5)$, the two conditions in the above definition are either both true or both false. We expect that this fails in more complicated examples. If we only assume one of these two conditions, only half of the cases in Theorem 1.11 are still true.*

Remark 7.12. *We will see in Section 8 that $S^3, \pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are Floer KO_G-split, while $+\Sigma(2,3,12n+5)$ is not Floer KO_G-split.*

Proof of Theorem 1.11: (1) When $\mu(Y_0) = 0$, $S(Y_0) = [(X,a,b)]$ for some space X and some integers a,b. For large integers M,N, we have the following:
(i) The space $\Sigma^{(8M-a)D}\Sigma^{(2N-b+1)H}X$ is even KO_G-split.
(ii) The space $\Sigma^{(8M-a+4)D}\Sigma^{(2N-b)H}X$ is odd KO_G-split.
Now consider $p = 4l+m$ for $m = 0,1,2,3$:

- For $p = 4l$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b-2l)]$ is odd KO_G-split by (ii).
- For $p = 4l+1$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b-2l+\frac{1}{2})]$ is even KO_G-split by (i).
- For $p = 4l+2$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b-2l)]$ is even KO_G-split by (i).
- For $p = 4l+3$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b-2l-2+\frac{1}{2})]$ is odd KO_G-split by (ii).

Similarly, we can prove that when $\mu(Y_0) = 1$, $\Sigma^{\frac{p}{2}H}S(Y_0)$ is even KO_G-split for $p = 4l+2$ and $4l+3$ while $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0)$ is odd KO_G-split for $p = 4l$ and $4l+1$.

Now repeat the proof of Theorem 1.6 for $k = 0$ or 4, using Proposition 7.9 instead of Proposition 6.3. Notice that the two sides of the same inequalities are either both integers or both half-integers. The inequalities are proved. □

8. Examples and Explicit bounds

In this section, we will prove Theorem 1.9 about the values of $\kappa o_i(S^3)$ and $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(r,6) = 1$. We will also use Corollary 1.12 to give some new bounds about the intersection forms of spin four manifolds with given boundaries.

8.1. **Basic Examples.**

If Y is a rational homology sphere admitting metric g with a positive scaler curvature, then by the arguments in [17], we obtain:

$$S(Y,\mathfrak{s}) = [(S^0, 0, n(Y,\mathfrak{s})/2)].$$

In particular, S^3 is Floer KO_G-split and $\kappa o_i(S^3) = 0$ for any $i \in \mathbb{Z}/8$.

In [18], Manolescu gave two examples of spaces of type SWF that are related to the spectrum class invariants of the Brieskorn spheres $\pm\Sigma(2,3,r)$. We recall the construction here.

Suppose that G acts freely on a finite G-CW complex Z, with the quotient space $Q = Z/G$. Let

$$\widetilde{Z} = ([0,1]\times Z)/(0,z) \sim (0,z') \text{ and } (1,z) \sim (1,z') \text{ for all } z,z' \in Z$$

denote the unreduced suspension of Z, where G acts trivially on the $[0,1]$ factor. We can take one of the two cone points (say $(0,z) \in \widetilde{Z}$) as the base point and view $\widetilde{Z}$ as a pointed G-space. It's easy to see that $\widetilde{Z}$ is of type SWF at level 0.

We want to compute $\kappa o(\Sigma^{kD}\tilde{Z})$ for $k = 0, 1, ..., 7$. It turns out that the method in [18] also works here. Namely, the inclusion $i : (\Sigma^{kD}\tilde{Z})^{S^1} = \Sigma^{kD}S^0 \to \Sigma^{kD}\tilde{Z}$ gives the long exact sequence:

$$... \to \widetilde{KO}_G(\Sigma^{kD}\tilde{Z}) \xrightarrow{i^*} KO_G(kD) \xrightarrow{p^*} KO^1_G(\Sigma^{kD}\tilde{Z}, (kD)^+) \to .... \qquad (40)$$

By exactness of the sequence, we have $\operatorname{Im}(i^*) = \ker(p^*)$. By definition, we have:

$$KO^1_G(\Sigma^{kD}\tilde{Z}, (kD)^+) \cong \widetilde{KO}^1_G(\Sigma^{kD}\Sigma Z_+) \cong \widetilde{KO}_G(\Sigma^{kD}Z_+).$$

By abuse of notation, we still use p^* to represent the map between $KO_G(kD)$ and $\widetilde{KO}_G(\Sigma^{kD}Z_+)$. Checking the maps in the exact sequence, one can see that the p^* is induced by the natural projection $p : \Sigma^{kD}Z_+ \to (kD)^+$. Since G acts freely on $\Sigma^{kD}Z_+$ away from the base point, we see that $\widetilde{KO}_G(\Sigma^{kD}Z_+) \cong \widetilde{KO}((\Sigma^{kD}Z_+)/G)$. Notice that $(Z \times kD)/G$ is a vector bundle over Q and $(\Sigma^{kD}Z_+)/G$ is the Thom space of this bundle. We are interested in two cases:

- $Z \cong G$, acting on itself via left multiplication.
- $Z \cong T \cong S^1 \times jS^1 \subset \mathbb{C} \times j\mathbb{C} \subset \mathbb{H}$ and G acts on T by left multiplication in $\mathbb{H}$.

The first case is easy since the isomorphism $\widetilde{KO}_G(\Sigma^{kD}Z_+) \cong \widetilde{KO}(S^k)$ is given by $i_1^* \circ r_0$, where $i_1 : S^k \to \Sigma^{k\mathbb{R}}Z_+$ is the standard inclusion and $r_0 : \widetilde{KO}_G(\Sigma^{kD}Z_+) \to \widetilde{KO}(\Sigma^{k\mathbb{R}}Z_+)$ is the restriction map (See Fact 2.4 in Section 2). It follows that $\operatorname{Im}(i^*) = \ker(p^*) = \ker(i_1^* \circ r_0 \circ p^*) = \ker(r)$, where $r : KO_G(kD) \to \widetilde{KO}(S^k)$ is the restriction map.

We know the structure of $\widetilde{KO}(S^k)$:

- $\widetilde{KO}(S^0) \cong KO(pt) \cong \mathbb{Z}$.
- $\widetilde{KO}(S^1) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the Hopf map in $\pi_3(S^2)$.
- $\widetilde{KO}(S^2) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the square of the Hopf map.
- $\widetilde{KO}(S^4) \cong \mathbb{Z}$, generated by $V_{\mathbb{H}} - 4$, where $V_{\mathbb{H}}$ is the quaternion Hopf bundle.
- $\widetilde{KO}(S^k) \cong 0$ for $k = 3, 5, 6, 7$.

Therefore, by the explicit description of $\eta(D), \lambda(D), c$ after Theorem 2.13. We get the following results about the kernel of $r : KO_G(kD) \to \widetilde{KO}(S^k)$.

- For $k = 0$, $\ker(r)$ is the submodule generated by $1 - D, A, B$.
- For $k = 1$, $\ker(r)$ is generated by $2\eta(D)$.
- For $k = 2$, $\ker(r)$ is generated by $2\eta(D)^2$ and $\gamma(D)^2c$.
- For $k = 4$, $\ker(r)$ is generated by $\lambda(D) - c, (1 - D)\lambda(D), A\lambda(D)$ and Ac.
- For $k = 3, 5, 6, 7$, $\ker(r) \cong KO_G(kD)$.

From this, we get:

Proposition 8.1. *$\kappa o(\Sigma^{kD}\tilde{G}) = 0$ for $k = 3, 4, 5, 6, 7$ and $\kappa o(\Sigma^{kD}\tilde{G}) = 1$ for $k = 0, 1, 2$.*

Now let's consider the case $Z \cong T$. We want to find $\ker(p^*)$ for $p^* : KO_G(kD) \to KO_G(\Sigma^{kD}T_+)$. Notice that $S^1 \subset G$ acts trivially on $(kD)^+$ and freely on T with $T/S^1 =$

S^1. We have $\widetilde{KO}_G(\Sigma^{kD}T_+) = \widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2)$. The space $(\Sigma^{kD}S^1_+)/\mathbb{Z}_2$ can be identified with:

$[0,1]\times(kD)^+/(0,x)\sim(1,-x)$ and $(t_1,\infty)\sim(t_2,\infty)$ for any $x\in(kD)^+$ and $t_1,t_2\in[0,1]$.

Consider the inclusion $i_2:\{0\}\times(kD)^+\to(\Sigma^{kD}S^1_+)/\mathbb{Z}_2$. Notice that $((\Sigma^{kD}S^1_+)/\mathbb{Z}_2)/(kD)^+ \cong S^{k+1}$. We get the long exact sequence:

$$... \to \widetilde{KO}(S^{k+1}) \xrightarrow{\delta} \widetilde{KO}(S^{k+1}) \to \widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2) \xrightarrow{i_2^*} \widetilde{KO}(S^k) \to ... \tag{41}$$

By checking the iterated mapping cone construction, which gives us this long exact sequence, it is not hard to prove that δ is induced by the map $f: S^{k+1}\to S^{k+1}$ with $\deg(f)=0$ for even k and $\deg(f)=2$ for odd k.

When $k=2,4,5,6$, we have $\widetilde{KO}(S^{k+1})=0$. Therefore, i_2^* is injective, which implies $i_1^*\circ r_0:\widetilde{KO}_G(\Sigma^{kD}T_+)\to\widetilde{KO}((kD)^+)$ is injective (i_1^* and r_0 are defined as in the case $Z\cong G$). We see that when $k=2,4,5,6$, just like the case $Z\cong G$, the kernel of p^* is the kernel of the restriction map $r: KO_G(kD)\to\widetilde{KO}(S^k)$. Thus, we get $\kappa o(\Sigma^{kD}\tilde{T})=\kappa o(\Sigma^{kD}\tilde{G})$ for $k=2,4,5,6$.

For $k=0$, consider $[0,1]$ as the subset $\{1+je^{i\theta}|\theta\in[0,\pi]\}\subset T$. The left endpoint is mapped to the right endpoint under the action of $-j\in G$. This embedding of $[0,1]$ gives us the following explicit description of the map $p^*: RO(G)\cong\widetilde{KO}_G(S^0)\to\widetilde{KO}_G(T_+)\cong KO_G(T)\cong KO(T/G)=KO(S^1)$.

Starting from a representation space V of G, we get an trivial bundle $V\times[0,1]$ over $[0,1]$. Identifying $(x,0)$ with $((-j)\circ x,1)$ for any $x\in V$, we get a bundle E over S^1. $[E]\in KO(S^1)$ is the image of $[V]\in RO(G)$ under p^*.

We know that $KO(S^1)$ is generated by the one dimensional trivial bundle [1] and the one dimensional nontrivial bundle $[m]$, subject to the relation 2([1]-[m])=0. Using the explicit description of p^*, we see that $p^*(1)=[1]$, $p^*(D)=[m]$ and $p^*(A)=p^*(B)=0$. Therefore, we get $\kappa o(\tilde{T})=2$.

Applying Corollary 6.4 for $S=\Sigma^{2D}\tilde{T}$, we get $\kappa o(\Sigma^{3D}\tilde{T})+1\geq\kappa o(\Sigma^{2D}\tilde{T})=1$. Applying Corollary 6.4 for $S=\Sigma^{3D}\tilde{T}$, we get $0=\kappa o(\Sigma^{4D}\tilde{T})+0\geq\kappa o(\Sigma^{3D}\tilde{T})$. Therefore, we see that $\kappa o(\Sigma^{3D}\tilde{T})=0$.

Applying Corollary 6.4 for $S=\Sigma^{2D}\tilde{T}$ and $S=\Sigma^{D}\tilde{T}$, we get $\kappa o(\Sigma^{D}\tilde{T})=1$ or 2.

For $k=7$, the map $\delta:\widetilde{KO}(S^8)\to\widetilde{KO}(S^8)$ is multiplication by 2. Since $\widetilde{KO}(S^7)=0$, we get $\widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2)=\mathbb{Z}_2$. This implies $p^*(2b_{8D}\cdot\gamma(D))=2p^*(b_{8D}\cdot\gamma(D))=0$. Therefore, $2b_{8D}\cdot\gamma(D)\in\ker(p^*)$ and $\kappa o(\Sigma^{7D}\tilde{T})=0$ or 1.

Lemma 8.2. *$\kappa o(\Sigma^{D}\tilde{T})=2$ and $\kappa o(\Sigma^{7D}\tilde{T})=1$.*

Proof. This can be proved directly using Gysin sequence. But here we use a different approach. In [18] and [20], Manolescu proved that $S(-\Sigma(2,3,11))=[(\tilde{T},0,1)]$, where $-\Sigma(2,3,11)$ is a negative oriented Brieskorn sphere. Therefore, by Definition 5.11 and Proposition 5.13, we get:

$$\kappa o_i(-\Sigma(2,3,11))=\kappa o(\Sigma^{(i+4)D}\tilde{T})+1-\beta_i^4.$$

In particular, $\kappa o_3(-\Sigma(2,3,11)) = \kappa o(\Sigma^{7D}\tilde{T}) - 2$ and $\kappa o_5(-\Sigma(2,3,11)) = \kappa o(\Sigma^{D}\tilde{T}) - 2$. Since $-\Sigma(2,3,11)$ bounds a smooth spin four manifold with intersection from $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (see [18]). We can apply Corollary 1.12 for $p = 0, q = 1$ and get $\kappa o_5(-\Sigma(2,3,11)) \geq 0$, which implies $\kappa o(\Sigma^{D}\tilde{T}) \geq 2$. We get $\kappa o(\Sigma^{D}\tilde{T}) = 2$ by our discussion before the lemma.

We can also apply Theorem 1.6 for $Y_0 = S^3, Y_1 = -\Sigma(2,3,11), p = 0, q = 1$ and $k = 2$. We have $\kappa o_3(-\Sigma(2,3,11)) \geq -1$ and $\kappa o(\Sigma^{7D}\tilde{T}) \geq 1$. Therefore, $\kappa o(\Sigma^{7D}\tilde{T}) = 1$ by our discussions before. □

We summarise our results in the following proposition.

Proposition 8.3. *$\kappa o(\Sigma^{kD}\tilde{T}) = 2$ for $k = 0, 1$; $\kappa o(\Sigma^{kD}\tilde{T}) = 1$ for $k = 2, 7$ and $\kappa o(\Sigma^{kD}\tilde{T}) = 0$ for $k = 3, 4, 5, 6$.*

Now we calculate $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(6,r) = 1$. Actually, the spectrum class invariants $S(\pm\Sigma(2,3,r))$ are given in [18].

Proposition 8.4 (Manolescu [18])**.** *We have the following results about $S(\pm\Sigma(2,3,r))$.*

- $S(\Sigma(2,3,12n-1)) = [(\tilde{G} \vee \underbrace{\Sigma G_+ \vee ... \vee \Sigma G_+}_{n-1}, 0, 0)]$.
- $S(-\Sigma(2,3,12n-1)) = [(\tilde{T} \vee \underbrace{\Sigma^2 G_+ \vee ... \vee \Sigma^2 G_+}_{n-1}, 0, 1)]$.
- $S(\Sigma(2,3,12n-5)) = [(\tilde{G} \vee \underbrace{\Sigma G_+ \vee ... \vee \Sigma G_+}_{n-1}, 0, 1/2)]$.
- $S(-\Sigma(2,3,12n-5)) = [(\tilde{T} \vee \underbrace{\Sigma^2 G_+ \vee ... \vee \Sigma^2 G_+}_{n-1}, 0, 1/2)]$.
- $S(\Sigma(2,3,12n+1)) = [(S^0 \vee \underbrace{\Sigma^{-1} G_+ \vee ... \vee \Sigma^{-1} G_+}_{n}, 0, 0)]$.[8]
- $S(-\Sigma(2,3,12n+1)) = [(S^0 \vee \underbrace{G_+ \vee ... \vee G_+}_{n}, 0, 0)]$.
- $S(\Sigma(2,3,12n+5)) = [(S^0 \vee \underbrace{\Sigma^{-1} G_+ \vee ... \vee \Sigma^{-1} G_+}_{n}, 0, -1/2)]$.
- $S(-\Sigma(2,3,12n+5)) = [(S^0 \vee \underbrace{G_+ \vee ... \vee G_+}_{n}, 0, 1/2)]$.

As we mentioned in Remark 7.12, $\pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are KO_G-split because of Example 7.8. While using the relations in Theorem 2.13 and Theorem 2.16, it is not hard to prove that the space $(8MD \oplus (2N+2)H)^+$ is not even KO_G-split for integers $M, N \gg 0$. This implies that $+\Sigma(2,3,12n+5)$ is not KO_G-split.

Since it's easy to see that wedging with a free G-space does not change the κo invariants, we don't need to consider those $\Sigma^l G_+$ factors. By Definition 5.11 and Proposition 5.13, we can use Proposition 8.1 and Proposition 8.3 to prove the results in Theorem 1.9 easily.

---

[8] Strictly speaking, by this we mean the spectrum class of $(\mathbb{H}^+ \vee \underbrace{\Sigma^3 G_+ \vee ... \vee \Sigma^3 G_+}_{n}, 0, 1)$.

8.2. **Explicit Bounds.** Now we use Corollary 1.12 and Proposition 3.1 to get explicit bounds on the intersection forms of spin 4-manifolds with boundary $\pm\Sigma(2,3,r)$.

Theorem 8.5. *Let W be an oriented, smooth spin 4-manifold with $\partial W = \pm\Sigma(2,3,r)$. Assume that the intersection form of W is $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p>1, q>0$.*[9] *If the mod 8 reduction of p is m, then we have $q-p\geq c_m$, where c_m are constants listed below. (Recall that the mod 2 reduction of p is the Rohklin invariant of the boundary.)*

	$m=0$	$m=2$	$m=4$	$m=6$
$\Sigma(2,3,12n-1)$	2	0	1	2
$-\Sigma(2,3,12n-1)$	3	(2)	(3)	3
$\Sigma(2,3,12n+1)$	(3)	1	(2)	(3)
$-\Sigma(2,3,12n+1)$	3	1	2	3

	$m=1$	$m=3$	$m=5$	$m=7$
$\Sigma(2,3,12n-5)$	1	2	3	3
$-\Sigma(2,3,12n-5)$	2	(1)	(2)	2
$\Sigma(2,3,12n+5)$	(2)	0	(1)	(2)
$-\Sigma(2,3,12n+5)$	2	3	4	4

Remark 8.6. *Some of the bounds in Theorem 8.5 can also be obtained by other methods. For example, the case $m=2$ for $\Sigma(2,3,12n+1)$ can be obtained using κ-invariant (see* [18]*). Also, some bounds can be obtained by filling method for small n. For example, the case $m=2,4$ for $-\Sigma(2,3,11)$ can be deduced from Theorem 1.2, using the fact that $\Sigma(2,3,11)$ bounds a spin 4-manifold with intersection form $2(-E_8)\oplus 2\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. However, the bounds that we put in the brackets in Theorem 8.5 appear to be new for general n.*

Proof. Since we can do surgeries on loops without changing intersection forms, we will always assume $b_1(W)=0$.

(1) Suppose $\Sigma(2,3,12n+1)$ bounds a spin 4-manifold with intersection form $8l(-E_8)\oplus(8l+2)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l>0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+1)$ to S^3 with the same intersection form. By Theorem 4.10, $\Sigma^{4lH}S(-\Sigma(2,3,12n+1))$ dominates $\Sigma^{8l+2}S(S^3)$. Since $S(-\Sigma(2,3,12n+1))=[(S^0\vee G_+\vee\ldots\vee G_+,0,0)]$ and $S(S^3)=[(S^0,0,0)]$, by Definition 4.3, we get a map:

$$f:\Sigma^{r\mathbb{R}+(4l+M)H+ND}(S^0\vee G_+\vee\ldots\vee G_+)\to\Sigma^{r\mathbb{R}+MH+(8l+2+N)D}S^0$$

for some $M,N\in\mathbb{Z}$. Restricting to the first factor of $S^0\vee G_+\vee\ldots\vee G_+$, we obtain:

$$g:\Sigma^{r\mathbb{R}+(4l+M)H+ND}S^0\to\Sigma^{r\mathbb{R}+MH+(8l+2+N)D}S^0,$$

which induces homotopy equivalence between the G-fixed point sets. This a contradiction with Proposition 3.1. The case $m=0$ for $\Sigma(2,3,12n+1)$ is proved.

(2) Suppose $\Sigma(2,3,12n+5)$ bounds a smooth spin manifold with intersection form $(8l+1)(-E_8)\oplus(8l+2)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l>0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+5)$ to S^3. As the previous case, this implies $\Sigma^{(4l+1/2)H}S(-\Sigma(2,3,12n+5))$ dominates $\Sigma^{(8l+2)D}S(S^3)$.

[9] It is easy to see that the conclusions are not true for $p=0,1$. For example, $\pm\Sigma(2,3,12n-1)$ bounds a spin manifold with intersection form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Since $\Sigma^{(4l+1/2)H}S(-\Sigma(2,3,12n+5)) = [(\Sigma^{4lH}S^0,0,0)]$, we get the contradiction as before. This proves the case $m=1$ for $\Sigma(2,3,12n+5)$.

(3) Suppose $-\Sigma(2,3,12n-1)$ bounds a spin 4-manifold with intersection form $(8l+2)(-E_8)\oplus(8l+3)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l\geq 0$. By Corollary 1.12, we get $4l+3<\kappa o_{3+8l}(-\Sigma(2,3,12n-1))+\beta^{8l+7}_{8l+3} = -1+4+4l$, which is a contradiction. This proves the case $m=2$ for $-\Sigma(2,3,12n-1)$.

Using similar method as (3), we can prove all the other cases except:

- $m=0$ for $\pm\Sigma(2,3,12n-1)$ and $-\Sigma(2,3,12n+1)$,
- $m=7$ for $\Sigma(2,3,12n-5)$ and $-\Sigma(2,3,12n+5)$,
- $m=1$ for $-\Sigma(2,3,12n-5)$.

(4) We need to introduce another approach in order to prove the rest of the cases. Consider the orbifold D^2-bundle over $S^2(2,3,r)$. This gives us an orbifold X' with boundary $+\Sigma(2,3,r)$. We have $b_2^+(X')=0, b_2^-(X)=1$ and X' has a unique spin structure $\mathfrak{t}$. Now suppose $-\Sigma(2,3,r)$ bounds a spin manifold X with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then we can glue X and X' together to get an oriented closed spin 4-orbifold. We have:

$$\mathrm{ind}_{\mathbb{C}}\not{D}(X\cup X') = p+\omega(\Sigma(2,3,r),X',\mathfrak{t}).$$

Here $\omega(\Sigma(2,3,r),X',\mathfrak{t})$ is the Fukumoto-Furuta invariant defined in [11]. Saveliev [26] proved that $\omega(\Sigma(2,3,r),X',\mathfrak{t}) = -\overline{\mu}(\Sigma(2,3,r)) = \overline{\mu}(-\Sigma(2,3,r))$, where $\overline{\mu}$ is the Neumann-Siebenmann invariant [22, 23]. In [11], Fukumoto and Furuta considered the finite dimensional approximation of the Seiberg-Witten equations on the orbifold $X\cup X'$ and constructed a stable Pin(2)-equivariant map: $(\frac{\mathrm{ind}_{\mathbb{C}}\not{D}(X\cup X')}{2}H)^+\to(b_2^+(X\cup X')D)^+$ which induces homotopy equivalence on the Pin(2)-fixed point set. (Recall that H and D are Pin(2)-representations defined in Section 2). Since $b_2^+(X\cup X')=q$ and $\mathrm{ind}_{\mathbb{C}}\not{D}(X\cup X') = p+\overline{\mu}(-\Sigma(2,3,r))$, we can apply Proposition 3.1 to get:

$$q-p\geq 3+\overline{\mu}(-\Sigma(2,3,r)) \text{ if } 0<p+\overline{\mu}(-\Sigma(2,3,r)) \text{ can be divided by 8.}$$

Similarly, suppose $\Sigma(2,3,r)$ bounds a spin 4-manifold X' with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We can consider $X'\cup(-X)$ and repeat the argument above. We get:

$$q-p\geq 2+\overline{\mu}(\Sigma(2,3,r)) \text{ if } 0<p+\overline{\mu}(\Sigma(2,3,r)) \text{ can be divided by 8.}$$

The invariants $\overline{\mu}(\pm\Sigma(2,3,r))$ were computed in [22, 23]:

$$\overline{\mu}(\pm\Sigma(2,3,12n-1)) = \overline{\mu}(\pm\Sigma(2,3,12n+1)) = 0,$$
$$\overline{\mu}(\Sigma(2,3,12n-5)) = \overline{\mu}(-\Sigma(2,3,12n+5) = 1,$$
$$\overline{\mu}(-\Sigma(2,3,12n-5)) = \overline{\mu}(\Sigma(2,3,12n+5) = -1.$$

Therefore, simple calculations prove the rest of the cases. □

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