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A COLIMIT OF TRACES OF REFLECTION GROUPS

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ABSTRACT. Li-Nadler proposed a conjecture about traces of Hecke categories, which implies the semistable part of the Betti geometric Langlands conjecture of Ben-Zvi-Nadler in genus 1. We prove a Weyl group analogue of this conjecture. Our theorem holds in the natural generality of reflection groups in Euclidean or hyperbolic space. As a corollary, we give an expression of the centralizer of a finite order element in a reflection group using homotopy theory.

1. INTRODUCTION

1.1. **Statement of the theorem.** Let W be a reflection group in Euclidean or hyperbolic space. For I a facet, denote by W_I the subgroup fixing I. For C a chamber, denote by $\mathcal{F}_C$ the category (or poset) of facets in $\overline{C}$ (:= the closure of C). We view W (with discrete topology) as a monoid in $\mathcal{S}$:= the ∞-category of topological spaces and denote its trace by $\mathrm{Tr}(W) \in \mathcal{S}$. Our main theorem is:

Theorem 1.1. *The natural map is fully-faithful:*

$$\mathrm{colim}_{I \in \mathcal{F}_C^{op}} \mathrm{Tr}(W_I) \hookrightarrow \mathrm{Tr}(W).$$

Moreover, denote by $W^f \subset W$ the subset consisting of elements of finite order, and let $\mathrm{Tr}(W)^f \subset \mathrm{Tr}(W)$ be those components in the image of the natural map $W^f \to W \to \mathrm{Tr}(W)$. Then the above map induces an equivalence in $\mathcal{S}$:

$$\mathrm{colim}_{I \in \mathcal{F}_C^{op}} \mathrm{Tr}(W_I) \xrightarrow{\simeq} \mathrm{Tr}(W)^f.$$

For a topological group G, it is not hard to see that $\mathrm{Tr}(G)$ is equivalent to the Borel construction $G/G := G \times_G EG$ for the adjoint action. So the above theorem is equivalent to

$$\mathrm{colim}_{I \in \mathcal{F}_C^{op}} W_I/W_I \hookrightarrow W/W, \qquad \mathrm{colim}_{I \in \mathcal{F}_C^{op}} W_I/W_I \xrightarrow{\simeq} W^f/W.$$

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We get the following expression of the centralizer $C_W(w)$ of a finite order element w:

Corollary 1.2. *Let $w \in W^f$, let J be a facet fixed by w (always exists by Proposition 2.3 (4)), and let C be a chamber with $\overline{C} \supset J$. Then there is an equivalence of H-groups:*

$$C_W(w) \simeq \Omega_w(\operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I),$$

where $\Omega_w(-)$ denotes the based loop space at w, and w is the image of $w \in W_J \to W_J/W_J \to \operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I$. In particular, there is an isomorphism of groups:

$$C_W(w) \simeq \pi_1(\operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I, w).$$

Note that the theorem/corollary also implies the non-trivial fact that $\operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I$ is a disjoint union of $K(\pi,1)$ spaces.

The idea of the proof of the theorem is to resolve the colimit using the geometry of the associated hyperplane arrangement (cf. the functor K at the start of Section 3) and reduce the calculation of the colimit to combinatorics using an ∞-categorical version of the Seifert-van Kampen theorem due to Lurie [Lur12, Theorem A.3.1].

Remark 1.3.

(1) For a topological group G acting on a topological space X, we denote by X/G the topological space $X \times_G EG$, where EG is a contractible space with free G action. It is not hard to see that $\mathrm{Tr}(G) \simeq G/G$ for the adjoint action of G (Proposition 2.4). Denote by $\bullet$ a single point. It is known that $\operatorname{colim}_{I\in\mathcal{F}_C^{op}} \bullet/W_I \simeq \bullet/W$ (see e.g. [Li18]). This equivalence sits inside Theorem 1.1 via the commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{I\in\mathcal{F}_C^{op}} \bullet/W_I & \xrightarrow{\sim} & \bullet/W \\ \big\downarrow & & \big\downarrow \\ \operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I & \xrightarrow{\sim} & W^f/W \end{array}$$

where the vertical maps take $\bullet$ to 1. A similar statement of the top equivalence for the Bruhat-Tits building was used to prove that the representation category of a p-adic group has global dimension $\leq \dim(C)$ (see e.g. Bernstein's lectures on representation of p-adic groups). It may be interesting to see the meaning of the bottom arrow in p-adic representation theory.

(2) Note that colim $\bullet/W_I \simeq \bullet/W$, while in general colim $\mathrm{Map}(S^1, \bullet/W_I) \simeq$ colim $W_I/W_I \not\simeq W/W \simeq \mathrm{Map}(S^1, \bullet/W)$. This reflects the fact that $\mathrm{Map}(S^1, -)$ does not preserve colimits; i.e., loop spaces are not calculated locally.

1.2. **Relation to geometric Langlands.** When $W = W_{\mathrm{aff}}$, the affine Weyl group of a simply-connected reductive group G, Theorem 1.1 confirms a Weyl group analogue of the following conjecture in [LN15]. Let G be a simply-connected reductive algebraic group, LG the loop group of G. Let C be an affine alcove. For each facet I of C (i.e., $I \subset \overline{C}$), denote by G_I the Levi of the parahoric subgroup of LG corresponding to I. Let $\mathcal{H}_I$ be the Hecke category of G_I, and let $\mathcal{H}_{\mathrm{aff}}$ be the affine Hecke category.

Conjecture 1.4 ([LN15, Claim 1.12])**.** *The natural map of ∞-categories*

$$\operatorname{colim}_{I\in\mathcal{F}_C^{op}} \mathrm{Tr}(\mathcal{H}_I) \longrightarrow \mathrm{Tr}(\mathcal{H}_{\mathrm{aff}})$$

is fully-faithful.

This conjecture comes from the consideration of geometric Langlands. Roughly speaking, the Betti geometric Langlands conjecture [BZN16] predicts the equivalence of two (∞)-categories: the automorphic category $\mathcal{A}_g$ and the spectral category $\mathcal{B}_g$. As explained in [LN15], the above conjecture implies that for genus $g=1$, one can embed the semistable automorphic category $\mathcal{A}_1^{ss} \subset \mathcal{A}_1$ fully-faithfully into $\mathcal{B}_1$ and hence implies this part of geometric Langlands. Note that W_I is the Weyl group of G_I, Weyl groups are specializations of Hecke algebra, and Hecke algebras are decategorifications of Hecke categories. Hence Theorem 1.1 confirms an easier analogue of Conjecture 1.4.

1.3. **Examples.** We give some examples of Theorem 1.1. Denote by $\mathfrak{S}_n$ the symmetric group on n letters:

(1) W is the Weyl group of a reductive algebraic group G. Then $\mathcal{F}_C^{op}$ has a final object O the origin, and $W_O = W$. Hence Theorem 1.1 holds trivially since the LHS is also $\mathrm{Tr}(W)$.

(2) W is the affine Weyl group of SL_2. $\mathcal{F}_C^{op}$ is the category $\bullet \leftarrow \bullet \rightarrow \bullet$. $\mathfrak{S}_2/\mathfrak{S}_2 \simeq \bullet/\mathfrak{S}_2 \coprod \bullet/\mathfrak{S}_2$

$$LHS = \operatorname{colim}\left(\mathfrak{S}_2/\mathfrak{S}_2 \leftarrow \bullet \rightarrow \mathfrak{S}_2/\mathfrak{S}_2\right) \simeq \operatorname{colim}\left(\bullet/\mathfrak{S}_2 \leftarrow \bullet \rightarrow \bullet/\mathfrak{S}_2\right) \coprod \operatorname{colim}\left(\bullet/\mathfrak{S}_2 \quad \bullet/\mathfrak{S}_2\right)$$

$\simeq \bullet/W \coprod \bullet/\mathfrak{S}_2 \coprod \bullet/\mathfrak{S}_2 \to W/W$.

The image consists of components labelled by $1, s_1, s_0$ (viewed as elements in W/W via the map $W \to W/W$) for s_1, s_0 two simple reflections in W. The map being fully-faithful reflects the fact that s_1 and s_0 are not conjugate in W, and the centralizer of each is $\mathfrak{S}_2$.

(3) W is the affine Weyl group of SL_3. Note that $\mathfrak{S}_3/\mathfrak{S}_3 \simeq \bullet/\mathfrak{S}_3 \coprod \bullet/\mathfrak{S}_2 \coprod \bullet/(\mathbb{Z}/3)$.

$$LHS = \operatorname{colim}\begin{array}{ccccc} & & \mathfrak{S}_3/\mathfrak{S}_3 & & \\ & \nearrow & \uparrow & \nwarrow & \\ \mathfrak{S}_2/\mathfrak{S}_2 & \nwarrow & \bullet & \nearrow & \mathfrak{S}_2/\mathfrak{S}_2 \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ \mathfrak{S}_3/\mathfrak{S}_3 & \leftarrow & \mathfrak{S}_2/\mathfrak{S}_2 & \rightarrow & \mathfrak{S}_3/\mathfrak{S}_3 \end{array} \simeq \operatorname{colim}\begin{array}{ccccc} & & \bullet/\mathfrak{S}_3 & & \\ & \nearrow & \uparrow & \nwarrow & \\ \bullet/\mathfrak{S}_2 & \nwarrow & \bullet & \nearrow & \bullet/\mathfrak{S}_2 \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ \bullet/\mathfrak{S}_3 & \longleftarrow & \bullet/\mathfrak{S}_2 & \longrightarrow & \bullet/\mathfrak{S}_3 \end{array}$$

$$\coprod \operatorname{colim}\begin{array}{ccccc} & & \bullet/\mathfrak{S}_2 & & \\ & \nearrow & & \nwarrow & \\ \bullet/\mathfrak{S}_2 & & & & \bullet/\mathfrak{S}_2 \\ \downarrow & & & & \downarrow \\ \bullet/\mathfrak{S}_2 & \longleftarrow & \bullet/\mathfrak{S}_2 & \longrightarrow & \bullet/\mathfrak{S}_2 \end{array} \coprod \operatorname{colim}\begin{array}{ccc} & \bullet/(\mathbb{Z}/3) & \\ \bullet/(\mathbb{Z}/3) & & \bullet/(\mathbb{Z}/3) \end{array}$$

$\simeq \bullet/W \coprod (\bullet/\mathfrak{S}_2 \times \partial C) \coprod (\bullet/(\mathbb{Z}/3))^{\coprod 3}$.

The second factor $\bullet/\mathfrak{S}_2 \times \partial C$ can be identified as the component labelled by reflections in W/W: let $s_I \in W$ be the reflection corresponding to a one-dimensional facet I of C. The component of s_I is equivalent to $\bullet/\mathfrak{S}_2 \times \partial C$. We find the centralizer $C_W(s_I) \simeq \Omega_{s_I}(\bullet/\mathfrak{S}_2 \times \partial C) \simeq \mathfrak{S}_2 \times \mathbb{Z}$. This agrees with the more familar presentation $C_W(s_I) \simeq \langle s_I \rangle \times X_*(Z(G_I)) \simeq \mathfrak{S}_2 \times \mathbb{Z}$. Also note that there is only one component of the form $\bullet/\mathfrak{S}_2 \times \partial C$, and this reflects the fact that all reflections are conjugate (unlike the previous case).

(4) Let W be the triangle group $(2,3,\infty)$. It is a reflection group in the hyperbolic plane.

$$LHS = \operatorname{colim} \quad \begin{array}{ccccc} & & \mathfrak{S}_2/\mathfrak{S}_2 & \longleftarrow & \bullet & \longrightarrow & \mathfrak{S}_2/\mathfrak{S}_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & (\mathfrak{S}_2\times\mathfrak{S}_2)/(\mathfrak{S}_2\times\mathfrak{S}_2) & \longleftarrow & \mathfrak{S}_2/\mathfrak{S}_2 & \longrightarrow & \mathfrak{S}_3/\mathfrak{S}_3 \end{array}$$

$$= \operatorname{colim}\{\ (\mathfrak{S}_2\times\mathfrak{S}_2)/(\mathfrak{S}_2\times\mathfrak{S}_2) \longleftarrow \mathfrak{S}_2/\mathfrak{S}_2 \longrightarrow \mathfrak{S}_3/\mathfrak{S}_3\ \}$$

$$= \bullet/W \coprod (\bullet/(\mathfrak{S}_2\times\mathfrak{S}_2))^{\amalg 3} \coprod \bullet/(\mathbb{Z}/3).$$

We see that in this case, $C_{W_I}(w) = C_W(w)$ for any $w \in W_I$.

2. Preliminaries

2.1. **Discrete groups generated by reflections.** References for this section are [Bou02, V] and [Vin88]. Let X be an Euclidean space $\mathbb{E}^n$ or hyperbolic space $\mathbb{H}^n$. Let $\mathfrak{H}$ be a collection of hyperplanes in X. Let W be the group generated by the orthogonal reflection along the hyperplanes $H \in \mathfrak{H}$. Assume that:

(1) For any $w \in W$ and $H \in \mathfrak{H}$, we have $w(H) \in \mathfrak{H}$.
(2) W provided with discrete topology acts properly on X.

Given two points x and y of X, denote by $R\{x,y\}$ the equivalence relation:

For any hyperplane $H \in \mathfrak{H}$, either $x \in H$ and $y \in H$ or x and y are strictly on the same side of H.

Definition 2.1.

(1) A *facet* of X is an equivalence class of the equivalence relation defined above.
(2) For two facets I, J, denote $I \leq J$ if $I \subset \overline{J}$. Then $\leq$ defines a partial order on the set of facets.
(3) A *chamber* C of X is a maximal elements of the partial order set of facets.
(4) For any facet J, denote by $\mathcal{F}_J$ the category corresponding to the poset $\{I | I \leq J\}$.
(5) The *star* of I is $X_I := \bigcup_{\{J|I\leq J\}} J \subset X$ and $W_I := \{w \in W |\ w|_I = \mathrm{id}\}$.

Proposition 2.2.

(1) *A facet is a polytope.*
(2) *For any chamber C, the closure $\overline{C}$ of C is a fundamental domain for the action of W on X; i.e., every orbit of W in X meets $\overline{C}$ in exactly one point.*
(3) *For I a facet, the group W_I is generated by the reflections fixing I.*
(4) *W_I acts on X_I with fundamental domain $X_I \cap \overline{C}$ for any chamber $C \geq I$.*
(5) *Let $w \in W, x \in I$. Then $w(x) = x$ if and only if $w|_I = \mathrm{id}$ if and only if $w(I) = I$.*

Proof. (1) By definition since each facet is an intersection of hyperplanes and half spaces. (2) See [Bou02, V.3.3, Theorem 2]. (3) See [Bou02, V.3.3, Proposition 2]. (4) Let $J \geq I$ be a facet in X_I. Then $w(J) \geq w(I) = I$; hence $w(J) \subset X_I$, and therefore W_J acts on X_I. For the second statement, note that W_I is also a reflection group on X. Hence by (2), we deduce that $X_I \cap \overline{C_I}$ is a fundamental domain of the W_I action on X_I, where C_I is any W_I-chamber (for the action of X). We can choose C_I to contain C, and then we just have to show that $X_I \cap \overline{C_I} = X_I \cap \overline{C}$.

Note that $\overline{C_I}$ is the union of the closures of the W-chambers contained in C_I and that such a closure intersects X_I only if its interior W-chambers are contained in X_I. But observe that any two W-chambers in X_I are connected by an element in W_I (a path between them close to some fixed point of I will cross only hyperplanes containing I), which means that $X_I \cap \overline{C_I} = \overline{C}$; hence the result follows. (5) See [Bou02, V.3.3, Proposition 1]. □

Proposition 2.3.

(1) *Let I be a facet. Then the star X_I is star-shaped with center c for any $c \in I$.*
(2) *Let w be a finite order isometry of X. Then the fixed point locus X^w is non-empty and convex.*
(3) *X^w, X_I, and $X_I^w := X^w \cap X_I$ are contractible for any $w \in W^f$ and facet I.*
(4) *$W^f = \{w \in W : w(I) = I$ for some facet $I\}$ as subsets of W.*

Proof. (1) Take any $x \in X_I$. Then by definition $x \in J$ for some $J \geq I$; hence $x \in \overline{J}$. Therefore the line segment $\overline{cx}$ is contained in $\overline{J}$, because $\overline{J}$ is convex. We conclude that $\overline{cx} \backslash \{c\} \subset J$; hence $\overline{cx} \subset I \cup J \subset X_I$. (2) Follows from [BH13, Corollary II.2.8]. (3) Follows from (1) and (2). (4) It is clear that RHS $\subset W^f$. Now take $w \in W^f$; by (2) w has a fixed point x. Then by Proposition 2.2, $w(I) = I$ for the facet I containing x. □

2.2. **Traces of monoids.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category such that all colimits exist in $\mathcal{C}$. Let $A \in \mathrm{Mon}(\mathcal{C})$ be a monoid in $\mathcal{C}$. The *trace* (*or Hochschild homology*) of A is by definition $\mathrm{Tr}(A) := A \otimes_{A \otimes A^{op}} A \in \mathcal{C}$, and there is a natural map $A \to \mathrm{Tr}(A)$. We view $\mathcal{S}$ as a symmetric monoidal ∞-category, where $\otimes$ is given by the Cartesian product.

Proposition 2.4. *Let G be a topological group. View $G \in \mathrm{Mon}(\mathcal{S})$. Then $\mathrm{Tr}(G) \simeq G/G := G \times_G EG$, where G acts on G by conjugation, and the map $G \to \mathrm{Tr}(G)$ is identified with the natural projection $G \to G/G$.*

Proof. We have an isomorphism $G \simeq G \otimes G^{op} \otimes_G \bullet$ as $G \otimes G^{op}$ modules, where G acts on $G \otimes G^{op}$ diagonally. Then $G \otimes_{G \otimes G^{op}} G \simeq G \otimes_{G \otimes G^{op}} (G \otimes G^{op} \otimes_G \bullet) \simeq G \otimes_G \bullet \simeq G/G$, where the action of G on G is the conjugation. One also checks that this isomorphism is compatible with the natural map from G. □

2.3. **Topological groupoid and open descent.** We denote a *topological groupoid* $\mathcal{G}$ to be the data consisting of a discrete group G acting properly discontinuously on a topogical space Y, and we use the notation $\mathcal{G} = [Y/G]$. Let $\mathcal{G}' = [Y'/G']$ be another topological groupoid. A *morphism* $F : \mathcal{G} \to \mathcal{G}'$ consists of the data (f, φ), where $f : Y \to Y'$ is a continous map and $\varphi : G \to G'$ is an injective homomorphism such that $f(a \cdot y) = \varphi(a) \cdot f(z)$ for all $a \in G, y \in Y$. We denote by TopGrpd the category of topological groupoids. A morphism F is an *open embedding* if the induced map $Y \times_G G' \to Y'$ is an open embedding. We denote by $\underline{Y}$ the underlying set of Y and $\underline{\mathcal{G}} := \underline{Y}/G \in \mathcal{S}$ (recall that G is assumed to be discrete). Also define $\mathcal{G}_h := Y/G \in \mathcal{S}$, the *homotopy type* of $\mathcal{G}$.

Remark 2.5. Our definition of topological groupoid is somewhat non-standard; in particular we require the group homomorphism φ above to be injective. This condition ensures that the morphism F is actually "representable"; hence we could define the notion of open embedding via base change.

Proposition 2.6. *Let $F = (f, \varphi) : \mathcal{G} \to \mathcal{G}'$ as above, assume that f is a local homeomorphism, and assume the induced map $\underline{F} : \underline{\mathcal{G}} \to \underline{\mathcal{G}}'$ is fully-faithful. Then F is an open embedding.*

Proof. The base changed map $Y \times_G G' \to Y'$ is a local homeomorphism since G and G' are discrete. The map on underlying sets $\underline{Y} \times_G G' \to \underline{Y}'$ is fully-faithful (i.e., injective), because fully-faithful maps between groupoids are stable under base change. These imply the map $Y \times_G G' \to Y'$ is an open embedding. □

Let $\mathcal{I}$ be a small category. We denote by $\mathcal{I}^{\triangleright}$ the category by adding one final object $*$ to $\mathcal{I}$. We say a functor $K : \mathcal{I}^{\triangleright} \to \mathcal{T}$ is a *colimit diagram* if the induced map $\operatorname{colim} K|_{\mathcal{I}} \to K(*)$ is an isomorphism.

Lemma 2.7. *Let $K : \mathcal{I}^{\triangleright} \to \mathcal{S}$ be a functor, and let $p : Y \to K(*)$ be a morphism in $\mathcal{S}$ which induces surjection on π_0. Denote by $K_p : \mathcal{I}^{\triangleright} \to \mathcal{S}$ the base changed functor via $I \mapsto K(I) \times_{K(*)} Y$. Then K is a colimit diagram if and only if K_p is a colimit diagram.*

Proof. The only if part follows from the fact that the colimit in $\mathcal{S}$ is stable under base change. For the if part, assume K_f is a colimit diagram, put $X \coloneqq K(*)$, and let $p^{[n]} : Y_X^{n+1} \to X$. Then $K_{p^{[n]}}$ is a colimit diagram for all $n \geq 0$, and $\{K_{p^n}\}_{n \geq -1}$ form an augmented simplicial functor, where $K_{p^{[-1]}} := K$. We have $X \simeq \operatorname{colim}_{[n] \in \Delta^{op}} Y_X^{n+1} \simeq \operatorname{colim}_{[n] \in \Delta^{op}} \operatorname{colim}_{I \in \mathcal{I}} K_{p^{[n]}}(I) \simeq \operatorname{colim}_{I \in \mathcal{I}} \operatorname{colim}_{[n] \in \Delta^{op}} K_{p^{[n]}}(I) \simeq \operatorname{colim}_{I \in \mathcal{I}} K(I)$. Hence K is a colimit diagram. Here we have used the fact that for any morphism $T \to S$ in $\mathcal{S}$, which induces surjection on π_0, we get an equivalence $\operatorname{colim}_{[n] \in \Delta^{op}} T_S^{n+1} \xrightarrow{\sim} S$, where $T_S^{n+1} := T \times_S T \times_S \times_S \cdots \times_S T$ with $(n+1)$ copies of T. □

Proposition 2.8 (∞-categorical Seifert-van Kampen theorem for topological groupoids)**.** *Let $K : \mathcal{I}^{\triangleright} \to$ TopGrpd be a functor, assume that all arrows in $\mathcal{I}^{\triangleright}$ go to open embeddings, and assume the induced functor $\underline{K} : \mathcal{I}^{\triangleright} \to \mathcal{S}$ is a colimit diagram. Then the induced functor $K_h : \mathcal{I}^{\triangleright} \to \mathcal{S}$ is a colimit diagram.*

Proof. Assume $K(*) = [Y/G]$, let $P : Y \to [Y/G]$ be the projection, and denote by K_P the base changed functor via $I \mapsto K(I) \times_{[Y/G]} Y = Y_I \times_{G_I} G$, for $K(I) = [Y_I/G_I]$. By Lemma 2.7, the proposition holds for K if and only if it holds for K_P. Hence we could assume that K takes value in Top the category of topological spaces. Then this is the ∞-categorical Seifert-van Kampen theorem [Lur12, Theorem A.3.1]. Note that the condition $(*)$ [Lur12, Theorem A.3.1] is equivalent to the condition that $\underline{K}$ is a colimit diagram. To see this, we have $\underline{K} = \coprod_{x \in \underline{K}(*)} I_x^{\triangleright}$, where $I_x^{\triangleright} : \mathcal{I}_x^{\triangleright} \to \mathcal{S}$ is the constant functor mapping to $\bullet \in \mathcal{S}$, and $\mathcal{I}_x$ is a category defined in [Lur12, Theorem A.3.1]. Then condition $(*)$ (that $|\operatorname{N}(\mathcal{I}_x)|$ is contractible for all $x \in \underline{K}(*)$) is equivalent to $I_x^{\triangleright}$ being a colimit diagram for all $x \in \underline{K}(*)$ and is equivalent to $\underline{K}$ being a colimit diagram. □

Remark 2.9 (Topological groupoids as topological stacks)*.* Denote by Top the category of topological spaces with continuous map. One can define a topogical stack as a functor $X : \mathrm{Top}^{op} \to \mathcal{S}$, satisfying certain descent and representibility conditions. Then Yoneda embedding (and regarding Set $\subset \mathcal{S}$) gives ι : Top $\hookrightarrow$ TopStack. One can define embedding ι' : TopGrpd $\to$ TopStack via $[Y/G] \to \operatorname{colim}_{\bullet \in \Delta^{op}} \iota(G^{\times \bullet} \times Y)$. In this case, $\underline{\mathcal{G}} = \iota'(\mathcal{G})(*)$, and $\mathcal{G}_h$ is also the homotopy type of $\iota'(\mathcal{G})$. Proposition 2.8 is most naturally presented in the context of topological stacks (with local homeomorphisms), but we shall not use this generality.

3. Proof of main theorem

Recall X^w, X^w_I as in Proposition 2.3. We define a functor $K : \mathcal{F}_C^{op,\triangleright} \to \mathrm{TopGrpd}$ by $I \mapsto [(\coprod_{w\in W_I} X^w_I)/W_I]$ and $* \mapsto [(\coprod_{w\in W^f} X^w)/W]$.

Lemma 3.1.

$$\underline{K(I)} = \coprod_{\{J|C\geq J\geq I\}} (\underline{J}\times W_J)/W_J, \textit{ and } \underline{K(*)} = \coprod_{\{J|C\geq J\}} (\underline{J}\times W_J)/W_J.$$

Proof. By Proposition 2.2(5), we see that $X^w = \coprod_{\{J|w(J)=J\}} J$. Hence, as sets, we have $X^w_I = \coprod_{\{J|J\geq I, w(J)=J\}} J$ and $\coprod_{w\in W_I} X^w_I = \coprod_{w\in W_I}\coprod_{\{J|J\geq I,w(J)=J\}} J = \coprod_{\{(J,w)|J\geq I,w(J)=J,w\in W_I\}} J\times\{w\} = \coprod_{\{(J,w)|J\geq I,w\in W_J\}} J\times\{w\} = \coprod_{\{J|J\geq I\}} J\times W_J$. Hence $\underline{K(I)} = (\coprod_{\{J|J\geq I\}} \underline{J}\times W_J)/W_I = \coprod_{\{J|C\geq J\geq I\}}(\underline{J}\times W_J)/W_J$, where the last equality is by Proposition 2.2(2). The second statement follows from a similar argument with the description of W^f as in Proposition 2.3(4). □

Lemma 3.2.
(1) *For any $I'\to I$ in $\mathcal{F}_C^{op,\triangleright}$, $\underline{K(I')}\to\underline{K(I)}$ is fully-faithful.*
(2) *$\underline{K}$ is a colimit diagram.*

Proof. (1) One checks that under the identification in Lemma 3.1, the map $\underline{K(I')}\to\underline{K(I)}$ is induced by the inclusion of indexing sets $\{J|C\geq J\geq I'\}\to\{J|C\geq J\geq I\}$.

(2) For any $J\leq C$, define $\underline{K}_J : \mathcal{F}_C^{op,\triangleright}\to\mathcal{S}$ by

$$\underline{K}_J(I) := \begin{cases} (\underline{J}\times W_J)/W_J & \text{if } I\leq J,\\ \emptyset & \text{otherwise.}\end{cases}$$

We see that $\mathrm{colim}_{\mathcal{F}_C^{op}} \underline{K}_J \simeq |\mathcal{F}_J^{op}|\times((\underline{J}\times W_J)/W_J) \simeq (\underline{J}\times W_J)/W_J \simeq \underline{K}_J(*)$. The second equivalence follows from the fact that the geometric realization $|\mathcal{F}_J^{op}|\simeq\overline{J}$ is contractible. Hence $\underline{K}_J$ is a colimit diagram, and $\underline{K}\simeq\coprod_{J\leq C}\underline{K}_J$ is also a colimit diagram. □

Proof of Theorem 1.1. By Proposition 2.4, it is equivalent to show that the natural map $\mathrm{colim}_{I\in\mathcal{F}^{op}} W_I/W_I \to W/W$ is fully-faithful and the image is W^f/W. We claim the functor K satisfies the assumption of Proposition 2.8. We first show that all arrows in $\mathcal{F}_C^{op,\triangleright}$ go to open embeddings. For any $I'\geq I$, the natural map $\coprod_{w\in W_{I'}} X^w_{I'} \to \coprod_{w\in W_I} X^w_I$ is an open embedding. Hence by Proposition 2.6 and Lemma 3.2(1), $K(I')\to K(I)$ is an open embedding, and $\underline{K}$ is a colimit diagram by Lemma 3.2(2). Hence we conclude that K_h is a colimit diagram by Proposition 2.8.

Now we have a commutative diagram in $\mathcal{S}$:

$$\begin{array}{ccc}
\operatorname{colim}_{I\in\mathcal{F}_C^{op}}(\coprod_{w\in W_I} X_I^w)/W_I & \xrightarrow[\sim]{K_h} & (\coprod_{w\in W^f} X^w)/W \\
\downarrow\sim & & \downarrow\sim \\
\operatorname{colim}_{I\in\mathcal{F}_C^{op}} W_I/W_I & \xrightarrow{p} & W^f/W \\
& \searrow^{q} & \downarrow i \\
& & W/W
\end{array}$$

The two vertical arrows are given by X_I^w (resp. X^w) $\mapsto \{w\}$; hence they are equivalences since X_I^w and X^w are contractible (Proposition 2.3). We conclude that p is an equivalence. Now i is fully-faithful by definition; hence q is fully-faithful. □

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