

New conformal field theory from $\mathcal{N} = (0, 2)$ Landau-Ginzburg model

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By studying the infra-red fixed point of an $\mathcal{N} = (0, 2)$ Landau-Ginzburg model, we find an example of modular invariant partition function beyond the ADE classification. This stems from the fact that a part of the left-moving sector is a new conformal field theory which is a variant of the parafermion model.

Dedicated to the academic achievements of Tohru Eguchi and Sung-Kil Yang

INTRODUCTION

A 2d CFT is endowed with an infinite-dimensional Lie algebra [1], and modular invariance further constrains its spectrum on the torus [2]. Consequently, a number of models have been exactly solved. (For instance, see [3].) In an RCFT, a modular invariant partition function consists of finitely many pairs I of left- and right-moving characters of chiral algebras $\mathcal{A} \otimes \overline{\mathcal{A}}$

$$Z = \sum_{(i, \bar{i}) \in I} N_{i\bar{i}} \chi_i^{\mathcal{A}} \otimes \overline{\chi_{\bar{i}}^{\overline{\mathcal{A}}}}. \quad (1)$$

If we write $M_{ij}^{\mathcal{A}} \circ \{\chi_j\}$ and $(M_{\bar{i}\bar{j}}^{\overline{\mathcal{A}}})^* \circ \{\overline{\chi_{\bar{j}}}\}$ for actions of $M \in \text{SL}(2, \mathbb{Z})$ on the spaces of the left and right-moving characters [4], then the modular invariance requires

$$M_{ij}^{\mathcal{A}} N_{\bar{j}\bar{i}} (M_{\bar{i}\bar{j}}^{\overline{\mathcal{A}}})^* = N_{i\bar{i}}.$$

As a result, modular invariant partition functions of $\text{SU}(2)_k$ WZNW models and unitary Virasoro minimal models admit the celebrated ADE classifications [5–8]. If a CFT is described by a non-chiral coset model [9] involving $\text{SU}(2)$ and $\text{U}(1)$, its modular invariant partition function fits into the ADE classification. As such, one can find modular invariant partition functions for parafermion (PF) models $\text{SU}(2)_k/\text{U}(1)_k$ [10] and $\mathcal{N} = 2$ minimal models (MMs) $(\text{SU}(2)_k \times \text{U}(1)_2)/\text{U}(1)_{k+2}$ [11]. Furthermore, with $\mathcal{N} = (2, 2)$ supersymmetry, Landau-Ginzburg (LG) models with ADE quasi-homogeneous superpotential are described by the MMs of corresponding ADE type in the infra-red (IR) limit [12, 13].

On the other hand, the class of $\mathcal{N} = (0, 2)$ LG models is much richer because firstly they are chiral in general and secondly there is more freedom due to the E - and J -terms [14, §6]. Therefore, it is natural to ask how IR CFTs incorporate the richness of $\mathcal{N} = (0, 2)$ LG models by encoding the information of the E - and J -terms.

In this article, we make a modest step towards understanding the LG/CFT correspondence with $\mathcal{N} = (0, 2)$

supersymmetry by studying the IR fixed point of a certain $\mathcal{N} = (0, 2)$ LG model along the line of [15]. We will obtain its modular invariant partition function, which turns out to be beyond the ADE classification. Careful analysis of the Hilbert space will show that a part of the left-moving sector is described by a new CFT which is a close cousin of the parafermion model.

LG/CFT CORRESPONDENCE

To begin with, we describe the $\mathcal{N} = (0, 2)$ LG model we focus on. It is a theory of two chiral multiplets ϕ_1, ϕ_2 and two Fermi multiplets ψ_1, ψ_2 with interactions determined by a superpotential

$$W = \psi_1(\phi_1^4 + \phi_2^2) + \psi_2\phi_1^2\phi_2. \quad (2)$$

The E -term is set to zero. This theory is called Class 2.b with $k = 4$ in [16].

Since the numbers of chiral and Fermi multiplets are equal, the vanishing of the gravitational anomaly $\text{Tr} \gamma_3 = \bar{c} - c = 0$ guarantees the equality of the left- and right-moving central charges. Furthermore, the c -extremization [17] calculates

$$c = \bar{c} = \frac{75}{27}, \quad (3)$$

where the \mathcal{R} -charges of all the multiplets are listed in the following table. There is also a left-moving $\text{U}(1)_\ell$

	ϕ_1	ϕ_2	ψ_1	ψ_2
$\text{U}(1)_{\mathcal{R}}$	$\frac{5}{27}$	$\frac{10}{27}$	$\frac{7}{27}$	$\frac{7}{27}$
$\text{U}(1)_\ell$	1	2	-4	-4

global symmetry with 't Hooft anomaly 27. Therefore, these data suggest that, in the IR fixed point, the right-moving sector is the $\mathcal{N} = 2$ MM₂₅ with level $k = 25$, and the left-moving sector is the $\text{U}(1)_{\frac{27}{2}}$ WZNW model with level $k = 27/2$ and a CFT of central charge $16/9$. It is tempting to identify the CFT of central charge $16/9$ with

the parafermion model PF_{25} of level 25 as in [15], and we will indeed write a modular invariant partition function using characters of PF_{25} in the next section. However, as we will see later, it is not exactly the PF_{25} , but a certain variant of the PF_{25} .

Let us extract more information about the IR CFT from the UV data. Since an elliptic genus is protected under the RG flow [18], it can be computed from the information of the LG model. We evaluate it in the NS-NS sector

$$\begin{aligned} \text{EG}(\tau, z) &= \text{Tr}_{\text{NSNS}} (-1)^F q^{L_0 - \frac{c}{24}} y^{J_0} e^{-\beta(\bar{L}_0 - \frac{1}{2}\bar{J}_0)} \\ &= q^{-\frac{25}{216}} \frac{\theta(y^{-4} q^{17/27}; q)^2}{\theta(y q^{5/54}; q) \theta(y^2 q^{5/27}; q)}. \end{aligned} \quad (4)$$

where $\theta(x; q) = \prod_{i=0}^{\infty} (1 - xq^i) (1 - q^{i+1}/x)$, and J_0 is the $U(1)_\ell$ charge. Note that $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$.

Among chiral primary states ($\bar{L}_0 = \bar{J}_0/2$) in the right-moving sector that contribute to the elliptic genus, the state subject to $L_0 = \mathfrak{q}/2$ in the left-moving sector form the topological heterotic ring \mathcal{H}_{top} [19, 20] where \mathfrak{q} is equal to the $U(1)_{\mathcal{R}}$ charge r_ϕ for a chiral field and $r_\psi - 1$ for a Fermi field. Since the numbers of chiral and Fermi multiplets are equal in the LG theory, it receives contributions only from chiral multiplets with $L_0 = \bar{J}_0/2$, which is isomorphic to the Jacobi ring of the J -term

$$\begin{aligned} \mathcal{H}_{\text{top}} &= \mathbb{C}[\phi_1, \phi_2] / (\phi_1^4 + \phi_2^2, \phi_1^2 \phi_2) \\ &\cong \text{Span}[\phi_1^i]_{i=0}^5 \oplus \text{Span}[\phi_2, \phi_1 \phi_2]. \end{aligned} \quad (5)$$

In fact, the holomorphic part of the stress-energy tensor [18, 21] is written as

$$\begin{aligned} T &= \sum_{a=1}^2 \left[\left(1 - \frac{r_{\phi_a}}{2}\right) \partial\phi_a \partial\bar{\phi}_a - \frac{r_{\phi_a}}{2} \phi_a \partial^2 \bar{\phi}_a \right] \\ &+ \sum_{a=1}^2 \left[\frac{i}{2} (1 + r_{\psi_a}) \psi_a \partial\bar{\psi}_a - \frac{i}{2} (1 - r_{\psi_a}) \partial\psi_a \bar{\psi}_a \right], \end{aligned} \quad (6)$$

and the OPE of a generator of \mathcal{H}_{top} with the stress-energy tensor shows that it is a primary state with $L_0 = \bar{J}_0/2$.

MODULAR INVARIANT PARTITION FUNCTION

Our goal is to find the modular invariant partition function of the IR CFT in the NS-NS sector as the following form

$$\begin{aligned} Z &= \text{Tr}_{\text{NSNS}} q^{L_0 - \frac{c}{24}} y^{J_0} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \bar{y}^{\bar{J}_0} \\ &= \sum_{\text{wts}} N_{\ell\bar{\ell}}^{\text{SU}(2)} N_{\nu\lambda\bar{\nu}}^{\text{U}(1)} \chi_{\ell,\nu}^{\text{PF}_{25}}(\tau) \chi_\lambda^{\text{U}(1)} \chi_{\bar{\ell},\bar{\nu}}^{\text{U}(1)} \chi_{\bar{\ell},\bar{\nu}}^{\text{MM}_{25}}(\bar{\tau}, \bar{w}) \end{aligned} \quad (7)$$

which is consistent with the elliptic genus (4), where $\bar{q} = e^{-2\pi i\bar{\tau}}$, $\bar{y} = e^{-2\pi i\bar{w}}$ and ‘wts’ stands for all the weights

labeled by $l, \bar{l}, \nu, \lambda, \bar{\nu}$. To obtain an elliptic genus from a partition function, we fix the right-moving sector to be chiral primary states ($\bar{L}_0 = \bar{J}_0/2$) only from which the elliptic genus receives contributions. Then, we insert $(-1)^F$ or equivalently $(-1)^{2(L_0 - \bar{L}_0)}$ in each term of the left-moving sector [22].

For this purpose, we shall find the modular invariant combination of $U(1)$ WZNW characters by following [15, 23]. In fact, the quadratic forms given by $U(1)$ levels are rationally equivalent

$$\text{diag} \left(\frac{27}{2}, 27 \right) = R^T \text{diag}(25, 2) R$$

where

$$R = \frac{1}{10} \begin{pmatrix} 2 & 10 \\ 25 & -10 \end{pmatrix}.$$

This rational equivalence gives rise to an identity among theta functions

$$\begin{aligned} &\chi_\mu^{\text{U}(1)_{25}}(\tau, 2u + 10v) (\chi_0^{\text{U}(1)_2} + \chi_2^{\text{U}(1)_2})(\tau, 25u - 10v) \\ &= \sum_{i=0}^{27 \times 10 - 1} \left\{ \chi_{2\mu+50i}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10\mu-20i}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) \right. \\ &\quad \left. + \chi_{2\mu+50i}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10\mu-20i+27 \times 10^2}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) \right\} \\ &\equiv \sum_{\lambda', \rho'} A_{\mu\lambda'\rho'} \chi_{\lambda'}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{\rho'}^{\text{U}(1)_{27 \times 10^2}}(\tau, v). \end{aligned}$$

Furthermore, there is another identity of theta functions

$$\begin{aligned} &\chi_\lambda^{\text{U}(1)_{\frac{27}{2}}}(\tau, 10u) \chi_\rho^{\text{U}(1)_{27}}(\tau, 10v) \\ &= \sum_{i_1, i_2 \in \mathbb{Z}_{10}} \chi_{10(\lambda+27i_1)}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{10(\rho+54i_2)}^{\text{U}(1)_{27 \times 10^2}}(\tau, v) \\ &\equiv \sum_{\lambda', \rho'} B_{\lambda\rho\lambda'\rho'} \chi_{\lambda'}^{\text{U}(1)_{\frac{27 \times 10^2}{2}}}(\tau, u) \chi_{\rho'}^{\text{U}(1)_{27 \times 10^2}}(\tau, v). \end{aligned}$$

From these identities, one can construct the $U(1)$ modular invariant tensor by

$$N_{\nu\lambda\bar{\nu}}^{\text{U}(1)} = \sum_{\lambda', \beta'} A_{\nu, \lambda', \beta'} B_{\lambda, \bar{\nu}, \lambda', \beta'}$$

which satisfies

$$(M_{\nu'\nu}^{\text{U}(1)_{25}})^* M_{\lambda'\lambda}^{\text{U}(1)_{27/2}} N_{\nu\lambda\bar{\nu}}^{\text{U}(1)} M_{\bar{\beta}\beta'}^{\text{U}(1)_{27}} = N_{\nu'\lambda'\bar{\beta}'}^{\text{U}(1)},$$

for all $M \in \text{SL}(2, \mathbb{Z})$. More explicitly, one can write

$$Z = \sum_{\text{wts}} N_{\ell\bar{\ell}}^{\text{SU}(2)} \chi_{\ell, 5m}^{\text{PF}_{25}}(\tau) \chi_{\frac{27m-5n}{2}}^{\text{U}(1)_{\frac{27}{2}}}(\tau, z) \cdot \bar{\chi}_{\bar{\ell}, n}^{\text{MM}_{25}}(\bar{\tau}, \bar{w}) \quad (8)$$

where the summation over weights runs $m \in \mathbb{Z}_{10}$, $n \in \mathbb{Z}_{54}$ and $\ell, \bar{\ell} \in \mathbb{Z}_{26}$.

Next, we need to determine the $SU(2)$ modular invariant tensor $N_{\ell\bar{\ell}}^{SU(2)}$. For the $SU(2)$ level $k = 25$, only the diagonal (type-A) combination $N_{\ell\bar{\ell}}^{SU(2)} = \delta_{\ell\bar{\ell}}$ is listed in the ADE classification [5–8]. However, with the diagonal $SU(2)$ combination, one can check that the partition function would not realize the elliptic genus (4).

To circumvent this situation, we need to relax some of the assumptions in [5–8] for the classification. We notice that the following matrix commutes with all the modular matrices $M^{SU(2)}$

$$N_{\ell\bar{\ell}}^{\text{nd}} = (\delta_{2,\ell} - \delta_{14,\ell} + \delta_{20,\ell})(\delta_{2,\bar{\ell}} - \delta_{14,\bar{\ell}} + \delta_{20,\bar{\ell}}) \\ + (\delta_{5,\ell} - \delta_{11,\ell} + \delta_{23,\ell})(\delta_{5,\bar{\ell}} - \delta_{11,\bar{\ell}} + \delta_{23,\bar{\ell}})$$

where the indices range $\ell, \bar{\ell} \in \mathbb{Z}_{26}$. Then, we set

$$N_{\ell\bar{\ell}}^{SU(2)} = \delta_{\ell\bar{\ell}} - \frac{1}{3}N_{\ell\bar{\ell}}^{\text{nd}}. \quad (9)$$

This clearly violates the assumption that $N_{i\bar{i}}$ in (1) are non-negative integer multiplicities, which has been adopted in the literature including [5–8]. However, if we use (9) in (8), the partition function is a formal series of (q, y, \bar{q}, \bar{x}) with non-negative integer coefficients and it is moreover consistent with the elliptic genus (4). We claim that it is the partition function of the IR CFT.

HILBERT SPACE AND A NEW CFT $\widetilde{\text{PF}}_{25}$

To demystify the multiplicities (9) with negative fractional numbers, let us investigate the Hilbert space of the IR CFT. To this end, we denote by $V_{\ell,m}^{\text{PF}_{25}}$ a highest weight representation of PF_{25} . In addition, by taking the direct sum of $s = 0$ and $s = 2$ weight of $U(1)_2$, we write by $V_{\ell,m}^{\text{MM}_{25}}$ a highest weight representation of MM_{25} in the NS sector. There are isomorphisms of irreducible modules

$$V_{\ell,m}^{\text{PF}_{25}} \cong V_{\ell,50-m}^{\text{PF}_{25}} \cong V_{25-\ell,m+25}^{\text{PF}_{25}} \\ V_{\ell,m}^{\text{MM}_{25}} \cong V_{\ell,54-m}^{\text{MM}_{25}} \cong V_{25-\ell,m+27}^{\text{MM}_{25}}.$$

First, we note an identity of the parafermion characters

$$3 = \sum_{m=0}^4 \chi_{2,10m}^{\text{PF}_{25}} - \chi_{14,10m}^{\text{PF}_{25}} + \chi_{20,10m}^{\text{PF}_{25}} \\ = \sum_{m=0}^4 \chi_{5,10m+5}^{\text{PF}_{25}} - \chi_{11,10m+5}^{\text{PF}_{25}} + \chi_{23,10m+5}^{\text{PF}_{25}}, \quad (10)$$

which counts the number of primary states $|\ell, m\rangle_{\text{PF}_{25}}$ with conformal dimension $h_{\ell,m}^{\text{PF}_{25}} = 2/27$ in PF_{25} :

$$|\ell, m\rangle_{\text{PF}_{25}} = |2, 0\rangle, |20, 20\rangle, |20, 30\rangle, \quad \text{or} \\ |\ell, m\rangle_{\text{PF}_{25}} = |23, 25\rangle, |5, 5\rangle, |5, 45\rangle. \quad (11)$$

Hence, roughly speaking, the non-diagonal part of (9) adds or eliminates a certain linear combination of these states to or from $V_{\ell,m}^{\text{PF}_{25}}$.

To see how the spectrum is organized, we compare the diagonal spectrum

$$\mathcal{H}_{\text{diag}} = \bigoplus_{\ell,m,n} V_{\ell,5m}^{\text{PF}_{25}} \otimes V_{\frac{27m-5n}{2}}^{U(1)_{27}} \otimes \overline{V}_{\ell,n}^{\text{MM}_{25}} \quad (12)$$

where $N_{\ell\bar{\ell}}^{SU(2)} = \delta_{\ell\bar{\ell}}$ with the information of the Hilbert space of the IR CFT obtained from the LG model. Note that the diagonal spectrum $\mathcal{H}_{\text{diag}}$ contains primary states of $\text{PF}_{25} \times U(1)_{\frac{27}{2}} \times \text{MM}_{25}$

$$|5s, 5s\rangle_{\text{PF}_{25}} \otimes | -s\rangle_{U(1)_{\frac{27}{2}}} \otimes |5s, -5s\rangle_{\text{MM}_{25}}, \\ |5s, 50-5s\rangle_{\text{PF}_{25}} \otimes | -s\rangle_{U(1)_{\frac{27}{2}}} \otimes |5s, -5s\rangle_{\text{MM}_{25}}, \quad (13)$$

which obey the condition $L_0 = \overline{J}_0/2 = \overline{L}_0$. Here we have $s = 0, 1, \dots, 5$ and the states in the first and second line are identical to the vacuum state when $s = 0$. Thus, there are ten primary states subject to the condition in $\mathcal{H}_{\text{diag}}$ whereas the topological heterotic ring (5) of the IR CFT is eight-dimensional as a vector space.

Hence, the diagonal spectrum (12) is not the actual Hilbert space. To realize the ring structure of \mathcal{H}_{top} in (5), let us suppose that ϕ_1 and ϕ_2 in \mathcal{H}_{top} respectively correspond to

$$|5, 5\rangle_{\text{PF}_{25}} + |5, 45\rangle_{\text{PF}_{25}} \quad \text{and} \\ |10, 10\rangle_{\text{PF}_{25}} - |10, 40\rangle_{\text{PF}_{25}}. \quad (14)$$

Here and in what follows, we suppress the parts of $U(1)_{27/2}$ and MM_{25} of (13). Then the fusion rule tells us that $\phi_1^2\phi_2$ corresponds to

$$|20, 20\rangle_{\text{PF}_{25}} - |20, 30\rangle_{\text{PF}_{25}} \quad (15)$$

in (13), which is decoupled from the spectrum due to the equation of motion. In addition, there is no generator in \mathcal{H}_{top} corresponding to

$$|5, 5\rangle_{\text{PF}_{25}} - |5, 45\rangle_{\text{PF}_{25}} \quad (16)$$

in (13). Thus, the IR CFT excludes these two states, (15) and (16), and the identification of (14) with ϕ_1 and ϕ_2 as well as the fusion rule indeed reproduces the topological heterotic ring (5).

On the other hand, one can show that $\phi_1^2\partial\phi_2 \sim -2\phi_1\phi_2\partial\phi_1$ is a primary in the IR CFT from the OPE with the stress-energy tensor (6) up to the equations of motion ($\partial W/\partial\phi_i = 0 = \partial W/\partial\psi_i$ and their complex conjugates). Moreover, $\phi_1^2\partial\phi_2$ and its descendants contribute to the elliptic genus by

$$(\chi_{|20,20\rangle - |20,30\rangle}^{\text{PF}_{25}} - 1)\chi_{-4}^{U(1)_{27/2}} = (\chi_{20,20}^{\text{PF}_{25}} - 1)\chi_{-4}^{U(1)_{27/2}} \\ = (q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + \dots)\chi_{-4}^{U(1)_{27/2}}.$$

Ignoring the $U(1)_{27/2}$ part, the subtraction by one means the omission of (15), and the primary $\phi_1^2 \partial \phi_2$ contributes to q^1 whereas the subsequent higher order terms count its descendants. This implies that although the IR CFT is not endowed with the parafermionic symmetry $SU(2)_{25}/U(1)_{25}$, it is still a character of a module of the Virasoro algebra in the left-moving sector. Similarly, it is easy to check from the OPE that $\phi_1^3 \partial \bar{\phi}_2 \sim -2\phi_2 \partial \bar{\phi}_1$ is also a primary in the IR CFT, and the contribution from its conformal family to the elliptic genus is $(\chi_{[5,5>-|5,45>}^{\text{PF}_{25}} - 1)\chi_{-1}^{U(1)_{27/2}}$.

Furthermore, an explicit computation using (10) shows that the elliptic genus (4) receives all the contributions from the part of $\ell = 5, n = -5$ in $\mathcal{H}_{\text{diag}}$ except the states (15) and (16), and their $U(1)_{27/2}$ descendants. Indeed, the Hilbert space is organized at the IR fixed point in such a way that the states (15) and (16) are excluded in the PF_{25} part but it preserves the Virasoro symmetry and the modular invariance. Denoting the CFT of central charge $16/9$ by $\widetilde{\text{PF}}_{25}$, the $\ell = 5, 20$ parts of the Hilbert space are isomorphic to the quotient spaces

$$\begin{aligned}\mathcal{H}_5^{\widetilde{\text{PF}}_{25}} &\cong \bigoplus_{m=0}^4 V_{5,10m+5}^{\text{PF}_{25}} / \mathbb{C}(|5, 5\rangle - |5, 45\rangle), \\ \mathcal{H}_{20}^{\widetilde{\text{PF}}_{25}} &\cong \bigoplus_{m=0}^4 V_{20,10m}^{\text{PF}_{25}} / \mathbb{C}(|20, 20\rangle - |20, 30\rangle),\end{aligned}$$

as vector spaces graded by L_0 .

In order to keep the modular invariance, one needs to arrange the primary states (11) of PF_{25} according to the non-diagonal part of (9):

$$\begin{aligned}\mathcal{H}_2^{\widetilde{\text{PF}}_{25}} &\cong \bigoplus_{m=0}^4 V_{2,10m}^{\text{PF}_{25}} / \mathbb{C}|2, 0\rangle, \\ \mathcal{H}_{23}^{\widetilde{\text{PF}}_{25}} &\cong \bigoplus_{m=0}^4 V_{23,10m+5}^{\text{PF}_{25}} / \mathbb{C}|23, 25\rangle, \\ \mathcal{H}_{14}^{\widetilde{\text{PF}}_{25}} &\cong \mathbb{C}(|20, 20\rangle + |20, 30\rangle) \oplus \bigoplus_{m=0}^4 V_{14,10m}^{\text{PF}_{25}}, \\ \mathcal{H}_{11}^{\widetilde{\text{PF}}_{25}} &\cong \mathbb{C}(|5, 5\rangle + |5, 45\rangle) \oplus \bigoplus_{m=0}^4 V_{11,10m+5}^{\text{PF}_{25}}.\end{aligned}\tag{17}$$

Here, \cong means an isomorphism as vector spaces graded by L_0 . For the other $\ell \neq 2, 5, 11, 14, 20, 23$, they are isomorphic to those of PF_{25}

$$\mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \cong \bigoplus_{m=0}^4 V_{\ell,10m+5(\ell \bmod 2)}^{\text{PF}_{25}}.$$

All in all, the Hilbert space of the IR CFT is then expressed as

$$\mathcal{H} = \bigoplus_{\ell, n} \mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \otimes V_{\frac{27\ell-5n}{2}}^{U(1)_{27/2}} \otimes \overline{V}_{\ell, n}^{\text{MM}_{25}},\tag{18}$$

whose generating function is (8) with (9). This explains the reason why the partition function (8) with (9) is a formal power series with non-negative integer coefficients. If we restrict the right-moving sector to be chiral primary states, we have

$$\mathcal{H}|_{L_0=\overline{J}_0/2} = \bigoplus_{\ell} \mathcal{H}_\ell^{\widetilde{\text{PF}}_{25}} \otimes V_{16\ell}^{U(1)_{27/2}},$$

which exactly reproduces the elliptic genus (4) by appropriately including signs. In fact, under the equations of motion, the conformal families of two primaries $\psi_i \partial \bar{\phi}_1$ ($i = 1, 2$) contribute to (4)

$$\begin{aligned}(\chi_{2,0}^{\text{PF}_{25}} - 1)\chi_5^{U(1)_{27/2}} \\ = (2q + 3q^2 + 6q^3 + 10q^4 + 18q^5 + \dots)\chi_5^{U(1)_{27/2}}.\end{aligned}\tag{19}$$

For $\ell = 23$, those of two primaries $\bar{\psi}_i \phi_1^2 (\partial \phi_2)^2$ ($i = 1, 2$) yield the contribution $(\chi_{23,25}^{\text{PF}_{25}} - 1)\chi_{17}^{U(1)_{27/2}}$. In addition, the conformal family of a primary $\psi_1 \psi_2$ combines the two irreducible characters of PF_{25} into one ‘‘irreducible’’ character of $\widetilde{\text{PF}}_{25}$

$$\begin{aligned}(1 + \chi_{14,20}^{\text{PF}_{25}} + \chi_{14,30}^{\text{PF}_{25}})\chi_8^{U(1)_{27/2}} \\ = (1 + 2q + 4q^2 + 10q^3 + 20q^4 + 38q^5 + \dots)\chi_8^{U(1)_{27/2}}.\end{aligned}$$

In a similar fashion, that of a primary $\bar{\psi}_1 \bar{\psi}_2 \phi_1 \phi_2 (\partial \phi_1)^2$ gives the contribution $(1 + \chi_{11,5}^{\text{PF}_{25}} + \chi_{11,45}^{\text{PF}_{25}})\chi_{14}^{U(1)_{27/2}}$. Hence, this provides a strong evidence that the graded vector spaces (17) are decomposed into modules of the Virasoro algebra and $\widetilde{\text{PF}}_{25}$ preserves the conformal symmetry. In conclusion, the $\mathcal{N} = (0, 2)$ LG model flows to

$$\left(\widetilde{\text{PF}}_{25} \times U(1)_{\frac{27}{2}}\right) \otimes \overline{\left(\frac{SU(2)_{25} \times U(1)_2}{U(1)_{27}}\right)},$$

and the modular invariant Hilbert space (18) on a torus is decomposed into modules of the left-moving Virasoro algebra and the right-moving $\mathcal{N} = 2$ super-Virasoro algebra.

DISCUSSIONS

We find the modular invariant partition function beyond the ADE classification [5–8] because a part of the left-moving sector is the new CFT $\widetilde{\text{PF}}_{25}$ obtained by breaking the parafermionic symmetry of PF_{25} . Certainly, more investigation needs to be carried out to understand $\widetilde{\text{PF}}_{25}$. In particular, it is desirable to determine two ‘‘irreducible’’ characters of the primaries $\psi_i \partial \bar{\phi}_1$ (resp. $\bar{\psi}_i \phi_1^2 (\partial \phi_2)^2$) in $\widetilde{\text{PF}}_{25}$ whose sum is equal to $\chi_{2,0}^{\text{PF}_{25}} - 1$ in (19) (resp. $\chi_{23,25}^{\text{PF}_{25}} - 1$).

In [16], $\mathcal{N} = (0, 2)$ LG models with the same left and right central charges ≤ 3 have been classified. In the classification of [16], IR CFTs of Class 2.a with superpotential

$$\psi_1(\phi_1^m + \phi_2^n) + \psi_2\phi_1\phi_2, \quad m, n \in \mathbb{Z}_{>0},$$

are described by diagonal modular pairing of PFs and $U(1)$ WZNW models in the left-moving-sector and $\mathcal{N} = 2$ MMs in the right-moving sector [15]. This is because their topological heterotic rings are simple and it does not contain a mixed generator like $\phi_1\phi_2$. Like in our example, the topological heterotic rings of the other classes in [16] are more complicated, and we observe that their elliptic genera cannot be realized by characters of PFs and $U(1)$ WZNW models except our example (2). (Another exception is Class 2.b with $k = 3$, but it is equivalent to $\mathcal{N} = (2, 2)$ MM of type E_7 .) It is expected that the left-moving sectors of IR CFTs would be unknown ones so that it requires further study to understand how J -terms of $\mathcal{N} = (0, 2)$ LG models are encoded in IR CFTs. It is also worth mentioning that the condition of the same left and right-moving central charges in [16] is rather special in $\mathcal{N} = (0, 2)$ LG models, and a vast class of general $\mathcal{N} = (0, 2)$ LG models are waiting to be investigated.

Since A.B. Zamolodchikov has identified the LG/CFT correspondence [24], it has given drastically new insights in quantum field theories and mathematical physics. This article just takes a peek at the LG/CFT correspondence with $\mathcal{N} = (0, 2)$ supersymmetry, but we hope that our example shows its fertility and will intensify further study on it.

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Notations

Here, we summarize convention and definitions necessary in this article. $U(1)_k$ and $SU(2)_k$ characters are given by

$$\begin{aligned} \chi_m^{U(1)_k}(\tau, z) &= \frac{\Theta_{m,k}(\tau, z)}{\eta(\tau)}, \\ \chi_\ell^{SU(2)_k}(\tau, z) &= \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{-(\ell+1),k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)}, \end{aligned}$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$ is the Dedekind eta-function and the theta function is defined as

$$\Theta_{m,k}(\tau, z) \equiv \sum_{n \in \mathbb{Z}} q^{k(n + \frac{m}{2k})^2} y^{k(n + \frac{m}{2k})}.$$

The weights of $U(1)_k$ and $SU(2)_k$ run over $m = 0, \dots, 2k - 1$ and $\ell = 0, \dots, k$, respectively. It is well-known that the modular group $SL(2, \mathbb{Z})$ is generated by T and S , and a T -transformation on characters of a chiral algebra \mathcal{A} is always diagonalizable

$$\chi_r^{\mathcal{A}}(\tau + 1) = e^{2\pi i(h_r - c/24)} \chi_r^{\mathcal{A}}(\tau),$$

where h_r is the conformal dimension of the corresponding highest weight state. Under the S -transformation, the characters of $\mathcal{A}_k = U(1)_k, SU(2)_k$ are transformed as

$$\chi_r^{A_k} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\frac{i\pi k z^2}{2\tau}} \sum_{r'} S_{rr'}^{A_k} \chi_{r'}^{A_k}(\tau, z),$$

where

$$\begin{aligned} S_{\ell, \ell'}^{SU(2)_k} &\equiv \sqrt{\frac{2}{k+2}} \sin \left(\frac{\pi(\ell+1)(\ell'+1)}{k+2} \right), \\ S_{m, m'}^{U(1)_k} &\equiv \frac{1}{\sqrt{2k}} e^{-2\pi i \frac{mm'}{2k}}. \end{aligned}$$

A character of a coset model \mathcal{A}/\mathcal{B} can be computed via a branching rule

$$V_\ell^{\mathcal{A}} = \bigoplus_m V_m^{\mathcal{B}} \oplus V_{\ell, m}^{\mathcal{A}/\mathcal{B}}$$

where $V_\ell^{\mathcal{A}}$ and $V_m^{\mathcal{B}}$ are highest weight representations of the chiral algebra \mathcal{A} and \mathcal{B} , respectively. By defining the string function $c_{\ell, m}^{(k)}$ [10]

$$\chi_\ell^{SU(2)_k}(\tau, z) = \sum_{m \in \mathbb{Z}_{2k}} c_{\ell, m}^{(k)}(\tau) \Theta_{m,k}(\tau, z),$$

a character of the parafermion is then expressed as

$$\chi_{\ell, m}^{\text{PF}_k}(\tau) = \eta(\tau) c_{\ell, m}^{(k)}(\tau),$$

where $\ell + m \in 2\mathbb{Z}$, and otherwise $\chi_{\ell, m}^{\text{PF}_k} = 0$. Note that the characters obey $\chi_{\ell, m}^{\text{PF}_k} = \chi_{\ell, 2k-m}^{\text{PF}_k} = \chi_{k-\ell, m+k}^{\text{PF}_k}$.

In addition, a character of the $\mathcal{N} = 2$ minimal model in the NS sector [11] is given by

$$\chi_{\ell,m}^{\text{MM}_k}(\tau, z) = \sum_{r \in \mathbb{Z}_{2k}} c_{\ell,r}^{(k)}(\tau) \Theta_{(k+2)r - km, k(k+2)} \left(\frac{\tau}{2}, \frac{z}{k+2} \right)$$

where the weights $s = 0, 2$ of $U(1)_2$ are summed. Note that the weights are subject to $\ell + m \in 2\mathbb{Z}$, and otherwise $\chi_{\ell,m}^{\text{MM}_k} = 0$. Note that the characters satisfy $\chi_{\ell,m}^{\text{MM}_k} = \chi_{\ell, 2(k+2) - m}^{\text{MM}_k} = \chi_{k - \ell, m + k + 2}^{\text{MM}_k}$.

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