ISING MODEL WITH CURIE-WEISS PERTURBATION

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Abstract. Consider the nearest-neighbor Ising model on $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ at inverse temperature $\beta \geq 0$ with free boundary conditions, and let $Y_n(\sigma) := \sum_{u \in \Lambda_n} \sigma_u$ be its total magnetization. Let X_n be the total magnetization perturbed by a critical Curie-Weiss interaction, i.e.,

$$\frac{dF_{X_n}}{dF_{Y_n}}(x) := \frac{\exp[x^2 / \left(2\langle Y_n^2\rangle_{\Lambda_n,\beta}\right)]}{\langle \exp[Y_n^2 / \left(2\langle Y_n^2\rangle_{\Lambda_n,\beta}\right)]\rangle_{\Lambda_n,\beta}},$$

where F_{X_n} and F_{Y_n} are the distribution functions for X_n and Y_n respectively. We prove that for any $d \geq 4$ and $\beta \in [0, \beta_c(d)]$ where $\beta_c(d)$ is the critical inverse temperature, any subsequential limit (in distribution) of $\{X_n/\sqrt{\mathbb{E}\left(X_n^2\right)} : n \in \mathbb{N}\}$ has an analytic density (say, f_X) all of whose zeros are pure imaginary, and f_X has an explicit expression in terms of the asymptotic behavior of zeros for the moment generating function of Y_n. We also prove that for any $d \geq 1$ and then for β small,

$$f_X(x) = K \exp(-C^4 x^4),$$

where $C = \sqrt{\Gamma(3/4)/\Gamma(1/4)}$ and $K = \sqrt{\Gamma(3/4)}/(4\Gamma(5/4)^{3/2})$. Possible connections between f_X and the high-dimensional critical Ising model with periodic boundary conditions are discussed.

1. Introduction

1.1. **Overview and motivation.** It has been known since [28] that the renormalized total magnetization in the critical Curie-Weiss model converges in distribution to a random variable with density $C_1 \exp(-C_2 x^4)$. See [8, 9] for various extensions of this classical result as the underlying single spin distribution at $\beta = 0$ is varied. Part of the current paper (when $d \geq 1$ and β is small) can be viewed as studying extensions of this classical result by including some nearest-neighbor interactions.

The effect of periodic and free boundary conditions for the high-dimensional Ising model in finite domains has been a long-standing debate in the physics literature; see, e.g., [30, 14, 32] for some recent results and references therein for many more studies. For the critical Ising model on $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ where $d > 4$ with periodic boundary conditions, the two-point function is conjectured in [24] to have a "plateau": it behaves like the infinite-volume counterpart for small distance before leveling off at an order of $n^{-d/2}$ for large distance. This plateau lower bound was proved in Theorem 1 of [24] and the upper bound remains open; the conjecture was numerically verified in [14]. Similar plateaus were proved for simple random walk for $d > 2$ in [33, 29, 32], for weakly self-avoiding walk for $d > 4$ in [29], and for nearest-neighbor percolation for $d \geq 11$ in [16]. On the contrary, it was proved in [5] that such a plateau does not exist in the high-dimensional critical Ising model with free boundary conditions.

Loosely speaking, the main conclusion in [24] was that a macroscopic (or bulk) quantity in the high-dimensional Ising model with periodic boundary conditions "parallels more closely the complete graph paradigm". For example, it was proved in [24] that the critical susceptibility on Λ_n in dimensions $d > 4$ with periodic boundary conditions has a lower bound $n^{d/2}$ (which is the order of the critical susceptibility in the Curie-Weiss model); while it behaves like n^2 for free boundary conditions (see [5] for a proof).

A main motivation of the current paper is to study the scaling limit of the critical Ising magnetization field in dimension $d > 4$ with periodic boundary conditions. The results in [24] suggest that this scaling limit is non-Gaussian, and on page 37 of [24], the author mentions that the macroscopic behavior in the high dimensional Ising model with periodic boundary conditions is expected to behave similarly to the Curie-Weiss model. So in this paper, we study the Ising model perturbed by a critical Curie-Weiss interaction. We prove that for any $d \geq 1$ and β small, the limit of the total magnetization in this perturbed model is the same as in the standard critical Curie-Weiss model, and we conjecture that this should be true for any $\beta \in [0, \beta_c(d))$ where $\beta_c(d)$ is the critical inverse temperature of the classical Ising model on $\mathbb{Z}^d$. For $d \geq 4$ and $\beta = \beta_c(d)$, we prove the partial result that any subsequential limit of this perturbed total magnetization is non-Gaussian and has an analytic density all of whose zeros are pure imaginary. We think that there may be a close connection between periodic boundary conditions and Curie-Weiss perturbation scaling limits in the high-dimensional Ising model (see Conjecture 1 and (32) of Section 1.3 below) and thus the analysis of the Curie-Weiss case may help understand the periodic boundary conditions case.

1.2. **Main results.** Consider the Ising model on Λ_n at inverse temperature $\beta \geq 0$ with free boundary conditions; let $\langle \cdot \rangle_{\Lambda_n,\beta}$ denote its expectation. See (128)-(129) (with $h = 0$) in the Appendix for explicit definitions. Let

$$Y_n(\sigma) := \sum_{u \in \Lambda_n} \sigma_u \tag{1}$$

be the total magnetization. For any $d \geq 1$ and $\beta \in [0, \beta_c(d))$, it is not hard to prove (see Lemma 4 below) that

$$\frac{Y_n}{\sqrt{\langle Y_n^2 \rangle_{\Lambda_n,\beta}}} \Longrightarrow \text{ a standard normal distribution as } n \to \infty, \tag{2}$$

where $\Longrightarrow$ denotes convergence in distribution. By the main results in [1, 11, 3], the last weak convergence also holds for any $d \geq 4$ and $\beta = \beta_c(d)$. For $d = 2$ and $\beta = \beta_c(2)$, it was proved in [4] that $Y_n/\sqrt{\langle Y_n^2 \rangle_{\Lambda_n,\beta}}$ converges in distribution to Y with

$$\lim_{x \to \infty} \frac{\ln \mathbb{P}(Y > x)}{x^{16}} = -c, \tag{3}$$

where $c \in (0, \infty)$. For $d = 3$ and $\beta = \beta_c(3)$, the limit behavior of $Y_n/\sqrt{\langle Y_n^2 \rangle_{\Lambda_n,\beta}}$ is unknown. For any $d \geq 2$ and $\beta > \beta_c(d)$, it is believed that

$$\frac{Y_n}{\sqrt{\langle Y_n^2 \rangle_{\Lambda_n,\beta}}} \Longrightarrow \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}, \tag{4}$$

where $\delta_{\pm 1}$ is the unit Dirac point measure at ± 1. A well-known result due to Lee and Yang [17] is that all zeros of the moment generating function of Y_n, $\langle \exp(zY_n) \rangle_{\Lambda_n,\beta}$ or equivalently of the partition function (see (129)), are pure imaginary; so we may assume that all the zeros are $\{\pm i\alpha_{j,n} : j \geq 1\}$ (listed with multiplicities, if any) such that

$$0 < \alpha_{1,n} \leq \alpha_{2,n} \leq \dots. \tag{5}$$

We consider a random variable X_n whose distribution is obtained from that of Y_n by adding a Curie-Weiss interaction:

$$\frac{dF_{X_n}}{dF_{Y_n}}(x) = \frac{\exp[x^2/(2\langle Y_n^2\rangle_{\Lambda_n,\beta})]}{\langle\exp[Y_n^2/(2\langle Y_n^2\rangle_{\Lambda_n,\beta})]\rangle_{\Lambda_n,\beta}}, \tag{6}$$

where F_{X_n} and F_{Y_n} are the distribution functions for X_n and Y_n respectively. Equivalently, X_n is the total magnetization for the perturbed Ising model with Hamiltonian

$$H_{\Lambda_n,\beta}(\sigma) := -\beta\sum_{\{u,v\}}\sigma_u\sigma_v - \frac{Y_n^2(\sigma)}{2\langle Y_n^2\rangle_{\Lambda_n,\beta}}, \qquad \sigma\in\{-1,+1\}^{\Lambda_n}. \tag{7}$$

When $\beta = 0$, the form of (7) means that this perturbed model is exactly the critical Curie-Weiss model. Let $\mathbb{E}_{\Lambda_n,\beta}$ denote the expectation for this perturbed Ising model. The **cumulants**, $u_n(X)$, of a random variable X are defined (if they exist) by

$$u_n(X) := \left.\frac{d^n \ln\mathbb{E}\exp(zX)}{dz^n}\right|_{z=0}. \tag{8}$$

Here is our main result.

Theorem 1. *For $d\geq 1$ and $\beta\in[0,\beta_c(d))$, and also for $d\geq 4$ and $\beta=\beta_c(d)$, any subsequential limit in distribution of $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n\in\mathbb{N}\}$ has an analytic density f_X all of whose zeros in $\mathbb{C}$ are pure imaginary. More precisely, suppose that there exists a subsequence $\{n_k : k\in\mathbb{N}\}$ and a random variable X such that*

$$X_{n_k}/\sqrt{\mathbb{E}_{\Lambda_{n_k},\beta}(X_{n_k}^2)} \Longrightarrow X \text{ as } k\to\infty. \tag{9}$$

Then X has a density function given by

$$f_X(x) = C_3 f_W(C_3 x), \tag{10}$$

where $C_3 := \sqrt{\mathbb{E}(W^2)}\in(0,\infty)$ and

$$f_W(x) = K\exp[-\kappa_1 x^4]\prod_{j\geq 1}\left[\left(1+\frac{x^2}{a_j^2}\right)\exp\left(-\frac{x^2}{a_j^2}\right)\right],\ x\in\mathbb{R}, \tag{11}$$

where

$$a_j := \lim_{k\to\infty}\alpha_{j,n_k}\left[-u_4(Y_{n_k})/4!\right]^{1/4} \text{ for each } j\geq 1, \tag{12}$$

$$\kappa_1 := 1-\frac{\sum_{j\geq 1}a_j^{-4}}{2}, \tag{13}$$

and $K\in(0,\infty)$ is such that f_W is a probability density function; $\{a_j : j\geq 1\}$ may be empty, finite or infinite.

Remark 1. *Theorem 1 treats the critical Curie-Weiss perturbation of the Ising model with $\beta\leq\beta_c(d)$. If the Curie-Weiss perturbation is subcritical, then like in the $\beta=0$ case, X is expected to be standard normal. Similarly, in the supercritical case, we expect X to be distributed as a sum of two Dirac point measures as in (4).*

Remark 2. *By Fatou's lemma and (41) below, we have*

$$\sum_{j\geq 1}a_j^{-4} = \sum_{j\geq 1}\liminf_{k\to\infty}\alpha_{j,n_k}^{-4}\left[-u_4(Y_{n_k})/4!\right]^{-1} \leq \liminf_{k\to\infty}\sum_{j\geq 1}\alpha_{j,n_k}^{-4}\left[-u_4(Y_{n_k})/4!\right]^{-1} = 2. \tag{14}$$

So (13) implies that $\kappa_1\in[0,1]$.

In [31], Yang and Lee argued that when $\beta < \beta_c(d)$, the partition function of the Ising model (as a function of $\exp(-2h)$ where h is the external field defined in the Appendix) should be nonzero in a neighborhood of $\exp(i0)$, which means that $\alpha_{1,n}$ should be uniformly (in n) bounded away from 0. This has been proved rigorously in [27] (see also Theorem A of [25]) when β is small but positive. It is not hard to show that $-u_4(Y_n)$ grows at least linearly in $|\Lambda_n|$. So we have the following corollary, with a complete proof presented in Section 3.3.

Corollary 1. *For any $d \geq 1$, there exists $\beta_1(d) \in (0, \beta_c(d)]$ such that for any $\beta \in [0, \beta_1(d))$, we have that as $n \to \infty$,*

$$\frac{X_n}{\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}} \Longrightarrow \text{ a random variable with density } C\exp(-C^4x^4)/\int_{-\infty}^{\infty}\exp(-t^4)dt, \tag{15}$$

where

$$C := \sqrt{\Gamma(3/4)/\Gamma(1/4)} \tag{16}$$

with Γ denoting the gamma function.

Remark 3. (10) *and* (11) *imply that there exist $C_4, C_5 \in (0, \infty)$ such that*

$$f_X(x) = C_4 - C_5x^4 + O(x^6) \text{ as } x \to 0. \tag{17}$$

(See, e.g., (105) *below for a derivation.) So any subsequential limit of $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n \in \mathbb{N}\}$ is not Gaussian. Theorem 1 and Corollary 1 also hold (with a similar proof) if Y_n is the total magnetization in Λ_n from the unique infinite-volume Gibbs measure (rather than from Λ_n with free boundary conditions).*

Remark 4. *We expect that $\beta_1(d) = \beta_c(d)$ should be valid in Corollary 1. In the Appendix, we prove that when $\beta < \beta_c(d)$, the limiting distribution (or measure) of Lee-Yang zeros has no mass in an arc containing* $\exp(i0)$ *of the unit circle. This is an indication that $\beta_1(d)$ ought to be $\beta_c(d)$.*

Remark 5. *When $d \geq 4$ and $\beta = \beta_c(d)$, one possible candidate for $f_X(x)$ is the symmetric mixture of two independent normal densities with opposite means $\pm\sqrt{2}/2$ and variance* $1/2$.

1.3. **Discussion about high-dimensional Ising model with periodic boundary conditions.** We now consider the Ising model on Λ_n at inverse temperature $\beta \geq 0$ with periodic boundary conditions. Let $\langle\cdot\rangle_{\Lambda_n,\beta,p}$ denote its expectation and $\bar{Y}_n := \sum_{u\in\Lambda_n}\sigma_u$ be its total magnetization, where the overbar is a reminder that we are using periodic boundary conditions. The infinite product representation for the moment generating function given in Lemma 2 below also holds for this case. So we have for any $d \geq 1$ and $\beta \geq 0$,

$$\left\langle \exp\left(z\frac{\bar{Y}_n}{\sqrt{\langle\bar{Y}_n^2\rangle_{\Lambda_n,\beta,p}}}\right)\right\rangle_{\Lambda_n,\beta,p} = \exp\left(\frac{z^2}{2}\right)\prod_{j\geq1}\left[\left(1+\frac{z^2}{\bar{\alpha}_{j,n}^2\langle\bar{Y}_n^2\rangle_{\Lambda_n,\beta,p}}\right)\exp\left(-\frac{z^2}{\bar{\alpha}_{j,n}^2\langle\bar{Y}_n^2\rangle_{\Lambda_n,\beta,p}}\right)\right], \tag{18}$$

where $0 < \bar{\alpha}_{1,n} \leq \bar{\alpha}_{2,n} \leq \dots$, $\{\pm i\bar{\alpha}_{j,n} : j \geq 1\}$ are all the zeros (listed with multiplicities) of $\langle\exp(z\bar{Y}_n)\rangle_{\Lambda_n,\beta,p}$, and $\sum_{j\geq1}1/\bar{\alpha}_{j,n}^2 \leq \langle\bar{Y}_n^2\rangle_{\Lambda_n,\beta,p}/2$. Combining (18) with the inequality

$|(1+y)\exp(-y)| \le \exp(2|y|)$ gives that

$$\left| \left\langle \exp\left(z \frac{\bar{Y}_n}{\sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}}} \right) \right\rangle_{\Lambda_n,\beta,p} \right| \le \exp\left(\frac{3|z|^2}{2} \right) \text{ for any } z \in \mathbb{C}. \tag{19}$$

Therefore, $\{\langle \exp(z\bar{Y}_n/\sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}}) \rangle_{\Lambda_n,\beta,p} : n \in \mathbb{N}\}$ is locally uniformly bounded. Suppose that

$$\bar{Y}_n/\sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}} \Longrightarrow \bar{Y} \text{ as } n \to \infty. \tag{20}$$

Then we have

$$\lim_{n\to\infty} \left\langle \exp\left(it\bar{Y}_n/\sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}} \right) \right\rangle = \mathbb{E}\exp(it\bar{Y}) \text{ for any } t \in \mathbb{R}. \tag{21}$$

So Vitali's theorem (see, e.g., Theorem B.25 of [10]) implies that

$$\lim_{n\to\infty} \left\langle \exp\left(z\bar{Y}_n/\sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}} \right) \right\rangle = \mathbb{E}\exp(z\bar{Y}) \text{ locally uniformly on } \mathbb{C}. \tag{22}$$

Theorem 7 of [22] and Theorem 2 below give that

$$\mathbb{E}\exp(z\bar{Y}) = \exp\left(\frac{z^2}{2}\right) \prod_{j\ge1} \left[\left(1 + \frac{z^2}{\bar{a}_j^2}\right) \exp\left(-\frac{z^2}{\bar{a}_j^2}\right) \right], \ \forall z \in \mathbb{C}, \tag{23}$$

where $\{\pm i\bar{a}_j : j \ge 1\}$ are all the zeros of $\mathbb{E}\exp(z\bar{Y})$ with $0 < \bar{a}_1 \le \bar{a}_2 \le \dots$. By Hurwitz's theorem (see Lemma 6 below), we have

$$\lim_{n\to\infty} \bar{\alpha}_{j,n} \sqrt{\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta,p}} = \bar{a}_j \text{ for each } j \ge 1. \tag{24}$$

In the rest of this subsection, we focus on $d > 4$ and $\beta = \beta_c := \beta_c(d)$. Here $d = 4$ is the upper critical dimension, and we exclude $d = 4$ since the corresponding critical behavior is subtle and various quantities may have logarithmic corrections. By a conditional result of [24], Theorem 2 there, it is expected that

$$\liminf_{n\to\infty} \bar{\alpha}_{1,n}^2 \langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta_c,p} < \infty \tag{25}$$

since otherwise (23) and (24) would imply that $\bar{Y}$ is standard normal.

Recall that $u_4(\bar{Y}_n)$ is the fourth cumulant of $\bar{Y}_n$. It is well-known (see Theorem 5 of [19] or (5.3) of [1]) that

$$0 \le -u_4(\bar{Y}_n) \le 3\left[\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta_c,p}\right]^2 \tag{26}$$

It is also known (see, e.g., (52)-(53) below for a proof) that if $\bar{Y}$ is not standard normal, then

$$\liminf_{n\to\infty} \frac{-u_4(\bar{Y}_n)}{\left[\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta_c,p}\right]^2} > 0. \tag{27}$$

So it is expected that there exists $C_6 \in (0,\infty)$ such that

$$\lim_{n\to\infty} \frac{-u_4(\bar{Y}_n)}{\left[\langle \bar{Y}_n^2 \rangle_{\Lambda_n,\beta_c,p}\right]^2} = C_6^{-4}. \tag{28}$$

To summarize, we formulate a conjecture relating $\bar{Y}$ to the asymptotic behavior of the $\bar{\alpha}_{j,n}$'s.

Conjecture 1. *For the critical Ising model on Λ_n with periodic boundary conditions and $d>4$. The following limits exist*

$$\bar{a}_j := \lim_{n\to\infty} \bar{\alpha}_{j,n}\sqrt{\langle \bar{Y}_n^2\rangle_{\Lambda_n,\beta_c,p}} = C_6 \lim_{n\to\infty} \bar{\alpha}_{j,n}\left[-u_4(\bar{Y}_n)\right]^{1/4} \text{ for each } j\geq 1. \tag{29}$$

Hence, we have

$$\bar{Y}_n/\sqrt{\langle \bar{Y}_n^2\rangle_{\Lambda_n,\beta_c,p}} \Longrightarrow \bar{Y} \text{ as } n\to\infty, \tag{30}$$

where $\bar{Y}$ has moment generating function

$$\mathbb{E}\exp(z\bar{Y}) = \exp\left(\frac{z^2}{2}\right)\prod_{j\geq 1}\left[\left(1+\frac{z^2}{\bar{a}_j^2}\right)\exp\left(-\frac{z^2}{\bar{a}_j^2}\right)\right],\ \forall z\in\mathbb{C}. \tag{31}$$

Comparing Theorem 1 and Conjecture 1, it is natural to ask whether

$$\lim_{n\to\infty}\alpha_{j,n}[-u_4(Y_n)]^{1/4} = \lim_{n\to\infty}\bar{\alpha}_{j,n}\left[-u_4(\bar{Y}_n)\right]^{1/4} \text{ for each } j \tag{32}$$

or at least whether the ratio of these two limits is the same constant for all $j\geq 1$. Even though the behavior of Y_n (under free boundary conditions) is quite different from that of $\bar{Y}_n$ (under periodic boundary conditions), the answer to the above question could be positive.

In Section 2, we use a Gaussian transform of X_n to obtain a new random variable, W_n, whose density is directly related to the moment generating function of Y_n. Then we use the asymptotics of Y_n to show that W_n and X_n (after appropriate rescalings) converge along the same subsequence and yield the same limit up to a linear transformation. In Section 3, we first prove that for any subsequential limiting distribution of the normalized X_n, F_X, the moment generating function of $\exp(-bx^2)dF_X(x)$ for each positive b has only pure imaginary zeros (see Proposition 1); this enables us to finish the proof of Theorem 1 by applying a complete classification result about such distributions from [20]. In the Appendix, we prove that the limiting distribution of Lee-Yang zeros has no mass in an arc containing $\exp(i0)$ of the unit circle for any $\beta<\beta_c(d)$.

2. A Gaussian transform and moment generating functions

2.1. A Gaussian transform. In order to study the limit of $X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}$, as a tool, we add to X_n a multiple of a standard normal random variable $N(0,1)$, which is independent of X_n. That is

$$W_n := X_n + \sqrt{\lambda_n}N(0,1), \tag{33}$$

where $\lambda_n\geq 0$ will be defined later.

To simply the notation, we define

$$\gamma_n := \frac{1}{2\langle Y_n^2\rangle_{\Lambda_n,\beta}}. \tag{34}$$

Then we have

$$\frac{dF_{X_n}}{dF_{Y_n}}(x) = \frac{\exp[\gamma_n x^2]}{\langle\exp[\gamma_n Y_n^2]\rangle_{\Lambda_n,\beta}}. \tag{35}$$

Lemma 1. *The random variable W_n has density*

$$f_{W_n}(x) = \frac{\exp[-x^2/(2\lambda_n)]}{\sqrt{2\pi\lambda_n}\langle\exp(\gamma_n Y_n^2)\rangle_{\Lambda_n,\beta}}\int_{-\infty}^{\infty}\exp\left(\frac{xy}{\lambda_n}\right)\exp\left[\left(\gamma_n-\frac{1}{2\lambda_n}\right)y^2\right]dF_{Y_n}(y). \tag{36}$$

Proof. This follows from (33), (35) and the convolution formula (see, e.g., Theorem 2.1.16 of [7]). □

In order to get rid of the exponential function with y^2 term in the integrand of (36), we fix λ_n once and for all:

$$\lambda_n := \langle Y_n^2 \rangle_{\Lambda_n,\beta}. \tag{37}$$

Then we have

$$f_{W_n}(x) = \frac{\exp[-x^2/(2\lambda_n)]}{\sqrt{2\pi\lambda_n}\langle \exp(\gamma_n Y_n^2)\rangle_{\Lambda_n,\beta}} \langle \exp[xY_n/\lambda_n]\rangle_{\Lambda_n,\beta} \,. \tag{38}$$

2.2. **Random variables of Lee-Yang type.** A random variable X is said to be of LY type if

(a) $X \overset{d}{=} -X$ (i.e., $\mathbb{P}(X \in B) = \mathbb{P}(-X \in B)$ for any Borel set B),
(b) $\mathbb{E}\exp(DX^2) < \infty$ for some $D > 0$,
(c) $\mathbb{E}\exp(zX)$ has only pure imaginary zeros.

The following theorem from [19] says that the moment generating function of a LY type random variable has a nice factorization formula.

Theorem 2 ([19])**.** *If X is of LY type, then*

$$\mathbb{E}\exp(zX) = \exp(bz^2)\prod_{j\geq 1}\left[1 + \frac{z^2}{\alpha_j^2}\right],\ z \in \mathbb{C}, \tag{39}$$

for some $b \geq 0$ and $0 < \alpha_1 \leq \alpha_2 \leq \dots$, with $\sum_{j\geq 1} 1/\alpha_j^2 < \infty$; $\{\alpha_j : j \geq 1\}$ may be empty, finite or infinite. Moreover,

$$u_2(X) = 2\left(b + \sum_{j\geq 1}\frac{1}{\alpha_j^2}\right), \tag{40}$$

$$u_{2m}(X) = (-1)^{m-1}\frac{(2m)!}{m}\sum_{j\geq 1}\frac{1}{\alpha_j^{2m}},\ m \geq 2 \text{ and } m \in \mathbb{N}. \tag{41}$$

2.3. **Asymptotics of Y_n and W_n.** A direct application of Theorem 2 expressing b using (40) gives

Lemma 2. *For any $d \geq 1$ and $\beta \geq 0$, we have that for any $z \in \mathbb{C}$,*

$$\langle \exp(zY_n)\rangle_{\Lambda_n,\beta} = \exp\left[z^2\langle Y_n^2\rangle_{\Lambda_n,\beta}/2\right]\prod_{j\geq 1}\left[\left(1 + \frac{z^2}{\alpha_{j,n}^2}\right)\exp\left(-\frac{z^2}{\alpha_{j,n}^2}\right)\right], \tag{42}$$

where $0 < \alpha_{1,n} \leq \alpha_{2,n} \leq \dots$, $\{\pm i\alpha_{j,n} : j \geq 1\}$ are all zeros (listed according to their multiplicities) of $\langle \exp(zY_n)\rangle_{\Lambda_n,\beta}$, and $\sum_{j\geq 1} 1/\alpha_{j,n}^2 \leq \langle Y_n^2\rangle_{\Lambda_n,\beta}/2$.

We define for $n \in \mathbb{N}$,

$$c_n := \frac{\sqrt{\langle Y_n^2\rangle_{\Lambda_n,\beta}}}{\sqrt{2\pi}\langle \exp[Y_n^2/(2\langle Y_n^2\rangle_{\Lambda_n,\beta})]\rangle_{\Lambda_n,\beta}}\left[\frac{4!}{-u_4(Y_n)}\right]^{1/4}, \tag{43}$$

$$d_n := \frac{1}{\langle Y_n^2\rangle_{\Lambda_n,\beta}}\left[\frac{-u_4(Y_n)}{4!}\right]^{1/4}. \tag{44}$$

Then we have

Lemma 3. *For any $d \geq 1$ and $\beta \geq 0$, the family of random variables $\{d_nW_n : n \in \mathbb{N}\}$ is tight — more precisely,*

$$c_n \exp(-x^4) \leq f_{d_nW_n}(x) \leq \frac{c_n}{1 + x^4/3} \text{ for any } x \in \mathbb{R} \tag{45}$$

with the constants c_n satisfying the bounds

$$\left[\int_{-\infty}^{\infty} \frac{1}{1+x^4/3} dx\right]^{-1} \le c_n \le \left[\int_{-\infty}^{\infty} \exp(-x^4) dx\right]^{-1} \text{ for any } n \in \mathbb{N}. \tag{46}$$

Proof. A direct computation using (38), Lemma 2, (43) and (44) gives that

$$f_{d_n W_n}(x) = \frac{1}{d_n} f_{W_n}\left(\frac{x}{d_n}\right) = c_n \prod_{j\ge 1}\left[1+\frac{x^2}{\alpha_{j,n}^2[-u_4(Y_n)/4!]^{1/2}}\right]\exp\left[-\frac{x^2}{\alpha_{j,n}^2[-u_4(Y_n)/4!]^{1/2}}\right]. \tag{47}$$

Now (45) follows by applying the two inequalities below to the exponential term in (47)

$$\exp(-y^2/2) \le (1+y)\exp(-y) \le \frac{1}{1+y^2/6} \text{ for any } y \ge 0. \tag{48}$$

Clearly, (45) implies that $\{d_n W_n : n \in \mathbb{N}\}$ is tight and (46). □

The following lemma contains some important properties of X_n and Y_n.

Lemma 4. *One has*

$$\frac{Y_n}{\sqrt{\langle Y_n^2\rangle_{\Lambda_n,\beta}}} \Longrightarrow N(0,1), \tag{49}$$

where $N(0,1)$ is a standard normal random variable in the following two situations: (i) any $d \ge 1$ and any $\beta \in [0, \beta_c(d))$, (ii) any $d \ge 4$ and $\beta = \beta_c(d)$. Furthermore, in those two situations, one has

$$|\langle \exp(zY_n)\rangle_{\Lambda_n,\beta}| \le \exp\left[|z|^2\langle Y_n^2\rangle_{\Lambda_n,\beta}/2\right],\ \forall z \in \mathbb{C}; \tag{50}$$

$$\lim_{n\to\infty} \frac{\langle Y_n^2\rangle_{\Lambda_n,\beta}}{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} = 0. \tag{51}$$

Proof. For $\beta < \beta_c(d)$, if Y_n were from a translation invariant system (see the second part of Remark 3), then (49) would be a standard result (see, e.g., [21]). For our current Y_n, we note that Lemma 2 and (48) imply that for any $x \in \mathbb{R}$

$$\exp(x^2/2)\exp\left[\frac{x^4 u_4(Y_n)}{24\lambda_n^2}\right] \le \langle \exp[xY_n/\sqrt{\lambda_n}]\rangle_{\Lambda_n,\beta} \le \exp(x^2/2)\frac{1}{1-x^4u_4(Y_n)/(72\lambda_n^2)}. \tag{52}$$

So (49) follows if one can show

$$\lim_{n\to\infty} \frac{u_4(Y_n)}{\lambda_n^2} = 0; \tag{53}$$

This follows from two observations. On the one hand, Aizenman's inequality for finite systems (Proposition 5.3 of [1]) and exponential decay of two-point functions when $\beta < \beta_c(d)$ (see [2]) imply that $u_4(Y_n)$ is at most of order $|\Lambda_n|$. On the other hand, the trivial exponential lower bound obtained using the Edwards-Sokal coupling (see, e.g., Theorems 1.16 and 3.1 of [15]),

$$\langle \sigma_x\sigma_y\rangle_{\Lambda_n,\beta} \ge \left(\frac{1-\exp(-2\beta)}{1+\exp(-2\beta)}\right)^{\|x-y\|_1},\ \forall x,y \in \Lambda_n, \tag{54}$$

implies that λ_n is at least of order $|\Lambda_n|$. For $d \ge 4$ and $\beta = \beta_c(d)$, (49) follows from [1, 11, 3]. The inequality (50) follows from (42). For (51), we have

$$\frac{\langle Y_n^2\rangle_{\Lambda_n,\beta}}{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} = \frac{\langle Y_n^2\rangle_{\Lambda_n,\beta}\,\langle \exp[Y_n^2/(2\langle Y_n^2\rangle_{\Lambda_n,\beta})]\rangle_{\Lambda_n,\beta}}{\langle Y_n^2\exp[Y_n^2/(2\langle Y_n^2\rangle_{\Lambda_n,\beta})]\rangle_{\Lambda_n,\beta}} := \frac{\langle \exp(V_n/2)\rangle_{\Lambda_n,\beta}}{\langle V_n\exp(V_n/2)\rangle_{\Lambda_n,\beta}}, \tag{55}$$

where $V_n := Y_n^2/\langle Y_n^2\rangle_{\Lambda_n,\beta}$. (49) and (50) imply (using the convergence of moments and the inequality $\exp(x) \geq \sum_{k=1}^{M} x^k/k!$ for any $x \geq 0$ and $M \in \mathbb{N}$) that

$$\lim_{n\to\infty} \langle \exp(V_n/2)\rangle_{\Lambda_n,\beta} = \infty. \tag{56}$$

For any $M > 0$, we get from (55) that

$$\frac{\langle Y_n^2\rangle_{\Lambda_n,\beta}}{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} \leq \frac{\langle 1[V_n \geq M]\exp(V_n/2)\rangle_{\Lambda_n,\beta} + \langle 1[V_n < M]\exp(V_n/2)\rangle_{\Lambda_n,\beta}}{M\langle 1[V_n \geq M]\exp(V_n/2)\rangle_{\Lambda_n,\beta}}, \tag{57}$$

where $1[\cdot]$ is the indicator function. This combined with (56) implies that

$$\limsup_{n\to\infty} \frac{\langle Y_n^2\rangle_{\Lambda_n,\beta}}{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} = 0, \tag{58}$$

which completes the proof of (51). □

We define

$$\tilde{W}_n := \frac{W_n}{\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}} = \frac{X_n + \sqrt{\lambda_n}N(0,1)}{\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}}. \tag{59}$$

Then by (51), we have

$$\mathbb{E}(\tilde{W}_n^2) = \frac{\mathbb{E}_{\Lambda_n,\beta}X_n^2 + \langle Y_n^2\rangle_{\Lambda_n,\beta}}{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} \to 1 \text{ as } n \to \infty. \tag{60}$$

So $\{\tilde{W}_n : n \in \mathbb{N}\}$ is tight. The following lemma compares the three families of random variables: $\{d_nW_n : n \in \mathbb{N}\}$, $\{\tilde{W}_n : n \in \mathbb{N}\}$, and $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta,f}(X_n^2)} : n \in \mathbb{N}\}$.

Lemma 5. *Suppose that there exists a subsequence of* $\mathbb{N}$, $\{n_k : k \in \mathbb{N}\}$, *such that*

$$d_{n_k}W_{n_k} \Longrightarrow W \textit{ as } k \to \infty \textit{ for some random variable } W, \tag{61}$$

then

$$\tilde{W}_{n_k} \Longrightarrow C_7W \textit{ and } X_{n_k}/\sqrt{\mathbb{E}_{\Lambda_{n_k},\beta}(X_{n_k}^2)} \Longrightarrow C_7W \textit{ as } k \to \infty, \tag{62}$$

where $C_7 := \sqrt{1/\mathbb{E}(W^2)} \in (0,\infty)$.

Remark 6. *Lemma 5 and the proof of Theorem 1 imply the following stronger result: if one of the three families of random variables* $\{d_nW_n : n \in \mathbb{N}\}$, $\{\tilde{W}_n : n \in \mathbb{N}\}$ *and* $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n \in \mathbb{N}\}$ *has a convergent (in distribution) subsequence, say along* $\{n_k : k \in \mathbb{N}\}$, *then the other two families also converge (in distribution) along the same subsequence with their limits equal to the former one up to a linear transformation; and all subsequential limits are not trivial (i.e., not the Dirac point measure at* 0*).*

Proof. $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n \in \mathbb{N}\}$ is tight since the second moments are uniformly bounded by 1. The converging together lemma (see p. 108 of [7]) and (59), (51) imply that $\{\tilde{W}_n : n \in \mathbb{N}\}$ and $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n \in \mathbb{N}\}$ have the same convergent subsequences and same corresponding limits. By the tightness of $\{\tilde{W}_n : n \in \mathbb{N}\}$, we can find a subsequence of $\{n_k : k \in \mathbb{N}\}$, $\{m_l : l \in \mathbb{N}\}$, such that

$$\tilde{W}_{m_l} \Longrightarrow C_7W \text{ for some } C_7 \geq 0 \text{ as } l \to \infty. \tag{63}$$

By the GHS inequality [13] and (51), we have that for all large n and any $x \in \mathbb{R}$,

$$\mathbb{E}\exp[x\tilde{W}_n] = \exp\left[x^2\lambda_n/\left(2\mathbb{E}_{\Lambda_n,\beta}(X_n^2)\right)\right]\mathbb{E}_{\Lambda_n,\beta}\exp\left(xX_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right) \leq \exp(x^2). \tag{64}$$

So we have by (60) that

$$1 = \lim_{l\to\infty} \mathbb{E}\left(\tilde{W}_{m_l}\right)^2 = \mathbb{E}(C_7 W)^2. \tag{65}$$

Therefore,

$$C_7 := \sqrt{1/\mathbb{E}(W^2)} \tag{66}$$

and all subsequential limit of $\{\tilde{W}_{n_k} : k \geq 1\}$ are the same and thus

$$\tilde{W}_{n_k} \Longrightarrow C_7 W \text{ as } k \to \infty. \tag{67}$$

We note that $W \neq 0$ follows from (45), (46). □

3. Proofs of the main results

3.1. **A proposition.**

The following proposition is a key step towards the proof of Theorem 1.

Proposition 1. *Suppose that*

$$X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} \Longrightarrow X \textit{ as } n \to \infty \textit{ for some random variable } X. \tag{68}$$

For any fixed $b > 0$, let Z be a random variable with distribution function F_Z given by

$$\frac{dF_Z}{dF_X}(x) = \frac{\exp(-bx^2)}{\mathbb{E}\exp[-bX^2]}, \tag{69}$$

where F_X is the distribution function of X. Then the random variable Z is of LY type.

Proof. To simply the notation, we denote

$$\tilde{X}_n := X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}. \tag{70}$$

Define a sequence of random variable Z_n (or more precisely, a sequence of distribution functions F_{Z_n}) by

$$\frac{dF_{Z_n}}{dF_{\tilde{X}_n}}(x) = \frac{\exp(-bx^2)}{\mathbb{E}_{\Lambda_n,\beta}\exp[-b\tilde{X}_n^2]}. \tag{71}$$

Then for any continuous and bounded function g, by the continuous mapping theorem (Theorem 3.2.10 of [7]), we have

$$\mathbb{E}g(Z_n) = \frac{\mathbb{E}_{\Lambda_n,\beta}\left[g(\tilde{X}_n)\exp[-b\tilde{X}_n^2]\right]}{\mathbb{E}_{\Lambda_n,\beta}\exp[-b\tilde{X}_n^2]} \to \frac{\mathbb{E}\left[g(X)\exp[-bX^2]\right]}{\mathbb{E}\exp[-bX^2]} \text{ as } n \to \infty. \tag{72}$$

Therefore,

$$Z_n \Longrightarrow Z \text{ as } n \to \infty. \tag{73}$$

Equation (35) implies that

$$dF_{\tilde{X}_n}(x) = dF_{X_n}\left(x\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right) = \frac{\exp[\gamma_n x^2 \mathbb{E}_{\Lambda_n,\beta}(X_n^2)]}{\langle\exp[\gamma_n Y_n^2]\rangle_{\Lambda_n,\beta}} dF_{Y_n}\left(x\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right). \tag{74}$$

Combining (71) and (74), we get

$$dF_{Z_n}(x) = \frac{\exp[\hat{\gamma}_n x^2 \mathbb{E}_{\Lambda_n,\beta}(X_n^2)]}{\mathbb{E}_{\Lambda_n,\beta}\exp[-b\tilde{X}_n^2]\langle\exp[\gamma_n Y_n^2]\rangle_{\Lambda_n,\beta}} dF_{Y_n}\left(x\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right), \tag{75}$$

where

$$\hat{\gamma}_n := \gamma_n\left(1 - \frac{b}{\gamma_n \mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right). \tag{76}$$

Let $\hat{X}_n$ be the total magnetization for the perturbed Ising model with Hamiltonian

$$\hat{H}_{\Lambda_n,\beta}(\sigma) := -\beta \sum_{\{u,v\}} \sigma_u \sigma_y - \hat{\gamma}_n Y_n^2(\sigma), \qquad \sigma \in \{-1,+1\}^{\Lambda_n}. \tag{77}$$

Then

$$\frac{dF_{\hat{X}_n}}{dF_{Y_n}}(x) = \frac{\exp[\hat{\gamma}_n x^2]}{\langle \exp[\hat{\gamma}_n Y_n^2]\rangle_{\Lambda_n,\beta}}. \tag{78}$$

Hence

$$dF_{\hat{X}_n}\left(x\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right) = \frac{\exp[\hat{\gamma}_n x^2 \mathbb{E}_{\Lambda_n,\beta}(X_n^2)]}{\langle \exp[\hat{\gamma}_n Y_n^2]\rangle_{\Lambda_n,\beta}} dF_{Y_n}\left(x\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}\right) \tag{79}$$

Equations (75) and (79) imply that

$$Z_n \overset{d}{=} \frac{\hat{X}_n}{\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)}}, \tag{80}$$

(where $\overset{d}{=}$ denotes equal in distribution) and

$$\mathbb{E}_{\Lambda_n,\beta} \exp[-b\tilde{X}_n^2] \langle \exp[\gamma_n Y_n^2]\rangle_{\Lambda_n,\beta} = \langle \exp[\hat{\gamma}_n Y_n^2]\rangle_{\Lambda_n,\beta}. \tag{81}$$

From the definition of $\hat{\gamma}_n$ in (76), γ_n in (34), and (51), we know that

$$\lim_{n\to\infty} \frac{\hat{\gamma}_n}{\gamma_n} = 1. \tag{82}$$

So the interactions in (77) are ferromagnetic for all large n. Therefore, by the Lee-Yang theorem [17], the moment generating function of $\hat{X}_n$ has only pure imaginary zeros. So (80) says that each Z_n is of LY type, and thus Z is of LY type by (73) and Theorem 7 of [22]. □

3.2. **Proof of Theorem 1.** For ease of reference, we state without proof the following lemma since the proof follows directly from Hurwitz's theorem (see, e.g., p. 4 of [18]).

Lemma 6. *Let f_n be a sequence of analytic functions on $\mathbb{C}$ such that all zeros (listed according to their multiplicities) of f_n are $\{\pm i\alpha_{j,n} : j \geq 1\}$ with $0 < \alpha_{1,n} \leq \alpha_{2,n} \leq \dots$. Suppose that f_n converge uniformly on compact subsets of $\mathbb{C}$ to an analytic function f, and all zeros (listed according to their multiplicities) of f are $\{\pm ia_j : j \geq 1\}$ with $0 < a_1 \leq a_2 \leq \dots$. Then we have*

$$\lim_{n\to\infty} \alpha_{j,n} = a_j \text{ for each } j \geq 1.$$

We are ready to prove Theorem 1.

Proof of Theorem 1. By the assumption of the theorem, we have

$$X_{n_k}/\sqrt{\mathbb{E}_{\Lambda_{n_k},\beta}(X_{n_k}^2)} \Longrightarrow X \text{ as } k \to \infty. \tag{83}$$

The converging together lemma (see p. 108 of [7]) and (59), (51), together with (37), imply that $\{\tilde{W}_n : n \in \mathbb{N}\}$ and $\{X_n/\sqrt{\mathbb{E}_{\Lambda_n,\beta}(X_n^2)} : n \in \mathbb{N}\}$ have the same convergent subsequences and same corresponding limits. So we have

$$\tilde{W}_{n_k} \Longrightarrow X \text{ as } k \to \infty. \tag{84}$$

Lemma 3 implies tightness and non-triviality of any subsequential limit; so there is a subsequence of $\{n_k : k \in \mathbb{N}\}$, $\{m_l : l \in \mathbb{N}\}$, and a random variable W such that

$$d_{m_l} W_{m_l} \Longrightarrow W := C_8 X \text{ as } l \to \infty \text{ for some } C_8 \in (0,\infty). \tag{85}$$

Let F_W be the distribution function of W. Proposition 1 implies that Theorem 2 of [20] can be applied to W and so that either

$$dF_W(x) = \frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0} \text{ for some } x_0 \in \mathbb{R}, \tag{86}$$

or F_W has a density of the form

$$f_W(x) = Kx^{2m}\exp[-\kappa_1 x^4 - \kappa_2 x^2]\prod_{j\geq 1}\left[\left(1+\frac{x^2}{a_j^2}\right)\exp\left(-\frac{x^2}{a_j^2}\right)\right], \tag{87}$$

where $K > 0$, $m = 0, 1, 2, \ldots$, $0 < a_1 \leq a_2 \leq \ldots$, $\sum_{j\geq 1} 1/a_j^4 < \infty$, $\kappa_1 > 0$ and $\kappa_2 \in \mathbb{R}$ (or $\kappa_1 = 0$ and $\kappa_2 + \sum_{j\geq 1} 1/a_j^2 > 0$); $\{a_j : j \geq 1\}$ may be empty, finite or infinite and the condition $\kappa_2 + \sum_{j\geq 1} 1/a_j^2 > 0$ is considered to be satisfied if $\sum_{j\geq 1} 1/a_j^2 = \infty$.

From (47), we know that for each fixed $n \in \mathbb{N}$,

$$f_{d_n W_n}(x) \text{ is decreasing on } [0, \infty) \tag{88}$$

since $(1 + y)\exp(-y)$ is so. Hence,

$$\mathbb{P}(W \in [0, a]) = \lim_{l\to\infty}\mathbb{P}(d_{m_l}W_{m_l} \in [0, a]) \geq \lim_{l\to\infty}\mathbb{P}(d_{m_l}W_{m_l} \in [a, 2a]) = \mathbb{P}(W \in [a, 2a]) \tag{89}$$

for any $a > 0$ and $a, 2a$ are points of continuity of F_W. Note that $W \neq 0$ (see Lemmas 3 and 5, and Remark 6). So F_W can not be of the form (86) because (86) would not satisfy (89) for all $a > 0$. That means F_W has a density of the form (87). In particular, this implies that the density of W evaluated at 0 is

$$f_W(0) = \lim_{\epsilon\downarrow 0}\frac{\lim_{l\to\infty}\int_0^\epsilon f_{d_{m_l}W_{m_l}}(x)dx}{\epsilon}. \tag{90}$$

Combining (90) with Lemma 3, we get that there exists a $c \in (0, \infty)$ such that the following limit exists:

$$\lim_{l\to\infty} c_{m_l} = c := f_W(0). \tag{91}$$

Clearly, $f_W(0) > 0$ implies that $m = 0$ in (87). Note that by (41)

$$\alpha_{j,n}^2\left[-u_4(Y_n)/4!\right]^{1/2} = \left[\frac{1}{2}\sum_{k\geq 1}\frac{\alpha_{j,n}^4}{\alpha_{k,n}^4}\right]^{1/2} \geq \left(\frac{j}{2}\right)^{1/2} \text{ for each } j \geq 1. \tag{92}$$

So (47), the Taylor series for $\ln(1 + y)$ when $|y| < 1$, and Fubini's theorem give

$$\begin{aligned} f_{d_n W_n}(x) &= c_n \exp\left\{\sum_{j\geq 1}\left[\ln\left(1+\frac{x^2}{\alpha_{j,n}^2\left[-u_4(Y_n)/4!\right]^{1/2}}\right) - \frac{x^2}{\alpha_{j,n}^2\left[-u_4(Y_n)/4!\right]^{1/2}}\right]\right\} \\ &= c_n \exp\left\{\sum_{j\geq 1}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}x^{2k}}{k\alpha_{j,n}^{2k}\left[-u_4(Y_n)/4!\right]^{k/2}}\right\} \\ &= c_n \exp\left\{\sum_{k=2}^{\infty}\frac{(-1)^{k-1}x^{2k}\sum_{j\geq 1}\alpha_{j,n}^{-2k}}{k\left[-u_4(Y_n)/4!\right]^{k/2}}\right\}, \ \forall x \in \mathbb{R} \text{ and } |x| < \left(\frac{1}{2}\right)^{1/4}. \end{aligned} \tag{93}$$

Since

$$\left(\sum_{j\geq 1}\frac{1}{\alpha_{j,n}^{2k}}\right)^{1/(2k)} \leq \left(\sum_{j\geq 1}\frac{1}{\alpha_{j,n}^4}\right)^{1/4} = \left(\frac{-2u_4(Y_n)}{4!}\right)^{1/4} \text{ for each } k \geq 2, \tag{94}$$

we have

$$0 \le \frac{\sum_{j\ge1} \alpha_{j,n}^{-2k}}{[-u_4(Y_n)/4!]^{k/2}} \le 2^{k/2} \text{ for each } k \ge 2. \tag{95}$$

Therefore, by a standard diagonalization argument, we can extract a subsequence of $\{m_l : l \in \mathbb{N}\}$, $\{p_q : q \in \mathbb{N}\}$, such that the following limits exist

$$\lim_{q\to\infty} \frac{\sum_{j\ge1} \alpha_{j,p_q}^{-2k}}{\left[-u_4(Y_{p_q})/4!\right]^{k/2}} := b_k \in [0, 2^{k/2}] \text{ for all } k \ge 2. \tag{96}$$

Combining (91), (93) and (96), we get

$$\lim_{q\to\infty} f_{d_{p_q}W_{p_q}}(x) = c\exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1} b_k x^{2k}}{k}\right\}, \ \forall x \in \mathbb{R} \text{ and } |x| < \left(\frac{1}{2}\right)^{1/4}. \tag{97}$$

So Lemma 3, (97), the dominated convergence theorem and (85) imply that for any $-(1/2)^{1/4} < y_1 < y_2 < (1/2)^{1/4}$,

$$\lim_{q\to\infty} \int_{y_1}^{y_2} f_{d_{p_q}W_{p_q}}(x)dx = c\int_{y_1}^{y_2} \exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1} b_k x^{2k}}{k}\right\} dx = \int_{y_1}^{y_2} f_W(x)dx. \tag{98}$$

Hence (noting that $f_W(x)$ is continuous, and the RHS of (97) is also continuous in x)

$$f_W(x) = c\exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1} b_k x^{2k}}{k}\right\}, \ \forall x \in \mathbb{R} \text{ and } |x| < \left(\frac{1}{2}\right)^{1/4}. \tag{99}$$

It follows from (47), (46) and (92) that $\{f_{d_nW_n} : n \in \mathbb{N}\}$ is uniformly bounded as n varies in each bounded subset of $\mathbb{C}$. Then Vitali's theorem and (97), (99) imply that

$$\lim_{q\to\infty} f_{d_{p_q}W_{p_q}}(x) = f_W(x) \text{ uniformly on compact subsets of } \mathbb{C}. \tag{100}$$

From (47), we know all zeros of $f_{d_{p_q}W_{p_q}}$ are

$$\{\pm i\alpha_{j,p_q}\left[-u_4(Y_{p_q})/4!\right]^{1/4} : j \ge 1\} \text{ with } 0 < \alpha_{1,p_q} \le \alpha_{2,p_q} \le \ldots; \tag{101}$$

from (87) and the fact that $m = 0$, we know all zeros of f_W are

$$\{\pm i a_j : j \ge 1\} \text{ with } 0 < a_1 \le a_2 \le \ldots. \tag{102}$$

Consequently, (100) and Hurwitz's theorem (see Lemma 6) imply that

$$\lim_{q\to\infty} \alpha_{j,p_q}\left[-u_4(Y_{p_q})/4!\right]^{1/4} = a_j \text{ for each } j \ge 1. \tag{103}$$

So (92), (96), (103) and the dominated convergence theorem give that

$$b_k = \sum_{j\ge1} \frac{1}{a_j^{2k}} \text{ for each } k > 2. \tag{104}$$

Combining this with (99) gives that

$$f_W(x) = c\exp\left\{-x^4 + \sum_{k=3}^{\infty} \frac{(-1)^{k-1} x^{2k} \sum_{j\ge1} a_j^{-2k}}{k}\right\}, \ \forall x \in \mathbb{R} \text{ and } |x| < \left(\frac{1}{2}\right)^{1/4}, \tag{105}$$

where we have used $b_2 = 2$. By (92) and (103), we have

$$a_j^2 \ge \left(\frac{j}{2}\right)^{1/2} \text{ for each } j \ge 1. \tag{106}$$

So applying the argument leading to (93) to the product in (87), we obtain (recall $m = 0$)

$$f_W(x) = K \exp\left[-\kappa_1 x^4 - \kappa_2 x^2\right] \exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1} x^{2k} \sum_{j\geq 1} a_j^{-2k}}{k}\right\}, \ |x| < \left(\frac{1}{2}\right)^{1/4}. \tag{107}$$

Comparing (105) and (107), we get

$$K = c, \kappa_2 = 0, \kappa_1 + \frac{\sum_{j\geq 1} a_j^{-4}}{2} = 1. \tag{108}$$

Finally, we show that $d_{n_k} W_{n_k}$ converges in distribution to W as $k \to \infty$. Suppose that there is another subsequence of $\{n_k : k \in \mathbb{N}\}$, $\{\tilde{m}_l : l \in \mathbb{N}\}$, and a random variable $\tilde{W}$ such that

$$d_{\tilde{m}_l} W_{\tilde{m}_l} \Longrightarrow \tilde{W} := C_9 X \text{ as } l \to \infty \text{ for some } C_9 \in (0, \infty), \tag{109}$$

where we have used (84). Then by (85) and (109), we have

$$\tilde{W} \overset{d}{=} \rho W \text{ for some } \rho \in (0, \infty). \tag{110}$$

The arguments leading to (100), (103) and (108) also imply that there exists a subsequence of $\{\tilde{m}_l : l \in \mathbb{N}\}$, $\{\tilde{p}_q : q \in \mathbb{N}\}$, such that

$$\lim_{q\to\infty} f_{d_{\tilde{p}_q} W_{\tilde{p}_q}}(x) = f_{\tilde{W}}(x) = \tilde{K} \exp[-\tilde{\kappa}_1 x^4] \prod_{j\geq 1} \left[\left(1 + \frac{x^2}{\tilde{a}_j^2}\right) \exp\left(-\frac{x^2}{\tilde{a}_j^2}\right)\right] \tag{111}$$

uniformly on compact subsets of $\mathbb{C}$ where

$$\lim_{q\to\infty} \alpha_{j,\tilde{p}_q} \left[-u_4(Y_{\tilde{p}_q})/4!\right]^{1/4} = \tilde{a}_j \text{ for each } j \geq 1, \tag{112}$$

$$\tilde{\kappa}_1 + \frac{\sum_{j\geq 1} \tilde{a}_j^{-4}}{2} = 1. \tag{113}$$

The relation between $\tilde{W}$ and W in (110), (87) and (108) give

$$f_{\tilde{W}}(x) = \frac{1}{\rho} f_W\left(\frac{x}{\rho}\right) = \frac{K}{\rho} \exp\left[-\kappa_1 \frac{x^4}{\rho^4}\right] \prod_{j\geq 1} \left[\left(1 + \frac{x^2}{\rho^2 a_j^2}\right) \exp\left(-\frac{x^2}{\rho^2 a_j^2}\right)\right], \ \forall x \in \mathbb{C}. \tag{114}$$

From (111) and (114), we get

$$\tilde{\kappa}_1 = \kappa_1/\rho^4, \tilde{a}_j = \rho a_j \text{ for each } j \geq 1. \tag{115}$$

Plugging this into (113) and using (108), we get $\rho = 1$. Therefore

$$d_{n_k} W_{n_k} \Longrightarrow W \text{ as } k \to \infty, \tag{116}$$

$$\lim_{k\to\infty} \alpha_{j,n_k} \left[-u_4(Y_{n_k})/4!\right]^{1/4} = a_j \text{ for each } j \geq 1. \tag{117}$$

This completes the proof of Theorem 1 by applying Lemma 5. □

3.3. **Proof of Corollary 1.** We conclude this section with a proof of Corollary 1.

Proof of Corollary 1. Let W be a random variable with density function

$$f_W(x) = \exp(-x^4) / \int_{-\infty}^{\infty} \exp(-t^4) dt, \ \forall x \in \mathbb{R}. \tag{118}$$

Then we have

$$\mathbb{E}(W^2) = \frac{\int_{-\infty}^{\infty} x^2 \exp(-x^4) dx}{\int_{-\infty}^{\infty} \exp(-x^4) dx} = \frac{\Gamma(3/4)}{\Gamma(1/4)}. \tag{119}$$

So to prove Corollary 1, by Theorem 1, it suffices to show that there exists $\beta_1(d) \in (0, \beta_c(d)]$ such that for any $\beta \in [0, \beta_1(d))$, we have that

$$\lim_{n\to\infty} \alpha_{1,n} \left[-u_4(Y_n)/4!\right]^{1/4} = \infty. \tag{120}$$

By [27] or Theorem A of [25], there exists $\beta_1(d) \in (0, \beta_c(d)]$ such that for any $\beta \in [0, \beta_1(d))$, there is $C_{10} = C_{10}(d, \beta) \in (0, \infty)$ satisfying

$$\alpha_{1,n} \geq C_{10} \text{ uniformly in } n. \tag{121}$$

Note that

$$\langle \exp(zY_n)\rangle_{\Lambda_n,\beta} = 0 \text{ is equivalent to } \sum_{\sigma\in\{-1,+1\}^{\Lambda_n}} \exp\left[\beta \sum_{\{u,v\}} \sigma_u\sigma_v + z \sum_{u\in\Lambda_n} \sigma_u\right] = 0. \tag{122}$$

We also note that

$$\langle \exp(zY_n)\rangle_{\Lambda_n,\beta} = (-1)^{k|\Lambda_n|} \langle \exp\left((z + ik\pi)Y_n\right)\rangle_{\Lambda_n,\beta} \text{ for any } k \in \mathbb{Z} \tag{123}$$

since Y_n and $|\Lambda_n|$ always have the same parity. So the arguments from (129) to (132) in the Appendix imply that exactly $|\Lambda_n|$ roots of $\langle \exp(zY_n)\rangle_{\Lambda_n,\beta}$ lie in the interval $i[0, \pi]$:

$$\alpha_{1,n} = \frac{\theta_{1,n}}{2}, \alpha_{2,n} = \frac{\theta_{2,n}}{2}, \dots, \alpha_{|\Lambda_n|,n} = \frac{\theta_{|\Lambda_n|,n}}{2} \text{ where } 0 < \theta_{1,n} \leq \theta_{2,n} \leq \dots \leq \theta_{|\Lambda_n|,n} < 2\pi. \tag{124}$$

Therefore, all roots of $\langle \exp(zY_n)\rangle_{\Lambda_n,\beta}$ are

$$\left\{(\pm i)(\theta_{l,n}/2 + k\pi) : l \in \{1, 2, \dots, |\Lambda_n|\}, k \in \{0, 1, 2, \dots\}\right\}. \tag{125}$$

This implies that for any $\beta \geq 0$, using (41), we have

$$-u_4(Y_n)/4! = \frac{1}{2}\sum_{j\geq 1} \alpha_{j,n}^{-4} = \frac{1}{2}\sum_{l=1}^{|\Lambda_n|}\sum_{k=0}^{\infty}\left(\frac{\theta_{l,n}}{2} + k\pi\right)^{-4} \geq \frac{|\Lambda_n|}{2}\sum_{k=0}^{\infty}(\pi + k\pi)^{-4} \geq C_{11}|\Lambda_n|, \tag{126}$$

where $C_{11} \in (0, \infty)$. Clearly, (120) follows from (121) and (126). □

Appendix A. Limit distribution of Lee-Yang zeros when $\beta < \beta_c(d)$

In this appendix, we prove that when $\beta < \beta_c(d)$, the limiting distribution of Lee-Yang zeros has no mass in an arc containing $\exp(i0)$ of the unit circle. This is stated as a conjecture in Section 1.3 of [6]. As can be seen from below, the proof follows essentially from Theorem 1.2 of [12] and Theorem 1.5 of [23]. We present a slightly different and relatively self-contained proof here which might be better suited to the context. We came up with this proof before we knew of the existence of [12].

Let $\Lambda \subseteq \mathbb{Z}^d$ be finite. The Ising model on Λ at inverse temperature $\beta \geq 0$ with free boundary conditions and external field $h \in \mathbb{R}$ is defined by the probability measure $\mathbb{P}_{\Lambda,\beta,h}$ on $\{-1, +1\}^\Lambda$ such that for each $\sigma \in \{-1, +1\}^\Lambda$

$$\mathbb{P}_{\Lambda,\beta,h}(\sigma) := \frac{\exp\left[\beta \sum_{\{u,v\}} \sigma_u\sigma_v + h \sum_{u\in\Lambda} \sigma_u\right]}{Z_{\Lambda,\beta,h}}, \tag{127}$$

where the first sum is over all nearest-neighbor edges in Λ, and $Z_{\Lambda,\beta,h}$ is the partition function that makes (127) a probability measure. In this appendix, we will consider complex $h \in \mathbb{C}$. A famous result due to Lee and Yang [17] is that $Z_{\Lambda,\beta,h} \neq 0$ if $h \notin i\mathbb{R}$ where $i\mathbb{R}$ denotes the pure imaginary axis. So the fraction in (127) is well-defined if $h \notin i\mathbb{R}$ but it could take a complex value, and thus $\mathbb{P}_{\Lambda,\beta,h}$ is a complex measure. Let

$\langle\cdot\rangle_{\Lambda,\beta,h}$ denote the expectation with respect to $\mathbb{P}_{\Lambda,\beta,h}$. For example, the magnetization at $u \in \Lambda$ is defined by

$$\langle\sigma_u\rangle_{\Lambda,\beta,h} := \frac{\sum_{\sigma\in\{-1,+1\}^\Lambda} \sigma_u \exp\left[\beta\sum_{\{u,v\}}\sigma_u\sigma_v + h\sum_{u\in\Lambda}\sigma_u\right]}{Z_{\Lambda,\beta,h}}. \tag{128}$$

Let $\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d$ and E_n be the set of all nearest-neighbor edges $\{u,v\}$ with $u, v \in \Lambda_n$. Note that

$$Z_{\Lambda_n,\beta,h} = \sum_{\sigma\in\{-1,+1\}^{\Lambda_n}} \exp\left[\beta\sum_{\{u,v\}\in E_n}\sigma_u\sigma_v + h\sum_{u\in\Lambda_n}\sigma_u\right] \tag{129}$$

$$= \exp\left[\beta|E_n| + h|\Lambda_n|\right] \sum_{\sigma\in\{-1,+1\}^{\Lambda_n}} \exp\left[\beta\sum_{\{u,v\}\in E_n}(\sigma_u\sigma_v - 1) + h\sum_{u\in\Lambda_n}(\sigma_u - 1)\right]. \tag{130}$$

Let $z = e^{-2h}$ throughout the appendix and write $Z_{\Lambda_n,\beta}(z) := Z_{\Lambda_n,\beta,h}$. Then the outmost sum in (130) is a polynomial of z with degree $|\Lambda_n|$. So by the fundamental theorem of algebra, $Z_{\Lambda_n,\beta}(z)$ has exactly $|\Lambda_n|$ complex roots. The Lee-Yang circle theorem [17] says that these $|\Lambda_n|$ roots are on the unit circle $\partial\mathbb{D}$ where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. So we may assume that these roots are

$$\exp(i\theta_{1,n}), \exp(i\theta_{2,n}), \ldots, \exp(i\theta_{|\Lambda_n|,n}) \text{ where } 0 < \theta_{1,n} \le \theta_{2,n} \le \cdots \le \theta_{|\Lambda_n|,n} < 2\pi. \tag{131}$$

Note that we have used the fact that $Z_{\Lambda_n,\beta}(1) > 0$. By the spin-flip symmetry, $Z_{\Lambda_n,\beta,-h} = Z_{\Lambda_n,\beta,h}$ for any $h \in \mathbb{C}$. As a result, those $|\Lambda_n|$ roots in (131) are symmetric with respect to the real-axis. Combining (130) and (131), we have

$$Z_{\Lambda_n,\beta}(z) = \exp[\beta|E_n| - (\ln z)|\Lambda_n|/2] \prod_{j=1}^{|\Lambda_n|}\left[z - \exp(i\theta_{j,n})\right], \ z \in \mathbb{C}\setminus(-\infty,0]. \tag{132}$$

Here we take out $(-\infty, 0]$ from $\mathbb{C}$ so that $h = -(\ln z)/2$ is analytic in this slit domain. Therefore,

$$\frac{\partial \ln Z_{\Lambda_n,\beta}(z)}{\partial z} = -\frac{|\Lambda_n|}{2z} + \sum_{j=1}^{|\Lambda_n|}\frac{1}{z - \exp(i\theta_{j,n})}, \ \forall z \in \mathbb{D}\setminus(-1,0]. \tag{133}$$

It is easy to see that

$$\left\langle\sum_{u\in\Lambda_n}\sigma_u\right\rangle_{\Lambda_n,\beta,h} = \frac{\partial \ln Z_{\Lambda_n,\beta,h}}{\partial h} = \frac{\partial \ln Z_{\Lambda_n,\beta}(z)}{\partial z}\frac{\partial z}{\partial h} = -2z\frac{\partial \ln Z_{\Lambda_n,\beta}(z)}{\partial z}, \ \forall z \in \mathbb{D}\setminus(-1,0]. \tag{134}$$

We define the **average magnetization density** in Λ_n by

$$m_{\Lambda_n}(z) := \frac{\left\langle\sum_{u\in\Lambda_n}\sigma_u\right\rangle_{\Lambda_n,\beta,h}}{|\Lambda_n|}, \ \forall z \in \mathbb{C}\setminus\partial\mathbb{D}, \tag{135}$$

where we have dropped the β dependence in $m_{\Lambda_n}(z)$. By (133) and (134), we have

$$m_{\Lambda_n}(z) = 1 - \frac{2z}{|\Lambda_n|}\sum_{j=1}^{|\Lambda_n|}\frac{1}{z - \exp(i\theta_{j,n})} = \frac{1}{|\Lambda_n|}\sum_{j=1}^{|\Lambda_n|}\frac{\exp(i\theta_{j,n}) + z}{\exp(i\theta_{j,n}) - z}, \ \forall z \in \mathbb{D}\setminus(-1,0]. \tag{136}$$

By using the definition of $\langle\cdot\rangle_{\Lambda,\beta,h}$ (see (128)), we know that $m_{\Lambda_n}(z)$ is a rational function of z with poles on $\partial\mathbb{D}$, and thus (136) holds for each $z\in\mathbb{D}$. We define the empirical distribution

$$\mu_n := \frac{1}{|\Lambda_n|}\sum_{j=1}^{|\Lambda_n|}\delta_{\exp(i\theta_{j,n})}, \tag{137}$$

where $\delta_{\exp(i\theta_{j,n})}$ is the unit Dirac point measure at $\exp(i\theta_{j,n})$. Then

$$m_{\Lambda_n}(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu_n(e^{i\theta}),\ \forall z\in\mathbb{D}. \tag{138}$$

Since μ_n's live on the unit circle, $\{\mu_n : n\in\mathbb{N}\}$ is tight. So there is a subsequence of $\mathbb{N}$, $\{n_k : k\in\mathbb{N}\}$, such that $\mu_{n_k}\Longrightarrow\mu$ as $k\to\infty$ where μ is some probability measure on $\partial\mathbb{D}$. We will see later that μ is actually unique. Therefore,

$$m_{\Lambda_{n_k}}(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu_{n_k}(e^{i\theta}) \to \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{i\theta}) \text{ as } k\to\infty,\ \forall z\in\mathbb{D}. \tag{139}$$

Note that

$$\Re m_{\Lambda_n}(z) = \int_{\partial\mathbb{D}} \Re\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu_n(e^{i\theta}) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|e^{i\theta}-z|^2}d\mu_n(e^{i\theta}) > 0,\ \forall z\in\mathbb{D}, \tag{140}$$

and $m_{\Lambda_n}(0)=1$. So m_{Λ_n} is a **Herglotz function**. See Section 8.4 of [26] for more details. From (138), we know

$$|m_{\Lambda_n}(z)| \le \frac{1+r}{1-r},\ \text{for any } z \text{ satisfying } |z|\le r<1. \tag{141}$$

So $\{m_{\Lambda_n} : n\in\mathbb{N}\}$ is locally uniformly bounded. It is well-known that

$$\lim_{n\to\infty} m_{\Lambda_n}(z) = \langle\sigma_0\rangle_{\mathbb{Z}^d,\beta,h},\ \forall z\in(0,1), \tag{142}$$

where $\langle\cdot\rangle_{\mathbb{Z}^d,\beta,h}$ is the expectation with respect to the unique infinite volume measure when $\beta\ge 0$ and $h>0$ (see, e.g., Proposition 3.29 and Theorem 3.46 of [10]). So by Vitali's theorem (see, e.g., Theorem B.25 of [10]),

$$m(z) := \lim_{n\to\infty} m_{\Lambda_n}(z) \tag{143}$$

exists locally uniformly on $\mathbb{D}$ and m is a Herglotz function. Comparing (139) and (143), we obtain

$$m(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{i\theta}),\ \forall z\in\mathbb{D}. \tag{144}$$

We define

$$m(z) := -m(1/z),\ \forall z\in\mathbb{C}\setminus\bar{\mathbb{D}}. \tag{145}$$

Since $\langle\sigma_x\rangle_{\Lambda_n,\beta,h} = -\langle\sigma_x\rangle_{\Lambda_n,\beta,-h}$ for any $x\in\Lambda_n$ and $h\in\mathbb{C}\setminus i\mathbb{R}$, we get

$$m(z) = \lim_{n\to\infty} m_{\Lambda_n}(z),\ \forall z\in\mathbb{C}\setminus\bar{\mathbb{D}}. \tag{146}$$

The following Stieltjes inversion formula on page 12 of [26] will be very import to our analysis of μ.

Theorem A (Stieltjes Inversion Formula)**.** *Let $\gamma := \{e^{it} : a<t<b\}$ be an open arc on $\partial\mathbb{D}$ with endpoints e^{ia} and e^{ib}, $0<b-a<2\pi$. Then*

$$\mu(\gamma) + \frac{1}{2}\mu(\{e^{ia}\}) + \frac{1}{2}\mu(\{e^{ib}\}) = \lim_{r\uparrow 1}\frac{1}{2\pi}\int_a^b \Re m(re^{i\theta})d\theta. \tag{147}$$

In particular, Theorem A implies the μ that we obtained from the subsequential limit is unique, that is, we have $\mu_n \Longrightarrow \mu$ as $n \to \infty$. We call μ the **limiting distribution of Lee-Yang zeros**. Note that μ is actually a function of β. Let $\beta_c(d)$ be the critical inverse temperature of the Ising model on $\mathbb{Z}^d$. We are ready to prove the main result about μ.

Theorem B. *For any $d \geq 1$ and any $\beta \in [0, \beta_c(d))$, there is $\epsilon > 0$ (which only depends on β and d) such that*

$$\mu\left(\{e^{it} : -\epsilon < t < \epsilon\}\right) = 0. \tag{148}$$

Proof. By Theorem A,

$$\mu\left(\{e^{it} : -\epsilon < t < \epsilon\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) = \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \Re m(re^{i\theta})d\theta \tag{149}$$

$$= \lim_{r\uparrow 1} \frac{1}{2\pi}\left[\int_{-\epsilon}^{0} \Re m(re^{i\theta})d\theta + \int_{0}^{\epsilon} \Re m(re^{i\theta})d\theta\right] \tag{150}$$

$$= \lim_{r\uparrow 1} \frac{1}{2\pi}\left[\int_{0}^{\epsilon} \Re m(re^{-i\theta})d\theta + \int_{0}^{\epsilon} \Re m(re^{i\theta})d\theta\right]. \tag{151}$$

By Theorem 1.5 of [23], we have that $m(z)$ is complex analytic in a neighborhood of $z = 1$. So we may pick ϵ small such that m is analytic in $D(1, 2\epsilon) := \{z \in \mathbb{C} : |z - 1| < 2\epsilon\}$. Then both $\Re m(re^{-i\theta})$ and $\Re m(re^{i\theta})$ are bounded if $re^{-i\theta}$ and $re^{i\theta}$ are in $D(1, \epsilon)$. The dominated converge theorem implies that

$$\mu\left(\{e^{it} : -\epsilon < t < \epsilon\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) \tag{152}$$

$$= \frac{1}{2\pi}\int_{0}^{\epsilon}\left[\lim_{r\uparrow 1} \Re m(re^{-i\theta}) + \lim_{r\uparrow 1} \Re m(re^{i\theta})\right] d\theta. \tag{153}$$

By (145) and (146), and continuity of m in $D(1, 2\epsilon)$, we have

$$\lim_{r\uparrow 1} \Re m(re^{-i\theta}) = -\lim_{r\uparrow 1} \Re m(r^{-1}e^{i\theta}) = -\lim_{r\uparrow 1} \Re m(re^{i\theta}). \tag{154}$$

Plugging this into (153), we get

$$\mu\left(\{e^{it} : -\epsilon < t < \epsilon\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) + \frac{1}{2}\mu\left(\{e^{i\epsilon}\}\right) = 0, \tag{155}$$

which concludes the proof of the theorem. □

Acknowledgements

The research of the second author was partially supported by NSFC grant 11901394 and that of the third author by US-NSF grant DMS-1507019. The authors thank two anonymous reviewers for useful comments and suggestions.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] M. Aizenman. Geometric analysis of φ^4 fields and Ising models. parts I and II. *Communications in Mathematical Physics*, 86(1):1–48, 1982.

[2] M. Aizenman, D. J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *Journal of Statistical Physics*, 47(3):343–374, 1987.

[3] M. Aizenman and H. Duminil-Copin. Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models. *Annals of Mathematics*, 194(1):163–235, 2021.

[4] F. Camia, C. Garban, and C. M. Newman. Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits. *Annales de l'IHP, Probabilités et Statistiques*, 52(1):146–161, 2016.

[5] F. Camia, J. Jiang, and C. M. Newman. The effect of free boundary conditions on the Ising model in high dimensions. *Probability Theory and Related Fields*, 181(1):311–328, 2021.

[6] I. Chio, C. He, A. L. Ji, and R. K. Roeder. Limiting measure of Lee–Yang zeros for the Cayley tree. *Communications in Mathematical Physics*, 370(3):925–957, 2019.

[7] R. Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[8] R. S. Ellis and C. M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 44(2):117–139, 1978.

[9] R. S. Ellis, C. M. Newman, and J. S. Rosen. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 51(2):153–169, 1980.

[10] S. Friedli and Y. Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction*. Cambridge University Press, 2017.

[11] J. Fröhlich. On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. *Nuclear Physics B*, 200(2):281–296, 1982.

[12] D. S. Greenstein. On the analytic continuation of functions which map the upper half plane into itself. *Journal of Mathematical Analysis and Applications*, 1:355–362, 1960.

[13] R. B. Griffiths, C. A. Hurst, and S. Sherman. Concavity of magnetization of an Ising ferromagnet in a positive external field. *Journal of Mathematical Physics*, 11(3):790–795, 1970.

[14] J. Grimm, E. M. Elçi, Z. Zhou, T. M. Garoni, and Y. Deng. Geometric explanation of anomalous finite-size scaling in high dimensions. *Physical Review Letters*, 118(11):115701, 2017.

[15] G. R. Grimmett. *The Random-Cluster Model*, volume 333. Springer, 2006.

[16] T. Hutchcroft, E. Michta, and G. Slade. High-dimensional near-critical percolation and the torus plateau. *arXiv preprint arXiv:2107.12971*, 2021.

[17] T.-D. Lee and C.-N. Yang. Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model. *Physical Review*, 87(3):410, 1952.

[18] M. Marden. *Geometry of polynomials*. Number 3. American Mathematical Soc., 1966.

[19] C. M. Newman. Inequalities for Ising models and field theories which obey the Lee-Yang theorem. *Communications in Mathematical Physics*, 41(1):1–9, 1975.

[20] C. M. Newman. Fourier transforms with only real zeros. *Proceedings of the American Mathematical Society*, 61(2):245–251, 1976.

[21] C. M. Newman. Normal fluctuations and the FKG inequalities. *Communications in Mathematical Physics*, 74(2):119–128, 1980.

[22] C. M. Newman and W. Wu. Lee–Yang property and Gaussian multiplicative chaos. *Communications in Mathematical Physics*, 369(1):153–170, 2019.

[23] S. Ott. Weak mixing and analyticity of the pressure in the Ising model. *Communications in Mathematical Physics*, 377(1):675–696, 2020.

[24] V. Papathanakos. *Finite-size effects in high-dimensional statistical mechanical systems: The Ising model with periodic boundary conditions. PhD Thesis* Princeton University, Princeton, New Jersey, 2006.

[25] H. Peters and G. Regts. Location of zeros for the partition function of the Ising model on bounded degree graphs. *Journal of the London Mathematical Society*, 101(2):765–785, 2020.

[26] M. Rosenblum and J. Rovnyak. *Topics in Hardy classes and univalent functions*. Springer Science & Business Media, 1994.

[27] D. Ruelle. Extension of the Lee-Yang circle theorem. *Physical Review Letters*, 26(6):303, 1971.

[28] B. Simon and R. B. Griffiths. The $(\phi^4)_2$ field theory as a classical Ising model. *Communications in Mathematical Physics*, 33(2):145–164, 1973.

[29] G. Slade. The near-critical two-point function and the torus plateau for weakly self-avoiding walk in high dimensions. *arXiv preprint arXiv:2008.00080*, 2020.

[30] M. Wittmann and A. Young. Finite-size scaling above the upper critical dimension. *Physical Review E*, 90(6):062137, 2014.

[31] C.-N. Yang and T.-D. Lee. Statistical theory of equations of state and phase transitions. I. Theory of condensation. *Physical Review*, 87(3):404, 1952.

[32] Z. Zhou, J. Grimm, Y. Deng, and T. M. Garoni. Random-length random walks and finite-size scaling on high-dimensional hypercubic lattices I: Periodic boundary conditions. *arXiv preprint arXiv:2008.00913*, 2020.

[33] Z. Zhou, J. Grimm, S. Fang, Y. Deng, and T. M. Garoni. Random-length random walks and finite-size scaling in high dimensions. *Physical Review Letters*, 121(18):185701, 2018.

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