CUTOFF BOLTZMANN EQUATION WITH POLYNOMIAL PERTURBATION NEAR MAXWELLIAN

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ABSTRACT. In this paper, we consider the cutoff Boltzmann equation near Maxwellian, we proved the global existence and uniqueness for the cutoff Boltzmann equation in polynomial weighted space for all $\gamma \in (-3, 1]$. We also proved initially polynomial decay for the large velocity in L^2 space will induce polynomial decay rate, while initially exponential decay will induce exponential rate for the convergence. Our proof is based on newly established inequalities for the cutoff Boltzmann equation and semigroup techniques. Moreover, by generalizing the $L^\infty_x L^1_v \cap L^\infty_{x,v}$ approach, we prove the global existence and uniqueness of a mild solution to the Boltzmann equation with bounded polynomial weighted $L^\infty_{x,v}$ norm under some small condition on the initial $L^1_x L^\infty_v$ norm and entropy so that this initial data allows large amplitude oscillations.

Keywords: Boltzmann equation; Global existence; Polynomial weighted space; Convergence to equilibrium.

AMS subject classifications: 35B40, 35Q20, 47D06.

CONTENTS

1. INTRODUCTION

The Boltzmann equation reads

$$\partial_t F + v \cdot \nabla_x F = Q(F,F), \quad F(0,x,v) = F_0(x,v), \tag{1}$$

where $F(t,x,v) \ge 0$ is a distributional functions of colliding particles which, at time $t > 0$ and position $x \in \mathbb{T}^3$, move with velocity $v \in \mathbb{R}^3$. We remark that the Boltzmann equation is one of the fundamental equations of mathematical physics and is a cornerstone of statistical physics. The Boltzmann collision operator Q is a bilinear operator which acts only on the velocity variable v, that is

$$Q(G,F)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(G'_* F' - G_* F) d\sigma dv_*.$$

Let us give some explanations on the collision operator.

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(1) We use the standard shorthand $F = F(v), G_* = G(v_*), F' = F(v'), G'_* = G(v'_*)$, where v', v'_* are given by
$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$
This representation follows the parametrization of set of solutions of the physical law of elastic collision:
$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$
(2) The nonnegative function $B(v - v_*, \sigma)$ in the collision operator is called the Boltzmann collision kernel. It is always assumed to depend only on $|v-v_*|$ and the deviation angle θ through $\cos\theta := \frac{v-v_*}{|v-v_*|}\cdot\sigma$.
(3) In the present work, our **basic assumptions on the kernel** B can be concluded as follows:
 (**A1**). The Boltzmann kernel B takes the form: $B(v-v_*,\sigma) = |v-v_*|^\gamma b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)$, where b is a nonnegative function.
 (**A2**). The angular function $b(\cos\theta)$ satisfies the Grad's cutoff assumption
$$K \le b(\cos\theta) \le K^{-1}, \quad K > 0.$$
 (**A3**). The parameter γ satisfies the condition $-3 < \gamma \le 1$.
 (**A4**). Without lose of generality, we may assume that $B(v-v_*,\sigma)$ is supported in the set $0 \le \theta \le \pi/2$, i.e. $\frac{v-v_*}{|v-v_*|}\cdot\sigma \ge 0$, otherwise B can be replaced by its symmetrized form:
$$\overline{B}(v-v_*,\sigma) = |v-v_*|^\gamma\big(b(\frac{v-v_*}{|v-v_*|}\cdot\sigma) + b(\frac{v-v_*}{|v-v_*|}\cdot(-\sigma))\big)1_{\frac{v-v_*}{|v-v_*|}\cdot\sigma\ge 0},$$
 where 1_A is the characteristic function of the set A.

Remark 1.1. *Generally, the case $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ correspond to so-called hard, Maxwellian, and soft potentials respectively.*

1.1. **Basic properties and the perturbation equation.** We recall some basic facts on the Boltzmann equation.

• **Conservation Law.** Formally if F is the solution to the Boltzmann equation (1) with initial data F_0, then it enjoys the conservation of mass, momentum and the energy, that is,
$$\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F(t,x,v)\varphi(v)\,dv\,dx = 0, \quad \varphi(v) = 1, v, |v|^2. \tag{2}$$
For simplicity, we introduce the normalization identities on the initial data F_0 which satisfies
$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F_0(x,v)\,dv\,dx = 1, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0(x,v)\,v\,dv\,dx = 0, \quad \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0(x,v)\,|v|^2\,dv\,dx = 3.$$
This means that the equilibrium associated to (1) will be the standard Gaussian function, i.e.
$$\mu(v) := (2\pi)^{-3/2}e^{-|v|^2/2},$$
which enjoys the same mass, momentum and energy as F_0. By the Boltzmann H-Theorem, the solution to the Boltzmann equation (1) satisfies
$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} F(t,x,v)\ln F(t,x,v)\,dv\,dx \le \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} F_0(x,v)\ln F_0(x,v)\,dv\,dx, \quad \forall t \ge 0. \tag{3}$$

• **Perturbation Equation.** In the perturbation framework, let f be the perturbation such that
$$F = \mu + f.$$
The Boltzmann equation (1) becomes
$$\partial_t f + v\cdot\nabla_x f = Q(\mu, f) + Q(f,\mu) + Q(f,f) := Lf + Q(f,f), \tag{4}$$
with the linearized operator $L = Q(\mu,\cdot) + Q(\cdot,\mu)$.

1.2. **Brief review of previous results.** In what follows we recall some known results on the Landau and Boltzmann equations with a focus on the topics under consideration in this paper, particularly on global existence and large-time behavior of solutions to the spatially inhomogeneous equations in the perturbation framework. For global solutions to the renormalized equation with large initial data, we mention the classical works [14, 15, 40, 50, 51, 13, 6]. For the stability of vacuum, see [41, 30, 12] for the Landau, cutoff and non-cutoff Boltzmann equation with moderate soft potential respectively.

We focus on the results in the perturbation framework. In the near Maxwellian framework, a key point is to characterize the dissipation property in the L^2 norm in v for the linearized operator and further control the trilinear term in an appropriate way. More precisely, for the cutoff Boltzmann equation, the corresponding linearized operator L_μ is self-adjoint and has the null space

$$L_\mu f = \frac{1}{\sqrt{\mu}}(Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)), \quad \mathrm{Null}(L_\mu) = \mathrm{Span}\{\sqrt{\mu}, \sqrt{\mu} v, \sqrt{\mu}|v|^2\},$$

and having the coercivity property

$$(f, L_\mu f) \le -C\|f\|^2_{L^2_{\gamma/2}}, \quad \forall f \in \mathrm{Null}(L_\mu)^\perp,$$

for some constant $C > 0$, such coercivity property is essential in the near Maxwellian framework.

In the near Maxwellian framework, global existence and large-time behavior of solutions to the spatially inhomogeneous equations is proved in [27, 28, 46, 47] for the cutoff Boltzmann equation and in [26] for the Landau equation. For the non-cutoff Boltzmann equation it is proved in [2, 3, 4, 5, 23, 24], see also [19] for a recent work. We also refer to [31, 32, 33, 18, 17, 20] for the Vlasov-Poisson/Maxwell-Boltzmann/Landau equation near Maxwellian. We remark here all these works above are based on the following decomposition

$$\partial_t f + v \cdot \nabla_x f = L_\mu f + \Gamma(f, f), \quad L_\mu f = \frac{1}{\sqrt{\mu}}(Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)), \quad \Gamma(f, f) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f),$$

which means the result are in $\mu^{-1/2}$ weighted space.

In the near Maxwellian framework, there are also several results for $L^\infty_{x,v}$ well-posedness results near Maxwellian. For the cutoff Boltzmann equation near Maxwellian, a $L^2 - L^\infty$ approach has been introduced in [48, 29] and apply to various contexts, see [38, 35] and the reference therein for example. Global $L^\infty_{x,v}$ well-posedness result near Maxwellian is proved for the Landau equation in [37] and for the non-cutoff Boltzmann equation in [8, 45]. The solution with large amplitude initial data is first proved in [16] under the assumption of small entropy, we also refer to [21] for the Boltzmann equation with large amplitude initial data in bounded domains and [52] for the relativistic Boltzmann equation with large amplitude initial data.

For inhomogeneous equations with polynomial weighted perturbation near Maxwellian, in Gualdani-Mischler-Mouhot [25] the authors first prove the global existence and large-time behavior of solutions with polynomial velocity weight for the cutoff Boltzmann equation with hard potential in $L^1_v L^\infty_x(1 + |v|^k), k > 2$, this method is generalized to the Landau equation in [10, 11]. The non-cutoff Boltzmann equation with hard potential is proved in [36, 7], the soft potential case is proved in [9]. The cutoff Boltzmann equation with soft potential is proved in this paper.

1.3. **Main results and notations.** Let us first introduce the function spaces and notations.
• For any $p \in [1, +\infty)$, $q \in \mathbb{R}$ the L^p_q norm is defined by

$$\|f\|^p_{L^p_q} := \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^{pq} dv,$$

where the Japanese bracket $\langle v \rangle$ is defined as $\langle v \rangle := (1 + |v|^2)^{1/2}$.

• For any $p \in [1,+\infty)$, $k \in \mathbb{R}, a > 0, b \in (0,2)$ the $L^p_{k,a,b}$ norm is defined by

$$\|f\|^p_{L^p_{k,a,b}} = \int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^{pq} e^{pa\langle v\rangle^b} dv.$$

• For real numbers m, l, we define the weighted Sobolev space H^m_l by

$$H^m_l := \{f(v) | |f|_{H^m_l} = |\langle\cdot\rangle^l \langle D\rangle^m f|_{L^2} < +\infty\},$$

where $a(D)$ is a pseudo-differential operator with the symbol $a(\xi)$ and it is defined as

$$(a(D)f)(v) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(v-u)\xi} a(\xi) f(u) du d\xi,$$

and we denote $H^m := H^m_0$. The weighted Sobolev space $H^m_{k,a,b}$ can be defined in a similar way.
• For function $f(x,v), x \in \mathbb{T}^3, v \in \mathbb{R}^3$, the norm $\|\cdot\|_{H^\alpha_x H^m_l}$ is defined as

$$\|f\|_{H^\alpha_x H^m_l} := \left(\int_{\mathbb{T}^3} \|\langle D_x\rangle^\alpha f(x,\cdot)\|^2_{H^m_l} dx\right)^{1/2}.$$

If $\alpha = 0$, $H^0_x H^m_l = L^2_x H^m_l$. The weighted Sobolev space $H^\alpha_x H^m_{k,a,b}$ can be defined in a similar way.
• We write $a \lesssim b$ indicate that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$. We use the notation $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$. We denote $C_{a_1,a_2,\cdots,a_n}$ by a constant depending on parameters $a_1, a_2, \cdots, a_n$. Moreover, we use parameter ϵ to represent different positive numbers much less than 1 and determined in different cases.
• We use (f,g) to denote the inner product of f,g in the v variable $(f,g)_{L^2_v}$ for short, we use $(f,g)_{L^2_k}$ to denote $(f, g\langle v\rangle^{2k})_{L^2_v}$.
• For any function f we define

$$\|f(\theta)\|_{L^1_\theta} := \int_{\mathbb{S}^2} f(\theta) d\sigma = 2\pi \int_0^\pi f(\theta)\sin\theta d\theta.$$

• Gamma function and Beta function are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0, \quad B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}, \quad p,q > 0.$$

We recall that Beta and Gamma functions fulfill the following properties:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad B(p,q) \sim q^{-p} \quad \text{if} \quad 0 < p \le 2. \tag{5}$$

• For the cross section $B(\cos\theta, |v-v_*|)$ with an angular cutoff, we will use the notation $Q^\pm$ defined as

$$Q^+(f,g) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(\cos\theta, |v_*-v|) f'_* g' dv_* d\sigma, \quad Q^-(f,g) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(\cos\theta, |v-v_*|) f_* g dv_* d\sigma. \tag{6}$$

Note that $Q(f,g) = Q^+(f,g) - Q^-(f,g)$.
• For the linearized operator L we have

$$\ker(L) = \text{span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}.$$

We define the projection onto $\ker(L)$ by

$$Pf := \left(\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f dv dx\right)\mu + \sum_{i=1}^3 \left(\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} v_i f dv dx\right) v_i\mu + \left(\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{|v|^2-3}{\sqrt{6}} f dv dx\right)\frac{|v|^2-3}{\sqrt{6}}\mu. \tag{7}$$

• For any $k \in \mathbb{R}, \gamma \in (-3,1]$, we define

$$\|f\|^2_{L^2_{k+\gamma/2,*}} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mu(v_*)|v-v_*|^\gamma |f(v)|^2 \langle v\rangle^{2k} dv dv_*.$$

It is easily seen that $\|f\|_{L^2_{k+\gamma/2,*}} \sim \|f\|_{L^2_{k+\gamma/2}}$.

• During the whole paper, we will denote N by

$$N=\begin{cases}2, & \gamma\in(-\frac{3}{2},1],\\ 3, & \gamma\in(-\frac{5}{2},-\frac{3}{2}],\\ 4, & \gamma\in(-3,-\frac{5}{2}].\end{cases} \tag{8}$$

• We let the multi-indices α and β be $\alpha=[\alpha_1,\alpha_2,\alpha_3]$, $\beta=[\beta_1,\beta_2,\beta_3]$ and define

$$\partial^\alpha_\beta:=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

• If each component of θ is not greater than that of the $\overline{\theta}$'s, we denote by $\theta\le\overline{\theta}$; $\theta<\overline{\theta}$ means $\theta\le\overline{\theta}$, and $|\theta|<|\overline{\theta}|$.
• For any $k\in\mathbb{R}, a>0, b\in(0,2)$, if $\gamma\in(-\frac{3}{2},1]$ we will denote

$$\|f\|^2_{X_k}:=\sum_{|\alpha|\le 2}\|\langle v\rangle^k\partial^\alpha f\|^2_{L^2_{x,v}},\quad \|f\|^2_{X_{k,a,b}}:=\sum_{|\alpha|\le 2}\|\partial^\alpha f\langle v\rangle^k e^{a\langle v\rangle^b}\|^2_{L^2_{x,v}}. \tag{9}$$

If $-3<\gamma\le-\frac{3}{2}$ we will denote the weight function $w(\alpha,\beta)$ as

$$w(|\alpha|,|\beta|)=\langle v\rangle^{k-a|\alpha|-b|\beta|+c},\quad b=7\max\{-\gamma,0\},\quad a=b+\min\{\gamma,0\}=6\max\{-\gamma,0\},\quad c=4b. \tag{10}$$

Note that $w(\alpha,\beta)=w(|\alpha|,|\beta|)$, the two notations will have the same meaning in the whole paper. We then define

$$\|f\|^2_{X_k}:=\sum_{|\alpha|+|\beta|\le N}C^2_{|\alpha|,|\beta|}\|w(\alpha,\beta)\partial^\alpha_\beta f\|^2_{L^2_{x,v}},\quad \|f\|^2_{X_{k,a,b}}:=\sum_{|\alpha|+|\beta|\le N}C^2_{|\alpha|,|\beta|}\|\partial^\alpha_\beta f\,w(\alpha,\beta)e^{a\langle v\rangle^b}\|^2_{L^2_{x,v}}, \tag{11}$$

where the constant $C_{|\alpha|,|\beta|}$ satisfies

$$C_{|\alpha|,|\beta|}\ll C_{|\alpha|,|\beta_1|},\quad \forall|\alpha|\ge0,\quad 0\le|\beta_1|<|\beta|,\quad C_{|\alpha|+1,||\beta|-1}\gg C_{|\alpha|,|\beta|},\quad \forall|\alpha|\ge0,\quad |\beta|\ge1, \tag{12}$$

and we will denote $Y_k:=X_{k+\gamma/2}, Y_{k,a,b}:=X_{k+\gamma/2,a,b}$. We also define

$$\|f\|^2_{Y_{k,*}}:=\sum_{|\alpha|\le2}\|\partial^\alpha f\|^2_{L^2_xL^2_{k+\gamma/2,*}},\quad \gamma\in(-\frac{3}{2},1],\quad \|f\|^2_{Y_{k,*}}:=\sum_{|\alpha|+|\beta|\le N}C^2_{|\alpha|,|\beta|}\|w(\alpha,\beta)\partial^\alpha_\beta f\|^2_{L^2_xL^2_{\gamma/2,*}},\quad \gamma\in(-3,-\frac{3}{2}]. \tag{13}$$

• Define $\bar{X}_0:=H^2_xL^2_v$ if $\gamma\in(-\frac{3}{2},1]$, $\bar{X}_0:=H^N_{x,v}$ if $\gamma\in(-3,-\frac{3}{2}]$.
• Define the relative entropy by

$$H(F(t)):=\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}F(t,x,v)\ln F(t,x,v)-\mu\ln\mu\,dv\,dx, \tag{14}$$

it is easily seen that $H(F(t))\ge0,\forall t\ge0, H(\mu)=0$.

1.4. **Main results.** We may now state our main results.

Theorem 1.2. *Consider the Cauchy problem*

$$\partial_t f+v\cdot\nabla_x f=Lf+Q(f,f),\quad \mu+f\ge0,\quad f(0)=f_0,\quad Pf_0=0.$$

- *Polynomial case: For any $k\ge6$, there exists a small constant $\epsilon_0>0$ such that for any initial data $f_0\in X_k$ satisfies*

$$\mu+f_0\ge0,\quad Pf_0=0,\quad \|f_0\|_{X_6}<\epsilon_0,\quad \|f_0\|_{X_k}<+\infty,$$

there exists a unique global solution $f\in L^\infty([0,+\infty),X_k)$ satisfies $F=\mu+f\ge0$. Moreover if $\gamma\in[0,1]$, we have

$$\|f(t)\|_{X_k}\lesssim e^{-\lambda t}\|f_0\|_{X_k},\quad \forall t\in(0,+\infty),$$

for some constant $\lambda > 0$. *If* $\gamma < 0$, *then for any* $6 \le k_1 < k$ *we have*

$$\|f(t)\|_{X_{k_1}} \lesssim t^{-\frac{|k-k_*|}{|\gamma|}} \|f_0\|_{X_k}, \quad \forall k_* \in (k_1, k), \quad \forall t \in (0, +\infty).$$

- *Exponential case: For any* $k \in \mathbb{R}, a > 0, b \in (0,2)$, *there exists a small constant* $\epsilon_0 > 0$ *such that for any* $f_0 \in X_{k,a,b}$ *satisfies*

$$\mu + f_0 \ge 0, \quad P f_0 = 0, \quad \|f_0\|_{X_{k,a,b}} < \epsilon_0,$$

there exists a unique global solution $f \in L^\infty([0,+\infty), X_{k,a,b})$ *satisfies* $F = \mu + f \ge 0$. *Moreover if* $\gamma \in [0,1]$, *we have*

$$\|f(t)\|_{X_{k,a,b}} \lesssim e^{-\lambda t} \|f_0\|_{X_{k,a,b}}, \quad \forall t \in (0, +\infty),$$

for some constant $\lambda > 0$. *If* $\gamma < 0$, *then for any* $0 < a_0 < a$ *we have*

$$\|f(t)\|_{X_{k,a_0,b}} \lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}} \|f_0\|_{X_{k,a,b}}, \quad \forall t \in (0, +\infty),$$

for some constant $\lambda > 0$.

Several comments on the results are in order:

• The choice of N is optimal in the sense that $N-2$ is the minimal integer such that

$$\|Q(f,f)\|_{H_v^{N-2}} \le C \|f\|^2_{H_4^{N-2}}.$$

• We emphasize that for global solutions in the polynomial weight space, we only need the small initial value of a given norm, more precisely we only require the smallness of X_6 instead of X_k.

By assuming the entropy is small we can prove the well-posedness for the Boltzmann equation with large amplitude initial data.

Theorem 1.3. *Consider the Cauchy problem*

$$\partial_t f + v \cdot \nabla_x f = Lf + Q(f,f), \quad \mu + f_0 \ge 0, \quad f(0) = f_0, \quad P f_0 = 0.$$

For any $\gamma \in (-3,1]$, *there exists a constant* $k_0 \ge 8$ *such that for all* $k \ge k_0$, *for any* $\beta \ge \max\{3, 3+\gamma\}$, *for any fixed constant* $M > 1$, *there exists a constant* $\epsilon_0 > 0$ *depends on* M, k, β *such that if*

$$\|\langle v\rangle^{k+\beta} f_0\|_{L^\infty_{x,v}} \le M, \quad H(F_0) + \|\langle v\rangle^k f_0\|_{L^1_x L^\infty_v} \le \epsilon_0, \tag{15}$$

then the Boltzmann solution has a unique global mild solution $f \in L^\infty([0,+\infty), L^\infty_{x,v}(\langle v\rangle^{k+\beta}))$ *satisfies that*

$$\|\langle v\rangle^{k+\beta} f(t)\|_{L^\infty_{x,v}} \le C M^2, \quad \forall t \ge 0,$$

where $C > 0$ *depends on* γ, k, β. *Moreover, for the case* $\gamma \ge 0$ *we have*

$$\|\langle v\rangle^{k+\beta} f(t)\|_{L^\infty_{x,v}} \le C e^{-\lambda t}, \quad \forall t \ge 0,$$

for some constants $C, \lambda > 0$. *For the case* $-3 < \gamma < 0$, *if we further assume* $\beta \ge 6$, *for any* $r \in (0,1)$ *we have*

$$\|\langle v\rangle^k f(t)\|_{L^\infty_{x,v}} \le C(1+t)^{-r}, \quad \forall t \ge 0,$$

for some constants $C > 0$.

• <u>*Comment on the solutions.*</u> It should be pointed out that initial data satisfying the smallness condition (15) are allowed to have large amplitude oscillations in the spatial variable. For instance, one may take

$$F_0(x,v) = \rho_0(x)\mu(v), \quad x \in \mathbb{T}^3 \times \mathbb{R}^3,$$

with $\rho_0(x) \ge 0, \rho_0 \in L^\infty_x, \rho_0 - 1 \in L^1_x, \rho_0 \ln \rho_0 - \rho_0 + 1 \in L^1_x$. It is easy to verify that (15) holds if $\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}$ is small. Even though $\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}$ is required to be small, initial data are allowed to have large amplitude oscillations.

1.5. **Strategies and ideas of the proof.** In this subsection, we will explain main strategies and ideas of the proof for our results.

We briefly talk on the semigroup method, this method is first initiated in [43] and extended into an abstract setting in a famous work by Gualdani-Mischler-Mouhot [25], see also its application in kinetic Fokker-Planck equation in [42]. The main idea for the L^2 case can be expressed briefly as follows: Taking the case $\gamma = 0$ in the homogeneous Boltzmann equation for example, first by existing results we have

$$(Lf,f)_{L^2(\mu^{-1/2})} \le -\lambda \|f\|_{L^2(\mu^{-1/2})}, \quad \|S_L(t) f_0\|_{L^2(\mu^{-1/2})} \le e^{-\lambda t}\|f_0\|^2_{L^2(\mu^{-1/2})},$$

for some constant $\lambda > 0$. If we can prove

$$(Lf,f)_{L^2_k} \le -C\|f\|^2_{L^2_k} + C_k\|f\|^2_{L^2}, \tag{16}$$

then define $A = M\chi_R, B = L - A$, where $\chi \in D(\mathbb{R})$ a truncation function which satisfies $\mathbb{1}_{[-1,1]} \le \chi \le \mathbb{1}_{[-2,2]}$ and we denote $\chi_a(\cdot) := \chi(\cdot/a)$ for some constant $a > 0$. Taking M, R large we have

$$(Bf,f)_{L^2_k} \le -C\|f\|^2_{L^2_k}, \quad \|Af\|_{L^2(\mu^{-1/2})} \le C\|f\|_{L^2_k},$$

which implies

$$\|S_B(t)f\|_{L^2_k} \le e^{-\lambda t}\|f_0\|_{L^2_k},$$

by Duhamel's formula

$$\|S_L(t)\|_{L^2_k \to L^2_k} \le \|S_B(t)\|_{L^2_k\to L^2_k} + \int_0^t \|S_L(s)\|_{L^2(\mu^{-1/2})\to L^2(\mu^{-1/2})}\|A\|_{L^2_k \to L^2(\mu^{-1/2})}\|S_B(t-s)\|_{L^2_k\to L^2_k} \le Ce^{-\lambda t}.$$

The rate of convergence for the linear operator L is established. Define a scalar product by

$$((f,g)) = (f,g)_{L^2_k} + \eta\int_0^{+\infty}(S_L(\tau)f, S_L(\tau)g)d\tau,$$

since

$$\int_0^{+\infty}(S_L(\tau)f, S_L(\tau)f)d\tau \le C\|f\|^2_{L^2_k}\int_0^{+\infty} e^{-2\lambda\tau}d\tau \le C\|f\|^2_{L^2_k},$$

we deduce $((\cdot,\cdot))$ is an equivalent norm to L^2_k. By

$$\int_0^{+\infty}(S_L(\tau)Lf, S_L(\tau)f)d\tau = \int_0^{+\infty}\frac{d}{d\tau}\|S_L(\tau)f\|^2_{L^2}d\tau = \lim_{\tau\to\infty}\|S_L(\tau)f\|^2_{L^2} - \|f\|^2_{L^2} = -\|f\|^2_{L^2},$$

which implies

$$((Lf,f)) = (Lf,f)_{L^2_k} + \eta\int_0^{\infty}(S_L(\tau)Lf, S_L(\tau)f)d\tau = -C_1\|f\|_{L^2_k} + (C_2-\eta)\|f\|_{L^2} \le -C_1\|f\|_{L^2_k},$$

by choosing a suitable η. The estimate for the linearized operator L in this equivalent norm allows us to combine with the nonlinear estimates to conclude the full convergence. In other words, one of the main works of this paper is to prove (16) plus some appropriate upper bounds for the nonlinear operator which consists with the lower bound in the linearized estimate.

We briefly describe one proof for (16). We observed and proved, for any $\gamma \in (-3,1], k > \max\{\gamma+3, 3\}$

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k}e^{-\frac{1}{2}|v'_*|^2}dv_*d\sigma \le \frac{c}{k^{\frac{\gamma+3}{4}}}\langle v\rangle^\gamma + C_k\langle v\rangle^{\gamma-2}, \quad \forall v\in\mathbb{R}^3, \tag{17}$$

where the constant c is independent of k. The proof of (17) can be seen in Lemma 6.2. By (17), under the cutoff assumption (A2), for the Q^+ term we have

$$
\begin{aligned}
|(Q^+(\mu,f),f\langle v\rangle^{2k})| \le & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma |f(v')| e^{-\frac{1}{2}\langle v_*'\rangle^2}|f(v)|\langle v\rangle^{2k} dv_* dv d\sigma \\
\le & C\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma |f(v')|^2\langle v'\rangle^{2k} e^{-\frac{1}{2}\langle v_*'\rangle^2} dv_* dv d\sigma\right)^{1/2} \\
& \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^{2k}}{\langle v'\rangle^{2k}} e^{-\frac{1}{2}\langle v_*'\rangle^2}|f(v)|^2\langle v\rangle^{2k} dv_* dv d\sigma\right)^{1/2} \\
\le & \frac{c_2}{k^{\frac{\gamma+3}{4}}}\|f\|^2_{L^2_{k+\gamma/2}} + C_k\|f\|^2_{L^2_{k+\gamma/2-1}},
\end{aligned}
\tag{18}
$$

for some constant $c_2>0$ independent of k, the $(Q^+(f,\mu),f\langle v\rangle^{2k})$ term can be estimated by the same way. For the Q^- term we easily compute

$$
-Q^-(f,\mu)-Q^-(\mu,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)(\mu(v_*)f(v)+\mu(v)f(v_*))dv_*d\sigma \le -c_1\langle v\rangle^\gamma f(v) + C\mu(v)\|f\|_{L^1_4},
$$

so we have

$$
-(Q^-(f,\mu)+Q^-(\mu,f),f\langle v\rangle^{2k}) \le -c_1\|f\|^2_{L^2_{k+\gamma/2}} + C_k\|f\|^2_{L^2_6}.
$$

Gathering the two terms we have

$$
(Lf,f\langle v\rangle^{2k}) \le -(c_1 - \frac{2c_2}{k^{\frac{\gamma+3}{4}}})\|f\|^2_{L^2_{k+\gamma/2}} + C_k\|f\|^2_{L^2_{k+\gamma/2-1}}.
$$

If k is larger than some constant k_0 such that $2c_2 < c_1 k_0^{\frac{\gamma+3}{4}}$, (16) is thus proved. To get an optimal k for the polynomial weight case, we will use the pre-post collisional change of variables to give more precise computation in this paper.

We remark here that (17) also serves an important point in the proof for Boltzmann equation with large amplitude initial data. In [16], the following inequality

$$
\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma \frac{e^{-\frac{1}{4}|v|^2}}{e^{-\frac{1}{4}|v'|^2}} e^{-\frac{1}{2}|v_*'|^2} dv_* d\sigma \le C_k\langle v\rangle^{\gamma-2},
$$

plays a key role in the proof for Boltzmann equation with large amplitude initial data for the $\mu^{-1/2}$ case. The inequality (17) can be seen as its polynomial version.

For the upper bound, we observed and proved such fact that

$$
\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k\langle v_*'\rangle^k} dv_* d\sigma \le C_k\langle v\rangle^\gamma, \quad \forall v\in\mathbb{R}^3,
\tag{19}
$$

which plays an essential role in the proof of the upper bound. Compared to the $\mu^{-1/2}$ case, if we replace $\langle v\rangle^k$ by $\mu^{-\frac{1}{2}} = e^{\frac{1}{4}|v|^2}$ in (19), we easily seen that

$$
\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma \frac{e^{\frac{1}{4}|v|^2}}{e^{\frac{1}{4}|v'|^2}e^{\frac{1}{4}|v_*'|^2}} dv_* d\sigma = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma e^{-\frac{1}{4}|v_*|^2} dv_* d\sigma \le C\langle v\rangle^\gamma.
$$

Now we explain the strategy of the proof for the Boltzmann equation with large amplitude initial data. For the polynomial weight case, let $f = \langle v\rangle^{-k}(F-\mu)$ in (1), then f satisfies

$$
\partial_t f + v\cdot\nabla_x f = L_k f + \Gamma_k(f,f),
$$

with

$$
L_k f = \langle v\rangle^k Q(f\langle v\rangle^{-k},\mu) + \langle v\rangle^k Q(\mu, f\langle v\rangle^{-k}), \quad \Gamma_k(f,f) = \langle v\rangle^k Q(f\langle v\rangle^{-k}, f\langle v\rangle^{-k}).
$$

For any $k \geq \max\{3, 3+\gamma\}$, like [16], instead of estimate the nonlinear term in the following way

$$|\langle v\rangle^\beta \Gamma_k(f,f)| \leq C\langle v\rangle^\gamma \|\langle v\rangle^\beta f\|_{L^\infty}^2,$$

we prove a new estimate

$$|\langle v\rangle^\beta \Gamma_k(f,f)| \leq C\langle v\rangle^\gamma \|\langle v\rangle^\beta f\|_{L^\infty}^{2-a}\left(\int_{\mathbb{R}^3}|f(t,x,v)|dv\right)^a,$$

for some constant $0 < a < 1$. Second, we observe that under the condition

$$H(F_0) + \|\langle v\rangle^k f_0\|_{L^1_x L^\infty_v} \leq \epsilon_0,$$

we can prove that $\int_{\mathbb{R}^3}|f(t,x,v)|dv$ will be small after some positive time even if it could be initially large. This observation is the key point to control the nonlinear term $\Gamma_k(f,f)$, we can finally obtain the uniform $L^\infty_{x,v}$ estimate under the smallness of $\|\langle v\rangle^k f_0\|_{L^1_x L^\infty_v}$ and $H(F_0)$ so that initial data is allowed to have large amplitude oscillations.

1.6. **Remark on the Cutoff assumption (A2).** Compared to the cutoff assumption (A2), in many text people define the cutoff assumption in the following way:

$$K \leq \int_{\mathbb{S}^2} b(\cos\theta)d\theta \leq K^{-1} \tag{20}$$

which is weaker than assumption (A2). But assumption (A2) plays an important in our proof, for example, if we want to use (17) to prove (18), the assumption (20) is not enough, we need to assume that (A2) holds.

In the proof of Theorem 1.2, we use the prepost-collisional change of variable instead of using inequalities like (17), thus under assumption (20), Theorem 1.2 is still true, actually Theorem 1.2 can be proved for the Boltzmann equation without cutoff, see [9].

But for Theorem 1.3 we need to assume that (A2) holds since technically we need to use inequality like (17) to prove it.

1.7. **Organization of the paper.** Technical tools and lemmas are listed in Section 2. Section 3 is devoted to the upper bounds and coercivity estimate on collision operator Q. In Section 4 we will prove estimates for the inhomogeneous equation. We obtain global well-posedness and rate of convergence in L^2 in Section 5. Section 6 and Section 7 are devoted to the proof for the Boltzmann equation with large amplitude initial data. In Section 6 we prove global well-posedness for the Boltzmann equation with large amplitude initial data and in Section 7 we establish rate of convergence to the equilibrium.

2. Preliminaries

In later analysis, we often use two types of change of variables below.

Lemma 2.1. *([1]) For any smooth function f we have*
(1) (Regular change of variables)

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f(v')d\sigma dv = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\frac{1}{\cos^{3+\gamma}(\theta/2)}|v-v_*|^\gamma f(v)d\sigma dv.$$

(2) (Singular change of variables)

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f(v')d\sigma dv_* = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\frac{1}{\sin^{3+\gamma}(\theta/2)}|v-v_*|^\gamma f(v_*)d\sigma dv_*.$$

Lemma 2.2. *For any smooth function f,g,h,b, for any constant $\gamma\in\mathbb{R}$, we have*

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f_* g h' dv dv_* d\sigma\right)^2$$
$$\le \left(\int_{\mathbb{S}^2} b(\cos\theta)\sin^{-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\right)^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f_*|^2|g|dvdv_*\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v'|^\gamma|g||h'|^2dvdv'.$$

Similarly

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f_* g h' dv dv_* d\sigma\right)^2$$
$$\le \left(\int_{\mathbb{S}^2} b(\cos\theta)\cos^{-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\right)^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f_*||g|^2dvdv_*\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v_*-v'|^\gamma|f_*||h'|^2dv_*dv'.$$

Proof. For the first inequality, by Cauchy-Schwarz inequality and singular change of variables

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f_* g h' dv dv_* d\sigma\right)^2$$
$$\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\sin^{-\frac{3+\gamma}{2}}\frac{\theta}{2}|v-v_*|^\gamma|f_*|^2|g|dvdv_*d\sigma\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\sin^{\frac{3+\gamma}{2}}\frac{\theta}{2}|v-v_*|^\gamma||g||h'|^2dvdv_*d\sigma$$
$$\le \left(\int_{\mathbb{S}^2} b(\cos\theta)\sin^{-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\right)^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f_*|^2|g|dvdv_*\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v'|^\gamma|g||h'|^2dvdv',$$

which finishes the proof. The second inequality can be proved similarly by regular change of variables. □

Lemma 2.3. *([49], Section 1.4)(Pre-post collisional change of variables) For smooth function F we have*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} F(v,v_*,v',v'_*)B(|v-v_*|,\cos\theta)dvdv_*d\sigma=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} F(v',v'_*,v,v_*)B(|v-v_*|,\cos\theta)dvdv_*d\sigma.$$

Lemma 2.4. *(Hardy-Littlewood-Sobolev inequality) ([39], Chapter 4) Let $1<p,r<+\infty$ and $0<\lambda<d$ with $1/p+\lambda/d+1/r=2$. Then there exists a constant $C(n,\lambda,p)$, such that for all smooth function f,h we have*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda}h(y)dxdy\le C(n,\lambda,p)\|f\|_{L^p}\|h\|_{L^r}.$$

The following can be seen as a weak version of Hardy-Littlewood-Sobolev inequality when $q=+\infty$ which is useful in the following proof.

Lemma 2.5. *Suppose $\gamma\in(-3,0)$, then for any smooth function f, the following estimate holds:*
If $\gamma\in(-\frac{3}{2},0)$, we have

$$\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f|(v_*)dv_*\lesssim\|f\|_{L^1}^{1+\frac{2\gamma}{3}}\|f\|_{L^2}^{-\frac{2\gamma}{3}}.$$

If $\gamma\in(-2,0)$, we have

$$\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f|(v_*)dv_*\lesssim\|f\|_{L^1}^{1+\frac{\gamma}{2}}\|f\|_{L^3}^{-\frac{\gamma}{2}}.$$

If $\gamma\in(-3,0)$, we have

$$\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f|(v_*)dv_*\lesssim\|f\|_{L^1}^{1+\frac{\gamma}{3}}\|f\|_{L^\infty}^{-\frac{\gamma}{3}}.$$

Proof. Assume f does not equal to 0 otherwise the estimate is trivial. Let $\lambda>0$ be a constant to be determined. We divide the integral into two regions $|v-v_*|\le\lambda$ and $|v-v_*|>\lambda$ we have

$$\int_{\mathbb{R}^3}|v-v_*|^\gamma|f|(v_*)dv_*\le\int_{|v-v_*|\le\lambda}|v-v_*|^\gamma|f|(v_*)dv_*+\int_{|v-v_*|>\lambda}|v-v_*|^\gamma|f|(v_*)dv_*.$$

The second part is bounded by

$$\int_{|v-v_*|>\lambda} |v-v_*|^\gamma |f|(v_*)dv_* \lesssim \lambda^\gamma \|f\|_{L^1}.$$

For the first part we divided it into three cases, for $\gamma \in (-\frac{3}{2}, 0)$, by Cauchy-Schwarz inequality we have

$$\int_{|v-v_*|\le\lambda} |v-v_*|^\gamma |f|(v_*)dv_* \le \left(\int_{|v-v_*|\le\lambda} |v-v_*|^{2\gamma} dv_*\right)^{\frac{1}{2}} \|f\|_{L^2} \lesssim \lambda^{\frac{3}{2}+\gamma} \|f\|_{L^2}.$$

For $\gamma \in (-2, 0)$, by Hölder's inequality we have

$$\int_{|v-v_*|\le\lambda} |v-v_*|^\gamma |f|(v_*)dv_* \le \left(\int_{|v-v_*|\le\lambda} |v-v_*|^{\frac{3}{2}\gamma} dv_*\right)^{\frac{2}{3}} \|f\|_{L^3} \lesssim \lambda^{2+\gamma} \|f\|_{L^3}.$$

For $\gamma \in (-3, 0)$, we have

$$\int_{|v-v_*|\le\lambda} |v-v_*|^\gamma |f|(v_*)dv_* \le \int_{|v-v_*|\le\lambda} |v-v_*|^\gamma dv_* \|f\|_{L^\infty} \lesssim \lambda^{3+\gamma} \|f\|_{L^\infty},$$

so the proof is ended by taking $\lambda = \|f\|_{L^1}^{\frac{2}{3}} \|f\|_{L^2}^{-\frac{2}{3}}, \|f\|_{L^1}^{\frac{1}{2}} \|f\|_{L^3}^{-\frac{1}{2}}, \|f\|_{L^1}^{\frac{1}{3}} \|f\|_{L^\infty}^{-\frac{1}{3}}$ respectively. □

Lemma 2.6. *For any smooth function f, g, for any $\gamma \in (-2, 0)$ we have*

$$\mathscr{R} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma |f(v_*)|^2 |g(v)|^2 dv dv_* \lesssim \min_{m+n=1} \{\|f\|_{H^m_{\gamma/2}}^2 \|g\|_{H^n_2}^2, \|f\|_{H^m_2}^2 \|g\|_{H^n_{\gamma/2}}^2\}.$$

Similarly For any $\gamma \in (-3, -2]$ we have

$$\mathscr{R} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma |f(v_*)|^2 |g(v)|^2 dv dv_* \lesssim \min_{m+n=2} \{\|f\|_{H^m_{\gamma/2}}^2 \|g\|_{H^n_2}^2, \|f\|_{H^m_2}^2 \|g\|_{H^n_{\gamma/2}}^2\}.$$

Proof. Without loss of generality we only prove that

$$\mathscr{R} \lesssim \min_{m+n=1} \{\|f\|_{H^m_{\gamma/2}}^2 \|g\|_{H^n_2}^2\}, \quad \gamma \in (-2, 0), \quad \mathscr{R} \lesssim \min_{m+n=2} \{\|f\|_{H^m_{\gamma/2}}^2 \|g\|_{H^n_2}^2\}, \quad \gamma \in (-3, -2].$$

For both $\gamma \in (-2, 0)$ and $\gamma \in (-3, -2]$, by

$$C\langle v_*\rangle^{-\gamma} \le \langle v-v_*\rangle^{-\gamma} \langle v\rangle^{-\gamma},$$

we compute

$$\begin{aligned}
\mathscr{R} &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma \langle v-v_*\rangle^{-\gamma} |f(v_*)|^2 \langle v_*\rangle^\gamma |g(v)|^2 \langle v\rangle^{-\gamma} dv dv_* \\
&\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (1+|v-v_*|^\gamma) |f(v_*)|^2 \langle v_*\rangle^\gamma |g(v)|^2 \langle v\rangle^{-\gamma} dv dv_* := I_1 + I_2.
\end{aligned}$$

For the I_1 term, for both $\gamma \in (-2, 0)$ and $\gamma \in (-3, -2]$ we easily compute

$$I_1 \lesssim \|f\|_{L^2_{\gamma/2}}^2 \|g\|_{L^2_{-\gamma/2}}^2 \lesssim \|f\|_{L^2_{\gamma/2}}^2 \|g\|_{L^2_2}^2.$$

For the I_2 term, when $\gamma \in (-2, 0)$ by Lemma 2.5 we have

$$\begin{aligned}
I_2 &\lesssim \left(\sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^\gamma |f(v_*)|^2 \langle v_*\rangle^\gamma dv_*\right) \|g\|_{L^2_{-\gamma/2}}^2 \lesssim \|f^2\|_{L^1_\gamma}^{1+\frac{\gamma}{2}} \|f^2\|_{L^3_\gamma}^{-\frac{\gamma}{2}} \|g\|_{L^2_2}^2 \\
&\lesssim \|f\|_{L^2_{\gamma/2}}^{2+\gamma} \|f\|_{L^6_{\gamma/2}}^{-\gamma} \|g\|_{L^2_2}^2 \lesssim \|f\|_{H^1_{\gamma/2}}^2 \|g\|_{L^2_2}^2,
\end{aligned}$$

similarly

$$I_2 \lesssim \left(\sup_{v_*\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^\gamma |g(v)|^2 \langle v\rangle^{-\gamma} dv\right) \|f\|_{L^2_{\gamma/2}}^2 \lesssim \|f\|_{L^2_{\gamma/2}}^2 \|g\|_{H^1_2}^2,$$

the case $\gamma \in (-2,0)$ is thus proved. When $\gamma \in (-3,-2]$, by Lemma 2.5 we have

$$I_2 \lesssim \left(\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma |f(v_*)|^2 \langle v_*\rangle^\gamma dv_*\right)\|g\|^2_{L^2_{-\gamma/2}} \lesssim \|f^2\|^{1+\frac{\gamma}{3}}_{L^1_\gamma}\|f^2\|^{-\frac{\gamma}{3}}_{L^\infty_\gamma}\|g\|^2_{L^2_2}$$
$$\lesssim \|f\|^{2+\frac{2\gamma}{3}}_{L^2_{\gamma/2}}\|f\|^{-\frac{2\gamma}{3}}_{L^\infty_{\gamma/2}}\|g\|^2_{L^2_2} \lesssim \|f\|^2_{H^2_{\gamma/2}}\|g\|^2_{L^2_2},$$

similarly

$$I_2 \lesssim \left(\sup_{v_*\in\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma |g(v)|^2 \langle v\rangle^{-\gamma} dv\right)\|f\|^2_{L^2_{\gamma/2}} \lesssim \|f\|^2_{L^2_{\gamma/2}}\|g\|^2_{H^2_2},$$

and by Hardy-Littlewood-Sobolev inequality

$$I_2 \lesssim \|f^2\|_{L^p_\gamma}\|g^2\|_{L^3_{-\gamma}} \lesssim \|f\|^2_{L^{2p}_{\gamma/2}}\|g\|^2_{L^6_{-\gamma/2}} \lesssim \|f\|^2_{H^1_{\gamma/2}}\|g\|^2_{H^1_2},$$

where $p = \frac{3}{5+\gamma}$ implies $2p \in [2,3)$. So the case $\gamma \in (-3,-2]$ is proved by combining the three cases. □

We will use the following representation of v' which can be proved directly. We have

$$\langle v'\rangle^2 = \langle v\rangle^2\cos^2\frac{\theta}{2} + \langle v_*\rangle^2\sin^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega, \quad \omega\perp(v-v_*), \quad v\cdot w = v_*\cdot w, \tag{21}$$

where $\omega = \frac{\sigma-(\sigma\cdot k)k}{|\sigma-(\sigma\cdot k)k|}$ with $k = \frac{v-v_*}{|v-v_*|}$. We have the following estimate for the term $\langle v'\rangle^k$.

Lemma 2.7. *For any constant $k \ge 4$ we have*

$$\langle v'\rangle^k = \sin^k\frac{\theta}{2}\langle v_*\rangle^k + R_1 + R_2, \quad |R_1| \le C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^{k-1}\langle v\rangle, \quad |R_2| \le C_k\langle v\rangle^k, \tag{22}$$

for some constant $C_k > 0$. We also have

$$\langle v'\rangle^k - \langle v\rangle^k\cos^k\frac{\theta}{2} = k\langle v\rangle^{k-2}\cos^{k-1}\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|(v\cdot\omega) + L_1 + L_2, \tag{23}$$

with

$$|L_1| \le C_k\sin^{k-2}\frac{\theta}{2}\langle v_*\rangle^k\langle v\rangle^2, \quad |L_2| \le C_k\langle v\rangle^{k-2}\langle v_*\rangle^4\sin^2\frac{\theta}{2},$$

for some constant $C_k > 0$, in particular

$$\langle v'\rangle^k = \cos^k\frac{\theta}{2}\langle v\rangle^k + Q_1, \quad |Q_1| \le C_k\langle v_*\rangle^k\langle v\rangle^{k-1}.$$

Proof. For simplicity, we prove the result for $\langle v'\rangle^{2k}$. By (21) and mean value theorem we have

$$\langle v'\rangle^{2k} - \langle v_*\rangle^{2k}\sin^{2k}\frac{\theta}{2} = k\int_0^1\left(\langle v_*\rangle^2\sin^2\frac{\theta}{2} + t(\langle v\rangle^2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega)\right)^{k-1}dt$$
$$\times\left(\langle v\rangle^2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega\right),$$

since $v\cdot w = v_*\cdot w$ which implies $|v-v_*||v\cdot\omega| \le |v||v_*\cdot\omega| + |v_*||v\cdot\omega| \le 2|v||v_*|$, so we have

$$\langle v\rangle^2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega \le \langle v\rangle^2 + 4\sin\frac{\theta}{2}|v||v_*| \le 3\langle v\rangle^2 + 2\sin^2\frac{\theta}{2}\langle v_*\rangle^2,$$

so if $2k-1 \ge 2$ we have

$$\langle v'\rangle^{2k} - \langle v_*\rangle^{2k}\sin^{2k}\frac{\theta}{2} \le C_k(\sin^2\frac{\theta}{2}\langle v_*\rangle^2 + \langle v\rangle^2)^{k-1}(\langle v\rangle^2 + \sin\frac{\theta}{2}|v_*||v|)$$
$$\le C_k(\sin^{2k-2}\frac{\theta}{2}\langle v_*\rangle^{2k-2} + \langle v\rangle^{2k-2})(\sin\frac{\theta}{2}\langle v_*\rangle + \langle v\rangle)\langle v\rangle$$
$$\le C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^{2k-1}\langle v\rangle + C_k\langle v\rangle^{2k},$$

(22) is thus proved. For (23), using (21) and the mean value theorem twice we have

$$
\begin{aligned}
&\langle v'\rangle^{2k}-\langle v\rangle^{2k}\cos^{2k}\frac{\theta}{2}-2k(\langle v\rangle^2\cos^2\frac{\theta}{2})^{k-1}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|(v\cdot\omega)\\
=&k(\langle v\rangle^2\cos^2\frac{\theta}{2})^{k-1}\langle v_*\rangle^2\sin^2\frac{\theta}{2}\\
&+k(k-1)\int_0^1(1-t)\left(\langle v\rangle^2\cos^2\frac{\theta}{2}+t(\langle v_*\rangle^2\sin^2\frac{\theta}{2}+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega)\right)^{k-2}dt\\
&\times\left(\langle v_*\rangle^2\sin^2\frac{\theta}{2}+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega\right)^2:=I_1+I_2.
\end{aligned}
$$

For I_1 we easily deduce

$$
I_1\le C_k\langle v\rangle^{2k-2}\langle v_*\rangle^2\sin^2\frac{\theta}{2},
$$

since $|v-v_*||v\cdot\omega|\le 2|v||v_*|$, we have

$$
\langle v_*\rangle^2\sin^2\frac{\theta}{2}+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|v-v_*|v\cdot\omega\le\langle v_*\rangle^2\sin^2\frac{\theta}{2}+4|v||v_*|\sin\frac{\theta}{2}\le 3\langle v_*\rangle^2\sin^2\frac{\theta}{2}+2\langle v\rangle^2,
$$

so we have

$$
\begin{aligned}
I_2\le&C_k(\langle v_*\rangle^2\sin^2\frac{\theta}{2}+4|v||v_*|\sin\frac{\theta}{2})^2(3\langle v_*\rangle^2\sin^2\frac{\theta}{2}+3\langle v\rangle^2)^{k-2}\\
\le&C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^2(\langle v_*\rangle^2+\langle v\rangle^2)(\langle v_*\rangle^{2k-4}\sin^{2k-4}\frac{\theta}{2}+\langle v\rangle^{2k-4})\\
\le&C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^4\langle v\rangle^2(\langle v_*\rangle^{2k-4}\sin^{2k-4}\frac{\theta}{2}+\langle v\rangle^{2k-4})\\
\le&C_k\sin^{2k-2}\frac{\theta}{2}\langle v_*\rangle^{2k}\langle v\rangle^2+C_k\langle v\rangle^{2k-2}\langle v_*\rangle^4\sin^2\frac{\theta}{2},
\end{aligned}
$$

if $2k\ge 4$, so the theorem is thus proved. □

Lemma 2.8. *For any smooth function g and f, if $\gamma>-3$ we have*

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma g_*f\,dv_*dv\le C\|g\|_{L^2_l}\|f\|^2_{L^2_l},
$$

with $l=\max\{\gamma+2,\frac{3}{2}\}$.

Proof. We split it into two cases $\gamma\ge 0$ and $\gamma<0$, for the case $\gamma\ge 0$, we easily compute

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma g_*f\,dv_*dv\lesssim\|f\|_{L^1_\gamma}\|g\|_{L^1_\gamma}\lesssim\|f\|_{L^2_{\gamma+2}}\|g\|_{L^2_{\gamma+2}}.
$$

For the case $\gamma<0$, by Hardy-Littlewood-Sobolev inequality we have

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma g_*f\,dv_*dv\lesssim\|f\|_{L^p}\|g\|_{L^p}\lesssim\|f\|_{L^2_{3/2}}\|g\|_{L^2_{3/2}},
$$

where $\frac{1}{p}=\frac{1}{2}(2+\frac{\gamma}{3})\in(\frac{1}{2},1)$ implies $p\in(1,2)$. The proof is thus finished by gathering the two cases. □

Lemma 2.9. *For any $\gamma>-3$, $k>\max\{3,3+\gamma\}$ we have*

$$
\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{-k}dv_*\le\frac{c}{k}\langle v\rangle^\gamma+C_k\langle v\rangle^{\gamma-2}.
$$

Proof. For the case $\gamma>0$, we have

$$
\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{-k}dv_*\le C\int_{\mathbb{R}^3}(|v|^\gamma+|v_*|^\gamma)\langle v_*\rangle^{-k}dv_*,
$$

It is easily seen that

$$
\int_{\mathbb{R}^3}|v_*|^\gamma\langle v_*\rangle^{-k}dv_*\le C_k\le\frac{c}{k}\langle v\rangle^\gamma+C_k\langle v\rangle^{\gamma-2},
$$

recall the theory of beta function (5) we have

$$\langle v\rangle^\gamma \int_{\mathbb{R}^3} \langle v_*\rangle^{-k} dv_* \le C\langle v\rangle^\gamma \int_0^{+\infty} r^2 \frac{1}{(1+r^2)^{k/2}} dr \le \frac{c}{k^{3/2}}\langle v\rangle^\gamma,$$

so the case $\gamma \in [0,1]$ is thus proved. For the case $\gamma \in (-3,0)$, If $|v| \le \frac{1}{2}$, we have $|v_*| + \frac{1}{2} \le 1 + |v - v_*|$, so we have

$$\int_{\mathbb{R}^3} |v - v_*|^\gamma \langle v_*\rangle^{-k} dv_* = \int_{\mathbb{R}^3} |v_*|^\gamma \langle v - v_*\rangle^{-k} dv_* \le C_k \int_{\mathbb{R}^3} |v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \le C_k \langle v\rangle^{\gamma-2}.$$

Consider now $|v| > \frac{1}{2}$ and split the integral into two regions $|v - v_*| > \langle v\rangle/4$ and $|v - v_*| \le \langle v\rangle/4$. For the first region we obtain

$$\int_{|v-v_*|>\frac{\langle v\rangle}{4}} |v - v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C\langle v\rangle^\gamma \int_{\mathbb{R}^3} \langle v_*\rangle^{-k} dv_* \le C\langle v\rangle^\gamma \int_0^\infty r^2 \frac{1}{(1+r^2)^{k/2}} dr \le \frac{c}{k^{3/2}}\langle v\rangle^\gamma.$$

For the second region, $|v| > 1/2$ and $|v - v_*| \le \langle v\rangle/4$ imply $|v_*| \ge \langle v\rangle/8$, hence

$$\int_{|v-v_*|\le\frac{\langle v\rangle}{4}} |v - v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \langle v\rangle^{-k} \int_{|v-v_*|\le\frac{\langle v\rangle}{4}} |v - v_*|^\gamma dv_* \le C_k \langle v\rangle^{-k+\gamma+3} \le \frac{c}{k}\langle v\rangle^\gamma + C_k\langle v\rangle^{\gamma-2},$$

so the theorem is thus proved. □

We introduce a change of variable which will be used frequently.

Lemma 2.10. *([44], Lemma A.1.) For any non negative function F in terms of $v_*, v, r = |v - v_*|, \theta, v', v'_*$, we have*

$$\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} F d\sigma dv_* = 4 \int_{\mathbb{R}^3} \frac{1}{|v' - v|} \int_{\{w: w\cdot(v'-v)=0\}} F \frac{1}{r} dw dv',$$

where r, θ, v'_*, v_* in the left hand side is changed to

$$r = \sqrt{|v' - v|^2 + |w|^2}, \quad \cos(\theta/2) = |w|/r, \quad v'_* = v + w, \quad v_* = v' + w,$$

respectively.

Lemma 2.11. *For any $-3 < \gamma \le 2$, for any constant $k > \max\{3, 3+\gamma\}$ we have*

$$I := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k \langle v'_*\rangle^k} dv_* d\sigma \le C_k \langle v\rangle^\gamma,$$

for all $v \in \mathbb{R}^d$. In fact for $\gamma \in [0,1]$ we can prove a stronger estimate

$$I := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k \langle v'_*\rangle^k} dv_* d\sigma \le \frac{c}{k}\langle v\rangle^\gamma + C_k \langle v\rangle^{\gamma-2}.$$

Proof. If $|v| \le 1$, since $\langle v_*\rangle \le \langle v'_*\rangle\langle v'\rangle$, we have

$$I \le 2^k \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^\gamma \frac{1}{\langle v_*\rangle^k} dv_* d\sigma \le C_k \langle v\rangle^{\gamma-2},$$

we focus on the case $|v| \ge 1$ later. We first prove the case $\gamma \in [0,1]$. By Lemma 2.10 and since $0 \le \gamma \le 1$, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \frac{1}{|v' - v|} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} \frac{1}{\sqrt{|v' - v|^2 + |w|^2}} (\sqrt{|v' - v|^2 + |w|^2})^\gamma \langle v + w\rangle^{-k} dw dv' \\ &\le \int_{\mathbb{R}^3} \frac{1}{|v' - v|^{2-\gamma}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} \langle v + w\rangle^{-k} dw dv'. \end{aligned}$$

We first split v into two parts

$$v=v_\perp+v_\parallel,\quad v_\perp=\frac{v\cdot(v'-v)}{|v-v'|^2}(v'-v),\quad |v|^2=|v_\perp|^2+|v_\parallel|^2,\quad v_\perp\perp v_\parallel,\quad v_\parallel\parallel w,\quad |v+w|^2=|w+v_\parallel|^2+|v_\perp|^2. \tag{24}$$

Then the integral becomes

$$\int_{\{w:w\cdot(v'-v)=0\}}\langle v+w\rangle^{-k}dw=\int_{\mathbb{R}^2}\frac{1}{(1+|w+v_\parallel|^2+|v_\perp|^2)^{k/2}}dw=\int_{\mathbb{R}^2}\frac{1}{(1+|w|^2+|v_\perp|^2)^{k/2}}dw,$$

making a change of variable $w=\sqrt{1+|v_\perp|^2}x$ we have

$$\int_{\mathbb{R}^2}\frac{1}{(1+|w|^2+|v_\perp|^2)^{k/2}}dw=(1+|v_\perp|^2)^{1-\frac{k}{2}}\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^{k/2}}dx=2\pi\frac{1}{k-2}(1+|v_\perp|^2)^{1-\frac{k}{2}},$$

so I turns to

$$I\le 2\pi\frac{1}{k-2}\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{2-\gamma}}\frac{\langle v\rangle^k}{\langle v'\rangle^k}\frac{1}{\langle v_\perp\rangle^{k-2}}dv',\quad |v_\perp|=\frac{|v\cdot(v'-v)|}{|v-v'|}.$$

Since $|v_\perp|\le|v|$, so we split it into three cases $|v_\perp|\ge(1-\frac{1}{k})|v|$, $|v_\perp|\le\frac{|v|}{k}$ and $\frac{1}{k}|v|\le|v_\perp|\le(1-\frac{1}{k})|v|$. For the case $|v_\perp|\ge(1-\frac{1}{k})|v|$ we have

$$\langle v\rangle^{k-2}\le(1+\frac{1}{k-1})^{k-2}\langle v_\perp\rangle^{k-2}\le e\langle v_\perp\rangle^{k-2},$$

together with Lemma 2.9 we have

$$I\le\frac{c}{k}\langle v\rangle^2\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{2-\gamma}}\frac{1}{\langle v'\rangle^k}dv'\le\frac{c}{k^2}\langle v\rangle^2\langle v\rangle^{\gamma-2}+C_k\langle v\rangle^2\langle v\rangle^{\gamma-4}\le\frac{c}{k}\langle v\rangle^\gamma+C_k\langle v\rangle^{\gamma-2}.$$

For the case $|v_\perp|\le\frac{1}{k}|v|$ we have

$$|v'|^2=|v-v'|^2+|v|^2+2v\cdot(v'-v)\ge|v-v'|^2+|v|^2-2|v-v'||v_\perp|\ge(1-\frac{1}{k})(|v-v'|^2+|v|^2),$$

which implies

$$\begin{aligned}I\le&\frac{c}{k}\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{2-\gamma}}\frac{\langle v\rangle^k}{(1+|v-v'|^2+|v|^2)^{k/2}}\frac{1}{\langle v_\perp\rangle^{k-2}}dv'\\ \le&\frac{c}{k}\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{2-\gamma}}\frac{1}{(1+\frac{|v-v'|^2}{1+|v|^2})^{k/2}}\frac{1}{\langle v_\perp\rangle^{k-2}}dv'.\end{aligned}$$

We make the change of variables

$$r=|v-v'|,\quad v\cdot(v'-v)=r|v|\cos\theta,\quad dv'=r^2\sin\theta drd\theta d\phi,\quad |v_\perp|^2=|v|^2\cos^2\theta,\quad |v_\parallel|^2=|v|^2\sin^2\theta. \tag{25}$$

So I turns to

$$I\le\frac{c}{k}2\pi\int_0^\infty r^\gamma\frac{1}{(1+\frac{r^2}{1+|v|^2})^{k/2}}\int_0^\pi\frac{1}{\langle|v|^2\cos^2\theta\rangle^{k-2}}\sin\theta drd\theta.$$

Making another change of variables

$$r=\sqrt{1+|v|^2}x,\quad y=|v|\cos\theta,\quad dy=|v|\sin\theta d\theta, \tag{26}$$

then by (5) we deduce

$$\begin{aligned}I\le&\frac{c}{k}2\pi\frac{1}{|v|}\langle v\rangle^{\gamma+1}\int_0^\infty x^\gamma\frac{1}{(1+|x|^2)^{k/2}}\int_{-|v|}^{|v|}\frac{1}{\langle y\rangle^{k-2}}dydx\\ \le&\frac{c}{k}2\pi\langle v\rangle^\gamma\int_0^\infty x^\gamma\frac{1}{(1+|x|^2)^{k/2}}dx\int_{-\infty}^{+\infty}\frac{1}{\langle y\rangle^{k-2}}dy\le\frac{c}{k}2\pi\langle v\rangle^\gamma.\end{aligned}$$

For the case $\frac{1}{k}|v| \le |v_\perp| \le (1-\frac{1}{k})|v|$ we have

$$\langle v\rangle^{k-2} \le k^{k-2}\langle v_\perp\rangle^{k-2} \le C_k\langle v_\perp\rangle^{k-2}, \quad |v'|^2 \ge |v-v'|^2+|v|^2-2|v-v'||v_\perp| \ge \frac{1}{k}(|v-v'|^2+|v|^2),$$

still make the same change of variables as (25) and (26) we have

$$\begin{aligned} I \le& C_k\langle v\rangle^2 \int_{\mathbb{R}^3} \frac{1}{|v'-v|^{2-\gamma}} \frac{1}{(1+|v-v'|^2+|v|^2)^{k/2}} dv' \\ \le& C_k\langle v\rangle^{2-k} \int_{\mathbb{R}^3} \frac{1}{|v'-v|^{2-\gamma}} \frac{1}{(1+\frac{|v-v'|^2}{1+|v|^2})^{k/2}} dv' \\ \le& C_k 2\pi\langle v\rangle^{2-k} \int_0^\infty r^\gamma \frac{1}{(1+\frac{r^2}{1+|v|^2})^{k/2}} dr d\theta \\ \le& C_k 2\pi\langle v\rangle^{2+\gamma-k} \int_0^\infty x^\gamma \frac{1}{(1+x^2)^{k/2}} dx \le \frac{c}{k}\langle v\rangle^\gamma + C_k\langle v\rangle^{\gamma-2}, \end{aligned}$$

the case $\gamma \in [0,1]$ is thus proved. For the case $\gamma \in (-3,0)$, since $\langle v\rangle^k \le C_k\langle v'\rangle^k + C_k\langle v'_*\rangle^k$, we have

$$I \le C_k \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{1}{\langle v'\rangle^k} dv_* d\sigma + C_k \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{1}{\langle v'_*\rangle^k} dv_* d\sigma := I_1 + I_2.$$

We only prove the term I_1 since the I_2 term is easily achieved by interchange v' and v'_*. For the I_1 term by Lemma 2.10 we have

$$\begin{aligned} I_1 \le& C_k \int_{\mathbb{R}^3} \frac{1}{|v'-v|} \frac{1}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} \frac{1}{\sqrt{|v'-v|^2+|w|^2}} (\sqrt{|v'-v|^2+|w|^2})^\gamma dw dv' \\ \le& C_k \int_{\mathbb{R}^3} \frac{1}{|v'-v|} \frac{1}{\langle v'\rangle^k} \int_{\mathbb{R}^2} \frac{1}{(|v'-v|^2+|w|^2)^{\frac{1-\gamma}{2}}} dw dv', \end{aligned}$$

by a change of variable $w = |v-v'|x$ we deduce

$$I_1 \le C_k \int_{\mathbb{R}^3} |v'-v|^\gamma \frac{1}{\langle v'\rangle^k} \int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^{\frac{1-\gamma}{2}}} dx dv' \le C_k \int_{\mathbb{R}^3} |v'-v|^\gamma \frac{1}{\langle v'\rangle^k} dv', \le C_k\langle v\rangle^\gamma,$$

since $\gamma < 0$ implies $1-\gamma > 1$ so $(1+|x|^2)^{-\frac{1-\gamma}{2}}$ is integrable, the case $\gamma \in (-3,0)$ is thus proved. For the case $\gamma \in [1,2]$, since $1 \le \gamma$, we have

$$\begin{aligned} I =& \int_{\mathbb{R}^3} \frac{1}{|v'-v|} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} \frac{1}{\sqrt{|v'-v|^2+|w|^2}} (\sqrt{|v'-v|^2+|w|^2})^\gamma \langle v+w\rangle^{-k} dw dv' \\ \le& C\int_{\mathbb{R}^3} \frac{1}{|v'-v|^{2-\gamma}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} \langle v+w\rangle^{-k} dw dv' \\ &+ C\int_{\mathbb{R}^3} \frac{1}{|v'-v|} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{w: w\cdot(v'-v)=0\}} |w|^{\gamma-1}\langle v+w\rangle^{-k} dw dv' := I_1 + I_2, \end{aligned}$$

since $\gamma \le 2$, the I_1 term is the same as the term I in the case $\gamma \in [0,1]$ so can be estimated by the same way. We now focus on the I_2 term, we have

$$\begin{aligned} \int_{\{w: w\cdot(v'-v)=0\}} |w|^{\gamma-1}\langle v+w\rangle^{-k} dw =& \int_{\mathbb{R}^2} \frac{|w|^{\gamma-1}}{(1+|w+v_\parallel|^2+|v_\perp|^2)^{k/2}} dw \\ \le& C\int_{\mathbb{R}^2} \frac{|w|^{\gamma-1}+|v_\parallel|^{\gamma-1}}{(1+|w|^2+|v_\perp|^2)^{k/2}} dw. \end{aligned}$$

Making the change of variable $|w| = \sqrt{1+|v_\perp|^2}x$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|w|^{\gamma-1}+|v_\|\|^{\gamma-1}}{(1+|w|^2+|v_\perp|^2)^{k/2}}dw \leq& |v_\||^{\gamma-1}(1+|v_\perp|^2)^{1-\frac{k}{2}}\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^{k/2}}dx\\ &+\langle v_\perp\rangle^{\gamma-1}(1+|v_\perp|^2)^{1-\frac{k}{2}}\int_{\mathbb{R}^2}\frac{|x|^\gamma}{(1+|x|^2)^{k/2}}dx\\ &\leq C_k\langle v\rangle^{\gamma-1}(1+|v_\perp|^2)^{1-\frac{k}{2}},\end{aligned}$$

we decude

$$I_2 \leq C_k\langle v\rangle^{\gamma-1}\int_{\mathbb{R}^3}\frac{1}{|v'-v|}\frac{\langle v\rangle^k}{\langle v'\rangle^k}\frac{1}{\langle v_\perp\rangle^{k-2}}dv',$$

it is easily seen that I_2 also be estimated by the same way as the term I in the case $\gamma\in[0,1]$, so the proof is thus finished. □

For the exponential weight case, we have a better estimate for the linearized operator.

Lemma 2.12. *For any $-3<\gamma\leq 1$, for any constant $a>0$ and $b\in(0,2)$ we have*

$$I := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}e^{-\frac{1}{2}|v'_*|^2}dv_*d\sigma \leq C_{a,b}\langle v\rangle^{\gamma-\frac{b(\gamma+3)}{4}},$$

for all $v\in\mathbb{R}^d$.

Proof. Since $\gamma-1\leq 0$, by Lemma 2.10 we have

$$\begin{aligned}I &= 4\int_{\mathbb{R}^3}\frac{1}{|v'-v|}\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}\int_{\{\omega:\omega\cdot(v'-v)=0\}}\frac{1}{\sqrt{|v'-v|^2+|w|^2}}(\sqrt{|v'-v|^2+|w|^2})^\gamma e^{-\frac{|v+w|^2}{2}}dwdv'\\ &\leq 4\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{\frac{3-\gamma}{2}}}\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}\int_{\{\omega:\omega\cdot(v'-v)=0\}}|w|^{\frac{\gamma-1}{2}}e^{-\frac{|v+w|^2}{2}}dwdv'.\end{aligned}$$

Recall the decomposition (24) we have

$$I \leq 4\int_{\mathbb{R}^3}\frac{1}{|v'-v|^{\frac{3-\gamma}{2}}}\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}e^{-\frac{|v_\perp|^2}{2}}\int_{\{\omega:\omega\cdot(v'-v)=0\}}|w|^{\frac{\gamma-1}{2}}e^{-\frac{|v_\|+w|^2}{2}}dwdv'.$$

Since $\frac{\gamma-1}{2}>-2$, we have

$$\int_{\{\omega:\omega\cdot(v'-v)=0\}}|w|^{\frac{\gamma-1}{2}}e^{-\frac{|v_\|+w|^2}{2}}dw = \int_{\mathbb{R}^2}|w-v_\||^{\frac{\gamma-1}{2}}e^{-\frac{|w|^2}{2}}dw \leq C\langle v_\|\rangle^{\frac{\gamma-1}{2}},$$

which implies

$$I \leq C\int_{\mathbb{R}^3}|v'-v|^{\frac{\gamma-3}{2}}\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}e^{-\frac{|v_\perp|^2}{2}}\langle v_\|\rangle^{\frac{\gamma-1}{2}}dv'.$$

If $|v|\leq 1$, since $\frac{\gamma-3}{2}>-3$ it is easily seen that

$$I \leq C_{a,b}\int_{\mathbb{R}^3}|v'-v|^{\frac{\gamma-3}{2}}\frac{1}{e^{a\langle v'\rangle^b}}dv' \leq C_{a,b}\leq C_{a,b}\langle v\rangle^{\gamma-\frac{b(\gamma+3)}{4}},$$

so we focus on the $|v|>1$ case. We split it into two case $|v_\perp|\geq\frac{|v|}{2}$ and $|v_\perp|<\frac{|v|}{2}$. For the case $|v_\perp|\geq\frac{|v|}{2}$ we easily compute

$$I \leq C\int_{\mathbb{R}^3}|v'-v|^{\frac{\gamma-3}{2}}\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}e^{-\frac{|v|^2}{8}}dv' \leq C_{a,b}e^{-\frac{|v|^2}{16}}\int_{\mathbb{R}^3}|v'-v|^{\frac{\gamma-3}{2}}\frac{1}{e^{a\langle v'\rangle^b}}dv' \leq C_{a,b}\langle v\rangle^{\gamma-b}.$$

For the case $|v_\perp| < \frac{|v|}{2}$ we have $|v_\parallel| \ge \frac{|v|}{2}$ which implies $\langle v_\parallel\rangle^{\frac{\gamma-1}{2}} \le C\langle v\rangle^{\frac{\gamma-1}{2}}$, so we have

$$I \le C\langle v\rangle^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}} e^{-\frac{|v_\perp|^2}{2}} dv'.$$

Still take the change of variables (25), since $|v'|^2 = |v-v'|^2 + |v|^2 + 2v\cdot(v'-v)$, I turns to

$$I \le C\langle v\rangle^{\frac{\gamma-1}{2}} \int_0^{+\infty}\int_0^{\pi} r^{\frac{\gamma+1}{2}} e^{a(1+|v|^2)^{\frac{b}{2}} - a(1+|v|^2+r^2+2r|v|\cos\theta)^{\frac{b}{2}}} e^{-\frac{|v|^2\cos^2\theta}{2}} \sin\theta dr d\theta.$$

Since $0<b<2$, there exists a constant $C_{a,b}$ such that

$$\frac{|x|^2}{4} + C_{a,b} > 4a\langle x\rangle^b, \quad \forall x\in\mathbb{R}, \quad x^{\frac{b}{2}} + y^{\frac{b}{2}} \ge (x+y)^{\frac{b}{2}}, \quad \forall x,y\ge 0,$$

which implies that

$$\frac{|v|^2\cos^2\theta}{4} + C_{a,b} \ge a(4+4|v|^2\cos^2\theta)^{\frac{b}{2}},$$

and

$$a(1+|v|^2+r^2+2r|v|\cos\theta)^{\frac{b}{2}} + a(4+4|v|^2\cos^2\theta)^{\frac{b}{2}} \ge a(1+|v|^2+r^2+2r|v|\cos\theta+4|v|^2\cos^2\theta)^{\frac{b}{2}} \ge a(1+|v|^2+\frac{1}{2}r^2)^{\frac{b}{2}}.$$

So we have

$$I \le C_{a,b}\langle v\rangle^{\frac{\gamma-1}{2}} \int_0^{+\infty}\int_0^{\pi} r^{\frac{\gamma+1}{2}} e^{a(1+|v|^2)^{\frac{b}{2}} - a(1+|v|^2+\frac{r^2}{2})^{\frac{b}{2}}} e^{-\frac{|v|^2\cos^2\theta}{4}} \sin\theta dr d\theta.$$

As before, taking another change of variables (26) we have

$$\begin{aligned} I \le& C_{a,b}\frac{1}{|v|}\langle v\rangle^{\gamma+1} \int_0^{+\infty} x^{\frac{\gamma+1}{2}} e^{a(1+|v|^2)^{\frac{b}{2}} - a(1+|v|^2+\frac{x^2(1+|v|^2)}{2})^{\frac{b}{2}}} dx \int_{-|v|}^{|v|} e^{-\frac{y^2}{4}} dy \\ \le& C_{a,b}\langle v\rangle^{\gamma} \int_0^{+\infty} x^{\frac{\gamma+1}{2}} e^{a(1+|v|^2)^{\frac{b}{2}}(1-(1+\frac{x^2}{2})^{\frac{b}{2}})} dx \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4}} dy \\ \le& C_{a,b}\langle v\rangle^{\gamma} \int_0^{+\infty} x^{\frac{\gamma+1}{2}} e^{a\langle v\rangle^b(1-(1+\frac{x^2}{2})^{\frac{b}{2}})} dx. \end{aligned}$$

We split it into to two case $x\ge 2$ and $x\in[0,2]$. For the case $x\ge 2$ we have $(1+\frac{x^2}{2})^{\frac{b}{2}} - 1 \ge C_b\langle x\rangle^b$ for some $C_b>0$. Together with $\langle v\rangle^b + \langle x\rangle^b \le 2\langle x\rangle^b\langle v\rangle^b$ implies

$$\begin{aligned} \int_2^{+\infty} x^{\frac{\gamma+1}{2}} e^{a\langle v\rangle^b(1-(1+\frac{x^2}{2})^{\frac{b}{2}})} dx \le& C_b \int_2^{+\infty} x^{\frac{\gamma+1}{2}} e^{-aC_b\langle v\rangle^b\langle x\rangle^b} dx \\ \le& C_b e^{-\frac{a}{2}C_b\langle v\rangle^b} \int_0^{+\infty} x^{\frac{\gamma+1}{2}} e^{-\frac{a}{2}C_b\langle x\rangle^b} dx \le C_{a,b}\langle v\rangle^{-b}. \end{aligned}$$

For the case $x\in[0,2]$, we have $(1+\frac{x^2}{2})^{\frac{b}{2}} - 1 \ge C_b x^2$ for some constant $C_b>0$. Making the change of variable $z = \langle v\rangle^{\frac{b}{2}} x$, since $\frac{\gamma+1}{2} > -1$ we have

$$\begin{aligned} \int_0^{2} x^{\frac{\gamma+1}{2}} e^{a\langle v\rangle^b(1-(1+\frac{x^2}{2})^{\frac{b}{2}})} dx \le& C_b \int_0^{+\infty} x^{\frac{\gamma+1}{2}} e^{-aC_b\langle v\rangle^b x^2} dx \\ \le& C_b\langle v\rangle^{-\frac{b(\gamma+3)}{4}} \int_0^{+\infty} z^{\frac{\gamma+1}{2}} e^{-aC_b|z|^b} dz \le C_{a,b}\langle v\rangle^{-\frac{b(\gamma+3)}{4}}, \end{aligned}$$

so the theorem is thus proved by gathering the terms together. □

We introduce the following lemma about relative entropy which will be used later.

Lemma 2.13. *([34], [16], Lemma 2.7) For any smooth function F satisfies (2) and (3), we have*

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\frac{|F(t,x,v)-\mu(v)|^2}{4\mu(v)}1_{|F(t,x,v-\mu(v)|\le\mu(v)}dvdx$$
$$+\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\frac{1}{4}|F(t,x,v)-\mu(v)|1_{|F(t,x,v)-\mu(v)|\ge\mu(v)}dvdx\le H(F_0),$$

where the relative entropy H is defined in (14).

The following estimate is established for the weight function $w(\alpha,\beta)$ defined in (10).

Lemma 2.14. *Suppose $|\alpha|,|\beta|$ non-negative integers, $k\in\mathbb{R}$, the weight function $w(\alpha,\beta)$ satisfies the following properties*

$$w(|\alpha|,|\beta|)\le w(|\alpha|,|\beta_1|),\quad w(|\alpha|,|\beta|)\le w(|\alpha_1|,|\beta|),\quad \forall 0\le|\alpha_1|<|\alpha|,\quad 0\le|\beta_1|<|\beta|,$$

and

$$w(|\alpha|,|\beta|)\le\langle v\rangle^{\min\{\gamma,0\}}w(|\alpha|+1,|\beta|-1),\quad \forall|\alpha|\ge0,\quad \forall|\beta|\ge1, \tag{27}$$

which implies

$$\langle v\rangle^k\le\min_{|\alpha|+|\beta|\le4}\{w(|\alpha|,|\beta|)\}.$$

We also have

$$\max_{|\alpha|+|\beta|=3}w^2(\alpha,\beta)\le w(0,2)w(1,2), \tag{28}$$

and

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)\le\min\{w(1,2)^{4/5}w(2,2)^{6/5},w(1,2)w(2,2),w(0,3)w(1,3)\}. \tag{29}$$

Proof. The first is just form the fact that $a,b\ge0$. (27) just follows form the fact that

$$k-a|\alpha|-b|\beta|+c\le k-a(|\alpha|+1)-b(|\beta|-1)+c+\min\{\gamma,0\}\quad\Longleftrightarrow\quad b-a+\min\{\gamma,0\}\ge0,$$

and we conclude form the definition of a and b. Then third statement is just by

$$\langle v\rangle^k=w(0,4)=\min_{|\alpha|+|\beta|\le4}\{w(|\alpha|,|\beta|)\}.$$

For equation (28) we easily compute

$$\max_{|\alpha|+|\beta|=3}w^2(\alpha,\beta)=w^2(3,0)=2k-6a+2c\le2k-a-4b+2c=w(0,2)w(1,2).$$

For equation (28) and(29), since $7a=6b$, we easily compute

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)=w^2(4,0)=2k-8a+2c\le2k-\frac{16}{5}a-4b+2c=w(1,2)^{4/5}w(2,2)^{6/5},$$

and

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)=w^2(4,0)=2k-8a+2c=2k-6b-a+2c=w(0,3)w(1,3)\le w(1,2)w(2,2),$$

so the proof of the lemma is thus finished. □

Next we prove a lemma related to the exponential weight case.

Lemma 2.15. *For any $a>0,b\in(0,2),k\ge0$, define $f(x):=a(1+x)^{\frac{b}{2}}-\frac{k}{2}\ln(1+x),x\ge0,f(0)=a$, then the following two statements holds*

$$f(c)\le f(d)+C_{k,a,b},\quad\forall0\le c\le d,\quad f(c+d)\le f(c)+f(d)+C_{k,a,b},\quad\forall c,d\ge0,$$

for some constant $C_{k,a,b}>0$ independent of c,d.

Proof. We easily compute that

$$f'(x) = a\frac{b}{2}(1+x)^{\frac{b}{2}-1} - \frac{k}{2}\frac{1}{1+x}, \quad x \in \mathbb{R}, \quad x \geq 0,$$

we easily we have there exists a constant e which may depend on k, a, b such that

$$f'(x) < 0, \quad \text{if} \quad x \leq e, \quad f'(x) \geq 0, \quad \text{if} \quad x \geq e,$$

so we have

$$\max_{0\leq c\leq d} f(c) - f(d) = f(0) - f(e) \leq C_{k,a,b}, \quad \text{if} \quad e > 0, \quad \max_{0\leq c\leq d} f(c) - f(d) = 0, \quad \text{if} \quad e \leq 0,$$

so the first statement is thus proved. For the second statement since

$$f''(x) = a\frac{b}{2}(\frac{b}{2}-1)(1+x)^{b/2-2} + \frac{k}{2}\frac{1}{(1+x)^2}, \quad x \geq 0,$$

Since $0 < b/2 < 1$, there exists a constant f which may depend on k, a, b such that

$$f''(x) \geq 0, \quad \text{if} \quad x \leq f, \quad f''(x) < 0, \quad \text{if} \quad x \geq f.$$

Thus for any $c, d \geq 0$, we easily compute that

$$f(c+d) - f(c) - f(d) + f(0) = \int_0^d \int_0^c f''(x+s)dsdx \leq \int_0^{\max\{f,0\}} \int_0^{\max\{f,0\}} |f''(x+s)|dsdx \leq C_{k,a,b},$$

so the second statement is thus proved since $f(0) = a$. □

Lemma 2.16. *For any constant $k \in \mathbb{R}, a > 0, b \in (0,2)$ we have*

$$\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}e^{a\langle v'_*\rangle^b}} \leq C_{k,a,b}\frac{\langle v\rangle^k}{\langle v_*\rangle^k\langle v'_*\rangle^k},$$

for some constant $C_{k,a,b} > 0$.

Proof. The case $k \leq 0$ is easy, since $b \in (0,2)$ we easily have

$$\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}e^{a\langle v'_*\rangle^b}} \leq 1 \leq \frac{\langle v\rangle^k}{\langle v_*\rangle^k\langle v'_*\rangle^k}.$$

We focus on the case $k \geq 0$ later. This is equivalent to prove that

$$a\langle v\rangle^b - k\ln\langle v\rangle \leq a\langle v'\rangle^b - k\ln\langle v'\rangle + a\langle v'_*\rangle^b - k\ln\langle v'_*\rangle + C_{k,a,b},$$

for some constant $C_{k,a,b}$ independent of v. By Lemma 2.15 we have

$$\begin{aligned} a\langle v\rangle^b - k\ln\langle v\rangle = f(|v|^2) &\leq f(|v'_*|^2 + |v'|^2) + C_{k,a,b} \\ &\leq f(|v'_*|^2) + f(|v'|^2) + C_{k,a,b} = a\langle v'\rangle^b - k\ln\langle v'\rangle + a\langle v'_*\rangle^b - k\ln\langle v'_*\rangle + C_{k,a,b}, \end{aligned}$$

where f is defined in Lemma 2.15, so the proof is thus finished. □

We also recall some basic interpolation on x, the proof is elementary and thus omitted.

Lemma 2.17. *For any non-negative integer m, n, for any function f, for any constant $k \in \mathbb{R}$ we have*

$$\|fg\|_{H^2_x} \lesssim \min_{m+n=2}\{\|f\|_{H^m_x}\|g\|_{H^n_x}\},$$

and

$$\|f\|_{L^\infty_x L^2_v} \lesssim \|f\|_{H^{8/5}_x L^2_v} \leq \|f\langle v\rangle^{\frac{3}{2}k}\|^{2/5}_{H^1_x L^2_v}\|f\langle v\rangle^{-k}\|^{3/5}_{H^2_x L^2_x} \lesssim \|f\langle v\rangle^{\frac{3}{2}k}\|_{H^1_x L^2_v} + \|f\langle v\rangle^{-k}\|_{H^2_x L^2_v}, \tag{30}$$

also we have

$$\|f\|_{L^3_x L^2_v} \lesssim \|f\|_{H^{1/2}_x L^2_v} \lesssim \|f\langle v\rangle^k\|_{L^2_x} + \|f\langle v\rangle^{-k}\|_{H^1_x}. \tag{31}$$

3. LINEARIZED AND NONLINEAR ESTIMATE FOR THE BOLTZMANN OPERATOR

In this chapter we will prove linearized and nonlinear estimate for the collision operator Q.

Lemma 3.1. *For any* $-3<\gamma\le 1$*, for any* $k\ge 4$*,* h,g *smooth, we have*

$$
\begin{aligned}
|(Q(h,\mu),g\langle v\rangle^{2k})| \le& \|b(\cos\theta)\sin^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}\|_{L^1_\theta}\|h\|_{L^2_{k+\gamma/2,*}}\|g\|_{L^2_{k+\gamma/2,*}}+C_k\|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}}\\
\le& \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|h\|_{L^2_{k+\gamma/2,*}}\|g\|_{L^2_{k+\gamma/2,*}}+C_k\|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}},
\end{aligned}
$$

for some constant $C_k>0$.

Proof. We first compute Q^+, by pre-post collisional change of variables and Lemma 2.7 we have

$$
\begin{aligned}
(Q^+(h,\mu),g\langle v\rangle^{2k}) =& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)h(v_*)\mu(v)g(v')\langle v'\rangle^{2k}dvdv_*d\sigma\\
=& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)\sin^k\frac{\theta}{2}h(v_*)\langle v_*\rangle^k\mu(v)g(v')\langle v'\rangle^k dvdv_*d\sigma\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)h(v_*)\mu(v)g(v')\langle v'\rangle^k R_1 dvdv_*d\sigma\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)h(v_*)\mu(v)g(v')\langle v'\rangle^k R_2 dvdv_*d\sigma:=\Gamma_1+\Gamma_2+\Gamma_3,
\end{aligned}
$$

where

$$
|R_1|\le C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^{k-1}\langle v\rangle,\quad |R_2|\le C_k\langle v\rangle^k.
$$

For the Γ_1 term, by the first inequality of Lemma 2.2

$$
\begin{aligned}
|\Gamma_1|\le& \int_{\mathbb{S}^2}b(\cos\theta)\sin^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|h_*|^2\langle v_*\rangle^{2k}\mu dvdv_*\right)^{1/2}\\
&\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v'|^\gamma\mu|g'|^2\langle v'\rangle^{2k}dvdv'\right)^{1/2}=\int_{\mathbb{S}^2}b(\cos\theta)\sin^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\|h\|_{L^2_{k+\gamma/2,*}}\|g\|_{L^2_{k+\gamma/2,*}}.
\end{aligned}
$$

For the Γ_2 term, since $\frac{3+\gamma}{2}\le 2$, by Lemma 2.2 we have

$$
\begin{aligned}
|\Gamma_2|\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b(\cos\theta)\sin^2\frac{\theta}{2}|v-v_*|^\gamma|h_*|\langle v_*\rangle^{k-1}\mu\langle v\rangle|g'|\langle v'\rangle^k dvdv_*d\sigma\\
\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b(\cos\theta)\sin^2\frac{\theta}{2}|v-v_*|^\gamma|h_*|\langle v_*\rangle^{k-1/2}\mu\langle v\rangle^2|g'|\langle v'\rangle^{k-1/2}dvdv_*d\sigma\\
\lesssim& \int_{\mathbb{S}^2}b(\cos\theta)\sin^{2-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|h_*|^2\langle v_*\rangle^{2k-1}\langle v\rangle^2\mu dvdv_*\right)^{1/2}\\
&\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v'|^\gamma\langle v\rangle^2\mu|g'|^2\langle v'\rangle^{2k-1}dvdv'\right)^{1/2}\lesssim\|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}}.
\end{aligned}
$$

For the Γ_3 term, since $k-1\ge 3$, by Lemma 2.7 we have

$$
\langle v'\rangle^{k-1}\le C_k\sin^2\frac{\theta}{2}\langle v_*\rangle^{k-1}+C_k\langle v\rangle^{k-1},
$$

thus we split Γ_3 into

$$
\begin{aligned}
|\Gamma_3|\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|h_*|\mu\langle v\rangle^k|g'|\langle v'\rangle^k dvdv_*d\sigma\\
\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b(\cos\theta)\sin^2\frac{\theta}{2}|v-v_*|^\gamma|h_*|\langle v_*\rangle^{k-1}\mu\langle v\rangle^k|g'|\langle v'\rangle dvdv_*d\sigma\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b(\cos\theta)|v-v_*|^\gamma|h_*|\mu\langle v\rangle^{2k-1}|g'|\langle v'\rangle dvdv_*d\sigma:=\Gamma_{31}+\Gamma_{32},
\end{aligned}
$$

the Γ_{31} term can be proved the same way as the Γ_2 term. For Γ_{32} term, by the regular change of variables and Lemma 2.8 if $k \ge 4$ we have

$$\begin{aligned}|\Gamma_{32}| &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma |h_*|\mu\langle v\rangle^{2k-1}|g'|\langle v'\rangle dv dv_* d\sigma \\ &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma |h_*|\langle v_*\rangle^{1/2}\mu\langle v\rangle^{2k-1/2}|g'|\langle v'\rangle^{1/2} dv dv_* d\sigma \\ &\lesssim \int_{\mathbb{S}^2} b(\cos\theta)\cos^{-(3+\gamma)}\frac{\theta}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v'-v_*|^\gamma|h_*|\langle v_*\rangle^{1/2}|g'|\langle v'\rangle^{1/2}dv'dv_* \\ &\lesssim \|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}},\end{aligned}$$

the Q^+ term is thus proved. For the Q^- part by Lemma 2.8 we have

$$\begin{aligned}|(Q^-(h,\mu),g\langle\cdot\rangle^{2k})| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)h(v_*)\mu(v)g(v)\langle v\rangle^{2k}dv dv_* d\sigma\right| \\ &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|h(v_*)||g(v)|dv dv_* d\sigma \lesssim \|h\|_{L^2_{k+\gamma/2-1/2}}\|g\|_{L^2_{k+\gamma/2-1/2}},\end{aligned}$$

the theorem is thus proved by combing the Q^+ and Q^- term. ☐

Lemma 3.2. *Suppose that $-3<\gamma\le 1$, then for any smooth function f we have*

$$(Q(\mu,f),f\langle v\rangle^{2k}) \le -\|b(\cos\theta)(1-\cos^{k-\frac{3+\gamma}{2}}\frac{\theta}{2})\|_{L^1_\theta}\|f\|^2_{L^2_{k+\gamma/2,*}} + C_k\|f\|^2_{L^2_{k+\gamma/2-1/2}},$$

for some constant $C_k>0$. In particular if $k\ge 4$ we have

$$(Q(\mu,f),f\langle v\rangle^{2k}) \le -\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{L^2_{k+\gamma/2,*}} + C_k\|f\|^2_{L^2_{k+\gamma/2-1/2}}.$$

Proof. We first compute Q^+, by pre-post collisional change of variables and Lemma 2.7 we have

$$\begin{aligned}(Q^+(\mu,f),f\langle v\rangle^{2k}) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)\mu(v_*)f(v)f(v')\langle v'\rangle^{2k}dv dv_* d\sigma \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)\cos^k\frac{\theta}{2}\mu(v_*)f(v)\langle v\rangle^k f(v')\langle v'\rangle^k dv dv_* d\sigma \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)\mu(v_*)f(v)g(v')\langle v'\rangle^k Q_1 dv dv_* d\sigma := \Gamma_1+\Gamma_2,\end{aligned}$$

where $|Q_1|\le C_k\langle v_*\rangle^k\langle v\rangle^{k-1}$. For the Γ_1 term, by the second inequality of Lemma 2.2 we have

$$|\Gamma_1| \le \int_{\mathbb{S}^2} b(\cos\theta)\cos^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma\mu(v_*)|f|^2\langle v\rangle^{2k}dv dv_*.$$

For the Γ_2 term, we compute

$$\begin{aligned}|\Gamma_2| &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma\mu_*\langle v_*\rangle^k|f|\langle v\rangle^{k-1}|f'|\langle v'\rangle^k dv dv_* d\sigma \\ &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma\mu_*\langle v_*\rangle^{k+1/2}|f|\langle v\rangle^{k-1/2}|f'|\langle v'\rangle^{k-1/2} dv dv_* d\sigma \\ &\lesssim \int_{\mathbb{S}^2} b(\cos\theta)\cos^{-\frac{3+\gamma}{2}}\frac{\theta}{2}d\sigma\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{k+1/2}\mu_*|f|^2\langle v\rangle^{2k-1}dv dv_*\right)^{1/2} \\ &\quad \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v'|^\gamma\langle v_*\rangle^{k+1/2}\mu_*|f'|^2\langle v'\rangle^{2k-1}dv dv'\right)^{1/2} \lesssim \|f\|^2_{L^2_{k+\gamma/2-1/2}}.\end{aligned}$$

It is easily seen that

$$(-Q^-(\mu,f),f\langle v\rangle^{2k}) = -\int_{\mathbb{S}^2} b(\cos\theta)d\sigma\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma\mu(v_*)|f|^2\langle v\rangle^{2k}dv dv_*,$$

so the proof is finished by gathering the three terms. □

Then we come to estimate the nonlinear operator Q^+.

Lemma 3.3. *For any $\gamma \in (-3,1]$, for smooth function f, g, h, recall N is defined in (8), for any $k \geq 4$ we have*

$$|(Q^+(f,g), h\langle v\rangle^{2k})| \leq C_k \min_{m+n=N-2}\{\|f\|_{H^m_{k+\gamma/2}}\|g\|_{H^n_4}\}\|h\|_{L^2_{k+\gamma/2}} + C_k \min_{m_1+n_1=N-2}\{\|g\|_{H^{m_1}_{k+\gamma/2}}\|f\|_{H^{n_1}_4}\}\|h\|_{L^2_{k+\gamma/2}},$$

where m, n, m_1, n_1 are nonnegative integers.

Proof. We first prove the case $-\frac{3}{2} < \gamma \leq 1$, we have

$$\begin{aligned}|(Q^+(f,g), h\langle v\rangle^{2k})| &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)|f(v'_*)||g(v')||h(v)|\langle v\rangle^{2k} dv dv_* d\sigma \\ &\leq \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v\rangle^{2k-6+\gamma}\langle v'\rangle^6\langle v'_*\rangle^6|f(v'_*)|^2|g(v')|^2 dv dv_* d\sigma\right)^{1/2} \\ &\quad \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{2\gamma} b(\cos\theta)\frac{\langle v\rangle^6}{\langle v'\rangle^6\langle v'_*\rangle^6}\langle v\rangle^{2k-\gamma}|h(v)|^2 dv dv_* d\sigma\right)^{1/2},\end{aligned}$$

Since $-3 < 2\gamma \leq 2$, by Lemma 2.11 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{2\gamma} b(\cos\theta)\frac{\langle v\rangle^6}{\langle v'\rangle^6\langle v'_*\rangle^6}\langle v\rangle^{2k-\gamma}|h(v)|^2 dv dv_* d\sigma \leq C_k\|h\|^2_{L^2_{k+\gamma/2}}.$$

Since $2k-6+\gamma \geq 0$, by pre-post collisional change of variables we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v\rangle^{2k-6+\gamma}\langle v'\rangle^6\langle v'_*\rangle^6|f(v'_*)|^2|g(v')|^2 dv dv_* d\sigma \\ =&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v'\rangle^{2k-6+\gamma}\langle v\rangle^6\langle v_*\rangle^6|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \\ \lesssim&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v\rangle^{2k+\gamma}\langle v_*\rangle^6|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v\rangle^6\langle v_*\rangle^{2k+\gamma}|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \\ \lesssim&\|f\|^2_{L^2_3}\|g\|^2_{L^2_{k+\gamma/2}} + \|g\|^2_{L^2_3}\|f\|^2_{L^2_{k+\gamma/2}},\end{aligned}$$

so the case $\gamma \in (-\frac{3}{2}, 1]$ is proved. For the case $\gamma \in (-\frac{5}{2}, -\frac{3}{2}]$, we have $-3 < \gamma - \frac{1}{2} < 0, -2 < \gamma + \frac{1}{2} < 0$, hence

$$\begin{aligned}|(Q^+(f,g), h\langle v\rangle^{2k})| &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)|f(v'_*)||g(v')||h(v)|\langle v\rangle^{2k} dv dv_* d\sigma \\ &\leq \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{\gamma+\frac{1}{2}} b(\cos\theta)\langle v\rangle^{2k-\frac{9}{2}}\langle v'\rangle^4\langle v'_*\rangle^4|f(v'_*)|^2|g(v')|^2 dv dv_* d\sigma\right)^{1/2} \\ &\quad \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{\gamma-\frac{1}{2}} b(\cos\theta)\frac{\langle v\rangle^4}{\langle v'\rangle^4\langle v'_*\rangle^4}\langle v\rangle^{2k+\frac{1}{2}}|h(v)|^2 dv dv_* d\sigma\right)^{1/2},\end{aligned}$$

still by Lemma 2.11 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{\gamma-\frac{1}{2}} b(\cos\theta)\frac{\langle v\rangle^4}{\langle v'\rangle^4\langle v'_*\rangle^4}\langle v\rangle^{2k+\frac{1}{2}}|h(v)|^2 dv dv_* d\sigma \leq C_k\|h\|^2_{L^2_{k+\gamma/2}},$$

similarly we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma+\frac{1}{2}}b(\cos\theta)\langle v'\rangle^{2k-\frac{9}{2}}\langle v\rangle^4\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2dvdv_*d\sigma\\ \lesssim&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma+\frac{1}{2}}b(\cos\theta)\langle v\rangle^{2k-\frac{1}{2}}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2dvdv_*d\sigma\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma+\frac{1}{2}}b(\cos\theta)\langle v\rangle^4\langle v_*\rangle^{2k-\frac{1}{2}}|f(v_*)|^2|g(v)|^2dvdv_*d\sigma.:=I_1+I_2.\end{aligned}$$

Since $\gamma+\frac{1}{2}\in(-2,0)$ by Lemma 2.6 we have

$$I_1\lesssim\min_{m+n=1}\{\|f\|^2_{H^m_4}\|g\|^2_{H^n_{k+\gamma/2}}\},\quad I_2\lesssim\min_{m+n=1}\{\|f\|^2_{H^m_{k+\gamma/2}}\|g\|^2_{H^n_4}\},$$

so the case $\gamma\in(-\frac{5}{2},-\frac{3}{2}]$ is proved. For the case $\gamma\in(-3,0)$ we have

$$\begin{aligned}|(Q^+(f,g),h\langle v\rangle^{2k})|\le&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)|f(v_*')||g(v')||h(v)|\langle v\rangle^{2k}dvdv_*d\sigma\\ \le&\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^{2k-4}\langle v'\rangle^4\langle v_*'\rangle^4|f(v_*')|^2|g(v')|^2dvdv_*d\sigma\right)^{1/2}\\ &\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\frac{\langle v\rangle^4}{\langle v'\rangle^4\langle v_*'\rangle^4}\langle v\rangle^{2k}|h(v)|^2dvdv_*d\sigma\right)^{1/2},\end{aligned}$$

still by Lemma 2.11 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\frac{\langle v\rangle^4}{\langle v'\rangle^4\langle v_*'\rangle^4}\langle v\rangle^{2k}|h(v)|^2dvdv_*d\sigma\le C_k\|h\|^2_{L^2_{k+\gamma/2}},$$

and by pre-post collisional change of variables we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v'\rangle^{2k-4}\langle v\rangle^4\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2dvdv_*d\sigma\\ \lesssim&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^{2k}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2dvdv_*d\sigma\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^4\langle v_*\rangle^{2k}|f(v_*)|^2|g(v)|^2dvdv_*d\sigma:=I_1+I_2.\end{aligned}$$

Since $\gamma\in(-3,-\frac{5}{2}]$ by Lemma 2.5 we have

$$I_1\lesssim\min_{m+n=2}\{\|f\|^2_{H^m_4}\|g\|^2_{H^n_{k+\gamma/2}}\},\quad I_2\lesssim\min_{m+n=2}\{\|f\|^2_{H^m_{k+\gamma/2}}\|g\|^2_{H^n_4}\},$$

thus the theorem is proved by gathering all the terms together. □

The estimate for the Q^- term is similar.

Lemma 3.4. *For any $\gamma\in(-3,1]$, for smooth function f,g,h, recall N is defined in (8), for any $k\ge4$ we have*

$$|(Q^-(f,g),h\langle v\rangle^{2k})|\le C_k\min_{m+n=N-2}\{\|f\|_{H^m_3}\|g\|_{H^n_{k+\gamma/2}}\}\|h\|_{L^2_{k+\gamma/2}},$$

where m,n are nonnegative integers.

Proof. The proof is similar to the Q^+ case, first for the case $\gamma\in(-\frac{3}{2},1]$ we have

$$\begin{aligned}|(Q^-(f,g),h\langle v\rangle^{2k})|\le&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)|f(v_*)||g(v)||h(v)|\langle v\rangle^{2k}dvdv_*d\sigma\\ \le&\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b(\cos\theta)\langle v\rangle^{2k+\gamma}\langle v_*\rangle^6|f(v_*)|^2|g(v)|^2dvdv_*d\sigma\right)^{1/2}\\ &\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{2\gamma}b(\cos\theta)\frac{1}{\langle v_*\rangle^6}\langle v\rangle^{2k-\gamma}|h(v)|^2dvdv_*d\sigma\right)^{1/2},\end{aligned}$$

it is easily seen that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\langle v\rangle^{2k+\gamma}\langle v_*\rangle^6|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \le C_k\|f\|_{L^2_3}^2\|g\|_{L^2_{k+\gamma/2}}^2,$$

since $2\gamma\in(-3,2]$ we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{2\gamma}b(\cos\theta)\frac{1}{\langle v_*\rangle^6}\langle v\rangle^{2k-\gamma}|h(v)|^2 dv dv_* d\sigma \le C_k\|h\|_{L^2_{k+\gamma/2}}^2,$$

so the case $\gamma\in(-\frac{3}{2},1]$ is thus proved. For the case $\gamma\in(-\frac{5}{2},-\frac{3}{2}]$, we have $-3<\gamma-\frac{1}{2}<0, -2<\gamma+\frac{1}{2}<0$, hence

$$|(Q^-(f,g),h\langle v\rangle^{2k})| \le \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma+\frac{1}{2}}b(\cos\theta)\langle v\rangle^{2k-\frac{1}{2}}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma-\frac{1}{2}}b(\cos\theta)\frac{1}{\langle v_*\rangle^4}\langle v\rangle^{2k+\frac{1}{2}}|h(v)|^2 dv dv_* d\sigma\right)^{1/2},$$

since $-3<\gamma-\frac{1}{2}<0$, it is easily seen that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma-\frac{1}{2}}b(\cos\theta)\frac{1}{\langle v_*\rangle^4}\langle v\rangle^{2k+\frac{1}{2}}|h(v)|^2 dv dv_* d\sigma \le C_k\|h\|_{L^2_{k+\gamma/2}}^2,$$

since $-2<\gamma-\frac{1}{2}<0$, by Lemma 2.6 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma+\frac{1}{2}}b(\cos\theta)\langle v\rangle^{2k-\frac{1}{2}}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \lesssim \min_{m+n=1}\{\|f\|_{H^m_4}^2\|g\|_{H^n_{k+\gamma/2}}^2\},$$

so the case $\gamma\in(-\frac{5}{2},-\frac{3}{2}]$ is proved. For the case $\gamma\in(-3,-\frac{5}{2}]$ we have

$$|(Q^-(f,g),h\langle v\rangle^{2k})| \le \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^{2k}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\frac{1}{\langle v_*\rangle^4}\langle v\rangle^{2k}|h(v)|^2 dv dv_* d\sigma\right)^{1/2},$$

it is easily seen that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\frac{1}{\langle v_*\rangle^4}\langle v\rangle^{2k}|h(v)|^2 dv dv_* d\sigma \le C_k\|h\|_{L^2_{k+\gamma/2}}^2,$$

and by Lemma 2.6

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^{2k}\langle v_*\rangle^4|f(v_*)|^2|g(v)|^2 dv dv_* d\sigma \lesssim \min_{m+n=2}\{\|f\|_{H^m_4}^2\|g\|_{H^n_{k+\gamma/2}}^2\},$$

the lemma is proved by gathering all the terms together. □

Corollary 3.5. *For any $\gamma\in(-3,1]$, for smooth function f,g,h, for any $k\ge 4$ we have*

$$|(Q(f,g),h\langle v\rangle^{2k})| \le C_k\min_{m+n=N-2}\{\|f\|_{H^m_{k+\gamma/2}}\|g\|_{H^n_4}+\|g\|_{H^m_{k+\gamma/2}}\|f\|_{H^n_4}\}\|h\|_{L^2_{k+\gamma/2}},$$

with m,n nonnegative integers.

Proof. By taking $m=m_1, n=n_1$ in Lemma 3.3 and Lemma 3.4 the proof is finished. □

Corollary 3.6. *For any $-3<\gamma\le 1$, for any $|\beta|\le 4$, for any f,g smooth we have*

$$|(Q(f,\partial_\beta\mu),g\langle v\rangle^{2k})|+|(Q(\partial_\beta\mu,f),g\langle v\rangle^{2k})| \le C_k\|f\|_{L^2_{k+\gamma/2}}\|g\|_{L^2_{k+\gamma/2}},$$

for some constant $C_k>0$.

Proof. By taking $m = n_1 = 0$ in Lemma 3.3 and Lemma 3.4 we have

$$|(Q(f,\partial_\beta\mu),g\langle v\rangle^{2k})|+|(Q(\partial_\beta\mu,f),g\langle v\rangle^{2k})|$$
$$\le C_k\|f\|_{L^2_{k+\gamma/2}}\|\partial_\beta\mu\|_{H^n_5}\|g\|_{L^2_{k+\gamma/2}}+C_k\|\partial_\beta\mu\|_{H^{m_1}_{k+\gamma/2}}\|f\|_{L^2_5}\|g\|_{L^2_{k+\gamma/2}}\le C_k\|f\|_{L^2_{k+\gamma/2}}\|g\|_{L^2_{k+\gamma/2}},$$

so the proof is ended. □

The estimate for the exponential weight case is similar.

Lemma 3.7. *For any $\gamma\in(-3,1]$, for smooth function f,g,h , for any $k\in\mathbb{R}$, $a>0$, $b\in(0,2)$ we have*

$$|(Q(f,g),h\langle v\rangle^{2k}e^{2a\langle v\rangle^b})|\le C_{k,a,b}\min_{m+n=N-2}\{\|f\|_{H^m_{k+\gamma/2,a,b}}\|g\|_{H^n_{k-96,a,b}}+\|g\|_{H^m_{k+\gamma/2,a,b}}\|f\|_{H^n_{k-96,a,b}}\}\|h\|_{L^2_{k+\gamma/2,a,b}},$$

with m,n nonnegative integers.

Proof. For the case $\gamma\in(-\frac{3}{2},1]$, by Lemma 2.16 we have

$$\langle v\rangle^{k-100}e^{a\langle v\rangle^b}\le C_{k,a,b}\langle v'_*\rangle^{k-100}e^{a\langle v'_*\rangle^b}\langle v'\rangle^{k-100}e^{a\langle v'\rangle^b},$$

which implies

$$|(Q^+(f,g),he^{2a\langle v\rangle^b}\langle v\rangle^{2k})$$
$$\le\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v'_*)||g(v')||h(v)|\langle v\rangle^{2k}e^{2a\langle v\rangle^b}dvdv_*d\sigma$$
$$\le C_{k,a,b}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v'_*)|\langle v'_*\rangle^{k-100}e^{a\langle v'_*\rangle^b}|g(v')|\langle v'\rangle^{k-100}e^{a\langle v'\rangle^b}|h(v)|\langle v\rangle^{k+100}e^{a\langle v\rangle^b}dvdv_*d\sigma$$
$$=(Q^+(f\langle v\rangle^{k-100}e^{a\langle v\rangle^b},g\langle v\rangle^{k-100}e^{a\langle v\rangle^b}),(h\langle v\rangle^{k-100}e^{a\langle v\rangle^b})\langle v\rangle^{200}),$$

taking $k=100$ in Lemma 3.3 the Q^+ part is proved. Similarly for the Q^- term we have

$$(Q^-(f,g),he^{2\alpha\langle v\rangle^\beta}\langle v\rangle^{2k})=(Q^-(f,g\langle v\rangle^{k-100}e^{\alpha\langle v\rangle^\beta}),(h\langle v\rangle^{k-100}e^{\alpha\langle v\rangle^\beta})\langle v\rangle^{200}),$$

we conclude by taking $k=100$ in Corollary 3.5. □

Corollary 3.8. *For any $-3<\gamma\le1$, for any $|\beta|\le4$, for any f,g smooth we have*

$$|(Q(f,\partial_\beta\mu),g\langle v\rangle^{2k}e^{a\langle v\rangle^b})|+|(Q(\partial_\beta\mu,f),g\langle v\rangle^{2k}e^{\alpha\langle v\rangle^\beta})|\le C_{k,a,b}\|f\|_{L^2_{k+\gamma/2,a,b}}\|g\|_{L^2_{k+\gamma/2,a,b}},$$

for some constant $C_{k,a,b}>0$.

Proof. The proof is similar as Lemma 3.6 thus omitted. □

Next we come to prove the linearized part for the exponential weight case.

Lemma 3.9. *For any $-3<\gamma\le1$, for any $k\in\mathbb{R}$, $a>0$, $b\in(0,2)$ and f smooth, we have*

$$|(Q^+(\mu,f),f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})|+|(Q^+(f,\mu),f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})|\le C_{k,a,b}\|f\|_{L^2_{k+\gamma/2,a,b}}\|f\|_{L^2_{k+\gamma/2-b(\gamma+3)/4,a,b}},$$

for some constant $C_k>0$.

Proof. For the first term, by Lemma 2.16 we have

$$\langle v\rangle^k e^{\frac{1}{2}a\langle v\rangle^b}\lesssim\langle v'_*\rangle^k e^{\frac{1}{2}a\langle v'_*\rangle^b}\langle v'\rangle^k e^{\frac{1}{2}a\langle v'\rangle^b},$$

together with Lemma 2.12 we have

$$
\begin{aligned}
&|(Q^+(f,\mu), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})|\\
\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)|\mu(v_*')||f(v')||f(v)|\langle v\rangle^{2k}e^{2a\langle v\rangle^b} dv dv_* d\sigma\\
\lesssim& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)e^{-\frac{1}{2}|v_*'|^2}\langle v_*'\rangle^k e^{\frac{1}{2}a\langle v_*'\rangle^b}|f(v')|\langle v'\rangle^k e^{\frac{1}{2}a\langle v'\rangle^b}|f(v)|\langle v\rangle^k e^{\frac{3}{2}a\langle v\rangle^b} dv dv_* d\sigma\\
\lesssim& \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)e^{-\frac{1}{2}|v_*'|^2}e^{a\langle v_*'\rangle^b}\langle v_*'\rangle^{2k}e^{2a\langle v'\rangle^b}\langle v'\rangle^{2k}|f(v')|^2 dv dv_* d\sigma\right)^{1/2}\\
&\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)\frac{e^{a\langle v\rangle^b}}{e^{a\langle v'\rangle^b}}e^{-\frac{1}{2}|v_*'|^2}e^{2a\langle v\rangle^b}\langle v\rangle^{2k}|f(v)|^2 dv dv_* d\sigma\right)^{1/2}\\
\lesssim& C_{k,a,b}\|f\|_{L^2_{k+\gamma/2,a,b}}\|f\|_{L^2_{k+\gamma/2-b(\gamma+3)/4,a,b}},
\end{aligned}
$$

the second term can be proved by changing v' and v_*', so the proof is thus finished. □

Lemma 3.10. *For any $-3<\gamma\le 1$, for any $k\ge 0, a>0, b\in(0,2)$, f smooth, we have*

$$
-(Q^-(\mu,f), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b}) - (Q^-(f,\mu), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b}) \le -C_1\|f\|^2_{L^2_{k+\gamma/2,a,b}} + C_{k,a,b}\|f\|^2_{L^2_3},
$$

for some constants $C_1, C_{k,a,b}>0$.

Proof. It is esaily seen that

$$
(-Q^-(\mu,f), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b}) = -\int_{\mathbb{S}^2} b(\cos\theta)d\sigma\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma\mu(v_*)|f|^2\langle v\rangle^{2k}e^{2a\langle v\rangle^b}dvdv_*,
$$

and by Lemma 2.8 we have

$$
\begin{aligned}
|(Q^-(f,\mu), f\langle\cdot\rangle^{2k}e^{2a\langle v\rangle^b})| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)f(v_*)\mu(v)f(v)\langle v\rangle^{2k}e^{2a\langle v\rangle^b}dvdv_*d\sigma\right|\\
&\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*)||f(v)|dvdv_*d\sigma \lesssim \|f\|^2_{L^2_3}.
\end{aligned}
$$

Gathering the two terms, the lemma is thus proved. □

Corollary 3.11. *For any $-3<\gamma\le 1$, for any $k\in\mathbb{R}, a>0, b\in(0,2)$ and f smooth, we have*

$$
(Q(\mu,f), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b}) + (Q(f,\mu), f\langle v\rangle^{2k}e^{2a\langle v\rangle^b}) \le -C_1\|f\|^2_{L^2_{k+\gamma/2,a,b}} + C_{k,a,b}\|f\|^2_{L^2},
$$

for some constants $C_1, C_{k,a,b}>0$.

Lemma 3.12. *For any $-3<\gamma\le 1$, f smooth, define*

$$
\bar{L}f = -v\cdot\nabla_x f + Q(\mu,f) + Q(f,\mu),
$$

then for any $k>4$ we have

$$
(\bar{L}f, f\langle v\rangle^{2k})_{L^2_{x,v}} \le -C\|f\|^2_{L^2_x L^2_{k+\gamma/2}} + C_k\|f\|^2_{L^2_x L^2_v}, \tag{32}
$$

for some constants $C, C_k>0$. For any $k\in\mathbb{R}, a>0, b\in(0,2)$ we have

$$
(\bar{L}f, f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})_{L^2_{x,v}} \le -C_1\|f\|^2_{L^2_x L^2_{k+\gamma/2,a,b}} + C_{k,a,b}\|f\|^2_{L^2_x L^2_v}, \tag{33}
$$

for some constants $C_1, C_{k,a,b}>0$. As a consequence for the solution to the inhomogeneous Boltzmann equation

$$
\partial_t f = \bar{L}f = -v\cdot\nabla_x f + Q(f,\mu) + Q(\mu,f), \quad f|_{t=0} = f_0.
$$

If $\gamma\in[0,1]$, for any $k>4$ we have

$$\|f-Pf\|_{L^2_xL^2_k}\lesssim e^{-\lambda t}\|f_0-Pf_0\|_{L^2_xL^2_k},$$

for some constant $\lambda>0$. If $\gamma\in(-3,0)$, for any $4<k_0<k$ we have

$$\|f-Pf\|_{L^2_xL^2_{k_0}}\lesssim\langle t\rangle^{-\frac{k-k_*}{|\gamma|}}\|f_0-Pf_0\|_{L^2_xL^2_k},\quad\forall k_*\in(k_0,k).$$

For the exponential weight, if $\gamma\in[0,1]$, for any $k\in\mathbb{R}, 0<a,b\in(0,2)$ we have

$$\|f-Pf\|_{L^2_xL^2_{k,a,b}}\lesssim e^{-\lambda t}\|f_0-Pf_0\|_{L^2_xL^2_{k,a,b}},$$

for some constant $\lambda>0$. If $\gamma\in(-3,0)$, for any $k\in\mathbb{R}, 0<a_0<a,b\in(0,2)$ we have

$$\|f-Pf\|_{L^2_xL^2_{k,a_0,b}}\lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}}\|f_0-Pf_0\|_{L^2_xL^2_{k,a,b}},$$

for some constant $\lambda>0$.

Proof. The exponential case (33) just follows from Corollary 3.11 above. We only prove the polynomial case (32). By Lemma 3.1 and 3.2 we have

$$(Lf,f\langle v\rangle^{2k})=-\|b(\cos\theta)(\sin^2\frac{\theta}{2}-\sin^{k-2}\frac{\theta}{2})\|_{L^1_\theta}\|f\|_{L^2_{k+\gamma/2,*}}+C_k\|f\|^2_{L^2_{k+\gamma/2-1/2}},$$

if $k>4$, then

$$\sin^2\frac{\theta}{2}-\sin^{k-2}\frac{\theta}{2}\geq\sin^2\frac{\theta}{2}(1-\sin^{k-4}\frac{\theta}{2})>0,$$

so the polynomial case follows by interpolation. Then we come to prove the convergence rate. By combing the results in [47, 46, 38, 22] we have

$$\|(I-P)S_L(t)f\|_{L^2_xL^2(\mu^{-1/2})}\lesssim e^{-\lambda t}\|(I-P)S_L(t)f\|_{L^2_xL^2(\mu^{-1/2})},\quad\text{if}\quad\gamma\geq0,$$

and

$$\|(I-P)S_L(t)f\|_{L^2_xL^2(\mu^{-1/2})}\lesssim e^{-\lambda t^{\frac{2}{2-\gamma}}}\|(I-P)S_L(t)f\|_{L^2_xL^2(\mu^{-3/4})},\quad\text{if}\quad-3<\gamma<0.$$

Define two operators $L=A+B$ with $A=M\chi_{R_1}$ and $B=L-M\chi_{R_1}$ for some $M,R_1>0$ large, where χ_{R_1} is the truncation function in ball with center zero and radius $R_1>0$. Denote S_L and S_B semigroups generated by L and B respectively. Then if M,R_1 is large we have

$$(Bf,f)_{L^2_xL^2_k}\lesssim-\|f\|_{L^2_xL^2_{k+\gamma/2}},\quad\|Af\|_{L^2_xL^2(\mu^{-3/4})}\lesssim\|f\|_{L^2_xL^2_v}.$$

For the case $\gamma\geq0$, for the polynomial case we have

$$\|S_B(t)f\|_{L^2_xL^2_k}\lesssim e^{-\lambda t}\|f\|_{L^2_xL^2_k}.$$

By Duhamel's principle we have

$$\|(I-P)S_L(t)\|_{L^2_xL^2_k\to L^2_xL^2_k}\lesssim\int_0^t\|(I-P)S_L(t-s)\|_{L^2_xL^2(\mu^{-1/2})\to L^2_xL^2(\mu^{-1/2})}\|A\|_{L^2_xL^2_k\to L^2_xL^2(\mu^{-1/2})}\|S_B(s)\|_{L^2_xL^2_k\to L^2_xL^2_k}ds$$
$$+\|(I-P)S_B(t)\|_{L^2_xL^2_k\to L^2_xL^2_k}\lesssim e^{-\lambda t},$$

and for the exponential weight case the proof for $\gamma\geq0$ is the same. For the case $\gamma<0$, we first prove the polynomial case, on one hand, we have

$$\|S_B(t)f\|_{L^2_xL^2_k}\lesssim\|f\|_{L^2_xL^2_k}.$$

On the other hand, for any $k_0<k$, for any $R>0$ we have

$$\frac{d}{dt}\|S_B(t)f\|^2_{L^2_xL^2_{k_0}}\leq-c\|S_B(t)f\|^2_{L^2_xL^2_{k_0+\gamma/2}}\leq-c\langle R\rangle^\gamma\|S_B(t)f\|^2_{L^2_xL^2_{k_0}}+C\langle R\rangle^{-2(k-k_0)+\gamma}\|S_B(t)f\|^2_{L^2_xL^2_k},$$

where we use the following interpolation

$$\langle R\rangle^{\gamma}\|f\|^2_{L^2_xL^2_{k_0}} \le \|f\|^2_{L^2_xL^2_{k_0+\gamma/2}} + \langle R\rangle^{-2(k-k_0)+\gamma}\|f\|^2_{L^2_xL^2_k}.$$

Integrating the differential inequality, we obtain

$$\begin{aligned}\|S_B(t)f\|^2_{L^2_xL^2_{k_0}} &\lesssim e^{-c\langle R\rangle^{\gamma}t}\|f\|^2_{L^2_xL^2_{k_0}} + \langle R\rangle^{-2(k-k_0)}\|f\|^2_{L^2_xL^2_k} \lesssim \inf_{R>0}(e^{-c\langle R\rangle^{\gamma}t} + \langle R\rangle^{-2(k-k_0)})\|f\|^2_{L^2_xL^2_k}\\ &\lesssim \langle t\rangle^{-\frac{2(k-k_*)}{|\gamma|}}\|f\|^2_{L^2_xL^2_k}, \quad \forall k_* \in (k_0,k),\end{aligned}$$

where we choose $\langle R\rangle = \langle t\rangle^{-1/\gamma}[\log(1+t)^{-\frac{2}{c}(k-k_0)}]^{1/\gamma}$. Moreover, thanks to Duhamel's formula

$$\begin{aligned}\|(I-P)S_L(t)\|_{L^2_xL^2_k\to L^2_xL^2_{k_0}} \lesssim& \|(I-P)S_B(t)\|_{L^2_xL^2_k\to L^2_xL^2_{k_0}}\\ &+\int_0^t \|(I-P)S_L(t-s)\|_{L^2_xL^2(\mu^{-3/4})\to L^2_xL^2(\mu^{-1/2})}\|A\|_{L^2_xL^2_k\to L^2_xL^2(\mu^{-3/4})}\|S_B(s)\|_{L^2_xL^2_k\to L^2_xL^2_{k_0}}\,ds\\ \lesssim& \langle t\rangle^{-\frac{k-k_*}{|\gamma|}}, \quad \forall k_*\in(k_0,k),\end{aligned}$$

so the proof for the polynomial case is thus finished. For the exponential weight case, if M, R_1 is large we have

$$(Bf,f)_{L^2_xL^2_{k,a,b}} \le -c\|f\|^2_{L^2_xL^2_{k+\gamma/2,a,b}}, \quad \|Af\|_{L^2_xL^2(\mu^{-3/4})} \lesssim \|f\|_{L^2_xL^2_v}.$$

So first we have

$$\|S_B(t)f\|_{L^2_xL^2_{k,a,b}} \lesssim \|f\|_{L^2_xL^2_{k,a,b}}.$$

On the other hand, for any $a_0 < a$, for any constant $R > 0$ we have

$$\frac{d}{dt}\|S_B(t)f\|^2_{L^2_xL^2_{k,a_0,b}} \le -c\|S_B(t)f\|^2_{L^2_xL^2_{k+\gamma/2,a_0,b}} \le -c\langle R\rangle^{\gamma}\|S_B(t)f\|^2_{L^2_xL^2_{k,a_0,b}} + \langle R\rangle^{\gamma}e^{-(a-a_0)\langle R\rangle^b}\|S_B(t)f\|^2_{L^2_xL^2_{k,a,b}},$$

where we use the following interpolation

$$\langle R\rangle^{\gamma}\|f\|^2_{L^2_xL^2_{k,a_0,b}} \le \|f\|^2_{L^2_xL^2_{k+\gamma/2,a_0,b}} + \langle R\rangle^{\gamma}e^{-(a-a_0)\langle R\rangle^b}\|f\|^2_{L^2_xL^2_{k,a,b}},$$

we can deduce

$$\begin{aligned}\|S_B(t)f\|^2_{L^2_xL^2_{k,a,b}} &\lesssim e^{-c\langle R\rangle^{\gamma}t}\|f\|^2_{L^2_xL^2_{k,a,b}} + e^{-(a-a_0)\langle R\rangle^b}\|f\|^2_{L^2_xL^2_{k,a,b}}\\ &\lesssim \inf_{R>0}(e^{-c\langle R\rangle^{\gamma}t} + e^{-(a-a_0)\langle R\rangle^b})\|f\|^2_{L^2_xL^2_{k,a,b}} \lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}}\|f\|^2_{L^2_xL^2_{k,a,b}},\end{aligned}$$

for some constant $\lambda > 0$, where we choose $\langle R\rangle = t^{\frac{1}{b-\gamma}}$. Using Duhamel's formula again we have

$$\begin{aligned}&\|(I-P)S_L(t)\|_{L^2_xL^2_{k,a,b}\to L^2_xL^2_{k,a_0,b}}\\ \lesssim&\int_0^t \|(I-P)S_L(t-s)\|_{L^2_xL^2(\mu^{-3/4})\to L^2_xL^2(\mu^{-1/2})}\|A\|_{L^2_xL^2_{k,a_0,b}\to L^2_xL^2(\mu^{-3/4})}\|S_B(s)\|_{L^2_xL^2_{k,a,b}\to L^2_xL^2_{k,a_0,b}}\,ds\\ &+\|(I-P)S_B(t)\|_{L^2_xL^2_{k,a,b}\to L^2_xL^2_{k,a_0,b}} \lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}},\end{aligned}$$

so the proof is thus finished. ∎

4. ESTIMATES FOR THE INHOMOGENEOUS EQUATION

In this section we prove the estimates for the inhomogeneous Boltzmann equation. Recall N is defined in (8), $w(\alpha,\beta)$ is defined in (10), $X_k, X_{k,a,b}$ is defined in (9) and (11), and $Y_k := X_{k+\gamma/2}$, $Y_{k,a,b} := X_{k+\gamma/2,a,b}$. We first prove an estimate for the nonlinear term.

Lemma 4.1. *Suppose f,g,h smooth function. For the polynomial weight case, for any $k \ge 4$ large, if $\gamma \in (-\frac{3}{2}, 1]$, then for any indices $|\alpha| \le 2$, we have*

$$|(\partial^\alpha Q(f,g), \partial^\alpha h \langle v\rangle^{2k})_{L^2_{x,v}}| \le C_k \|f\|_{H^2_x L^2_4} \|g\|_{Y_k} \|h\|_{Y_k} + C_k \|g\|_{H^2_x L^2_4} \|f\|_{Y_k} \|h\|_{Y_k},$$

for some constant $C_k > 0$. If $\gamma \in (-3, -\frac{3}{2}]$, for any indices $|\alpha| + |\beta| \le N$ we have

$$|(\partial^\alpha_\beta Q(f,g), \partial^\alpha_\beta h w^2(\alpha,\beta))_{L^2_{x,v}}| \le C_k \|f\langle v\rangle^4\|_{H^N_{x,v}} \|g\|_{Y_k} \|h\|_{Y_k} + C_k \|g\langle v\rangle^4\|_{H^N_{x,v}} \|f\|_{Y_k} \|h\|_{Y_k},$$

for some constant $C_k > 0$. For the exponential weight case, for any $k \in \mathbb{R}$, $a > 0$, $b \in (0,2)$, if $\gamma \in (-\frac{3}{2}, 1]$, then for any indices $|\alpha| \le 2$, we have

$$|(\partial^\alpha Q(f,g), \partial^\alpha h \langle v\rangle^{2k} e^{2a\langle v\rangle^b})_{L^2_{x,v}}| \le C_k \|f\|_{X_{k,a,b}} \|g\|_{Y_{k,a,b}} \|h\|_{Y_{k,a,b}} + C_k \|g\|_{X_{k,a,b}} \|f\|_{Y_{k,a,b}} \|h\|_{Y_{k,a,b}},$$

for some constant $C_{k,a,b} > 0$. If $\gamma \in (-3, -\frac{3}{2}]$ for any indices $|\alpha| + |\beta| \le N$ we still have

$$|(\partial^\alpha_\beta Q(f,g), \partial^\alpha_\beta h w^2(\alpha,\beta) e^{2a\langle v\rangle^b})_{L^2_{x,v}}| \le C_{k,a,b} \|f\|_{X_{k,a,b}} \|g\|_{Y_{k,a,b}} \|h\|_{Y_{k,a,b}} + C_{k,a,b} \|g\|_{X_{k,a,b}} \|f\|_{Y_{k,a,b}} \|h\|_{Y_{k,a,b}},$$

for some constant $C_{k,a,b} > 0$.

Proof. We first prove the polynomial case, for the case $\gamma \in (-\frac{3}{2}, 1]$, by

$$(\partial^\alpha Q(f,g), \partial^\alpha h \langle v\rangle^{2k})_{L^2_{x,v}} = \sum_{\alpha_1 \le \alpha} (Q(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} g), \partial^\alpha h \langle v\rangle^{2k})_{L^2_{x,v}},$$

together with Lemma 3.5 we have

$$\begin{aligned}
&(Q(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} g), \partial^\alpha h \langle v\rangle^{2k})_{L^2_{x,v}} \\
\le& C_k \int_{\mathbb{T}^3} \|\partial^{\alpha-\alpha_1} f\|_{L^2_4} \|\partial^{\alpha_1} g\|_{L^2_{k+\gamma/2}} \|\partial^\alpha h\|_{L^2_{k+\gamma/2}} + \|\partial^{\alpha_1} g\|_{L^2_4} \|\partial^{\alpha-\alpha_1} f\|_{L^2_{k+\gamma/2}} \|\partial^\alpha h\|_{L^2_{k+\gamma/2}} dx.
\end{aligned}$$

By symmetry we only estimate the first term, we easily compute

$$\begin{aligned}
\int_{\mathbb{T}^3} \|\partial^{\alpha-\alpha_1} f\|_{L^2_4} \|\partial^{\alpha_1} g\|_{L^2_{k+\gamma/2}} \|\partial^\alpha h\|_{L^2_{k+\gamma/2}} dx \lesssim& \|\partial^{\alpha-\alpha_1} f\|_{H^{2-|\alpha-\alpha_1|}_x L^2_4} \|\partial^{\alpha_1} g\|_{H^{|\alpha-\alpha_1|}_x L^2_{k+\gamma/2}} \|\partial^\alpha h\|_{L^2_x L^2_{k+\gamma/2}} \\
\lesssim& \|f\|_{H^2_x L^2_4} \|g\|_{H^2_x L^2_{k+\gamma/2}} \|h\|_{H^2_x L^2_{k+\gamma/2}},
\end{aligned}$$

so the case $\gamma \in (-\frac{3}{2}, 1]$ is proved. We then prove the case $(-3, -\frac{3}{2}]$, we only prove the case $\gamma \in (-3, -\frac{5}{2}]$, the case $\gamma \in (-\frac{5}{2}, -\frac{3}{2}]$ can be proved similarly. First we have

$$(\partial^\alpha_\beta Q(f,g), \partial^\alpha h w^2(\alpha,\beta))_{L^2_{x,v}} = \sum_{\alpha_1 \le \alpha, \beta_1 \le \beta} (Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g), \partial^\alpha_\beta h w^2(\alpha,\beta))_{L^2_{x,v}},$$

together with Lemma 3.5 we have

$$\begin{aligned}
&(Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g), \partial^\alpha_\beta h w^2(\alpha,\beta))_{L^2_{x,v}} \\
\le& C_k \int_{\mathbb{T}^3} \min_{m+n=2} \{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f\|_{H^m_4} \|\partial^{\alpha_1}_{\beta_1} g w(\alpha,\beta)\|_{H^n_{\gamma/2}}\} \|\partial^\alpha_\beta h w(\alpha,\beta)\|_{L^2_{\gamma/2}} dx \\
&+ C_k \int_{\mathbb{T}^3} \min_{m+n=2} \{\|\partial^{\alpha_1}_{\beta_1} g\|_{H^n_4} \|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f w(\alpha,\beta)\|_{H^m_{\gamma/2}}\} \|\partial^\alpha_\beta h w(\alpha,\beta)\|_{L^2_{\gamma/2}} dx.
\end{aligned}$$

By symmetry we only need to prove that

$$\int_{\mathbb{T}^3} \min_{m+n=2} \{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} f\|_{H^m_4} \|\partial^{\alpha_1}_{\beta_1} g w(\alpha,\beta)\|_{H^n_{\gamma/2}}\} \|\partial^\alpha_\beta h w(\alpha,\beta)\|_{L^2_{\gamma/2}} dx \lesssim \|\langle v\rangle^4 f\|_{H^4_{x,v}} \|g\|_{Y_k} \|h\|_{Y_k}, \quad \forall \alpha_1 \le \alpha, \beta_1 \le \beta.$$

First we split it into three cases, $|\alpha-\alpha_1|+|\beta-\beta_1|\le 2$, $|\alpha-\alpha_1|+|\beta-\beta_1|=3$, $|\alpha-\alpha_1|+|\beta-\beta_1|=4$. For the case $|\alpha-\alpha_1|+|\beta-\beta_1|\le 2$, take $m=2, n=0$ we have

$$\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^m_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{\gamma/2}}\}\|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_{\gamma/2}}dx$$
$$\le\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^c_xH^2_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^d_xL^2_{\gamma/2}}\|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}},$$

with c,d nonnegative integers satisfying $c+d=2$. Taking

$$c=2-|\beta-\beta_1|-|\alpha-\alpha_1|,\quad d=|\beta-\beta_1|+|\alpha-\alpha_1|,$$

such that

$$|\beta-\beta_1|+|\alpha-\alpha_1|+2+c=4,\quad |\alpha_1|+|\beta_1|+d=|\alpha|+|\beta|,$$

so first we have

$$\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^2_xH^2_4}\le\|\langle v\rangle^4 f\|_{H^4_{x,v}},\quad \|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}\le\|h\|_{Y_k}.$$

Using the fact that

$$|\alpha_1|+|\beta_1|=|\alpha|+|\beta|\quad \&\quad |\beta_1|\le|\beta|\quad\Rightarrow\quad w(\alpha,\beta)\le w(\alpha_1,\beta_1),$$

we deduce

$$\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^d_xL^2_{\gamma/2}}=\sum_{|d|\le|\beta-\beta_1|+|\alpha-\alpha_1|}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^d_xL^2_{\gamma/2}}\le\|g\|_{Y_k}.$$

For the case $|\alpha-\alpha_1|+|\beta-\beta_1|=3$, by $\|fg\|_{L^2_x}\le\|f\|_{L^6_x}\|g\|_{L^3_x}$ and taking $m=0, n=2$ we have

$$\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^m_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{\gamma/2}}\}\|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_{\gamma/2}}dx$$
$$\le\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{L^6_xL^2_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{L^3_xH^2_{\gamma/2}}\|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}$$
$$\le\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^1_xL^2_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{L^3_xH^2_{\gamma/2}}\|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}.$$

Since $|\alpha-\alpha_1|+|\beta-\beta_1|=3$, we have

$$\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^1_xH^a_4}\le\|\langle v\rangle^4 f\|_{H^4_{x,v}},\quad \|\partial^\alpha_\beta hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}\le\|h\|_{Y_k}.$$

For the g term, since $|\alpha|+|\beta|\ge 3$, we split it into two cases $|\alpha|+|\beta|=3$ and $|\alpha|+|\beta|=4$. For the case $|\alpha-\alpha_1|+|\beta-\beta_1|=|\alpha|+|\beta|=3$ we have $|\alpha_1|=|\beta_1|=0$, by (28)

$$\max_{|\alpha|+|\beta|=3}w^2(\alpha,\beta)\le w(0,2)w(1,2),$$

together with (31) we have

$$\|gw(\alpha,\beta)\|_{L^3_xH^2_{\gamma/2}}\le\|gw(0,2)\|_{L^2_xH^2_{\gamma/2}}+\|gw(1,2)\|_{H^1_xH^2_{\gamma/2}}\lesssim\|g\|_{Y_k}.$$

For the case $|\alpha-\alpha_1|+|\beta-\beta_1|=3, |\alpha|+|\beta|=4$, this time we have $|\alpha_1|=1, |\beta_1|=0$ or $|\alpha_1|=0, |\beta_1|=1$. For the first case by (29)

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)\le w(1,2)w(2,2),$$

together with (31) we have

$$\|\partial^{\alpha_1}gw(\alpha,\beta)\|_{L^3_xH^2_{\gamma/2}}\lesssim\|gw(1,2)\|_{H^1_xH^2_{\gamma/2}}+\|gw(2,2)\|_{H^2_xH^2_{\gamma/2}}\lesssim\|g\|_{Y_k}.$$

For the second case by (29)

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)\le w(0,3)w(1,3),$$

together with (31) we deduce

$$\|\partial_{\beta_1}gw(\alpha,\beta)\|_{L^3_xH^2_{\gamma/2}}\lesssim\|gw(0,3)\|_{L^2_xH^3_{\gamma/2}}+\|gw(1,3)\|_{H^1_xH^3_{\gamma/2}}\lesssim\|g\|_{Y_k}.$$

Finally for $|\alpha-\alpha_1|+|\beta-\beta_1|=4$, we have $|\alpha_1|=|\beta_1|=0$, taking $m=0,n=2$ we have

$$\begin{aligned}&\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{H^m_4}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{\gamma/2}}\}\|\partial^{\alpha}_{\beta}hw(\alpha,\beta)\|_{L^2_{\gamma/2}}dx\\ \le&\|\partial^{\alpha}_{\beta}f\|_{L^2_xL^2_4}\|gw(\alpha,\beta)\|_{L^\infty_xH^2_{\gamma/2}}\|\partial^{\alpha}_{\beta}hw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}\\ \le&\|\langle v\rangle^4f\|_{H^4_{x,v}}\|gw(\alpha,\beta)\|_{L^\infty_xH^2_{\gamma/2}}\|h\|_{Y_k},\end{aligned}$$

by (29)

$$\max_{|\alpha|+|\beta|=4}w^2(\alpha,\beta)\le w(1,2)^{4/5}w(2,2)^{6/5},$$

together with (30) we deduce

$$\|gw(\alpha,\beta)\|_{L^\infty_xH^2_{\gamma/2}}\lesssim\|gw(1,2)\|_{H^1_xH^2_{\gamma/2}}+\|gw(2,2)\|_{H^2_xH^2_{\gamma/2}}\lesssim\|g\|_{Y_k},$$

so the polynomial case is thus proved by gathering all the case. For the exponential weight case, for $\gamma\in(-\frac{3}{2},1]$ by Lemma 3.7 we have

$$\begin{aligned}&(Q(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}g),\partial^{\alpha}h\langle v\rangle^{2k}e^{2a\langle v\rangle^b})_{L^2_{x,v}}\\ \le&C_{k,a,b}\int_{\mathbb{T}^3}\|\partial^{\alpha-\alpha_1}f\|_{L^2_{k,a,b}}\|\partial^{\alpha_1}g\|_{L^2_{k+\gamma/2,a,b}}\|\partial^{\alpha}h\|_{L^2_{k+\gamma/2,a,b}}+\|\partial^{\alpha_1}g\|_{L^2_{k,a,b}}\|\partial^{\alpha-\alpha_1}f\|_{L^2_{k+\gamma/2,a,b}}\|\partial^{\alpha}h\|_{L^2_{k+\gamma/2,a,b}}dx.\end{aligned}$$

For the first term we have

$$\begin{aligned}&\int_{\mathbb{T}^3}\|\partial^{\alpha-\alpha_1}f\|_{L^2_{k,a,b}}\|\partial^{\alpha_1}g\|_{L^2_{k+\gamma/2,a,b}}\|\partial^{\alpha}h\|_{L^2_{k+\gamma/2,a,b}}dx\\ \lesssim&\|\partial^{\alpha-\alpha_1}f\|_{H^{2-|\alpha-\alpha_1|}_xL^2_{k,a,b}}\|\partial^{\alpha_1}g\|_{H^{|\alpha-\alpha_1|}_xL^2_{k+\gamma/2,a,b}}\|\partial^{\alpha}h\|_{L^2_xL^2_{k+\gamma/2,a,b}}\\ \lesssim&\|f\|_{H^2_xL^2_{k,a,b}}\|g\|_{H^2_xL^2_{k+\gamma/2,a,b}}\|h\|_{H^2_xL^2_{k+\gamma/2,a,b}}\lesssim\|f\|_{X_{k,a,b}}\|g\|_{Y_{k,a,b}}\|h\|_{Y_{k,a,b}},\end{aligned}$$

and the second term follows by symmetry. For the case $\gamma\in(-3,-\frac{3}{2}]$, for simplicity we only prove the case $\gamma\in(-3,-\frac{5}{2}]$, by Lemma 3.7 we have

$$\begin{aligned}&(Q(\partial^{\alpha_1}_{\beta_1}f,\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g),\partial^{\alpha}_{\beta}hw^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}\\ \le&C_{k,a,b}\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}fw(\alpha,\beta)\|_{H^m_{-96,a,b}}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{\gamma/2,a,b}}\}\|\partial^{\alpha}_{\beta}hw(\alpha,\beta)\|_{L^2_{\gamma/2,a,b}}dx\\ &+C_{k,a,b}\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{-96,a,b}}\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}fw(\alpha,\beta)\|_{H^m_{\gamma/2,a,b}}\}\|\partial^{\alpha}_{\beta}hw(\alpha,\beta)\|_{L^2_{\gamma/2,a,b}}dx.\end{aligned}$$

By symmetry we only need to prove that

$$\begin{aligned}&\int_{\mathbb{T}^3}\min_{m+n=2}\{\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}fw(\alpha,\beta)\|_{H^m_{-96,a,b}}\|\partial^{\alpha_1}_{\beta_1}gw(\alpha,\beta)\|_{H^n_{\gamma/2,a,b}}\}\|\partial^{\alpha}_{\beta}hw(\alpha,\beta)\|_{L^2_{\gamma/2.a,b}}dx\\ \lesssim&\|f\|_{X_{k,a,b}}\|g\|_{Y_{k,a,b}}\|h\|_{Y_{k,a,b}},\quad\forall\alpha_1\le\alpha,\beta_1\le\beta.\end{aligned}$$

By (10) we have

$$w(\alpha,\beta)\langle v\rangle^{-96}\le w(\alpha_2,\beta_2),\quad\forall|\alpha|+|\beta|\le4,\quad\forall|\alpha_2|+|\beta_2|\le4,$$

so for any nonnegative integer m,n satisfies $m+n+|\alpha-\alpha_1|+|\beta-\beta_1|\le4$ we have

$$\|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}fw(\alpha,\beta)\|_{H^m_xH^n_{-96,a,b}}\le\|f\|_{X_{k,a,b}},$$

the remaining proof is the same as the polynomial case thus omitted, so the proof is thus finished. □

Then we come to prove estimate for the linearized part, we first prove the polynomial case.

Lemma 4.2. *For any smooth function f, if $\gamma \in (-\frac{3}{2}, 1]$, for any indices $|\alpha| \le 2$ we have*

$$|(\partial^\alpha Q(f,\mu), \partial^\alpha g \langle v\rangle^{2k})_{L^2_{x,v}}| \le \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha f\|_{L^2_x L^2_{k+\gamma/2,*}}\|\partial^\alpha g\|_{L^2_x L^2_{k+\gamma/2,*}} + C_k\|\partial^\alpha f\|_{L^2_x L^2_{k+\gamma/2-1/2}}\|\partial^\alpha g\|_{L^2_x L^2_{k+\gamma/2-1/2}},$$

for some constant $C_k > 0$. For the case $\gamma \in (-3, -\frac{3}{2}]$, for any indices $|\alpha| + |\beta| \le N$ we have

$$\begin{aligned}|(\partial^\alpha_\beta Q(f,\mu), \partial^\alpha_\beta g w^2(\alpha,\beta))_{L^2_{x,v}}| \le& \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\\ &+ C_k\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\\ &+ C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{L^2_x L^2_{\gamma/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}}.\end{aligned}$$

Proof. Since $\partial^\alpha\mu = 0$ if $|\alpha| > 0$, the case $\gamma \in (-\frac{3}{2}, 1]$ is just Lemma 3.1 and thus omitted. For the case $\gamma \in (-3, -\frac{3}{2}]$, by

$$(\partial^\alpha_\beta Q(f,\mu), \partial^\alpha_\beta g w^2(\alpha,\beta))_{L^2_{x,v}} = \sum_{\beta_1\le\beta}(Q(\partial^\alpha_{\beta_1} f, \partial_{\beta-\beta_1}\mu), \partial^\alpha_\beta g w^2(\alpha,\beta))_{L^2_{x,v}},$$

we split it into two cases. For the case $\beta_1 = \beta$, by Lemma 3.1 we have

$$\begin{aligned}|(Q(\partial^\alpha_\beta f,\mu), \partial^\alpha_\beta g w^2(\alpha,\beta))_{L^2_{x,v}}| \le& \|b(\cos\theta)\sin^{k-2}\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2,*}}\\ &+ C_k\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2-1/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{k+\gamma/2-1/2}}.\end{aligned}$$

For the case $\beta_1 < \beta$, by Corollary 3.6 we have

$$|(Q(\partial^\alpha_{\beta_1} f, \partial_{\beta-\beta_1}\mu), \partial^\alpha_\beta g w^2(\alpha,\beta))_{L^2_{x,v}}| \le C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}},$$

so the proof is thus finished since $w(\alpha,\beta) \le w(\alpha,\beta_1)$ if $|\beta_1| \le |\beta|$. ☐

Lemma 4.3. *For any smooth function f, for any $k \ge 4$. If $\gamma \in (-\frac{3}{2}, 1]$, for any indices $|\alpha| \le 2$ we have*

$$(\partial^\alpha Q(\mu, f), \partial^\alpha f\langle v\rangle^{2k})_{L^2_{x,v}} \le -\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha f\|^2_{L^2_x L^2_{k+\gamma/2,*}} + C_k\|\partial^\alpha f\|^2_{L^2_x L^2_{k+\gamma/2-1/2}},$$

for some constant $C_k \ge 0$. If $\gamma \in (-3, -\frac{3}{2}]$, for any indices $|\alpha| + |\beta| \le N$ we have

$$\begin{aligned}(\partial^\alpha_\beta Q(\mu, f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}} \le& -\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} + C_k\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2-1/2}}\\ &+ C_k\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{L^2_x L^2_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_x L^2_{\gamma/2}},\end{aligned}$$

for some constant $C_k \ge 0$.

Proof. Since $\partial^\alpha\mu = 0$ if $|\alpha| > 0$, the case $\gamma \in (-\frac{3}{2}, 1]$ is just Lemma 3.2 and thus omitted. For the case $\gamma \in (-3, -\frac{3}{2}]$, first we have

$$(\partial^\alpha_\beta Q(\mu, f), \partial^\alpha f w^2(\alpha,\beta))_{L^2_{x,v}} = \sum_{\beta_1\le\beta}(Q(\partial_{\beta-\beta_1}\mu, \partial^\alpha_{\beta_1} f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}},$$

we split it into two cases. For the case $\beta_1 = \beta$, by Lemma 3.2 we have

$$(Q(\mu, \partial^\alpha_\beta f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}} \le -\|b(\cos\theta)\sin^2\frac{\theta}{2}\|_{L^1_\theta}\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2,*}} + C_k\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_x L^2_{\gamma/2-1/2}}.$$

For the case $|\beta_1| < |\beta|$, by Corollary 3.6 we have

$$|(Q(\partial_{\beta-\beta_1}\mu, \partial^\alpha_{\beta_1} f), \partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}| \le C_k\|\partial^\alpha_{\beta_1} f w(\alpha,\beta)\|_{L^2_{\gamma/2}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_{\gamma/2}},$$

so the lemma is thus proved since $w(\alpha,\beta) \le w(\alpha,\beta_1)$ if $|\beta_1| \le |\beta|$. ☐

Then we prove the estimate for the exponential weight case.

Lemma 4.4. *For any smooth function f, for any $k\in\mathbb{R}, a>0, b\in(0,2)$. If $\gamma\in(-\frac{3}{2},1]$, for any indices $|\alpha|\le 2$ we have*

$$(\partial^\alpha Q(f,\mu),\partial^\alpha f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})_{L^2_{x,v}}+(\partial^\alpha Q(\mu,f),\partial^\alpha f\langle v\rangle^{2k}e^{2a\langle v\rangle^b})_{L^2_{x,v}}\le -C_1\|\partial^\alpha f\|^2_{L^2_xL^2_{k+\gamma/2,a,b}}+C_{k,a,b}\|\partial^\alpha f\|^2_{L^2_{x,v}},$$

for some constants $C_1, C_{k,a,b}>0$. If $\gamma\in(-3,-\frac{3}{2}]$, for any indices $|\alpha|+|\beta|\le N$ we have

$$\begin{aligned}&(\partial^\alpha_\beta Q(f,\mu),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}+(\partial^\alpha_\beta Q(\mu,f),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}\\ \le&-C_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xL^2_{\gamma/2,a,b}}+C_{k,a,b}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}}+C_{k,a,b}\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1} f w(\alpha,\beta_1)\|_{L^2_xL^2_{\gamma/2,a,b}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2,a,b}},\end{aligned}$$

for some constants $C_1, C_{k,a,b}>0$.

Proof. Since $\partial^\alpha\mu=0$ if $|\alpha|>0$, the case $\gamma\in(-\frac{3}{2},1]$ is just Corollary 3.11 and thus omitted. For the case $\gamma\in(-3,-\frac{3}{2}]$, first we have

$$\begin{aligned}&(\partial^\alpha_\beta Q(f,\mu),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}+(\partial^\alpha_\beta Q(\mu,f),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}\\ =&\sum_{\beta_1\le\beta}(Q(\partial^\alpha_{\beta_1}f,\partial_{\beta-\beta_1}\mu),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}+(Q(\partial_{\beta-\beta_1}\mu,\partial^\alpha_{\beta_1}f),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}},\end{aligned}$$

we split it into two cases. For the case $\beta_1=\beta$, by Corollary 3.11 we have

$$\begin{aligned}&(Q(\partial^\alpha_\beta f,\mu),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}+(Q(\mu,\partial^\alpha_\beta f),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}\\ \le&-C_1\|\partial^\alpha_\beta f w(\alpha,\beta)\|^2_{L^2_xL^2_{\gamma/2,a,b}}+C_{k,a,b}\|\partial^\alpha_\beta f\|^2_{L^2_{x,v}}.\end{aligned}$$

For the case $\beta_1<\beta$, by Corollary 3.8 we have

$$\begin{aligned}&|(Q(\partial^\alpha_{\beta_1}f,\partial_{\beta-\beta_1}\mu),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}|+|(Q(\partial_{\beta-\beta_1}\mu,\partial^\alpha_{\beta_1}f),\partial^\alpha_\beta f w^2(\alpha,\beta)e^{2a\langle v\rangle^b})_{L^2_{x,v}}|\\ \le&C_{k,a,b}\sum_{\beta_1<\beta}\|\partial^\alpha_{\beta_1}f w(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2,a,b}}\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2,a,b}},\end{aligned}$$

so the lemma is thus proved since $w(\alpha,\beta)\le w(\alpha,\beta_1)$ if $\beta_1\le\beta$. □

For the transport term $v\cdot\nabla_x f$ we need the following estimate.

Lemma 4.5. *Suppose $|\beta|>0$, for any smooth function f we have*

$$(\partial^\alpha_\beta(v\cdot\nabla_x f),\partial^\alpha_\beta f w^2(\alpha,\beta))_{L^2_{x,v}}\le C(\sum_{|\beta_2|=|\beta|-1,|\alpha_2|=|\alpha|+1}\|\partial^{\alpha_2}_{\beta_2}f w(\alpha_2,\beta_2)\|_{L^2_xL^2_{\gamma/2}})\|\partial^\alpha_\beta f w(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}.$$

Proof. By (27) and

$$\partial_{v_i}(v\cdot\nabla_x f)=\partial_{x_i}f+v\cdot\nabla_x\partial_{v_i}f,\quad\forall i=1,2,3,$$

the theorem is thus proved. □

Recall the definition of $\bar{X}_0$. $\bar{X}_0:=H^2_xL^2_v$ if $\gamma\in(-\frac{3}{2},1]$, $\bar{X}_0=H^N_{x,v}$, if $\gamma\in(-3,-\frac{3}{2}]$, and $Y_{k,*}$ is defined in (13). Gathering the estimates above, we obtian following estimate.

Lemma 4.6. *For any smooth function f,g,h smooth, for the polynomial case, for any $k>4$ large, for the nonlinear term we have*

$$|(Q(f,g),h)_{X_k}|\lesssim(\|f\|_{X_4}\|g\|_{Y_k}+\|f\|_{Y_k}\|g\|_{X_4})\|h\|_{Y_k}.\tag{34}$$

For the linearized term we have

$$|(Q(f,\mu),g)_{X_k}|\le\|b(\cos\theta)\sin^{k-2-\frac{k-4}{3}}\frac{\theta}{2}\|_{L^1_\theta}\|f\|_{Y_{k,*}}\|g\|_{Y_{k,*}}+C_k\|f\|_{Y_{k-1/2}}\|g\|_{Y_{k-1/2}},\tag{35}$$

and

$$(Q(\mu,f),f)_{X_k} \le -\|b(\cos\theta)\sin^{2+\frac{k-4}{3}}\frac{\theta}{2}\|_{L^1_\theta}\|f\|^2_{Y_{k,*}} + C_k\|f\|^2_{Y_{k-1/2}}. \tag{36}$$

In particular gathering the two terms we have

$$(Q(\mu,f),f)_{X_k} + (Q(f,\mu),f)_{X_k} \le -\|b(\cos\theta)\sin^2\frac{\theta}{2}(\sin^{\frac{k-4}{3}}\frac{\theta}{2} - \sin^{\frac{2(k-4)}{3}}\frac{\theta}{2})\|_{L^1_\theta}\|f\|^2_{Y_{k,*}} + C_k\|f\|^2_{Y_{k-1/2}}. \tag{37}$$

For the exponential weight case, for the nonlinear term we have

$$|(Q(f,g),h)_{X_{k,a,b}}| \lesssim (\|f\|_{X_{k,a,b}}\|g\|_{Y_{k,a,b}} + \|f\|_{Y_{k,a,b}}\|g\|_{X_{k,a,b}})\|h\|_{Y_{k,a,b}}, \tag{38}$$

for the linearized term we have

$$(Q(\mu,f),f)_{X_{k,a,b}} + (Q(f,\mu),f)_{X_{k,a,b}} \le -C_2\|f\|^2_{Y_{k,a,b}} + C_k\|f\|^2_{\bar{X}_0}. \tag{39}$$

For the $v\cdot\nabla_x f$ term, for the polynomial case we have

$$|(-v\cdot\nabla_x f,f)_{X_k}| \le \frac{1}{4}\|b(\cos\theta)\sin^2\frac{\theta}{2}(\sin^{\frac{k-4}{3}}\frac{\theta}{2} - \sin^{\frac{2(k-4)}{3}}\frac{\theta}{2})\|_{L^1_\theta}\|f\|^2_{Y_{k,*}}, \tag{40}$$

for the exponential weight case we have

$$|(-v\cdot\nabla_x f,f)_{X_{k,a,b}}| \le \frac{C_2}{4}\|f\|^2_{Y_{k,a,b}}. \tag{41}$$

Proof. (34) and (38) can be proved by summing on $|\alpha|+|\beta|\le N$ ($|\alpha|\le 2$ if $\gamma\in(-\frac{3}{2},1]$) in Lemma 4.1. For any $\eta>0$ we have

$$\begin{aligned} &C^2_{|\alpha|,|\beta|}C_k\sum_{|\beta_1|<|\beta|}\|\partial^\alpha_{\beta_1}fw(\alpha,\beta_1)\|_{L^2_{\gamma/2}}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{L^2_{\gamma/2}}\\ \le &\sum_{|\beta_1|<|\beta|}\frac{C_kC_{|\alpha|,|\beta|}}{\eta}\|\partial^\alpha_{\beta_1}fw(\alpha,\beta_1)\|_{L^2_{\gamma/2}}\eta C_{|\alpha|,|\beta|}\|\partial^\alpha_\beta g w(\alpha,\beta)\|_{\gamma/2}. \end{aligned}$$

By (12) we have $C_{|\alpha|,|\beta|}\ll C_{|\alpha|,|\beta_1|}$, so (35) follows from summing on $|\alpha|+|\beta|\le N$ ($|\alpha|\le 2$ if $\gamma\in(-\frac{3}{2},1]$) in Lemma 4.2 and taking suitable small constants $\eta>0$ such that

$$\eta \ll \|b(\cos\theta)(\sin^{k-2-\frac{k-4}{3}}\frac{\theta}{2} - \sin^{k-2}\frac{\theta}{2})\|_{L^1_\theta}, \quad \frac{C_kC_{|\alpha|,|\beta|}}{\eta} \ll C_{|\alpha|,|\beta_1|},$$

the estimate (36) and (39) can be proved similarly. For the $v\cdot\nabla_x f$ term, if $\gamma\in(-\frac{3}{2},1]$, it is easily seen that

$$(v\cdot\nabla_x\partial^\alpha f,\partial^\alpha f\langle v\rangle^{2k})_{L^2_{x,v}} = 0.$$

For the case $\gamma\in(-3,-\frac{3}{2}]$ we have

$$\begin{aligned} &C^2_{|\alpha|,|\beta|}C_k(\sum_{|\beta_2|=|\beta|-1,|\alpha_2|=|\alpha|+1}\|\partial^{\alpha_2}_{\beta_2}fw(\alpha_2,\beta_2)\|_{L^2_xL^2_{\gamma/2}})\|\partial^\alpha_\beta fw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}\\ \le &(\frac{C_{|\alpha|,|\beta|}C_k}{\eta}\sum_{|\beta_2|=|\beta|-1,|\alpha_2|=|\alpha|+1}\|\partial^{\alpha_2}_{\beta_2}fw(\alpha_2,\beta_2)\|_{L^2_xL^2_{\gamma/2}})\eta C_{|\alpha|,|\beta|}\|\partial^\alpha_\beta fw(\alpha,\beta)\|_{L^2_xL^2_{\gamma/2}}. \end{aligned}$$

By (12) we have $C_{|\alpha|,|\beta|}\ll C_{|\alpha|+1,|\beta|-1}$, so (40) follows by summing on $|\alpha|+|\beta|\le N$ in Lemma 4.5 and taking suitable η such that

$$\frac{C_{|\alpha|,|\beta|}C_k}{\eta} \ll C_{|\alpha|+1,|\beta|-1}, \quad \eta \ll \frac{1}{2}\|b(\cos\theta)\sin^2\frac{\theta}{2}(\sin^{\frac{k-4}{3}}\frac{\theta}{2} - \sin^{\frac{2(k-4)}{3}}\frac{\theta}{2})\|_{L^1_\theta}.$$

And (41) can be proved similarly. □

Taking $g=f$ in Lemma 4.6 we can easily obtain the following estimate.

Corollary 4.7. *Suppose that $-3 < \gamma \le 1$, f smooth. For the polynomial case, for any $k > 4$ large, there exists constants $c_0, C_k > 0$ such that*

$$(Q(\mu+f,\mu+f),f)_{X_k} \le -2c_0\|f\|_{Y_k}^2 + C_k\|f\|_{Y_{k-1/2}}^2 + C_k\|f\|_{\bar{X}_4}\|f\|_{Y_k}^2$$
$$\le -c_0\|f\|_{Y_k}^2 + C_k\|f\|_{\bar{X}_0}^2 + C_k\|f\|_{X_4}\|f\|_{Y_k}^2.$$

For the exponential weight case, for any $k \in \mathbb{R}$, $a > 0$, $b \in (0,2)$ we have

$$(Q(\mu+f,\mu+f),f)_{X_{k,a,b}} \le -c_0\|f\|_{Y_{k,a,b}}^2 + C_{k,a,b}\|f\|_{\bar{X}_0}^2 + C_{k,a,b}\|f\|_{X_{k,a,b}}\|f\|_{Y_{k,a,b}}^2,$$

for some constants $c_0, C_{k,a,b} > 0$.

Corollary 4.8. *Suppose $\gamma \in (-3,1]$. For any smooth function f, suppose f is the solution of*

$$\partial_t f = \bar{L}f := -v\cdot\nabla_x f + Q(f,\mu) + Q(\mu,f), \quad f|_{t=0} = f_0.$$

If $\gamma \in [0,1]$, for any $k > 4$ we have

$$\|f - Pf\|_{X_k} \lesssim e^{-\lambda t}\|f_0 - Pf_0\|_{X_k},$$

for some constant $\lambda > 0$. If $\gamma \in (-3,0)$, for any $4 < k_0 < k$ we have

$$\|f - Pf\|_{X_{k_0}} \lesssim \langle t\rangle^{-\frac{k-k_*}{|\gamma|}}\|f_0 - Pf_0\|_{X_k}, \quad \forall k_* \in (k_0,k).$$

For the exponential weight, if $\gamma \in [0,1]$, for any $k \in \mathbb{R}, 0 < a, b \in (0,2)$ we have

$$\|f - Pf\|_{X_{k,a,b}} \lesssim e^{-\lambda t}\|f_0 - Pf_0\|_{X_{k,a,b}},$$

for some constant $\lambda > 0$. If $\gamma \in (-3,0)$, for any $k \in \mathbb{R}, 0 < a_0 < a, b \in (0,2)$ we have

$$\|f - Pf\|_{X_{k,a_0,b}} \lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}}\|f_0 - Pf_0\|_{X_{k,a,b}},$$

for some constant $\lambda > 0$.

Proof. The proof is similar as Lemma 3.12 thus omitted. □

Corollary 4.9. *Denote $Z_k = X_{k-\gamma/2}$, then we have*

$$\|Q(f,g)\|_{Z_k} \lesssim \|f\|_{X_4}\|g\|_{Y_k} + \|f\|_{Y_k}\|g\|_{X_4}.$$

Proof. It's easily seen that Z_k is the dual of Y_k with respect to X_k, so the corollary follows by (34) in Lemma 4.6. □

5. Global existence and convergence

The proof of local existence is standard once we have established estimates in Lemma 4.6, we refer to [7] for example.

Theorem 5.1. *(Local existence) For any $k > 4$, there exists $\epsilon_0, \epsilon_1, T > 0$ such that if $f_0 \in X_k$ and*

$$\|f_0\|_{X_k} \le \epsilon_0, \quad \mu + f_0 \ge 0,$$

then the Cauchy problem

$$\partial_t f + v\cdot\nabla_x f = Q(\mu+f,\mu+f), \quad f|_{t=0} = f_0(x,v),$$

admits a unique weak solution $f \in L^\infty([0,T];X_k)$ satisfying

$$\|f\|_{L^\infty([0,T];X_k)} \le \epsilon_1, \quad \mu + f \ge 0, \quad \|f\|_{L^2([0,T];Y_k)} \le \epsilon_1.$$

Theorem 5.2. *Recall* $\bar{L}f = -v\cdot\nabla_x f + Q(f,\mu) + Q(\mu,f)$*. For any function* f *satisfies* $Pf = 0$*, for any* $k \ge 6$*, define the norm* $|||f|||_{X_k}$ *and the associate scalar product* $((f,g))_{X_k}$ *by*

$$|||f|||_{X_k}^2 = \eta\|f\|_{X_k}^2 + \int_0^\infty \|S_{\bar{L}}(\tau)f\|_{\bar{X}_0}^2 d\tau, \quad ((f,g))_{X_k} = \eta(f,g)_{X_k} + \int_0^\infty (S_{\bar{L}}(\tau)f, S_{\bar{L}}(\tau)g)_{\bar{X}_0} d\tau.$$

Similarly for any $k \in \mathbb{R}$*,* $a > 0$*,* $b \in (0,2)$*, define the norm* $|||f|||_{X_{k,a,b}}$ *and the associate scalar product* $((f,g))_{X_{k,a,b}}$ *by*

$$|||f|||_{X_{k,a,b}}^2 = \eta\|f\|_{X_{k,a,b}}^2 + \int_0^\infty \|S_{\bar{L}}(\tau)f\|_{\bar{X}_0}^2 d\tau, \quad ((f,g))_{X_{k,a,b}} = \eta(f,g)_{X_{k,a,b}} + \int_0^\infty (S_{\bar{L}}(\tau)f, S_{\bar{L}}(\tau)g)_{\bar{X}_0} d\tau.$$

Then there exists some $\eta > 0$*, such that the norm* $|||\cdot|||_{X_k}$ $(|||\cdot|||_{X_{k,a,b}})$ *is equivalent to* $\|\cdot\|_{X_k}$ $(\|\cdot\|_{X_{k,a,b}})$ *on the space* $\{f \in X_k | Pf = 0\}$ $(\{f \in X_{k,a,b} | Pf = 0\})$*. Moreover there exists some constants* $C, K > 0$ *such that any smooth solution to the following equation*

$$\partial_t f = \bar{L}f + Q(f,f), \quad f(0) = f_0, \quad Pf_0 = 0, \tag{42}$$

satisfies for the polynomial case

$$\frac{d}{dt}|||f|||_{X_k}^2 \le (C|||f|||_{X_6} - K)\|f\|_{Y_k}^2, \tag{43}$$

and for the exponential weight case

$$\frac{d}{dt}|||f|||_{X_{k,a,b}}^2 \le (C|||f|||_{X_{k,a,b}} - K)\|f\|_{Y_{k,a,b}}^2. \tag{44}$$

As a consequence, if $|||f_0|||_{X_6} \le \frac{K}{2C}$*, then there exists a global solution* $f \in L^\infty([0,\infty), X_6), \mu + f \ge 0$ *to the Boltzmann equation* (42)*. Moreover for any* $k \ge 6$*, if we assume* $\|f_0\|_{X_k} < +\infty$*, then for the case* $\gamma \in [0,1]$ *we have*

$$|||f|||_{X_k} \lesssim e^{-\lambda t}|||f|||_{X_k},$$

for some constant $\lambda > 0$*. For the case* $\gamma \in (-3,0)$*, for any* $6 \le k_1 < k$ *we have*

$$|||f|||_{X_{k_1}} \lesssim \langle t\rangle^{\frac{k-k_*}{|\gamma|}}|||f|||_{X_k}, \quad \forall k_* \in (k_1,k).$$

For the exponential weight case if we assume $|||f_0|||_{X_{k,a,b}} \le \frac{K}{2C}$*, then there exists a global solution* $f \in L^\infty([0,\infty), X_{k,a,b})$*,* $\mu + f \ge 0$ *to the Boltzmann equation* (42)*. Moreover for the case* $\gamma \in [0,1]$

$$|||f|||_{X_{k,a,b}} \lesssim e^{-\lambda t}|||f|||_{X_{k,a,b}},$$

for some constant $\lambda > 0$*. For the case* $\gamma \in (-3,0)$*, for any* $0 < a_0 < a$ *we have*

$$|||f|||_{X_{k,a_0,b}} \lesssim e^{-\lambda t^{\frac{b}{b-\gamma}}}|||f|||_{X_{k,a,b}},$$

for some constant $\lambda > 0$*.*

Proof. During the proof, we will denote $X = X_k$, $Y = Y_k$ for the polynomial weight case and $X = X_{k,a,b}$, $Y = Y_{k,a,b}$ for the exponential weight case. Since $k \ge 6$ for the polynomial weight case, by Corollary 4.8, for both cases we have

$$\|S_{\bar{L}}(\tau)f\|_{\bar{X}_0} \le \theta(\tau)\|f\|_X, \quad \lim_{\tau\to\infty}\theta(\tau) = 0, \quad \int_0^\infty \theta^2(\tau)d\tau < +\infty,$$

for some function $\theta(\tau)$, which implies

$$\int_0^\infty \|S_{\bar{L}}(\tau)f\|_{\bar{X}_0}^2 d\tau \lesssim \|f\|_X^2 \int_0^\infty \theta^2(\tau)d\tau,$$

the equivalence between two norms is thus proved. Then we compute

$$\frac{d}{dt}\frac{1}{2}|||f(t)|||_X^2 = \eta(Q(\mu+f,\mu+f),f)_X + \int_0^\infty (S_{\bar{L}}(\tau)\bar{L}f, S_{\bar{L}}(\tau)f)_{\bar{X}_0} d\tau + \int_0^\infty (S_{\bar{L}}(\tau)Q(f,f), S_{\bar{L}}(\tau)f)_{\bar{X}_0} d\tau.$$

We will estimate the terms separately, first by Corollary 4.7 we have for the polynomial weight case

$$(Q(\mu+f,\mu+f),f)_X \le -c_0\|f\|_Y^2 + C_k\|f\|_{\bar{X}_0}^2 + C_k\|f\|_{X_4}\|f\|_Y^2,$$

and for the exponential weight case

$$(Q(\mu+f,\mu+f),f)_X \le -c_0\|f\|_Y^2 + C_k\|f\|_{\bar{X}_0}^2 + C_k\|f\|_X\|f\|_Y^2.$$

Recall that

$$\|S_L(\tau)f(t)\|_{\bar{X}_0} \le \theta(\tau+t)\|f_0\|_X, \quad \lim_{\tau\to\infty}\theta(\tau+t)=0, \quad \forall t\ge 0.$$

For the second term we have

$$\int_0^\infty (S_{\bar{L}}(\tau)\bar{L}f, S_{\bar{L}}(\tau)f)_{\bar{X}_0}d\tau = \int_0^\infty \frac{d}{d\tau}\|S_{\bar{L}}(\tau)f(t)\|_{\bar{X}_0}^2 d\tau = \lim_{\tau\to\infty}\|S_{\bar{L}}(\tau)f(t)\|_{\bar{X}_0}^2 - \|f(t)\|_{\bar{X}_0}^2 = -\|f(t)\|_{\bar{X}_0}^2.$$

For the last term we have

$$\int_0^\infty (S_L(\tau)Q(f,f), S_L(\tau)f)_{\bar{X}_0}d\tau \le \int_0^\infty \|S_L(\tau)Q(f,f)\|_{\bar{X}_0}\|S_L(\tau)f\|_{\bar{X}_0}d\tau.$$

For the case $\gamma\in[0,1]$, by Corollary 4.8 and Corollary 4.9 we have

$$\int_0^\infty \|S_{\bar{L}}(\tau)Q(f,f)\|_{\bar{X}_0}\|S_L(\tau)f\|_{\bar{X}_0}d\tau \lesssim \|Q(f,f)\|_{X_5}\|f\|_{X_5}\int_0^\infty e^{-\lambda\tau}d\tau \lesssim \|Q(f,f)\|_{Z_6}\|f\|_{Y_6} \lesssim \|f\|_{X_6}\|f\|_{Y_6}^2.$$

For the case $\gamma\in(-3,0)$, since $6>k_1+\frac{|\gamma|}{2}$ for some $k_1>4$, by Corollary 4.8 and Corollary 4.9 so we have

$$\begin{aligned}\int_0^\infty \|S_{\bar{L}}(\tau)Q(f,f)\|_{\bar{X}_0}\|S_{\bar{L}}(\tau)f\|_{\bar{X}_0}d\tau &\lesssim \|Q(f,f)\|_{X_{6+|\gamma|/2}}\|f\|_{X_{6-|\gamma|/2}}\int_0^\infty \langle t\rangle^{-\frac{6+|\gamma|/2-k_1+6-|\gamma|/2-k_1}{|\gamma|}}d\tau\\ &\lesssim \|Q(f,f)\|_{Z_6}\|f\|_{Y_6} \lesssim \|f\|_{X_6}\|f\|_{Y_6}^2,\end{aligned}$$

by taking a suitable η and combining all the terms, (43) and (44) is thus proved. For the global existence and convergence rate, if $|||f_0|||_{X_6}\le\frac{K}{2C}$, then

$$\frac{d}{dt}|||f|||_{X_6}^2 \le (C|||f|||_{X_6}-K)\|f\|_{Y_6}^2,$$

we deduce that $|||f|||_{X_6}$ is decreasing over time for all $t\ge 0$. Together with the local existence we know that there exists a global solution $f\in L^\infty((0,\infty),X_6)$. Now we come to prove the convergence rate, for the polynomial case, for all $k\ge 6$ we have

$$\frac{d}{dt}|||f|||_{X_k}^2 \le (C|||f|||_{X_6}-K)\|f\|_{Y_k}^2 \le -\frac{K}{2}\|f\|_{Y_k}^2.$$

Thus the convergence rate can be proved similarly as Lemma 3.12, the exponential weight case can be proved similarly. □

6. Global existence for the Boltzmann equation with large amplitude initial data

In this section we prove the global existence for the Boltzmann equation with large amplitude initial data. We first prove some useful lemmas.

Lemma 6.1. *For any $\gamma\in(-3,1]$, $k>\max\{3+\gamma,3\}$, $\epsilon>0$ small enough we have*

$$\int_{\{|v-v_*|>\frac{\langle v\rangle}{\epsilon}\cup|v-v_*|<\epsilon\langle v\rangle\}}|v-v_*|^\gamma\langle v_*\rangle^{-k}dv_* \le C_{k,\epsilon}\langle v\rangle^\gamma, \quad \lim_{\epsilon\to 0}C_{k,\epsilon}=0,$$

for any $v\in\mathbb{R}^d$.

Proof. If $|v| \le \frac{1}{2}$, then $|v_*| + \frac{1}{2} \le 1 + |v - v_*|$, so we have

$$\int_{\{|v-v_*|>\frac{\langle v\rangle}{\epsilon} \cup |v-v_*|<\epsilon\langle v\rangle\}} |v-v_*|^\gamma \langle v_*\rangle^{-k} dv_* = \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon} \cup |v_*|<\epsilon\langle v\rangle\}} |v_*|^\gamma \langle v-v_*\rangle^{-k} dv_*$$
$$\le C_k \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon} \cup |v_*|<\epsilon\langle v\rangle\}} |v_*|^\gamma \langle v_*\rangle^{-k} dv_*.$$

We easily compute that

$$C_k \int_{\{|v_*|<\epsilon\langle v\rangle\}} |v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \int_{\{|v_*|<\epsilon\langle v\rangle\}} |v_*|^\gamma dv_* \le C_k \epsilon^{\gamma+3} \langle v\rangle^{\gamma+3} \le C_k \epsilon^{\gamma+3} \langle v\rangle^\gamma,$$

and

$$C_k \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon}\}} |v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \int_{\{|v_*|>\frac{1}{\epsilon}\}} |v_*|^{\gamma-k} dv_* \le C_k \epsilon^{k-\gamma-3} \le C_k \epsilon^{k-\gamma-3} \langle v\rangle^\gamma,$$

so the case $|v| \le \frac{1}{2}$ is thus proved. Consider now $|v| > 1/2$, we split the integral into two regions $|v - v_*| > \langle v\rangle/\epsilon$ and $|v - v_*| \le \epsilon\langle v\rangle$. For the first region, since $|v| \le \frac{1}{2}|v_*|$ implies $|v - v_*| \ge \frac{1}{2}|v_*|$, so we have

$$\int_{\{|v-v_*|>\frac{\langle v\rangle}{\epsilon}\}} |v-v_*|^\gamma \langle v_*\rangle^{-k} dv_* = \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon}\}} |v_*|^\gamma \langle v-v_*\rangle^{-k} dv_*$$
$$\le C_k \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon}\}} |v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \int_{\{|v_*|>\frac{\langle v\rangle}{\epsilon}\}} |v_*|^{-k+\gamma} dv_* \le C_k \epsilon^{k-\gamma-3} \langle v\rangle^{-k+\gamma+3} \le C_k \epsilon^{k-\gamma-3} \langle v\rangle^\gamma.$$

For the second region, since $|v| > 1/2$ and $|v - v_*| \le \epsilon\langle v\rangle$ imply $|v_*| \ge \langle v\rangle/4$, hence

$$\int_{|v-v_*|\le\epsilon\langle v\rangle} |v-v_*|^\gamma \langle v_*\rangle^{-k} dv_* \le C_k \langle v\rangle^{-k} \int_{|v-v_*|\le\epsilon\langle v\rangle} |v-v_*|^\gamma dv_* \le C_k \epsilon^{3+\gamma} \langle v\rangle^{-k+\gamma+3},$$

so the theorem is thus proved by gathering all the terms together. □

For the linearized part of the polynomial case we are able to prove a better estimate.

Lemma 6.2. *For any $-3 < \gamma \le 1$, for any constant $k > \max\{3, 3+\gamma\}$ we have*

$$I := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{1}{2}|v_*'|^2} dv_* d\sigma \le \frac{c}{k^{\frac{\gamma+3}{4}}} \langle v\rangle^\gamma + C_k \langle v\rangle^{\gamma-2},$$

for some constant $c > 0$ (independent of k) and for all $v \in \mathbb{R}^d$. Moreover for any $\epsilon > 0$ small we have

$$J = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} 1_{\{|v-v'|>\frac{\langle v\rangle}{\epsilon} \cup |v-v'|<\epsilon\langle v\rangle\}} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{1}{2}|v_*'|^2} dv_* d\sigma \le C_{k,\epsilon} \langle v\rangle^\gamma, \quad \lim_{\epsilon\to 0} C_{k,\epsilon} = 0.$$

We also have

$$K := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{1}{2}|v_*'|^2} \langle v'\rangle^{-2} dv_* d\sigma \le C_k \langle v\rangle^{\gamma-2}.$$

Proof. Since $\gamma - 1 \le 0$, by Lemma 2.10 we have

$$I = 4\int_{\mathbb{R}^3} \frac{1}{|v'-v|} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{\omega:\omega\cdot(v'-v)=0\}} \frac{1}{\sqrt{|v'-v|^2+|w|^2}} (\sqrt{|v'-v|^2+|w|^2})^\gamma e^{-\frac{|v+w|^2}{2}} dw dv'$$
$$\le 4\int_{\mathbb{R}^3} \frac{1}{|v'-v|^{\frac{3-\gamma}{2}}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} \int_{\{\omega:\omega\cdot(v'-v)=0\}} |w|^{\frac{\gamma-1}{2}} e^{-\frac{|v+w|^2}{2}} dw dv'.$$

Recall the decomposition (24) we have

$$I \le 4\int_{\mathbb{R}^3} \frac{1}{|v'-v|^{\frac{3-\gamma}{2}}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{|v_\perp|^2}{2}} \int_{\{\omega:\omega\cdot(v'-v)=0\}} |w|^{\frac{\gamma-1}{2}} e^{-\frac{|v_\parallel+w|^2}{2}} dw dv'.$$

Since $\frac{\gamma-1}{2} > -2$, we have

$$\int_{\{\omega:\omega\cdot(v'-v)=0\}} |w|^{\frac{\gamma-1}{2}} e^{-\frac{|v_\parallel+w|^2}{2}} dw = \int_{\mathbb{R}^2} |w-v_\parallel|^{\frac{\gamma-1}{2}} e^{-\frac{|w|^2}{2}} dw \le C\langle v_\parallel\rangle^{\frac{\gamma-1}{2}},$$

hence

$$I \le C\int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{|v_\perp|^2}{2}} \langle v_\parallel\rangle^{\frac{\gamma-1}{2}} dv'.$$

Similarly we have

$$J \le C\int_{\{|v-v'|>\frac{\langle v\rangle}{\epsilon}\cup|v-v'|<\epsilon\langle v\rangle\}} |v'-v|^{\frac{\gamma-3}{2}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{|v_\perp|^2}{2}} \langle v_\parallel\rangle^{\frac{\gamma-1}{2}} dv'.$$

We split into two regions $|v|\le 1$ and $|v|>1$. If $|v|\le 1$, since $\frac{\gamma-3}{2} > -3$, hence

$$I \le C_k\int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{1}{\langle v'\rangle^k} dv' \le C_k\langle v\rangle^{\frac{\gamma-3}{2}} \le C_k\langle v\rangle^{-100}.$$

Similarly by Lemma 6.1 we have

$$J \le C_k\int_{\{|v-v'|>\frac{\langle v\rangle}{\epsilon}\cup|v-v'|<\epsilon\langle v\rangle\}} |v'-v|^{\frac{\gamma-3}{2}} \frac{1}{\langle v'\rangle^k} dv' \le C_{k,\epsilon}\langle v\rangle^{\gamma}.$$

For the case $|v|>1$, since $|v_\perp|\le|v|$, we split it into two cases $|v_\perp|>\frac{|v|}{k}$ and $|v_\perp|\le\frac{|v|}{k}$. For the case $|v_\perp|>\frac{|v|}{k}$ we have

$$I \le C_k\int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{|v|^2}{2k^2}} dv' \le C_k e^{-\frac{|v|^2}{4k^2}} \int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \langle v'\rangle^{-k} dv' \le C_k\langle v\rangle^{-100},$$

and by Lemma 6.1

$$J \le C_k e^{-\frac{|v|^2}{4k^2}} \int_{\{|v-v'|>\frac{\langle v\rangle}{\epsilon}\cup|v-v'|<\epsilon\langle v\rangle\}} |v'-v|^{\frac{\gamma-3}{2}} \frac{1}{\langle v'\rangle^k} dv' \le C_{k,\epsilon}\langle v\rangle^{\gamma}.$$

For the case $|v_\perp|\le\frac{|v|}{k}$, $|v|>1$ we have

$$|v'|^2 = |v-v'|^2+|v|^2+2v\cdot(v'-v) \ge |v-v'|^2+|v|^2-2|v-v'||v_\perp| \ge (1-\frac{1}{k})(|v-v'|^2+|v|^2),$$

and since

$$(1+\frac{1}{k})^k \le e, \quad |v_\parallel|^2 \ge |v|^2-\frac{1}{k^2}|v|^2 \ge \frac{1}{2}|v|^2,$$

we deduce

$$\begin{aligned}
I \le& C\int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{\langle v\rangle^k}{\langle v'\rangle^k} e^{-\frac{|v_\perp|^2}{2}} \langle v_\parallel\rangle^{\frac{\gamma-1}{2}} dv' \\
\le& C\langle v\rangle^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{\langle v\rangle^k}{\langle |v-v'|^2+|v|^2\rangle^k} e^{-\frac{|v_\perp|^2}{2}} dv' \\
\le& C\langle v\rangle^{\frac{\gamma-1}{2}} \int_{\mathbb{R}^3} |v'-v|^{\frac{\gamma-3}{2}} \frac{1}{(1+\frac{|v-v'|^2}{1+|v|^2})^{\frac{k}{2}}} e^{-\frac{|v_\perp|^2}{2}} dv'.
\end{aligned}$$

If we take the change of variables (25) we have

$$I \le C2\pi\langle v\rangle^{\frac{\gamma-1}{2}} \int_0^\infty r^{\frac{\gamma+1}{2}} \frac{1}{(1+\frac{r^2}{1+|v|^2})^{\frac{k}{2}}} \int_0^\pi e^{-\frac{|v|^2\cos^2\theta}{2}} \sin\theta dr d\theta.$$

Taking another change of variables (26), recall (5) we deduce

$$
\begin{aligned}
I \le & C\frac{1}{|v|}\langle v\rangle^{\gamma+1}\int_0^\infty x^{\frac{\gamma+1}{2}}\frac{1}{(1+|x|^2)^{\frac{k}{2}}}\int_{-|v|}^{|v|} e^{-\frac{|y|^2}{2}}dydx\\
\le & C\langle v\rangle^{\gamma}\int_0^\infty x^{\frac{\gamma+1}{2}}\frac{1}{(1+|x|^2)^{\frac{k}{2}}}dx\int_{-\infty}^{+\infty} e^{-\frac{|y|^2}{2}}dy\\
\le & C\langle v\rangle^{\gamma}\int_0^\infty z^{\frac{\gamma-1}{4}}\frac{1}{(1+z)^{\frac{k}{2}}}dx\\
\le & C\langle v\rangle^{\gamma}k^{-\frac{\gamma+3}{4}},
\end{aligned}
$$

so the term I is estimated. By (26) and (25) we have $x=\frac{|v-v'|}{\langle v\rangle}$, which implies

$$
\{|v-v'|>\frac{\langle v\rangle}{\epsilon}\cup|v-v'|<\epsilon\langle v\rangle\}=\{x\le\epsilon\cup x\ge\frac{1}{\epsilon}\},
$$

thus for J we have

$$
J\le C\frac{1}{|v|}\langle v\rangle^{\gamma+1}\left(\int_0^\epsilon+\int_{\frac{1}{\epsilon}}^\infty\right)x^{\frac{\gamma+1}{2}}\frac{1}{(1+|x|^2)^{\frac{k}{2}}}\int_{-|v|}^{|v|}e^{-\frac{|y|^2}{2}}dydx\le C_{k,\epsilon}\langle v\rangle^{\gamma},
$$

so the proof for J is thus finished. For the K term since

$$
K=\langle v\rangle^{-2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}\frac{\langle v\rangle^{k+2}}{\langle v'\rangle^{k+2}}e^{-\frac{1}{2}|v_*'|^2}dv_*d\sigma,
$$

the estimate for term K just follows by the estimate for term I. □

We introduce the mild solution to the Boltzmann equation. For any $k\ge0$, let $f(t,x,v)=\langle v\rangle^{-k}(F(t,x,v)-\mu(v))$ in (1), then f satisfies

$$
\partial_t f+v\cdot\nabla_v f+L_k f=\Gamma_k(f,f), \tag{45}
$$

where

$$
L_k f:=\langle v\rangle^k Q(\mu,f\langle v\rangle^{-k})+\langle v\rangle^k Q(f\langle v\rangle^{-k},\mu),
$$

and

$$
\Gamma_k^{\pm}(f,f):=\langle v\rangle^k Q^{\pm}(f\langle v\rangle^{-k},f\langle v\rangle^{-k}),\quad \Gamma_k(f,f):=\Gamma_k^+(f,f)-\Gamma_k^-(f,f).
$$

We also have

$$
L_k f=K_k f-\nu(v)f,\quad \nu(v):=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\mu(v_*)dv_*d\sigma\ \sim\langle v\rangle^{\gamma},
$$

where $K_k:=K_{2,k}-K_{1,k}$ is defined as

$$
(K_{1,k}f)(v):=\mu(v)\langle v\rangle^k\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)f(v_*)\langle v_*\rangle^{-k}dv_*d\sigma,
$$

and

$$
\begin{aligned}
(K_{2,k}f)(v):=&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^k f(v_*')\langle v_*'\rangle^{-k}\mu(v')dv_*d\sigma\\
&+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^k\mu(v_*')f(v')\langle v'\rangle^{-k}dv_*d\sigma\\
=&2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{\gamma}b(\cos\theta)\langle v\rangle^k\mu(v_*')f(v')\langle v'\rangle^{-k}dv_*d\sigma.
\end{aligned}
$$

Thus the mild solution of (45) is given by

$$f(t,x,v) = e^{-\nu(v)t} f_0(x-vt,v) + \int_0^t e^{-\nu(v)(t-s)} (K_k f)(s, x-v(t-s), v) ds$$
$$+ \int_0^t e^{-\nu(v)(t-s)} \Gamma_k(f,f)(s, x-v(t-s), v) ds. \tag{46}$$

If we define $l_k(v,v')$ is the kernel with respect to K_k such that

$$K_k f(v) = \int_{\mathbb{R}^3} l_k(v,v') f(v') dv'.$$

For the kernel l_k we have the following estimate.

Lemma 6.3. *For any $\gamma \in (-3,1]$, for any $k > \max\{3, 3+\gamma\}$ we have*

$$\int_{\mathbb{R}^3} |l_k(v,v')| dv' \le \frac{c}{k^{\frac{\gamma+3}{4}}} \langle v \rangle^\gamma + C_k \langle v \rangle^{\gamma-2}, \quad \int_{\mathbb{R}^3} |l_k(v,v')| \langle v' \rangle^{-2} dv' \le C_k \langle v \rangle^{\gamma-2},$$

for some constant $C_k > 0$. Moreover, for $\epsilon > 0$ small enough, we have

$$\int_{\{|v-v'| > \frac{\langle v \rangle}{\epsilon} \cup |v-v'| < \epsilon \langle v \rangle\}} |l_k(v,v')| dv' \le C_{k,\epsilon} \langle v \rangle^\gamma, \quad \lim_{\epsilon \to 0} C_{k,\epsilon} = 0.$$

Proof. It is easily seen that

$$\int_{\mathbb{R}^3} |l_k(v,v')| dv' \le 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) \frac{\langle v \rangle^k}{\langle v' \rangle^k} e^{-\frac{1}{2}|v'_*|^2} dv_* d\sigma$$
$$+ \mu(v) \langle v \rangle^k \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) \langle v_* \rangle^{-k} dv_* d\sigma,$$

we easily conclude by Lemma 2.9, Lemma 6.1 and Lemma 6.2. □

For the mild solution f, we have the following lemma on local existence.

Lemma 6.4. *(Local existence) Suppose $\gamma \in (-3,1]$, $F_0 = \mu + f_0 \ge 0$. For any $k > \max\{3, 3+\gamma\}$ suppose $\|\langle v \rangle^k f_0\|_{L^\infty} < +\infty$. Then there exists a positive time*

$$t_1 = C_k (1 + \|\langle v \rangle^k f_0\|_{L^\infty})^{-1},$$

such that the Boltzmann equation (1) has a unique mild solution $F = \mu + f \ge 0$ in $[0, t_1]$ satisfies

$$\sup_{0 \le s \le t_1} \|\langle v \rangle^k f(s)\|_{L^\infty} \le 2 \|\langle v \rangle^k f_0\|_{L^\infty}.$$

Proof. The proof is similar to Proposition 2.1 in [16] thus omitted. □

We give an upper bound for the nonlinear term.

Lemma 6.5. *Let $\gamma \in (-3,1]$, for any $k > \max\{3, 3+\gamma\}$, $\alpha \ge 0$, for any $s \ge 0, y \in \mathbb{T}^3$, for any smooth function f it holds that*

$$|\langle v \rangle^\alpha \Gamma_k^-(f,f)(s,y,v)| \le C\nu(v) \|\langle v \rangle^\alpha f(s)\|_{L^\infty_{x,v}} \|f(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv' \right)^{\frac{p-1}{2p}},$$

similarly

$$|\langle v \rangle^\alpha \Gamma_k^+(f,f)(s,y,v)| \le C\nu(v) \|\langle v \rangle^\alpha f(s)\|_{L^\infty_{x,v}} \|\langle v \rangle f(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv' \right)^{\frac{p-1}{2p}},$$

for some constant $p > 1$ close to 1 which only depends on γ (such p is fixed and used later).

Proof. Fix $p>1$ close enough to 1 and $\epsilon>0$ small enough such that

$$-3<p\gamma<\frac{3}{2},\quad \frac{4(p-1)}{p+1}\le 1,\quad \frac{p-1}{2p}\le\frac{1}{2}+\frac{\gamma}{6},\quad -3<p\gamma+\epsilon\gamma\le -2,\quad \epsilon\frac{2p}{p-1}\le 1. \tag{47}$$

For the term $\Gamma_k^-(f,f)$, we easily compute

$$\begin{aligned}|\langle v\rangle^\alpha\Gamma_k^-(f,f)(s,y,v)| \le& C\|\langle v\rangle^\alpha f(s)\|_{L^\infty_{x,v}}\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v\rangle^{-k}f(s,y,v_*)dv_*\\ \le& C\|\langle v\rangle^\alpha f(s)\|_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|v-v_*|^{p\gamma}\langle v_*\rangle^{-k}dv_*\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^3}\langle v_*\rangle^{-k}f(s,y,v_*)^{\frac{p}{p-1}}dv_*\right)^{\frac{p-1}{p}}\\ \le& C\|\langle v\rangle^\alpha f(s)\|_{L^\infty_{x,v}}\nu(v)\left(\int_{\mathbb{R}^3}\langle v_*\rangle^{-2k}dv_*\right)^{\frac{p-1}{2p}}\left(\int_{\mathbb{R}^3}f(s,y,v_*)^{\frac{2p}{p-1}}dv_*\right)^{\frac{p-1}{2p}}\\ \le& C\nu(v)\|\langle v\rangle^\alpha f(s)\|_{L^\infty_{x,v}}\|f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}.\end{aligned}$$

For the term $\Gamma_k^+(f,f)$, since $\langle v\rangle^\alpha\le C_\alpha\langle v_*\rangle^\alpha+C_\alpha\langle v_*'\rangle^\alpha$, we have

$$\begin{aligned}|\langle v\rangle^\alpha\Gamma_k^+(f,f)(s,y,v)| \le& C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^k}{\langle v'\rangle^k\langle v_*'\rangle^k}|f(s,y,v_*')\langle v'\rangle^\alpha f(s,y,v')|dv_*d\sigma\\ &+C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^k}{\langle v'\rangle^k\langle v_*'\rangle^k}|\langle v_*'\rangle^\alpha f(s,y,v_*')f(s,y,v')|dv_*d\sigma := I_1+I_2.\end{aligned}$$

Without loss of generality we only prove I_2 in the following, we have

$$\begin{aligned}I_2=&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^k}{\langle v'\rangle^k\langle v_*'\rangle^k}|\langle v_*'\rangle^\alpha f(s,y,v_*')f(s,y,v')|dv_*d\sigma\\ \le& C\|\langle v\rangle^\alpha f(s)\|_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{p\gamma+\epsilon p}\frac{\langle v\rangle^{pk}}{\langle v'\rangle^{pk}\langle v_*'\rangle^{pk}}dvd\sigma\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{-\epsilon\frac{p}{p-1}}|f(s,y,v')|^{\frac{p}{p-1}}dv_*d\sigma\right)^{1-\frac{1}{p}}.\end{aligned}$$

By Lemma 2.10, for any function g we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{-\epsilon\frac{p}{p-1}}|g(v')|dv_*d\sigma\\ \le& 4\int_{\mathbb{R}^3}\frac{1}{|v'-v|}|g(v')|\int_{\{w:w\cdot(v'-v)=0\}}\frac{1}{\sqrt{|v'-v|^2+|w|^2}}\left(\sqrt{|v'-v|^2+|w|^2}\right)^{-\epsilon\frac{p}{p-1}}dwdv'\\ \le& 4\int_{\mathbb{R}^3}\frac{1}{|v'-v|}|g(v')|\int_{\mathbb{R}^2}\frac{1}{(|v'-v|^2+|w|^2)^{\frac{1+\epsilon\frac{p}{p-1}}{2}}}dwdv'.\end{aligned}$$

By a change of variable $w=|v-v'|x$, and since $1+\epsilon\frac{p}{p-1}>1$, the integral is integrable and we have

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^{-\epsilon\frac{p}{p-1}}|g(v')|dv_*d\sigma \le& C\int_{\mathbb{R}^3}|v'-v|^{-\epsilon\frac{p}{p-1}}|g(v')|\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^{\frac{1+\epsilon\frac{p}{p-1}}{2}}}dxdv'\\ \le& C\int_{\mathbb{R}^3}|v'-v|^{-\epsilon\frac{p}{p-1}}|g(v')|dv'.\end{aligned} \tag{48}$$

So using (48) and Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{-\epsilon\frac{p}{p-1}} |f(s,y,v')|^{\frac{p}{p-1}} dv_* d\sigma \\
\le & C\left(\int_{\mathbb{R}^3} |v-v'|^{-\epsilon\frac{2p}{p-1}} \langle v'\rangle^{-4} dv'\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \langle v'\rangle^{4} |f(s,y,v')|^{\frac{2p}{p-1}} dv'\right)^{\frac{1}{2}} \\
\le & C\langle v\rangle^{-\epsilon\frac{p}{p-1}} \|\langle v\rangle^4 |f|^{\frac{p+1}{p-1}}\|^{1/2}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{1}{2}} \\
\le & C\langle v\rangle^{-\epsilon\frac{p}{p-1}} \|\langle v\rangle^{\frac{4(p-1)}{p+1}} |f|\|^{\frac{p+1}{2(p-1)}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 2.11 we have

$$
\left(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^{p\gamma+\epsilon p} \frac{\langle v\rangle^{pk}}{\langle v'\rangle^{pk}\langle v'_*\rangle^{pk}} dv d\sigma\right)^{\frac{1}{p}} \lesssim \langle v\rangle^{\gamma+\epsilon}.
$$

Gathering the terms two we have

$$
I_2 \le C\langle v\rangle^{\gamma+\epsilon}\langle v\rangle^{-\epsilon} \|\langle v\rangle^{\alpha} f(s)\|_{L^\infty_{x,v}} \|\langle v\rangle f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}},
$$

the proof is thus finished. □

For any $\beta \ge 0$, let $h(t,x,v) = \langle v\rangle^{\beta} f(t,x,v)$, where f is a solution to (45). it is easily seen that h satisfies

$$
\partial_t h + v\cdot\nabla_x h + \nu(v) h = \Gamma_{k+\beta}(h,h) + K_{k+\beta} h, \tag{49}
$$

with

$$
K_{k+\beta} h(v) = \langle v\rangle^{\beta} K_k(\langle v\rangle^{-\beta} h)(v), \quad \Gamma_{k+\beta}(h,h) = \langle v\rangle^{\beta}\Gamma_k(\langle v\rangle^{-\beta} h, \langle v\rangle^{-\beta} h) = \langle v\rangle^{\beta}\Gamma_k(f,f).
$$

For the kernel of $K_{k+\beta}$ we have

$$
K_{k+\beta} f(v) = \int_{\mathbb{R}^3} l_{k+\beta}(v,v') f(v') dv', \quad l_{k+\beta}(v,v') = l_k(v,v')\langle v\rangle^{\beta}\langle v'\rangle^{-\beta},
$$

it is easily seen that $l_{k+\beta}$ still satisfies Lemma 6.5, with C_k replaced by $C_{k+\beta} = C_{k,\beta}$. The mild solution to (49) is given by

$$
\begin{aligned}
h(t,x,v) =& e^{-\nu(v)t} h_0(x-vt,v) + \int_0^t e^{-\nu(v)(t-s)} (K_{k+\beta} h)(s, x-v(t-s), v) ds \\
&+ \int_0^t e^{-\nu(v)(t-s)} \Gamma_{k+\beta}(h,h)(s,x-v(t-s),v) ds.
\end{aligned} \tag{50}
$$

For the mild solution h we have the following estimate.

Lemma 6.6. *Suppose f and h satisfy (45) and (49). For any $\gamma \in (-3,1]$, there exists a constant $k_0 > \max\{3, 3+\gamma\}$ such that for any $k \ge k_0$, $\beta \ge \max\{3, 3+\gamma\}$ it holds that*

$$
\begin{aligned}
\sup_{0\le s\le t} \|h(s)\|_{L^\infty_{x,v}} \le & C_{k,\beta}(\|h_0\|_{L^\infty_{x,v}} + \|h_0\|^2_{L^\infty_{x,v}} + \sqrt{H(F_0)} + H(F_0)) \\
&+ C_{k,\beta} \sup_{t_1\le s\le t, y\in\mathbb{T}^3} \left\{ \|h(s)\|^{\frac{3p+1}{2p}}_{L^\infty_{x,v}} \int_{\mathbb{R}^3} \left(|f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\},
\end{aligned}
$$

for some constant $C_{k,\beta} \ge 1$, where t_1 is defined in Lemma 6.4.

Proof. By (50) we have

$$|h(t,x,v)| \le e^{-\nu(v)t}\|h_0\|_{L^\infty_{x,v}} + \int_0^t e^{-\nu(v)(t-s)}|(K_{k+\beta}h)(s,x-v(t-s),v)|ds$$
$$+\int_0^t e^{-\nu(v)(t-s)}|\Gamma_{k+\beta}(h,h)(s,x-v(t-s),v)|ds := e^{-\nu(v)t}\|h_0\|_{L^\infty_{x,v}} + J_2 + J_3.$$

For the J_3 term, since $\beta \ge 1$, by Lemma 6.5 we have

$$|\Gamma_{k+\beta}(h,h)(s,y,v)| = |\langle v\rangle^\beta \Gamma_k(f,f)(s,y,v)| \le C_{k,\beta}\langle v\rangle^\gamma \sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\},$$

hence

$$\begin{aligned}J_3 \le& C_{k,\beta}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\\ \le& C_{k,\beta}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.\end{aligned}$$

For the J_2 term, denote $\tilde{x} = x - v(t-s)$, we have

$$J_2 \le \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')h(s,\tilde{x},v')|dv'ds,$$

by (50) again we have

$$\begin{aligned}J_2 \le& \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|e^{-\nu(v')s}|h_0(\tilde{x}-v's,v')|dv'ds\\ &+\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}|\Gamma_{k+\beta}(h,h)|(\tau,\tilde{x}-v'(s-\tau),v')d\tau dv'ds\\ &+\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')l_{k+\beta}(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\ :=& J_{21}+J_{22}+J_{23}.\end{aligned}$$

For the J_{21} term by Lemma 6.3 we have

$$J_{21} \le C_{k,\beta}\|h_0\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \le C_{k,\beta}\|h_0\|_{L^\infty_{x,v}}.$$

For the J_{22} term by Lemma 6.5 we have

$$\begin{aligned}J_{22} \le& C_{k,\beta}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma d\tau dv'ds\\ \le& C_{k,\beta}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\\ \le& C_{k,\beta}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.\end{aligned}$$

For term J_{23}, we first split it into two parts $|v|\le N$ and $|v|\ge N$ for some constant $N>0$ large to be fixed later. For the case $|v|\ge N$, by Lemma 6.3 we have

$$\int_{\mathbb{R}^3}|l_{k+\beta}(v',v'')|dv'' \le \frac{c}{(k+\beta)^{\frac{\gamma+3}{4}}}\langle v'\rangle^\gamma + C_{k,\beta}\langle v'\rangle^{\gamma-2},$$

which implies

$$\int_0^s e^{-\nu(v')(s-\tau)}\int_{\mathbb{R}^3}|l_{k+\beta}(v',v'')|dv''d\tau \le \frac{c}{k^{\frac{\gamma+3}{4}}}+C_{k,\beta}\langle v'\rangle^{-2}.$$

Since $|v|\ge N$ implies $\langle v\rangle^{-2}\le \frac{1}{N^2}$, using Lemma 6.3 again we have

$$\begin{aligned}&\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}\int_{\mathbb{R}^3}|l_{k+\beta}(v',v'')|dv''d\tau dv'\\ \le&\frac{c}{k^{\frac{\gamma+3}{4}}}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|dv'+C_{k,\beta}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|\langle v'\rangle^{-2}dv'\le \frac{c^2}{k^{\frac{\gamma+3}{2}}}\langle v\rangle^{\gamma}+C_{k,\beta}\langle v\rangle^{\gamma-2}\le \langle v\rangle^{\gamma}\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N^2}\right),\end{aligned}\tag{51}$$

we deduce

$$J_{23}\le\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N^2}\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^{\gamma}ds\le\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N^2}\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}.$$

For the case $|v|\le N$, since $l_{k+\beta}(v,v')$ is unbounded, by Lemma 6.3 we have for any N,k,β we can find a bounded compact support function $l_{k,N,\beta}$ such that

$$l_{k,N,\beta}(v,v'):=l_{k+\beta}(v,v')1_{\frac{\langle v\rangle}{C_{k,N,\beta}}\le|v-v'|\le C_{k,N,\beta}\langle v\rangle},\quad \int_{\mathbb{R}^3}|l_{k+\beta}(v,v')-l_{k,N,\beta}(v,v')|dv'\le\frac{C_{k,\beta}}{N}\langle v\rangle^{\gamma},\quad\forall v\in\mathbb{R}^3,\tag{52}$$

for some large constant $C_{k,N,\beta}>0$. By

$$\begin{aligned}l_{k+\beta}(v,v')l_{k+\beta}(v',v'')=&(l_{k+\beta}(v,v')-l_{k,N,\beta}(v,v'))l_{k+\beta}(v',v'')+l_{k,N,\beta}(v,v')(l_{k+\beta}(v',v'')-l_{k,N,\beta}(v',v''))\\&+l_{k,N,\beta}(v,v')l_{k,N,\beta}(v',v''),\end{aligned}\tag{53}$$

we split J_{23} into three terms respectively. For the first term we have

$$\begin{aligned}&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|(l_{k+\beta}(v,v')-l_{k,N,\beta}(v,v'))l_{k+\beta}(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\ \le&\frac{C_{k,\beta}}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^{\gamma}ds\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^{\gamma}d\tau\le\frac{C_{k,\beta}}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}},\end{aligned}$$

the second term can be estimated similarly. For the third term, by (52) we have $l_{k,N,\beta}(v,v')$ and $l_{k,N,\beta}(v',v'')$ is supported in

$$\frac{\langle v\rangle}{C_{k,N,\beta}}\le|v-v'|\le C_{k,N,\beta}\langle v\rangle,\quad\frac{\langle v'\rangle}{C_{k,N,\beta}}\le|v'-v''|\le C_{k,N,\beta}\langle v'\rangle,$$

since $|v|\le N$, which implies $l_{k,N,\beta}(v',v'')l_{k,N,\beta}(v',v'')$ is supported in $|v|\le N,|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}$ for some constant $C'_{k,N,\beta}>0$. We split it into two parts, $\tau\in[s-\lambda,s]$ and $\tau\in[0,s-\lambda]$, where $\lambda>0$ is a small constant to be fixed later. For the case $\tau\in[s-\lambda,s]$, since $|v'|\le C'_{k,N,\beta}$, we have

$$\begin{aligned}&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N,\beta}(v,v')l_{k,N,\beta}(v',v'')|\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\ \le&C_{k,N,\beta}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^{\gamma}ds\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^{\gamma}d\tau\\ \le&C_{k,N,\beta}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}(1-e^{-\nu(v')\lambda})\le C_{k,N,\beta}\lambda\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}.\end{aligned}$$

For the case $\tau\in[0,s-\lambda]$, first we have

$$l_{k,N,\beta}(v,v')\le C_{k,N,\beta},\quad l_{k,N,\beta}(v',v'')\le C_{k,N,\beta},\quad\frac{1}{C_{k,N,\beta}}\le\nu(v)\le C_{k,N,\beta},\quad\frac{1}{C_{k,N,\beta}}\le\nu(v')\le C_{k,N,\beta},\tag{54}$$

and by Lemma 2.13 we have

$$
\begin{aligned}
&\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}} |h(\tau,\tilde{x}-v'(s-\tau),v'')|dv'dv''\\
=&\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}} \frac{|F(\tau,\tilde{x}-v'(s-\tau),v'')-\mu(v'')|}{\langle v''\rangle^{k+\beta}}dv'dv''\\
\le& C_{k,N,\beta}\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}} \frac{|F(\tau,\tilde{x}-v'(s-\tau),v'')-\mu(v'')|}{\sqrt{\mu(v'')}} I_{\{|F(\tau,\tilde{x}-v'(s-\tau),v'')-\mu(v'')|\le\mu(v'')\}}dv'dv''\\
&+C_{k,N,\beta}\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}} |F(\tau,\tilde{x}-v'(s-\tau),v'')-\mu(v'')|I_{\{|F(\tau,\tilde{x}-v'(s-\tau),v'')-\mu(v'')|\ge\mu(v'')\}}dv'dv''\\
\le& C_{k,N,\beta}\frac{1}{(s-\tau)^{\frac{3}{2}}}\left(\int_{\mathbb{T}^3}\int_{|v''|\le C'_{k,N,\beta}} \frac{|F(\tau,y,v'')-\mu(v'')|^2}{\mu(v'')} I_{\{|F(\tau,y,v'')-\mu(v'')|\le\mu(v'')\}}dv''dy\right)^{\frac{1}{2}}\\
&+C_{k,N,\beta}\frac{1}{(s-\tau)^3}\int_{\mathbb{T}^3}\int_{|v''|\le C'_{k,N,\beta}} |F(\tau,y,v'')-\mu(v'')|I_{\{|F(\tau,y,v'')-\mu(v'')|\ge\mu(v'')\}}dv''dy\\
\le& C_{k,N,\beta}\frac{1}{(s-\tau)^{\frac{3}{2}}}\sqrt{H(F_0)}+C_{k,N,\beta}\frac{1}{(s-\tau)^3}H(F_0),
\end{aligned}
\tag{55}
$$

where we have made a change of variable $y=\tilde{x}-v'(s-\tau)$. Since $s-\tau\ge\lambda$, so we have

$$
\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N,\beta}(v,v')l_{k,N,\beta}(v',v'')|\int_0^{s-\lambda} e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau,v''))|dv''d\tau dv'ds\\
\le& C_{k,N,\beta}\int_0^t e^{-c(t-s)}\int_0^{s-\lambda}e^{-c(s-\tau)}\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}} |h(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\
\le& C_{k,N,\beta}\lambda^{-\frac{3}{2}}\sqrt{H(F_0)}+C_{k,N,\beta}\lambda^{-3}H(F_0).
\end{aligned}
$$

Gathering all the terms and taking supremum we have

$$
\begin{aligned}
\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\le& C_{k,\beta}\|h_0\|_{L^\infty_{x,v}}+\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N}+C_{k,N,\beta}\lambda\right)\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\\
&+C_{k,N,\beta}\lambda^{-\frac{3}{2}}\sqrt{H(F_0)}+C_{k,N,\beta}\lambda^{-3}H(F_0)+C_{k,\beta}\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\int_{\mathbb{R}^3}\left(|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}
$$

First fix $\beta\ge 0$, then choose k large, then let N be sufficiently large and finally let λ be sufficiently small such that

$$
\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N}+C_{k,N,\beta}\lambda\le\frac{1}{2},
$$

which implies

$$
\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\le C_{k,\beta}(\|h_0\|_{L^\infty_{x,v}}+\sqrt{H(F_0)}+H(F_0))+C_{k,\beta}\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\int_{\mathbb{R}^3}\left(|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\},
$$

using Lemma 6.4 we have

$$
\sup_{0\le s\le t_1,y\in\mathbb{T}^3}\left\{\|h(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\int_{\mathbb{R}^3}\left(|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\le C\sup_{0\le s\le t_1}\|h(s)\|_{L^\infty_{x,v}}^2\le C\|h_0\|_{L^\infty_{x,v}}^2,
$$

so the proof is thus finished. □

Lemma 6.7. *Suppose $\gamma \in (-3,1]$ and $k,\beta > \max\{3, 3+\gamma\}$, then for any smooth function f and h satisfy* (45) *and* (49) *we have*

$$\int_{\mathbb{R}^3} |f(t,x,v)| dv \le \int_{\mathbb{R}^3} e^{-\nu(v)t} |f_0(x-vt,v)| dv + C_{k,N,\beta}\lambda^{-\frac{3}{2}}\sqrt{H(F_0)} + C_{k,N,\beta}\lambda^{-3} H(F_0)$$
$$+ C_{k,\beta}(\lambda + \frac{1}{N^{\frac{\beta-3}{2}}})(\sup_{0\le s\le t} \|h(s)\|_{L^\infty_{x,v}} + \sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}})$$
$$+ C_{k,N,\beta}\lambda^{-3}(\sqrt{H(F_0)} + H(F_0))^{1-\frac{1}{p}} \sup_{0\le s\le t} \|h(s)\|^{1+\frac{1}{p}}_{L^\infty_{x,v}},$$

where $\lambda > 0, N \ge 1$ are to be chosen later. Recall that $p > 1$ is defined in (47).

Proof. By (46) we have

$$\int_{\mathbb{R}^3} |f(t,x,v)| dv \le \int_{\mathbb{R}^3} e^{-\nu(v)t}|f_0(x-vt,v)| dv + \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |(K_k f)(s, x-v(t-s), v)| dv ds$$
$$+ \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} |\Gamma_k(f,f)(s,x-v(t-s),v)| dv ds$$
$$:= \int_{\mathbb{R}^3} e^{-\nu(v)t}|f_0(x-vt,v)| dv + H_1 + H_2.$$

For the term H_1, recall

$$h(t,x,v) = \langle v\rangle^\beta f(t,x,v), \quad l_{k+\beta}(v,v') = l_k(v,v')\langle v\rangle^\beta \langle v'\rangle^{-\beta},$$

hence

$$H_1 \le \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta} \int_{\mathbb{R}^3} |l_{k+\beta}(v,v') h(s, x-v(t-s), v')| dv' dv ds.$$

We split it into two case $s \in [t-\lambda, t]$ and $s \in [0, t-\lambda]$, where λ is a small constant to be fixed later. For the case $s \in [t-\lambda, t]$, since $\beta - \gamma > 3$ we have

$$\int_{t-\lambda}^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3} |l_{k+\beta}(v,v')h(s,x-v(t-s),v')| dv' dv ds$$
$$\le \sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}} \int_{t-\lambda}^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')| dv' dv ds$$
$$\le C_{k,\beta}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}} \int_{t-\lambda}^t\int_{\mathbb{R}^3}\langle v\rangle^{-\beta}\langle v\rangle^{\gamma} dv ds \le C_{k,\beta}\lambda \sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}.$$

For the case $s \in [0, t-\lambda]$, we split it into two cases $|v| \ge N$ and $|v| \le N$ for some large constant N to be fixed later. For the case $|v| \ge N$, since $\beta > 3$ we have

$$\int_0^{t-\lambda}\int_{|v|\ge N} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')h(s,x-v(t-s),v')| dv' dv ds$$
$$\le \sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t\int_{|v|\ge N} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')| dv' dv ds$$
$$\le C_{k,\beta}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \int_{|v|\ge N}\langle v\rangle^{-\beta} dv ds \le C_{k,\beta}\frac{1}{N^{\beta-3}}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}.$$

For the case $|v| \le N$, using decomposition (52) we split it into two terms respectively. For the first term since $\beta > 3$ we have

$$\int_0^{t-\lambda}\int_{|v|\le N} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v') - l_{k,N,\beta}(v,v')||h(s,x-v(t-s),v')| dv' dv ds$$
$$\le C_{k,\beta}\frac{1}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_{\mathbb{R}^3}\langle v\rangle^{-\beta} dv ds \le C_{k,\beta}\frac{1}{N}\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}.$$

For the last term since $|v| \le N$, by (54) and (55) we have

$$\int_0^{t-\lambda}\int_{|v|\le N} e^{-\nu(v)(t-s)}\langle v\rangle^{-\beta}\int_{\mathbb{R}^3}|l_{k,N,\beta}(v,v')||h(s,x-v(t-s),v')|dv'dvds$$
$$\le C_{k,N,\beta}\int_0^{t-\lambda} e^{c(t-s)}\int_{|v|\le N}\int_{|v'|\le C'_{k,N,\beta}}|h(s,x-v(t-s),v')|dv'dvds \le C_{k,N,\beta}\lambda^{-\frac{3}{2}}\sqrt{H(F_0)}+C_{k,N,\beta}\lambda^{-3}H(F_0).$$

Then we come to the H_2 term, we have

$$\begin{aligned}|H_2| \le& \int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}|\Gamma_k^-(f,f)(s,x-v(t-s),v)|dvds\\ &+\int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}|\Gamma_k^+(f,f)(s,x-v(t-s),v)|dvds := H_{21}+H_{22}.\end{aligned}$$

For the H_{21} term, we split it into four terms for some constant $\lambda, N>0$ to be fixed later

$$\begin{aligned}H_{21} \le& C\int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{-k}|f(s,x-v(t-s),v_*)||f(s,x-v(t-s),v)|dv_*dvds\\ \le& C\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{-k-\beta}\langle v\rangle^{-\beta}|h(s,x-v(t-s),v_*)|dv_*dvds\\ =& C\sup_{0\le s\le t}\|h(s)\|_{L^\infty_{x,v}}\left(\int_{t-\lambda}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}+\int_0^{t-\lambda}\int_{|v|\ge N}\int_{\mathbb{R}^3}+\int_0^{t-\lambda}\int_{\mathbb{R}^3}\int_{|v_*|\ge N}+\int_0^{t-\lambda}\int_{|v|\le N}\int_{|v_*|\le N}\right)\{\cdots\}dv_*dvds\\ :=& I_1+I_2+I_3+I_4.\end{aligned}$$

For the term I_1, since $\beta, k > \max\{3, 3+\gamma\}$

$$I_1 \le C\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}\int_{t-\lambda}^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v_*\rangle^{-k-\beta}\langle v\rangle^{-\beta}dv_*dvds \le C_{k,\beta}\lambda\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}.$$

For the term I_2, since $\beta, k > \max\{3, 3+\gamma\}$

$$I_2 \le C\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_{|v|\ge N}\langle v\rangle^{-\beta}dv \le C_{k,\beta}\frac{1}{N^{\beta-3}}\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}.$$

For the term I_3, since $\beta, k > \max\{3, 3+\gamma\}$

$$\begin{aligned}I_3 \le& C\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}\int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\int_{|v_*|\ge N}|v-v_*|^\gamma\langle v_*\rangle^{-k-\beta}\langle v\rangle^{-\beta}dv_*dvds\\ \le& C\frac{1}{N^\beta}\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}\int_0^t\int_{\mathbb{R}^3} e^{-\nu(v)(t-s)}\int_{|v_*|\ge N}|v-v_*|^\gamma\langle v_*\rangle^{-k}\langle v\rangle^{-\beta}dv_*dvds\\ \le& C_k\frac{1}{N^\beta}\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_{\mathbb{R}^3}\langle v\rangle^{-\beta}dv \le C_{k,\beta}\frac{1}{N^\beta}\sup_{0\le s\le t}\|h(s)\|^2_{L^\infty_{x,v}}.\end{aligned}$$

For the I_4 term, since $\beta, k > \max\{3, 3+\gamma\}$, similar as (55) we have

$$\begin{aligned}
I_4 \leq & C \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \int_0^{t-\lambda} e^{-c(t-s)} \int_{|v|\leq N}\int_{|v_*|\leq N} |v-v_*|^\gamma \langle v_*\rangle^{-k-\beta}\langle v\rangle^{-\beta} |h(s, x-v(t-s), v_*)| dv_* dv ds \\
\leq & \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \int_0^{t-\lambda} e^{-c(t-s)} \left(\int_{|v|\leq N}\int_{|v_*|\leq N} |v-v_*|^{\gamma p}\langle v_*\rangle^{-kp-\beta p}\langle v\rangle^{-\beta p} dv_* dv\right)^{\frac{1}{p}} \\
& \left(\int_{|v|\leq N}\int_{|v_*|\leq N} |f(s, x-v(t-s), v_*)|^{\frac{p}{p-1}} dv_* dv\right)^{1-\frac{1}{p}} \\
\leq & C_{k,N,\beta} \sup_{0\leq s\leq t} \|h(s)\|^{1+\frac{1}{p}}_{L^\infty_{x,v}} \left(\int_{|v|\leq N}\int_{|v_*|\leq N} |f(s, x-v(t-s), v_*)| dv_* dv\right)^{1-\frac{1}{p}} \\
\leq & C_{k,N,\beta}\lambda^{-3}(\sqrt{H(F_0)} + H(F_0))^{1-\frac{1}{p}} \sup_{0\leq s\leq t} \|h(s)\|^{1+\frac{1}{p}}_{L^\infty_{x,v}},
\end{aligned}$$

where p is defined in (47). For the H_{22} term, we split it into four terms for some constant $\lambda, N > 0$ to be fixed later

$$\begin{aligned}
H_{22} \leq & C \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^k \langle v'_*\rangle^k} |f(s, x-v(t-s), v'_*)||f(s, x-v(t-s), v')| dv_* d\sigma dv ds \\
\leq & C \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^{k+\beta} \langle v'_*\rangle^{k+\beta}} |h(s, x-v(t-s), v'_*)||h(s, x-v(t-s), v')| dv_* d\sigma dv ds \\
= & C\left(\int_{t-\lambda}^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} + \int_0^{t-\lambda}\int_{|v|\geq N}\int_{\mathbb{R}^3} + \int_0^{t-\lambda}\int_{\mathbb{R}^3}\int_{|v_*|\geq N} + \int_0^{t-\lambda}\int_{|v|\leq N}\int_{|v_*|\leq N}\right)\{\cdots\} dv_* dv ds := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the I_1 term, since $\beta, k > \max\{3, 3+\gamma\}$, by Lemma 2.11

$$I_1 \leq C \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}} \int_{t-\lambda}^t \int_{\mathbb{R}^3} \langle v\rangle^{-\beta+\gamma} dv ds \leq C_{k,\beta}\lambda \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}}.$$

For the I_2 term, since $\beta, k > \max\{3, 3+\gamma\}$ still by Lemma 2.11

$$I_2 \leq C \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}} \int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \int_{|v|\geq N} \langle v\rangle^{-\beta} dv \leq C_{k,\beta} \frac{1}{N^{\beta-3}} \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}}.$$

For the I_3 term, since $\beta, k > \max\{3, 3+\gamma\}$, since $\langle v_*\rangle \leq \langle v'\rangle\langle v'_*\rangle$, still by Lemma 2.11 apply for $k+\beta-\frac{\beta-3}{2}$ we have

$$\begin{aligned}
I_3 \leq & C \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}} \int_0^t \int_{\mathbb{R}^3} e^{-\nu(v)(t-s)} \int_{|v_*|\geq N} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^{k+\beta-\frac{\beta-3}{2}}\langle v'_*\rangle^{k+\beta-\frac{\beta-3}{2}}\langle v_*\rangle^{\frac{\beta-3}{2}}} dv_* dv ds \\
\leq & C_{k,\beta} \frac{1}{N^{\frac{\beta-3}{2}}} \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}} \int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \int_{\mathbb{R}^3} \langle v\rangle^{-\beta+\frac{\beta-3}{2}} dv \leq C_{k,\beta} \frac{1}{N^{\frac{\beta-3}{2}}} \sup_{0\leq s\leq t} \|h(s)\|^2_{L^\infty_{x,v}}.
\end{aligned}$$

For the I_4 term, if $\beta, k > \max\{3, 3+\gamma\}$, similar as (55) we have

$$\begin{aligned}
I_4 \leq & C \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \int_0^{t-\lambda} e^{-c(t-s)} \int_{|v|\leq N}\int_{|v_*|\leq N}\int_{\mathbb{S}^2} |v-v_*|^\gamma \frac{\langle v\rangle^k}{\langle v'\rangle^{k+\beta}\langle v'_*\rangle^{k+\beta}} |h(s, x-v(t-s), v')| dv_* dv ds \\
\leq & C_{k,N,\beta} \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \int_0^{t-\lambda} e^{-c(t-s)} \left(\int_{|v|\leq N}\int_{|v_*|\leq N}\int_{\mathbb{S}^2} |v-v_*|^{\gamma p} \frac{\langle v\rangle^{pk}}{\langle v'\rangle^{pk}\langle v'_*\rangle^{pk}} d\sigma dv_* dv\right)^{\frac{1}{p}} \\
& \left(\int_{|v|\leq N}\int_{|v_*|\leq N}\int_{\mathbb{S}^2} |f(s, x-v(t-s), v')|^{\frac{p}{p-1}} d\sigma dv_* dv\right)^{1-\frac{1}{p}} \\
\leq & C_{k,N,\beta} \sup_{0\leq s\leq t} \|h(s)\|_{L^\infty_{x,v}} \int_0^{t-\lambda} e^{-c(t-s)} \left(\int_{|v'|\leq 2N}\int_{|v'_*|\leq 2N}\int_{\mathbb{S}^2} |f(s, x-v(t-s), v')|^{\frac{p}{p-1}} d\sigma dv'_* dv'\right)^{1-\frac{1}{p}} \\
\leq & C_{k,N,\beta} \lambda^{-3} (\sqrt{H(F_0)} + H(F_0))^{1-\frac{1}{p}} \sup_{0\leq s\leq t} \|h(s)\|^{1+\frac{1}{p}}_{L^\infty_{x,v}},
\end{aligned}$$

since

$$\{|v|\leq N, |v_*|\leq N, \sigma\in\mathbb{S}^2\} \quad \subset \quad \{|v'|\leq 2N, |v'_*|\leq 2N, \sigma\in\mathbb{S}^2\},$$

and p is defined in (47). The theorem is thus proved by gathering all the terms. □

Then we come to the proof for the global existence for the Boltzmann equation with large amplitude initial data.

Proof. (**Proof of Theorem 1.3**) Fix $\beta, k > \max\{3, 3+\gamma\}$ satisfies the assumption in Lemma 6.6 and Lemma 6.7. By the assumption in Theorem 1.3 we have $\|h_0\|_{L^\infty_{x,v}} \leq M$. We first assume that

$$\|h(t)\|_{L^\infty_{x,v}} \leq 2A_0 := 2C_{k,\beta}(2M^2 + \sqrt{H(F_0)} + H(F_0)), \tag{56}$$

where $C_{k,\beta}$ is defined in Lemma 6.6. By Lemma 6.6 and the priori assumption (56) we have

$$\|h(t)\|_{L^\infty_{x,v}} \leq A_0 + C_{k,\beta}(2A_0)^{\frac{3p+1}{2p}} \cdot \sup_{t_1\leq s\leq t, y\in\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |f(s,y,\eta)| d\eta\right)^{\frac{p-1}{2p}}. \tag{57}$$

Since $x\in\mathbb{T}^3$, for any $t\geq t_1$ we have

$$\begin{aligned}
\int_{\mathbb{R}^3} e^{-\nu(t)} |f_0(x-vt, v)| dv \leq & \left(\int_{|v|\geq\lambda} + \int_{|v|\leq\lambda}\right) |f_0(x-vt, v)| dv \\
\leq & C\|w_\beta f_0\|^{\frac{3}{\beta}}_{L^\infty_{x,v}} \|f_0\|^{1-\frac{3}{\beta}}_{L^1_x L^\infty_v} + \frac{C}{t_1^3}\|f_0\|_{L^1_x L^\infty_v} \\
\leq & CM^{\frac{3}{\beta}} \|f_0\|^{1-\frac{3}{\beta}}_{L^1_x L^\infty_v} + CM^3 \|f_0\|_{L^1_x L^\infty_v},
\end{aligned}$$

where we have chosen $\lambda = \|w_\beta f_0\|^{\frac{1}{\beta}}_{L^\infty_{x,v}} \|f_0\|^{-\frac{1}{\beta}}_{L^1_x L^\infty_v}$ and t_1 is defined in Lemma 6.4. By Lemma 6.7 and the priori assumption (56) we have

$$\begin{aligned}
\sup_{t_1\leq s\leq t, y\in\mathbb{T}^3} \int_{\mathbb{R}^3} |f(s,y,\eta)| d\eta \leq & CM^3 \|f_0\|_{L^1_x L^\infty_v} + CM^{\frac{3}{\beta}} \|f_0\|^{1-\frac{3}{\beta}}_{L^1_x L^\infty_v} + C_{k,N,\beta}\lambda^{-\frac{3}{2}} \sqrt{H(F_0)} + C_{k,N,\beta}\lambda^{-3} H(F_0) \\
& + C_{k,\beta}(\lambda + \frac{1}{N^{\frac{\beta-3}{2}}})(2A_0)^2 + C_{k,N,\beta}\lambda^{-3}(\sqrt{H(F_0)} + H(F_0))^{1-\frac{1}{p}} (2A_0)^{1+\frac{1}{p}}.
\end{aligned}$$

First choose N large then choose λ small, finally choose $H(F_0)$ and $\|f_0\|_{L^1_x L^\infty_v}$ small we deduce

$$4C_{k,\beta} A_0^{\frac{p+1}{2p}} \sup_{t_1\leq s\leq t} \left(\int_{\mathbb{R}^3} |f(s,y,\eta)| d\eta\right)^{\frac{p-1}{2p}} \leq \frac{1}{2}, \tag{58}$$

together with (57) implies that

$$\|h(t)\|_{L^\infty_{x,v}} \le \frac{7}{4}A_0, \quad \forall t \ge 0,$$

hence we have closed the priori assumption (56), the proof for the global existence is thus finished. $\square$

7. Convergence rate for the Boltzmann equation with large amplitude initial data

In this section, we consider the time-decay estimates for the global solution we obtained in Section 6. In the whole section we are under the assumption in Theorem 1.3 such that all the results in Section 6 remain true. In this section we will denote f the solution to (45) and denote h the solution to (49).

7.1. **Convergence rate for Hard Potentials.** In this subsection we consider the decay estimate for hard potential. We first recall a lemma on the convolution of semigroups.

Lemma 7.1. *For $\lambda_2 > \lambda_1 > 0, t > 0$. we have*

$$\int_0^t e^{-\lambda_1(t-s)}e^{-\lambda_2 s}ds = \int_0^t e^{-\lambda_1 s}e^{-\lambda_2(t-s)}ds = e^{-\lambda_1 t}\int_0^t e^{-(\lambda_2-\lambda_1)s}ds \le \frac{1}{\lambda_2-\lambda_1}e^{-\lambda_1 t}.$$

For the case $\gamma \ge 0$, we first prove that the linearized equation converges. We consider the following linearized equation

$$\xi_t + v\cdot\nabla_x\xi + \nu(v)\xi + K_k\xi = 0, \quad \xi(0,x,v) = \xi_0(x,v). \tag{59}$$

For the linearized equation (59), the corresponding mild solution is

$$\xi(t,x,v) = e^{-\nu(v)t}\xi_0(x-vt,v) + \int_0^t e^{-\nu(v)(t-s)}(K_k\xi)(s,x-v(t-s),v)ds. \tag{60}$$

Lemma 7.2. *There exists a $k_0 \ge 6$ large such that for any $k \ge k_0, \gamma \in [0,1]$, suppose $\xi(t)$ is the solution to the linearized equation (59), we have*

$$\|\xi(t)\|_{L^\infty_{x,v}} \le C_k e^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}}, \quad \forall t \ge 0,$$

for some constants $C_k, \lambda_2 > 0$.

Proof. The proof is similar to Lemma 6.6. First (60) implies

$$|\xi(t,x,v)| \le e^{-\nu(v)t}\|\xi_0\|_{L^\infty_{x,v}} + \int_0^t e^{-\nu(v)(t-s)}|(K_k\xi)(s,x-v(t-s),v)|ds := e^{-\nu(v)t}\|\xi_0\|_{L^\infty_{x,v}} + J_2.$$

For the J_2 term, denote $\tilde{x} = x - v(t-s)$, we have

$$J_2 \le \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')\xi(s,\tilde{x},v')|dv'ds.$$

by (60) again we have

$$\begin{aligned} J_2 \le & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|e^{-\nu(v')s}|\xi_0(\tilde{x}-v's,v')|dv'ds \\ &+ \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_k(v,v')l_k(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|\xi(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds := J_{21}+J_{22}. \end{aligned}$$

Denote $\lambda_2 := \min\{\lambda_1, \nu(v)|v \in \mathbb{R}^d\}$, where λ_1 is the exponential convergence rate for the linearized semi-group in L^2 proved in Lemma 3.12. For the J_{21} term we have

$$J_{21} \le C_k e^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}}\int_0^t e^{-\frac{\nu(v)}{2}(t-s)}\langle v\rangle^\gamma ds \le C_k e^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}}. \tag{61}$$

For the J_{22} term, if $|v| \geq N$, by (51) we have

$$
\begin{aligned}
J_{22} \leq & \sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_k(v,v')l_k(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}e^{-\frac{\lambda_2}{2}\tau}dv''d\tau dv'ds \\
\leq & e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\frac{\nu(v)}{2}(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_k(v,v')l_k(v',v'')|\int_0^s e^{-\frac{\nu(v')}{2}(s-\tau)}dv''d\tau dv'ds \\
\leq & \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N^2}\right)e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}.
\end{aligned}
$$

For the case $|v|\leq N$, similarly as decomposition (53) we have, for any $N>0, k\geq 6$ we can find a bounded compact support function $l_{k,N}$ such that

$$
l_{k,N}(v,v') := l_k(v,v')1_{\frac{\langle v\rangle}{C_{k,N}}\leq|v-v'|\leq C_{k,N}\langle v\rangle}, \quad \int_{\mathbb{R}^3}|l_k(v,v')-l_{k,N}(v,v')|dv' \leq \frac{C_k}{N}\langle v\rangle^\gamma, \quad \forall v\in\mathbb{R}^3, \tag{62}
$$

for some large constant $C_{k,N}>0$. By

$$
l_k(v,v')l_k(v',v'') = (l_k(v,v')-l_{k,N}(v,v'))l_k(v',v'')+l_{k,N}(v,v')(l_k(v',v'')-l_{k,N}(v',v''))+l_{k,N}(v,v')l_{k,N}(v',v''), \tag{63}
$$

we split J into three terms respectively. For the first term we have

$$
\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|(l_k(v,v')-l_{k,N}(v,v'))l_k(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|\xi(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds \\
\leq & \frac{C_k}{N}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma e^{-\frac{\lambda_2}{2}\tau}d\tau \\
\leq & \frac{C_k}{N}e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\},
\end{aligned}
$$

and the second term can be estimated similarly. For the last term, since $l_{k,N}(v',v'')l_{k,N}(v',v'')$ is supported in $|v|\leq N, |v'|\leq C'_{k,N}, |v''|\leq C'_{k,N}$ for some constant $C'_{k,N}>0$. We again split it into two cases, $\tau\in[s-\lambda,s]$ and $\tau\in[0,s-\lambda]$, where $\lambda>0$ is a small constant to be fixed later. For the case $\tau\in[s-\lambda,s]$, since $|v'|\leq C'_{k,N}$, we have

$$
\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N}(v,v')l_{k,N}(v',v'')|\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}|\xi(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds \\
\leq & C_{k,N}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma e^{-\frac{\lambda_2}{2}\tau}d\tau \\
\leq & C_{k,N}e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\frac{\nu(v)}{2}(t-s)}\langle v\rangle^\gamma ds\int_{s-\lambda}^s e^{-\frac{\nu(v')}{2}(s-\tau)}\langle v'\rangle^\gamma d\tau \\
\leq & C_{k,N}e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}(1-e^{-\frac{\nu(v')}{2}\lambda}) \leq C_{k,N}\lambda e^{-\frac{\lambda_2}{2}t}\sup_{0\leq\tau\leq t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}.
\end{aligned}
$$

For the case $\tau\in[0,s-\lambda]$ since

$$
l_{k,N}(v,v')\leq C_{k,N}, \quad l_{k,N}(v',v'')\leq C_{k,N}, \quad \frac{1}{C_{k,N}}\leq\nu(v)\leq C_{k,N}, \quad \frac{1}{C_{k,N}}\leq\nu(v')\leq C_{k,N}, \tag{64}
$$

denote $\xi_1(t,x,v)=\langle v\rangle^{-\frac{7}{4}}\xi(t,x,v)$, since $k-\frac{7}{4}>4$, by Lemma 3.12 apply for ξ_1 we have

$$\begin{aligned}&\int_{|v'|\le C'_{k,N},|v''|\le C'_{k,N}}|\xi(\tau,\tilde{x}-v'(s-\tau),v'')|dv'dv''\\ \le& C_{k,N}\int_{|v'|\le C'_{k,N},|v''|\le C'_{k,N}}|\xi_1(\tau,\tilde{x}-v'(s-\tau),v'')|dv'dv''\\ \le& C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}\left(\int_{\mathbb{T}^3}\int_{|v''|\le C'_{k,N}}|\xi_1(\tau,y,v'')|^2dv''dy\right)^{\frac{1}{2}}\\ \le& C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}\|\xi_1(\tau)\|_{L^2_{x,v}}\le C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}e^{-\lambda_1\tau}\|\xi_1(0)\|_{L^2_{x,v}}\le C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}e^{-\lambda_1\tau}\|\xi_0\|_{L^\infty_{x,v}},\end{aligned}$$

where we have made a change of variable $y=\tilde{x}-v'(s-\tau)$. Since $s-\tau\ge\lambda$, we have

$$\begin{aligned}&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N}(v,v')l_{k,N}(v',v'')|\int_0^{s-\lambda}e^{-\nu(v')(s-\tau)}|\xi(\tau,\tilde{x}-v'(s-\tau,v''))|dv''d\tau dv'ds\\ \le& C_{k,N}\int_0^t e^{-\lambda_2(t-s)}\int_0^{s-\lambda}e^{-\lambda_2(s-\tau)}\int_{|v'|\le C'_{k,N},|v''|\le C'_{k,N}}|\xi(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\ \le& C_{k,N}\lambda^{-\frac{3}{2}}\|\xi_0\|_{L^\infty_{x,v}}\int_0^t e^{-\lambda_2(t-s)}\int_0^{s-\lambda}e^{-\lambda_2(s-\tau)}e^{-\lambda_1\tau}d\tau ds\le C_{k,N}\lambda^{-\frac{3}{2}}e^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}}.\end{aligned}$$

Gathering all the terms we have

$$\|\xi(t)\|_{L^\infty_{x,v}}\le C_{k,N,\lambda}e^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}}+\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N}+C_{k,N}\lambda\right)e^{-\frac{\lambda_2}{2}t}\sup_{0\le\tau\le t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}.$$

First choose k large, then let N be sufficiently large and finally let λ be sufficiently small such that

$$\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N}+C_{k,N}\lambda\le\frac{1}{2}.$$

Multiply both side by $e^{\frac{\lambda_2}{2}t}$ and taking supremum we have

$$\sup_{0\le\tau\le t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\}\le C_k\|\xi_0\|_{L^\infty_{x,v}}+\frac{1}{2}\sup_{0\le\tau\le t}\{e^{\frac{\lambda_2}{2}\tau}\|\xi(\tau)\|_{L^\infty_{x,v}}\},$$

which implies

$$\|\xi(t)\|_{L^\infty_{x,v}}\le C_ke^{-\frac{\lambda_2}{2}t}\|\xi_0\|_{L^\infty_{x,v}},$$

so the convergence for the linear semigroup is thus proved. □

Denote $g(t,x,v)=\langle v\rangle^{-2}h(t,x,v)=\langle v\rangle^{\beta-2}f(t,x,v)$, it is easily seen that g satisfies

$$\partial_tg+v\cdot\nabla_xg+\nu(v)g=\Gamma_{k+\beta-2}(g,g)+K_{k+\beta-2}g. \tag{65}$$

We first prove the rate of convergence for g.

Lemma 7.3. *There exists a $k_0\ge 8$ large such that for any $\gamma\in[0,1]$, $k\ge k_0$, $\beta>\max\{3,3+\gamma\}$, suppose g is the solution to the nonlinear equation (65), we have*

$$\|g(t)\|_{L^\infty_{x,v}}\le C_{k,\beta}e^{-\frac{\lambda_2}{4}t}\|g\|_{L^\infty_{x,v}},\quad\forall t\ge 0,$$

for some constant $C_{k,\beta}>0$, where λ_2 is defined in Lemma 7.2.

Proof. Denote the solution to the linearized equation

$$\partial_t g + v\cdot\nabla_x g + \nu(v) g = K_{k+\beta-2} g, \quad g(0,x,v) = g_0(x,v),$$

by $g(t) = V(t)g_0$. By Lemma 7.2 we have

$$\|V(t)g\|_{L^\infty_{x,v}} \le C_{k,\beta} e^{-\frac{\lambda_2}{2}t}\|g\|_{L^\infty_{x,v}}.$$

By Duhamel's principle we have

$$g(t) = V(t)g_0 + \int_0^t V(t-s)\{\Gamma_{k+\beta-2}(g,g)(s)\}ds = V(t)g_0 + \int_0^t V(t-s)\{\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\}ds.$$

We easily compute

$$\|g(t)\|_{L^\infty_{x,v}} \le C_{k,\beta} e^{-\frac{\lambda_2}{2}t}\|g_0\|_{L^\infty_{x,v}} + \left\|\int_0^t V(t-s)\{\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\}ds\right\|_{L^\infty_{x,v}}.$$

For the second term, using Duhamel's principle again we have

$$\begin{aligned}\int_0^t V(t-s)\{\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\}ds =& \int_0^t e^{-\nu(v)(t-s)}\{\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\}ds\\ &+\int_0^t\int_s^t e^{-\nu(v)(t-\tau)}K_{k+\beta-2}\{V(\tau-s)(\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s))\}d\tau ds\\ =&\int_0^t e^{-\nu(v)(t-s)}\{\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\}ds\\ &+\int_0^t\int_0^\tau e^{-\nu(v)(t-\tau)}K_{k+\beta-2}\{V(\tau-s)(\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s))\}dsd\tau := A_1 + A_2.\end{aligned}$$

For the term A_1, since $\beta-2\ge 1$, by Lemma 6.5 we have

$$\begin{aligned}|A_1| \le& C_{k,\beta}\int_0^t e^{-\nu(v)(t-s)}\nu(v)\|g(s)\|_{L^\infty_{x,v}}^{\frac{3p+1}{2p}}\sup_{y\in\mathbb{T}^3}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}ds\\ \le& C_{k,\beta}\int_0^t e^{-\nu(v)(t-s)}\nu(v)e^{-\frac{\lambda_2}{4}s}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\left[e^{\frac{\lambda_2}{4}s}\|g(s)\|_{L^\infty_{x,v}}\right]\|g(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}ds\\ \le& C_{k,\beta}e^{-\frac{\lambda_2}{4}t}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\left[e^{\frac{\lambda_2}{4}s}\|g(s)\|_{L^\infty_{x,v}}\right]\|g(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.\end{aligned}$$

For the A_2 term, since $\gamma\le 1$ and $\frac{p+1}{2p}\ge\frac{1}{2}$ we have

$$\|\langle v\rangle^\gamma g(s)\|_{L^\infty_{x,v}} \le \|\langle v\rangle^2 g(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\|g(s)\|_{L^\infty_{x,v}}^{\frac{p-1}{2p}} = \|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\|g(s)\|_{L^\infty_{x,v}}^{\frac{p-1}{2p}},$$

together with Lemma 6.5 we have

$$\begin{aligned}\|V(\tau-s)(\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s))\|_{L^\infty_{x,v}} \le& Ce^{-\frac{\lambda_2}{2}(\tau-s)}\|\langle v\rangle^{\beta-2}\Gamma_k(f,f)(s)\|_{L^\infty_{x,v}}\\ \le& Ce^{-\frac{\lambda_2}{2}(\tau-s)}\|\langle v\rangle^\gamma g(s)\|_{L^\infty_{x,v}}\|g(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\sup_{y\in\mathbb{T}^3}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\\ \le& Ce^{-\frac{\lambda_2}{2}(\tau-s)}\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\|g(s)\|_{L^\infty_{x,v}}\sup_{y\in\mathbb{T}^3}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}},\end{aligned}$$

which implies

$$
\begin{aligned}
|A_2| \le &\int_0^t \int_0^\tau e^{-\nu(v)(t-\tau)} \int_{\mathbb{R}^3} |l_{k+\beta-2}(v,v')| dv' \| V(\tau-s)(\langle v\rangle^{\beta-2}\Gamma(f,f)(s))\|_{L^\infty_{x,v}} ds d\tau \\
\le & C_{k,\beta} \int_0^t \int_0^\tau e^{-\nu(v)(t-\tau)} \nu(v) e^{-\frac{\lambda_2}{2}(\tau-s)} \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \|g(s)\|_{L^\infty_{x,v}} \sup_{y\in\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} ds d\tau \\
\le & C_{k,\beta} \int_0^t \int_0^\tau e^{-\nu(v)(t-\tau)} \nu(v) e^{-\frac{\lambda_2}{2}(\tau-s)} e^{-\frac{\lambda_2}{4}s} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{ \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\} ds d\tau \\
\le & C_{k,\beta} e^{-\frac{\lambda_2}{4}t} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{ \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\}.
\end{aligned}
$$

Gathering the terms and taking supremum we have

$$
\begin{aligned}
\sup_{0\le s\le t} \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \le & C_{k,\beta} \|g_0\|_{L^\infty_{x,v}} + C_{k,\beta} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{ \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\} \\
\le & C_{k,\beta} \|g_0\|_{L^\infty_{x,v}} + C_{k,\beta} \sup_{0\le s\le t_1} \|h(s)\|^2_{L^\infty_{x,v}} + C_{k,\beta} \sup_{0\le s\le t} \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \\
& \times \sup_{t_1\le s\le t, y\in\mathbb{T}^3} \left\{ \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\} \\
\le & C_{k,\beta} M^4 + C_{k,\beta} \sup_{0\le s\le t} \left[e^{\frac{\lambda_2}{4}s} \|g(s)\|_{L^\infty_{x,v}}\right] \sup_{t_1\le s\le t, y\in\mathbb{T}^3} \left\{ \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\},
\end{aligned}
$$

together with (58) we deduce

$$
e^{\frac{\lambda_2}{4}t} \|g(t)\|_{L^\infty_{x,v}} \le 2C_{k,\beta} M^4,
$$

so the lemma is thus proved. □

Proof. (**Proof of Theorem 1.3**) Finally we come to prove the rate of convergence for h. By (50) we have

$$
\begin{aligned}
|h(t,x,v)| \le & e^{-\nu(v)t} \|h_0\|_{L^\infty_{x,v}} + \int_0^t e^{-\nu(v)(t-s)} |(K_{k+\beta} h)(s, x - v(t-s), v)| ds \\
& + \int_0^t e^{-\nu(v)(t-s)} |\Gamma_{k+\beta}(h,h)(s, x-v(t-s), v)| ds := e^{-\nu(v)t} \|h_0\|_{L^\infty_{x,v}} + J_2 + J_3.
\end{aligned}
$$

For the J_3 term by Lemma 6.5 we have

$$
\begin{aligned}
J_3 \le & C_{k,\beta} \int_0^t e^{-\nu(v)(t-s)} \langle v\rangle^\gamma \|h(s)\|_{L^\infty_{x,v}} \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \sup_{y\in\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} ds \\
\le & C_{k,\beta} e^{-\frac{\lambda_2}{4}t} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{ \left[e^{\frac{\lambda_2}{4}s} \|h(s)\|_{L^\infty_{x,v}}\right] \|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}} \right\}.
\end{aligned}
$$

For the J_2 term, by (50) again we have

$$
\begin{aligned}
J_2 \le & \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{k+\beta}(v,v')| e^{-\nu(v')s} |h_0(\tilde{x} - v's, v')| dv' ds \\
& + \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} |l_{k+\beta}(v,v')| \int_0^s e^{-\nu(v')(s-\tau)} |\Gamma_{k+\beta}(h,h)| (\tau, \tilde{x} - v'(s-\tau), v') d\tau dv' ds \\
& + \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |l_{k+\beta}(v,v') l_{k+\beta}(v',v'')| \int_0^s e^{-\nu(v')(s-\tau)} |h(\tau, \tilde{x} - v'(s-\tau), v'')| dv'' d\tau dv' ds \\
& := J_{21} + J_{22} + J_{23}.
\end{aligned}
$$

For the J_{21} term, similarly as (61)

$$J_{21} \le C_k e^{-\frac{\lambda_2}{2}t}\|h_0\|_{L^\infty_{x,v}} \int_0^t e^{-\frac{\nu(v)}{2}(t-s)}\langle v\rangle^\gamma ds \le C_k e^{-\frac{\lambda_2}{2}t}\|h_0\|_{L^\infty_{x,v}}.$$

For the J_{22} term by Lemma 6.5 we have

$$\begin{aligned} J_{22} \le & C_{k,\beta}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma e^{-\frac{\lambda_2}{4}\tau} \\ & \times \sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\left[e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}d\tau dv' ds \\ \le & C_{k,\beta}e^{-\frac{\lambda_2}{4}t}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\left[e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}. \end{aligned}$$

For the term J_{23}, we split into two case $|v|\ge N$ and $|v|\le N$, for the case $|v|\ge N$, by (51) we have

$$\begin{aligned} J_{23} \le & \sup_{0\le s\le t}\{e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k+\beta}(v,v')l_{k+\beta}(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}e^{-\frac{\lambda_2}{4}\tau}dv'' d\tau dv' ds \\ \le & \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N^2}\right)e^{-\frac{\lambda_2}{4}t}\sup_{0\le s\le t}\{e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\}. \end{aligned}$$

For the case $|v|\le N$, by (52) and (53) we split J_{23} into three terms respectively. For the first term we have

$$\begin{aligned} & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|(l_{k+\beta}(v,v')-l_{k,N,\beta}(v,v'))l_{k+\beta}(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv'' d\tau dv' ds \\ \le & \frac{C_{k,\beta}}{N}\sup_{0\le s\le t}\{e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma e^{-\frac{\lambda_2}{4}\tau}d\tau \le e^{-\frac{\lambda_2}{4}t}\frac{C_{k,\beta}}{N}\sup_{0\le s\le t}\{e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\}, \end{aligned}$$

the second term can be estimated similarly. For the third term, we have $l_{k,N,\beta}(v',v'')l_{k,N,\beta}(v',v'')$ is supported in $|v|\le N, |v'|\le C'_{k,N,\beta}, |v''|\le C'_{k,N,\beta}$, for some constant $C'_{k,N,\beta}>0$. By Lemma 7.2 we have

$$\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv'dv'' \le C_{k,N,\beta}\|g(\tau)\|_{L^\infty_{x,v}} \le C_{k,N,\beta}e^{-\frac{\lambda_2}{4}\tau}\|g_0\|_{L^\infty_{x,v}} \le C_{k,N,\beta}e^{-\frac{\lambda_2}{4}\tau}\|h_0\|_{L^\infty_{x,v}},$$

which implies

$$\begin{aligned} & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N,\beta}(v,v')l_{k,N,\beta}(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|h(\tau,\tilde{x}-v'(s-\tau,v''))|dv'' d\tau dv' ds \\ \le & C_{k,N,\beta}\int_0^t e^{-\lambda_2(t-s)}\int_0^s e^{-\lambda_2(s-\tau)}\int_{|v'|\le C'_{k,N,\beta},|v''|\le C'_{k,N,\beta}}|h(\tau,\tilde{x}-v'(s-\tau),v'')|dv'' d\tau dv' ds \\ \le & C_{k,N,\beta}\|h_0\|_{L^\infty_{x,v}}\int_0^t e^{-\lambda_2(t-s)}\int_0^s e^{-\lambda_2(s-\tau)}e^{-\frac{\lambda_2}{4}\tau}d\tau ds \le C_{k,N,\beta}e^{-\frac{\lambda_2}{4}t}\|h_0\|_{L^\infty_{x,v}}. \end{aligned}$$

Gathering all the terms we have

$$\begin{aligned} \|h(t)\|_{L^\infty_{x,v}} \le & C_{k,\beta,N}e^{-\frac{\lambda_2}{4}t}\|h_0\|_{L^\infty_{x,v}} + \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N}\right)e^{-\frac{\lambda_2}{4}t}\sup_{0\le s\le t}\{e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\} \\ & + C_{k,\beta}e^{-\frac{\lambda_2}{4}t}\sup_{0\le s\le t, y\in\mathbb{T}^3}\left\{\left[e^{\frac{\lambda_2}{4}s}\|h(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}. \end{aligned}$$

Taking suitable $k,\beta,N>0$ such that

$$\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_{k,\beta}}{N}\le\frac{1}{4},$$

together with (58) we conclude that

$$e^{\frac{\lambda_2}{4}t}\|h(t)\|_{L^\infty_{x,v}} \le 4C_{k,\beta}M^4,$$

for some constant $C_{k,\beta} > 0$. The rate of convergence for $\gamma \in [0,1]$ is thus proved. □

7.2. **Convergence rate for the case $-3 < \gamma < 0$.** Next we come to prove the convergence for the case $-3 < \gamma < 0$. Before going to the proof, we first prove some useful lemmas.

Lemma 7.4. *For any $a, b, k > 0$ we have*

$$\sup_{0<x\le b} x^k e^{-ax} \le C_{b,k}(1+a)^{-k},$$

for some constant $C_{b,k} > 0$. As a consequence we have for $-3 < \gamma \le 0$, for all $t > 0, k \ge 0$

$$\sup_{v\in\mathbb{R}^d}\{e^{-\nu(v)t}\nu(v)^k\} \le C(1+t)^{-k},$$

for some constant $C > 0$.

Proof. Take $f(x) = x^k e^{-ax}$, we have

$$f'(x) = kx^{k-1}e^{-ax} - ax^k e^{-ax} = x^{k-1}(k-ax)e^{-ax},$$

so $f'(x) = 0$ when $x = \frac{k}{a}$. We easily deduce that

$$\sup_{0\le x\le b} f(x) = f(\frac{k}{a}) = \frac{k^k}{a^k}e^{-k}, \quad \text{if} \quad \frac{k}{a} \le b, \quad \sup_{0\le x\le b} f(x) = f(b) = b^k e^{-ab}, \quad \text{if} \quad \frac{k}{a} > b,$$

so the first statement is thus proved, the second statement just from the fact that $\gamma \le 0$ implies that $0 < \nu(v) \le C$ for some constant $C \ge 0$. □

Lemma 7.5. *If $-3 < \gamma < 0$, then for any $0 < r < 1$ we have*

$$\int_0^t e^{-\nu(v)(t-s)}\nu(v)(1+s)^{-r}ds \le C_r(1+t)^{-r}, \quad \forall v \in \mathbb{R}^d, \quad \forall t \ge 0,$$

for some constant $C_r > 0$ independent of v.

Proof. We split the integral into two parts, $0 \le s \le \frac{t}{2}$ and $\frac{t}{2} \le s \le t$. For the first part we have $t - s \ge \frac{t}{2}$, so by Lemma 7.4 we have

$$\int_0^{\frac{t}{2}} e^{-\nu(v)(t-s)}\nu(v)(1+s)^{-r}ds \le C\int_0^{\frac{t}{2}}(1+t-s)^{-1}(1+s)^{-r}ds \le C(1+t)^{-1}\int_0^{\frac{t}{2}}(1+s)^{-r}ds \le C_r(1+t)^{-r}.$$

For the second part we have $s \ge \frac{t}{2}$, this time we have

$$\int_{\frac{t}{2}}^t e^{-\nu(v)(t-s)}\nu(v)(1+s)^{-r}ds \le C_r(1+t)^{-r}\int_{\frac{t}{2}}^t e^{-\nu(v)(t-s)}\nu(v)ds \le C_r(1+t)^{-r}(1-e^{-\nu(v)\frac{t}{2}}) \le C_r(1+t)^{-r},$$

the proof is thus finished by gathering the two cases. □

Remark 7.6. *If we directly use $(1+t-s)^{-1}$ instead of $e^{-\nu(v)(t-s)}\nu(v)$, we will have*

$$\int_0^t (1+t-s)^{-1}(1+s)^{-r}ds \le C_r(1+t)^{-r}\log(1+t),$$

where an extra term $\log(1+t)$ occurs.

Lemma 7.7. *If $\gamma \in (-3,0)$, for any $l \ge 2, k \ge 3$ we have*

$$\|\langle v\rangle^l\Gamma_k(f,f)\|_{L^2} \le C\|f\|_{L^2}\|\langle v\rangle^l f\|_{L^2}^{1+\frac{\gamma}{3}}\|\langle v\rangle^l f\|_{L^\infty}^{-\frac{\gamma}{3}} \le C\|f\|_{L^2}\|\langle v\rangle^{2l}f\|_{L^\infty}^{\frac{p+1}{2p}}\|f\|_{L^1}^{\frac{p-1}{2p}},$$

where p is defined in (47).

Proof. For the $\Gamma_k^+(f,f)$ term we prove by duality, for any smooth function h we have

$$|(\langle v\rangle^l\Gamma_k^+(f,f),h)| \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^k}{\langle v_*'\rangle^k\langle v'\rangle^k}b(\cos\theta)|f(v_*')||f(v')||h(v)|\langle v\rangle^l dvdv_*d\sigma$$
$$\le\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*')|^2|f(v')|^2\langle v\rangle^{2l+\gamma}dvdv_*d\sigma\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^{2k}}{\langle v_*'\rangle^{2k}\langle v'\rangle^{2k}}b(\cos\theta)|h(v)|^2\langle v\rangle^{-\gamma}dvdv_*d\sigma\right)^{1/2}.$$

By Lemma 2.11 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\frac{\langle v\rangle^{2k}}{\langle v_*'\rangle^{2k}\langle v'\rangle^{2k}}b(\cos\theta)|h(v)|^2\langle v\rangle^{-\gamma}dvdv_*d\sigma\le\|h\|_{L^2}.$$

Since $2l+\gamma\ge0$, by pre-post collisional change of variables we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*')|^2|f(v')|^2\langle v\rangle^{2l+\gamma}dvdv_*d\sigma$$
$$\le C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*)|^2|f(v)|^2\langle v_*\rangle^{2l+\gamma}dvdv_*d\sigma$$
$$+C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*)|^2|f(v)|^2\langle v\rangle^{2l+\gamma}dvdv_*d\sigma:=I_1+I_2.$$

Without loss of generality we only compute I_1, by Lemma 2.5 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*)|^2|f(v)|^2\langle v_*\rangle^{2l+\gamma}dvdv_*d\sigma$$
$$\le C\|f\|_{L^2}^2\left(\sup_{v\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^\gamma|f(v_*)|^2\langle v_*\rangle^{2l+\gamma}dv_*\right)\le C\|f\|_{L^2}^2\|f^2\|_{L^1_{2l}}^{1+\frac{\gamma}{3}}\|f^2\|_{L^\infty_{2l}}^{-\frac{\gamma}{3}}\le C\|f\|_{L^2_l}^2\|f\|_{L^2_l}^{2+\frac{2\gamma}{3}}\|f\|_{L^\infty_l}^{-\frac{2\gamma}{3}},\quad(66)$$

the $\Gamma_k^+(f,f)$ term is proved by duality. For the $\Gamma_k^-(f,f)$ term, we easily compute

$$(\langle v\rangle^l\Gamma_k^-(f,f),h)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)f(v_*)\langle v_*\rangle^{-k}\langle v\rangle^l f(v)h(v)dvdv_*d\sigma$$
$$\le\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma b(\cos\theta)|f(v_*)|^2|f(v)|^2\langle v\rangle^{2l+\gamma}dvdv_*d\sigma\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\langle v_*\rangle^{-2k}b(\cos\theta)|h(v)|^2\langle v\rangle^{-\gamma}dvdv_*d\sigma\right)^{1/2}.$$

The first term is the same as (66), for the second term we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|^\gamma\langle v_*\rangle^{-2k}b(\cos\theta)|h(v)|^2\langle v\rangle^{-\gamma}dvdv_*d\sigma\le\|h\|_{L^2},$$

so the first inequality is thus proved. For the last inequality by (47) we have $\frac{p-1}{2p}\le\frac12+\frac{\gamma}{6}$, together with $l\ge2$ we have

$$\|\langle v\rangle^l f\|_{L^2}^{1+\frac{\gamma}{3}}\le C\|f\|_{L^1}^{\frac12+\frac{\gamma}{6}}\|\langle v\rangle^{2l}f\|_{L^\infty}^{\frac12+\frac{\gamma}{6}}\le C\|f\|_{L^1}^{\frac{p-1}{2p}}\|\langle v\rangle^{2l}f\|_{L^\infty}^{1+\frac{\gamma}{3}-\frac{p-1}{2p}},$$

the lemma is thus proved. □

We first prove that the converge in $L^2_xL^2_v$, in fact the $L^2_xL^2_v$ convergence for the linearized semigroup is proved in Lemma 3.12, so we only need to prove the convergence for the nonlinear equation.

Lemma 7.8. *Suppose f the solution to (45), then there exists $k_0>\max\{3,3+\gamma\}$ such that if $k\ge k_0$ large, $\beta\ge6$ then we have*

$$\|f(t)\|_{L^2_xL^2_v}\le4C_{k,\beta}M^4(1+t)^{-r_1},\quad\forall t\ge0,$$

for some constants $C_{k,\beta}, r_1 > 1$.

Proof. Denote the solution to the linearized equation

$$\partial_t \xi + v \cdot \nabla_x \xi + \nu(v)\xi = K_k \xi, \quad \xi(0,x,v) = \xi_0(x,v),$$

by $\xi(t) = V(t)\xi_0$. By Lemma 3.12 we have

$$\|V(t)\xi\|_{L^2_x L^2_v} \lesssim \langle t\rangle^{-\frac{l}{|\gamma|}} \|\xi\|_{L^2_x L^2_3}, \quad \forall l \in (0,3).$$

By Duhamel's principle we have

$$f(t) = V(t) f_0 + \int_0^t V(t-s)\{\Gamma_k(f,f)(s)\} ds,$$

which implies

$$\|f(t)\|_{L^2_x L^2_v} \le C(1+t)^{-r_1} \|\langle v\rangle^3 f_0\|_{L^2_x L^2_v} + C\int_0^t (1+t-s)^{-r_1} \|\langle v\rangle^3 \Gamma_k(f,f)(s)\|_{L^2_x L^2_v} ds,$$

for some constant $r_1 > 1$. By Lemma 7.7 we have

$$\|\langle v\rangle^3 \Gamma_k(f,f)(s)\|_{L^2_x L^2_v} \le C\|f(s)\|_{L^2_x L^2_v} \|\langle v\rangle^6 f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \sup_{y\in\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}},$$

which implies

$$\begin{aligned}
&\int_0^t (1+t-s)^{-r_1} \|\langle v\rangle^3 \Gamma_k(f,f)(s)\|_{L^2_x L^2_v} ds \\
\le & C\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_1} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{[(1+s)^{r_1}\|f(s)\|_{L^2_x L^2_v}] \|\langle v\rangle^6 f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}}\right\} ds \\
\le & C(1+t)^{-r_1} \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{[(1+s)^{r_1}\|f(s)\|_{L^2_x L^2_v}] \|\langle v\rangle^6 f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}$$

Gathering the two terms together we easily have

$$\begin{aligned}
\sup_{0\le s\le t} [(1+s)^{r_1}\|f(s)\|_{L^2_x L^2_v}] \le & C_k \|\langle v\rangle^3 f_0\|_{L^2_x L^2_v} \\
& + C_k \sup_{0\le s\le t, y\in\mathbb{T}^3} \left\{[(1+s)^{r_1}\|f(s)\|_{L^2_x L^2_v}] \|\langle v\rangle^6 f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}}\right\} \\
\le & C_{k,\beta} M^4 + C_k \sup_{0\le s\le t} [(1+s)^{r_1}\|f(s)\|_{L^2_x L^2_v}] \\
& \times \sup_{t_1\le s\le t, y\in\mathbb{T}^3} \left\{\|\langle v\rangle^6 f(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}} \left(\int_{\mathbb{R}^3} |f(s,y,v')| dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}$$

If $\beta \ge 6$, by (58) we deduce

$$\|f(t)\|_{L^2_x L^2_v} \le 2C_{k,\beta} M^4 (1+t)^{-r_1}, \quad \forall t \ge 0,$$

the proof is thus finished. □

Proof. (**Proof of Theorem 1.3**) Recall (46), the corresponding mild solution is

$$\begin{aligned}
|f(t,x,v)| \le & e^{-\nu(v)t} f_0(x-vt,v) + \int_0^t e^{-\nu(v)(t-s)} |(K_k f)(s,x-v(t-s),v)| ds \\
& + \int_0^t e^{-\nu(v)(t-s)} |\Gamma_k(f,f)(s,x-v(t-s),v)| ds := J_1 + J_2 + J_3.
\end{aligned}$$

For any $r \in (0,1)$ fixed, for the J_1 term by Lemma 7.4 we have

$$|J_1| \le e^{-\nu(v)t}\nu(v)^r|\nu(v)^{-r}f_0(x-vt,v)| \le C(1+t)^{-r}\|\nu^{-r}f_0\|_{L^\infty_{x,v}} \le CM(1+t)^{-r}.$$

For the J_3 term, by Lemma 6.5 we have

$$\begin{aligned}
J_3 \le & C_k\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma\|f(s)\|_{L^\infty}\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\sup_{y\in\mathbb{T}^3}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}ds\\
\le & C_k\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma(1+s)^{-r}ds\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}]\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\\
\le & C_k(1+t)^{-r}\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}]\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}$$

For the J_2 term, denote $\tilde{x} = x - v(t-s)$ we have

$$J_2 \le \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')f(s,\tilde{x},v')|dv'ds.$$

by (46) again we have

$$\begin{aligned}
J_2 \le & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|e^{-\nu(v')s}|f_0(\tilde{x}-v's,v')|dv'ds\\
& + \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}|\Gamma_k(f,f)|(\tau,\tilde{x}-v'(s-\tau),v')d\tau dv'ds\\
& + \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_k(v,v')l_k(v',v'')|\int_0^s e^{-\nu(v')(s-\tau)}|f(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds := J_{21}+J_{22}+J_{23}.
\end{aligned}$$

For the J_{21} term by Lemma 7.4 and Lemma 7.5 we have

$$\begin{aligned}
J_{21} \le & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|e^{-\nu(v')s}\nu(v')^r|\nu(v')^{-r}f_0(\tilde{x}-v's,v')|dv'ds\\
\le & C\|\nu^{-r}f_0\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|(1+s)^{-r}dv'ds\\
\le & C_k\|\nu^{-r}f_0\|_{L^\infty_{x,v}}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma(1+s)^{-r}dv'ds\\
\le & C_k\|\nu^{-r}f_0\|_{L^\infty_{x,v}}(1+t)^{-r} \le C_kM(1+t)^{-r}.
\end{aligned}$$

For the J_{22} term, by Lemma 6.5 we have

$$\begin{aligned}
J_{22} \le & \int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}|\Gamma_k(f,f)|(\tau,\tilde{x}-v'(s-\tau),v')d\tau dv'ds\\
\le & C_k\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}|l_k(v,v')|\int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma(1+\tau)^{-r}d\tau dv'ds\\
& \sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{\left[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\\
\le & C_k(1+t)^{-r}\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{\left[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|_{L^\infty_{x,v}}^{\frac{p+1}{2p}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}$$

For the J_{23} term, we again split it into two parts $|v| \le N$ and $|v| \ge N$. For the case $|v| \ge N$ we have $\langle v\rangle^{-2} \le \frac{1}{N^2}$, by Lemma 6.3 and Lemma 7.5 we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |l_k(v,v')| \int_0^s e^{-\nu(v')(s-\tau)} \int_{\mathbb{R}^3} |l_k(v',v'')||f(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau \\
&\le \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\} \int_{\mathbb{R}^3} |l_k(v,v')| \int_0^s e^{-\nu(v')(s-\tau)} \int_{\mathbb{R}^3} |l_k(v',v'')|(1+\tau)^{-r} dv''d\tau \\
&\le \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\} \int_{\mathbb{R}^3} |l_k(v,v')| \left(\frac{c}{k^{\frac{\gamma+3}{4}}} + C_k\langle v'\rangle^{-2}\right) \int_0^s e^{-\nu(v')(s-\tau)} \langle v'\rangle^\gamma (1+\tau)^{-r} dv''d\tau dv' \\
&\le \langle v\rangle^\gamma \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}} + \frac{C_k}{N^2}\right)(1+s)^{-r} \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\},
\end{aligned}$$

which implies

$$\begin{aligned}
J_{23} &\le \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}} + \frac{C_k}{N^2}\right) \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\} \int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma (1+s)^{-r} ds \\
&\le \left(\frac{c^2}{k^{\frac{\gamma+3}{2}}} + \frac{C_k}{N^2}\right)(1+t)^{-r} \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\}.
\end{aligned}$$

For the case $|v| \le N$, by the decomposition (63), we split it into three terms respectively. For the first term by Lemma 7.5 we have

$$\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |(l_k(v,v') - l_{k,N}(v,v'))l_k(v',v'')| \int_0^s e^{-\nu(v')(s-\tau)}|f(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds \\
\le& \frac{C_k}{N} \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\} \int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds \int_0^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma (1+\tau)^{-r} d\tau \\
\le& \frac{C_k}{N}(1+t)^{-r} \sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\},
\end{aligned}$$

the second term can be estimated similarly. For the last term, since $l_{k,N}(v',v'')l_{k,N}(v',v'')$ is supported in $|v| \le N, |v'| \le C'_{k,N}, |v''| \le C'_{k,N}$ for some constant $C'_{k,N} > 0$. We again split it into two cases, $\tau \in [s-\lambda, s]$ and $\tau \in [0, s-\lambda]$, where $\lambda > 0$ is a small constant to be fixed later. For the case $\tau \in [s-\lambda, s]$, we first prove that

$$\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma (1+\tau)^{-r} d\tau \lesssim C_r\lambda(1+s)^{-r}. \tag{67}$$

Similarly as Lemma 7.5, we split the integral into two cases, $0 \le \tau \le \frac{s}{2}$ and $\frac{s}{2} \le \tau \le s$. For the first case we have $s-\tau \ge \frac{s}{2}$, so by Lemma 7.4 we have

$$\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma (1+\tau)^{-r} d\tau \le C\int_{s-\lambda}^s (1+s-\tau)^{-1}(1+\tau)^{-r} d\tau \le C(1+s)^{-1}\int_{s-\lambda}^s d\tau \le C_r\lambda(1+s)^{-r}.$$

For the second case we have $\tau \ge \frac{s}{2}$, since $|v'| \le C'_{k,N}$, this time we have

$$\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma (1+\tau)^{-r} d\tau \le C_r(1+s)^{-r}\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma d\tau \le C_r(1+s)^{-r}(1-e^{-\nu(v')\lambda}) \le C_r\lambda(1+s)^{-r},$$

so (67) is proved by gathering the two cases. Then we have

$$\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N}(v,v')l_{k,N}(v',v'')|\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}|f(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\
\le& C_{k,N}\sup_{0\le\tau\le t}\{(1+\tau)^r\|f(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma ds\int_{s-\lambda}^s e^{-\nu(v')(s-\tau)}\langle v'\rangle^\gamma(1+\tau)^{-r}d\tau\\
\le& C_{k,N}\lambda\sup_{0\le\tau\le t}\{(1+\tau)^r\|f(\tau)\|_{L^\infty_{x,v}}\}\int_0^t e^{-\nu(v)(t-s)}\langle v\rangle^\gamma(1+s)^{-r}ds\\
\le& C_{k,N}\lambda(1+t)^{-r}\sup_{0\le\tau\le t}\{(1+\tau)^r\|f(\tau)\|_{L^\infty_{x,v}}\}.
\end{aligned}$$

For the case $\tau\in[0,s-\lambda]$, by (64) and Lemma 7.8 we have

$$\begin{aligned}
&\int_{|v'|\le C'_{k,N},|v''|\le C'_{k,N}}|f(\tau,\tilde{x}-v'(s-\tau),v'')|dv'dv''\\
\le& C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}\left(\int_{\mathbb{T}^3}\int_{|v''|\le C'_{k,N}}|f(\tau,y,v'')|^2dv''dy\right)^{\frac{1}{2}}\le C_{k,N}\frac{1}{(s-\tau)^{\frac{3}{2}}}\|f(\tau)\|_{L^2_xL^2_v}\le C_{k,N,\beta}\lambda^{-\frac{3}{2}}M^4(1+\tau)^{-r},
\end{aligned}$$

where we have made a change of variable $y=\tilde{x}-v'(s-\tau)$. So we have

$$\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|l_{k,N}(v,v')l_{k,N}(v',v'')|\int_0^{s-\lambda} e^{-\nu(v')(s-\tau)}|f(\tau,\tilde{x}-v'(s-\tau,v''))|dv''d\tau dv'ds\\
\le& C_{k,N}\int_0^t e^{-\lambda_2(t-s)}\int_0^s e^{-\lambda_2(s-\tau)}\int_{|v'|\le C'_{k,N},|v''|\le C'_{k,N}}|f(\tau,\tilde{x}-v'(s-\tau),v'')|dv''d\tau dv'ds\\
\le& C_{k,N,\beta}\lambda^{-\frac{3}{2}}M^4\int_0^t e^{-\lambda_2(t-s)}\int_0^s e^{-\lambda_2(s-\tau)}(1+\tau)^{-r}d\tau ds\le C_{k,N,\beta}\lambda^{-\frac{3}{2}}M^4(1+t)^{-r}.
\end{aligned}$$

Gathering all the terms we have

$$\begin{aligned}
\|f(t)\|_{L^\infty_{x,v}}\le& C_{k,\beta,N}\lambda^{-\frac{3}{2}}M^4(1+t)^{-r}+\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N}+C_{k,N}\lambda\right)(1+t)^{-r}\sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\}\\
&+C_k(1+t)^{-r}\sup_{0\le s\le t,y\in\mathbb{T}^3}\left\{\left[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\right]\|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}\\
\le& C_{k,\beta,N}\lambda^{-\frac{3}{2}}M^4(1+t)^{-r}+\left(\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N}+C_{k,N}\lambda\right)(1+t)^{-r}\sup_{0\le s\le t}\{(1+s)^r\|f(s)\|_{L^\infty_{x,v}}\}\\
&+C_k(1+t)^{-r}\sup_{0\le s\le t}[(1+s)^r\|f(s)\|_{L^\infty_{x,v}}]\sup_{t_1\le s\le t,y\in\mathbb{T}^3}\left\{\|h(s)\|^{\frac{p+1}{2p}}_{L^\infty_{x,v}}\left(\int_{\mathbb{R}^3}|f(s,y,v')|dv'\right)^{\frac{p-1}{2p}}\right\}.
\end{aligned}$$

Taking suitable k,N,λ such that

$$\frac{c^2}{k^{\frac{\gamma+3}{2}}}+\frac{C_k}{N}+C_{k,N}\lambda\le\frac{1}{4},$$

together with (58) we conclude

$$\|f(t)\|_{L^\infty_{x,v}}\le 4C_{k,\beta}M^4(1+t)^{-r},\quad\forall t\ge 0.$$

The rate of convergence is thus proved for the case $-3<\gamma<0$. □

Acknowledgments. The author would thanks to Lingbing He and Yong Wang for fruitful talks on the paper. The author is supported by grants from Beijing Institute of Mathematical Sciences and Applications and Yau Mathematical Science Center, Tsinghua University.

Declarations of interest: None.

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