THE KINETIC FOKKER-PLANCK EQUATION WITH GENERAL FORCE

CHUQI CAO

ABSTRACT. We consider the kinetic Fokker-Planck equation with a class of general force. We prove the existence and uniqueness of a positive normalized equilibrium (in the case of a general force) and establish some exponential rate of convergence to the equilibrium (and the rate can be explicitly computed). Our results improve results about classical force to general force case. Our result also improve the rate of convergence for the Fitzhugh-Nagumo equation from non-quantitative to quantitative explicit rate.

1. INTRODUCTION

In this paper, we consider the kinetic Fokker-Planck (KFP for short) equation with general force and confinement

$$\partial_t f = \mathscr{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(\nabla_v W(v) f), \tag{1}$$

for a density function $f = f(t,x,v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$, with

$$V(x) = \frac{\langle x\rangle^\gamma}{\gamma}, \quad \gamma \geq 1, \quad W(v) = \frac{\langle v\rangle^\beta}{\beta}, \quad \beta \geq 2,$$

where $\langle x\rangle^2 := 1 + |x|^2$, and the Fitzhugh-Nagumo equation

$$\partial_t f := \mathscr{L} f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \partial^2_{vv} f, \tag{2}$$

with

$$A(x,v) = ax - bv, \quad B(x,v) = v(v-1)(v-c) + x,$$

for some $a, b, c > 0$. The evolution equations are complemented with an initial datum

$$f(0,x,v) = f_0(x,v) \quad \text{on} \quad \mathbb{R}^{2d}.$$

It's easily seen that both equations are mass conservative, that is

$$\mathscr{M}(f(t,\cdot)) = \mathscr{M}(f_0),$$

where we define the mass of f by

$$\mathscr{M}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v)\,dx\,dv.$$

When G satisfies

$$\mathscr{L} G = 0, \quad \mathscr{M}(G) = 1, \quad G > 0,$$

we say that G is a positive normalized steady state.
For a given weight function m, we will denote $L^p(m) = \{f \mid fm \in L^p\}$ the associated Lebesgue

C. Cao: Yau Mathematical Science Center and Beijing Institute of Mathematical Sciences and Applications, Tsinghua University, Beijing, China.
Email address: chuqicao@gmail.com
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space and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$ the associated norm, for $p=2$ we use $(f,g)_{L^2(m)}$ to denote the associate scalar product.
With these notations, we can introduce the main result of this paper.

Theorem 1.1. *(1) When $2 \le \beta, 1 \le \gamma$, there exist a weight function $m > 0$ and a positive normalized steady state $G \in L^1(m)$ such that for any initial datum $f_0 \in L^p(m)$, $p \in [1,\infty]$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck equation (1) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^p(m)} \le Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^p(m)},$$

for some constant $C, \lambda > 0$.
(2) The same conclusion holds for the kinetic Fitzhugh-Nagumo equation (2).

In the results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 1.2. *Theorem 1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$ and $W(v)$ behaves like $\langle v\rangle^\beta$, that is for any $V(x)$ satisfying*

$$C_1\langle x\rangle^\gamma \le V(x) \le C_2\langle x\rangle^\gamma, \ \ \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \le x\cdot\nabla_x V(x) \le C_4|x|\langle x\rangle^{\gamma-1}, \ \ \forall x \in B_R^c,$$

and

$$|D_x^n V(x)| \le C_n\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d, \quad \forall n \ge 2,$$

for some constant $C_i > 0$, $R > 0$, where B_r denotes the ball centered at origin with radius R and $B_r(x_0)$ denotes the ball centered at x_0 with radius R. Similar estimates hold for $W(v)$.

We prove Theorem 1.1 by proving the following theorem, which gives convergence result for more general KFP type models.

Theorem 1.3. *Consider the following equation*

$$\partial_t f := \mathcal{L}f = \mathrm{div}_x(A(x,v)f) + \mathrm{div}_v(B(x,v)f) + K\Delta_v f, \tag{3}$$

with $K > 0$ constant, assume $A(x,v), B(x,v) \in C^1$ and

$$A(x,v) = -v + \Phi(x),$$

where $\Phi(x)$ is Lipschitz

$$|\Phi(x) - \Phi(y)| \le M|x-y|,$$

for some $M > 0$. We also assume that there exists $W(x,v)$ such that

$$\nabla_v W(x,v) = B(x,v),$$

define

$$\phi_2(m) = v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + \frac{1}{2}\mathrm{div}_x\Phi(x) + K\frac{|\nabla_v m|^2}{m^2} + K\frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}\mathrm{div}_v B(x,v).$$

If we can find a weight function m and a function $H \geq 1$ such that the following four conditions hold
(C1)(Lyapunov condition) For some $\alpha, b > 0$ there holds

$$\mathscr{L}^* m \leq -\alpha m + b,$$

(C2)for some constants $C_1, C_2, C_3 > 0$ we have

$$-C_1 H \leq \phi_2(m) \leq -C_2 H + C_3,$$

(C3)For any integer $n \geq 1$ fixed, for any $\epsilon > 0$ small, we can find a constant $C_{\epsilon,n}$ such that

$$\sum_{k=1}^{n} |D_x^k \Phi(x)| + \sum_{k=1}^{n} |D_{x,v}^k B(x,v)| \leq C_{n,\epsilon} + \epsilon H,$$

(C4)For some constant $C_4 > 0$ there holds

$$\frac{\Delta_{x,v} m}{m} \geq -C_4.$$

Then there exists a positive normalized steady state G such that

$$\| f(t,\cdot) - \mathscr{M}(f_0) G\|_{L^1(m)} \leq C e^{-\lambda t} \| f_0 - \mathscr{M}(f_0) G\|_{L^1(m)},$$

for some $C, \lambda > 0$. In addition, for any $p \in [1,\infty]$, if

$$\varphi_p(m) \leq -a + M \mathbb{1}_{B_R},$$

for some constants $a, M, R > 0$, where

$$\varphi_p(m) = v \cdot \frac{\nabla_x m}{m} - \Phi(x) \cdot \frac{\nabla_x m}{m} + (1 - \frac{1}{p}) \mathrm{div}_x \Phi(x) + 2K(1 - \frac{1}{p}) \frac{|\nabla_v m|^2}{m^2}$$
$$+ K(\frac{2}{p} - 1) \frac{\Delta_v m}{m} - B(x,v) \cdot \frac{\nabla_v m}{m} + (1 - \frac{1}{p}) \mathrm{div}_v B(x,v),$$

then we have

$$\| f(t,\cdot) - \mathscr{M}(f_0) G\|_{L^q(m)} \leq C e^{-\lambda t} \| f_0 - \mathscr{M}(f_0) G\|_{L^q(m)}.$$

for all $q \in [1,p]$.

Remark 1.4. *In fact $\phi_2(m)$ satisfies*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (f(\mathscr{L} g) + g(\mathscr{L} f)) m^2 = -2K \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v f \cdot \nabla_v g m^2 + 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} f g \phi_2(m) m^2,$$

and $\varphi_p(m)$ satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{sign} f |f|^{p-1} \mathscr{L} f m^p = -K \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v (m f)|^2 |f|^{p-2} m^{p-2} + \int_{\mathbb{R}^d \times \mathbb{R}^d} |f|^p \varphi_p(m) m^p.$$

the computation can be found in Appendix B. Condition (C2)-(C4) ensure some regularity for the solution, which we will see in Section 3.

Remark 1.5. *For the kinetic Fokker-Planck equation with general force (1), we can take*

$$W(x,v) = F(v) + v \cdot \nabla_x V(x),$$

and

$$m = e^{\lambda H_1}, \quad H_1 = |v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle, \quad H = \langle v \rangle^\beta + \langle x \rangle^{\gamma - 1} + 1,$$

for some $\lambda,\epsilon > 0$ small. For the kinetic Fitzhugh-Nagumo equation (2), we can take

$$m = e^{\lambda(|x|^2+|v|^2)}, \quad H = |v|^4 + |x|^2 + 1, \quad W = \frac{1}{4}|v|^4 - \frac{1}{3}(1+c)v^3 + \frac{1}{2}|v|^2 + x\cdot v,$$

for some constant $\lambda > 0$, the computation can be found in Section 7 below.

Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, kinetic Fokker-Planck equation is it's kinetic version. The Fitzhugh-Nagumo model is a simplified 2D version of the Hodgkin-Huxley model. The Hodgkin-Huxley model is a mathematical model that describes how action potentials in neurons are initiated and propagated. Alan Hodgkin and Andrew Huxley described the model in 1952 to explain the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. They received the 1963 Nobel Prize in Physiology or Medicine for this work. There are only few mathematical analysis for the Fitzhugh-Nagumo equation and Hodgkin-Huxley equation now. For the Fitzhugh-Nagumo equation (2), an exponentially and non-quantitative rate to convergence has been proved recently in [16], our method improves the result to a quantitative rate. The convergence for the Hodgkin-Huxley model is still open.

If $\beta = 2$, equation (1) turns to the classical KFP equation

$$\partial_t f = \mathscr{L} f := -v\cdot\nabla_x f + \nabla_x V(x)\cdot\nabla_v f + \Delta_v f + \mathrm{div}_v(vf).$$

In this case

$$G = Z^{-1}e^{-W}, \quad W = \frac{v^2}{2} + V(x), \quad Z \in \mathbb{R}_+,$$

is an explicit steady state. There are many classical results for this equation on the case $\gamma \geq 1$, where there is an exponentially decay. The proofs using various versions of hypocoercivity methods. We refer the interested readers to [18, 5, 6, 11, 8, 9, 4]. And in [15] the authors extend the convergence to larger spaces. For the weak confinement case $\gamma \in (0,1)$, polynomial or sub-geometric convergence results are proved in [1, 3, 7]. We also emphasize that our results for kinetic Fokker-Planck equation with general force and confinement are to our knowledge new.

A natural question would be what happened for the kFP equation (1) if $\beta < 2$. When $\beta < 2$, the main difficulty is to find a Lyapunov function m for the KFP equation. But when $\beta \in (1,2)$, I believe we can find a weight function m_1 satisfies weak Lyapunov condition such that

$$\mathscr{L}^* m_1 \leq -\alpha\phi(m_1) + b,$$

where $\phi(x) = |x|^q, q \in (0,1)$, which would imply a polynomial decay for the KFP equation. The Harris condition can be proved by following the proof of the paper and carefully deal with the norms. When $\beta < 1$, whether we can find a weak Lyapunov condition is left open.

All the existing hypocoercivity frameworks [9, 5, 6, 11, 4] work in the same weight space $L^2(G^{-1/2})$, where G is the equilibrium, these proofs implicitly require that the equilibrium G satisfies

$$\mathscr{T} G = 0, \quad \mathscr{C} G = 0,$$

where $\mathscr{T}$ is the transport part of the equation and $\mathscr{C}$ is the remaining part defined by

$$\mathscr{T} := -v\cdot\nabla_x f + \nabla_x V(x)\cdot\nabla_v f, \quad \mathscr{C} := \mathscr{L} - \mathscr{T},$$

which is not satisfied in the case $\beta \neq 2$. So the KFP equation with general force and confinement does not fit into the existing hypocoercivity framework and new method is needed.

We carry out all of our proofs using variations of Harris's Theorem for Markov semigroup. Harris's Theorem was first shown in the paper [10] where Harris gave conditions for existence and uniqueness of a steady state for Markov processes. It was then pushed forward by Meyn and Tweedie in [17] to show exponential convergence to equilibrium. The author's former paper with J. A. Cañizo, J. Evans, H. Yoldaş [2] use this Harris's Theorem to prove hypocoercivity for linear BGK and linear Boltzmann equation. A similar work is by Mattingly, Higham and Stuart [14], in the paper the authors prove similar result for the KFP equation ($\beta = 2$) using Harris method in a non-quantitative way, and this paper extends it to quantitative way.

One advantage of the Harris method is that it directly yields convergence for a wide range of initial conditions (there are many choice of m and H in Theorem 1.3, for example we can also take $m = e^{\lambda(|x|^2+|v|^2+1)^{1/2}}$, $H = |v|^2 + 1$ for the Fitzhugh-Nagumo equation (2)), while previous proofs of convergence to equilibrium mainly use some strongly weighted L^2 or H^1 norms (typically with a weight which is the inverse of a Gaussian). The Harris method also gives existence of stationary solutions under general conditions; in some cases these are explicit and easy to find, but in other cases they can be nontrivial and non-explicit, it has the potential to be extended to other equations with non-explicit steady state where trend to equilibrium has not yet been shown. In [12] the author uses this method to show the existence of equilibrium and convergence to the equilibrium for the fractional Fokker-Planck equation. Also the Harris method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as Krein Rutman theorem.

Here we briefly introduce the main idea of the paper. The paper uses Harris method to prove convergence. Roughly speaking, Harris method says that Lyapunov function plus positivity condition on a large ball implies $L^1(m)$ convergence for some weight function m. The Lyapunov function is easy to find so we mainly focus on positivity. The positivity proof mainly contains three steps. First we prove that f is above a constant on a point, then using regularity estimates to prove local continuity of the solution, thus we obtain that f is above a constant in a small ball. Finally we use the spreading of positivity lemma which says if $f \geq \delta > 0$ in $[0, t) \times B_r(x_0, v_0)$, then $f \geq K\delta$ in $[\frac{t}{2}, t) \times B_{\alpha r}(x_0, v_0)$ for any $\alpha > 1$ and some $K > 0$. Thus the convergence in $L^1(m)$ is proved. We then use the Duhamel's formula and regularity estimate to prove convergence $p \in (1, \infty]$.

Let us end the introduction by describing the plan of the paper. In Section 2, we introduce Harris Theorem. In Section 3 we present the proof of a regularization estimate on $S_{\mathscr{L}}$. In Section 4 we prove the spreading of positivity. In Section 5 we prove the positivity of the equation. In Section 6 we prove Theorem 1.3 in the case of $L^p(m)$ with general p. In section 7, we compute the Lyapunov function for the two equations. Finally in Appendix we prove some useful lemmas.

2. Harris Theorem and existence of steady state

In this section we introduce a stronger version of Harris-Doeblin theorem and the existence of steady state

Theorem 2.1. *(Harris-Doeblin Theorem) We consider a Markov semigroup $S_{\mathscr{L}}(t)$ with generator $\mathscr{L}$ and define $S_t := S_{\mathscr{L}}(t)$, we assume that*

(H1)(Lyapunov condition) There exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $|x| \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathscr{L}^* m \le -\alpha m + b,$$

(H2)(Harris condition) For any $R > 0$, there exist a constant $T = T(R) > 0$ and a positive, nonzero measure $\mu = \mu(R)$ such that

$$S_T f \ge \mu \int_{B_R} f, \quad \forall f \in L^1(m), \quad f \ge 0, \quad \|f\|_{L^1(m)} \le 1,$$

where B_R denotes the ball centered at origin with radius R. Suppose the Markov semigroup S_t on $L^1(m)$ satisfies (H1) and (H2). Then there exist some constants $C \ge 1$ and $a < 0$ such that

$$\|S_t f\|_{L^1(m)} \le C e^{at} \|f\|_{L^1(m)}, \quad \forall t \ge 0, \quad \forall f \in L^1(m), \quad \mathscr{M}(f) = 0.$$

Proof. The proof of classical Harris-Doeblin theorem is well-known, the proof of this stronger version is by S. Mischler and J. A. Cañizo in their unpublished notes, interested readers can find it in Appendix A. □

Remark 2.2. *We make a little weaker assumption in the statement of Harris condition*

$$\|f\|_{L^1(m)} \le 1,$$

This additional assumption will be helpful in the proof of Theorem 5.2 equation (11).

Remark 2.3. *In fact of Harris-Doeblin Theorem is a little stronger than the version in [17] because this version do not require a minimum of T for all R, in this version it may happen that*

$$T(R) \to 0, \text{ as } R \to \infty,$$

while in [17] they require a minimum $t_* > 0$ for all $R > 0$.

The Lyapunov condition also provides a sufficient condition for the existence of an invariant measure (for the dual semigroup).

Theorem 2.4. *Any mass conserving positive Markov semigroup (S_t) which fulfills the above Lyapunov condition has at least one invariant borelian measure $G \in M^1(m)$, where M^1 is the space of measures.*

Proof. Step 1. We prove that (S_t) is a bounded semigroup. For $f_0 \in M^1(m)$, we define $f_t := S_{\mathscr{L}}(t) f_0$, and we easily compute

$$\frac{d}{dt} \int |f_t| m \le \int |f_t| \mathscr{L}^* m \le \int |f_t| (-\alpha m + b).$$

Using the mass conservation and positivity, integrating the above differential inequality, we get

$$\begin{aligned} \int |f_t| m \le & e^{-\alpha t} \int |f_0| m + \frac{b}{\alpha}(1 - e^{-\alpha t}) \int |f_0| \\ \le & \max(1, \frac{b}{\alpha}) \int |f_0| m, \forall t \ge 0, \end{aligned}$$

so that (S_t) is bounded in $M^1(m)$.
Step 2. We prove the existence of a steady state, more precisely, we start proving that there exists

a positive and normalized steady state $G \in M^1(m)$. For the equivalent norm $|||\cdot|||$ defined on $M^1(m)$ by

$$|||f||| := \sup_{t>0} \|S_{\mathscr{L}}(t)f\|_{M^1(m)},$$

we have $|||S_{\mathscr{L}}(t)f||| \le |||f|||$ for all $t \ge 0$, that is the semigroup $S_{\mathscr{L}}$ is a contraction semigroup on $(M^1(m), |||\cdot|||)$. There exists $R > 0$ large enough such that the intersection of the closed hyperplane $\{f \in M^1(m); \mathscr{M}(f) = 1\}$ and the closed ball of radius R in $(M^1(m), |||\cdot|||)$ is a convex, non-empty subset. Then consider the closed, weakly * compact convex set

$$\mathbb{K} := \{f \in M^1(m); |||f||| \le R, f \ge 0, \mathscr{M}(f) = 1\},$$

Since $S_{\mathscr{L}}(t)$ is a linear, weakly * continuous, contraction in $(M^1(m), |||\cdot|||)$ and $\mathscr{M}(S_{\mathscr{L}}(t)f) = \mathscr{M}(f)$ for all $t \ge 0$, we see that $\mathbb{K}$ is stable under the action of the semigroup. Therefore we apply the Markov-Kakutani fixed point theorem and we conclude that there exists $G \in \mathbb{K}$ such that $S_{\mathscr{L}}(t)G = G$ for all $t \ge 0$. Therefore we have in particular $G \in D(\mathscr{L})$ and $\mathscr{L}G = 0$. □

3. REGULARIZATION PROPERTY OF $S_{\mathscr{L}}$

The aim of this section is to establish the following short time regularization property. This proof for L^2 regularity for short time is first seen in [9], then it is generalized in [15] for general L^p regularity by using Nash's inequality. The proofs in [9, 15] assume that $H \le C$ for some constant $C > 0$ and we generalize it to general function H. In the whole section, m and H refers to the one defined in Theorem 1.3.

Theorem 3.1. *Consider the weight function m defined in Theorem 1.3 satisfying condition (C1)-(C4), then there exist $\eta, C > 0$ such that*

$$\|S_{\mathscr{L}}(t)f\|_{L^2(m)} \le \frac{C}{t^{\frac{5d+2}{4}}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

In addition, for any integer $k > 0$ there exist some $\alpha(k), C(k) > 0$ such that

$$\|S_{\mathscr{L}}(t)f\|_{H^k(m)} \le \frac{C}{t^{\alpha}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta],$$

as a consequence we have

$$\|S_{\mathscr{L}}(t)f\|_{C^{2,\delta}} \le \frac{C}{t^{\zeta}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta],$$

for some $\delta \in (0,1), \zeta > 0$.

We start with some elementary lemmas.

Lemma 3.2. *For $f_t = S_{\mathscr{L}}(t)f_0$, define an energy functional*

$$\mathscr{F}(t, f_t) := A\|f_t\|^2_{L^2(m)} + at\|\nabla_v f_t\|^2_{L^2(m)} + 2ct^2(\nabla_v f_t, \nabla_x f_t)_{L^2(m)} + bt^3\|\nabla_x f_t\|^2_{L^2(m)}, \tag{4}$$

with $a, b, c > 0, c \le \sqrt{ab}$ and A large enough. Then there exist $\eta > 0$ such that

$$\frac{d}{dt}\mathscr{F}(t, f_t) \le -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^2\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for all $t \in [0, \eta]$ and some $L > 0, C > 0$, as a consequence, we have

$$\|S_{\mathscr{L}} f_0\|_{H^1(m)} \le Ct^{-\frac{3}{2}}\|f_0\|_{L^2(m)},$$

for all $t \in [0,\eta]$, similarly we have

$$\|S_{\mathscr{L}} f_0\|_{H^k(m)} \le C t^{-\frac{3k}{2}} \|f_0\|_{L^2(m)}.$$

Remark 3.3. *We need to note here that if we consider*

$$\mathscr{F}^*(t,f_t) := A\|f_t\|^2_{L^2(m)} + at^2\|\nabla_v f_t\|^2_{L^2(m)} + 2ct^4(\nabla_v f_t, \nabla_x f_t)_{L^2(m)} + bt^6\|\nabla_x f_t\|^2_{L^2(m)},$$

then by the same proof we have

$$\frac{d}{dt}\mathscr{F}^*(t,f_t) \le -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for all $t \in [0,\eta]$, for some $L > 0, C > 0$. This version will be useful in the later proof.

Proof. We only prove the case $k = 1$, for $k > 2$, the proof is similar but long, we put it in Appendix C. First by Theorem 1.3 and Remark 1.4 we have

$$(f, \mathscr{L} g)_{L^2(m)} + (g, \mathscr{L} f)_{L^2(m)} = -2K(\nabla_v f, \nabla_v g)_{L^2(m)} + (f, g\phi_2(m))_{L^2(m)},$$

for any $f, g \in L^2(m)$, without loss of generality we will assume $K = 1$. By condition (C2), we have

$$\frac{d}{dt}\|f\|^2_{L^2(m)} = (f, \mathscr{L} f)_{L^2(m)} \le -\|\nabla_v f\|^2_{L^2(m)} - C_1\|f\|^2_{L^2(mH^{1/2})} + C_2\|f\|^2_{L^2(m)}.$$

We compute

$$\partial_{x_i}\mathscr{L} f = \mathscr{L}\partial_{x_i} f + \partial_{x_i}\Phi\cdot\nabla_x f + \partial_{x_i} B\cdot\nabla_v f + \partial_{x_i}\mathrm{div}_x\Phi f + \partial_{x_i}\mathrm{div}_v B f, \tag{5}$$

by condition (C3)

$$|\partial_{x_i}\Phi| + |\partial_{x_i} B| + |\partial_{x_i}\mathrm{div}_x\Phi| + |\partial_{x_i}\mathrm{div}_v B| \le \epsilon H + C,$$

for some $C > 0$, we have

$$\begin{aligned}
\frac{d}{dt}\|\partial_{x_i} f\|^2_{L^2(m)} =&(\partial_{x_i} f, \mathscr{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f, \partial_{x_i}\Phi\cdot\nabla_x f + \partial_{x_i} B\cdot\nabla_v f)_{L^2(m)}\\
&+ (\partial_{x_i} f, \partial_{x_i}\mathrm{div}_x\Phi f + \partial_{x_i}\mathrm{div}_v B f)_{L^2(m)}\\
\le& -\|\nabla_v(\partial_{x_i} f)\|^2_{L^2(m)} - C_1\|\partial_{x_i} f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{x_i} f\|^2_{L^2(m)}\\
&+ \epsilon(\|\nabla_v f\|^2_{L^2(mH^{1/2})} + \|\nabla_x f\|^2_{L^2(mH^{1/2})} + \|f\|^2_{L^2(mH^{1/2})})\\
&+ C(\|\nabla_v f\|^2_{L^2(m)} + \|\nabla_x f\|^2_{L^2(m)} + \|f\|^2_{L^2(m)}).
\end{aligned}$$

Summing over $i = 1,2,3,...,n$, we get

$$\begin{aligned}
\frac{d}{dt}\|\nabla_x f\|^2_{L^2(m)} \le& -\|\nabla_v\nabla_x f\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_x f\|^2_{L^2(mH^{1/2})} + C\|\nabla_x f\|^2_{L^2(m)}\\
&+ C\|\nabla_v f\|^2_{L^2(mH^{1/2})} + C\|f\|^2_{L^2(mH^{1/2})},
\end{aligned}$$

for some $C > 0$. Similarly using

$$\partial_{v_i}\mathscr{L} f = \mathscr{L}\partial_{v_i} f - \partial_{x_i} f + \partial_{v_i} B\cdot\nabla_v f + \partial_{v_i}\mathrm{div}_v B f, \tag{6}$$

and since

$$|\partial_{v_i} B| + |\partial_{v_i}\mathrm{div}_v B| \le \epsilon H + C,$$

we have

$$\frac{d}{dt}\|\partial_{v_i} f\|^2_{L^2(m)} =(\partial_{v_i} f,\mathscr{L}\partial_{v_i} f)_{L^2(m)} - (\partial_{x_i} f,\partial_{v_i} f)_{L^2(m)} + (\partial_{v_i} f,\partial_{v_i} B\cdot\nabla_v f)_{L^2(m)}$$
$$+ (\partial_{v_i} f,\partial_{v_i}\mathrm{div}_v B f)_{L^2(m)}$$
$$\le -\|\nabla_v(\partial_{v_i} f)\|^2_{L^2(m)} - C_1\|\partial_{v_i} f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{v_i} f\|^2_{L^2(m)} + \epsilon\|\nabla_v f\|^2_{L^2(mH^{1/2})}$$
$$+ C\|\nabla_v f\|^2_{L^2(m)} - (\partial_{x_i} f,\partial_{v_i} f)_{L^2(m)} + C\|f\|^2_{L^2(mH^{1/2})}.$$

Summing over $i = 1,2,...,d$ we get

$$\frac{d}{dt}\|\nabla_v f\|^2_{L^2(m)} \le -\|\nabla_v^2 f\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_v f\|^2_{L^2(mH^{1/2})} + C\|f\|^2_{L^2(mH^{1/2})}.$$
$$+ C\|\nabla_v f\|^2_{L^2(m)} - (\nabla_v f,\nabla_x f)_{L^2(m)}.$$

For the crossing term, using (5), (6) and condition (C2) and (C3), we have

$$\frac{d}{dt}2(\partial_{v_i} f,\partial_{x_i} f)_{L^2(m)} =(\partial_{v_i} f,\mathscr{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{v_i} f,\partial_{x_i}\Phi\cdot\nabla_x f + \partial_{x_i} B\cdot\nabla_v f)_{L^2(m)}$$
$$+ (\partial_{v_i} f,\partial_{x_i}\mathrm{div}_x\Phi f + \partial_{x_i}\mathrm{div}_v B f)_{L^2(m)}$$
$$+ (\partial_{x_i} f,\mathscr{L}\partial_{v_i} f)_{L^2(m)} - (\partial_{x_i} f,\partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f,\partial_{v_i} B\cdot\nabla_v f)_{L^2(m)}$$
$$+ (\partial_{x_i} f,\partial_{v_i}\mathrm{div}_v B f)_{L^2(m)},$$

We split into two parts, for the first part we compute

$$(\partial_{v_i} f,\mathscr{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f,\mathscr{L}\partial_{v_i} f)_{L^2(m)} - \|\partial_{x_i} f\|^2_{L^2(m)}$$
$$= -2(\nabla_v(\partial_{x_i} f),\nabla(\partial_{v_i} f))_{L^2(m)} + (\partial_{x_i} f,\phi_2(m)\partial_{v_i} f)_{L^2(m)} - \|\partial_{x_i} f\|^2_{L^2(m)}$$
$$\le -2(\nabla_v(\partial_{x_i} f),\nabla(\partial_{v_i} f))_{L^2(m)} - \|\partial_{x_i} f\|^2_{L^2(m)} + C(|\nabla_v f|,|\nabla_x f|)_{L^2(mH^{1/2})},$$

for the second part we have

$$(\partial_{v_i} f,\partial_{x_i}\Phi\cdot\nabla_x f + \partial_{x_i} B\cdot\nabla_v f)_{L^2(m)} + (\partial_{x_i} f,\partial_{v_i} B\cdot\nabla_v f)_{L^2(m)}$$
$$+ (\partial_{v_i} f,\partial_{x_i}\mathrm{div}_x\Phi f + \partial_{x_i}\mathrm{div}_v B f)_{L^2(m)} + (\partial_{x_i} f,\partial_{v_i}\mathrm{div}_v B f)_{L^2(m)}$$
$$\le C\|\nabla_v f\|^2_{L^2(mH^{1/2})} + C(|\nabla_v f|,|\nabla_x f|)_{L^2(mH^{1/2})} + C(|f|,|\nabla_x f|)_{L^2(mH^{1/2})} + C\|f\|^2_{L^2(mH^{1/2})}.$$

Gathering the two terms, and summing over i we get

$$\frac{d}{dt}2(\nabla_v f,\nabla_x f)_{L^2(m)} \le -2(\nabla_v\nabla_x f,\nabla_v^2 f)_{L^2(m)} - \|\nabla_x f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})}$$
$$+ C(|\nabla_v f|,|\nabla_x f|)_{L^2(mH^{1/2})} + C(|f|,|\nabla_x f|)_{L^2(mH^{1/2})} + C\|f\|^2_{L^2(mH^{1/2})}.$$

For the very definition of $\mathscr{F}$ in (4), we easily compute

$$\frac{d}{dt}\mathscr{F}(t,f_t) =A\frac{d}{dt}\|f_t\|^2_{L^2(m)} + at\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(m)} + 2ct^2\frac{d}{dt}(\nabla_v f_t,\nabla_x f_t)_{L^2(m)}$$
$$+ bt^3\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(m)} + a\|\nabla_v f_t\|^2_{L^2(m)} + 4ct(\nabla_v f_t,\nabla_x f_t)_{L^2(m)}$$
$$+ 3bt^2\|\nabla_x f_t\|^2_{L^2(m)}.$$

Gathering all the inequalities above together, we have

$$\frac{d}{dt}\mathscr{F}(t,f_t) \le T_1 + T_2 + T_3,$$

with

$$\begin{aligned}T_1 =&(a-A+Cat)\|\nabla_v f_t\|^2_{L^2(m)} + (3bt^2 - ct^2 + Cbt^3)\|\nabla_x f_t\|^2_{L^2(m)}\\&+ (4ct-at)(\nabla_v f_t, \nabla_x f_t)_{L^2(m)} + C_2 A\|f_t\|^2_{L^2(m)}\\ \leq& -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^2\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},\end{aligned}$$

for some $L, C > 0$, if $c > 6b$, $A \gg a, b, c$ and $0 < \eta$ small. For the term T_2 we have

$$\begin{aligned}T_2 =&(-at\|\nabla_v^2 f_t\|^2_{L^2(m)} - bt^3\|\nabla_v\nabla_x f_t\|^2_{L^2(m)} - 2ct^2(\nabla_v\nabla_x f_t, \nabla_v^2 f_t)_{L^2(m)})\\ \leq& -\frac{a}{2}t\|\nabla_v^2 f_t\|^2_{L^2(m)} - \frac{b}{2}t^3\|\nabla_v\nabla_x f_t\|^2_{L^2(m)},\end{aligned}$$

by our choice on taking $ab \gg c^2$. For the term T_3

$$\begin{aligned}T_3 =& -\frac{C_1}{2}bt^3\|\nabla_x f_t\|^2_{L^2(mH^{1/2})} + \left(-\frac{C_1}{2}at + Cbt^3 + Cct^2\right)\|\nabla_v f_t\|^2_{L^2(mH^{1/2})}\\&+ Cct^2(|\nabla_v f_t|, |\nabla_x f_t|)_{L^2(mH^{1/2})} + Cct^2(|f|, |\nabla_x f|)_{L^2(mH^{1/2})}\\&+ (-C_1A + Cbt^3 + Cat + Cct^2)\|f_t\|^2_{L^2(mH^{1/2})}\\ \leq& -K\|f_t\|^2_{L^2(mH^{1/2})} - Kt\|\nabla_v f_t\|^2_{L^2(mH^{1/2})} - Kt^3\|\nabla_x f_t\|^2_{L^2(mH^{1/2})},\end{aligned}$$

for some $K > 0$ by taking $A \gg a \gg b, c$ and $ab \gg c^2$. So by taking A large and $0 < \eta$ small ($t \in [0, \eta]$), we conclude to

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t, f_t) \leq& -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^2\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},\\&- M_1(\|f_t\|^2_{L^2(mH^{1/2})} + t\|\nabla_v f_t\|^2_{L^2(mH^{1/2})} + t^3\|\nabla_x f_t\|^2_{L^2(mH^{1/2})}\\&+ t\|\nabla_v^2 f_t\|^2_{L^2(m)} + t^3\|\nabla_v\nabla_x f_t\|^2_{L^2(m)})\end{aligned}$$

for some $L, C, M_1 > 0$. In particular

$$\frac{d}{dt}\mathscr{F}(t, f_t) \leq -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^2\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

□

Lemma 3.4. *We have*

$$\|\nabla_{x,v}(f_t m)\|^2_{L^2} \leq \|\nabla_{x,v} f_t\|^2_{L^2(m)} + C\|f_t\|^2_{L^2(m)},$$

for some constant C.

Proof. We have

$$\begin{aligned}\|\nabla_{x,v}(f_t m)\|^2_{L^2} =&\|m\nabla_{x,v} f_t\|^2_{L^2} + \|f_t\nabla_{x,v} m\|^2_{L^2} + 2(f_t\nabla_{x,v} m, m\nabla_{x,v} f_t)_{L^2}\\ =&\|\nabla_{x,v} f_t\|^2_{L^2(m)} + \|f_t\nabla_{x,v} m\|^2_{L^2} - \frac{1}{2}(f_t^2, \Delta_{x,v}(m^2))_{L^2}\\ =&\|\nabla_{x,v} f_t\|^2_{L^2(m)} + (f_t^2, |\nabla_{x,v} m|^2 - \frac{1}{2}\Delta_{x,v}(m^2))_{L^2}\\ =&\|\nabla_{x,v} f_t\|^2_{L^2(m)} - (f_t^2, m\Delta_{x,v} m)_{L^2},\end{aligned}$$

by condition (C4)

$$\frac{\Delta_{x,v} m}{m} \geq C,$$

for some constant C, we are done. □

Lemma 3.5. *(Nash's inequality) For any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \le C_d \|f\|_{L^1}^{\frac{2}{d}} \|\nabla_v f\|_{L^2}.$$

For the proof we refer to [13], Section 8.13 for instance. □

Lemma 3.6. *There exist $\lambda > 0$ such that*

$$\frac{d}{dt}\|f\|_{L^1(m)} \le \lambda \|f\|_{L^1(m)}, \tag{7}$$

which implies

$$\|f_t\|_{L^1(m)} \le C e^{\lambda t} \|f_0\|_{L^1(m)}.$$

In particular we have

$$\|f_t\|_{L^1(m)} \le C\|f_0\|_{L^1(m)}, \quad \forall t \in [0,\eta], \tag{8}$$

for some constant $C > 0$.

Proof. It' s an immediate consequence of the Lyapunov condition (C1). □

Now we come to the proof of Theorem 3.1.

Proof. (**Proof of Theorem 3.1.**) We define

$$\mathscr{G}(t, f_t) = B\|f_t\|_{L^1(m)}^2 + t^Z \mathscr{F}^*(t, f_t),$$

with $B, Z > 0$ to be fixed and $\mathscr{F}^*$ defined in Remark 3.3. We choose $t \in [0,\eta]$, η small enough such that $(a+b+c)Z\eta^{Z+1} \le \frac{1}{2}L\eta^Z$ (a, b, c, L are also defined Remark 3.3). By Lemma 5, Remark 3.3 and Lemma 3.6, we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{G}(t, f_t) \le& \lambda B\|f_t\|_{L^1(m)}^2 + Z t^{Z-1}\mathscr{F}^*(t, f_t) \\
&- L t^Z (\|\nabla_v f_t\|_{L^2(m)}^2 + t^4 \|\nabla_x f_t\|_{L^2(m)}^2) + C t^Z \|f_t\|_{L^2(m)}^2 \\
\le& \lambda B\|f_t\|_{L^1(m)}^2 + C t^{Z-1}\|f_t\|_{L^2(m)}^2 - \frac{L}{2} t^Z (\|\nabla_v f_t\|_{L^2(m)}^2 + t^4 \|\nabla_x f_t\|_{L^2(m)}^2),
\end{aligned}$$

where λ is defined in Lemma 3.6. Nash's inequality and Lemma 3.4 imply

$$\|f_t m\|_{L^2} \le C\|fm\|_{L^1}^{\frac{2}{d+2}} \|\nabla_{x,v}(f_t m)\|_{L^2}^{\frac{d}{d+2}} \le C\|f_t m\|_{L^1}^{\frac{2}{d+2}} (\|\nabla_{x,v} f m\|_{L^2} + C\|f_t m\|_{L^2})^{\frac{d}{d+2}}.$$

Using Young's inequality, we have

$$\|f_t\|_{L^2(m)}^2 \le C_\epsilon t^{-\frac{5}{2}d} \|f\|_{L^1(m)}^2 + \epsilon t^5 (\|\nabla_{x,v} f_t\|_{L^2(m)}^2 + C\|f_t\|_{L^2(m)}^2).$$

Taking ϵ small such that $C\epsilon\eta^5 \le \frac{1}{2}$, we deduce

$$\|f_t\|_{L^2(m)}^2 \le 2C_\epsilon t^{-\frac{5}{2}d} \|f\|_{L^1(m)}^2 + 2\epsilon t^5 \|\nabla_{x,v} f_t\|_{L^2(m)}^2.$$

Taking ϵ small we have

$$\frac{d}{dt}\mathscr{G}(t, f_t) \le \lambda B\|f_t\|_{L^1(m)}^2 + C_1 t^{Z-1-\frac{5}{2}d} \|f_t\|_{L^1(m)}^2,$$

for some $C_1 > 0$. Choosing $Z = 1 + \frac{5}{2}d$, and using (8), we deduce

$$\forall t \in [0,\eta], \quad \mathscr{G}(t, f_t) \le \mathscr{G}(0, f_0) + C_2\|f_0\|_{L^1(m)}^2 \le C_3\|f_0\|_{L^1(m)}^2,$$

which proves

$$\|S_{\mathscr{L}}(t)f\|_{L^2(m)} \le \frac{C}{t^{\frac{5d+2}{4}}}\|f\|_{L^1(m)}, \quad \forall t \in [0,\eta].$$

together with Lemma 3.2 ends the proof. □

4. Spreading of positivity

In this section, we will use the notation

$$\bar{B}_r(x_0, v_0) = \{(x,v) \in \mathbb{R}^d \times \mathbb{R}^d : |v - v_0| \le r, |x - x_0| \le r^3\},$$

and $\bar{B}_r$ will stand for $\bar{B}_r(x_0, v_0)$. Before proving the theorem on spreading of positivity, we first prove a useful lemma.

Lemma 4.1. *Define $X_t(x_0, v_0)$ (abbreviated X_t in the sequel) in this way, consider the ordinary differential equation*

$$\frac{dx}{dt} = v_0 + \Phi(x),$$

and denote by $X_t(x_0, v_0)$ the solution to this ordinary differential equation at time t with $x(0) = x_0$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x) - \Phi(y)| \le M|x - y|, \quad \forall x, y \in \mathbb{R}^d.$$

with loss of generality we assume $M \ge 1$. Then we have, for any $(x_0, v_0) \in \mathbb{R}^d$ fixed, $t \in [0, \min\{\frac{log2}{M}, 1\})$, we have

$$|X_t - x_0| \le t(M+1)^2(|v_0| + |x_0| + |\Phi(0)|),$$

Proof. Since $\Phi(x)$ is Lipschitz, the existence and uniqueness of X_t is satisfied. First by the definition of X_t we have

$$\frac{d|X_t|}{dt} \le |\frac{dX_t}{dt}| \le |v_0| + M(|X_t| + |\Phi(0)|),$$

by Grönwall's lemma we have

$$|X_t| \le e^{Mt}|x_0| + \frac{1}{M}(e^{Mt} - 1)(|v_0| + M|\Phi(0)|),$$

since $M \ge 1$, so for $t \in (0, \frac{log2}{M})$ we have

$$|X_t| \le 2|x_0| + |v_0| + M|\Phi(0)|,$$

so

$$\begin{aligned} |\frac{dX_t}{dt}| \le |v_0| + M(|X_t| + |\Phi(0)|) &\le (M+1)|v_0| + 2M|x_0| + M^2|\Phi(0)| \\ &\le (M+1)^2(|v_0| + |x_0| + |\Phi(0)|), \end{aligned}$$

for any $t \in (0, \frac{log2}{M})$, the lemma is thus proved. □

Use this X_t, we come to construct a subsolution which is useful in our proof.

Lemma 4.2. *Define operator $\mathscr{L}$ as*

$$\mathscr{L} = \frac{\partial}{\partial t} + (v + \Phi(x)) \cdot \nabla_x - \Delta_v,$$

where $\Phi(x)$ is Lipschitz

$$|\Phi(x) - \Phi(y)| \le M|x - y|, \quad \forall x, y \in \mathbb{R}^d.$$

Then for any $(x_0, v_0) \in \mathbb{R}^d$ *fixed, define* $V = (M+1)^2(\Phi(0) + |x_0| + |v_0|)$, *then for any* $r > 0$, $0 < \tau < \min\{1, r^3/2V, log2/M, 1/20M\}$,$\alpha > 1$, $\delta > 0$, *there exist constants* $\lambda > \alpha, K > 0$ *which only depend on* r^2/τ, V, M, α *(independent of* δ*) and a function* ϕ *such that*

$$\mathscr{L}\phi \le 0, \quad \text{in } [0,\tau) \times (\bar{B}_{\lambda r} \setminus \bar{B}_r),$$

and ϕ *satisfies some boundary conditions*

$$\phi \le 0, \quad \text{on } t = 0, \quad \phi \le \delta \quad \text{on } [0,\tau) \times \partial\bar{B}_r, \quad \phi \le 0 \quad \text{on } [0,\tau) \times \partial\bar{B}_{\lambda r},$$

while

$$\phi \ge K\delta \quad \text{on } [\frac{\tau}{2}, \tau) \times (\bar{B}_{\alpha r} \setminus \bar{B}_r).$$

Proof. This proof is similar to the proof in [18] Appendix A. 22. For $t \in (0,\tau]$ and $(x, v) \in \mathbb{R}^d \setminus \bar{B}_r$ let

$$Q(t,x,v) = a\frac{|v-v_0|^2}{2t} - b\frac{(v-v_0, x - X_t(x_0,v_0))}{t^2} + c\frac{|x - X_t(x_0,v_0)|^2}{2t^3},$$

where $a, b, c > 0$ will be chosen later on, and we define $X_t(x_0, v_0)$ (abbreviated X_t in the sequel) in this way, consider the ordinary differential equation

$$\frac{dx}{dt} = v_0 + \Phi(x),$$

and denote by $X_t(x_0, v_0)$ the solution to this ordinary differential equation at time t with $x(0) = x_0$. Let further

$$\phi(t,x,v) = \delta e^{-\mu Q(t,x,v)} - \epsilon,$$

where $\mu, \epsilon > 0$ will be chosen later on. Let us assume $b^2 < ac$, so that Q is a positive definite quadratic form in the two variables $v - v_0$ and $x - X_t$. Then

$$\mathscr{L}\phi = -\mu\delta e^{-\mu Q}\mathscr{A}(Q),$$

where

$$\mathscr{A}(Q) = \partial_t Q + (v + \Phi(x)) \cdot \nabla_x Q - \Delta_v Q + \mu|\nabla_v Q|^2.$$

By computation,

$$\begin{aligned}
\mathscr{A}(Q) =& -a\frac{|v-v_0|^2}{2t^2} + 2b\frac{(v-v_0, x-X_t)}{t^3} - 3c\frac{|x-X_t|^2}{2t^4} \\
&+ b\frac{(v-v_0, v_0+\Phi(X_t))}{t^2} - c\frac{(x-X_t, v_0+\Phi(X_t))}{t^3} \\
&- b\frac{(v-v_0, v+\Phi(x))}{t^2} + c\frac{(x-X_t, v+\Phi(x))}{t^3} - a\frac{d}{t} \\
&+ \mu|a\frac{v-v_0}{t} - b\frac{x-X_t}{t^2}|^2 \\
=& \mathscr{B}(\frac{v-v_0}{t}, \frac{x-X_t}{t^2}) + c\frac{(x-X_t, \Phi(x)-\Phi(X_t))}{2t^3} \\
&- b\frac{(v-v_0, \Phi(x)-\Phi(X_t))}{t^2} - a\frac{d}{t},
\end{aligned}$$

where $\mathscr{B}$ is a quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ with matrix $P \otimes I_n$,

$$P = \begin{pmatrix} \mu a^2 - \frac{a}{2} - b & -\mu ab + b + \frac{c}{2} \\ -\mu ab + b + \frac{c}{2} & \mu b^2 - \frac{3c}{2} \end{pmatrix}$$

If a, b, c are given, then as $\mu \to \infty$

$$\begin{cases} \mathrm{tr} P = \mu(a^2+b^2) + O(1), \\ \det P = \mu(\frac{3ab^2}{2} + abc - b^3 - \frac{3a^2c}{2}) + O(1). \end{cases}$$

Both quantities are positive if $b \geq 2a$ and $ac \geq b^2$, for example we can take $b = 2a, c > 12b$, then as $\mu \to \infty$

$$\begin{cases} \mathrm{tr} P = 5\mu a^2 + O(1), \\ \det P = \mu(\frac{1}{2}a^2c - a^2b) + O(1) \geq \mu\frac{1}{3}a^2c + O(1). \end{cases}$$

the eigenvalues of M are of order $5\mu a^2$ and $\frac{c}{15}$. So we may choose a, b, c and μ so that

$$\mathscr{B}(\frac{v-v_0}{t}, \frac{x-X_t}{t^2}) \geq \frac{c}{20}(\frac{|v-v_0|^2}{t^2} + \frac{|x-X_t|^2}{t^4}).$$

If $\tau \leq \frac{1}{20M}$, we have

$$c\frac{|(x-X_t, \Phi(x)-\Phi(X_t))|}{2t^3} \leq \frac{Mt}{2}c\frac{|x-X_t|^2}{t^4} \leq \frac{c}{40}\frac{|x-X_t|^2}{t^4},$$

gathering the two terms we have

$$\mathscr{B}(\frac{v-v_0}{t}, \frac{x-X_t}{t^2}) + c\frac{(x-X_t, \Phi(x)-\Phi(X_t))}{2t^3} \geq \frac{c}{40}(\frac{|v-v_0|^2}{t^2} + \frac{|x-X_t|^2}{t^4}).$$

If $\tau \leq 1$

$$-b\frac{(v-v_0, \Phi(x)-\Phi(X_t))}{t^2} - a\frac{d}{t} \geq -bM^2\frac{|x-X_t|^2}{t^4} - b\frac{|v-v_0|^2}{t^2} - \frac{bd}{2t},$$

gathering the two terms, suppose $c \geq 80(M+1)^2 b$, we have

$$\mathscr{A}(Q) \geq \frac{bd}{2t}[\frac{c}{40bd}(\frac{|v-v_0|^2}{t} + \frac{|x-X_t|^2}{t^3}) - 1].$$

Recall that $(x, v) \notin \bar{B}_r$, so

- either $|v-v_0| \geq r$, and then $\mathring{A}(Q) \geq \frac{bd}{2t}[\frac{c}{40bd}r^2/\tau - 1]$, which is positive if $c \geq 40bd\frac{\tau}{r^2}$;
- or $|x-x_0| \geq r^3$, and then by Lemma 4.1, for any $\tau \leq \min\{1, r^3/(2V), log2/M\}$

$$\begin{aligned} \frac{|x-X_t|^2}{t^2} &\geq \frac{|x-x_0|^2}{t^2} - \frac{|X_t-x_0|^2}{t^2} \\ &\geq \frac{|x-x_0|^2}{t^2} - ((M+1)^2(|v_0|+|x_0|+|\Phi(0)|))^2 \geq \frac{r^6}{\tau^2} - V^2 \geq \frac{r^6}{2\tau^2}, \end{aligned}$$

so $\mathscr{A}(Q) \geq \frac{bd}{2t}[\frac{c}{40bd}\frac{r^6}{2\tau^3} - 1]$, which is positive as soon as $c \geq 80bd(\frac{\tau}{r^2})^3$. so it's OK to take

$$c = b\max\{12, 80(M+1)^2, 80d(\frac{\tau}{r^2})^3, 40d\frac{\tau}{r^2}\}.$$

To summarize: under our assumptions there is a way to choose the constants a, b, c, μ, depending only on $d, M, V, r^2/\tau$, satisfying $c > b > a$ and $ac > b^2$, so that

$$\mathscr{L}\phi \leq 0, \quad \text{in } [0,\tau) \times (\bar{B}_{\lambda r} \setminus \bar{B}_r),$$

as soon as $0 < \tau < \min(1, r^3/2V, log2/M, 1/20M)$. Recall that

$$\phi(t,x,v) = \delta e^{-\mu Q(t,x,v)} - \epsilon.$$

The boundary condition at $t = 0$ is obvious since $e^{-\mu Q(t,x,v)}$ vanishes identically at $t = 0$ (more rigorously, $e^{-\mu Q(t,x,v)}$ can be extended by continuity by 0 at $t = 0$). The condition is also true on $[0,\tau)\times\partial\bar{B}_r$ since $\phi \le \delta$. It remains to prove it on $[0,\tau)\times\partial\bar{B}_{\lambda r}$. For that we estimate Q from below, since $c > 12b, b = 2a$, it's easily to seen that for any $(t,x,v) \in [0,\tau)\times\partial\bar{B}_{\lambda r}$

$$Q(t,x,v) \ge \frac{a}{4}(\frac{|v-v_0|^2}{t} + \frac{|x-X_t|^2}{t^3}) \ge \frac{a}{4}\min(\frac{\lambda^2 r^2}{\tau}, \frac{\lambda^6 r^6}{2\tau^3}) \ge \frac{a\lambda^2}{8}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}).$$

Thus if we choose

$$\epsilon = \delta\exp(-\frac{\mu a\lambda^2}{8}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})),$$

we make sure that $\phi = \delta e^{-\mu Q} - \epsilon \le 0$ on $[0,\tau)\times\partial\bar{B}_{\lambda r}$. We finally come to prove the last thing, indeed, if $t \ge \tau/2$ and $(x,v) \in \bar{B}_{\alpha r}\setminus\bar{B}_r$ then

$$Q(t,x,v) \le 2c(\frac{|v-v_0|^2}{t} + \frac{|x-X_t|^2}{t^3}) \le 2c(\frac{2\alpha^2 r^2}{\tau} + \frac{9\alpha^6 r^6}{\tau^3}) \le 22\alpha^6 c\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

take λ such that

$$22\times 8\times 2\alpha^6 c\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}) = a\lambda^2\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

so we can find $K > 0$ such that

$$\phi \ge \delta[\exp(-22\alpha^6 a\mu c\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})) - \exp(-\frac{\mu a\lambda^2}{8}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}))] \ge K\delta,$$

on $[\frac{\tau}{2},\tau)\times(\bar{B}_{\alpha r}\setminus\bar{B}_r)$. Recall the relationship between a and c, we conclude that λ, K depends only on $r^2/\tau, M, V, \alpha$, the proof is thus ended. □

Theorem 4.3. *Let $f(t,x,v)$ be a classical nonnegative solution of*

$$\partial_t f - \Delta_v f = -(v+\Phi(x))\cdot\nabla_x f + A(t,x,v)\cdot\nabla_v f + C(t,x,v)f,$$

in $[0,T)\times\mathbb{R}^d$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x)-\Phi(y)| \le M|x-y|, \quad \forall x,y\in\mathbb{R}^d,$$

suppose further that

$$A(t,x,v) = \nabla_v W(x,v),$$

for some $W(x,v)$. Define

$$D(t,x,v) = -\frac{1}{4}|A(t,x,v)|^2 - \frac{1}{2}\mathrm{div}_v A(t,x,v) + \frac{1}{2}(v+\Phi(x))\cdot A(t,x,v) + C(t,x,v),$$

Then for any (x_0,v_0) fixed, $r > 0$, $0 < \tau < \min(1, r^3/2V, log2/M, 1/20M)$, $\alpha > 1$, $\delta > 0$, there exists $\lambda > 0$ only depends on $r^2/\tau, \alpha, M, V$ (independent of δ) such that if $f \ge \delta > 0$ in $[0,\tau)\times\bar{B}_r(x_0,v_0)$, then $f \ge K\delta$ in $[\tau/2,\tau)\times\bar{B}_{\alpha r}(x_0,v_0)$ where K also depends on $\|D\|_{L^\infty(\bar{B}_{\lambda r}(x_0,v_0))}$ and $\|W\|_{L^\infty(\bar{B}_{\lambda r}(x_0,v_0))}$.

Remark 4.4. *Former proofs of spreading of positivity such as Theorem A.19 in [18] assumes that A and C are uniformly bounded, by assuming*

$$A(t,x,v) = \nabla_v W(x,v)$$

we generalize this theorem to unbounded cases.

Proof. We first start by a taking $f = hG$, with $G = e^{-\frac{1}{2}W(x,v)}$, we have

$$\nabla_x G = -\frac{1}{2}\nabla_x W(x,v)G, \quad \nabla_v G = -\frac{1}{2}\nabla_v W(x,v)G,$$

and

$$\Delta_v G = \frac{1}{4}|\nabla_v W(x,v)|^2 - \frac{1}{2}\Delta_v W(x,v),$$

the equation turns to

$$\begin{aligned} G\partial_t h =& G\Delta_v h + 2\nabla_v h\cdot\nabla_v G + h\Delta_v G - G(v+\Phi(x))\cdot\nabla_x h - h(v+\Phi(x))\cdot\nabla_x G \\ &+ GA(t,x,v)\cdot\nabla_v h + \nabla_v G\cdot A(t,x,v)h + C(t,x,v)Gh. \end{aligned}$$

By the definition of G we have

$$2\nabla_v G\cdot\nabla_v h = -GA(t,x,v)\cdot\nabla_v h,$$

so the equation turns to

$$\partial_t h = \Delta_v h - (v+\Phi(x))\cdot\nabla_x h + D(t,x,v)h,$$

with

$$\begin{aligned} D(t,x,v) =& \frac{1}{4}|\nabla_v W(x,v)|^2 - \frac{1}{2}\Delta_v W(x,v) + \frac{1}{2}(v+\Phi(x))\cdot\nabla_v W(x,v) \\ &- \frac{1}{2}\nabla_v W(x,v)\cdot A(t,x,v) + C(t,x,v) \\ =& -\frac{1}{4}|A(t,x,v)|^2 - \frac{1}{2}\mathrm{div}_v A(t,x,v) + \frac{1}{2}(v+\Phi(x))\cdot A(t,x,v) \\ &+ C(t,x,v). \end{aligned}$$

Take the λ from Lemma 4.2. Then take $\bar{D} = \|D\|_{L^\infty(\bar{B}_{\lambda r}(x_0,v_0))}$, and $\bar{E} = e^{\|W\|_{L^\infty(\bar{B}_{\lambda r}(x_0,v_0))}}$, then we have

$$h \geq \frac{\delta}{\bar{E}}, \quad \text{in } [0,\tau)\times\bar{B}_r.$$

Let $g(t,x,v) = e^{\bar{D}t}h(t,x,v)$, then $g\geq h$ and $\mathscr{L}g\geq 0$ in $(0,\tau)\times\bar{B}_{\lambda r}(x_0,v_0)$, where

$$\mathscr{L} = \frac{\partial}{\partial t} + (v+\Phi(x))\cdot\nabla_x - \Delta_v,$$

by Lemma 4.2, we can find a ϕ such that

$$\mathscr{L}\phi \leq 0, \quad \text{in } [0,\tau)\times(\bar{B}_{\lambda r}\setminus\bar{B}_r),$$

and

$$\phi\leq 0, \quad \text{on } t=0, \quad \phi\leq\frac{\delta}{\bar{E}} \quad \text{on } [0,\tau)\times\partial\bar{B}_r, \quad \phi\leq 0 \quad \text{on } [0,\tau)\times\partial\bar{B}_{\lambda r},$$

while

$$\phi\geq K\frac{\delta}{\bar{E}} \quad \text{on} \quad [\frac{\tau}{2},\tau)\times(\bar{B}_{\alpha r}\setminus\bar{B}_r).$$

So ϕ is a subsolution to g, then by maximum principle we have

$$g\geq\phi\geq K\frac{\delta}{\bar{E}} \quad \text{on} \quad [\frac{\tau}{2},\tau)\times(\bar{B}_{\alpha r}\setminus\bar{B}_r),$$

which implies

$$h\geq ge^{-\tau\bar{D}}\geq K\frac{\delta}{\bar{E}}e^{-\tau\bar{D}} \quad \text{on} \quad [\frac{\tau}{2},\tau)\times(\bar{B}_{\alpha r}\setminus\bar{B}_r).$$

Taking back to f we have

$$f \geq \frac{1}{\bar{E}} h \geq K \frac{\delta}{\bar{E}^2} e^{-\tau \bar{D}} \quad \text{on} \quad [\frac{\tau}{2}, \tau) \times (\bar{B}_{\alpha r} \setminus \bar{B}_r),$$

we conclude the theorem. □

5. Convergence in $L^1(m)$

In this section we prove the Harris condition (H2) for Theorem 1.3, which would imply the convergence for $p = 1$. Before the proof of the theorem, we first prove a useful lemma.

Lemma 5.1. *For any $R > 0$, there exist $\gamma, \rho > 0$ such that for any $t, R > 0$, there exists $(x_0, v_0) \in B_\rho$ such that*

$$f(t, x_0, v_0) \geq \gamma \int_{B_R} f_0.$$

γ, ρ does not depend on f_0, t, while x_0, v_0 may depend on f_0, t

Proof. From conservation of mass, we classically show that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t, x, v) \, dx \, dv = 0,$$

so we have

$$\|S_{\mathscr{L}}(t)\|_{L^1 \to L^1} \leq 1, \quad \forall t \geq 0, \tag{9}$$

Define the splitting of the operator $\mathscr{L}$ by

$$\mathscr{B} = \mathscr{L} - \mathscr{A}, \quad \mathscr{A} = M \chi_R(x, v),$$

with $M, R > 0$ large, where χ is the cut-off function such that $\chi(x, v) \in [0,1]$, $\chi(x, v) \in C^\infty$, $\chi(x, v) = 1$ when $|x|^2 + |v|^2 \leq 1$, $\chi(x, v) = 0$ when $|x|^2 + |v|^2 \geq 2$, and $\chi_R = \chi(x/R, v/R)$. We define $f_{\mathscr{B}}(t) := S_{\mathscr{B}}(t) f_0$, by the Lyapunov function condition (H1) and taking M, R large

$$\frac{d}{dt} \int |f_{\mathscr{B}}(t)| m \leq \int |f_{\mathscr{B}}(t)| \mathscr{B}^* m \leq \int |f_{\mathscr{B}}(t)| (-\alpha m + b - M \chi_R) \leq -\lambda \int |f_{\mathscr{B}}(t)| m.$$

for some $\lambda > 0$, we have

$$\|S_{\mathscr{B}}(t)\|_{L^1(m) \to L^1(m)} \leq e^{-\lambda t}, \quad \forall t \geq 0. \tag{10}$$

By Duhamel's formula

$$S_{\mathscr{L}} = S_{\mathscr{B}} + S_{\mathscr{B}} * \mathscr{A} S_{\mathscr{L}},$$

we directly deduce from (9) and (10) that

$$\|S_{\mathscr{L}}(t)\|_{L^1(m) \to L^1(m)} \leq A, \quad \forall t \geq 0,$$

for some $A > 0$. We fix $R > 0$ and take $g_0 = f_0 \mathbb{1}_{B_R} \in L^1(\mathbb{R}^d)$ such that supp $g_0 \subset B_R$, ,denote $g_t = S_{\mathscr{L}} g_0$, $f_t = S_{\mathscr{L}} f_0$, then we have

$$\int_{\mathbb{R}^d} g_t = \int_{\mathbb{R}^d} g_0 = \int_{B_R} g_0 = \int_{B_R} f_0.$$

Define

$$m_1(R) = \max\{|m(x)|, x \in B_R\}, \quad m_2(R) = \min\{|m(x)|, x \in B_R^c\},$$

We can see both $m_1, m_2 \to \infty$ as $R \to \infty$, moreover, since there exists $A > 0$ such that

$$\int_{\mathbb{R}^d} g_t m \le A \int_{\mathbb{R}^d} g_0 m \le A m_1(R) \int_{B_R} g_0.$$

For any $\rho > 0$, we write

$$\begin{aligned}\int_{B_\rho} g_t &= \int_{\mathbb{R}^d} g_t - \int_{B_\rho^c} g_t \\ &\ge \int_{\mathbb{R}^d} g_0 - \frac{1}{m_2(\rho)} \int_{\mathbb{R}^d} g_t m \\ &\ge \int_{\mathbb{R}^d} g_0 - \frac{A m_1(R)}{m_2(\rho)} \int_{B_R} g_0 \ge \frac{1}{2} \int_{B_R} g_0,\end{aligned}$$

by taking $m_2(\rho) = 2Am(R)$. As a consequence, for any $t > 0$, there exists $(x_0, v_0) \in B_\rho$ which may depend on g_0, t such that

$$g(t, x_0, v_0) \ge \frac{1}{|B_\rho|} \int_{B_\rho} g_t := \gamma \int_{B_R} g_0.$$

By the maximum principle we have

$$f(t, x_0, v_0) \ge g(t, x_0, v_0) \ge \gamma \int_{B_R} g_0 = \gamma \int_{B_R} f_0.$$

□

Recall

$$\bar{B}_r(x_0, v_0) = \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d : |v - v_0| \le r, |x - x_0| \le r^3\},$$

Then we come to prove our main theorem

Theorem 5.2. *Under the assumption of Theorem 1.3. The equation (3) defined in Theorem 2 satisfies the Harris condition: For any $R > 0$, there exist a constant $T = T(R) > 0$ and a positive, nonzero measure $\mu = \mu(R)$ such that*

$$S_T f_0 \ge \mu \int_{B_R} f_0, \quad \forall f_0 \in L^1(m), \quad f \ge 0, \quad \|f_0\|_{L^1(m)} \le 1,$$

As a consequence, Theorem 1.3 is proved.

Proof. By Lemma 5.1 we have there exist $\gamma, \rho > 0$ such that for any $t, R > 0$, there exists $(x_0, v_0) \in B_\rho$ such that

$$f(t, x_0, v_0) \ge \gamma \int_{B_R} f_0.$$

where γ, ρ does not depend on f_0 and t while x_0, v_0 may depend on f_0, t. By Lemma 4.1 we have

$$\|f\|_{C^{2,\delta}} \le C \|f_0\|_{L^1(m)}, \quad \forall t \in [\frac{\eta}{2}, \eta],$$

in particular

$$\|\nabla_x f\|_{L^\infty} + \|\nabla_v f\|_{L^\infty} + \|\Delta_x f\|_{L^\infty} + \|f\|_{L^\infty} \le C \|f_0\|_{L^1(m)}, \quad \forall t \in [\frac{\eta}{2}, \eta],$$

and by equation

$$\partial_t f := \mathscr{L} f = \partial_x (A(x, v) f) + \partial_v (B(x, v) f) + \Delta_v f,$$

we have

$$\|\nabla_x f\|_{L^\infty(\Omega)} + \|\nabla_v f\|_{L^\infty(\Omega)} + \|\partial_t f\|_{L^\infty(\Omega)} \le C\|f_0\|_{L^1(m)} \le C, \quad \forall t \in [\frac{\eta}{2}, \eta], \tag{11}$$

for $\Omega = B_{2\rho}$ and some constant $C > 0$. So for every $R > 0$, there exist $t_1, t_2, r_0, \rho, \gamma > 0$ which do not depend on f_0 and $(x_0, v_0) \in B_\rho$ which may depend on f_0, such that for all $t \in (t_1, t_2)$, we have

$$f(t,x,v) \ge \frac{\gamma}{2} 1_{B_{r_0}(x_0,v_0)} \int_{B_R} f_0,$$

where $B_{r_0}(x_0, v_0)$ denotes the ball centered at (x_0, v_0) with radius r_0. Take $r_1 = \min\{(\frac{r_0}{2}, (\frac{r_0}{2})^{\frac{1}{3}}\}$ such that $\bar{B}_{r_1}(x_0, v_0) \subset B_{r_0}(x_0, v_0)$, then we have

$$f(t,x,v) \ge \frac{\gamma}{2} 1_{\bar{B}_{r_1}(x_0,v_0)} \int_{B_R} f_0.$$

Take $\alpha = \max\{\frac{2\rho}{r_1}, (\frac{2\rho}{r_1})^{\frac{1}{3}}, 1\}$ large such that $B_{2\rho}(x_0, v_0) \subset \bar{B}_{\alpha r_1}(x_0, v_0)$. Define

$$\tau = \min(t_2 - t_1, 1, \frac{r_1^3}{2V}, \frac{log2}{M}, \frac{1}{20M}).$$

Using Theorem 4.3, we have

$$f(t,x,v) \ge \frac{\gamma}{2} 1_{\bar{B}_{\alpha r_1}(x_0,v_0)} \int_{B_R} f_0 \ge K\frac{\gamma}{2} 1_{B_{2\rho}(x_0,v_0)} \int_{B_R} f_0, \quad \forall t \in (t_2 - \frac{\tau}{2}, t_2)$$

since $(x_0, v_0) \in B_\rho$ implies that $B_\rho \subset B_{2\rho}(x_0, v_0)$, we have

$$f(t,x,v) \ge K\frac{\gamma}{2} 1_{B_\rho(0,0)} \int_{B_R} f_0,$$

for any $t \in (t_2 - \frac{\tau}{2}, t_2)$. So by define $\mu(R) = K\frac{\lambda}{2} 1_{B_\rho(0,0)}, T(R) = t_2 - \frac{\tau}{4}$, we conclude the Harris condition (H2). □

Then by Theorem 2.1, we have proved

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \le Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

which is Theorem 1.3 in the case $p = 1$.

Remark 5.3. *If we replace f in the proof by the normalized steady state G, we can deduce that $G > 0$.*

6. Proof of $L^p(m)$ convergence

In the last section, we have proved Theorem 1.3 in the case of $p = 1$, now we prove it for general p, which will complete the proof of Theorem 1.3. In this section, $A \lesssim B$ will denote $A \le CB$ for some (explicitly computable) constant $C > 0$. First make the splitting

$$\mathcal{B} = \mathcal{L} - \mathcal{A}, \quad \mathcal{A} = M_1 \chi_{R_1}(x,v),$$

since

$$\varphi_p(m) \le -C + M\mathbb{1}_{B_R},$$

by Remark 1.4 it's easily seen that we can take M_1, R_1 such that

$$\|S_{\mathcal{B}}(t)\|_{L^p(m)\to L^p(m)} \lesssim e^{-at}, \tag{12}$$

and by Lyapunov condition we have

$$\|S_{\mathscr{B}}(t)\|_{L^1(m)\to L^1(m)} \lesssim e^{-at}, \tag{13}$$

for some $a>0$. Before going to the proof of our main theorem, we need two last deduced results.

Lemma 6.1. *We have*

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^p(m)\to L^p(m)} \lesssim e^{-at}, \quad \forall t\geq 0,$$

and

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^1(m)\to L^1(m)} \lesssim e^{-at}, \quad \forall t\geq 0,$$

and

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^1(m)\to L^p(m)} \lesssim t^{-\alpha}e^{-at}, \quad \forall t\geq 0,$$

for $\alpha=\frac{5d+2}{4}$ *and some* $a>0$.

Proof. The first two inequalities are obtained obviously by (12), (13) and the property of $\mathscr{A}$. For the third inequality we split it into two parts, $t\in(0,\eta]$ and $t>\eta$, where η is defined in Theorem 3.1. When $t\in(0,\eta]$, we have $e^{-at}\geq e^{-a\eta}$, by Theorem 3.1, we have

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^1(m)\to L^p(m)} \lesssim t^{-\alpha} \lesssim t^{-\alpha}e^{-at}, \quad \forall t\in(0,\eta],$$

for some $a>0$. When $t\geq\eta$, by Theorem 3.1, we have

$$\|S_{\mathscr{B}}(\eta)\|_{L^1(m)\to L^p(m)} \lesssim \eta^{\alpha} \lesssim 1,$$

and by (12)

$$\|S_{\mathscr{B}}(t-\eta)\|_{L^p(m)\to L^p(m)} \lesssim e^{-a(t-\eta)} \lesssim e^{-at},$$

and

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^1(m)\to L^p(m)} \lesssim e^{-at} \lesssim t^{-\alpha}e^{-at}, \quad \forall t>\eta,$$

the proof is ended by combining the two cases above. □

In the following $\mathscr{U}*\mathscr{V}$ denotes the convolution of two operators valued function $\mathscr{U},\mathscr{V}$ defined by

$$(\mathscr{U}*\mathscr{V})(t)=\int_0^t \mathscr{U}(s)\mathscr{V}(t-s)ds,$$

and we set $\mathscr{U}^{(*0)}=I, \mathscr{U}^{(*1)}=\mathscr{U}$ and for any $k\geq 2$, $\mathscr{U}^{(*k)}=\mathscr{U}^{(*(k-1))}*\mathscr{U}$.

Lemma 6.2. *let* X,Y *be two Banach spaces,* $S(t)$ *a semigroup such that for all* $t\geq 0$ *and some* $0<a$ *we have*

$$\|S(t)\|_{X\to X}\leq C_X e^{-at}, \ \ \|S(t)\|_{Y\to Y}\leq C_Y e^{-at},$$

and for some $0<\alpha$*, we have*

$$\|S(t)\|_{X\to Y}\leq C_{X,Y}t^{-\alpha}e^{-at}.$$

Then we can have that for all integer $n>0$

$$\|S^{(*n)}(t)\|_{X\to X}\leq C_{X,n}t^{n-1}e^{-at},$$

similarly

$$\|S^{(*n)}(t)\|_{Y\to Y}\leq C_{Y,n}t^{n-1}e^{-at},$$

and

$$\|S^{(*n)}(t)\|_{X\to Y}\leq C_{X,Y,n}t^{n-\alpha-1}e^{-at}.$$

In particular for $\alpha+1<n$*, and for any* $a^*<a$

$$\|S^{(*n)}(t)\|_{X\to Y}\leq C_{X,Y,n}e^{-a^*t}.$$

Proof. See Lemma 2.5 in [16]. □

Then we come to the final proof.

Proof. (**Proof of Theorem 1.3.**) Remember that we already proved

$$\|S_{\mathscr{L}}(I-\Pi)(t)\|_{L^1(m)\to L^1(m)} \lesssim e^{-at},$$

where I is the identity operator and Π is a projection operator defined by

$$\Pi(f) = \mathscr{M}(f)G.$$

First, Iterating the Duhamel's formula we split it into 3 terms

$$\begin{aligned} S_{\mathscr{L}}(I-\Pi) =&(I-\Pi)\{S_{\mathscr{B}} + \sum_{l=1}^{n-1}(S_{\mathscr{B}}\mathscr{A})^{(*l)} * (S_{\mathscr{B}})\} \\ &+ (S_{\mathscr{B}}(t)\mathscr{A})^{(*(n-1))} * \{(I-\Pi)S_{\mathscr{L}}\} * (\text{Å}S_{\mathscr{B}}(t)), \end{aligned}$$

and we will estimate them separately. By (12) the first term is thus estimated. For the second term, still using (9), we get

$$\|S_{\mathscr{B}}(t)\mathscr{A}\|_{L^p(m)\to L^p(m)} \lesssim e^{-at},$$

by Lemma 6.2, we have

$$\|(S_{\mathscr{B}}(t)\mathscr{A})^{(*l)}\|_{L^p(m)\to L^p(m)} \lesssim t^{l-1}e^{-at},$$

together with (12) the second term is estimated. For the last term by Hölder's inequality we have

$$\|I\|_{L^p(m)\to L^1(mG)} \lesssim 1$$

with

$$G = e^{-(|v|^2+|x|^2)}$$

(there are many choice of G) so we have

$$\|\mathscr{A}S_{\mathscr{B}}(t)\|_{L^p(m)\to L^1(m)} \lesssim e^{-at}.$$

By Lemma 6.1 and 6.2, we have

$$\|(S_{\mathscr{B}}\mathscr{A})^{(*(n-1))}(t)\|_{L^1(m)\to L^p(m)} \lesssim t^{n-\alpha-2}e^{-at},$$

finally recall

$$\|S_{\mathscr{L}}(t)(I-\Pi)\|_{L^1(m)\to L^1(m)} \lesssim e^{-at}.$$

Taking $n > \alpha+2$ the third term is estimated, thus the proof is ended by gathering the inequalities above. □

7. Proof of Main Theorem

This section we come to prove Theorem 1.1. Recall that we have proved Theorem 1.3 in the last section, the only thing remain to prove is to find a weight function m and a function $H \geq 1$ in Theorem 1.3.

Theorem 7.1. *Denote $\mathscr{L}$ the operator of the kinetic Fitzhugh-Nagumo equation (2), then there exist a weight function m and a function $H \geq 1$ satisfies Theorem 1.3.*

Proof. We recall the kinetic Fitzhugh-Nagumo equation

$$\partial_t f := \mathscr{L} f = \partial_x (A(x,v) f) + \partial_v (B(x,v) f) + \partial^2_{vv} f,$$

with

$$A(x,v) = ax - bv, \quad B(x,v) = v(v-1)(v-c) + x,$$

by a change of variable $w = bv$, the equation is changed to

$$\partial_t f := \mathscr{L} f = \partial_x (A(x,v) f) + \partial_v (B(x,v) f) + \frac{1}{b^2}\partial^2_{vv} f,$$

with

$$A(x,v) = ax - v, \quad B(x,v) = \frac{1}{b^3} v(v-b)(v-bc) + x,$$

we have

$$\mathscr{L}^* f = -A(x,v)\partial_x f - B(x,v)\partial_v f + \frac{1}{b^2}\partial_{vv} f,$$

for some $a, b, c > 0$. This time we have

$$\begin{aligned}\frac{\mathscr{L}^* m}{m} =& v \cdot \frac{\nabla_x m}{m} - ax \cdot \frac{\nabla_x m}{m} + \frac{1}{b^2}\frac{\Delta_v m}{m} \\ & - (\frac{1}{b^3} v(v-b)(v-bc) + x) \cdot \frac{\nabla_v m}{m}.\end{aligned}$$

We can take $m = e^{\frac{r}{2}(|x|^2 + |v|^2)}$, with $r > 0$ to be fixed later, then we have

$$\frac{\nabla_x m}{m} = rx, \quad \frac{\nabla_v m}{m} = rv, \quad \frac{\Delta_v m}{m} = r + r^2 |v|^2,$$

we then compute

$$\begin{aligned}\frac{\mathscr{L}^* m}{m} =& rx \cdot v - ar|x|^2 + \frac{r}{b^2} + \frac{r^2}{b^2}|v|^2 \\ & - \frac{1}{b^3}|v|^2 (v-b)(v-bc) - rx \cdot v \\ =& -ar|x|^2 - \frac{1}{b^3}|v|^4 + M_1 v^3 + M_2 |v|^2 + M_3,\end{aligned}$$

for some constant $M_1, M_2, M_3 > 0$, so the Lyapunov condition (C1)

$$\mathscr{L}^* m \leq -\alpha m + b,$$

is satisfied for some $\alpha, b > 0$, similarly

$$\begin{aligned}\phi_2(m) =& v \cdot \frac{\nabla_x m}{m} - ax \cdot \frac{\nabla_x m}{m} + \frac{a}{2} + \frac{1}{b^2}\frac{|\nabla_v m|^2}{m^2} + \frac{1}{b^2}\frac{\Delta_v m}{m} \\ & - (\frac{1}{b^3} v(v-b)(v-bc) + x) \cdot \frac{\nabla_v m}{m} + \frac{1}{2b^3}(3v^2 + 2b(1+c)v + b^2 c),\end{aligned}$$

and this time we have

$$\begin{aligned}\phi_2(m) =& r x\cdot v - ar|x|^2 + \frac{a}{2} + \frac{r^2}{b^2}|v|^2 + \frac{r}{b^2} + \frac{r^2}{b^2}|v|^2\\ &- \frac{1}{b^3}|v|^2(v-b)(v-bc) - rx\cdot v + \frac{1}{2b^3}(3v^2+2b(1+c)v+b^2c)\\ =& -ar|x|^2 - \frac{1}{b^3}|v|^4 + K_1v^3 + K_2|v|^2 + K_3 v + K_4,\end{aligned}$$

for some constants K_1, K_2, K_3, K_4, if we take

$$H = |v|^4 + |x|^2 + 1,$$

it's easily seen that we have

$$-C_1 H \le \phi_2(m) \le -C_2 H + C_3,$$

for some $C_1, C_2, C_3 > 0$, which is just condition (C2). And it's easily seen that for any integer $n \ge 2$ fixed, for any $\epsilon > 0$ small, we can find a constant $C_{\epsilon,n}$ such that

$$\sum_{k=1}^{n} |D_x^k(ax)| + \sum_{k=1}^{n} |D_{x,v}^k(\frac{1}{b^3}v(v-b)(v-bc)+x)| \le P_1|v|^2 + P_2|v| + P_3 \le C_{n,\epsilon} + \epsilon H,$$

with $P_1, P_2, P_3 > 0$ constant, so condition (C3) is also satisfied. Since

$$\frac{\Delta_{x,v} m}{m} = 2r + r^2|v|^2 + r^2|x|^2 \ge 0,$$

all the conditions of Theorem 1.3 is satisfied, we finally compute

$$\begin{aligned}\varphi_\infty(m) =& v\cdot\frac{\nabla_x m}{m} - ax\cdot\frac{\nabla_x m}{m} + a + \frac{2}{b^2}\frac{|\nabla_v m|^2}{m^2} - \frac{1}{b^2}\frac{\Delta_v m}{m}\\ &- (\frac{1}{b^3}v(v-b)(v-bc)+x)\cdot\frac{\nabla_v m}{m} + \frac{1}{b^3}(3v^2+2b(1+c)v+b^2c).\end{aligned}$$

We have

$$\begin{aligned}\varphi_\infty(m) =& r x\cdot v - ar|x|^2 + a + \frac{2r^2}{b^2}|v|^2 - \frac{r}{b^2} - \frac{r^2}{b^2}|v|^2\\ &- \frac{1}{b^3}|v|^2(v-b)(v-bc) - rx\cdot v + \frac{1}{b^3}(3v^2+2b(1+c)v+b^2c)\\ =& -ar|x|^2 - \frac{1}{b^3}|v|^4 + K_1v^3 + K_2|v|^2 + K_3v + K_4,\end{aligned}$$

for some constants K_1, K_2, K_3, K_4, it's easily seen that

$$\varphi_\infty(m) \le -C + M\mathbb{1}_{B_R},$$

for some $C, M, R > 0$, the proof is finished. □

Then we come to find a weight function m and a function $H \ge 1$ for the kinetic Fokker-Planck equation with general force.

Theorem 7.2. *Denote $\mathscr{L}$ the operator of the kinetic Fokker-Planck equation (1), then there exist a weight function m and a function H satisfies Theorem 1.3.*

Proof. First we have

$$\mathscr{L}^* f = v\cdot\nabla_x f - \nabla_x V(x)\cdot\nabla_v f + \Delta_v f - \nabla_v W(v)\cdot\nabla_v f,$$

denote

$$H_1 = \frac{|v|^2}{2} + V(x) + \epsilon v\cdot\nabla_x\langle x\rangle, \quad m = e^{\lambda H_1},$$

so we have

$$\frac{\mathscr{L}^* m}{m} = \lambda(v\cdot\nabla_x H_1 - \nabla_x V(x)\cdot\nabla_v H_1 + \Delta_v H_1 + \lambda|\nabla_v H_1|^2 - \nabla_v W(v)\cdot\nabla_v H_1).$$

We easily compute

$$\nabla_v H_1 = v + \epsilon\nabla_x\langle x\rangle, \quad \nabla_x H_1 = \nabla_x V(x) + \epsilon v\cdot\nabla_x^2\langle x\rangle, \quad \Delta_v H_1 = d,$$

and since

$$\nabla_x^2\langle x\rangle \le CI,$$

where I is the $d\times d$ identity matrix, we have

$$\begin{aligned}\frac{\mathscr{L}^* m}{m} =&\lambda(v\cdot\nabla_x V(x) + \epsilon v\nabla_x\langle x\rangle^2 v - \nabla_x V(x)\cdot v - \nabla_x V(x)\cdot\nabla_x\langle x\rangle + d\\ &+\lambda|v+\epsilon\nabla_x\langle x\rangle|^2 - \epsilon\nabla_v W(v)\cdot\nabla_x\langle x\rangle - \nabla_v W(v)\cdot v\\ \le& C(\lambda^2|v|^2 + \lambda|\nabla W(v)|) - \lambda\nabla_x V(x)\cdot\nabla_x\langle x\rangle - \lambda\nabla_v W(v)\cdot v,\end{aligned}$$

for some constant take $\lambda > 0$ small, we conclude

$$\mathscr{L}^* m \le -C_1 H m + C_2,$$

for some constant $C_1, C_2 > 0$, with $H = \langle v\rangle^\beta + \langle x\rangle^{\gamma-1} + 1$, then the Lyapunov condition (C1) follows. For the second inequality, by Lemma B.1 we have

$$\begin{aligned}\phi_2(m) =&\lambda(v\cdot\nabla_x H_1 + \nabla_x V(x)\cdot\nabla_v H_1 + \Delta_v H_1\\ &+2\lambda|\nabla_v H_1|^2 - \nabla_v W(v)\cdot\nabla_v H_1) + \frac{1}{2}\Delta_v W(v),\end{aligned}$$

we compute

$$\begin{aligned}\phi_2(m) =&\lambda(v\cdot\nabla_x V(x) + \epsilon v\nabla_x\langle x\rangle v - \nabla_x V(x)\cdot v - \nabla_x V(x)\cdot\nabla_x\langle x\rangle + d\\ &+2\lambda|v+\epsilon\nabla_x\langle x\rangle|^2 - \epsilon\nabla_v W(v)\cdot\nabla_x\langle x\rangle - \nabla_v W(v)\cdot v) + \frac{1}{2}\Delta_v W(v)\\ \le& C(\lambda^2|v|^2 + \lambda|\nabla W(v)| + |\Delta_x W(v)|) - \lambda\nabla_x V(x)\cdot\nabla_x\langle x\rangle - \lambda\nabla_v W(v)\cdot v,\end{aligned}$$

and we still have

$$-C_1 H \le \phi_2(m) \le -C_2 H + C_3,$$

for some constant $C_1, C_2, C_3 > 0$, thus condition(C2) is proved. It's easily seen that for any integer $n\ge 2$ fixed, for any $\epsilon > 0$ small, we can find a constant $C_{\epsilon,n}$ such that

$$\sum_{k=1}^n |D_x^k\nabla_x V(x)| + \sum_{k=1}^n |D_{x,v}^k\nabla_v W(v)| = \sum_{k=2}^{n+1}|D_x^k V(x)| + \sum_{k=2}^{n+1}|D_{x,v}^k W(v)| \le C_{n,\epsilon} + \epsilon H,$$

by the definition of $V(x)$ and $W(v)$, so condition (C3) is also satisfied. For the last condition

$$\frac{\Delta_{x,v} m}{m} = \lambda^2|\nabla_x H_1|^2 + \lambda^2|\nabla_v H_1|^2 + \lambda\Delta_x H_1 + \lambda\Delta_v H_1.$$

For the term $\Delta_x H_1$ we compute

$$\Delta_x H_1 = \nabla_x V(x) + \epsilon v \cdot \nabla \Delta \langle x \rangle \geq -L_1 - L_2 |v|,$$

for some constant $L_1, L_2 > 0$, and

$$|\nabla_v H_1|^2 = |v + \epsilon \nabla_x \langle x \rangle|^2 \geq \frac{|v|^2}{2} - L_3,$$

for some constant $L_3 > 0$, since

$$\Delta_v H_1 = d \geq 0,$$

we conclude that

$$\frac{\Delta_{x,v} m}{m} \geq C,$$

for some constant C, so all the conditions are satisfied. We finally compute

$$\begin{aligned} \varphi_\infty(m) =& \lambda(v \cdot \nabla_x V(x) + \epsilon v \nabla_x \langle x \rangle v - \nabla_x V(x) \cdot v - \nabla_x V(x) \cdot \nabla_x \langle x \rangle - d \\ &+ \lambda |v + \epsilon \nabla_x \langle x \rangle|^2 - \epsilon \nabla_v W(v) \cdot \nabla_x \langle x \rangle - \nabla_v W(v) \cdot v) + \Delta_v W(v) \\ \leq& C(\lambda^2 |v|^2 + \lambda |\nabla W(v)| + |\Delta_x W(v)|) - \lambda \nabla_x V(x) \cdot \nabla_x \langle x \rangle - \lambda \nabla_v W(v) \cdot v, \end{aligned}$$

it's easily seen that

$$\varphi_\infty(m) \leq -C + M \mathbb{1}_{B_R},$$

for some $C, M, R > 0$, the proof is thus finished. □

Appendix A. Proof for the Harris-Doeblin Theorem

In this section, we prove the Harris-Doeblin Theorem 2.1. The whole proof is in this section is by S. Mischler and J. A. Cañizo in their unpublished notes.

Lemma A.1. *(Doeblin's variant). Under assumption (H2) in Theorem 2.1, if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \geq \frac{4}{m(R)} \|f\|_{L^1(m)}, \quad \mathcal{M}(f) = 0, \quad \|f\|_{L^1(m)} \leq 1, \tag{14}$$

we then have

$$\|S_T f\|_{L^1} \leq (1 - \frac{\langle \mu(R) \rangle}{2}) \|f\|_{L^1},$$

where

$$\langle \mu \rangle = \int_{\mathbb{R}^d} \mu, \quad m(R) := \min\{|m(x)|, x \in B_R^c\},$$

Proof. From the hypothesis (14), we have

$$\begin{aligned} \int_{B_R} f_\pm &= \int_{\mathbb{R}^d} f_\pm - \int_{B_R^c} f_\pm \\ &\geq \frac{1}{2} \int_{\mathbb{R}^d} |f| - \frac{1}{m(R)} \int_{\mathbb{R}^d} |f| m \geq \frac{1}{4} \int_{\mathbb{R}^d} |f|, \end{aligned}$$

since

$$\|f_\pm\|_{L^1(m)} \leq \|f\|_{L^1(m)}.$$

Together with (H2), we get

$$S_T f_\pm \geq \frac{\mu(R)}{4} \int_{\mathbb{R}^d} |f| := \eta,$$

We deduce

$$|S_T f| \le |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T |f| - 2\eta,$$

and next

$$\int_{\mathbb{R}^d} |S_T f| \le \int_{\mathbb{R}^d} S_T |f| - 2\eta = \int_{\mathbb{R}^d} (|f| - \frac{\mu(R)}{2} \int_{\mathbb{R}^d} |f|),$$

which is nothing but the announced estimate. □

Then we come to the proof of Theorem 2.1.

Proof. **Proof of Theorem 2.1**. We split the proof in several steps. In Step 1-6 we will assume $\|f_0\|_{L^1(m)} \le 1$.

Step 1. We fix $f_0 \in L^1(m), \mathcal{M}(f) = 0$, and we denote $f_t := S_t f_0$. From (H1), we have

$$\frac{d}{dt} \|f\|_{L^1(m)} \le -\alpha \|f_t\|_{L^1(m)} + b\|f_t\|_{L^1}, \quad \forall t \ge 0,$$

which implies

$$\|S_t f\|_{L^1(m)} \le e^{-\alpha t} \|f_0\|_{L^1(m)} + (1 - e^{-\alpha t}) \frac{b}{\alpha} \|f_0\|_{L^1}, \quad \forall t \ge 0,$$

We fix $R > 0$ large enough such that $\frac{8b}{\alpha} \le m(R)$, then take $T = T(R)$ and $\mu = \mu(R)$, define

$$\gamma := e^{-\alpha T} \in (0,1), \quad K := (1 - e^{-\alpha T}) \frac{b}{\alpha},$$

Then we have $K/A \le (1-\gamma)/2$ with $A := m(R)/4$, and we have

$$\|S_T f\|_{L^1(m)} \le \gamma \|f_0\|_{L^1(m)} + K\|f_0\|_{L^1}. \tag{15}$$

We also recall that

$$\|S_T f\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall t \ge 0. \tag{16}$$

We define

$$\|f\|_\beta = \|f_0\|_{L^1} + \beta \|f_0\|_{L^1(m)},$$

with $\beta > 0$ to be fixed later, and we observe that the following alternative holds

$$\|f_0\|_{L^1(m)} \le A\|f_0\|_{L^1}, \tag{17}$$

or

$$\|f_0\|_{L^1(m)} > A\|f_0\|_{L^1}. \tag{18}$$

Step 2. By Lemma A.1 that under condition (17), there holds

$$\|S_T f_0\|_{L^1} \le \gamma_1 \|f_0\|_{L^1}, \quad \gamma_1 \in (0,1), \tag{19}$$

and more precisely $\gamma_1 := 1 - \langle \mu \rangle /2$, which is nothing but the conclusion of Lemma A.1.

Step 3. We claim that under condition (17), there holds

$$\|S_T f_0\|_\beta \le \gamma_2 \|f_0\|_\beta, \quad \gamma_2 := \max(\frac{\gamma_1 + 1}{2}, \gamma), \tag{20}$$

for $\beta > 0$ small enough. Indeed, using (15) and (20), we compute

$$\begin{aligned} \|S_T f_0\|_\beta =& \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ \le& (\gamma_1 + K\beta) \|f_0\|_{L^1} + \gamma\beta \|f_0\|_{L^1(m)}, \end{aligned}$$

and we take $\beta > 0$ such that $\gamma_1 + K\beta \le \gamma_2$.
Step 4. We claim that under condition (18), there holds

$$\|S_T f_0\|_{L^1(m)} \le \gamma_3 \|f_0\|_{L^1(m)}, \quad \gamma_3 := \frac{\gamma+1}{2}. \tag{21}$$

Indeed we compute

$$\|S_T f_0\|_{L^1(m)} \le \gamma \|f_0\|_{L^1(m)} + \frac{K}{A}\|f_0\|_{L^1(m)} = \gamma_3 \|f_0\|_{L^1(m)}.$$

Step 5. We claim that under condition (18), there holds

$$\|S_T f_0\|_\beta \le \gamma_4 \|f_0\|_\beta, \quad \gamma_4 := \frac{\gamma_3 + 1/\beta}{1+1/\beta}. \tag{22}$$

Indeed, using (16) and (21), we compute

$$\begin{aligned}\|S_T f_0\|_\beta &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le \|f_0\|_{L^1} + \gamma_3 \beta \|f_0\|_{L^1(m)} \\ &\le (1-\epsilon)\|f_0\|_{L^1} + (\epsilon + \gamma_3\beta)\|f_0\|_{L^1(m)},\end{aligned}$$

and we choose $\epsilon \in (0,1)$ such that $1-\epsilon = \epsilon/\beta + \gamma_3$.
Step 6. By gathering (20) and (22), we have

$$\|S_T f_0\|_\beta \le \gamma_5 \|f_0\|_\beta, \quad \gamma_5 := \max(\gamma_2, \gamma_4) \in (0,1),$$

for some well chosen $\beta > 0$. By iteration, we get

$$\|S_{nT} f_0\|_\beta \le \gamma_5^n \|f_0\|_\beta,$$

and we then conclude there exist some constants $C \ge 1$ and $a < 0$ such that

$$\|S_t f\|_{L^1(m)} \le C e^{at} \|f\|_{L^1(m)}, \quad \forall t \ge 0, \quad \forall f \in L^1(m), \|f\|_{L^1(m)} \le 1, \quad \mathscr{M}(f) = 0.$$

Step 7. (Linearity argument) For general f, we can always find $\lambda > 0$ such that $\|\frac{1}{\lambda} f\|_{L^1(m)} \le 1$, since S_t is linear we have

$$\|S_t f\|_{L^1(m)} = \lambda \|S_t(\frac{1}{\lambda} f)\|_{L^1(m)} \le \lambda C e^{at} \|\frac{1}{\lambda} f\|_{L^1(m)} = C e^{at} \|f\|_{L^1(m)}, \quad \forall t \ge 0,$$

for all $f \in L^1(m)$, $\quad \mathscr{M}(f) = 0$. □

Appendix B. Computation for $\phi_2(m)$ and $\varphi_p(m)$

Lemma B.1. *Define*

$$\partial_t f := \mathscr{L} f = \mathrm{div}_x(A(x,v)f) + \mathrm{div}_v(B(x,v)f) + K\Delta_v f,$$

with

$$A(x,v) = -v + \Phi(x),$$

and $K > 0$ a constant, then for any weight function m we have

$$\int (f(\mathscr{L} g) + g(\mathscr{L} f)) m^2 = -2K \int \nabla_v f \cdot \nabla_v g m^2 + 2\int f g \phi_2(m) m^2, \tag{23}$$

with

$$\phi_2(m) = v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + \frac{1}{2}\mathrm{div}_x\Phi(x) + K\frac{|\nabla_v m|^2}{m^2}$$
$$+K\frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}\mathrm{div}_v B(x,v).$$

Also we have for $p\in[1,\infty]$

$$\int sign f|f|^{p-1}\mathscr{L} f m^p = -K\int|\nabla_v(mf)|^2|f|^{p-2}m^{p-2} + \int |f|^p\varphi_p(m)m^p, \tag{24}$$

with

$$\varphi_p(m) = v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + (1-\frac{1}{p})\mathrm{div}_x\Phi(x) + 2K(1-\frac{1}{p})\frac{|\nabla_v m|^2}{m^2}$$
$$+K(\frac{2}{p}-1)\frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + (1-\frac{1}{p})\mathrm{div}_v B(x,v),$$

where we use $\int f$ in place of $\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,dx\,dv$ for short.

Proof. Define

$$\mathscr{T} f = -v\cdot\nabla_x f,$$

we have

$$\int f(\mathscr{T} g)m^2 + \int(\mathscr{T} f)gm^2 = \int\mathscr{T}(fg)m^2 = -\int fg\mathscr{T}(m^2) = -2\int fgm^2\frac{\mathscr{T} m}{m},$$

for the term with operator Δ we have

$$\int(f\Delta_v g + \Delta_v f g)m^2 = -\int\nabla_v(fm^2)\cdot\nabla_v g + \nabla_v(gm^2)\cdot\nabla_v f$$
$$= -2\int\nabla_v f\cdot\nabla_v g m^2 + \int fg\Delta_v(m^2)$$
$$= -2\int\nabla_v f\cdot\nabla_v g m^2 + 2\int fg(|\nabla_v m|^2 + \Delta_v m m).$$

For the other terms, using integration by parts we deduce

$$\int f\mathrm{div}_v(B(x,v)g)m^2 + g\mathrm{div}_v(B(x,v)f)m^2$$
$$= \int fB(x,v)\cdot\nabla_v g m^2 + gB(x,v)\cdot\nabla_v f m^2 + 2\mathrm{div}_v B(x,v)fgm^2$$
$$= -\int fg\nabla_v\cdot(B(x,v)m^2) + 2\mathrm{div}_v B(x,v)fgm^2$$
$$= \int -2fgB(x,v)\cdot\frac{\nabla_v m}{m}m^2 + \mathrm{div}_v B(x,v)fgm^2.$$

Similarly

$$\int f\mathrm{div}_x(\Phi(x)g)m^2 + g\mathrm{div}_x(\Phi(x)f)m^2$$
$$= \int -2fg\Phi(x)\cdot\frac{\nabla_v m}{m}m^2 + \mathrm{div}_x\Phi(x)fgm^2,$$

so (23) are proved by combining the terms above. For (24) we compute

$$C_1 := \int \operatorname{sign} f|f|^{p-1} m^p \Delta_v f = -\int \nabla_v(\operatorname{sign} f|f|^{p-1} m^p)\cdot \nabla_v f$$
$$= \int -(p-1)|\nabla_v f|^2|f|^{p-2} m^p - \frac{1}{p}\int \nabla_v |f|^p \cdot \nabla_v(m^p).$$

Using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we deduce

$$\begin{aligned} C_1 =& -(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^{p-2} + (p-1)\int |\nabla_v m|^2|f|^p m^{p-2} \\ &+ \frac{2(p-1)}{p^2}\int \nabla_v(|f|^p)\cdot\nabla_v(m^p) - \frac{1}{p}\int \nabla_v(|f|^p)\cdot\nabla_v(m^p) \\ =& -(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^p + (p-1)\int |\nabla_v m|^2|f|^p m^{p-2} \\ &- \frac{p-2}{p^2}\int |f|^p \Delta_v m^p. \end{aligned}$$

Using that $\Delta_v m^p = p\Delta_v m\, m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we obtain

$$\begin{aligned} C_1 =& -(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^{p-2} \\ &+ \int |f|^p m^p\left[\left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v m|^2}{m^2}\right]. \end{aligned}$$

For the other terms we have

$$\begin{aligned} &\int \operatorname{sign} f|f|^{p-1}\operatorname{div}_v(B(x,v)f)m^p \\ &= -\int \frac{1}{p}|f|^p \operatorname{div}_v(B(x,v)m^p) + \operatorname{div}_v B(x,v)|f|^p m^p \\ &= \int -|f|^p B(x,v)\cdot\frac{\nabla_v m}{m}m^p + (1-\frac{1}{p})\operatorname{div}_v B(x,v) f g m^p, \end{aligned}$$

similarly

$$\begin{aligned} &\int \operatorname{sign} f|f|^{p-1}\operatorname{div}_x(A(x,v)f)m^p \\ &= \int -|f|^p A(x,v)\cdot\frac{\nabla_x m}{m}m^p + (1-\frac{1}{p})\operatorname{div}_x A(x,v) f g m^p \\ &= \int |f|^p (v-\Phi(x))\cdot\frac{\nabla_x m}{m}m^p + (1-\frac{1}{p})\operatorname{div}_x \Phi(x) f g m^p. \end{aligned}$$

Gathering all the terms (24) is proved. □

Appendix C. General proof for Lemma 3.2

Lemma C.1. *For any $k>0$ integer, there exist $C,\eta>0$ such that for all $t\in[0,\eta]$,*

$$\|S_{\mathscr{L}} f_0\|_{H^k(m)} \le C t^{-\frac{3k}{2}} \|f_0\|_{L^2(m)}.$$

Proof. The case $k=1$ is proved in Lemma 3.2, we then prove it for general k, recall that in Lemma 3.2, define

$$\mathscr{F}(t,f_t):=A\|f_t\|^2_{L^2(m)}+at\|\nabla_v f_t\|^2_{L^2(m)}+2ct^2(\nabla_v f_t,\nabla_x f_t)_{L^2(m)}+bt^3\|\nabla_x f_t\|^2_{L^2(m)},$$

by choice of A,a,b,c,η, we have proved

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t,f_t)\le&-L(\|\nabla_v f_t\|^2_{L^2(m)}+t^2\|\nabla_x f_t\|^2_{L^2(m)})+C\|f_t\|^2_{L^2(m)},\\&-M_1(\|f_t\|^2_{L^2(mH^{1/2})}+t\|\nabla_v f_t\|^2_{L^2(mH^{1/2})}+t^3\|\nabla_x f_t\|^2_{L^2(mH^{1/2})}\\&+t\|\nabla^2_v f_t\|^2_{L^2(m)}+t^3\|\nabla_v\nabla_x f_t\|^2_{L^2(m)}),\end{aligned}$$

for some $L,C,M_1>0$. We then come to prove the case $k\ge 2$, for simplicity we will denote

$$\mathrm{div}_x\Phi+\mathrm{div}_v B=D,$$

recall

$$\mathscr{L}f=\mathrm{div}_x(Af)+\mathrm{div}_v(Bf)+\Delta_v f=-v\cdot\nabla_x f+\Phi\cdot\nabla_x f+B\cdot\nabla_v f+Df+\Delta_v f$$

so we have

$$\begin{aligned}\partial_{x_j}\partial_{x_i}\mathscr{L}f=&\mathscr{L}\partial_{x_j}\partial_{x_i}f+\partial_{x_j}\Phi\cdot\nabla_x\partial_{x_i}f+\partial_{x_j}B\cdot\nabla_v\partial_{x_i}f+\partial_{x_j}D\partial_{x_i}f+\partial_{x_j}\partial_{x_i}\Phi\cdot\nabla_x f\\&+\partial_{x_j}\partial_{x_i}B\cdot\nabla_v f+\partial_{x_j}\partial_{x_i}Df+\partial_{x_i}\Phi\cdot\nabla_x\partial_{x_j}f+\partial_{x_i}B\cdot\nabla_v\partial_{x_j}f+\partial_{x_i}D\partial_{x_j}f\\\le&\mathscr{L}\partial_{x_j}\partial_{x_i}f+(\epsilon H+C)(|\nabla^2_x f|+|\nabla_x\nabla_v f|+|\nabla_x f|+|\nabla_v f|+|f|)\end{aligned}$$

Summing over i,j we have

$$\begin{aligned}\frac{d}{dt}\|\nabla^2_x f\|^2_{L^2(m)}\le&-\|\nabla_v\nabla^2_x f\|^2_{L^2(m)}-C_1\|\nabla^2_x f\|^2_{L^2(mH^{1/2})}+C_2\|\nabla^2_x f\|^2_{L^2(m)}\\&+\epsilon\|\nabla^2_x f\|^2_{L^2(mH^{1/2})}+C\|\nabla^2_x f\|^2_{L^2(m)}+C(\|\nabla_x\nabla_v f\|^2_{L^2(mH^{1/2})}+\|\nabla_x f\|^2_{L^2(mH^{1/2})}\\&+\|\nabla_v f\|^2_{L^2(mH^{1/2})}+\|f\|^2_{L^2(mH^{1/2})})\\\le&-\|\nabla_v\nabla^2_x f\|^2_{L^2(m)}-\frac{C_1}{2}\|\nabla^2_x f\|^2_{L^2(mH^{1/2})}+C\|\nabla^2_x f\|^2_{L^2(m)}\\&+C(\|\nabla_x\nabla_v f\|^2_{L^2(mH^{1/2})}+\|\nabla_x f\|^2_{L^2(mH^{1/2})}+\|\nabla_v f\|^2_{L^2(mH^{1/2})}+\|f\|^2_{L^2(mH^{1/2})})\end{aligned}$$

similarly we have

$$\begin{aligned}\partial_{v_j}\partial_{x_i}\mathscr{L}f=&\mathscr{L}\partial_{v_j}\partial_{x_i}f+\partial_{x_i}\Phi\cdot\nabla_x\partial_{v_j}f+\partial_{x_i}B\cdot\nabla_v\partial_{v_j}f+\partial_{x_i}\mathrm{div}\,_x\Phi\partial_{v_j}f+\partial_{x_i}\mathrm{div}_v B\partial_{v_j}f\\&+\partial_{v_j}\partial_{x_i}B\cdot\nabla_v f+\partial_{v_j}\partial_{x_i}\mathrm{div}_v Bf-\partial_{x_j}\partial_{x_i}f+\partial_{v_j}B\cdot\nabla_v\partial_{x_i}f+\partial_{v_j}\mathrm{div}_v B\partial_{x_i}f,\\\le&\mathscr{L}\partial_{x_j}\partial_{x_i}f-\partial_{x_j}\partial_{x_i}f+(\epsilon H+C)(|\nabla^2_v f|+|\nabla_x\nabla_v f|+|\nabla_x f|+|\nabla_v f|+|f|)\end{aligned}$$

Summing over i,j we have

$$\begin{aligned}\frac{d}{dt}\|\nabla_x\nabla_v f\|^2_{L^2(m)}\le&-\|\nabla^2_v\nabla_x f\|^2_{L^2(m)}-C_1\|\nabla_x\nabla_v f\|^2_{L^2(mH^{1/2})}+C_2\|\nabla_x\nabla_v f\|^2_{L^2(m)}\\&+\epsilon(\|\nabla^2_v f\|^2_{L^2(mH^{1/2})}+\|\nabla_x\nabla_v f\|^2_{L^2(mH^{1/2})})+C(\|\nabla^2_v f\|^2_{L^2(m)}+\|\nabla_x\nabla_v f\|^2_{L^2(m)})\\&+C(\|\nabla_x f\|^2_{L^2(mH^{1/2})}+\|\nabla_v f\|^2_{L^2(mH^{1/2})}+\|f\|^2_{L^2(mH^{1/2})})-(|\nabla_x\nabla_v f|,|\nabla^2_x f|)_{L^2(m)}\end{aligned}$$

similarly

$$\begin{aligned}\partial_{v_j}\partial_{v_i}\mathscr{L}f =&\mathscr{L}\partial_{v_i}\partial_{v_j}f-\partial_{x_i}\partial_{v_j}f+\partial_{v_i}B\cdot\nabla_v\partial_{v_j}f+\partial_{v_i}\mathrm{div}_v B\partial_{v_j}f,\\ &-\partial_{v_i}\partial_{x_j}f+\partial_{v_j}\partial_{v_i}B\cdot\nabla_v f+\partial_{v_j}\partial_{v_i}\mathrm{div}_v Bf+\partial_{v_j}B\cdot\nabla_v\partial_{v_i}f+\partial_{v_j}\mathrm{div}_v B\partial_{v_i}f,\\ \le&\mathscr{L}\partial_{v_j}\partial_{v_i}f-\partial_{x_i}\partial_{v_j}f-\partial_{v_i}\partial_{x_j}f+(\epsilon H+C)(|\nabla_v^2 f|+|\nabla_v f|+|f|)\end{aligned}$$

Summing over i, j we have

$$\begin{aligned}\frac{d}{dt}\|\nabla_v^2 f\|_{L^2(m)}^2\le&-\|\nabla_v^3 f\|_{L^2(m)}^2-C_1\|\nabla_v^2 f\|_{L^2(mH^{1/2})}^2+C_2\|\nabla_v^2 f\|_{L^2(m)}^2\\ &+\epsilon\|\nabla_v^2 f\|_{L^2(mH^{1/2})}^2+C(\|\nabla_v^2 f\|_{L^2(m)}^2+\|\nabla_x\nabla_v f\|_{L^2(m)}^2)\\ &+C(\|\nabla_v f\|_{L^2(mH^{1/2})}^2+\|f\|_{L^2(mH^{1/2})}^2)\end{aligned}$$

and we define

$$(\nabla_x^2 f,\nabla_x\nabla_v f)_{L^2(m)}:=\sum_{i=1}^n\sum_{j=1}^n(\partial_{x_i}\partial_{x_j}f,\partial_{x_i}\partial_{v_j}f)_{L^2(m)}$$

so we have

$$\begin{aligned}&(\partial_{x_i}\partial_{x_j}f,\partial_{x_i}\partial_{v_j}\mathscr{L}f)_{L^2(m)}+(\partial_{x_i}\partial_{x_j}\mathscr{L}f,\partial_{x_i}\partial_{v_j}f)_{L^2(m)}\\ \le&(\partial_{x_i}\partial_{x_j}f,\mathscr{L}\partial_{x_i}\partial_{v_j}f)_{L^2(m)}+(\mathscr{L}\partial_{x_i}\partial_{x_j}f,\partial_{x_i}\partial_{v_j}f)_{L^2(m)}-\|\partial_{x_i}\partial_{x_j}f\|_{L^2(m)}^2\\ &+C(\|\nabla_v^2 f\|_{L^2(mH^{1/2})}^2+\|\nabla_x\nabla_v f\|_{L^2(mH^{1/2})}^2+\|\nabla_x f\|_{L^2(mH^{1/2})}^2\\ &+\|\nabla_v f\|_{L^2(mH^{1/2})}^2+\|f\|_{L^2(mH^{1/2})}^2)+C\|\nabla_x^2 f\|_{L^2(mH^{1/2})}(\|\nabla_v^2 f\|_{L^2(mH^{1/2})}\\ &+\|\nabla_x\nabla_v f\|_{L^2(mH^{1/2})}+\|\nabla_x f\|_{L^2(mH^{1/2})}+\|\nabla_v f\|_{L^2(mH^{1/2})}+\|f\|_{L^2(mH^{1/2})}).\end{aligned}$$

Summing over i, j we have

$$\begin{aligned}&\frac{d}{dt}2(\nabla_x^2 f,\nabla_x\nabla_v f)_{L^2(m)}\\ \le&2(\nabla_v\nabla_x^2 f,\nabla_v^2\nabla_x f)_{L^2(m)}-\|\nabla_x^2 f\|_{L^2(m)}^2\\ &+C(\|\nabla_v^2 f\|_{L^2(mH^{1/2})}^2+\|\nabla_x\nabla_v f\|_{L^2(mH^{1/2})}^2+\|\nabla_x f\|_{L^2(mH^{1/2})}^2\\ &+\|\nabla_v f\|_{L^2(mH^{1/2})}^2+\|f\|_{L^2(mH^{1/2})}^2)+C\|\nabla_x^2 f\|_{L^2(mH^{1/2})}(\|\nabla_v^2 f\|_{L^2(mH^{1/2})}\\ &+\|\nabla_x\nabla_v f\|_{L^2(mH^{1/2})}+\|\nabla_x f\|_{L^2(mH^{1/2})}+\|\nabla_v f\|_{L^2(mH^{1/2})}+\|f\|_{L^2(mH^{1/2})}).\end{aligned}$$

Then we prove the case $k=2$, define

$$\mathscr{F}_1(t,f_t)=a_1t^4\|\nabla_v^2 f_t\|_{L^2(m)}^2+a_1t^4\|\nabla_v\nabla_x f_t\|_{L^2(m)}^2+2b_1t^5(\nabla_x\nabla_v f_t,\nabla_x^2 f_t)_{L^2(m)}+c_1t^6\|\nabla_x^2 f_t\|_{L^2(m)}^2$$

where $a_1\gg b_1\gg c_1, a_1c_1\gg b_1^2$ we have

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t,f_t)=&a_1t^4\frac{d}{dt}\|\nabla_v^2 f_t\|_{L^2(m)}^2+a_1t^4\frac{d}{dt}\|\nabla_v\nabla_x f_t\|_{L^2(m)}^2+2b_1t^5\frac{d}{dt}(\nabla_x\nabla_v f_t,\nabla_x^2 f_t)_{L^2(m)}\\ &+c_1t^6\frac{d}{dt}\|\nabla_x^2 f_t\|_{L^2(m)}^2+4a_1t^3\|\nabla_v^2 f_t\|_{L^2(m)}^2+4a_1t^3\|\nabla_v\nabla_x f_t\|_{L^2(m)}^2\\ &+5b_1t^4(\nabla_x\nabla_v f_t,\nabla_x^2 f_t)_{L^2(m)}+6c_1t^5\|\nabla_x^2 f_t\|_{L^2(m)}^2\\ :=&T_4+T_5+T_6\end{aligned}$$

where

$$\begin{aligned}
T_4 \le & -\frac{C_1}{2} c_1 t^6 \|\nabla_x^2 f_t\|^2_{L^2(mH^{1/2})} + \left(-\frac{C_1}{2} a_1 t^4 + C b_1 t^5 + C c_1 t^6\right) (\|\nabla_v^2 f_t\|^2_{L^2(mH^{1/2})} \\
& + \|\nabla_v \nabla_x f_t\|^2_{L^2(mH^{1/2})}) + C b_1 t^5 \|\nabla_x^2 f\|_{L^2(mH^{1/2})} (\|\nabla_v^2 f\|_{L^2(mH^{1/2})} + \|\nabla_x \nabla_v f\|_{L^2(mH^{1/2})} \\
& + \|\nabla_x f\|_{L^2(mH^{1/2})} + \|\nabla_v f\|_{L^2(mH^{1/2})} + \|f\|_{L^2(mH^{1/2})}) \\
& + C(a_1 + b_1 + c_1) t^4 (\|f_t\|^2_{L^2(mH^{1/2})} + \|\nabla_x f_t\|^2_{L^2(mH^{1/2})} + \|\nabla_v f_t\|^2_{L^2(mH^{1/2})}) \\
\le & -\frac{C_1}{4} c_1 t^6 \|\nabla_x^2 f_t\|^2_{L^2(mH^{1/2})} - \frac{C_1}{4} a_1 t^4 (\|\nabla_v^2 f_t\|^2_{L^2(mH^{1/2})} + \|\nabla_x \nabla_v f_t\|^2_{L^2(mH^{1/2})}) \\
& + C(a_1 + b_1 + c_1) t^4 (\|f_t\|^2_{L^2(mH^{1/2})} + \|\nabla_x f_t\|^2_{L^2(mH^{1/2})} + \|\nabla_v f_t\|^2_{L^2(mH^{1/2})})
\end{aligned}$$

and

$$\begin{aligned}
T_5 = & - a_1 t^4 \|\nabla_v^3 f_t\|^2_{L^2(m)} - a_1 t^4 \|\nabla_v^2 \nabla_x f_t\|^2_{L^2(m)} - c_1 t^6 \|\nabla_v \nabla_x^2 f_t\|^2_{L^2(m)} - 2 b_1 t^5 (\nabla_v \nabla_x^2 f_t, \nabla_v^2 \nabla_x f_t)_{L^2(m)} \\
\le & -\frac{a_1}{2} t^4 \|\nabla_v^3 f_t\|^2_{L^2(m)} - \frac{a_1}{2} t^4 \|\nabla_v^2 \nabla_x f_t\|^2_{L^2(m)} - \frac{c_1}{2} t^6 \|\nabla_v \nabla_x^2 f_t\|^2_{L^2(m)}
\end{aligned}$$

and

$$\begin{aligned}
T_6 = & C(a_1 + b_1 + c_1) t^3 (\|\nabla_v^2 f_t\|^2_{L^2(m)} + \|\nabla_v \nabla_x f_t\|^2_{L^2(m)}) + (6 c_1 t^5 - b_1 t^5 + C b_1 t^6) \|\nabla_x^2 f_t\|^2_{L^2(m)} \\
& + (C a_1 t^4 + 5 b_1 t^4) (|\nabla_v \nabla_x f_t|, |\nabla_x^2 f_t|)_{L^2(m)} \\
\le & C(\frac{a_1^2}{b_1} + a_1 + b_1 + c_1) t^3 (\|\nabla_v^2 f_t\|^2_{L^2(m)} + \|\nabla_x \nabla_v f_t\|^2_{L^2(m)}) - \frac{b_1}{2} t^5 \|\nabla_x^2 f_t\|^2_{L^2(m)},
\end{aligned}$$

so that define

$$\mathscr{H}_2(t) = \mathscr{F}(t) + \mathscr{F}_2(t),$$

Recall that we already have

$$\begin{aligned}
\frac{d}{dt} \mathscr{F}(t, f_t) \le & - L(\|\nabla_v f_t\|^2_{L^2(m)} + t^2 \|\nabla_x f_t\|^2_{L^2(m)}) + C \|f_t\|^2_{L^2(m)}, \\
& - M_1 (\|f_t\|^2_{L^2(mH^{1/2})} + t \|\nabla_v f_t\|^2_{L^2(mH^{1/2})} + t^3 \|\nabla_x f_t\|^2_{L^2(mH^{1/2})} \\
& + t \|\nabla_v^2 f_t\|^2_{L^2(m)} + t^3 \|\nabla_v \nabla_x f_t\|^2_{L^2(m)}),
\end{aligned}$$

by taking $M_1 \gg a_1, b_1, c_1, \frac{a_1^2}{b_1}$ and η small we conclude

$$\begin{aligned}
\mathscr{H}_2(t) \le & - M_2 t^6 (\|\nabla_v^3 f\|^2_{L^2(m)} + \|\nabla_v^2 \nabla_x f\|^2_{L^2(m)} + \|\nabla_v \nabla_x^2 f\|^2_{L^2(m)} \\
& + \|\nabla_v^2 f\|^2_{L^2(mH^{1/2})} + \|\nabla_v \nabla_x f\|^2_{L^2(mH^{1/2})} + \|\nabla_x^2 f\|^2_{L^2(mH^{1/2})} + \|\nabla_v f\|^2_{L^2(mH^{1/2})} \\
& + \|\nabla_x f\|^2_{L^2(mH^{1/2})} + \|f\|^2_{L^2(mH^{1/2})}) + C \|f_0\|^2_{L^2(m)} \\
\le & C \|f_0\|^2_{L^2(m)}
\end{aligned}$$

for some $M_2 > 0$, so the proof for case $k = 2$ is thus finished. For general k, we briefly describe how to prove it. We prove by induction, suppose that for $k-1$ we alreaday have

$$\begin{aligned}
\mathscr{H}_{k-1} = & a_0 \|f_0\|^2_{L^2(m)} + \sum_{j=1}^{k-1} (a_{j,1} t^{3j-2} (\sum_{i=1}^{j} \|\nabla_v^i \nabla_x^{j-i} f_t\|^2_{L^2(m)}) + 2 a_{j,2} t^{3j-1} (\nabla_v \nabla_x^{j-1} f_t, \nabla_x^j f_t)_{L^2(m)} \\
& + a_{j,3} t^{3j} \|\nabla_x^j f_t\|^2_{L^2(m)})
\end{aligned}$$

for some $a_0, a_{j,1}, a_{j,2}, a_{j,3} > 0, a_{j,2}^2 \ll a_{j,1}a_{j,3}$ and for all $t \in [0,\eta]$ satisfies that

$$\frac{d}{dt}\mathscr{H}_{k-1} \le -M_{k-1}t^{3k-3}\sum_{i=0}^{k-1}\sum_{j=0}^{j}\|\nabla_v^j\nabla_x^{i-j}f_t\|_{L^2(mH^{1/2})} - M_{k-1}t^{3k-3}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|_{L^2(m)}^2 + C\|f_0\|_{L^2(m)}^2$$

for some constant $M_{k-1}, C > 0$, note that we have already proved this for $k = 1, 2$, we then try to prove the same thing for k. Recall condition (C3)

$$\sum_{k=1}^{n}|D_x^k\Phi(x)| + \sum_{k=1}^{n}|D_{x,v}^kB(x,v)| \le C_{n,\epsilon} + \epsilon H,$$

for $n \ge 1$ we can compute

$$\begin{aligned}\frac{d}{dt}\|\nabla_v^n\nabla_x^{k-n}f\|_{L^2(m)}^2 \le &-\|\nabla_v^{n+1}\nabla_x^{k-n}f\|_{L^2(m)}^2 - C_1\|\nabla_v^n\nabla_x^{k-n}f\|_{L^2(mH^{1/2})}^2 + C_2\|\nabla_v^n\nabla_x^{k-n}f\|_{L^2(m)}^2\\ &+\epsilon\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(mH^{1/2})}^2 + C\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(m)}^2\\ &+C(|\nabla_x^kf|, \sum_{i=1}^{k}|\nabla_v^i\nabla_x^{k-i}f|)_{L^2(m)} + \mathscr{R}\end{aligned}$$

Sum over $n = 1$ to k we have

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(m)}^2 \le &-\sum_{i=1}^{k}\|\nabla_v^{i+1}\nabla_x^{k-i}f\|_{L^2(m)}^2 - \frac{C_1}{2}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(mH^{1/2})}^2\\ &+C\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(m)}^2 + C(|\nabla_x^kf|, \sum_{i=1}^{k}|\nabla_v^i\nabla_x^{k-i}f|)_{L^2(m)} + \mathscr{R}\end{aligned}$$

where $\mathscr{R}$ denotes the lower order derivative term such that

$$\mathscr{R} \le C\sum_{i=0}^{k-1}\sum_{j=0}^{i}\|\nabla_v^j\nabla_x^{i-j}f\|_{L^2(mH^{1/2})}$$

is estimated by the case $k-1$, for the crossing term define

$$(\nabla_v\nabla_x^{k-1}f, \nabla_x^kf)_{L^2(m)} := \sum_{1\le i_1,\dots,i_k\le d}(\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}f, \partial_{x_{i_1}}\cdots\partial_{x_{i_{k-1}}}\partial_{v_{i_k}}f)_{L^2(m)}$$

we have

$$\begin{aligned}\frac{d}{dt}2(\nabla_v\nabla_x^{k-1}f, \nabla_x^kf)_{L^2(m)} =&2(\nabla_v^2\nabla_x^{k-1}f, \nabla_v\nabla_x^kf)_{L^2(m)} - \|\nabla_x^kf\|_{L^2(m)}^2 + C\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f\|_{L^2(mH^{1/2})}^2\\ &+C(|\nabla_x^kf|, \sum_{i=1}^{k}|\nabla_v^i\nabla_x^{k-i}f|)_{L^2(mH^{1/2})}\\ &+C(|\nabla_x^kf|, \sum_{i=0}^{k-1}\sum_{j=0}^{i}|\nabla_v^j\nabla_x^{i-j}f|)_{L^2(mH^{1/2})} + \mathscr{R}\end{aligned}$$

and

$$\frac{d}{dt}\|\nabla_x^k f\|^2_{L^2(m)} \le -\|\nabla_v \nabla_x^k f\|^2_{L^2(m)} - C_1\|\nabla_x^k f\|^2_{L^2(mH^{1/2})} + C_2\|\nabla_x^k f\|^2_{L^2(m)}$$

$$+\epsilon\|\nabla_x^k f\|^2_{L^2(mH^{1/2})} + C\|\nabla_x^k f\|^2_{L^2(m)} + C\sum_{i=1}^{k}\|\nabla_v^i \nabla_x^{k-i} f\|^2_{L^2(mH^{1/2})} + \mathscr{R}$$

$$\le -\|\nabla_v \nabla_x^k f\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_x^k f\|^2_{L^2(mH^{1/2})} + C\|\nabla_x^k f\|^2_{L^2(m)}$$

$$+ C\sum_{i=1}^{k}\|\nabla_v^i \nabla_x^{k-i} f\|^2_{L^2(mH^{1/2})} + \mathscr{R}$$

define

$$\mathscr{F}_k[t,f_t] = a_{k,1}t^{3k-2}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(m)} + 2a_{k,2}t^{3k-1}(\nabla_v\nabla_x^{k-1}f_t, \nabla_x^k f_t)_{L^2(m)} + a_{k,3}t^{3k}\|\nabla_x^k f_t\|^2_{L^2(m)}$$

where $M_{k-1} \gg a_{k,1} \gg a_{k,2} \gg a_{k,3}, M_{k-1} \gg \frac{a_{k,1}^2}{a_{k,2}}$ and $a_{k,1}a_{k,3} \gg a_{k,2}^2$, so gathering the terms together we have

$$\frac{d}{dt}\mathscr{F}_k \le T_{k,1} + T_{k,2} + T_{k,3}$$

for the $k+1$ order term $T_{k,1}$we have

$$T_{k,1} = -a_{k,1}t^{3k-2}\sum_{i=1}^{k}\|\nabla_v^{i+1}\nabla_x^{k-i}f_t\|^2_{L^2(m)} - a_{k,3}t^{3k}\|\nabla_v\nabla_x^k f_t\|^2_{L^2(m)} + 2a_{k,2}t^{3k-1}(\nabla_v^2\nabla_x^{k-1}f_t, \nabla_v\nabla_x^k f_t)_{L^2(m)}$$

$$\le -\frac{a_{k,1}}{2}t^{3k-2}\sum_{i=1}^{k}\|\nabla_v^{i+1}\nabla_x^{k-i}f_t\|^2_{L^2(m)} - \frac{a_{k,3}}{2}t^{3k}\|\nabla_v\nabla_x^k f_t\|^2_{L^2(m)}$$

$$\le -\frac{a_{k,3}}{2}t^{3k}\sum_{i=1}^{k+1}\|\nabla_v^i\nabla_x^{k+1-i}f_t\|^2_{L^2(m)}$$

for the term $T_{k,2}$

$$T_{k,2} = \left(-\frac{C_1}{2}a_{k,1}t^{3k-2} + Ca_{k,2}t^{3k-1} + Ca_{k,3}t^{3k}\right)\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(mH^{1/2})}$$

$$-\frac{C_1}{2}a_{k,3}t^{3k}\|\nabla_x^k f_t\|^2_{L^2(mH^{1/2})} + Ca_{k,2}t^{3k-1}(|\nabla_x^k f_t|, \sum_{i=1}^{k}|\nabla_v^i\nabla_x^{k-i}f_t|)_{L^2(mH^{1/2})}$$

$$+ Ca_{k,2}t^{3k-1}(|\nabla_x^k f_t|, \sum_{i=0}^{k-1}\sum_{j=0}^{i}|\nabla_v^j\nabla_x^{i-j}f_t|)_{L^2(mH^{1/2})} + Ct^{3k-2}\mathscr{R}$$

$$\le -\frac{C_1}{4}a_{k,3}t^{3k}\|\nabla_x^k f_t\|^2_{L^2(mH^{1/2})} - \frac{C_1}{4}a_{k,1}t^{3k-2}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(mH^{1/2})} + Ct^{3k-2}\mathscr{R}$$

$$\le -\frac{C_1}{4}a_{k,3}t^{3k}\sum_{i=0}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(mH^{1/2})} + Ct^{3k-2}\mathscr{R}$$

and

$$T_{k,3} \leq -(a_{k,2}t^{3k-1} - Ca_{k,3}t^{3k} - 3ka_{k,3}t^{3k-1})\|\nabla_x^k f_t\|^2_{L^2(m)} + Ca_{k,1}t^{3k-3}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(m)}$$

$$+ (Ca_{k,1} + Ca_{k,2}t^{3k-2})(|\nabla_x^k f_t|, \sum_{i=1}^{k}|\nabla_v^i\nabla_x^{k-i}f_t|)_{L^2(m)}$$

$$\leq -\frac{a_{k,2}}{2}t^{3k-1}\|\nabla_x^k f_t\|^2_{L^2(m)} + C(a_{k,1} + \frac{a_{k,1}^2}{a_{k,2}})t^{3k-3}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(m)}$$

Recall that

$$\frac{d}{dt}\mathscr{H}_{k-1} \leq -M_{k-1}t^{3k-3}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\|\nabla_v^j\nabla_x^{i-j}f_t\|_{L^2(mH^{1/2})} - M_{k-1}t^{3k-3}\sum_{i=1}^{k}\|\nabla_v^i\nabla_x^{k-i}f_t\|^2_{L^2(m)} + C\|f_0\|^2_{L^2(m)}$$

so by choosing suitable $a_{k,1}, a_{k,2}, a_{k,3}, \frac{a_{k,1}^2}{a_{k,2}} \ll M_{k-1}, \eta > 0$, define

$$\mathscr{H}_k := \mathscr{H}_{k-1} + \mathscr{F}_k$$

we have

$$\frac{d}{dt}\mathscr{H}_k \leq -M_k t^{3k}\sum_{i=0}^{k}\sum_{j=0}^{i}\|\nabla_v^j\nabla_x^{i-j}f_t\|_{L^2(mH^{1/2})} - M_k t^{3k}\sum_{i=1}^{k+1}\|\nabla_v^i\nabla_x^{k+1-i}f_t\|^2_{L^2(m)} + C\|f_0\|^2_{L^2(m)}$$

we end the proof by induction.

□

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E-mail address: chuqicao@gmail.com