

GEOMETRIC QUANTIZATION AND QUANTUM MOMENT MAPS ON COADJOINT ORBITS AND KÄHLER-EINSTEIN MANIFOLDS

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ABSTRACT. Deformation quantization and geometric quantization on Kähler manifolds give the mathematical description of the algebra of quantum observables and the Hilbert spaces respectively, where the later forms a representation of quantum observables asymptotically via Toeplitz operators. When there is a Hamiltonian G -action on a Kähler manifold, there are associated symmetries on both the quantum algebra and representation aspects. We show that in nice cases of coadjoint orbits and Kähler-Einstein manifolds, these symmetries are strictly compatible (not only asymptotically).

1. INTRODUCTION

Quantizing a classical mechanical system on $(X = T^*\mathbb{R}^n, \omega = \sum dx^j \wedge dp_j)$ is amount to treating position x^j and moment p_j as operators $\hat{x}^j = x^j \cdot$ and $\hat{p}_j = i\hbar \frac{\partial}{\partial x^j}$ on $L^2(\mathbb{R}^n)$ respectively, thus realizing the uncertainty principle $[\hat{x}^j, \hat{p}_k] = i\hbar \delta_k^j$. In fact, this gives an action of $C^\infty(X) [[\hbar]]$ on $L^2(\mathbb{R}^n)$ with a non-commutative algebra structure \star_\hbar on $C^\infty(X) [[\hbar]]$. $L^2(\mathbb{R}^n)$ can also be realized as $\Gamma_{L^2}(X, L) \cap \text{Ker}(\nabla_{\mathcal{P}_\mathbb{R}})$, the space of L^2 -sections of the trivial line bundle L , equipped with a unitary connection ∇ with curvature $-i\omega$, over X which are constant along the real polarization $\mathcal{P}_\mathbb{R} = \left\langle \frac{\partial}{\partial p_j} \text{'s} \right\rangle$, namely those sections which are independent of p_j 's.

Quantizing a physical system should be independent of the choice of polarizations. If we use the complex polarization $\mathcal{P}_\mathbb{C} = \left\langle \frac{\partial}{\partial \bar{z}_j} \text{'s} \right\rangle$ on $X = \mathbb{C}^n$ with $z_j = x^j + ip_j$, then

$$\Gamma_{L^2}(X, L) \cap \text{Ker}(\nabla_{\mathcal{P}_\mathbb{C}}) = \Gamma_{L^2}(X, L) \cap \text{Ker}(\bar{\partial}) = H_{L^2}^0(X, L)$$

is the space of L^2 -holomorphic sections of the trivial line bundle L . This is a representation of polynomials on $X = \mathbb{C}^n$ under the Wick product. Explicitly, z_j and \bar{z}_j act as operators $z_j \cdot$ and $\hbar \frac{\partial}{\partial \bar{z}_j}$ respectively. This action can also be obtained via the Toeplitz operators

$$\begin{aligned} T : C^\infty(X) &\rightarrow \text{End}(H_{L^2}^0(X, L)) \\ f &\mapsto T_f = \Pi \circ M_f, \end{aligned}$$

which is given by the multiplication M_f followed by the orthogonal projection $\Pi : \Gamma_{L^2}(X, L) \rightarrow H_{L^2}^0(X, L)$ with respect to the volume form $i^n \cdot e^{-|z|^2/\hbar} dz^1 d\bar{z}^1 \cdots dz^n d\bar{z}^n$ on \mathbb{C}^n . By turning \hbar to a formal variable, we obtain the star product for smooth functions on \mathbb{C}^n .

This construction can be generalized to quantization of Kähler manifolds. In the rest of this paper, we will let (X, ω, J) denote a compact Kähler manifold with prequantum line bundle (L, ∇) satisfying $\nabla^2 = -i\omega$. The Hilbert spaces are holomorphic sections $H^0(X, L^m)$ of positive tensor powers of L . Given $f \in C^\infty(X)$, one can take multiplication operator $M_f : L^2(X, L^m) \rightarrow L^2(X, L^m)$ giving a representation of commutative algebra $C^\infty(X)$ on $L^2(X, L^m)$. To obtain an action on $H^0(X, L^m)$ one takes the projections to define Toeplitz operator $T^{(m)} : C^\infty(X) \rightarrow \text{End}(H^0(X, L^m))$ as

$$(1.1) \quad T_f^{(m)} := T^{(m)}(f) := \Pi \circ M_f,$$

where $\Pi : L^2(X, L^m) \rightarrow H^0(X, L^m)$ denotes the orthogonal projection to holomorphic sections. The Toeplitz operators is closed under composition only in the asymptotic sense as $\hbar = \frac{1}{m} \rightarrow 0$. By the result of [5], there is a sequence of bidifferential operators $C_l : C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X)$ such that for any $f, g \in C^\infty(X)$ there is an asymptotic expansion as $m \rightarrow \infty$

$$(1.2) \quad T_f^{(m)} \circ T_g^{(m)} \sim \sum_{l \geq 0} T_{C_l(f,g)}^{(m)} m^{-l},$$

with $C_0 = f \cdot g$ the ordinary product and $C_1(f, g) - C_1(g, f) = \{f, g\}$ the Poisson bracket ([20, Theorem 4.5.]). Furthermore, these operators C_l 's can be put together into an associative \star -product via the formula

$$(1.3) \quad f \star g := \sum_l \hbar^l \cdot C_l(f, g),$$

as a deformation quantization $(C^\infty(X)[[\hbar]], \star)$ which is known as the Berezin-Toeplitz star product. The non-commutative algebra $(C^\infty(X)[[\hbar]], \star)$, the Hilbert space $\mathcal{H}_m = H^0(X, L^m)$ and the operators $T_f^{(m)}$'s are called the deformation quantization, geometric quantization and Berezin-Toeplitz quantization respectively.

The quantization of X is a much more complicated, but important question, especially in representation theory. As when (X, ω, J) has a Hamiltonian G -symmetry with moment map $\mu : X \rightarrow \mathfrak{g}^*$, then its geometric quantization $\mathcal{H} := H^0(X, L)$ would be a G -representation $G \rightarrow GL(\mathcal{H})$. This representation is itself very important, a closely related question is the Guillemin-Sternberg conjecture on symmetry commutes with geometric quantization, which there has been extensive studies, e.g. [12, 17, 21, 23]. The associated Lie algebra representation is denoted as

$$(1.4) \quad \beta : \mathfrak{g} \rightarrow gl(\mathcal{H}).$$

And one expects most interesting representations should come from such a quantization process.

By pulling back linear functions on \mathfrak{g} via μ , we have

$$\mu^* : \mathfrak{g} \rightarrow C^\infty(X).$$

Composing with the Toeplitz operator $T : C^\infty(X) \rightarrow gl(\mathcal{H})$, we obtain a map

$$-iT \circ \mu^* : \mathfrak{g} \rightarrow gl(\mathcal{H}).$$

It is natural to compare this with the representation β . It turns out that they do not agree with each other even in the case of X being coadjoint orbits, unless we replace μ by the Karabegov moment map $\tilde{\mu}$.

Definition 1.1. *A map $\tilde{\mu} : X \rightarrow \mathfrak{g}^*$ is called a Karabegov moment map if it is the moment map associated to the 2-form $\tilde{\omega}$ given by*

$$(1.5) \quad \tilde{\omega} := \omega + i \cdot Ric_X,$$

i.e. $\tilde{\mu}$ is G -equivariant and satisfies $\iota_v \tilde{\omega} = d\tilde{\mu}^(v)$ for all $v \in \mathfrak{g}^1$.*

The main result of this paper is that in nice cases such as coadjoint orbits or Hamiltonian Kähler-Einstein manifolds the representation β can be obtained via the composition of the Toeplitz quantization $T = T^{(1)}$ with Karabegov moment map $\tilde{\mu}$.

Theorem 1.2 (= Theorem 2.4). *If $X = O_\xi$ is an integral coadjoint orbit of G , then*

$$-iT \circ \tilde{\mu}^* = \beta.$$

¹We use the same notation for an element in \mathfrak{g} and its associated vector field on X .

We show that this is also true for any Kähler-Einstein manifolds with G -symmetry.

Theorem 1.3 (=Theorem 2.2). *If X is a compact Kähler-Einstein manifold with G -symmetry, then*

$$-iT \circ \tilde{\mu}^* = \beta.$$

Remark 1.4. *The above assumption implies that the Einstein constant is positive and the G -symmetry is Hamiltonian.*

Our main theorems say the following diagram commutes:

$$(1.6) \quad \begin{array}{ccc} & C^\infty(X) & \\ -i\tilde{\mu}^* \nearrow & & \searrow T \\ \mathfrak{g} & \xrightarrow{\beta} & \mathfrak{gl}(\mathcal{H}), \end{array}$$

which means the compatibility between the quantum symmetries. The map β describes the infinitesimal quantum symmetry on \mathcal{H} . The left arrow describes the “quantum symmetry” on the algebra of observables. The quantum moment map $\tilde{\mu}^*$ has close relation with the Fedosov quantization scheme will be explained in details in §3.

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2. MAIN THEOREMS

In this section, we prove our theorems by differential geometric and Lie theoretic computations.

We take the following convention: For every smooth function f on X , the Hamiltonian vector field X_f is defined by $\iota_{X_f}(\omega) = df$, and there is

$$\{f, g\} := \omega(X_g, X_f).$$

This convention guarantees that the map $f \mapsto X_f$ is a Lie algebra homomorphism:

$$X_{\{f, g\}} = [X_f, X_g].$$

2.1. Hamiltonian Kähler-Einstein manifold. When (X, ω, J) is Kähler-Einstein with $i\text{Ric}_X = \lambda\omega$, Karabegov moment map is given by $\tilde{\mu} = (1 + \lambda)\mu$.

Lemma 2.1 (Tuynman [22], see also [4, Proposition 4.1.]). *Let (X, ω, J, L) be Kähler manifold with a pre-quantum line bundle, then we have*

$$\Pi \circ \nabla_{X_f} = \frac{i}{2} \Pi \circ T_{\Delta}(f)$$

acting on holomorphic sections $H^0(X, L)$, where Δ is the Laplacian with respect to the Kähler metric.

Theorem 2.2. *Let (X, ω, J, L) be Kähler manifold with a pre-quantum line bundle together with a G -symmetry. Suppose further that X is Kähler-Einstein, then we have*

$$\beta = -iT \circ \tilde{\mu}^*.$$

Proof. For any $w \in \mathfrak{g}$, let $f_w := \langle \mu, w \rangle$ denote the pullback of w via the moment map μ . $JX_{f_w} = \nabla f_w$ is a holomorphic Killing vector field on (X, ω, J) , and using the theorem by Matsushima (see e.g. [2, Theorem 6.16.]) there is a bijection between eigenspace $E_{2\lambda}$ of Δ and space of holomorphic Killing vector fields by taking their gradient vector fields. That is $\Delta f_w = -2\lambda f_w$.

Notice $\langle \tilde{\mu}, w \rangle = (1 + \lambda)f_w$, and therefore using the Lemma 2.1 we have

$$-iT \circ \tilde{\mu}^* = -iT(f_w - \frac{1}{2}\Delta(f_w)) = Q_f,$$

where $Q_f = \Pi \circ (\nabla_{X_f} - if)$. Finally we have $[Q_{f_w}, Q_{f_v}] = Q_{\{f_w, f_v\}} = Q_{f_{[w, v]}}$ due to the fact that the vector field X_{f_v} comes from the G -action which preserves the Kähler structure (ω, J) , and therefore we obtain the representation β of \mathfrak{g} on $H^0(X, L)$ by sending $w \mapsto Q_{f_w}$. \square

2.2. Coadjoint orbits. In this section, we recall the notions and results for coadjoint orbits that are necessary for the proof of our second main theorem.

2.2.1. Coadjoint orbits as symplectic manifolds. We consider a compact connected Lie group G with a fixed choice of maximal torus T . Without loss of generality, we assume that \mathfrak{g} has no Abelian factor. We let \mathfrak{g}^* be the dual of \mathfrak{g} equipped with coadjoint action Ad^* . We let the non-degenerated negative definite Killing form on \mathfrak{g} to be $\kappa(\cdot, \cdot)$ which identifies \mathfrak{g} and \mathfrak{g}^* , and we will abuse our notation by treating κ as a pairing on \mathfrak{g}^* as well.

We denote $X := \mathcal{O}_\xi$ the coadjoint orbit through $\xi \in \mathfrak{g}^*$, which can be identified with G/G_ξ where G_ξ denotes the stabilizer subgroup of ξ with Lie subalgebra \mathfrak{g}_ξ . X is equipped with the natural Kirillov-Kostant-Souriau symplectic form (KKS form in short) given by the formula $\omega_\xi(\eta_1, \eta_2) := \langle \xi, [\eta_1, \eta_2] \rangle$, for $\eta_1, \eta_2 \in \mathfrak{g}/\mathfrak{g}_\xi \cong T_o(G/G_\xi)$ at the identity $o \in G/G_\xi$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between \mathfrak{g} and \mathfrak{g}^* . The natural G -action $G \curvearrowright X = G/G_\xi$ is Hamiltonian and its moment map is given by the natural embedding $\mu_\xi : \mathcal{O}_\xi \hookrightarrow \mathfrak{g}^*$.

2.2.2. Root space decomposition. We let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of \mathfrak{g} and similarly for $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ for any subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We let \mathcal{R} to be the set of complex roots $\alpha \in i\mathfrak{t}^* \subset \mathfrak{t}_\mathbb{C}^*$, where \mathfrak{t} is the Lie-algebra of T . We further choose a set of simple roots $\mathcal{S} = \{\alpha_1, \dots, \alpha_r\}$ which specifies a decomposition $\mathcal{R} = \mathcal{R}_+ \sqcup \mathcal{R}_-$ into positive and negative roots, as well as a choice of fundamental Weyl chamber \mathcal{C}_+ . We have the root space decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \mathcal{R}_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Given a root α , we define the co-root $H_\alpha \in \mathfrak{t}_\mathbb{C}$ to α by taking $\kappa(H_\alpha, \cdot) := \frac{2\alpha}{\kappa(\alpha, \alpha)}$, which is the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying $\alpha(H_\alpha) = 2$.

For computations on the homogeneous space $X = G/G_\xi$, we further introduce the notation $\mathcal{R}_\xi := \{\alpha \in \mathcal{R} \mid \kappa(\xi, \alpha) = 0\}$ and the set of complementary root $\mathcal{R}_\xi^c := \mathcal{R} \setminus \mathcal{R}_\xi$ as in [1], and similarly $\mathcal{R}_{\xi,+} = \mathcal{R}_\xi \cap \mathcal{R}_+$ and $\mathcal{R}_{\xi,+}^c := \mathcal{R}_+ \setminus \mathcal{R}_{\xi,+}$. Notice that \mathcal{R}_ξ and $\mathcal{R}_{\xi,+}$ only depends on the smallest closed strata containing ξ in the fundamental Weyl chamber \mathcal{C}_+ from its definition.

2.2.3. Kähler structures on coadjoint orbits. For any $\xi \in \mathfrak{g}^*$, letting $\xi^\vee \in \mathfrak{g}$ be the corresponding element via the identification by κ , we have $\mathfrak{g}_\xi = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_{\xi,+}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \overline{(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})}$. Therefore, we have the natural identification $\mathfrak{g}/\mathfrak{g}_\xi = \mathfrak{g}_\xi^c := \bigoplus_{\alpha \in \mathcal{R}_{\xi,+}^c} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \overline{(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})}$, together with a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_\xi \oplus \mathfrak{g}_\xi^c$. There is a unique G -invariant integrable complex structure J on \mathcal{O}_ξ with

$$T_e^{1,0}(G/G_\xi) = \bigoplus_{\alpha \in \mathcal{R}_{\xi,+}^c} \mathfrak{g}_\alpha, \quad T_e^{0,1}(G/G_\xi) = \bigoplus_{\alpha \in \mathcal{R}_{\xi,+}^c} \mathfrak{g}_{-\alpha}.$$

This J together with ω_ξ gives a G -invariant Kähler structure on $X = \mathcal{O}_\xi$. We write g_ξ for the metric tensor associated to this Kähler structure.

2.2.4. Pre-quantum line bundles on coadjoint orbits. Without loss of generality, we take $\xi \in i\mathcal{C}_+$ with integral symplectic class $[\omega_\xi] \in H^2(X, 2\pi\mathbb{Z})$. This allows us to define the associated unitary line bundle L_ξ on $X = G/G_\xi$ equipped with a unitary connection ∇^ξ on L_ξ such that the curvature form $(\nabla^\xi)^2 = -i\omega_\xi$. This gives a pre-quantum line bundle (L_ξ, ∇^ξ) on X with a natural G -equivariant structure given by left translations.

2.3. Toeplitz quantization on coadjoint orbits. Using the results from [3, 6–8], the Ricci form $\text{Ric}_\xi = -iF_K$ of X can be computed and is given by

$$(2.1) \quad \text{Ric}_\xi(X, Y) = \langle \delta_{2\xi}, [X, Y] \rangle,$$

for $X, Y \in \mathfrak{g}/\mathfrak{g}_\xi \cong T_o(G/G_\xi)$, where $\delta_\xi = \sum_{\alpha \in \mathcal{R}_\xi^c} \frac{1}{2}\alpha$. As a consequence, we find that X is Kähler-Einstein with Einstein constant λ exactly when $\xi = \frac{2}{\lambda}\delta_\xi$. For X equipped with other Kähler metrics, it would be interesting to ask whether the identity in Theorem 2.2 still hold for these metrics.

In the case of coadjoint orbit $X = G/G_\xi$, the Karabegov form in equation (1.5) is explicitly given by $\tilde{\omega}_\xi = \omega_\xi + \omega_{2\delta_\xi}$. The Karabegov moment map is given by $\tilde{\mu}_\xi = \mu_\xi + \mu_{2\delta_\xi}$, where μ_ξ and $\mu_{2\delta_\xi}$ are moment map on G/G_ξ with respect to symplectic form ω_ξ and $\omega_{2\delta_\xi}$ respectively.

2.3.1. Peter-Weyl theorem. Using the Peter-Weyl theorem, we have a decomposition of the space of complex valued square integrable functions

$$L^2(G) = \overline{\sum_{\rho \in \hat{G}} V_\rho^* \otimes V_\rho}$$

as $G \times G$ modules, where \hat{G} is the set of irreducible representations of G . Explicitly, an element $w \otimes v$ is treated as a function given by $f_{w,v}(g) := \langle w, \rho(g)(v) \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between V_ρ^* and V_ρ . Taking the right G_ξ -invariant part we obtain the corresponding decomposition

$$(2.2) \quad L^2(X) = \overline{\sum_{\rho \in \hat{G}} V_\rho^* \otimes V_\rho^{G_\xi}}$$

as left G -modules, where $V_\rho^{G_\xi}$ refers to the subspace fixed by G_ξ . We consider the Laplace-Beltrami operator $\Delta = dd^* + d^*d$ acting on smooth function $C^\infty(X)$, which is invariant under the left G -action and hence acting on the individual component $V_\rho^* \otimes V_\rho^{G_\xi}$. This action is computed explicitly in [25] for the Kähler-Einstein metric, we modify its proof for the Kähler metric g_ξ . We take a basis e_1, \dots, e_l of $\mathfrak{g}_\xi^c \cong T_o(G/G_\xi)$, and obtain a local coordinate near $[g] \in G/G_\xi$ by the following map:

$$(2.3) \quad (x_1, \dots, x_l) \mapsto [g \cdot \exp(x_1 e_1 + \dots + x_l e_l)]$$

for every $g \in G$. Here \exp denotes the Lie-theoretic exponential map, and notice that this may not coincide with the Riemannian exponential map because the Kähler metric g_ξ is not naturally reductive metric in general.

Lemma 2.3. *There exists a linear map $\Omega : V_\rho \rightarrow V_\rho$, such that for the function $f_{w,v}$ on G/G_ξ , we have $\Delta f_{w,v} = f_{w,\Omega v}$. Explicitly, Ω is given by*

$$\Omega(v) = \sum_{i,j=1}^l a^{ij} d\rho(e_i) \circ d\rho(e_j)(v),$$

where $a_{ij} := g_{\xi,o}(e_i, e_j)$'s are the matrix coefficients associated to metric $g_{\xi,o}$ on \mathfrak{g}_ξ^c at identity $o \in G/G_\xi$, and a^{ij} 's are their inverse matrix coefficients. Here $d\rho$ denotes the Lie algebra representation associated to ρ .

Proof. With the identification $dl_g : \mathfrak{g}_\xi^c \cong T_g(G/G_\xi)$ via left translation $l_g : X \rightarrow X$, we use the exponential coordinates from equation (2.3), and compute $(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j})|_g \in \mathfrak{g}_\xi^c$ for arbitrary i, j . Using the Baker-Campbell-Hausdorff formula we notice that we can write $g \cdot \exp(x_i e_i + x_j e_j) = g \cdot \exp(x_j e_j - \frac{x_i x_j}{2} [e_j, e_i] + o(\|x\|^2)) \exp(x_i e_i)$, where $\|x\|$ refers to the norm on the tangent space $T_o(G/G_\xi)$ given by $g_{\xi,o}$. We have

$$dl_{g^{-1}} \left(\frac{\partial}{\partial x_j} \Big|_{g \exp(x_i e_i)} \right) = \left(e_j - \frac{x_i}{2} [e_j, e_i] + o(\|x\|) \right)^\# \Big|_{\exp(x_i e_i)},$$

where $X^\#$ refers to the vector field generated by the left action for $X \in \mathfrak{g}$. We have

$$dl_{g^{-1}} \left((\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}) \Big|_g \right) = (\nabla_{e_i} (e_j - \frac{x_i}{2} [e_j, e_i])^\#)_o = (\nabla_{e_i} e_j^\#)_o + \frac{1}{2} [e_i, e_j]^\#$$

at $o \in G/G_\xi$. Making use of the formula [1, Proposition 5.2.] we have that

$$(2.4) \quad dl_{g^{-1}} \left((\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}) \Big|_g \right) = U(e_i, e_j),$$

where $U(\cdot, \cdot) : \mathfrak{g}_\xi^c \times \mathfrak{g}_\xi^c \rightarrow \mathfrak{g}_\xi^c$ defined by $2g_{\xi,o}(U(e_i, e_j), e_k) = g_{\xi,o}([e_k, e_i]_{\mathfrak{g}_\xi^c}, e_j) + g_{\xi,o}(e_i, [e_k, e_j]_{\mathfrak{g}_\xi^c})$ (Here $X_{\mathfrak{g}_\xi^c}$ refers to the component of $X \in \mathfrak{g}$ in \mathfrak{g}_ξ^c with respect to the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_\xi \oplus \mathfrak{g}_\xi^c$).

Therefore, we use the formula $\Delta f = \sum_{i,j} g^{ij} (\frac{\partial}{\partial x_i} (\frac{\partial}{\partial x_j} f) - (\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j})(f))$ in local coordinates to compute $\Delta f_{w,v}$ at the point $g \in G/G_\xi$. We have

$$\begin{aligned} \frac{\partial}{\partial x_i} (\frac{\partial}{\partial x_j} f) &= \frac{\partial^2}{\partial x_i \partial x_j} (f(g \cdot \exp(x_i e_i + x_j e_j))) = \frac{\partial^2}{\partial x_i \partial x_j} \langle w, \rho(g)(\rho(\exp(x_i e_i + x_j e_j))(v)) \rangle \\ &= \langle w, \rho(g)(d\rho(e_i)(d\rho(e_j)(v)) - \frac{1}{2} d\rho([e_i, e_j])(v)) \rangle. \end{aligned}$$

Using earlier calculation from (2.4) we get $((\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} f)|_g) = \langle w, \rho(g)(d\rho(U(e_i, e_j))(v)) \rangle$, we obtain

$$\begin{aligned} 2 \sum_{i,j} a^{ij} g_{\xi,o}(U(e_i, e_j), e_k) &= \sum_{i,j} a^{ij} g_{\xi,o}([e_k, e_i]_{\mathfrak{g}_\xi^c}, e_j) + \sum_{i,j} a^{ij} g_{\xi,o}(e_i, [e_k, e_j]_{\mathfrak{g}_\xi^c}) \\ &= 2 \operatorname{tr}(\operatorname{ad}(e_k)|_{\mathfrak{g}_\xi^c}) = 0. \end{aligned}$$

Combining with the fact that $a^{ij}[e_i, e_j] = 0$ we obtain the desired identity. \square

With the Karabegov moment map $\tilde{\mu}_\xi : X \rightarrow \mathfrak{g}^*$, we have a natural embedding $\tilde{\mu}_\xi^* : \mathfrak{g} \rightarrow \mathfrak{g} \otimes (\mathfrak{g}^*)^{G_\xi} \hookrightarrow L^2(X)$ as a left G -submodule which is given by

$$(2.5) \quad \tilde{\mu}_\xi(g) = \langle w, \operatorname{Ad}^*(g)(\xi + 2\delta_\xi) \rangle.$$

We have the following theorem saying geometric quantization $\beta : \mathfrak{g} \rightarrow \operatorname{End}(H^0(X, L_\xi))$ is given by composition of the Toeplitz quantization with the Karabegov moment map for coadjoint orbits.

Theorem 2.4. *For an integral coadjoint orbit $(X_\xi = G/G_\xi, \omega_\xi, J, L_\xi)$ with pre-quantum line bundle equipped with the natural Hamiltonian G -symmetry by left-translation, we have*

$$\beta = -iT \circ \tilde{\mu}_\xi^*.$$

Proof. Making use of the Lemma 2.3, we compute $\Omega \cdot \xi$, or equivalently $\Omega \cdot \xi^\vee$ via the identification induced by the Killing form. For each $\alpha \in \mathcal{R}_+$, we choose triple $iH_\alpha, X_\alpha, Y_\alpha$ lying in \mathfrak{g} such that

$$\begin{aligned} [iH_\alpha, X_\alpha] &= 2Y_\alpha \\ [iH_\alpha, Y_{-\alpha}] &= -2X_\alpha \\ [X_\alpha, Y_{-\alpha}] &= iH_\alpha, \end{aligned}$$

where H_α is the coroot as in §2.2.2. Therefore $\{X_\alpha, Y_\alpha\}_{\alpha \in \mathcal{R}_\xi^c}$ form a basis for \mathfrak{g}_ξ^c . From the discussion in [1, Chapter 3 §7] about the Riemannian metric, we notice that this is an orthogonal basis with $\|X_\alpha\|^2 = \|Y_\alpha\|^2 = \xi(iH_\alpha)$. Therefore we have

$$\begin{aligned} \Omega(\xi^\vee) &= \sum_{\alpha \in \mathcal{R}_\xi^c} \xi(iH_\alpha)^{-1} ([X_\alpha, [X_\alpha, \xi^\vee]] + [Y_\alpha, [Y_\alpha, \xi^\vee]]) \\ &= \sum_{\alpha \in \mathcal{R}_\xi^c} \xi(iH_\alpha)^{-1} \alpha(\xi^\vee) ([X_\alpha, -Y_\alpha] + [Y_\alpha, X_\alpha]) \\ &= \sum_{\alpha \in \mathcal{R}_\xi^c} \frac{\kappa(\alpha, \alpha)}{2i} (-2iH_\alpha) = -2 \sum_{\alpha \in \mathcal{R}_\xi^c} \alpha^\vee. \end{aligned}$$

Therefore we have $\Omega(\xi) = -4\delta_\xi$.

By writing $\langle w, \tilde{\mu}_\xi \rangle = f_{w,\xi} + 2f_{w,\delta_\xi}$ for any $w \in \mathfrak{g}$, we therefore have

$$-iT(\langle w, \tilde{\mu}_\xi \rangle) = -iT(f_{w,\xi} + 2f_{w,\delta_\xi}) = -iT(f_{w,\xi} - \frac{1}{2}\Delta(f_{w,\xi})) = Q_{f_{w,\xi}},$$

using Lemma 2.1, where $Q_f = \Pi \circ (\nabla_{X_f} - if)$ as in §3. Making use of the fact that $[Q_{f_{w,\xi}}, Q_{f_{v,\xi}}] = Q_{\{f_{w,\xi}, f_{v,\xi}\}}$ again as in the proof of Theorem 2.2, and the fact that $\{f_{w,\xi}, f_{v,\xi}\} = f_{[w,v],\xi}$ we obtain the identity $[Q_{f_{w,\xi}}, Q_{f_{v,\xi}}] = Q_{f_{[w,v],\xi}}$. As a result, one obtains the Lie algebra representation β via $w \mapsto Q_{f_{w,\xi}}$. \square

3. QUANTIZATION VIA TOEPLITZ OPERATORS

In this section, we given an explanation of how $\tilde{\mu}$ is related to quantum moment map in Fedosov deformation quantization.

3.1. Berezin-Toeplitz quantization with Hamiltonian G -action.

3.1.1. Karabegov moment map. In this subsection, we explain the definition *Karabegov moment map* as a variation of quantum moment map in deformation quantization. Recall that a *quantum moment map* is a Lie algebra homomorphism $\mu_\hbar^* : \mathfrak{g} \rightarrow (C^\infty(X)[[\hbar]], \frac{i}{\hbar}[\cdot, \cdot]_\star)$, such that for any $v \in \mathfrak{g}$:

$$v(f) = \frac{i}{\hbar} [\mu_\hbar^*(v), f]_\star.$$

Remark 3.1. *The normalization factor $\frac{i}{\hbar}$ guarantees that $\lim_{\hbar \rightarrow 0} \frac{i}{\hbar}[\cdot, \cdot]_\star = \{\cdot, \cdot\}$. With this normalization, $-i\hbar \cdot \mu_\hbar^*$ is a Lie algebra homomorphism if we take the bracket $[\cdot, \cdot]_\star$ on $C^\infty(X)[[\hbar]]$. In particular, $\mu_\hbar|_{\hbar=0} = \mu$. This also explains the coefficient $-i$ in the commutative diagram (1.6) (There we set $\hbar = 1$).*

It is shown in [13, 18] that when a deformation quantization is induced by a Fedosov connection $D_F = \nabla^{\mathcal{W}} + \frac{1}{\hbar}[\gamma, -]_{\star}$ satisfying

$$(3.1) \quad \nabla^{\mathcal{W}}\gamma + \frac{1}{\hbar}\gamma \star \gamma = -\omega + \hbar\alpha_1 + \hbar^2\alpha_2 + \cdots = -\omega + \Omega.$$

Then a quantum moment map must satisfy the following equation

$$(3.2) \quad \iota_v(\omega - \Omega) = d\mu_{\hbar}^*(v).$$

A deformation quantization on a Kähler manifold X is called of Wick type (also known as separation of variables) if all the bi-differential operators $C_l(f, g)$ take holomorphic and anti-holomorphic derivatives of f and g respectively. It is shown in [14] that to every Wick type star product, there is an associated closed formal $(1, 1)$ -form $-\frac{1}{\hbar}\omega + \alpha_1 + \alpha_2\hbar + \alpha_3\hbar^2 + \cdots$ known as the *Karabegov form*, which gives rise to a one-one correspondence.

In [10], it is shown that there is a family of Fedosov connections induced from quantization of L_{∞} structure on Kähler manifolds, such that the formal closed $(1, 1)$ -form $-\omega + \Omega$ in equation (3.1) is exactly the Karabegov form of the associated star product. The Berezin-Toeplitz quantization is a Wick type deformation quantization whose Karabegov form is $-\frac{1}{\hbar}\omega - i \cdot \text{Ric}_X$. Since this formal $(1, 1)$ -form only has two terms in the \hbar power expansion, we can turn the formal variable to any complex number without the convergence issue. In particular, equation (3.2) with $1/\hbar = 1$ gives the definition of Karabegov map (Definition 1.1).

The formal variable $1/\hbar$ plays the role of the tensor power m of the prequantum line bundle L . It is natural to define a family of Karabegov moment maps $\tilde{\mu}_m$ associated to the form $\omega + \frac{i}{m} \cdot \text{Ric}_X$, $m \in \mathbb{N}$. By considering $L^{\otimes m}$ and let $\mathcal{H}_m := H_{L^2}^0(X, L^{\otimes m})$ when X being coadjoint orbit or compact Kähler-Einstein, we have following commutative diagrams:

$$\begin{array}{ccc} & C^{\infty}(X) & \\ -i \cdot \tilde{\mu}_m^* \nearrow & & \searrow T \\ \mathfrak{g} & \xrightarrow{\beta_m} & \mathfrak{gl}(\mathcal{H}_m) \end{array}$$

in these two cases.

For general Kähler manifolds, the above diagram only commutes in an asymptotic sense:

$$\|\beta + iT \circ \tilde{\mu}_m^*\| = O(m^{-\infty}).$$

Here $\|\cdot\|$ denotes the operator norm. Equivalently, for any $k > 0$, there exists a $C_k > 0$, such that $\|\beta + iT \circ \tilde{\mu}_m^*\| \leq C_k m^{-k}$.

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