

Systolic inequalities for K3 surfaces via stability conditions

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Abstract

We introduce the notions of categorical systoles and categorical volumes of Bridgeland stability conditions on triangulated categories. We prove that for any projective K3 surface X , there exists a constant C depending only on the rank and discriminant of $NS(X)$, such that

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

holds for any stability condition on $\mathcal{D}^b\text{Coh}(X)$. This is an algebro-geometric generalization of a classical systolic inequality on two-tori. We also discuss applications of this inequality in symplectic geometry.

Key words. Derived categories, Bridgeland stability conditions, systolic inequalities, Calabi–Yau manifolds, mirror symmetry.

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1 Introduction

Let (M, g) be a Riemannian manifold. Its *systole* $\text{sys}(M, g)$ is defined to be the least length of a non-contractible loop in M . In 1949, Charles Loewner proved that

$$\text{sys}(\mathbb{T}^2, g)^2 \leq \frac{2}{\sqrt{3}} \text{vol}(\mathbb{T}^2, g)$$

holds for any Riemannian metric g on a two-torus \mathbb{T}^2 . There are various generalizations of Loewner's torus systolic inequality. We refer to [Kat07] for a survey on the rich subject of systolic geometry.

The first goal of the present article is to propose a new generalization of Loewner's torus systolic inequality from the perspective of *Calabi–Yau geometry*. We start with an observation in the case of a two-torus. Suppose that the torus is flat $\mathbb{T}_\tau^2 \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, and is equipped with the standard complex structure $\Omega = dz$ and symplectic structure $\omega = dx \wedge dy$. Then the shortest non-contractible loops must be straight lines, therefore are *special Lagrangian submanifolds* with respect to the complex and symplectic structures. Under these assumptions, Loewner's torus systolic inequality can be interpreted as:

$$\inf_{\text{sLag } L \subset \mathbb{T}_\tau^2} \left| \int_L dz \right|^2 \leq \frac{1}{\sqrt{3}} \left| \int_{\mathbb{T}_\tau^2} dz \wedge d\bar{z} \right| \quad \text{for all } \tau \in \mathbb{H}. \quad (1)$$

The key observation is that the quantities in both sides of this inequality can be generalized to any Calabi–Yau manifold.

We propose the following definition of *systole of a Calabi–Yau manifold*, with respect to its complex and symplectic structures.

Definition 1.1. *Let Y be a Calabi–Yau manifold, equipping with a symplectic form ω and a holomorphic top form Ω . Then its systole is defined to be*

$$\text{sys}(Y, \omega, \Omega) := \inf \left\{ \left| \int_L \Omega \right| : L \text{ is a compact special Lagrangian in } (Y, \omega, \Omega) \right\}.$$

With this definition, we propose the following question that naturally generalizes inequality (1) to any Calabi–Yau manifold.

Question 1.2. *Let Y be a Calabi–Yau manifold and ω be a symplectic form on Y . Does there exist a constant $C = C(Y, \omega) > 0$ such that*

$$\text{sys}(Y, \omega, \Omega)^2 \leq C \cdot \left| \int_Y \Omega \wedge \bar{\Omega} \right|$$

holds for any holomorphic top form Ω on Y ?

Here we treat the Calabi–Yau manifolds topologically so that the complex structures Ω can vary. Note that the choice of the symplectic structure is not important in the case of two-tori, since any one-dimensional submanifold is Lagrangian in \mathbb{T}^2 . However, in higher dimensions, the notion of Lagrangian submanifolds certainly depends on the choice of the symplectic structure. Therefore

the systolic constant C should depend on the symplectic structure in general, unlike the case of \mathbb{T}^2 . Also, note that the ratio $|\int_L \Omega|^2 / |\int_Y \Omega \wedge \overline{\Omega}|$ has been considered in the context of *attractor mechanism* in physics [DRY06, KS14, Moo98], which is of independent interest.

The second goal of the present article is to introduce the definitions of categorical systoles and categorical volumes of Bridgeland stability conditions on triangulated categories.

Definition 1.3 (see Definition 2.3). *Let \mathcal{D} be a triangulated category and $\sigma = (\mathcal{Z}, \mathcal{P})$ be a Bridgeland stability condition on \mathcal{D} . Its systole is defined to be*

$$\text{sys}(\sigma) := \min\{|\mathcal{Z}_\sigma(E)| : E \text{ is a } \sigma\text{-semistable object in } \mathcal{D}\}.$$

Definition 1.4 (see Definition 2.10). *Let $\{E_i\}$ be a basis of the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$, and $\sigma = (\mathcal{Z}, \mathcal{P})$ be a Bridgeland stability condition on \mathcal{D} . Its volume is defined to be*

$$\text{vol}(\sigma) := \left| \sum_{i,j} \chi^{i,j} \mathcal{Z}(E_i) \overline{\mathcal{Z}(E_j)} \right|,$$

where $(\chi^{i,j}) = (\chi(E_i, E_j))^{-1}$ is the inverse matrix of the Euler pairings.

The motivations of these definitions stem from the correspondence between flat surfaces and stability conditions, and the conjectural description of stability conditions on the Fukaya categories of Calabi–Yau manifolds. We refer to Section 2 for more details. Note that under these correspondences, $\text{sys}(\sigma)$ is the categorical generalization of $\text{sys}(Y, \omega, \Omega)$ in Definition 1.1, and $\text{vol}(\sigma)$ is the categorical generalization of the holomorphic volume $|\int_Y \Omega \wedge \overline{\Omega}|$.

As a sanity check of these definitions, we prove the following categorical analogue of Loewner’s torus systolic inequality.

Theorem 1.5 (see Theorem 3.1). *Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(E)$ be the derived category of an elliptic curve E . Then*

$$\text{sys}(\sigma)^2 \leq \frac{1}{\sqrt{3}} \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}(\mathcal{D})$.

We then propose the following algebro-geometric analogue of Question 1.2, which is the higher-dimensional generalization of Theorem 1.5.

Question 1.6. *Let X be a Calabi–Yau manifold and Ω be a complex structure on X . Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(X, \Omega)$ be its derived category of coherent sheaves. Does there exist a constant $C = C(X, \Omega) > 0$ such that*

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^\dagger(\mathcal{D})$? Here $\text{Stab}^\dagger(\mathcal{D})$ denotes the distinguished connected component of $\text{Stab}(\mathcal{D})$ that contains geometric stability conditions.

Note that Questions 1.2 and 1.6 are related via the *mirror symmetry conjecture*, which is a conjectural duality between algebraic geometry and symplectic geometry. We refer to Section 5 for more discussions on this.

The third goal, which is the main result of the present article, is to give an affirmative answer to Question 1.6 for any complex projective K3 surface. A priori there is no reason to believe that Question 1.2 and Question 1.6 have affirmative answers in general. The following theorem is the first evidence that the natural categorical generalization of systolic inequality is possible for higher-dimensional Calabi–Yau manifolds.

Theorem 1.7 (see Theorem 4.1). *Let X be a complex projective K3 surface. Then*

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

for any $\sigma \in \text{Stab}^\dagger(\mathcal{D}^b\text{Coh}(X))$, where

$$C = \frac{((\rho + 2)!)^2 |\text{disc NS}(X)|}{2^\rho} + 4.$$

Here ρ and disc denote the rank and the discriminant of the Néron–Severi group $\text{NS}(X)$, respectively.

Moreover, when the K3 surface is of Picard rank one, we can use a different method to get a better systolic bound.

Theorem 1.8 (see Theorem 4.2). *Let X be a K3 surface of Picard rank one, with $\text{NS}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Then*

$$\text{sys}(\sigma)^2 \leq 4(n + 1) \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^\dagger(\mathcal{D}^b\text{Coh}(X))$.

Finally, we remark that one can also define a categorical generalization of systole using only the *spherical objects* (Remark 4.8), which we denoted by $\text{sys}_{\text{sph}}(\sigma)$. However, as we prove in Proposition 4.9, the ratio $\text{sys}_{\text{sph}}(\sigma)^2/\text{vol}(\sigma)$ is unbounded in general.

Related work

After the first version of the present article was posted online, Haiden [Hai18] proves a systolic inequality for certain higher-dimensional symplectic torus, and Pacini [Pac19] proposes a higher-dimensional generalization of extremal lengths and complex systolic inequalities.

Organization

In Section 2, we recall the definition of Bridgeland stability conditions and introduce the notions of categorical systole and categorical volume. In Section 3, we give an affirmative answer to Question 1.6 for elliptic curves by proving Theorem 1.5. In Section 4, we give an affirmative answer to Question 1.6 for any K3 surface by proving Theorem 1.7. In Section 5, we discuss some directions for future studies.

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2 Categorical systoles and categorical volumes

2.1 Bridgeland stability conditions

In the seminal work [Bri07], Bridgeland introduced the notion of stability conditions on triangulated categories. We recall the definition and some basic properties of Bridgeland stability conditions.

Throughout the article, a triangulated category \mathcal{D} is essentially small, linear over \mathbb{C} , and is of finite type. The last condition means that for any pair of objects $E, F \in \mathcal{D}$, the \mathbb{C} -vector space $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(E, F[i])$ is of finite-dimensional. The Euler form χ on the Grothendieck group $K(\mathcal{D})$ is given by the alternating sum

$$\chi(E, F) := \sum_i (-1)^i \dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}}(E, F[i]).$$

The numerical Grothendieck group $\mathcal{N}(\mathcal{D}) := K(\mathcal{D})/K(\mathcal{D})^{\perp\chi}$ is defined to be the quotient of $K(\mathcal{D})$ by the null space of the Euler pairing χ . We assume that \mathcal{D} is *numerically finite*, that is, $\mathcal{N}(\mathcal{D})$ is of finite rank. One large class of examples of such triangulated categories is provided by the bounded derived category of coherent sheaves $\mathcal{D}^b \text{Coh}(X)$ of a smooth projective variety X .

Definition 2.1 ([Bri07]). *A (full numerical) stability condition $\sigma = (\mathcal{Z}, \mathcal{P})$ on a triangulated category \mathcal{D} consists of:*

- a group homomorphism $\mathcal{Z}: \mathcal{N}(\mathcal{D}) \rightarrow \mathbb{C}$, and
- a collection of full additive subcategories $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ of \mathcal{D} ,

such that:

1. If $0 \neq E \in \mathcal{P}(\phi)$, then $\mathcal{Z}(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$.
2. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
3. If $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$, then $\text{Hom}(A_1, A_2) = 0$.
4. For every $0 \neq E \in \mathcal{D}$, there exists a (unique) collection of distinguished triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{k-1} & \longrightarrow & E \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & B_1 & & B_2 & & & & B_k & & \\
 & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
 & & & & & & & & & &
 \end{array}$$

such that $B_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \dots > \phi_k$. Denote $\phi_\sigma^+(E) := \phi_1$ and $\phi_\sigma^-(E) := \phi_k$. The mass of E is defined to be $m_\sigma(E) := \sum_i |\mathcal{Z}(B_i)|$.

5. (Support property [KS08]) There exists a constant $C > 0$ and a norm $\|\cdot\|$ on $\mathcal{N}(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$\|E\| \leq C|\mathcal{Z}(E)|$$

for any semistable object E .

The group homomorphism \mathcal{Z} is called the *central charge*, and the nonzero objects in $\mathcal{P}(\phi)$ are called the *semistable objects of phase ϕ* . The additive subcategories $\mathcal{P}(\phi)$ actually are abelian, and the simple objects of $\mathcal{P}(\phi)$ are said to be *stable*.

The space of (full numerical) Bridgeland stability conditions on \mathcal{D} is denoted by $\text{Stab}(\mathcal{D})$. There is a nice topology on $\text{Stab}(\mathcal{D})$ introduced by Bridgeland, which is induced by the generalized distance:

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\sigma_2}^-(E) - \phi_{\sigma_1}^-(E)|, |\phi_{\sigma_2}^+(E) - \phi_{\sigma_1}^+(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty].$$

The forgetful map

$$\text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}), \quad \sigma = (\mathcal{Z}, \mathcal{P}) \mapsto \mathcal{Z}$$

is a local homeomorphism [Bri07, KS08]. Hence $\text{Stab}(\mathcal{D})$ is a complex manifold.

There are two natural group actions on the space of Bridgeland stability conditions $\text{Stab}(\mathcal{D})$ which commute with each other [Bri07, Lemma 8.2]. Firstly, the group of autoequivalences $\text{Aut}(\mathcal{D})$ acts on $\text{Stab}(\mathcal{D})$ as isometries with respect to the generalized metric: Let $\Phi \in \text{Aut}(\mathcal{D})$ be an autoequivalence, define

$$\sigma = (\mathcal{Z}, \mathcal{P}) \mapsto \Phi \cdot \sigma := (\mathcal{Z} \circ [\Phi]^{-1}, \mathcal{P}'),$$

where $[\Phi]$ is the induced automorphism on $\mathcal{N}(\mathcal{D})$, and $\mathcal{P}'(\phi) := \Phi(\mathcal{P}(\phi))$.

Secondly, the universal cover $\widetilde{\text{GL}^+(2, \mathbb{R})}$ also admits a natural group action on $\text{Stab}(\mathcal{D})$. Recall that $\widetilde{\text{GL}^+(2, \mathbb{R})}$ is isomorphic to the group of pairs (T, f) , where $T \in \text{GL}^+(2, \mathbb{R})$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, such that their induced maps on $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0} \cong \mathbb{R}/2\mathbb{Z} \cong S^1$ coincide. Let $g = (T, f) \in \widetilde{\text{GL}^+(2, \mathbb{R})}$, define

$$\sigma = (\mathcal{Z}, \mathcal{P}) \mapsto \sigma \cdot g := (T^{-1} \circ \mathcal{Z}, \mathcal{P}''),$$

where $\mathcal{P}''(\phi) := \mathcal{P}(f(\phi))$. Note that the subgroup $\mathbb{C} \subset \widetilde{\text{GL}^+(2, \mathbb{R})}$ acts freely and transitively on $\text{Stab}(\mathcal{D})$. Let $z \in \mathbb{C}$, then its action on $\text{Stab}(\mathcal{D})$ is

$$\sigma = (\mathcal{Z}, \mathcal{P}) \mapsto \sigma \cdot z := (\exp(-i\pi z)\mathcal{Z}, \mathcal{P}'''),$$

where $\mathcal{P}'''(\phi) := \mathcal{P}(\phi + \text{Re}(z))$.

2.2 Categorical systoles

We first recall some results and conjectures that motivate our definition of categorical systoles.

In a striking series of work by Gaiotto–Moore–Neitzke [GMN13], Bridgeland–Smith [BS15] and Haiden–Katzarkov–Kontsevich [HKK17], the connections between stability conditions and Teichmüller theory have been established. One of the main results in this direction is the following theorem.

Theorem 2.2 ([HKK17], Theorem 5.2 and 5.3). *Let S be a marked surface of finite type, $\mathcal{M}(S)$ be the space of marked flat structures on S , and $\mathrm{Fuk}(S)$ be the Fukaya category of S . Then there is a natural map*

$$\mathcal{M}(S) \rightarrow \mathrm{Stab}(\mathrm{Fuk}(S))$$

which is bianalytic onto its image, which is a union of connected components. Under this correspondence, saddle connections on flat surfaces are corresponded to semistable objects, and their lengths are corresponded to the absolute values of central charges.

Recall that the systole of a flat surface is defined to be the length of its shortest saddle connection. Based on the correspondence established in Theorem 2.2, we propose the following definition of systole of a Bridgeland stability condition.

Definition 2.3. *Let σ be a Bridgeland stability condition on a triangulated category \mathcal{D} . Its systole is defined to be*

$$\mathrm{sys}(\sigma) := \min\{|\mathcal{Z}_\sigma(E)| : E \text{ is a } \sigma\text{-semistable object in } \mathcal{D}\}.$$

Remark 2.4. *The support property of stability conditions guarantees that the systole $\mathrm{sys}(\sigma)$ is a positive number, and is attained by the absolute value of the central charge of a stable object.*

Another source of motivation of Definition 2.3 is a conjectural description of stability conditions on the Fukaya categories of Calabi–Yau manifolds, proposed by Bridgeland and Joyce.

Conjecture 2.5 ([Bri09], Section 2.4 and [Joy15], Conjecture 3.2). *Let Y be a Calabi–Yau manifold equipping with a symplectic form ω , and $\mathcal{D}^\pi \mathrm{Fuk}(Y, \omega)$ be the derived Fukaya category of Y . For any holomorphic top form Ω on Y , there exists a natural Bridgeland stability condition on $\mathcal{D}^\pi \mathrm{Fuk}(Y, \omega)$, such that the central charges are given by the period integrals along Lagrangians*

$$\mathcal{Z}(L) = \int_L \Omega,$$

and semistable objects are given by special Lagrangian submanifolds with respect to Ω and ω .

If we assume this conjecture to be true, then the definition of systole of (Y, ω, Ω) in Definition 1.1 coincides with the definition of categorical systole when $\mathcal{D} = \mathcal{D}^\pi \text{Fuk}(Y, \omega)$ and the stability is given by the holomorphic top form Ω as described in Conjecture 2.5.

We summarize these correspondences in the following table.

Surface S	Calabi–Yau (Y, ω)	Triangulated category
abelian differentials	holomorphic top forms	stability conditions
saddle connections	special Lagrangians	semistable objects
lengths	period integrals	central charges
$\text{sys}(S)$	$\text{sys}(Y, \omega, \Omega)$	$\text{sys}(\sigma)$
	$ \int_Y \Omega \wedge \bar{\Omega} $	$\text{vol}(\sigma)$

The definition of categorical volume $\text{vol}(\sigma)$, which is also inspired by Conjecture 2.5, will be discussed in the next subsection.

Now we study how the categorical systole changes under the natural group actions on $\text{Stab}(\mathcal{D})$.

Lemma 2.6. *Let σ be a Bridgeland stability condition on \mathcal{D} . Then*

1.

$$\begin{aligned} \text{sys}(\sigma) &= \min\{|\mathcal{Z}_\sigma(E)| : E \text{ is a } \sigma\text{-stable object in } \mathcal{D}\} \\ &= \min\{m_\sigma(E) : 0 \neq E \in \mathcal{D}\} \end{aligned}$$

Recall that $m_\sigma(E)$ is the mass of E with respect to σ (Definition 2.1).

2. For any autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$, $\text{sys}(\Phi \cdot \sigma) = \text{sys}(\sigma)$.

3. For any complex number $z = x + iy \in \mathbb{C}$, $\text{sys}(\sigma \cdot z) = \exp(y\pi) \cdot \text{sys}(\sigma)$.

4. For any $g = (T, f) \in \widetilde{\text{GL}^+(2, \mathbb{R})}$, write $T^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}^+(2, \mathbb{R})$, then

$$\text{sys}(\sigma \cdot g) \leq |t_1|(1 + |t_2|) \cdot \text{sys}(\sigma),$$

where $t_1 = \frac{(a+d)+i(c-b)}{2} \neq 0$ and $t_2 = \frac{(a-d)+i(b+c)}{(a+d)+i(c-b)}$. Note that $|t_2| < 1$.

Proof. The first three statements follow easily from the definition. To prove the last statement, we use a similar idea in the proof of [Hai18, Theorem 4.3]. One can verify by direct computation that

$$\mathcal{Z}_{\sigma \cdot g}(E) = t_1(\mathcal{Z}_\sigma(E) + t_2 \overline{\mathcal{Z}_\sigma(E)}) \quad (2)$$

holds for any σ and E . By Remark 2.4, there exists a σ -stable object E such that $\text{sys}(\sigma) = |\mathcal{Z}_\sigma(E)|$. Since the actions by $\widetilde{\text{GL}^+(2, \mathbb{R})}$ do not change the set of stable objects, E is also $\sigma \cdot g$ -stable. Therefore,

$$\begin{aligned} |t_1|(1 + |t_2|) \cdot \text{sys}(\sigma) &= |t_1|(1 + |t_2|) \cdot |\mathcal{Z}_\sigma(E)| \\ &\geq |\mathcal{Z}_{\sigma \cdot g}(E)| \geq \text{sys}(\sigma \cdot g) \end{aligned}$$

□

Proposition 2.7. *The function*

$$\text{sys}: \text{Stab}(\mathcal{D}) \longrightarrow \mathbb{R}_{>0}, \quad \sigma \mapsto \text{sys}(\sigma)$$

is continuous on $\text{Stab}(\mathcal{D})$.

Proof. Let $\sigma_1, \sigma_2 \in \text{Stab}(\mathcal{D})$ be two Bridgeland stability conditions with $d(\sigma_1, \sigma_2) < \epsilon$. By Remark 2.4, $\text{sys}(\sigma_1) = m_{\sigma_1}(E)$ for some object $E \in \mathcal{D}$. Thus

$$\log \text{sys}(\sigma_1) = \log m_{\sigma_1}(E) > \log m_{\sigma_2}(E) - \epsilon \geq \log \text{sys}(\sigma_2) - \epsilon.$$

Similarly, we have $\log \text{sys}(\sigma_2) > \log \text{sys}(\sigma_1) - \epsilon$. Hence $\log \text{sys}(\sigma)$ is a continuous function on $\text{Stab}(\mathcal{D})$, and so is $\text{sys}(\sigma)$. \square

Let us consider some basic examples of categorical systoles of algebraic stability conditions. In Section 3 and Section 4, we will be dealing with non-algebraic stability conditions on derived categories of coherent sheaves.

Example 2.8 (Derived category of representations of A_2 -quiver). *Let Q be the A_2 -quiver $(\cdot \rightarrow \cdot)$ and $\mathcal{D} = \mathcal{D}^b(\text{Rep}(Q))$ be the bounded derived category of the category of representations of Q . Let E_1 and E_2 be the simple objects in $\text{Rep}(Q)$ with dimension vectors $\underline{\dim}(E_1) = (1, 0)$ and $\underline{\dim}(E_2) = (0, 1)$. There is one more indecomposable object E_3 that fits into the exact sequence*

$$0 \rightarrow E_2 \rightarrow E_3 \rightarrow E_1 \rightarrow 0.$$

Let $\sigma = (\mathcal{Z}, \mathcal{P})$ be a stability condition on \mathcal{D} with $z_1 := \mathcal{Z}(E_1)$ and $z_2 := \mathcal{Z}(E_2)$. Suppose that $\mathcal{P}(0, 1] = \text{Rep}(Q)$. Then

- *When $\arg(z_1) < \arg(z_2)$, the only σ -stable objects are E_1 and E_2 up to shifts. Thus $\text{sys}(\sigma) = \min\{|z_1|, |z_2|\}$.*
- *When $\arg(z_1) > \arg(z_2)$, the only σ -stable objects are E_1 , E_2 and E_3 up to shifts. Thus $\text{sys}(\sigma) = \min\{|z_1|, |z_2|, |z_1 + z_2|\}$.*

Therefore, in order to compute the categorical systoles of stability conditions in different chambers, one needs to compute the central charges of different sets of dimensional vectors. Note that this is not the case for the derived categories of elliptic curves and K3 surfaces, see Section 3 and Section 4.

Example 2.9 (3-Calabi–Yau categories arising from marked Riemann surfaces). *There is a 3-Calabi–Yau category $\mathcal{D}_{S,M}$ (up to equivalence) associated to each marked Riemann surface (S, M) , see for instance [BS15, Section 9]. The main theorem in [BS15] states that*

$$\text{Stab}(\mathcal{D}_{S,M})/\text{Aut}(\mathcal{D}_{S,M}) \cong \text{Quad}(S, M), \quad (3)$$

where $\text{Quad}(S, M)$ is the moduli space of certain meromorphic quadratic differentials on S with simple zeros. By Lemma 2.6 (2), the function sys on $\text{Stab}(\mathcal{D})$ given by categorical systole descends to a function on the quotient $\text{Stab}(\mathcal{D})/\text{Aut}(\mathcal{D})$. Under the equivalence (3), the categorical systole of a stability condition on $\mathcal{D}_{S,M}$ is given by the minimum among the period integrals $\int \sqrt{\phi}$ along saddle connections of the corresponding quadratic differential.

2.3 Categorical volumes

We recall the notion of *categorical volumes* of Bridgeland stability conditions introduced in [FKY17]. It is the categorical analogue of the holomorphic volume $\left| \int_Y \Omega \wedge \overline{\Omega} \right|$ of a compact Calabi–Yau manifold Y with holomorphic top form Ω .

Let Y be a compact Calabi–Yau manifold of dimension n , and $\{A_i\}$ be a basis of the torsion-free part of $H_n(X, \mathbb{Z})$. Then one can rewrite the holomorphic volume of Y as

$$\left| \int_Y \Omega \wedge \overline{\Omega} \right| = \left| \sum_{i,j} \gamma^{i,j} \int_{A_i} \Omega \int_{A_j} \overline{\Omega} \right|,$$

where $(\gamma^{i,j}) = (A_i \cdot A_j)^{-1}$ is the inverse matrix of the intersection pairings.

Recall that in Conjecture 2.5, the period integral $\int \Omega$ should give the central charge of a Bridgeland stability condition on the derived Fukaya category $\mathcal{D}^{\text{Fuk}}(Y)$ that is associated to the holomorphic top form Ω . This motivates the following definition.

Definition 2.10 ([FKY17]). *Let $\{E_i\}$ be a basis of the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ and let $\sigma = (\mathcal{Z}, \mathcal{P})$ be a Bridgeland stability condition on \mathcal{D} . Its volume is defined to be*

$$\text{vol}(\sigma) := \left| \sum_{i,j} \chi^{i,j} \mathcal{Z}(E_i) \overline{\mathcal{Z}(E_j)} \right|,$$

where $(\chi^{i,j}) = (\chi(E_i, E_j))^{-1}$ is the inverse matrix of the Euler pairings.

One can easily check that the above definition is independent of the choice of the basis $\{E_i\}$. It is important to note that unlike categorical systoles, the categorical volume of a stability condition $\sigma = (\mathcal{Z}, \mathcal{P})$ depends only on its central charge. The proof of the following lemma is straightforward.

Lemma 2.11. *Let σ be a Bridgeland stability condition on \mathcal{D} . Then*

1. *For any autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$, $\text{vol}(\Phi \cdot \sigma) = \text{vol}(\sigma)$.*
2. *For any complex number $z = x + iy \in \mathbb{C}$, $\text{vol}(\sigma \cdot z) = \exp(2y\pi) \cdot \text{vol}(\sigma)$.*

Now we recall some computations of categorical volumes in [FKY17] which will be used in the later sections.

Example 2.12 (Elliptic curves). *Let $\mathcal{D} = \mathcal{D}^b \text{Coh}(E)$ be the derived category of coherent sheaves on an elliptic curve E . Let $\sigma = (\mathcal{Z}, \mathcal{P})$ be a stability condition on \mathcal{D} with central charge*

$$\mathcal{Z}_\sigma(F) = -\deg(F) + (\beta + i\omega) \cdot \text{rk}(F),$$

where $\beta \in \mathbb{R}$ and $\omega > 0$. Choose $\{\mathcal{O}_x, \mathcal{O}_E\}$ as a basis of the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$, where \mathcal{O}_x is a skyscraper sheaf and \mathcal{O}_E is the structure sheaf. Then the categorical volume of σ is

$$\begin{aligned} \text{vol}(\sigma) &= |\mathcal{Z}_\sigma(\mathcal{O}_x) \overline{\mathcal{Z}_\sigma(\mathcal{O}_E)} - \mathcal{Z}_\sigma(\mathcal{O}_E) \overline{\mathcal{Z}_\sigma(\mathcal{O}_x)}| \\ &= 2\omega > 0. \end{aligned}$$

Example 2.13 (K3 surfaces). Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be the derived category of coherent sheaves on a K3 surface X . Let $\sigma = (\mathcal{Z}, \mathcal{P})$ be a stability condition on \mathcal{D} with central charge

$$\mathcal{Z}_\sigma(v) = (\exp(\beta + i\omega), v),$$

where (\cdot, \cdot) is the Mukai pairing on the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$, and $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega^2 > 0$ (c.f. Section 4). Let $\{v_i\}$ be a basis of $\mathcal{N}(\mathcal{D})$. Then the categorical volume of σ is

$$\begin{aligned} \text{vol}(\sigma) &= \left| \sum_{i,j} \chi^{i,j} \mathcal{Z}_\sigma(v_i) \overline{\mathcal{Z}_\sigma(v_j)} \right| \\ &= \left| \sum_{i,j} \chi^{i,j} \cdot (\exp(\beta + i\omega), v_i) \cdot (\exp(\beta - i\omega), v_j) \right| \\ &= \left| \sum_{i,j} (\exp(\beta + i\omega), v_i) \cdot \chi^{i,j} \cdot (v_j, \exp(\beta - i\omega)) \right| \quad (\text{Serre duality}) \\ &= \left| \left(\exp(\beta + i\omega), \exp(\beta - i\omega) \right) \right| \quad (\text{compatibility btw Euler/Mukai pairings}) \\ &= 2\omega^2 > 0. \end{aligned}$$

Note that the same computation shows that $\sum_{i,j} \chi^{i,j} \mathcal{Z}_\sigma(v_i) \mathcal{Z}_\sigma(v_j) = 0$.

For any $g = (T, f) \in \text{GL}^+(2, \mathbb{R})$, one can compute the categorical volume of the stability condition $\sigma \cdot g$ following the same idea in the proof of Lemma 2.6 (4). Write $T^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}^+(2, \mathbb{R})$. Define $t_1 = \frac{(a+d)+i(c-b)}{2} \neq 0$ and $t_2 = \frac{(a-d)+i(b+c)}{(a+d)+i(c-b)}$. By Equation (2), we have

$$\begin{aligned} \text{vol}(\sigma \cdot g) &= \left| \sum_{i,j} \chi^{i,j} \mathcal{Z}_{\sigma \cdot g}(v_i) \overline{\mathcal{Z}_{\sigma \cdot g}(v_j)} \right| \\ &= |t_1|^2 \left| \sum_{i,j} \chi^{i,j} (\mathcal{Z}_\sigma(v_i) + t_2 \overline{\mathcal{Z}_\sigma(v_i)}) (\overline{\mathcal{Z}_\sigma(v_j)} + \overline{t_2} \mathcal{Z}_\sigma(v_j)) \right| \\ &= |t_1|^2 (1 + |t_2|^2) \cdot \text{vol}(\sigma). \end{aligned}$$

We use $\chi^{i,j} = \chi^{j,i}$ in the last step, which follows from the fact that \mathcal{D} is a 2-Calabi–Yau category therefore its Euler pairing is symmetric.

We should note that although the categorical volume can be defined for stability conditions on any triangulated category \mathcal{D} , its geometric meaning is not clear unless \mathcal{D} comes from compact Calabi–Yau geometry. Below is an example of a Calabi–Yau triangulated category for which the categorical volume vanishes for some stability conditions (on the wall of marginal stability conditions).

Example 2.14 (3-Calabi–Yau category of the A_2 -quiver). Let \mathcal{D} be the 3-Calabi–Yau category constructed from the Ginzburg 3-Calabi–Yau dg-algebra associated to the A_2 -quiver [Gin06, Kel11]. The numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ is generated by two spherical objects S_1, S_2 .

Let $\sigma = (\mathcal{Z}, \mathcal{P})$ be a stability condition on \mathcal{D} with $z_1 = \mathcal{Z}(S_1)$ and $z_2 = \mathcal{Z}(S_2)$. Then its categorical volume is

$$\text{vol}(\sigma) = |z_1 \bar{z}_2 - z_2 \bar{z}_1| = 2|\text{Im}(z_1 \bar{z}_2)|,$$

which vanishes if $z_1 \bar{z}_2 \in \mathbb{R}$. This can happen if z_1 and z_2 are of the same phase, i.e., the stability condition σ sits on a wall in $\text{Stab}(\mathcal{D})$.

3 Systolic inequality for elliptic curves: a sanity check

In this section, we give an affirmative answer to Question 1.6 in the case of elliptic curves. This provides a sanity check for our definitions of categorical systole and categorical volume.

Theorem 3.1. *Let $\mathcal{D} = \mathcal{D}^b \text{Coh}(E)$ be the derived category of an elliptic curve E . Then*

$$\text{sys}(\sigma)^2 \leq \frac{1}{\sqrt{3}} \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}(\mathcal{D})$.

One can consider this inequality as the *mirror* of Loewner's torus systolic inequality in the introduction. We refer to Section 5 for more discussions related to mirror symmetry.

Proof. By [Bri07, Theorem 9.1], the $\widetilde{\text{GL}}^+(2; \mathbb{R})$ -action on the space of Bridgeland stability conditions $\text{Stab}(\mathcal{D})$ of an elliptic curve is free and transitive. Therefore

$$\text{Stab}(\mathcal{D}) \cong \widetilde{\text{GL}}^+(2; \mathbb{R}) \cong \mathbb{C} \times \mathbb{H}.$$

By Lemma 2.6 and Lemma 2.11, the systolic ratio

$$\frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)}$$

is invariant under the free \mathbb{C} -action on the space of stability conditions. Hence we only need to compute the ratio on the quotient space $\text{Stab}(\mathcal{D})/\mathbb{C} \cong \mathbb{H}$. The quotient space $\text{Stab}(\mathcal{D})/\mathbb{C} \cong \mathbb{H}$ can be parametrized by the *normalized stability conditions* as follows. Let $\tau = \beta + i\omega \in \mathbb{H}$ where $\beta \in \mathbb{R}$ and $\omega > 0$. The associated normalized stability condition σ_τ is given by:

- Central charge: $\mathcal{Z}_\tau(F) = -\text{deg}(F) + \tau \cdot \text{rk}(F)$.
- For $0 < \phi \leq 1$, the (semi)stable objects $\mathcal{P}_\tau(\phi)$ are the slope-(semi)stable coherent sheaves whose central charges lie in the ray $\mathbb{R}_{>0} \cdot e^{i\pi\phi}$.
- For other $\phi \in \mathbb{R}$, define $\mathcal{P}_\tau(\phi)$ by the property $\mathcal{P}_\tau(\phi + 1) = \mathcal{P}_\tau(\phi)[1]$.

Note that there is no wall-crossing phenomenon in the elliptic curve case, i.e., the set of all Bridgeland (semi)stable objects is the same for any stability condition. This makes the computation of categorical systole easier.

To compute the systole of σ_τ , by Lemma 2.6 (1), we need to know the central charges of all the stable objects of σ_τ , which are the slope-stable coherent sheaves. Recall that if F is a slope-stable coherent sheaf on E , then it is either a vector bundle or a torsion sheaf. The slope-stable vector bundles on an elliptic curve E are well-understood, see for instance [Ati57, Pol03]. In particular, we have the following facts:

- Let F be an indecomposable vector bundle of rank r and degree d on an elliptic curve E . Then F is slope-stable if and only if d and r are relatively prime.
- Fix a point $x \in E$. For every rational number $\mu = \frac{d}{r}$, where $r > 0$ and $(d, r) = 1$, there exists a unique slope-stable vector bundle V_μ of rank r and $\det(V_\mu) \cong \mathcal{O}_E(dx)$.

Hence the categorical systole is

$$\begin{aligned} \text{sys}(\sigma_\tau) &= \min \left\{ 1, | -d + \tau r |: (d, r) = 1 \text{ and } r > 0 \right\} \\ &= \left\{ | -d + \tau r |: (d, r) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \right\} \\ &= \lambda_1(L_\tau), \end{aligned}$$

where $\lambda_1(L_\tau)$ denotes the least length of a nonzero element in the lattice $L_\tau = \langle 1, \tau \rangle$.

On the other hand, the categorical volume of σ_τ has been computed in Example 2.12, which is equals to 2ω . Thus

$$\sup_{\tau \in \mathbb{H}} \frac{\text{sys}(\sigma_\tau)^2}{\text{vol}(\sigma_\tau)} = \frac{1}{2} \cdot \sup_{\tau \in \mathbb{H}} \frac{\lambda_1(L_\tau)^2}{\omega}.$$

Note that ω is the area of the parallelogram spanned by 1 and τ . Hence the quantity $\sup_{\tau \in \mathbb{H}} \lambda_1(L_\tau)^2/\omega$ is the so-called *Hermite constant* γ_2 of lattices in \mathbb{R}^2 . It is classically known that the Hermite constant is given by $\gamma_2 = \frac{2}{\sqrt{3}}$ (see for instance [Cas97]). This concludes the proof. \square

4 Systolic inequalities for K3 surfaces

This section is devoted to prove the following main results of the present article.

Theorem 4.1. *Let X be a complex projective K3 surface. Then*

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

for any $\sigma \in \text{Stab}^\dagger(\mathcal{D}^b\text{Coh}(X))$, where

$$C = \frac{((\rho + 2)!)^2 |\text{disc NS}(X)|}{2^\rho} + 4.$$

Here ρ and disc denote the rank and the discriminant of the Néron–Severi group $\text{NS}(X)$, respectively.

Theorem 4.2. *Let X be a K3 surface of Picard rank one, with $\text{NS}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Then*

$$\text{sys}(\sigma)^2 \leq 4(n+1) \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^\dagger(\mathcal{D}^b\text{Coh}(X))$.

4.1 Reduction to lattice-theoretic problems

We start with recalling some standard notations. Let X be a smooth complex projective K3 surface and $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be its derived category. Sending an object $E \in \mathcal{D}$ to its Mukai vector $v(E) = \text{ch}(E)\sqrt{\text{td}(X)}$ identifies the numerical Grothendieck group of \mathcal{D} with the lattice

$$\mathcal{N}(\mathcal{D}) \cong H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}).$$

The Mukai pairing on $\mathcal{N}(\mathcal{D})$ is given by

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

Since the Mukai pairing on $\mathcal{N}(\mathcal{D})$ is non-degenerate, any group homomorphism $\mathcal{Z} : \mathcal{N}(\mathcal{D}) \rightarrow \mathbb{C}$ can be written as $\mathcal{Z}(v) = (\Omega, v)$ for a unique $\Omega \in \mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$. This defines a map

$$\pi : \text{Stab}(\mathcal{D}) \rightarrow \mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$$

which sends a stability condition $\sigma = (\mathcal{Z}_\sigma, \mathcal{P}_\sigma)$ to the unique element in $\mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$ associated to its central charge $\mathcal{Z}_\sigma : \mathcal{N}(\mathcal{D}) \rightarrow \mathbb{C}$.

Define

$$\Delta^+(\mathcal{D}) := \{\delta = (r, D, s) \in \mathcal{N}(\mathcal{D}) : \delta^2 = -2 \text{ and } r > 0\},$$

$$\mathcal{K}(\mathcal{D}) := \{\Omega = \exp(\beta + i\omega) \in \mathcal{N}(\mathcal{D}) \otimes \mathbb{C} : \beta, \omega \in \text{NS}(X) \otimes \mathbb{R}, \omega^2 > 0\},$$

$$\mathcal{L}(\mathcal{D}) := \{\Omega \in \mathcal{K}(\mathcal{D}) : (\Omega, \delta) \notin \mathbb{R}_{\leq 0} \text{ for all } \delta \in \Delta^+(\mathcal{D})\}.$$

Now we recall the description of the distinguished connected component $\text{Stab}^\dagger(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ by Bridgeland.

Theorem 4.3 ([Bri08]). *Let X be a complex projective K3 surface and $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$.*

1. *Let $V(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ be the subset consisting of geometric stability conditions constructed via tilting of $\text{Coh}(X)$ in [Bri08, Section 6]. Then the map π restricts to give a homeomorphism*

$$\pi|_{V(\mathcal{D})} : V(\mathcal{D}) \rightarrow \mathcal{L}(\mathcal{D}).$$

2. *Let $U(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ be the open subset consisting of all geometric stability conditions. For any $\sigma \in U(\mathcal{D})$, there exists a unique $g \in \widetilde{\text{GL}}^+(2, \mathbb{R})$ such that $\sigma \cdot g \in V(\mathcal{D})$.*

3. Let $\text{Stab}^\dagger(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ be the connected component containing $U(\mathcal{D})$. Every stability condition in $\text{Stab}^\dagger(\mathcal{D})$ is mapped into the closure of $U(\mathcal{D})$ by some autoequivalence of \mathcal{D} .

The following proposition allows us to compute the categorical systole of a stability condition on \mathcal{D} using *only* its central charge.

Proposition 4.4. *Let $\sigma = (\mathcal{Z}_\sigma, \mathcal{P}_\sigma) \in \text{Stab}^\dagger(\mathcal{D})$. Then*

$$\text{sys}(\sigma) = \min\{|\mathcal{Z}_\sigma(v)| : 0 \neq v \in \mathcal{N}(\mathcal{D}), v^2 = (v, v) \geq -2\}.$$

Proof. Let $v = mv_0 \in \mathcal{N}(\mathcal{D})$ be a Mukai vector, where $m \in \mathbb{Z}_{>0}$ and v_0 is primitive. A result of Bayer and Macrì [BM14, Theorem 6.8], which is based on a previous result of Toda [Tod08], says that if $v_0^2 \geq -2$, then there exists a σ -semistable object with Mukai vector v for *any* $\sigma \in \text{Stab}^\dagger(\mathcal{D})$. Hence

$$\begin{aligned} \text{sys}(\sigma) &= \min\{|\mathcal{Z}_\sigma(E)| : E \text{ is a } \sigma\text{-semistable object in } \mathcal{D}\} \\ &\leq \min\{|\mathcal{Z}_\sigma(v)| : 0 \neq v \in \mathcal{N}(\mathcal{D}), v^2 \geq -2\}. \end{aligned}$$

On the other hand, for any stable object E ,

$$\begin{aligned} v(E)^2 &= -\chi(E, E) = -\text{hom}^0(E, E) + \text{hom}^1(E, E) - \text{hom}^2(E, E) \\ &= -2 + \text{hom}^1(E, E) \geq -2. \end{aligned}$$

Hence

$$\begin{aligned} \text{sys}(\sigma) &= \min\{|\mathcal{Z}_\sigma(E)| : E \text{ is a } \sigma\text{-stable object in } \mathcal{D}\} \\ &\geq \min\{|\mathcal{Z}_\sigma(v)| : 0 \neq v \in \mathcal{N}(\mathcal{D}), v^2 \geq -2\}. \end{aligned}$$

This concludes the proof. \square

We can now reduce the categorical systolic inequality to a lattice-theoretic problem. To prove Theorem 4.1 and 4.2, one needs to find an upper bound of the systolic ratio

$$\frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)}$$

for all $\sigma \in \text{Stab}^\dagger(\mathcal{D})$. Since the systolic ratio is a continuous function and is invariant under the actions of autoequivalences, by Theorem 4.3 (3), it suffices to find an upper bound of the systolic ratio on $U(\mathcal{D})$.

By Theorem 4.3 (2), any element in $U(\mathcal{D})$ can be written as $\sigma \cdot g$ for some $\sigma \in V(\mathcal{D})$ and $g \in \widetilde{\text{GL}^+(2, \mathbb{R})}$. By Lemma 2.6 (4) and the computations in Example 2.13, we have

$$\frac{\text{sys}(\sigma \cdot g)^2}{\text{vol}(\sigma \cdot g)} \leq \frac{(1 + |t_2|)^2}{1 + |t_2|^2} \cdot \frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)} < 4 \cdot \frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)}$$

since $|t_2| < 1$ (recall the notations in Lemma 2.6). Therefore, it is enough to find an upper bound of the systolic ratio on $V(\mathcal{D})$.

By Theorem 4.3 (1), the central charges of stability conditions in $V(\mathcal{D})$ are of the form

$$\mathcal{Z}_\sigma(v) = (\exp(\beta + i\omega), v)$$

where $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ and $\omega^2 > 0$. By Example 2.13, the volume of the stability conditions of this form is $2\omega^2$. On the other hand, the categorical systole can be computed by Proposition 4.4 given the central charge:

$$\begin{aligned} \text{sys}(\sigma) &= \min\{|\mathcal{Z}_\sigma(v)| : 0 \neq v \in \mathcal{N}(\mathcal{D}), v^2 = (v, v) \geq -2\} \\ &= \min_{\substack{(r, D, s) \neq (0, 0, 0) \\ D^2 - 2rs \geq -2}} \left\{ \left| \left(s + \beta \cdot D + \frac{1}{2}(\beta^2 - \omega^2)r \right) + i(\omega \cdot (D + r\beta)) \right| \right\} \end{aligned}$$

Hence, Theorem 4.1 and 4.2 can be obtained by proving the following lattice-theoretic statements.

Proposition 4.5. *For any $\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega^2 > 0$ and any $C > \frac{(\rho+2)! \sqrt{|\text{discNS}(X)|}}{2^{(\rho+1)/2}}$, there exists $(r, D, s) \in \mathcal{N}(\mathcal{D}) \setminus \{(0, 0, 0)\}$ satisfying*

1. $|\omega \cdot (D + r\beta)| \leq C\sqrt{\omega^2}$;
2. $|s + \beta \cdot D + \frac{1}{2}(\beta^2 - \omega^2)r| \leq \sqrt{2\omega^2}$;
3. $D^2 - 2rs \geq -2$.

Proposition 4.6. *For any $\beta \in \mathbb{R}$ and $\omega > 0$, there exists $(r, d, s) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$ not all zero satisfying*

1.
$$\frac{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2r|^2}{4n\omega^2} < n + 1;$$

2. $nd^2 - rs \geq -1$.

We will prove these two propositions in the next two subsections.

4.2 Systolic inequality for K3 surfaces of Picard rank one

We prove Proposition 4.6 in this subsection. Note that the method in this proof does not work for K3 surfaces with higher Picard rank, due to the indefiniteness of the intersection pairing on $\text{NS}(X)$ for $\rho(X) > 1$.

In order to find such triple (r, d, s) in Proposition 4.6 for small ω , we need the following technical lemma.

Lemma 4.7. *For any real number β and any $0 < \omega < \frac{1}{\sqrt{n}}$, there exists integers (r, d, s) such that:*

$$\begin{aligned} 1 &\leq r \leq \frac{1}{\sqrt{n\omega}}, \\ |s + 2n\beta d + n\beta^2 r| &< \sqrt{n\omega}, \\ 0 &\leq nd^2 - sr \leq n. \end{aligned}$$

Proof. Let $l = \lfloor \frac{1}{\sqrt{n\omega}} \rfloor + 1$. For each $1 \leq j \leq l$, choose $d_j \in \mathbb{Z}$ such that

$$-\frac{1}{2} < d_j + j\beta \leq \frac{1}{2}.$$

Consider the real numbers $\{2n\beta d_j + n\beta^2 j\}_{1 \leq j \leq l}$ modulo 1. There is at least a pair $(2n\beta d_j + n\beta^2 j, 2n\beta d_k + n\beta^2 k)$ has distance less than or equals to $1/l$ modulo 1. Say $j > k$ without loss of generality. We choose $r = j - k$, $d = d_j - d_k$, and choose s to be the integer closest to $-2n\beta d - n\beta^2 r$. Then

$$1 \leq r \leq \lfloor \frac{1}{\sqrt{n\omega}} \rfloor$$

and

$$|s + 2n\beta d + n\beta^2 r| \leq \frac{1}{\lfloor \frac{1}{\sqrt{n\omega}} \rfloor + 1} < \sqrt{n\omega}.$$

Let $\epsilon = s + 2n\beta d + n\beta^2 r$. Then

$$\begin{aligned} nd^2 - sr &= nd^2 - (-2n\beta d - n\beta^2 r + \epsilon)r \\ &= n(d + r\beta)^2 - r\epsilon. \end{aligned}$$

We have

$$(d + r\beta)^2 = ((d_j + j\beta) - (d_k + k\beta))^2 < 1$$

and

$$|r\epsilon| < \frac{1}{\sqrt{n\omega}} \cdot \sqrt{n\omega} = 1.$$

Hence $-1 < nd^2 - sr < n + 1$. Since it is an integer, thus $0 \leq nd^2 - sr \leq n$. \square

We can now prove Proposition 4.6.

Proof of Proposition 4.6. If $\omega \geq \frac{1}{\sqrt{n}}$, one can simply take the class of skyscraper sheaves $(r, d, s) = (0, 0, 1)$ and check that it satisfies the required conditions. If $\omega < \frac{1}{\sqrt{n}}$, we choose (r, d, s) as in Lemma 4.7. Then it satisfies $sr < nd^2 + 1$ and

$$\begin{aligned} \frac{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2 r|^2}{4n\omega^2} &= \frac{1}{4n} \left(\frac{s + 2n\beta d + n\beta^2 r}{\omega} + n\omega r \right)^2 + (nd^2 - rs) \\ &< \frac{1}{4n} (\sqrt{n} + \sqrt{n})^2 + n = n + 1. \end{aligned}$$

\square

Remark 4.8 (Spherical systole). *The notion of spherical objects in a triangulated category was introduced by Seidel and Thomas [ST01]. These objects are the categorical analogue of Lagrangian spheres in derived Fukaya categories. One can define the spherical systole of a Bridgeland stability condition σ on a triangulated category \mathcal{D} as the minimum among the masses of spherical objects:*

$$\text{sys}_{\text{sph}}(\sigma) := \min\{m_\sigma(S) : S \text{ is a spherical object in } \mathcal{D}\}.$$

It is not hard to show that the notions of spherical systole and categorical systole coincide for stability conditions on elliptic curves. However, this is not true for the derived categories of K3 surfaces. The following proposition shows that the systolic inequality does not hold for spherical systole on K3 surfaces.

Proposition 4.9. *Let X be a K3 surface of Picard rank one and $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be its derived category of coherent sheaves. Then*

$$\sup_{\sigma \in \text{Stab}^\dagger(\mathcal{D})} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} = +\infty.$$

Proof. Let $\text{NS}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Let $\omega H \in \text{NS}(X) \otimes \mathbb{R}$ be an ample class. Then $\mathcal{Z} = (\exp(i\omega H), -)$ gives the central charge of a stability condition in $\text{Stab}^\dagger(\mathcal{D})$ if $\omega > 1$ ([Bri08, Lemma 6.2]). By Proposition 4.4 and Example 2.13,

$$\sup_{\sigma \in \text{Stab}^\dagger(\mathcal{D})} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} \geq \sup_{\omega > 1} \frac{\min\{|\langle \exp(i\omega H), v \rangle|^2 : v^2 = -2\}}{4n\omega^2}.$$

Here we use the fact that the Mukai vector of a spherical object S satisfies $v(S)^2 = -\text{hom}^0(S, S) + \text{hom}^1(S, S) - \text{hom}^2(S, S) = -2$.

Let $v = (r, dH, s) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}$ be a vector satisfying $v^2 = 2nd^2 - 2rs = -2$. Then $rs \neq 0$ and r, s are both positive or both negative. Hence

$$\begin{aligned} \sup_{\sigma \in \text{Stab}^\dagger(\mathcal{D})} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} &\geq \sup_{\omega > 1} \frac{\min\{|\langle \exp(i\omega H), v \rangle|^2 : v^2 = -2\}}{4n\omega^2} \\ &= \sup_{\omega > 1} \min_{nd^2 - rs = -1} \frac{1}{4n} \left(\frac{s}{\omega} + n\omega r \right)^2 - 1 \\ &\geq \sup_{\omega > 1} \frac{1}{4n} \left(\frac{1}{\omega} + n\omega \right)^2 - 1 = +\infty. \end{aligned}$$

□

Remark 4.10. *In the proof of Proposition 4.6, we do not make use of the Mukai vectors of spherical objects. Recall that*

$$\text{sys}(\sigma) = \min\{|\mathcal{Z}_\sigma(v)| : v^2 \geq -2, v \neq 0\},$$

but we only use the Mukai vectors that satisfy $0 \leq v^2 \leq 2n$ to prove the systolic inequality (c.f. Lemma 4.7).

4.3 Systolic inequality for general K3 surfaces

We prove Proposition 4.5 in this subsection. The proof is based on a classical result on the existence of integer points by Minkowski.

Theorem 4.11 (Minkowski). *Every convex set in \mathbb{R}^n which is symmetric with respect to the origin and has volume greater than 2^n contains a non-zero integer point.*

Proof of Proposition 4.5. We fix an identification $\text{NS}(X) \cong \mathbb{Z}^\rho$. For any $\omega \in \text{NS}(X) \otimes \mathbb{R}$ with $\omega^2 > 0$, the intersection pairing restricts on $\langle \omega \rangle^\perp \subset \mathbb{R}^\rho$ is negative definite by Hodge index theorem. Choose

$$D_1, \dots, D_{\rho-1} \in \langle \omega \rangle^\perp \subset \mathbb{R}^\rho$$

such that $D_i \cdot D_j = 0$ for all $i \neq j$, and $D_i^2 = -1$ for all i .

Consider the following vectors in $\mathbb{R}^{\rho+2} \cong \mathbb{R} \oplus (\text{NS}(X) \otimes \mathbb{R}) \oplus \mathbb{R}$:

$$\begin{aligned} v_1 &= \sqrt{2} \cdot (0, D_1, -\beta \cdot D_1), \\ v_2 &= \sqrt{2} \cdot (0, D_2, -\beta \cdot D_2), \\ &\dots \\ v_{\rho-1} &= \sqrt{2} \cdot (0, D_{\rho-1}, -\beta \cdot D_{\rho-1}), \\ v_\rho &= \sqrt{2} \cdot \left(\frac{1}{\sqrt{\omega^2}}, \frac{-\beta}{\sqrt{\omega^2}}, \frac{\beta^2 + \omega^2}{2\sqrt{\omega^2}} \right), \\ v_{\rho+1} &= C \cdot \left(0, \frac{\omega}{\sqrt{\omega^2}}, \frac{-\beta \cdot \omega}{\sqrt{\omega^2}} \right), \\ v_{\rho+2} &= \sqrt{2} \cdot (0, 0, \sqrt{\omega^2}), \end{aligned}$$

where C is any number larger than $\frac{(\rho+2)! \sqrt{|\text{discNS}(X)|}}{2^{(\rho+1)/2}}$.

Consider the following convex set in $\mathbb{R}^{\rho+2}$ which is symmetric with respect to the origin:

$$\mathcal{C} := \left\{ \sum_{i=1}^{\rho+2} e_i v_i : \sum_{i=1}^{\rho+2} |e_i| \leq 1 \right\}.$$

One can check that the volume of \mathcal{C} is greater than $2^{\rho+2}$. Therefore to prove Proposition 4.5, it suffices to show that any vector in \mathcal{C} satisfies the following conditions:

1. $|\omega \cdot (D + r\beta)| \leq C\sqrt{\omega^2}$;
2. $|s + \beta \cdot D + \frac{1}{2}(\beta^2 - \omega^2)r| \leq \sqrt{2\omega^2}$;
3. $D^2 - 2rs \geq -2$.

Observe that the vectors $v_1, v_2, \dots, v_{\rho+2}$ satisfy all three conditions. Since the first two conditions are linear in $(r, D, s) \in \mathbb{R}^{\rho+2}$, hence are satisfied by all the vectors in \mathcal{C} .

Let $(r, D, s) = \sum_{i=1}^{\rho+2} e_i v_i \in \mathcal{C}$. Then

$$\begin{aligned} D^2 - 2rs &= e_{\rho+1}^2 C^2 - 2(e_1^2 + \dots + e_\rho^2) - 4e_\rho e_{\rho+2} \\ &\geq -2(e_1^2 + \dots + e_{\rho-1}^2) - 2(e_\rho + e_{\rho+2})^2 \\ &\geq -2 \end{aligned}$$

since $\sum_{i=1}^{\rho+2} |e_i| \leq 1$. This concludes the proof of Proposition 4.5 by Minkowski's theorem. \square

5 Future studies

Applications in symplectic geometry

Recall the two questions we proposed in the introduction.

Question 5.1 (see Question 1.2). *Let Y be a Calabi–Yau manifold and ω be a symplectic form on Y . Does there exist a constant $C = C(Y, \omega) > 0$ such that*

$$\text{sys}(Y, \omega, \Omega)^2 \leq C \cdot \left| \int_Y \Omega \wedge \bar{\Omega} \right|$$

holds for any holomorphic top form Ω on Y ?

Question 5.2 (see Question 1.6). *Let X be a Calabi–Yau manifold and Ω be a complex structure on X . Let $\mathcal{D} = \mathcal{D}^b\text{Coh}(X, \Omega)$ be its derived category of coherent sheaves. Does there exist a constant $C = C(X, \Omega) > 0$ such that*

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^\dagger(\mathcal{D})$?

In the present article, we give an affirmative answer to Question 5.2 for any complex projective K3 surface X , and find an explicit systolic constant C which depends only on the rank and discriminant of $\text{NS}(X)$. Now we discuss how this result can be used to answer Question 5.1, via the homological mirror symmetry conjecture proposed by Kontsevich.

Conjecture 5.3 ([Kon95]). *For any Calabi–Yau manifold with a symplectic structure (Y, ω) , there exists a Calabi–Yau manifold with a complex structure (X, Ω) such that there is an equivalence between triangulated categories*

$$\mathcal{D}^b\text{Fuk}(Y, \omega) \cong \mathcal{D}^b\text{Coh}(X, \Omega).$$

The conjecture has been proved in several cases, see for instance [PZ98, Sei15, She15]. In particular, for any K3 surface Y and a symplectic form ω on Y , it is expected that there exists a K3 surface X with a complex structure Ω such that the above equivalence of categories holds.

Corollary 5.4. *Assume Conjecture 2.5 and 5.3 hold for all K3 surfaces. Also assume that the space of stability conditions on $\mathcal{D}^b\text{Coh}(X)$ is connected for any projective K3 surface X . Then Question 5.1 has an affirmative answer for any K3 surface Y .*

Proof. Conjecture 2.5 holds for $\mathcal{D}^b\text{Fuk}(Y, \omega)$ implies that for any holomorphic 2-form Ω_Y on Y , there is an associated Bridgeland stability condition σ_{Ω_Y} on the derived Fukaya category of (Y, ω) such that

$$\text{sys}(Y, \omega, \Omega_Y) = \text{sys}(\sigma_{\Omega_Y}) \quad \text{and} \quad \left| \int_Y \Omega_Y \wedge \bar{\Omega}_Y \right| = \text{vol}(\sigma_{\Omega_Y}).$$

Assuming the validity of Conjecture 5.3 for K3 surface (Y, ω) , then there exists a K3 surface with a complex structure (X, Ω) such that

$$\mathcal{D}^b\text{Fuk}(Y, \omega) \cong \mathcal{D}^b\text{Coh}(X, \Omega).$$

Therefore σ_{Ω_Y} induces a Bridgeland stability condition on $\mathcal{D}^b\text{Coh}(X, \Omega)$. By Theorem 1.7 and the assumption that $\text{Stab}(\mathcal{D}^b\text{Coh}(X))$ is connected, we have

$$\text{sys}(\sigma_{\Omega_Y})^2 \leq C \cdot \text{vol}(\sigma_{\Omega_Y}),$$

where $C = \frac{((\rho+2)!)^2 |\text{disc NS}(X)|}{2^\rho} + 4$. Hence, the inequality

$$\text{sys}(Y, \omega, \Omega_Y)^2 \leq C \cdot \left| \int_Y \Omega_Y \wedge \bar{\Omega}_Y \right|$$

holds for any holomorphic 2-form Ω_Y on Y . □

Systolic inequality for Calabi–Yau threefolds

The existence of Bridgeland stability conditions on quintic Calabi–Yau threefolds is proved by Li recently [Li19]. It would be interesting to investigate whether the categorical systolic inequality

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

continues to hold for Calabi–Yau threefolds. Note that the categorical volumes of geometric stability conditions on quintic Calabi–Yau threefolds near the “large volume limit” were computed in [FKY17, Section 4.4]. Hence the main difficulty lies in determining the categorical systoles. It would be nice if properties similar to Proposition 4.4 hold for Calabi–Yau threefolds.

Categorical systole as topological Morse function

It is proved by Akrouit [Akr03] that the systole of Riemann surfaces is a *topological Morse function* on the Teichmüller space. In fact, Akrouit shows that a “generalized systolic function” defined locally as the minimum of a finite number of functions with positive definite Hessians is a topological Morse function, and then apply a result of Wolpert [Wol87] that length functions have positive definite Hessians with respect to the Weil–Petersson metric on the Teichmüller space.

Motivated by the correspondence between flat surfaces and stability conditions, it would be interesting to show that the categorical systole is a topological Morse function on the space of Bridgeland stability conditions, and then deduce some topological properties of the quotient space $\text{Stab}(\mathcal{D})/\text{Aut}(\mathcal{D})$. Since there also is a categorical analogue of Weil–Petersson metric on the space of stability conditions studied in [FKY17], one might be able to follow the same approach as Akrouit’s proof. We hope to come back to this question in the future.

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