

FUKAYA'S CONJECTURE ON WITTEN'S TWISTED A_∞ -STRUCTURES

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ABSTRACT. Wedge product on deRham complex of a Riemannian manifold M can be pulled back to $H^*(M)$ via explicit homotopy, constructed using Green's operator, to give higher product structures. We prove Fukaya's conjecture which suggests that Witten deformation of these higher product structures have semiclassical limits as operators defined by counting gradient flow trees with respect to Morse functions, which generalizes the remarkable Witten deformation of deRham differential from a statement concerning homology to one concerning real homotopy type of M . Various applications of this conjecture to mirror symmetry are also suggested by Fukaya in [6].

1. INTRODUCTION

It is well known that the differential graded algebra $(\Omega^*(M), d, \wedge)$ on a smooth manifold M determine real homotopy type of M (if $\pi_1(M) = 0$), a simplified homotopic classification of manifolds founded by Quillen [13] and Sullivan [14]. If M is compact oriented Riemannian manifold, Hodge decomposition of the Laplacian Δ enables us to represent the cohomology of M by the finite dimensional kernel $\Omega^*(M)_0 \subset \Omega^*(M)$ of Δ . The real homotopy type can be captured by homotopic pull back of the wedge product to $\Omega^*(M)_0$, giving a structure of A_∞ algebra via the homological perturbation lemma in [12].

On the other hand, equipping M with a Morse-Smale function f allows us to study the cohomology of the manifold by the Morse complex CM_f^* , which is a finite dimensional vector space freely generated by critical points of f equipped with the Morse differential δ defined by counting gradient flow lines of f . Higher product structures can be introduced to enhance the Morse complex to the Morse A_∞ (pre)-category defined as in [1, 5], involving A_∞ products $\{m_k^{Morse}\}_{k \in \mathbb{Z}_+}$ defined by counting gradient trees.

In the paper [6] by Fukaya, he conjectured that the A_∞ product structures from these two constructions can be related to each others via technique of Witten deformation, a differential geometric approach suggested in an influential paper [15] by Witten to relate Hodge theory to Morse theory by deforming the exterior differential operator d with

$$d_f := e^{-\lambda f} d e^{\lambda f} = d + \lambda df \wedge,$$

by a Morse function f with large parameter $\lambda \in \mathbb{R}^+$. In this paper, we prove this conjecture by Fukaya.

This machinery plays an important role in understanding SYZ transformation of open strings datum and providing a geometric explanation for Kontsevich's Homological Mirror Symmetry (Abbrev. HMS) as pointed out by Fukaya in [6].

More precisely, given a Morse-Smale function f , we can define the Witten's twisted Laplace operator by

$$(1.1) \quad \Delta_f := d_f d_f^* + d_f^* d_f.$$

Witten argued that if we consider eigenvalues of the operator Δ_f lying inside an interval $[0, 1)$, the sum of corresponding eigensubspaces

$\Omega^*(M, \lambda)_{sm} \subset \Omega^*(M)$ could be identified with the Morse complex CM_f^* via a linear map

$$(1.2) \quad \phi = \Phi^{-1} : CM_f^* \rightarrow \Omega^*(M, \lambda)_{sm}.$$

For any critical point q of f , the eigenform $\phi(q)$ will concentrate near q when λ is large enough. Furthermore, the Witten differential d_f is also identified with the Morse differential m_1^{Morse} via ϕ . The original proof can be found in [9, 10, 11] while readers may see [16] for a detailed introduction.

In order to incorporate the product structure, we are forced to consider more than one Morse function as the Leibniz rule associated to twisted differential is given by

$$d_{g+h}(\alpha \wedge \beta) = d_g(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d_h(\beta).$$

This leads to the notation of the differential graded (dg) category $DR_\lambda(M)$, with objects being smooth functions on M . The corresponding morphism complex between two objects f_i and f_j is given by the Witten twisted complex $\Omega_{ij}^*(M, \lambda) = (\Omega^*(M), d_{f_{ij}})$, where $f_{ij} = f_j - f_i$. When f_{ij} satisfies the Morse-Smale condition, we can define $\Omega_{ij}^*(M, \lambda)_{sm}$ and a homotopy retraction $P_{ij} : \Omega_{ij}^*(M, \lambda) \rightarrow \Omega_{ij}^*(M, \lambda)_{sm}$ using explicit homotopy $H_{ij} = d_{f_{ij}}^* G_{ij}$, where G_{ij} is the twisted Green's operator. We can pull back the wedge product via the homotopy, making use of homological perturbation lemma in [12], to give a Witten's deformed A_∞ (pre)¹-category $DR_\lambda(M)_{sm}$ with A_∞ structure $\{m_k(\lambda)\}_{k \in \mathbb{Z}_+}$.

For instant, suppose we have smooth functions f_0, f_1, f_2 and f_3 such that their pairwise differences are Morse-Smale, with $\varphi_{ij} \in \Omega_{ij}^*(M, \lambda)_{sm}$, the higher product

$$m_3(\lambda) : \Omega_{23}^*(M, \lambda)_{sm} \otimes \Omega_{12}^*(M, \lambda)_{sm} \otimes \Omega_{01}^*(M, \lambda)_{sm} \rightarrow \Omega_{03}^*(M, \lambda)_{sm}$$

¹Roughly speaking, an A_∞ pre-category allows morphisms and A_∞ -operations only defined for a subcollection of objects, usually called a generic subcollection, and requiring A_∞ relation to hold once it is defined. Algebraic construction can be used to construct an honest A_∞ category consisting of essentially the same amount of informations, and therefore we will restrict ourself to A_∞ pre-category.

is defined by

$$(1.3) \quad m_3(\lambda)(\varphi_{23}, \varphi_{12}, \varphi_{01}) \\ = P_{03}(H_{13}(\varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01}) + P_{03}(\varphi_{23} \wedge H_{02}(\varphi_{12} \wedge \varphi_{01})).$$

In general $m_k(\lambda)$ will be given by a combinatorial formula involving summation over directed planar trees with k inputs and 1 output, with wedge product \wedge being applied at vertices and the homotopy operator H_{ij} being applied at internal edges.

Fukaya's conjecture says that the A_∞ structure $\{m_k(\lambda)\}_{k \in \mathbb{Z}_+}$, defined using twisted Green's operators, has leading order given by $\{m_k^{Morse}\}_{k \in \mathbb{Z}_+}$, defined by counting gradient flow trees, via the isomorphism ϕ .

Conjecture (Fukaya [6]). *For generic (see Definition 6) sequence of functions $\vec{f} = (f_0, \dots, f_k)$, with corresponding sequence of critical points $\vec{q} = (q_{01}, q_{12}, \dots, q_{(k-1)k})$, namely, q_{ij} is a critical point of f_{ij} , we have*

$$(1.4) \quad \Phi(m_k(\lambda)(\phi(\vec{q}))) = m_k^{Morse}(\vec{q}) + \mathcal{O}(\lambda^{-1/2}).$$

Theorem (Main Theorem). *Fukaya's conjecture is true.*

As A_∞ relations of $\{m_k(\lambda)\}_{k \in \mathbb{Z}_+}$ are obvious from their algebraic constructions while those of $\{m_k^{Morse}\}_{k \in \mathbb{Z}_+}$ require studies for boundaries of moduli spaces of gradient flow trees (see e.g. [1, 5]), we obtain an alternative proof for A_∞ relations of $\{m_k^{Morse}\}_{k \in \mathbb{Z}_+}$ as an corollary.

The original proof in [9, 10, 11, 16] is exactly the case $k = 1$, involving detailed estimate of operator d_{ij} along gradient flow lines. Starting from $k \geq 3$, our theorem involves the semi-classical analysis of the Witten twisted Green operator which is not needed in the $k = 1$ case.

Our Main Theorem for $k = 2$ involves three functions f_0, f_1, f_2 , having q_{01}, q_{12}, q_{02} being critical points of f_{01}, f_{12}, f_{02} respectively, and can be proven using the analytical techniques in [9, 11] as the Green's operator G_{ij} does not appear in the definition of $m_2(\lambda)$. We compute the leading order term in the matrix coefficients of $m_2(\lambda)$, which is essentially the integral

$$(1.5) \quad \int_M m_2(\lambda)(\phi(q_{01}), \phi(q_{12})) \wedge \frac{* \phi(q_{02})}{\|\phi(q_{02})\|^2}.$$

First, we make use of the global a priori estimate of the form $\phi(q_{ij}) \sim \mathcal{O}(e^{\lambda \rho(q_{ij}, \cdot)})$ (lemma 17), with ρ being the Agmon distance defined in definition 14, to cut off the integrand to neighborhoods of gradient trees appeared in m_2^{Morse} . After cutting off the integrand, we need to compute the leading order contribution from each gradient tree. The WKB approximation (lemma 20) of the eigenforms $\phi(q_{ij})$ is used to compute the leading order contribution of (1.5).

When $k \geq 3$, what we need is an WKB approximation of G_{ij} along a gradient flow line of f_{ij} in §4. More precisely, we need to study the local

behaviour of the inhomogeneous Witten Laplacian equation of the form

$$(1.6) \quad \Delta_{ij} \zeta_E = d_{ij}^*(e^{-\lambda\psi_S\nu})$$

along a gradient flow line segment of f_{ij} from x_S (starting point) to x_E (ending point), and obtain an approximation of ζ_E of the form

$$\zeta_E \sim e^{-\lambda\psi_E} \lambda^{1/2} (\omega_{E,0} + \omega_{E,1} \lambda^{-1/2} + \dots).$$

The *key* step in our proof is to determine ψ_E from ψ_S and detailed construction is given in §4. A naive guess $\tilde{\psi}_E(x) := \inf_y (\psi_S(y) + \rho(y, x))$ captures the desired behaviors of ψ_E near x_E but is singular along a hypersurface U_S containing x_S . Singularity of $\tilde{\psi}_E$ prohibits us to solve Equation (1.6) iteratively in order of λ^{-1} .

In order to solve (1.6) iteratively, we consider the minimal configuration in variational problem associated to $\inf_y (\psi_S(y) + \rho(y, x))$ and find that the point y is forced to lie on U_S , with a unique geodesic joining to x which realizes $\rho(y, x)$, for those x closed enough to x_E . This family of geodesics $\{\gamma_y\}_{y \in U_S}$ gives a foliation of a neighborhood of the flow line segment. Therefore we can use $\psi_E(\gamma_y(t)) = \psi_S(y) + t$ as an extension of $\tilde{\psi}_E$ across U_S and solve the Equation (1.6) iteratively.

The analytic results for G_{ij} (§4) can be used to give the proof for $k = 3$ case. The proof of the general case is similar to the $k = 3$ case, but with more involved combinatorics.

This paper consists of two parts. The first part involves the setup and definitions in §2 and the proof in §3 modulo technical analysis. The second part is a study of Witten twisted Green operator in §4 which is used in previous sections.

2. SETTING

In this section, we introduce the definitions and notations we need and state our main theorem. We begin with recalling the definition of Morse category.

2.1. Morse category. The Morse category $Morse(M)$ has the same class of objects being smooth functions $f : M \rightarrow \mathbb{R}$, with the space of morphisms between two objects given by

$$Hom_{Morse(M)}^*(f_i, f_j) = CM^*(f_{ij}) = \bigoplus_{q \in Crit(f_{ij})} \mathbb{C} \cdot e_q.$$

It is the Morse complex which is defined when f_{ij} satisfying the following Morse-Smale condition.

Definition 1. A Morse function f_{ij} is said to satisfy the Morse-Smale condition if V_p^+ and V_q^- intersecting transversally for any two critical points $p \neq q$ of f_{ij} .

It is graded by the Morse index of corresponding critical point q , which is the dimension of unstable submanifold V_q^- . The Morse category $Morse(M)$ is an A_∞ -category equipped with higher products m_k^{Morse} for every $k \in \mathbb{Z}_+$, or simply denoted by m_k , which are given by counting gradient flow trees. To describe that, we first need some terminologies about directed trees.

2.1.1. *Directed trees.*

Definition 2. A trivalent directed k -leafed tree T means an embedded tree in \mathbb{R}^2 , together with the following data:

- (1) a finite set of vertices $V(T)$;
- (2) a set of internal edges $E(T)$;
- (3) a set of k semi-infinite incoming edges $E_{in}(T)$;
- (4) a semi-infinite outgoing edge e_{out} .

Every vertex is required to be trivalent, having two incoming edges and one outgoing edge.

For simplicity, we will call it a k -tree. They are identified up to orientation preserving continuous map of \mathbb{R}^2 preserving the vertices and edges. Therefore, the topological class for k -trees will be finite.

Given a k -tree, by fixing the anticlockwise orientation of \mathbb{R}^2 , we have cyclic ordering of all the semi-infinite edges. We can label connected components of $\mathbb{R}^2 \setminus T$ by integers $0, \dots, k$ in anticlockwise ordering, inducing a labelling on edges such that edge e labelled with ij will be lying between components i and j with the unique normal to e pointing in component i . The labelling can be fixed uniquely by requiring the outgoing edge to be labelled by $0k$. For example, there are two different topological types for 3-tree, with corresponding labelings for their edges as shown in the following figure.

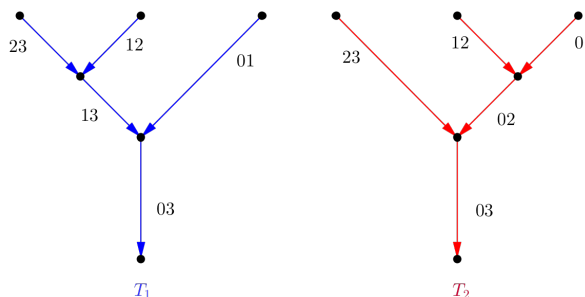


FIGURE 1. two different types of 3-trees

Notations 3. A pair (e, v) , with e being an edge (either finite or semi-infinite) and v being an adjacent vertex, is called a flag. The unique vertex attached to the outgoing semi-infinite edge is called the root vertex.

For the purpose of Morse homology, we need the following notation of metric trees.

Definition 4. A metric k -tree \tilde{T} is a k -tree together with a length function $l : E(T) \rightarrow (0, +\infty)$.

Metric k -trees are identified up to homeomorphism preserving the length functions. The space of metric k -trees has finite number of components, with each component corresponding to a topological type T . The component corresponding to T , denoted by $\mathcal{S}(T)$, is a copy of $(0, +\infty)^{|E(T)|}$, where $|E(T)|$ is the number of internal edges and equals to $d - 2$. The space $\mathcal{S}(T)$ can be partially compactified to a manifold with corners $(0, +\infty]^{|E(T)|}$, by allowing the length of internal edges going to be infinity. In particular, it has codimension-1 boundary

$$\partial\overline{\mathcal{S}(T)} = \coprod_{T=T' \sqcup T''} \mathcal{S}(T') \times \mathcal{S}(T''),$$

where the equation $T = T' \sqcup T''$ means splitting the tree T into T' and T'' at an internal edge.

2.1.2. *Morse A_∞ structure.* We are going to describe the product m_k of the Morse category. First of all, one may notice that the morphisms between two objects f_i and f_j is only defined when f_{ij} is Morse. Given a sequence $\vec{f} = (f_0, \dots, f_k)$ such that all the difference f_{ij} 's are Morse, with a sequence of points $\vec{q} = (q_{01}, \dots, q_{(k-1)k}, q_{0k})$ such that q_{ij} is a critical point of f_{ij} , we have the following definition of gradient flow tree.

Definition 5. A gradient flow tree Γ of \vec{f} with endpoints at \vec{q} is a continuous map $\mathbf{f} : \tilde{T} \rightarrow M$ such that it is a upward gradient flow lines of f_{ij} when restricted to the edge labelled ij , the incoming edge $i(i+1)$ begins at the critical point $q_{i(i+1)}$ and the outgoing edge $0k$ ends at the critical point q_{0k} .

We use $\mathcal{M}(\vec{f}, \vec{q})$ to denote the moduli space of gradient trees (in the case $k = 1$, the moduli of gradient flow line of a single Morse function has an extra \mathbb{R} symmetry given by translation in the domain. We will use this notation for the reduced moduli, that is the one after taking quotient by \mathbb{R}). It has a decomposition according to topological types

$$\mathcal{M}(\vec{f}, \vec{q}) = \coprod_T \mathcal{M}(\vec{f}, \vec{q})(T).$$

This space can be endowed with smooth manifold structure if we put generic assumption on the Morse sequence, which will be discibed as follows. For an incoming critical point $q_{i(i+1)}$, with corresponding stable submanifold $V_{q_{i(i+1)}}^+$, we define a map

$$\mathbf{f}_{T, i(i+1)} : V_{q_{i(i+1)}}^+ \times \mathcal{S}(T) \rightarrow M.$$

Fixing a point x in $V_{q_{i(i+1)}}^+$ together with a metric tree \tilde{T} , we need to determine a point in M . First, suppose v is the vertex connected to the edge labelled $i(i+1)$, there is a unique sequence of internal edges (e_1, \dots, e_{k-2}) connecting v to the root vertex v_r . To determine the image of x under our function, we apply Morse gradient flow with respect to Morse function associated to e_j 's for time $l(e_j)$ to x consecutively according to the sequence (e_1, \dots, e_{k-2}) .

The maps are then put together to give a map

$$(2.1) \quad \mathbf{f}_T : V_{q_{0k}}^- \times V_{q_{(k-1)k}}^+ \times \cdots \times V_{q_{01}}^+ \times \mathcal{S}(T) \rightarrow \prod_{k+1} M,$$

where we use the embedding $\iota : V_{q_{0k}}^- \rightarrow M$ for the first component.

Definition 6. A Morse sequence \vec{f} is said to be generic if the image of \mathbf{f}_T intersect transversally with the diagonal submanifold $\Delta \cong M \hookrightarrow M^{k+1}$, for any sequence of critical point \vec{q} and any topological type T .

When the sequence is generic, the moduli space $\mathcal{M}(\vec{f}, \vec{q})$ is of dimension

$$\dim_{\mathbb{R}}(\mathcal{M}(\vec{f}, \vec{q})) = \deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2,$$

where $\deg(q_{ij})$ is the Morse index of the critical point. Therefore, we can define m_k^{Morse} , or simply denoted by m_k , using the signed count $\#\mathcal{M}(\vec{f}, \vec{q})$ of points in $\dim_{\mathbb{R}}(\mathcal{M}(\vec{f}, \vec{q}))$ when it is of dimension 0. In order to have a signed count, we have to get an orientation of the space $\mathcal{M}(\vec{f}, \vec{q})$. We will come to that later in definition 33.

We now give the definition of the higher products in the Morse category.

Definition 7. Given a generic Morse sequence \vec{f} with sequence of critical points \vec{q} , we define

$$m_k : CM_{k(k-1)}^* \otimes \cdots \otimes CM_{01}^* \rightarrow CM_{0k}^*$$

given by

$$(2.2) \quad \langle m_k(q_{(k-1)k}, \dots, q_{01}), q_{0k} \rangle = \#\mathcal{M}(\vec{f}, \vec{q}),$$

when

$$\deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + k - 2 = 0.$$

Otherwise, the m_k is defined to be zero.

One may notice m_k^{Morse} can only be defined when \vec{f} is a Morse sequence satisfying the generic assumption in definition 6. The Morse category is indeed a A_∞ pre-category instead of an honest category. We will not go into detail about the algebraic problem on getting an honest category from this structures. For details about this, readers may see [1, 5].

2.2. Witten's twisted deRham category. Given a compact oriented Riemannian manifold M , we can construct the deRham category $DR_\lambda(M)$ depending on λ . Objects of the category are again smooth functions, while the space of morphisms between f_i and f_j is

$$\mathrm{Hom}_{DR_\lambda(M)}^*(f_i, f_j) = \Omega^*(M),$$

with differential $d + \lambda df_{ij} \wedge$, where $f_{ij} := f_j - f_i$. The composition of morphisms is defined to be the wedge product of differential forms on M . This composition is associative and hence the resulted category is a dg category. We denote the complex corresponding to $\mathrm{Hom}_{DR_\lambda(M)}^*(f_i, f_j)$ by $\Omega_{ij}^*(M, \lambda)$ and the differential $d + \lambda df_{ij}$ by d_{ij} .

To relate $DR_\lambda(M)$ and $Morse(M)$, we need to apply homological perturbation to $DR_\lambda(M)$. Fixing two functions f_i and f_j , we consider the Witten Laplacian

$$\Delta_{ij} = d_{ij}d_{ij}^* + d_{ij}^*d_{ij},$$

where $d_{ij}^* = d^* + \lambda \nu_{\nabla f_{ij}}$. We denote the span of eigenspaces with eigenvalues contained in $[0, 1)$ by $\Omega_{ij}^*(M, \lambda)_{sm}$ as before.

If the function f_{ij} satisfying the Morse-Smale assumption 1, it is proven by Laudenbach [4] that the closure $\overline{V_q^+}$ and $\overline{V_q^-}$ have a structure of submanifold with conical singularities. Using this result, one can define the following map as in [16]

$$\Phi = \Phi_{ij} : \Omega_{ij}^*(M, \lambda)_{sm} \rightarrow CM^*(f_{ij})$$

given by

$$(2.3) \quad \Phi(\alpha) = \sum_{p \in \mathrm{Crit}(f_{ij})} \int_{V_p^-} e^{\lambda f_{ij}} \alpha$$

which is an isomorphism identifying d_{ij} with Morse differential m_1 when λ large enough.

Remark 8. *This identification gives a connection on the family of vector space $\Omega_{ij}^*(M, \lambda)_{sm}$ parametrized by λ by declaring the basis e_p associated to critical point of f_{ij} to be flat. Equivalently, it is same as defining*

$$\nabla_{\frac{\partial}{\partial \lambda}} \alpha(\lambda) = P_{ij}(e^{-\lambda f_{ij}} \frac{\partial}{\partial \lambda} e^{\lambda f_{ij}} \alpha(\lambda)),$$

for $\alpha(\lambda) \in \Omega_{ij}^*(M, \lambda)_{sm}$.

It is natural to ask whether the product structures of two categories are related as $\lambda \rightarrow \infty$, and the answer is definite. The first observation is that the Witten's approach indeed produces an A_∞ category, denoted by $DR_\lambda(M)_{sm}$, with A_∞ structure $\{m_k(\lambda)\}_{k \in \mathbb{Z}_+}$. It has the same class of objects as $DR_\lambda(M)$. However, the space of morphisms between two objects f_i, f_j is taken to be $\Omega_{ij}^*(M, \lambda)_{sm}$, with $m_1(\lambda)$ being the restriction of d_{ij} to the eigenspace $\Omega_{ij}^*(M, \lambda)_{sm}$.

The natural way to define $m_2(\lambda)$ for any three objects f_0, f_1 and f_2 is the operation given by

$$\Omega_{12}^*(M, \lambda)_{sm} \otimes \Omega_{01}^*(M, \lambda)_{sm} \xrightarrow{\wedge} \Omega_{02}^*(M, \lambda) \xrightarrow{P_{02}} \Omega_{02}^*(M, \lambda)_{sm},$$

where $P_{ij} : \Omega_{ij}^*(M, \lambda) \rightarrow \Omega_{ij}^*(M, \lambda)_{sm}$ is the orthogonal projection.

Notice that $m_2(\lambda)$ is not associative, and we need a $m_3(\lambda)$ to record the non-associativity. To do this, let us consider the Green's operator G_{ij}^0 corresponding to Witten Laplacian Δ_{ij} . We let

$$(2.4) \quad G_{ij} = (I - P_{ij})G_{ij}^0$$

and

$$(2.5) \quad H_{ij} = d_{ij}^* G_{ij}.$$

Then H_{ij} is a linear operator from $\Omega_{ij}^*(M, \lambda)$ to $\Omega_{ij}^{*-1}(M, \lambda)$ and we have

$$d_{ij} H_{ij} + H_{ij} d_{ij} = I - P_{ij}.$$

Namely $\Omega_{ij}^*(M, \lambda)_{sm}$ is a homotopy retract of $\Omega_{ij}^*(M, \lambda)$ with homotopy operator H_{ij} . Suppose f_0, f_1, f_2 and f_3 are smooth functions on M and let $\varphi_{ij} \in \Omega_{ij}^*(M, \lambda)_{sm}$, the higher product

$$m_3(\lambda) : \Omega_{23}^*(M, \lambda)_{sm} \otimes \Omega_{12}^*(M, \lambda)_{sm} \otimes \Omega_{01}^*(M, \lambda)_{sm} \rightarrow \Omega_{03}^*(M, \lambda)_{sm}$$

is defined by

$$(2.6) \quad m_3(\lambda)(\varphi_{23}, \varphi_{12}, \varphi_{01}) = P_{03}(H_{13}(\varphi_{23} \wedge \varphi_{12}) \wedge \varphi_{01}) + P_{03}(\varphi_{23} \wedge H_{02}(\varphi_{12} \wedge \varphi_{01})).$$

In general, construction of $m_k(\lambda)$ can be described using k -tree. For $k \geq 2$, we decompose $m_k(\lambda) := \sum_T m_k^T(\lambda)$, where T runs over all topological types of k -trees.

$$m_k^T(\lambda) : \Omega_{(k-1)k}^*(M, \lambda)_{sm} \otimes \cdots \otimes \Omega_{01}^*(M, \lambda)_{sm} \rightarrow \Omega_{0k}^*(M, \lambda)_{sm}$$

is an operation defined along the directed tree T by

- (1) applying wedge product \wedge to each interior vertex;
- (2) applying homotopy operator H_{ij} to each internal edge labelled ij ;
- (3) applying projection P_{0k} to the outgoing semi-infinite edge.

The following graph shows the operation associated to the unique 2-tree.

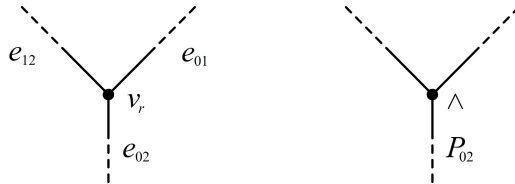


FIGURE 2. The unique 2-tree and the corresponding assignment of operators for defining $m_2(\lambda)$.

The higher products $\{m_k(\lambda)\}_{k \in \mathbb{Z}_+}$ satisfies the generalized associativity relation which is the so called A_∞ relation. One may treat the A_∞ product as a pullback of the wedge product under the homotopy retract $P_{ij} : \Omega_{ij}^*(M, \lambda) \rightarrow \Omega_{ij}^*(M, \lambda)_{sm}$. This proceed is called the homological perturbation lemma. For details about this construction, readers may see [12]. As a result, we obtain an A_∞ pre-category $DR_\lambda(M)_{sm}$.

Finally, we restate our Main Theorem with the notations from this section.

Theorem 9 (Main Theorem). *Given f_0, \dots, f_k satisfying generic assumption 6, with $q_{ij} \in CM^*(f_{ij})$ be corresponding critical points, there exist $\lambda_0 > 0$ and $C_0 > 0$, such that $\phi = \Phi^{-1} : CM^*(f_{ij}) \rightarrow \Omega_{ij}^*(M, \lambda)_{sm}$ are isomorphism for all $i \neq j$ when $\lambda < \lambda_0$. Furthermore, then we have*

$$\begin{aligned} & \Phi(m_k(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01}))) \\ &= m_k^{Morse}(q_{(k-1)k}, \dots, q_{01}) + R(\lambda), \end{aligned}$$

with $|R(\lambda)| \leq C_0 \lambda^{-1/2}$.

Remark 10. *The constants C_0 and λ_0 depend on the functions f_0, \dots, f_k . In general, we cannot choose fixed constants that the above statement holds true for all $m_k(\lambda)$ and all sequences of functions.*

Remark 11. *The relation with SYZ Mirror Symmetry is as follows. If we consider the cotangent bundle T^*M of a manifold M which equips the canonical symplectic form ω_{can} , and take $L_i = \Gamma_{df_i}$ to be the Lagrangian sections. Then a critical poin q_{ij} of f_{ij} can be identified with $q_{ij} \in L_i \pitchfork L_j$. Applying the theorem of Fukaya-Oh [7], the Morse A_∞ operation m_k^{Morse} is equivalent to Floer theoretical A_∞ operations counting holomorphic disks. In the simplest situation concerning (T^*M, ω_{can}) , the Witten's twisted deRham category $DR_\lambda(M)_{sm}$ is related to the Floer theory on (T^*M, ω_{can}) via our Main Theorem 9 and Fukaya-Oh's theorem. In more general situation, one expect that these correspondence will be part of the ingredients for realizing HMS geometrically.*

3. PROOF OF MAIN THEOREM

In the proof, we fix a generic sequence \vec{f} of $k+1$ functions, with corresponding sequence of critical points \vec{q} . First of all, we have

$$\deg(m_k(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01}))) = \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) - k + 2,$$

so $\langle m_k(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01}), \phi(q_{0k})) \rangle$ is non-trivial only when the equality

$$(3.1) \quad \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) - k + 2 = \deg(q_{0k})$$

holds, which is exactly the condition for m_k^{Morse} in the Morse category to be non-trivial. We will therefore assume condition (3.1) and consider the integral

$$\int_M \langle m_k(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01})), \frac{\phi(q_{0k})}{\|\phi(q_{0k})\|^2} \rangle vol_g.$$

Recall that each directed tree T gives an operation $m_k^T(\lambda)$ and $m_k(\lambda) = \sum_T m_k^T(\lambda)$ which is also the case in Morse category. Therefore, we just have to consider each $m_k^T(\lambda)$ separately.

3.1. Results for a single Morse function. We start with stating the results on Witten deformation for a single Morse function f_{ij} . These results come from [16] and [11], with a few modifications to fit our content. We will assume that the Morse function f_{ij} we are dealing with satisfy the Morse-Smale assumption 1.

Under this assumption, one can prove the following spectral gap in the twisted deRham complex.

Lemma 12. *For each f_{ij} , there is $\lambda_0 > 0$ and constants $c, C > 0$ such that we have*

$$\text{Spec}(\Delta_{ij}) \cap [ce^{-c\lambda}, C\lambda^{1/2}) = \emptyset,$$

for $\lambda > \lambda_0$.

Furthermore, we have the following theorem on Witten deformation on the level of chain complexes.

Theorem 13 ([11, 16]). *The map $\Phi = \Phi_{ij} : \Omega_{ij}^*(M, \lambda)_{sm} \rightarrow CM_{ij}^*$ in equation (2.3) is a chain isomorphism for λ large enough.*

We will denote the inverse by $\phi = \phi_{ij}$ and investigate the asymptotic behaviour of $\phi(q)$ for a critical point q of f_{ij} for understanding the asymptotic of the product structure. We will need the following Agmon distance ρ_{ij} for this purpose.

Definition 14. *For a Morse function f_{ij} , the Agmon distance ρ_{ij} , or simply denoted by ρ , is the distance function with respect to the degenerated Riemannian metric $\langle \cdot, \cdot \rangle_{f_{ij}} = |df_{ij}|^2 \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the background metric.*

Readers may see [8] for its basic properties. The Agmon distance is closely related to the Witten's Laplacian, or more precisely the corresponding Green's operator associated to it by the following lemma in [10].

Lemma 15. *Let $\gamma \subset \mathbb{C}$ to be a subset whose distance from $\text{Spec}(\Delta_{ij})$ is bounded below by a constant. For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon} > 0$ such that*

$$(3.2) \quad \|\nabla^j((z - \Delta_{ij})^{-1}u)\|_{C^0(V)} \leq C_{j,\epsilon} e^{-\lambda(\rho_{ij}(x_0, y_0) - \epsilon)} \|u\|_{W^{k_j, 2}(U)},$$

for all $\lambda > \lambda_0$ and $u \in C_c^0(U)$, where $W^{k,p}$ refers to the Sobolev norm.

We will also need modified version of the resolvent estimate for G_{ij} , which can be obtained by applying the original resolvent estimate to the the formula

$$(3.3) \quad G_{ij}(u) = \oint_{\gamma} z^{-1}(z - \Delta_{ij})^{-1}u.$$

Lemma 16. *For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon} > 0$ such that*

$$(3.4) \quad \|\nabla^j(G_{ij}u)\|_{C^0(V)} \leq C_{j,\epsilon} e^{-\lambda(\rho_{ij}(x_0, y_0) - \epsilon)} \|u\|_{W^{k_j, 2}(U)},$$

for all $\lambda < \lambda_0$ and $u \in C_c^0(U)$, where $W^{k,p}$ refers to the Sobolev norm.

For a critical point q of f_{ij} , $\phi(q)$, treated as the eigenform associated to the critical point, has certain exponential decay measured by the Agmon distance from the critical point q .

Lemma 17. *For any ϵ , there exists $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for $\lambda > \lambda_0$, we have*

$$(3.5) \quad \phi(q) = \mathcal{O}_{\epsilon}(e^{-\lambda(\psi_q(x) - \epsilon)}),$$

and same estimate holds for the derivatives of $\phi_{ij}(q)$ as well. Here \mathcal{O}_{ϵ} refers to the dependence of the constant on ϵ and $\psi_q(x) = \rho_{ij}(q, x) + f_{ij}(q)$.

Remark 18. *We write $g_q^+ = \psi_q - f_{ij}$, which is a nonnegative function with zero set V_q^+ that is smooth and Bott-Morse in a neighborhood W of $V_q^+ \cup V_q^-$. Similarly, we write $g_q^- = \psi_q + f_{ij}$ which is a nonnegative function with zero set V_q^- and is smooth and Bott-Morse in W .*

In that case, we can write

$$\begin{aligned} \phi(q) &= \mathcal{O}_{\epsilon}(e^{-\lambda(g_q^+ - \epsilon)}), \\ *\phi(q)/\|\phi(q)\|^2 &= \mathcal{O}_{\epsilon}(e^{-\lambda(g_q^- - \epsilon)}). \end{aligned}$$

Furthermore, we notice that the normalized basis $\phi(q)/\|\phi(q)\|$'s are almost orthonormal basis in the following sense.

Lemma 19. *There is a $C, c > 0$ and λ_0 such that when $\lambda > \lambda_0$, we will have*

$$\left\langle \frac{\phi(p)}{\|\phi(p)\|}, \frac{\phi(q)}{\|\phi(q)\|} \right\rangle = \delta_{pq} + Ce^{-c\lambda}.$$

3.1.1. *WKB approximation for eigenforms $\phi(q)$.* Restricting our attention to a small enough neighborhood W containing $V_q^+ \cup V_q^-$, the above decay estimate of eigenforms $\phi(q)$ from [11] can be improved from an error of order $\mathcal{O}_{\epsilon}(e^{\epsilon\lambda})$ to $\mathcal{O}(\lambda^{-\infty})$.

Lemma 20. *There is a WKB approximation of the eigenform $\phi(q)$ of the form*

$$(3.6) \quad \phi(q) \sim \lambda^{\frac{\deg(q)}{2}} e^{-\lambda\psi_q} (\omega_{q,0} + \omega_{q,1}\lambda^{-1/2} + \dots).$$

Remark 21. *The precise meaning of this WKB approximation is given in section 4.6. Roughly speaking, it is in the sense of C^∞ approximation in order of λ on every compact subset of W .*

Furthermore, the integral of the leading order term $\omega_{q,0}$ in the normal direction to the stable submanifold V_q^+ is computed in [11].

Lemma 22. *Fixing any point $x \in V_q^+$ and $\chi \equiv 1$ around x compactly supported in W , we take any closed submanifold (possibly with boundary) $NV_{q,x}^+$ of W intersecting transversally with V_q^+ at x . We have*

$$\lambda^{\frac{\deg(q)}{2}} \int_{NV_{q,x}^+} e^{-\lambda g_q^+} \chi \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1}).$$

Similarly, we also have

$$\frac{\lambda^{\frac{\deg(q)}{2}}}{\|\phi_{ij}(q)\|^2} \int_{NV_{q,x}^-} e^{-\lambda g_q^-} \chi * \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1}),$$

for any point $x \in V_q^-$, with $NV_{q,x}^-$ intersecting transversally with V_q^- .

So far we have been considering a fixed Morse function f_{ij} . From now on, we will consider a fixed generic sequence \vec{f} with corresponding sequence of critical points \vec{q} as in the beginning of section 3.

Notations 23. *We use q_{ij} to denote a fixed critical point of f_{ij} . The eigenform $\phi(q_{ij})$ associated to q_{ij} is abbreviated by ϕ_{ij} .*

3.2. Proof of m_3 . We will use the result in the previous section to localize the integral

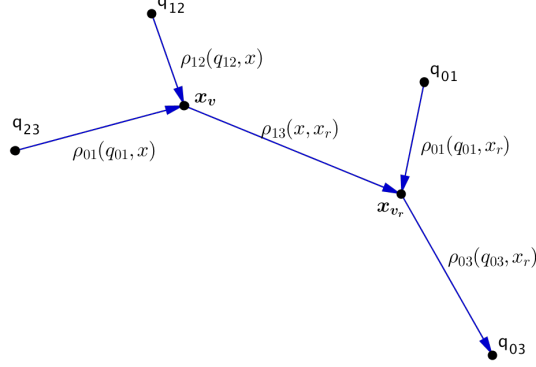
$$(3.7) \quad \int_M m_k(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01})) \wedge \frac{*\phi(q_{0k})}{\|\phi(q_{0k})\|^2}$$

to gradient flow trees, when the degree condition (3.1) holds. We begin with the $m_3(\lambda)$ case which involves less combinatorics to illustrate the analytic argument.

3.2.1. A priori estimate for $m_3(\lambda)$ case. There are two 3-leafed directed trees, which are denoted by T_1 and T_2 . We simply consider $m_3^{T_1}(\lambda)$ for T_1 which is the tree shown in figure 2.1.1 and relate this operation to counting gradient trees of type T_1 . T_1 has two interior vertices, which are denoted by v and v_r as in the figure. According to the combinatorics of T_1 , we define $\vec{\rho}_{T_1} : M^{|V(T_1)|} \rightarrow \mathbb{R}_+$ which is given by

$$\begin{aligned} & \vec{\rho}_{T_1}(x_v, x_{v_r}) \\ &= \rho_{13}(x_v, x_{v_r}) + \rho_{01}(x_{v_r}, q_{01}) + \rho_{12}(x_v, q_{12}) + \rho_{23}(x_v, q_{23}) + \rho_{03}(x_{v_r}, q_{03}). \end{aligned}$$

Roughly speaking, it is the length of the geodesic tree of type T_1 with interior vertices x_v, x_{v_r} and end points of semi-infinite edges e_{ij} 's laying on q_{ij} 's as shown in the following figure.



Making use of the fact that $\rho_{ij}(x, y) \geq f_{ij}(x) - f_{ij}(y)$ with equality holds iff y is connected to x through a (generalized) flow line of f_{ij} , we notice that $\vec{\rho}_{T_1}(x_v, x_{v_r}) \geq A = f_{03}(q_{03}) - f_{01}(q_{01}) - f_{12}(q_{12}) - f_{23}(q_{23})$ and the equality holds if and only if (x_v, x_{v_r}) are interior vertices of a gradient flow tree of the type T_1 . Here we only have gradient flow trees instead of generalized gradient trees since we assume the sequence of Morse function f satisfying generic assumption 6.

From the lemma 17 and 15, we notice the integrand

$$(3.8) \quad \int_M m_3^{T_1}(\phi_{23}, \phi_{12}, \phi_{01}) \wedge \frac{*\phi_{03}}{\|\phi_{03}\|^2} = \int_M H_{13}(\phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge \frac{*\phi_{03}}{\|\phi_{03}\|^2},$$

can be controlled by $e^{-\lambda(\rho_{T_1} - A)}$ in the following sense.

More precisely, fixing two points $x_v, x_{v_r} \in M$ and ϵ small enough, lemma 16 holds for operator G_{13} and hence H_{13} with U and V being balls centering at x_v and x_{v_r} (with respect to ρ_{13}) of radius r_1 . If we have two cut off functions χ and χ_r supported in $B(x_v, r_1)$ and $B(x_{v_r}, r_1)$ respectively, then we have

$$\|\chi_r H_{13}(\chi \phi_{23} \wedge \phi_{12})\|_{L^\infty} \leq C_\epsilon e^{-\lambda(\psi_{q_{23}}(x_v) + \psi_{q_{12}}(x_v) + \rho_{13}(x_v, x_{v_r}) - 2r_1 - 3\epsilon)}$$

for those large enough λ . Here the decay factors $\psi_{q_{23}}(x_v)$ and $\psi_{q_{12}}(x_v)$ come from the a priori estimate in lemma 17 for the input forms ϕ_{23} and ϕ_{12} respectively, while the decay factor $\rho_{13}(x_v, x_{v_r})$ comes from the resolvent estimate lemma 16. Combining with the decay estimates for ϕ_{01} and $\frac{*\phi_{03}}{\|\phi_{03}\|^2}$, we obtain

$$\|\chi_r H_{13}(\chi \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge \frac{*\phi_{03}}{\|\phi_{03}\|^2}\|_{L^\infty(M)} \leq C_\epsilon e^{-\lambda(\vec{\rho}_{T_1}(x_v, x_{v_r}) - A - 4r_1 - 5\epsilon)}$$

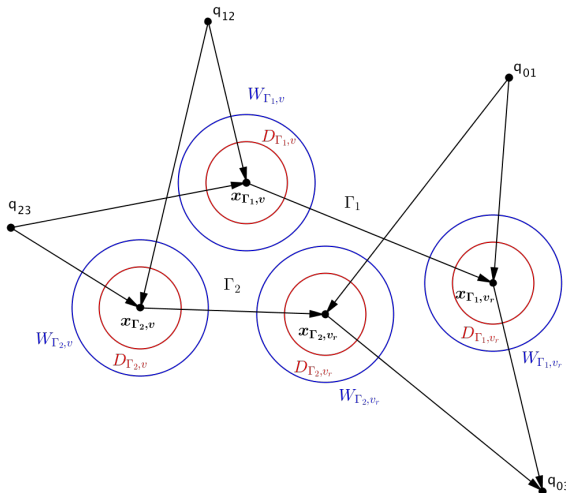


FIGURE 3. Cut off of integral near gradient trees

where x_v, x_{v_r} are the center of balls chosen for taking the cut off χ, χ_r as above.

We assume there are gradient trees $\Gamma_1, \dots, \Gamma_l$ of the type T_1 . For each tree Γ_i , we take open neighborhoods $D_{\Gamma_i, v}$ and $W_{\Gamma_i, v}$ of interior vertices $x_{\Gamma_i, v}$ with $\overline{D_{\Gamma_i, v}} \subset W_{\Gamma_i, v}$, and similarly D_{Γ_i, v_r} and W_{Γ_i, v_r} for x_{Γ_i, v_r} . The following Figure 3.2.1 illustrates the situation.

Since $\vec{\rho}_{T_1}$ is a continuous function in (x_v, x_{v_r}) attaining minimum value A exactly on internal vertices $(x_{\Gamma_i, v}, x_{\Gamma_i, v_r})$ of gradient trees Γ_i 's, we can assume there is a constant C such that $\vec{\rho}_{T_1} \geq A + C$ in $M^{V(T_1)} \setminus \cup_i D_{\Gamma_i}$, where $D_{\Gamma_i} = D_{\Gamma_i, v} \times D_{\Gamma_i, v_r}$. If $\vec{B}(\vec{x}, r_1) = B(x_v, r_1) \times B(x_{v_r}, r_1)$ is away from the D_{Γ_i} , we will have

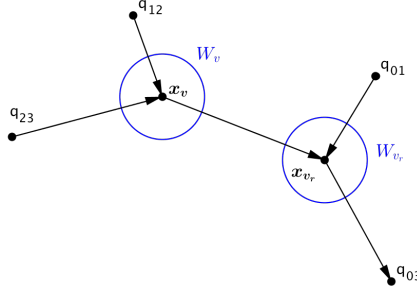
$$\|\chi_r H_{13}(\chi \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge \frac{* \phi_{03}}{\|\phi_{03}\|^2}\|_{L^\infty(M)} \leq C e^{-\lambda(\frac{C}{2})}.$$

Therefore, we can take cut off functions $\chi_{\Gamma_i, v}, \chi_{\Gamma_i, v_r}$ associating to each tree Γ_i , with support in $W_{\Gamma_i, v}, W_{\Gamma_i, v_r}$ and equal to 1 on $\overline{D_{\Gamma_i, v}}, \overline{D_{\Gamma_i, v_r}}$ respectively, to get

$$\begin{aligned} & \int_M m_3^{T_1}(\phi_{23}, \phi_{12}, \phi_{01}) \wedge \frac{* \phi_{03}}{\|\phi_{03}\|^2} \\ &= \sum_i \int_M \{\chi_{\Gamma_i, v_r} H_{13}(\chi_{\Gamma_i, v} \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge \frac{* \phi_{03}}{\|\phi_{03}\|^2}\} + \mathcal{O}(e^{-\lambda(\frac{C}{2})}). \end{aligned}$$

This localizes the integral computing $m_3^{T_1}$ to gradient trees of type T_1 . Notice that the neighborhood D_{Γ_i} and W_{Γ_i} can be chosen to be arbitrarily small in the previous argument.

3.2.2. *WKB method for m_3 .* In this section, we introduce the WKB method which allows us to compute the explicitly the leading order contribution in $m_3^{T_1}$. We fix a gradient tree Γ of type T_1 as in the section 3.2.1, with interior vertices $x_{\Gamma,v}$ and x_{Γ,v_r} . Since the gradient tree Γ is fixed, we tend to omit the dependence on Γ in our notations. We take neighborhoods W_v and W_{v_r} of x_v and x_{v_r} respectively, with cut off functions χ_v and χ_{v_r} supported in W_v and W_{v_r} respectively as shown in the following figure.



As $x_v \in V_{q_{12}}^+ \cap V_{q_{23}}^+$, we can assume that the WKB approximations

$$\phi_{12} \sim \lambda^{\frac{\deg(q_{12})}{2}} e^{-\lambda\psi_{12}} (\omega_{12,0} + \omega_{12,1}\lambda^{-1/2} + \dots),$$

and

$$\phi_{23} \sim \lambda^{\frac{\deg(q_{23})}{2}} e^{-\lambda\psi_{23}} (\omega_{23,0} + \omega_{23,1}\lambda^{-1/2} + \dots)$$

hold in W_v , by taking a smaller W_v if necessary using the lemma 17. What we need will be a similar WKB approximation for the term

$$H_{13}(\chi_v \phi_{23} \wedge \phi_{12}),$$

in the neighborhood W_{v_r} . Here we state a WKB lemma needed for the homotopy operators H_{ij} , appearing in the higher products $m_k(\lambda)$. The proof will occupy the whole section 4.

WKB for homotopy operator. We give the setup of the lemma. Let $\gamma(t)$ be a flow line of $\nabla f_{ij}/|\nabla f_{ij}|_{\rho_{ij}}$ starts at $\gamma(0) = x_S$ and $\gamma(T) = x_E$ for a fixed $T > 0$. We consider an input form ζ_S defined in a neighborhood W_S of x_S . Suppose we are given a WKB approximation of ζ_S in W_S , which is an approximation of ζ_S according to order of λ of the form

$$(3.9) \quad \zeta_S \sim e^{-\lambda\psi_S} (\omega_{S,0} + \omega_{S,1}\lambda^{-1/2} + \omega_{S,2}\lambda^{-1} + \dots)$$

(The precise meaning of this infinite series approximation can be found in section 4.6). We further assume that $g_S = \psi_S - f_{ij}$ is a nonnegative Bott-Morse function in W_S with zero set V_S . We consider the equation

$$(3.10) \quad \Delta_{ij}\zeta_E = (I - P_{ij})d_{ij}^*(\chi_S\zeta_S),$$

where χ_S is a cutoff function compactly supported in W_S , $P_{ij} : \Omega_{ij}^*(M, \lambda) \rightarrow \Omega_{ij}^*(M, \lambda)_{sm}$ is the projection. We want to have a WKB approximation of $\zeta_E = H_{ij}(\chi_S \zeta_S)$

Lemma 24. *For $\text{supp}(\chi_S)$ small enough, there is a WKB approximation of ζ_E in a small enough neighborhood W_E of E , of the form*

$$(3.11) \quad \zeta_E \sim e^{-\lambda\psi_E} \lambda^{-1/2} (\omega_{E,0} + \omega_{E,1} \lambda^{-1/2} + \dots).$$

Furthermore, the function $g_E := \psi_E - f_{ij}$ is a nonnegative function which is Bott-Morse in W_E with zero set $V_E = (\bigcup_{-\infty < t < +\infty} \sigma_t(V_S)) \cap W_E$ which is closed in W_E , where σ_t is the time t flow of $\nabla f_{ij} / |\nabla f_{ij}|^2$ (normalized according to $|df_{ij}|^2 \langle \cdot, \cdot \rangle$).

WKB for m_3 (cont'd). We apply lemma 4.1 with Morse function f_{13} , input form $\zeta_S = \phi_{23} \wedge \phi_{12}$, starting vertex $x_S = x_v$, ending vertex $x_E = x_{v_r}$, with neighborhood $W_S = W_v$ and $W_E = W_{v_r}$. (This can be done by shrinking W_v and W_{v_r} if necessary). As a result, we obtain the WKB approximation

$$H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \sim \lambda^{\frac{\deg(q_{23}) + \deg(q_{12}) - 1}{2}} e^{-\lambda\psi_{13}} (\omega_{13,0} + \omega_{13,1} \lambda^{-1/2} + \dots),$$

by taking $\psi_E = \psi_{13}$ and $\omega_{E,i} = \omega_{13,i}$ in the lemma.

In order to compute

$$\int_M m_3^{T_1}(\lambda, \vec{\chi}_\Gamma) \wedge \frac{* \phi_{03}}{\|\phi_{03}\|^2} = \int_M \chi_{v_r} H_{13}(\chi_v \phi_{23} \wedge \phi_{12}) \wedge \phi_{01} \wedge \frac{* \phi_{03}}{\|\phi_{03}\|^2}$$

up to an error of order $\mathcal{O}(\lambda^{-1/2})$, we can simply compute the integral

$$(3.12) \quad \lambda^{\frac{\deg(q_{23}) + \deg(q_{12}) + \deg(q_{01}) - 1}{2}} \int_M \{ \chi_{v_r} (e^{-\lambda\psi_{13}} \omega_{13,0}) \wedge (e^{-\lambda\psi_{01}} \omega_{01,0}) \wedge \\ \wedge (\lambda^{-\frac{\deg(q_{03})}{2}} \frac{e^{-\lambda\psi_{03}} * \omega_{03,0}}{\|\phi_{03}\|^2}) \} \\ = \frac{1}{\|\phi_{03}\|^2} \int_M \{ \chi_{v_r} (e^{-\lambda(\psi_{13} + \psi_{01} + \psi_{03})} \omega_{13,0} \wedge \omega_{01,0} \wedge * \omega_{03,0}) \}.$$

We first take a look on the exponential decay factor $e^{-\lambda(\psi_{13} + \psi_{01} + \psi_{03})}$ in the integral. We define g_{13} , g_{01}^+ and g_{03}^- by

$$\begin{aligned} \psi_{13} &= g_{13} + f_{13}, \\ \psi_{01} &= g_{01}^+ + f_{01}, \\ \psi_{03} &= g_{03}^- - f_{03}. \end{aligned}$$

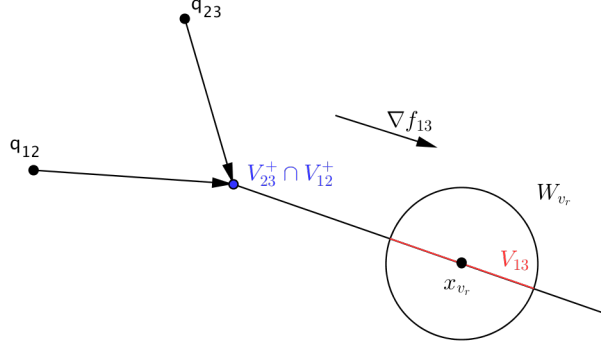
Therefore we have exponential decay being

$$e^{-\lambda(g_{13} + g_{01}^+ + g_{03}^-)}$$

of the integrand.

We recall in remark 18 that g_{01}^+ , g_{12}^+ , g_{23}^+ and g_{03}^- are Bott-Morse with absolute minimums on V_{01}^+ , V_{12}^+ , V_{23}^+ and V_{03}^- respectively. We also recall

from lemma 24 that g_{13} will be a Bott-Morse in W_{v_r} with absolute minimum denoted by V_{13} (colored red in the following figure), which is the submanifold $(\bigcup_{-\infty < t < +\infty} \sigma_t(V_{23}^+ \cap V_{12}^+)) \cap W_{v_r}$ flowed out from $V_{23}^+ \cap V_{12}^+$ (colored blue in the following figure), under the flow of $\frac{\nabla f_{13}}{|\nabla f_{13}|^2}$ which is denoted by σ_t .



The generic assumption 6 of the sequence \vec{f} indicates that $\{x_{v_r}\} = V_{13} \cap V_{01}^+ \cap V_{03}^-$ transversally at x_{v_r} , which means $e^{-\lambda(g_{13} + g_{01}^+ + g_{03}^-)}$ concentrating at x_{v_r} . The leading order contribution will only depends on the value of $\omega_{13,0} \wedge \omega_{01,0} \wedge * \omega_{03,0}$ at the point x_{v_r} , which will be computed in the upcoming section 3.2.3.

3.2.3. Explicit computations for m_3 . We will need the following lemma which will be proven in section 4.8.

Lemma 25. *Let M be a n -dimensional manifold and S be a k -dimensional submanifold in M , with a neighborhood B of S which can be identified as the normal bundle $\pi : NS \rightarrow S$. Suppose $\varphi : B \rightarrow \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set S and $\beta \in \Omega^*(B)$ has vertical compactly support along the fiber of π , we have*

$$\pi_*(e^{-\lambda\varphi(x)}\beta) = \left(\frac{\lambda}{2\pi}\right)^{(n-k)/2} (\iota_{\text{vol}(\nabla^2\varphi)}\beta)|_V (1 + \mathcal{O}(\lambda^{-1})),$$

where π_* is the integration along fiber. Here $\text{vol}(\nabla^2\varphi)$ stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2\varphi$ along fibers of π .

We find from the above lemma that the leading order contribution in the above integral (3.12) depend only on values of $\omega_{13,0}$, $\omega_{01,0}$ and $*\omega_{03,0}$ at the point x_{v_r} . We use the normal bundle $NV_{13} \oplus NV_{23}^+ \oplus NV_{03}^-$ at x_{v_r} to parametrize a neighborhood of x_{v_r} . Making use of the above lemma 25, we can split the integral as follows for computing leading order contribution.

We have

$$\begin{aligned}
 & \int_M \chi_{v_r} e^{-\lambda(g_{13}+g_{01}^++g_{03}^-)} \omega_{13,0} \wedge \omega_{01,0} \wedge * \omega_{03,0} \\
 = & \pm \left(\int_{NV_{13,x_{v_r}}} e^{-\lambda g_{13}} \chi_{v_r} \omega_{13,0} \right) \left(\int_{NV_{23,x_{v_r}}^+} e^{-\lambda g_{23}^+} \chi_{v_r} \omega_{23,0} \right) \cdot \\
 & \left(\int_{NV_{03,x_{v_r}}^-} e^{-\lambda g_{03}^-} \chi_{v_r} * \omega_{03,0} \right) (1 + \mathcal{O}(\lambda^{-1})),
 \end{aligned}$$

where the sign depends on whether the orientations of $NV_{13} \oplus NV_{01}^+ \oplus NV_{03}^-$ and TM at the point x_{v_r} match or not. We will compute the above two integrals one by one. We obtain equality

$$\lambda^{\frac{\deg(q_{01})}{2}} \int_{NV_{01,x_{v_r}}^+} e^{-\lambda g_{01}^+} \chi_{v_r} \omega_{01,0} = 1 + \mathcal{O}(\lambda^{-1}),$$

and

$$\frac{\lambda^{\frac{\deg(q_{03})}{2}}}{\|\phi_{03}\|^2} \left(\int_{NV_{03,x_{v_r}}^-} e^{-\lambda g_{03}^-} \chi_{v_r} * \omega_{03,0} \right) = 1 + \mathcal{O}(\lambda^{-1})$$

from the lemma 22. Moreover, we have an equality

$$\lambda^{\frac{\deg(q_{23})+\deg(q_{12})-1}{2}} \int_{NV_{13,x_{v_r}}} e^{\lambda g_{13}} \chi_{v_r} \omega_{13,0} = (1 + \mathcal{O}(\lambda^{-1/2})).$$

This depends on the fact that

$$\begin{aligned}
 & \lambda^{\frac{\deg(q_{23})+\deg(q_{12})}{2}} \int_{N(V_{23}^+ \cap V_{12}^+)_{x_v}} e^{-\lambda(g_{23}^++g_{12}^+)} \chi_v \omega_{23,0} \wedge \omega_{12,0} \\
 = & \left(\lambda^{\frac{\deg(q_{23})}{2}} \int_{N(V_{23}^+)_{x_v}} e^{-\lambda g_{23}^+} \chi_v \omega_{23,0} \right) \left(\lambda^{\frac{\deg(q_{12})}{2}} \int_{N(V_{12}^+)_{x_v}} e^{-\lambda g_{12}^+} \chi_v \omega_{12,0} \right) (1 + \mathcal{O}(\lambda^{-1})) \\
 = & 1 + \mathcal{O}(\lambda^{-1}),
 \end{aligned}$$

and the following lemma.

Lemma 26. *Using same notations in lemma 24 and suppose χ_S and χ_E are cut off function supported in W_S and W_E respectively, then we have*

$$(3.13) \quad \int_{N(V_E)_{v_E}} e^{-\lambda g_E} \chi_E \omega_{E,0} = \left(\int_{N(V_S)_{v_S}} e^{-\lambda g_S} \chi_S \omega_{S,0} \right) (1 + \mathcal{O}(\lambda^{-1/2})).$$

Furthermore, suppose $\omega_{S,0}(x_S) \in \bigwedge^{top} N(V_S)_{x_S}^*$, we have $\omega_{E,0}(x_E) \in \bigwedge^{top} N(V_E)_{x_E}^*$.

Putting the above together, we get the following

$$(3.14) \quad m_3^{T_1}(\lambda, \vec{\chi}_\Gamma) = \pm(1 + \mathcal{O}(\lambda^{-1/2})),$$

where the sign depends on matching the orientations of $NV_{13} \oplus NV_{01}^+ \oplus NV_{03}^-$ and TM at the point x_{v_r} . The proof for $m_3(\lambda)$ is completed and we move on to the $m_k(\lambda)$ case for any k . The proof is essentially the same as the

$m_3(\lambda)$ case except involving more combinatorics and notations.

3.3. Proof of m_k .

3.3.1. *A priori estimates for m_k .* We fix a k -leafed tree T and consider the operation corresponding to it, denoted by $m_k^T(\lambda)$. We try to relate this operation to counting of gradient trees of type T . We have the function $\vec{\rho}_T : M^{|V(T)|} \rightarrow \mathbb{R}_+$ defined according to the combinatorics of T given by

$$(3.15) \quad \vec{\rho}_T(\vec{x}) = \sum_{e_{ij} \in E(T)} \rho_{ij}(x_S(e_{ij}), x_E(e_{ij})) + \sum_{i=0}^{k-1} \rho_{i(i+1)}(q_{i(i+1)}, x_E(e_{i(i+1)})) + \rho_{0k}(q_{0k}, x_S(e_{0k})).$$

Here the variables \vec{x} are labelled by the vertices of T . ($x_S(e)$ and $x_E(e)$ refer to the variables corresponding to vertices which are starting point and endpoint of the edge e respectively.) Recall that $E(T)$ is the set of internal edges of T and each interior edge e has a unique label by two integers as e_{ij} , corresponding to the Morse function $f_{ij} = f_j - f_i$. The notation ρ_{ij} refers to the Agmon distance corresponding to the Morse function f_{ij} .

$\vec{\rho}_T(\vec{x})$ is the length function of a geodesic tree (may not be unique) with topological type T , with interior vertices \vec{x} and semi-infinite edges ending on critical points q_{ij} . Similar to the case of $m_3(\lambda)$, we have the following lemma.

Lemma 27. *The function $\vec{\rho}_T$ is bounded below by $A = f_{01}(q_{01}) + \cdots + f_{(k-1)k}(q_{(k-1)k}) - f_{0k}(q_{0k})$, and it attains minimum at \vec{x} if and only if \vec{x} is the vector consisting of interior vertices of a gradient flow tree of \vec{f} of type T ended at corresponding critical points \vec{q} .*

Proof. The proof relies on the fact (see [11]) that we have

$$|f_{ij}(x) - f_{ij}(y)| \leq \rho_{ij}(x, y),$$

if f_{ij} is a Morse function on M , and $\rho_{ij}(x, y)$ is the Agmon distance. Furthermore, the equality $f_{ij}(x) - f_{ij}(y) = \rho_{ij}(x, y)$ forces the geodesic from y to x to be a generalized integral curve of ∇f_{ij} . We apply this fact to each term in (3.15) and the result follows. \square

Similar to the $m_3(\lambda)$ case, every gradient flow tree $\Gamma \in \mathcal{M}(\vec{f}, \vec{q})(T)$ is associated with a unique minimum point $\vec{x}_\Gamma \in M^{|V(T)|}$ of $\vec{\rho}_T$. For each tree, we take a covering W_Γ of \vec{x}_Γ , given by a product $W_\Gamma = \prod_{v \in V(T)} W_{\Gamma, v}$, where each $W_{\Gamma, v}$ is an open subsets in M containing x_v such that all $W_{\Gamma, v}$'s are disjoint from each other. If we further take $D_\Gamma = \prod_{v \in V(T)} D_{\Gamma, v}$ such that $\overline{D_{\Gamma, v}} \subset W_{\Gamma, v}$, we have a constant $C > 0$ such that $\vec{\rho}_T \geq A + C$ on $M^{|V(T)|} \setminus D_\Gamma$. We are going to show that the integral (3.7) can be localized.

We take a finite covering of M with balls $\{B(x, r)\}_{B(x, r) \in \mathcal{J}}$ of radius r centering at x , with a partition of unity $\{\chi_B\}_{B \in \mathcal{J}}$ subordinating to it. We choose a covering $\{B_r(\vec{x})\}_{B \in \mathcal{I}}$ of $M^{|V(T)|}$ given by product $B_r(\vec{x}) = \prod_{v \in V(T)} B(x_v, r)$, where $B(x_v, r) \in \mathcal{J}$. We decompose $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $B \in \mathcal{I}_2$ are those having empty intersection with $\overline{D_\Gamma}$, and $B \in \mathcal{I}_1$ satisfying $\overline{B} \subset W_\Gamma$. These can be achieved by choosing r small enough.

We can take cut off functions subordinate to the covering $\{B\}_{\mathcal{I}}$, given by product of functions χ_B on M . We write $\vec{\chi}_B = \prod_{v \in V(T)} \chi_{B(x_v, r)}$ for the function supported in B . We will use $\vec{\chi}_B$ to cut off the following integral

$$(3.16) \quad \int_M m_k^T(\lambda)(\phi(q_{(k-1)k}), \dots, \phi(q_{01})) \wedge \frac{*\phi_{0k}}{\|\phi_{0k}\|^2}.$$

Recall that the $m_k^T(\lambda)$ is defined using wedge product and the homotopy operator H_{ij} , following the combinatorics of the tree T . We cut off the operation $m_k^T(\lambda)$ using the function $\chi_{B(x_v, r)}$ whenever taking wedge product at the vertex v . We will write $m_k^T(\lambda, \vec{\chi})$ for the integral after cutting off by $\vec{\chi}$. Therefore we have

$$(3.17) \quad m_k^T(\lambda)(\phi(\vec{q})) = \sum_{B \in \mathcal{I}_1} m_k^T(\lambda, \vec{\chi}_B)(\phi(\vec{q})) + \sum_{B \in \mathcal{I}_2} m_k^T(\lambda, \vec{\chi}_B)(\phi(\vec{q})),$$

where $m_k^T(\lambda, \vec{\chi}_{\overline{B}})(\phi(\vec{q}))$ stand for A_∞ operation after cutting off by $\vec{\chi}_{\overline{B}}$. Recall that there is a unique root vertex v_r associated to the direct tree \overline{T} , and obtain the following lemma by applying the resolvent estimate 16 and the estimate (17).

Lemma 28. *For any $\epsilon > 0$, there exist positive $r(\epsilon)$, and $\lambda(\epsilon)$ such that*

$$(3.18) \quad \|m_k^T(\lambda, \vec{\chi}_B)(\phi(\vec{q})) \wedge \frac{*\phi_{0k}}{\|\phi_{0k}\|^2}\|_{L^\infty(M)} = \mathcal{O}_{r, \epsilon}(e^{-\lambda(\vec{\rho}_T(\vec{x}) - A - \epsilon)})$$

for $\lambda < \lambda(\epsilon)$, if we take the covering of radius $r < r(\epsilon)$. Here \vec{x} is the center of the ball B .

The proof is essentially the same as the case for $m_3(\lambda)$. Similarly, we can have

$$\sum_{B \in \mathcal{I}_2} \int_M m_k^T(\lambda, \vec{\chi}_B) \wedge \frac{*\phi_{0k}}{\|\phi_{0k}\|^2} = \mathcal{O}_{r, \epsilon}(e^{-\lambda(\frac{C}{2})}),$$

for λ large enough. It follows from the fact that $\vec{\rho}_T(\vec{x}) \geq A + C$ for those covering in \mathcal{I}_2 . This result basically says that the integral $m_k^T(\lambda)$ can be localized to gradient flow tree using the cut off mentioned above. To summarize, we have the following proposition.

Proposition 29. *For each gradient flow tree Γ , there is a sequence of cutoff functions $\{\vec{\chi}_\Gamma\}$ which is supported in W_Γ and satisfy $\vec{\chi}_\Gamma \equiv 1$ on $\overline{D_\Gamma}$ such*

that

$$(3.19) \quad \int_M m_k^T(\lambda)(\phi(\vec{q})) \wedge \frac{*\phi_{0k}}{\|\phi_{0k}\|^2} = \sum_{\Gamma \in \mathcal{M}(\vec{f}, \vec{q})(T)} \int_M m_k^T(\lambda, \vec{\chi}_\Gamma)(\phi(\vec{q})) \wedge \frac{*\phi_{0k}}{\|\phi_{0k}\|^2} + \mathcal{O}(e^{-\lambda(\frac{C}{2})}),$$

for λ enough enough.

Remark 30. In the above argument, the neighborhood W_Γ can be chosen to be arbitrary small. We will obtain a smaller constant C if we shrink the neighborhood W_Γ .

After localizing the integral, we move on to the section concerning WKB approximation which helps to compute of the leading order contribution of $m_k^T(\lambda, \vec{\chi}_\Gamma)$.

3.3.2. WKB method for m_k . We consider a gradient tree Γ of type T , with k semi-infinite incoming edges. Recall in section 2.1.1 that each edge in T is assigned with a label by two integer ij . We will use ij to represent an edge in T and denote the corresponding edge in the gradient tree Γ by e_{ij} . The vertex in the gradient tree corresponding to v in T will be denoted by x_v . We again omit the dependence on Γ in our notations as it is fixed. We are going to associate $\phi_{(ij,v)} \in \Omega_{ij}^*(M, \lambda)$, together with its WKB approximation

$$\phi_{(ij,v)} \sim e^{-\lambda\psi_{(ij,v)}} \lambda^{r_{(ij,v)}} (\omega_{(ij,v),0} + \omega_{(ij,v),1} + \dots)$$

in some neighborhood W_v of x_v to each flag (ij, v) as shown in the following figure 4. We also fix cut off functions χ_v 's supported in W_v and restrict our attention to integral $m_k^T(\lambda, \vec{\chi})(\vec{q})$, using the arguments in section 3.2.1.

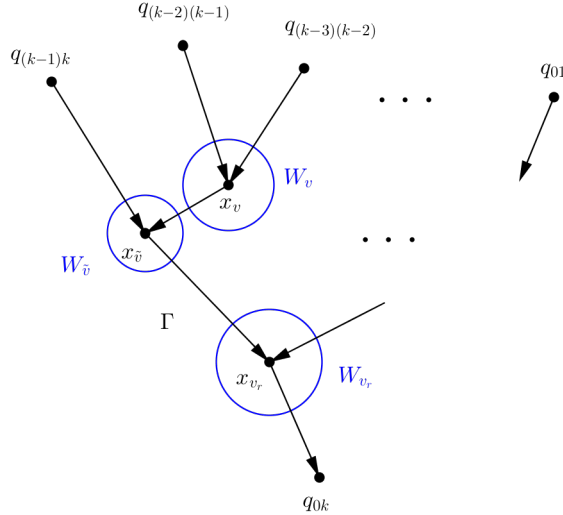


FIGURE 4

We define the followings inductively.

- (1) for a semi-infinite incoming edge $i(i+1)$ and its ending vertex v , we take $\phi_{(i(i+1),v)}$ to be the input eigenform $\phi_{i(i+1)}$, with its the WKB approximation in W_v as in lemma 20. We also let $g_{(i(i+1),v)} = \psi_{(i(i+1),v)} - f_{i(i+1)}$. We choose W_v small enough such that the WKB approximation of all input eigenforms associated to edges connected to v holds in W_v ;
- (2) for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting il at v as shown in figure 5, we

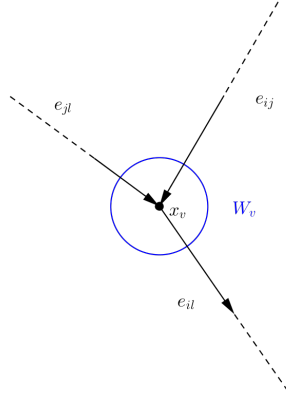


FIGURE 5

take $\phi_{(il,v)} = \phi_{(jl,v)} \wedge \phi_{(ij,v)}$. The WKB expression of $\phi_{(il,v)}$ comes from the expression of $\phi_{(jl,v)} \wedge \phi_{(ij,v)}$, which means

$$\begin{aligned} \psi_{(il,v)} &= \psi_{(ij,v)} + \psi_{(jl,v)}, \\ \omega_{(e_{il,v}),n} &= \sum_{m+m'=n} \omega_{(jl,v),m} \wedge \omega_{(ij,v),m'}, \\ r_{(il,v)} &= r_{(jl,v)} + r_{(ij,v)}. \end{aligned}$$

We also let $g_{(il,v)} = g_{(ij,v)} + g_{(jl,v)}$;

- (3) for an internal edge ij with its starting vertex v_S and ending vertex v_E as shown in figure 6, we take the WKB approximation in lemma 24 of $\phi_{(ij,v_E)} = H_{ij}(\chi_{v_S} \phi_{(ij,v_S)})$ in W_{v_E} by taking $\text{supp}(\chi_{v_S})$ and W_{v_E} small enough for applying the lemma if necessary. We also let $g_{(ij,v_E)} = \psi_{(ij,v_E)} - f_{ij}$ and $r_{(ij,v_E)} = r_{(ij,v_S)} - \frac{1}{2}$.
- (4) for the semi-infinite outgoing edge $0k$ with the root vertex v_r , we take $\phi_{(0k,v_r)}$ to be the eigenform ϕ_{0k} , with its the WKB approximation in lemma 20. We also let $g_{(0k,v_r)} = \psi_{(0k,v_r)} + f_{0k}$.

Remark 31. We need to choose the size of cut off $\text{supp}(\chi_{\Gamma,v})$ appearing in the previous section 3.2.1 at each internal vertex v small enough for apply lemma 24.

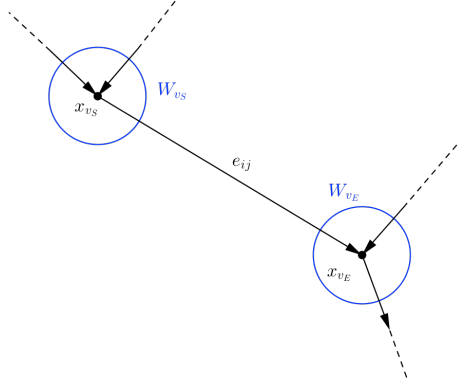


FIGURE 6

From the definition of $m_k^T(\lambda, \vec{\chi}_\Gamma)$, we see that

$$\int_M m_k^T(\lambda, \vec{\chi}_\Gamma)(\phi_{(k-1)k}, \dots, \phi_{01}) \wedge \frac{* \phi_{0k}}{\|\phi_{0k}\|^2} = \int_M \phi_{(jk, v_r)} \wedge \phi_{(0j, v_r)} \wedge \frac{* \phi_{(0k, v_r)}}{\|\phi_{(0k, v_r)}\|^2},$$

if three edges $0j$, jk and $0k$ are meeting at the root vertex v_r . Applying lemma 20 to input eigenforms and lemma 24 to homotopy operators H_{ij} along internal edges e_{ij} 's, we prove that each WKB approximation

$$\phi_{(ij, v)} \sim e^{-\lambda \psi_{(ij, v)}} \lambda^{r_{(ij, v)}} (\omega_{(ij, v), 0} + \omega_{(ij, v), 1} + \dots)$$

is an C^∞ approximation with error $e^{-\lambda \psi_{(ij, v)}} \mathcal{O}(\lambda^{-\infty})$. Therefore, we can replace ϕ 's by first term in its WKB approximation for computing the leading order contribution. We obtain

$$\begin{aligned} (3.20) \quad & \langle m_k^T(\lambda, \vec{\chi}_\Gamma)(\phi_{(k-1)k}, \dots, \phi_{01}), \frac{\phi_{0k}}{\|\phi_{0k}\|^2} \rangle \\ &= \left\{ \lambda^{r_{(jk, v_r)} + r_{(0j, v_r)} + r_{(0k, v_r)}} \int_M e^{-\lambda(\psi_{(jk, v_r)} + \psi_{(0j, v_r)} + \psi_{(0k, v_r)})} \right. \\ & \quad \left. \chi_{v_r}(\omega_{(jk, v_r), 0} \wedge \omega_{(0j, v_r), 0} \wedge \frac{* \omega_{(0k, v_r), 0}}{\|\phi_{0k}\|^2}) \right\} (1 + \mathcal{O}(\lambda^{-1})). \end{aligned}$$

3.3.3. Explicit computation for m_k . The argument of the general case is similar to the case $k = 3$, with more combinatorics involved. As in section 3.3.2, we fix a gradient tree Γ of type T . Similar to the previous section, we may drop the dependence of Γ in our notations. We are going to show that

$$(3.21) \quad \int_M m_k^T(\lambda, \vec{\chi}_\Gamma) \wedge \frac{* \phi_{0k}}{\|\phi_{0k}\|^2} = \pm(1 + \mathcal{O}(\lambda^{-1/2})),$$

where the sign agrees with that associated to the gradient tree Γ in Morse category. We begin with some notations associated to Γ .

Notations 32. Given a gradient tree Γ , we inductively associate to each flag (ij, v) an oriented closed submanifold $V_{(ij, v)} \subset W_v$ by specifying orientation of its normal bundle. We require:

- (1) for each semi-infinite incoming edge $i(i+1)$ with ending vertex v , we let $V_{(i(i+1), v)} := V_{q_{i(i+1)}}^+ \cap W_v$, where $V_{q_{i(i+1)}}^+$ is the stable submanifold of $f_{i(i+1)}$ from the critical point $q_{i(i+1)}$ with the chosen orientation $\nu_{(i(i+1), v)}$ equals to that in the Morse category;
- (2) for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting e_{il} at v as in the section 5. We let $V_{(il, v)} = V_{(ij, v)} \cap V_{(jl, v)}$ (the intersections is transversal from the generic assumption) and $\nu_{(il, v)} = \nu_{(jl, v)} \wedge \nu_{(ij, v)}$, if $\nu_{(ij, v)}$ and $\nu_{(jl, v)}$ are two corresponding orientation forms;
- (3) for an internal edge ij with its starting vertex v_S and ending vertex v_E , we define $V_{(ij, v_E)}$ to be V_E obtained from applying lemma 24 to the homotopy operator H_{ij} . The orientation form $\nu_{(ij, v_E)}$ is chosen such that $[\nu_{(ij, v_E)}] = [df_{ij} \wedge \nu_{(ij, v_S)}]$, under the identification by flow of ∇f_{ij} ;
- (4) for the semi-infinite incoming edge $0k$ with root vertex v_r , we let $V_{(0k, v_r)} := V_{q_{0k}}^- \cap W_{v_r}$, where $V_{q_{0k}}^-$ is the unstable submanifold of f_{0k} from critical point q_{0k} with the chosen orientation $\nu_{(0k, v_r)}$ equal to that in the Morse category;

We further choose an isomorphism and projection map for every flag (ij, v)

$$(3.22) \quad \begin{array}{ccc} W_v & \xrightarrow{\cong} & NV_{(ij, v)} \\ \pi_{(e, v)} \downarrow & & \pi_{NV_{(ij, v)}} \downarrow \\ V_{(ij, v)} & \xlongequal{\quad} & V_{(ij, v)} \end{array}$$

by further shrinking W_v suitably.

We can therefore assign a sign to the gradient tree Γ in the following way.

Definition 33. For a generic sequence of Morse function \vec{f} with corresponding critical points $q_{01}, \dots, q_{(k-1)k}, q_{0k}$ satisfying the degree condition (3.1), with a gradient tree Γ , we define

$$(3.23) \quad \text{sign}(\Gamma) = \text{sign}\left(\frac{\nu_{(jk, v_r)} \wedge \nu_{(0j, v_r)} \wedge \nu_{(0k, v_r)}}{\text{vol}_g}\right),$$

where $0j, jk$ and $0k$ are edges joining the root vertex v_r as in section 5, $\nu_{(ij, v)}$ is the orientation of normal bundle defined in notation 32 and $\nu_{(0k, v_r)}$ is the orientation of chosen for $V_{q_{0k}}^-$.

We are going to argue that

$$\int_{N(V_{(ij, v)})_{x_v}} (e^{-\lambda g_{(ij, v)}} \lambda^{r_{(ij, v)}} \chi_v \omega_{(ij, v), 0}) = (1 + \mathcal{O}(\lambda^{-1/2})),$$

for any flag (ij, v) except the outgoing edge $0k$, where r_v is the number of internal edge before the vertex v . This can be seen inductively along the tree T . We see that:

- (1) it is true for the semi-infinite incoming edge $i(i+1)$ by lemma 22;
- (2) for an internal edge il with its starting vertex v and assume ij and jl are two incoming edges meeting il at v , we have

$$\begin{aligned}
& \lambda^{r(i,v)} \int_{N(V(i,v))_{xv}} e^{-\lambda g(i,v)} \chi_v \omega(i,v),_0 \\
\equiv & \lambda^{r(jl,v)+r(ij,v)} \int_{N(V(jl,v) \cap V(ij,v))_{xv}} e^{-\lambda(g(jl,v)+g(ij,v))} \chi_v \omega(jl,v),_0 \wedge \omega(ij,v),_0 \\
\equiv & (\lambda^{r(jl,v)} \int_{N(V(jl,v))_{xv}} e^{-\lambda g(jl,v)} \chi_v \omega(jl,v),_0) (\lambda^{r(ij,v)} \int_{N(V(ij,v))_{xv}} e^{-\lambda g(ij,v)} \chi_v \omega(ij,v),_0) \\
\equiv & 1,
\end{aligned}$$

modulo an error of order $\mathcal{O}(\lambda^{-1/2})$;

- (3) for an internal edge ij with its starting vertex v_S and ending vertex v_E , we make use of the lemma 26 as before.

We can now calculate the leading contribution from the integral (3.20). Recall that we have

$$(3.24) \quad \psi_{(0j,v_r)} + \psi_{(jk,v_r)} - f_{0k} = g_{(0j,v_r)} + g_{(jk,v_r)}.$$

Therefore we obtain

$$\begin{aligned}
& \lambda^{r(0j,v_r)+r(jk,v_r)+r(0k,v_r)} \left\{ \int_M e^{-\lambda(\psi_{(0j,v_r)}+\psi_{(jk,v_r)}+\psi_{(0k,v_r)})} \right. \\
& \quad \left. \chi_{v_r} \cdot (\omega_{(jk,v_r),0} \wedge \omega_{(0j,v_r),0} \wedge \frac{* \omega_{(0k,v_r),0}}{\|\phi_{0k}\|^2}) \right\} \\
= & \lambda^{r(0j,v_r)+r(jk,v_r)+r(0k,v_r)} \left\{ \int_M e^{-\lambda(g_{(0j,v_r)}+g_{(jk,v_r)}+g_{(0k,v_r)})} \right. \\
& \quad \left. \chi_{v_r} (\omega_{(jk,v_r),0} \wedge \omega_{(0j,v_r),0} \wedge \frac{* \omega_{(0k,v_r),0}}{\|\phi_{0k}\|^2}) \right\} \\
= & \pm(1 + \mathcal{O}(\lambda^{-1/2})),
\end{aligned}$$

which means

$$(3.25) \quad m_k^T(\lambda, \vec{\chi}_\Gamma) = \pm(1 + \mathcal{O}(\lambda^{-1/2})).$$

The sign \pm comes from matching the orientation $[\nu_{(jk,v_r)} \wedge \nu_{(0j,v_r)} \wedge \nu_{(0k,v_r)}]$ against that of vol_g , which agrees with the sign in Morse category. This completes the proof of our Main Theorem.

4. WKB FOR GREEN OPERATOR

In lemma 16, we have a rough estimate for the twisted Green operator by a Morse function f , or the homotopy operator $H_f = d_f^* G_f (I - P_f)$, with an error of order $\mathcal{O}(e^{\lambda\epsilon})$. In a neighborhood of gradient flow line segment of f , we are going to improve this results to estimate with error $\mathcal{O}(\lambda^{-\infty})$. This is done by the WKB method for *inhomogeneous* Laplace equation (3.10).

We study the local behavior of the homotopy operator H_f along a normalized gradient flow line segment

$$\begin{aligned} \gamma : [0, T] &\longrightarrow M, \\ \frac{d\gamma}{dt} &= \frac{\nabla f}{|\nabla f|_f}, \\ \gamma(0) = x_S &\quad , \quad \gamma(T) = x_E, \end{aligned}$$

as shown in the following figure. We consider the relation

$$\zeta_E = H_f(\chi_S \zeta_S).$$

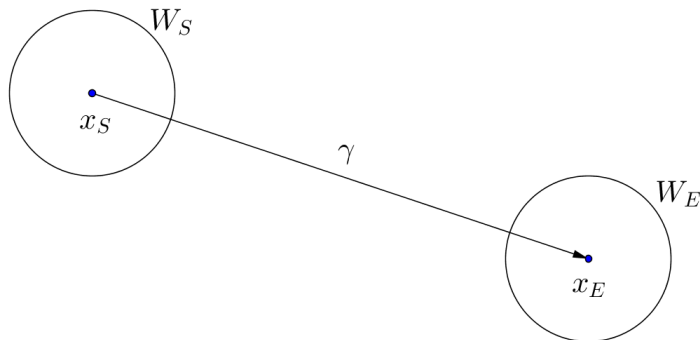
Suppose we have a WKB approximation of ζ_S in W_S of the form

$$(4.1) \quad \zeta_S \sim e^{-\lambda\psi_S} (\omega_{S,0} + \omega_{S,1}\lambda^{-1/2} + \omega_{S,2}\lambda^{-1} + \dots),$$

we need to establish a similar expression

$$(4.2) \quad \zeta_E \sim \lambda^{-1/2} e^{-\lambda\psi_E} (\omega_{E,0} + \omega_{E,1}\lambda^{-1/2} + \dots),$$

of ζ_E in a some open neighborhood W_E of x_E .



The key step is to determine ψ_E , which is given in the following subsection. As a first trial, we consider the function

$$\tilde{\psi}_E(x) := \inf_{y \in W_S} \{\psi_S(y) + \rho_f(y, x)\},$$

since $e^{-\lambda\tilde{\psi}_E}$ is the expected exponential decay suggested by the resolvent estimate in lemma 16.

$\tilde{\psi}_E$ is not the correct function since it is singular along a hypersurface U_S through x_S , and cannot be used for the iteration process as we keep on differentiating it.

In the coming section 4.1, we will solve the minimal configuration in variational problem associated to $\inf_{y \in W_S} (\psi_S(y) + \rho_f(y, x))$ and find that the point y is forced to lie on U_S , with a unique geodesic joining to x which realizes $\rho(y, x)$, for those x closed enough to x_E . These family of geodesics $\{\gamma_y\}_{y \in U_S}$ will give a foliation of a neighborhood of γ . Therefore we can use $\psi_E(\gamma_y(t)) = \psi_S(y) + t$ as an extension of $\tilde{\psi}_E$ across U_S . We then use ψ_E in the iteration similar to classical WKB approximation to obtain the above expansion 4.2.

4.1. The phase function ψ_E . We apply variational method to study the function $\psi(x)$. Fixing $x \in M$, we take $\alpha(\epsilon, t) := \alpha_\epsilon(t) : (-\epsilon_0, \epsilon_0) \times [0, 1] \rightarrow M \setminus \text{Crit}(f)$ such that $\alpha_\epsilon(1) \equiv x$ for all ϵ . To minimize the functional

$$L(\epsilon) = \psi_S(\alpha_\epsilon(0)) + \int_0^1 |\partial_t \alpha_\epsilon|_f dt,$$

we take derivatives and get

Lemma 34. (*First variation formula*)

$$(4.3) \quad \frac{dL}{d\epsilon} = \langle \tilde{\nabla} \psi_S(\alpha_\epsilon), \partial_\epsilon \alpha_\epsilon \rangle_f |_{t=0} + \int_0^1 \frac{1}{|\partial_t \alpha|_f} \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_t \alpha \rangle_f dt.$$

Here $\tilde{\nabla}$ is the Levi-Civita connection corresponding to the Agmon metric $\langle \cdot, \cdot \rangle_f$ in definition 14.

If we assume α_0 is a geodesic (with respect to twisted metric $|df|^2 g$) with $|\alpha'_0(t)|_f \equiv \text{const.}$, the Euler-Lagrange equation for $L(\epsilon)$ is

$$\left. \frac{dL}{d\epsilon} \right|_{\epsilon=0} = \langle \tilde{\nabla} \psi_S(\alpha_0) - \frac{\alpha'_0}{|\alpha'_0|_f}, \partial_\epsilon \alpha \rangle_f \Big|_{\epsilon=0} = 0.$$

Since $\partial_\epsilon \alpha(0, 0)$ can be chosen arbitrarily, we have

$$(4.4) \quad \left(\tilde{\nabla} \psi_S(\alpha_0) - \frac{\alpha'_0}{|\alpha'_0|_f} \right) \Big|_{t=0} = 0.$$

Such an equation restricts the possibility of the starting point $\alpha_0(0)$, namely, we have

$$|\nabla \psi_S| = |\nabla f|,$$

at $\alpha_0(0)$, or equivalently, $|\tilde{\nabla} \psi_S|_f = 1$.

Definition 35.

$$U_S := \{|\tilde{\nabla} \psi_S|_f = 1\} \cap W_S.$$

If α_0 is a local extrema of L with $\alpha_0(0) \in W_S$, it forces $\alpha_0(0) \in U_S$. To obtain nice properties of U_S , we are going to assume the following throughout the whole section.

Assumption 36. We assume $g_S : W_S \rightarrow \mathbb{R}_{\geq 0}$, defined by $g_S = \psi_S - f$, be a smooth Bott-Morse function in W_S with zero set V_S such that $v_S \in V_S$.

Lemma 37. U_S is a hypersurface containing V_S if $\dim(V_S) < \dim(M)$ (we shrink W_S if necessary). Otherwise, it is simply $V_S = W_S$.

Proof. Since we have $\nabla g_S \equiv 0$ on V_S and hence $|\nabla \psi_S| = |\nabla f|$ on V_S . This gives $V_S \subset U_S$. Moreover, U_S can be defined by the equation

$$\Phi(x) = 2\langle \nabla f(x), \nabla g_S(x) \rangle + |\nabla g_S(x)|^2 = 0.$$

If $v \in T_p M$ where $p \in V_S$, then we have

$$\begin{aligned} \nabla_v \Phi(p) &= 2\nabla^2 f(p)(v, \nabla g_S(p)) + 2\nabla^2 g_S(p)(v, \nabla f(p)) + 2\nabla^2 g_S(p)(v, \nabla g_S(p)) \\ &= 2\nabla^2 g_S(p)(v, \nabla f(p)), \end{aligned}$$

since $\nabla g_S(p) = 0$ on V_S . As g_S is a Bott-Morse function with critical set V_S , $\nabla^2 g_S(p)$ is nondegenerate when restricted to the orthogonal complement of $T_p V_S$ in $T_p M$. Therefore, there exists v such that $\nabla_v \Phi(p) \neq 0$. \square

We are going to parametrize a neighborhood of γ by $U_S \times (-\delta, T + \delta)$ such that $U_S \times \{0\} \rightarrow M$ is the embedding and $\nu_S \times [0, T]$ is γ . ψ_E is defined to be the coordinate function corresponding to the last variable.

Motivated from equation (4.4), we define a transversal vector field on U_S which is the initial tangent vector for minimizer of L .

Definition 38. We define a vector field $\nu \in \Gamma(U_S, T_M)$ transversal to U_S (shrinking W_S if necessary) by

$$(4.5) \quad \nu := \frac{\nabla \psi_S}{|\nabla \psi_S|_f} = \tilde{\nabla} \psi_S.$$

Notice that $\nu = \frac{\nabla f}{|\nabla f|_f} = \tilde{\nabla} f$ on V_S .

It follows from the Euler-Lagrange equation (4.4) that any local extrema α of L will have $\alpha(0) \in U_S$ and $\alpha'(0) = \nu(\alpha(0))$. For convenience, we assume that γ is extended to gradient flow line defined on (a, b) containing $[0, T]$.

Definition 39. We define a map

$$(4.6) \quad \sigma : W_0 \subset U_S \times (a, b) \rightarrow M,$$

given by

$$\sigma(u, t) = \text{e}\tilde{\text{x}}p_u(t\nu),$$

where W_0 is a suitable neighborhood of γ where the exponential map $\text{e}\tilde{\text{x}}p$ with respect to the Agmon Riemannian metric is well defined.

Lemma 40. Restricting to a small open neighborhood of $\{x_S\} \times [0, b)$, σ is a diffeomorphism onto its image containing γ .

This is achieved by showing there is no "conjugate point" along $\gamma(t)$ for certain type of geodesic family, and using the fact that γ being a global minimizer of functional L . Lemma 40 enable us to construct ψ_E needed for WKB approximation in a neighborhood $U_S \times (-\delta, b)$ (take a small enough δ and shrink U_S if necessary) of γ where σ is a diffeomorphism.

Definition 41. We define ψ_E on $\sigma(U_S \times (-\delta, b))$ by

$$(4.7) \quad \psi_E(\sigma(u, t)) = \psi_S(u) + t,$$

for $(u, t) \in U_S \times (-\delta, b)$.

4.2. Proof of lemma 40. We begin with the second variation formula of L . We assume $\alpha : (-\epsilon_0, \epsilon_0) \times [0, l] \rightarrow M$ is a family such that $\alpha_0(t)$ is arc-length parametrized geodesic (with respect to twisted metric $|df|^2g$) satisfying the condition

$$\left(\tilde{\nabla} \psi_S(\alpha) - \frac{\partial_t \alpha}{|\partial_t \alpha|_f} \right) \Big|_{\epsilon=0}^{t=0} = 0.$$

From the first variation formula

$$\frac{dL}{d\epsilon} = \langle \tilde{\nabla} \psi_S(\alpha_\epsilon(0)), \partial_\epsilon \alpha_\epsilon(0) \rangle_f + \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \frac{\partial_t \alpha}{|\partial_t \alpha|_f} \rangle_f dt,$$

we obtain

Lemma 42. (Second variation formula)

$$(4.8) \quad \begin{aligned} \frac{d^2 L}{d\epsilon^2} \Big|_{\epsilon=0} &= \langle \tilde{\nabla}_\epsilon \tilde{\nabla} \psi_S, \partial_\epsilon \alpha \rangle_f \Big|_{t=0} + \langle \tilde{\nabla} \psi_S, \tilde{\nabla}_\epsilon \partial_\epsilon \alpha \rangle_f \Big|_{t=0} + \langle \tilde{\nabla}_\epsilon \partial_\epsilon \alpha, \partial_t \alpha \rangle_f \Big|_0^l \\ &+ \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \tilde{\nabla}_t \partial_\epsilon \alpha \rangle_f + \langle \tilde{R}(\partial_\epsilon \alpha, \partial_t \alpha) \partial_\epsilon \alpha, \partial_t \alpha \rangle_f - \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_t \alpha \rangle_f^2 dt, \end{aligned}$$

where the right hand side is evaluated at $\epsilon = 0$. Here \tilde{R} is the curvature tensor with respect to $\langle \cdot, \cdot \rangle_f$.

If we further impose the condition that $\partial_\epsilon \alpha(\epsilon, l) \equiv 0$ for all ϵ , we have

$$(4.9) \quad \begin{aligned} \frac{d^2 L}{d\epsilon^2} \Big|_{\epsilon=0} &= \langle \tilde{\nabla}_\epsilon \tilde{\nabla} \psi_S, \partial_\epsilon \alpha \rangle_f \Big|_{t=0} \\ &+ \int_0^l \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \tilde{\nabla}_t \partial_\epsilon \alpha \rangle_f + \langle \tilde{R}(\partial_\epsilon \alpha, \partial_t \alpha) \partial_\epsilon \alpha, \partial_t \alpha \rangle_f - \langle \tilde{\nabla}_t \partial_\epsilon \alpha, \partial_t \alpha \rangle_f^2 ds. \end{aligned}$$

Therefore we consider the bilinear form I associated to the above quadratic form.

Definition 43.

$$(4.10) \quad I(X, Y) = \tilde{\nabla}^2 \psi_S(X, Y)(0) + \int_0^l \langle \tilde{R}(X, \partial_t \alpha) Y, \partial_t \alpha \rangle_f dt \\ + \int_0^l \langle \tilde{\nabla}_t X - \langle \tilde{\nabla}_t X, \partial_t \alpha \rangle_f \partial_t \alpha, \tilde{\nabla}_t Y - \langle \tilde{\nabla}_t Y, \partial_t \alpha \rangle_f \partial_t \alpha \rangle_f dt,$$

for vector fields X, Y on α_0 , $X(l) = 0 = Y(l)$.

For any such vector field X , we can find a family of curve α_ϵ satisfying the assumptions $\partial_\epsilon \alpha(\epsilon, l) \equiv 0$, with $\partial_\epsilon \alpha = X$. The same holds for piecewise smooth vector field with the same initial condition.

Proof of lemma 40. The proof depends on the fact that γ is an absolute minimum of L among the set of path α 's in $M \setminus Crit(f)$ with $\alpha(0) \in W_S$, and contradiction will occur if differential is singular along $\{x_S\} \times [0, b)$. The argument is a modification of the standard argument of geodesic beyond conjugate point is never length minimizing.

First, we notice that $d\sigma_{(x_S, t_0)}(0, \frac{\partial}{\partial t}) = \gamma'(t)$ for a fixed $t_0 \in [0, b)$. We have to compute $d\sigma_{(x_S, t_0)}(v, 0)$ for arbitrary $(v, 0) \in T_{(x_S, t_0)}(W_0)$. We claim that $\partial_\epsilon \alpha(0, t_0)$ can never be parallel to $\partial_t \alpha(0, t_0)$ for $v \neq 0$.

Taking a curve $\beta(\epsilon)$ in U_S with $\beta(0) = x_S$ and $\beta'(0) = v$, we can construct a family of arc-length parametrized geodesic α_ϵ by taking exponential map

$$\alpha(\epsilon, t) = \text{exp}_{\beta(\epsilon)}(t\nu).$$

We have $\partial_\epsilon \alpha(0, t) = d\sigma_{(x_S, t)}(v, 0)$ with $\partial_\epsilon \alpha$ being a Jacobi field on α_0 . Suppose the contrary that $\partial_\epsilon \alpha(0, t_0) = c\partial_t \alpha(0, t_0)$ for some constant c , then we must have $\tilde{\nabla}_t \partial_\epsilon \alpha(0, t_0) \neq 0$, otherwise we must have $\partial_\epsilon \alpha \equiv c\partial_t \alpha$ which contradicts $v \neq 0$.

We argue that we can construct a path from U_S to the point $\sigma(v_S, t_0 + \delta)$ which gives a smaller value of L comparing to the gradient flow line γ from v_S to the point $\sigma(v_S, t_0 + \delta)$. We will denote $l = t_0 + \delta$ to fit our previous discussion.

We construct the path by defining a variational vector field Y_η on γ , depending on a small $\eta > 0$ to be fixed. We take a vector field $Z(t)$ such that $Z(0) = 0$, $Z(l) = 0$, $\langle Z, \partial_t \rangle_f \equiv 0$ on $[t_0, l]$ and $Z(t_0) = -\tilde{\nabla}_t \partial_\epsilon(0, t_0)$. We define a piecewise smooth vector field

$$Y_\eta(t) := \begin{cases} \partial_\epsilon \alpha + \eta Z & \text{if } t \in [0, t_0], \\ \chi \langle \partial_\epsilon \alpha, \partial_t \alpha \rangle_f \partial_t \alpha + \eta Z & \text{if } t \in [t_0, l], \end{cases}$$

where χ is a cut off function in $[t_0, l]$ with $\chi(t_0) = 1$ and $\chi = 0$ in a neighborhood of l . Notice that $\tilde{\nabla}_t \langle \partial_\epsilon \alpha, \partial_t \alpha \rangle_f = 0$ from the fact that $|\partial_t \alpha|_f \equiv 1$.

A direct computation shows

$$I(Y_\eta, Y_\eta) = -2\eta |\tilde{\nabla}_t \partial_\epsilon \alpha(0, t_0)|_f^2 + 2\eta^2 I(Z, Z).$$

We have $I(Y_\eta, Y_\eta) < 0$ for η small enough.

By taking the family of curves β_ϵ corresponding to Y_η , we obtain

$$\left. \frac{d^2 L_\beta}{d\epsilon^2} \right|_{\epsilon=0} < 0,$$

where $L_\beta(\epsilon) = L(\beta(\epsilon))$. For small enough ϵ , $\beta_\epsilon(t)$ will be a curve from U_S to $\sigma(0, l)$ which gives a smaller value of L comparing to $\beta_0 = \gamma$. This is impossible because we have

$$L_\beta(\epsilon) \geq f(\sigma(0, l))$$

and the lower bound is attained at γ .

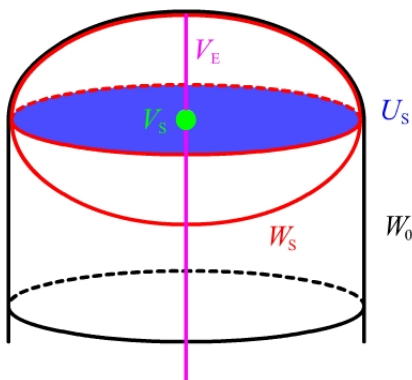
As a conclusion, we can show that σ gives a local diffeomorphism onto its image by shrinking W_0 if necessary. Therefore it is injective in a contractible neighborhood of the gradient flow line γ . \square

Under the identification σ , we use the coordinate u_1, \dots, u_{n-1} for U_S and use (u_1, \dots, u_{n-1}, t) , or simply (u, t) , as coordinate for image of W_0 under σ . By shrinking W_0 if necessary, we assume that W_0 is a coordinate chart through the map σ . This justifies the definition 41 of ψ_E as a smooth function on $\sigma(W_0) \subset M$.

4.3. Properties of ψ_E . We are going to study the first and second derivatives of ψ_E which is necessary for having a WKB approximation for the equation (3.10). We define

$$V_E := \sigma((V_S \times (-\delta, b)) \cap W_0) \subset \sigma(W_0)$$

as shown in the following picture.



Lemma 44. *In W_0 , we have*

$$\tilde{\nabla}\psi_E = d\sigma_* \frac{\partial}{\partial t}.$$

In particular, we have $\nabla\psi_E = \nabla f$ on V_E and $|\nabla\psi_E| = |\nabla f|$.

Proof. We first consider the subset $t \in [0, b)$ in W_0 . We let $\beta(\epsilon)$ be a curve in U_S such that $\beta(0) = u$ and

$$\alpha(\epsilon, t) = \exp_{\beta(\epsilon)}(t\nu) = \sigma(\beta(\epsilon), t).$$

Notice that we have $\psi_E(\alpha_\epsilon(t)) = L(\alpha_\epsilon|_{[0,t]})$. Applying the first variation formula, we have

$$\begin{aligned} \langle \tilde{\nabla}\psi_E(\alpha_\epsilon(t)), \partial_\epsilon \alpha_\epsilon(t) \rangle_f \Big|_{\epsilon=0} &= \frac{dL}{d\epsilon} \Big|_{\epsilon=0} \\ &= \langle \partial_t \alpha(0, t), \partial_\epsilon \alpha(0, t) \rangle_f. \end{aligned}$$

As $\partial_\epsilon \alpha(0, t)$ can be chosen arbitrarily, we get

$$\tilde{\nabla}\psi_E(u, t) = \partial_t \alpha(0, t) = d\sigma_{(u,t)} \frac{\partial}{\partial t}.$$

The same argument works for $t \in (-\delta, 0]$ by taking

$$L(\alpha_\epsilon|_{[t,0]}) = \psi_S(\alpha_\epsilon(0)) + \int_t^0 |\partial_t \alpha_\epsilon|_f dt.$$

Furthermore, we have $|\tilde{\nabla}\psi_E(u, t)|_f^2 = |d\sigma_{(u,t)} \frac{\partial}{\partial t}|_f^2 = 1$ which gives $|\nabla\psi_E(u, t)| = |\nabla f|$. Finally, as we know $\nabla\psi_S = \nabla f$ on V_S and flow lines of ∇f are geodesic after reparametrizations, we get $\nabla\psi_E = \nabla f$ on V_E . \square

We now consider second derivatives of $g_E = \psi_E - f$.

Lemma 45. *By choosing a small enough W_0 , we have*

- (1) $g_E \geq 0$ and
- (2) g_E is a Bott-Morse function with critical set $V_E = \{g_E = 0\}$.

Proof. The previous lemma implies that $\nabla g_E = 0$ on V_E . We are going to show $\nabla^2 g_E$ is positive definite in the normal bundle of V_E . Fixing any $t \in [0, b)$, we consider the submanifold $U_t = \sigma(U_S \times \{t\} \cap W_0)$. There is an isomorphism between the normal bundle of $V_t = \sigma(V_S \times \{t\} \cap W_0)$ in U_t and normal bundle of V_E in W_0 . Therefore we restrict g_E to U_t and consider its Hessian.

We abuse the notations and write $u : W_0 \rightarrow U_S$ as the projection map. We take $h = g_E - g_S \circ u$. We have $h \geq 0$ on U_t by definition of ψ_E and $\nabla h = 0 = h$ on V_t . Therefore we have h is positive semi-definite on the normal bundle of V_t in U_t . Moreover, we have $\nabla^2(g_S \circ u) = (\nabla^2 g_S) \circ u$ on V_S being positive definite in the normal bundle.

By choosing δ small enough, we can assume that $\nabla^2 g_E > 0$ along V_E and hence the result follows. \square

Next, we consider the second order derivatives for $\Psi = \psi_E - \psi_S = g_E - g_S$ defined on W_S .

Lemma 46. *By choosing small enough neighborhood W_S of v_S if necessary, we have*

- (1) $\Psi \leq 0$ on W_S and
- (2) Ψ is a Bott-Morse function with critical set $U_S = \{\Psi = 0\} \subset W_S$.

Proof. We first have $\nabla\Psi = 0$ on U_S because $\nabla\psi_E = \nabla\psi_S$ on U_S . If we consider $\nabla^2\Psi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ on V_S , then we have $\nabla^2g_E(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$ and $\nabla^2g_S(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) > 0$. Therefore, there exists an neighborhood U of V_S in U_S so that

$$\nabla^2\Psi(x)(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) < 0$$

for all $x \in U$. Choosing W_S small enough will achieve the desired result. \square

Remark 47. *We can extend the function Ψ from W_S to W_0 to be a non-negative function with critical set U_S which is also an absolute maximum. This is for our convenience in later arguments.*

4.4. The WKB iteration. After knowing these properties of ψ_E , we will describe the iteration procedure to define $\omega_{E,i}$ inductively.

First, by lemma 44, we have $|df|^2 = |d\psi_E|^2$ and hence the expansion

$$\begin{aligned} e^{\lambda\psi_E} \Delta_f e^{-\lambda\psi_E} &= \Delta + \lambda M_f + \lambda(\mathcal{L}_{\nabla\psi_E} - \mathcal{L}_{\nabla\psi_E}^*) \\ &= \Delta + \lambda(2\mathcal{L}_{\nabla\psi_E} - M_{g_E}), \end{aligned}$$

where $M_{g_E} = \mathcal{L}_{\nabla g_E} + \mathcal{L}_{\nabla g_E}^*$. Following [11], we let

$$\mathcal{T} = 2\mathcal{L}_{\nabla\psi_E} - M_{g_E},$$

and consider the following equation

$$(\Delta + \mathcal{T}\lambda)(\mu_0(\lambda) + \mu_1(\lambda) + \dots) = e^{\lambda\Psi} \nu,$$

order by order in λ where $\mu_i(\lambda)$ is a function (depending on λ). We often write μ_i to simplify our notations. The first equation to be solved is

$$(4.11) \quad \lambda\mathcal{T}\mu_0(\lambda) = e^{\lambda\Psi} \nu.$$

In order to solve the above equation involving $\mathcal{L}_{\nabla\psi_E}$, we need a map τ describing the flow of $\nabla\psi_E$. It is given by renormalising σ such that $d\tau_*(\frac{\partial}{\partial t}) = \nabla\psi_E$ and is of the form

$$(4.12) \quad \tau : W \subset U_S \times (-\infty, +\infty) \rightarrow M,$$

with the same image as σ . We can also assume that $W \cap \{u\} \times \mathbb{R}$ is a connected open interval.

Notations 48. *We use (u_1, \dots, u_{n-1}, t) as coordinate of $\tau(W)$ from now on. For simplicity, we also let $u_n = t$ and $\dot{u} = (u_1, \dots, u_{n-1})$.*

For the iteration process, we restrict our attention to

$$\Omega_0^*(W) = \{\beta \in \Omega^*(W) \mid \overline{\text{supp}(\beta)} \cap (U_S \times (-\infty, t_0]) \text{ compact for all } t_0\},$$

for the definition of the following integral operator.

Definition 49. We let $I : \Omega_0^*(W) \rightarrow \Omega_0^*(W)$ given by

$$(4.13) \quad I(\phi) := \int_{-\infty}^0 e^{\int_s^0 \frac{1}{2} \tau_\epsilon^*(M_{g_E}) d\epsilon} \tau_s^*(\phi) ds,$$

where $\tau_s(u, t) = \tau(u, t + s)$ is the flow of $\nabla \psi_E$ for time s .

To solve (4.11), we put

$$(4.14) \quad \mu_0 = \frac{1}{2\lambda} I(e^{\lambda\Psi} \nu).$$

Then it can be checked that μ_0 is the solution to (4.11). The second equation to be solved is

$$(4.15) \quad \lambda \mathcal{T} \mu_1 = -\Delta \mu_0.$$

Again, we put

$$\mu_1 = -\frac{1}{2\lambda} I(\Delta \mu_0).$$

In general, we have the transport equation for $l \geq 0$

$$(4.16) \quad \mathcal{T} \mu_{l+1} = -\lambda^{-1} \Delta \mu_l.$$

This gives

$$(4.17) \quad \mu_{l+1} = -\frac{1}{2\lambda} I(\Delta \mu_l).$$

as solutions in W .

4.5. Estimate of the WKB iteration. In this section, we are going to obtain norm estimates for μ_l 's. We consider terms appearing in the iteration which are essentially of the form

$$(4.18) \quad I^j \left(e^{\lambda\Psi} \left(\prod_{\alpha} \nabla_{\alpha} \Psi \right) \beta \right)$$

with $j \geq 0$ and $\beta \in \Omega_0^*(W)$, where I^j is the composition of I for j times. Here each $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index such that

$$\nabla_{\alpha} \Psi = \nabla_{\frac{\partial}{\partial u_1}}^{\alpha_1} \dots \nabla_{\frac{\partial}{\partial u_{n-1}}}^{\alpha_{n-1}} \nabla_{\frac{\partial}{\partial u_n}}^{\alpha_n} \Psi.$$

With

$$m(\alpha) := \max\{0, 2 - \alpha_n\},$$

we have

$$(4.19) \quad \nabla^j \left(\prod_{\alpha} \nabla_{\alpha} \Psi \right) |_{U_S} \equiv 0,$$

for $j \leq \sum_{\alpha} m(\alpha)$ from lemma 46.

Remark 50. *Different choices of order of taking differentiation in definition of ∇_α will result in a difference involving the curvature of (M, g) , however, the order of vanishing in equation (4.19) remains unchanged and hence the following estimates hold for any such choice.*

The counting of vanishing order along U_S is needed for applying the following semi-classical approximation lemma 51, appearing in [3].

Lemma 51. *Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 with coordinates x_1, \dots, x_n . Let $\varphi : U \rightarrow \mathbb{R}_{\geq 0}$ be a Morse function with unique minimum $\varphi(0) = 0$ in U . Let $\tilde{x}_1, \dots, \tilde{x}_n$ be a Morse coordinates near 0 such that*

$$\varphi(x) = \frac{1}{2}(\tilde{x}_1^2 + \dots + \tilde{x}_n^2).$$

For every compact subset $K \subset U$, there exists a constant $C = C_{K,N}$ such that for every $u \in C^\infty(U)$ with $\text{supp}(u) \subset K$, we have

$$(4.20) \quad \left| \int_K e^{-\lambda\varphi(x)} u - \left(\frac{\lambda}{2\pi}\right)^{n/2} \left(\sum_{k=0}^{N-1} \frac{\lambda^{-k}}{2^k k!} \tilde{\Delta}^k \left(\frac{u}{\mathfrak{S}}\right)(0) \right) \right| \leq C \lambda^{-n/2-N} \sum_{|\alpha| \leq 2N+n+1} \sup |\partial^\alpha u|,$$

where

$$\tilde{\Delta} = \sum \frac{\partial^2}{\partial \tilde{x}_j^2}, \quad \mathfrak{S} = \pm \det \left(\frac{d\tilde{x}}{dx} \right),$$

and $\mathfrak{S}(0) = (\det \nabla^2 \varphi(0))^{1/2}$.

In particular, if u vanishes at 0 up to order L , then we can take $N = \lceil L/2 \rceil$ and get

$$\left| \int_K e^{-\lambda\varphi(x)} u \right| \leq C \lambda^{-n/2 - \lceil L/2 \rceil}.$$

From the above, we observe the following lemma.

Lemma 52. *Let $L_{\tilde{u}}$ be the line interval along t direction with fixed \tilde{u} coordinates, we have the norm estimate*

$$\left(\int_{L_{\tilde{u}}} |\nabla_\alpha (e^{\lambda\Psi})|^{2k} \right)^{\frac{1}{2k}} \leq C_{\alpha,k} \lambda^{\frac{\alpha_n}{2} - \frac{1}{2k+1}}$$

for any multi-index α and $k \in \mathbb{Z}_{\geq 0}$.

Motivated by the above lemma, we consider a filtration

$$\dots \subset F^{-s} \subset \dots \subset F^{-1} \subset F^0 \subset F^1 \subset F^2 \subset \dots \subset F^s \subset \dots \subset \Omega_0^*(W)$$

of the space of differential forms on $\Omega_0^*(W)$ which is defined as follows.

Definition 53. $\phi \in \Omega_0^*(W)$ is in F^s if for any compact subset $K \subset W$ and integers $j, k \in \mathbb{Z}_+$, we have

$$\|\nabla_\alpha \phi\|_{L^{2k}(K \cap L)} \leq C_{\alpha,k,K} \lambda^{\frac{\alpha_n + s}{2} - \frac{1}{2k+1}},$$

for any line $L = L_{\hat{u}}$.

The Lemma 52 simply means $e^{\lambda\Psi} \in F^0$.

Proposition 54. *We have $\nabla F^s \subset F^{s+1}$ and $F^s \cdot F^r \subset F^{r+s}$, where \cdot denotes wedge product of forms.*

Proof. The first property is trivial. For the relation $F^s \cdot F^r \subset F^{r+s}$, we fix $j \in \mathbb{Z}_+$ and a compact subset K . For $\phi \in F^r$ and $\psi \in F^s$, we first observe that

$$\nabla_\alpha(\phi \wedge \psi) = \sum_{\beta+\theta=\alpha} (\nabla_\beta\phi) \wedge (\nabla_\theta\psi).$$

Then the Hölder inequality implies that

$$\begin{aligned} \|(\nabla_\beta\phi) \wedge (\nabla_\theta\psi)\|_{L^{2k}(K \cap L)} &\leq C \|\nabla_\beta\phi\|_{L^{2k+1}(K \cap L)} \|\nabla_\theta\psi\|_{L^{2k+1}(K \cap L)} \\ &\leq C \lambda^{\frac{\beta_n+s}{2} - \frac{1}{2k+2}} \cdot \lambda^{\frac{\theta_n+r}{2} - \frac{1}{2k+2}} \\ &\leq C \lambda^{\frac{\alpha_n+r+s}{2} - \frac{1}{2k+1}} \end{aligned}$$

and the result follows. \square

Lemma 55. *For $\phi \in F^s$, we have*

$$\begin{aligned} I(\phi) &\in F^s, \\ \Delta I(\phi) &\in F^{s+1}. \end{aligned}$$

Proof. To simplify the notations, we only prove the statement for functions as we can fix a basis (independent of λ) for differential forms in W , and estimate the coefficient functions. The Christoffel symbols appearing in differentiating the basis will be independent of λ and not affecting the following estimates. For the same reason, let us simply pick a flat metric in u_i 's coordinates for simplicity. In that case, we can write $\Delta = \sum_i \nabla_i^2$.

We first consider the operator ∇_n^2 , and we will have $2\nabla_n I(\phi) = M_{g_E} \phi$ where M_{g_E} is acting as scalar multiplication by function. Therefore we have

$$\|\nabla_\alpha(\nabla_n^2 I(\phi))\|_{L^{2k}(K \cap L)} = \|\nabla_\alpha \nabla_n(M_{g_E} \phi)\|_{L^{2k}(K \cap L)} \leq C_{\alpha,k,K} \lambda^{\frac{\alpha_n+s+1}{2} - \frac{1}{2k+1}}.$$

This will imply $(\nabla_n^2 I(\phi)) \in F^{s+1}$.

Next, we consider the operator ∇_i^2 for $i < n$. Fixing a multi-index α and using the result $I(\phi) \in F^s$, we have

$$\|\nabla_\alpha \nabla_i^2(I\phi)\|_{L^{2k}(K \cap L)} \leq C_{\alpha,k,K} \lambda^{\frac{\alpha_n+s}{2} - \frac{1}{2k+1}},$$

which gives $\nabla_i^2(I\phi) \in F^s \subset F^{s+1}$.

It remains to show that $I(\phi) \in F^s$ which requires estimates of the term $\nabla_\alpha I(\phi)$. There are two cases to be considered, the first case is $\alpha_n \neq 0$. In case $\alpha_n \neq 0$, we can cancel the integral operator with one of the ∇_n , which gives

$$\|\nabla_\alpha I(\phi)\|_{L^{2k}(K \cap L)} = \frac{1}{2} \|\nabla_{\hat{\alpha}}(M_{g_E} \phi)\|_{L^{2k}(K \cap L)} \leq C_{\alpha,k,K} \lambda^{\frac{\alpha_n+s-1}{2} - \frac{1}{2k+1}},$$

where $\hat{\alpha}$ refers to the multi-index by letting $\hat{\alpha}_n = \alpha_n - 1$.

In the second case we assume that $\alpha_n = 0$, and therefore we can commute all the ∇_α with the integral operator I . We let $Q(\dot{u}, t, s) = e^{\int_s^0 \frac{1}{2} \tau \varepsilon^* (M_{g_E}) d\tau}$ as a function and write $I(\phi)(\dot{u}, t) = \int_{-\infty}^0 Q(\dot{u}, t, s) \phi(\dot{u}, t + s) ds$. Therefore we have

$$\nabla_\alpha(I(\phi)) = \sum_{\beta+\theta=\alpha} \int_{-\infty}^0 \nabla_\beta(Q(\dot{u}, t, s)) \nabla_\theta \phi(\dot{u}, t + s) ds,$$

and

$$\|\nabla_\alpha(I(\phi))\|_{L^{2k}(K \cap L)} \leq C_{\alpha,k,K} \sum_{\theta \subset \alpha} \left| \int_{-\infty}^0 \nabla_\theta \phi(\dot{u}, t + s) ds \right| \leq C_{\alpha,k,K} \lambda^{\frac{\alpha_n+s}{2} - \frac{1}{2}}.$$

Combining the two cases will give $I(\phi) \in F^s$. \square

Remark 56. *Using the above lemma, we can show that the $\mu_l(\lambda)$'s appearing in the iteration equation (4.17) will satisfy $\mu_l(\lambda) \in F^{-l-2}$. In particular, we can get an explicit estimate as*

$$\|\nabla^j \mu_l(\lambda)\|_{L^2(K)} \leq C_{j,K} \lambda^{\frac{j-l-2}{2} - \frac{1}{4}},$$

for all j and compact subset $K \subset W$.

4.6. A priori estimate. We make use of the WKB iteration to construct the WKB expansion and prove that it does give a desired approximate to the solution in the rest of section 4. Before that, we obtain an a priori estimate for the solution in this subsection.

We consider the equation

$$(4.21) \quad \Delta_f \zeta_E = (I - P_f) d_f^*(\chi_S \zeta_S)$$

in W , where $\zeta_S \in \Omega^*(W_S)$ is the input form depending on λ and $\chi_S \in C_c^\infty(W_S)$ is some cut off function to be chosen later. We assume ζ_S has a WKB approximation on W_S of the form

$$(4.22) \quad \zeta_S \sim e^{-\lambda \psi_S} (\omega_{S,0} + \omega_{S,1} \lambda^{-1/2} + \omega_{S,2} \lambda^{-1} + \dots),$$

where $\omega_{S,i} \in \Omega^*(W_S)$ and $\psi_S = f + g_S$. It is an approximation in the sense that

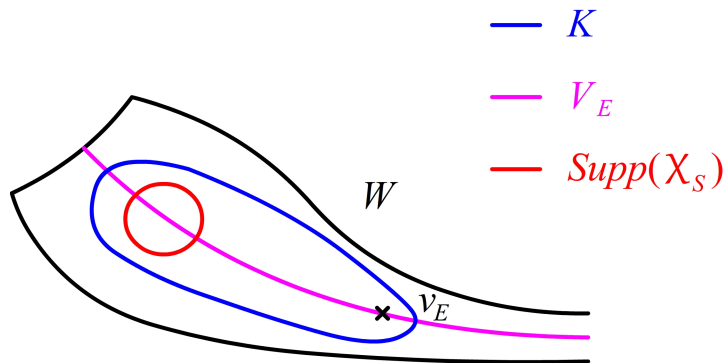
$$(4.23) \quad \|e^{\lambda \psi_S} \zeta_S - \left(\sum_{i=0}^N \omega_{S,i} \lambda^{-i/2} \right)\|_{L^\infty(W_S)}^2 \leq C_N \lambda^{-N-1},$$

for N large enough, where C_N is a constant depending on N . We also require similar norm estimates for its derivatives

$$(4.24) \quad \|e^{\lambda \psi_S} \nabla^j (\zeta_S - e^{-\lambda \psi_S} \left(\sum_{i=0}^N \omega_{S,i} \lambda^{-i/2} \right))\|_{L^\infty(W_S)}^2 \leq C_{j,N} \lambda^{-N-1+2j},$$

with $C_{j,N}$ depending on j, N .

We want to get a similar expansion for ζ_E , using the iteration defined in the section 4.4. We consider any small enough compact neighborhood $K \subset W$ of the flow line γ with $\chi \equiv 1$ on K . χ_S is chosen so that $\text{supp}(\chi_S) \subset K$. The following figure illustrates the situation.



For small enough K , we have an a priori estimate of ζ_E in K , the technique is similar to the method used for eigenform in [9].

Lemma 57. *For small enough $\text{supp}(\chi_S)$ and K , and any $j \in \mathbb{Z}_+$, there exists $\lambda_{j,0} > 0$ such that for any $\lambda > \lambda_{j,0}$, we have*

$$(4.25) \quad \|e^{\lambda\psi_E} \nabla^j \zeta_E\|_{L^\infty(K)}^2 \leq C_j \lambda^{N_j},$$

where N_j is an positive integer depending on j .

In order to prove the above lemma, we need to know certain special properties about χ and our chosen compact set K . Letting $\tilde{\psi} := \inf_{y \in \text{supp}(\chi_S)} \{\psi_S + \rho_f(y, x)\}$, we have the following lemma.

Lemma 58. *There exists $\epsilon > 0$ such that for all K small enough, we have*

$$(4.26) \quad \tilde{\psi}(x) + \rho(y, x) \geq \psi_E(y) + \epsilon,$$

for all $y \in K$ and $x \in \text{supp}(\nabla\chi)$.

Proof. Using the fact that $\psi_E = f$ on V_E and choosing K small enough such that $|\psi_E - f| \leq \epsilon$ on K , we can simply prove

$$\tilde{\psi}(x) + \rho(y, x) \geq f(y) + \epsilon,$$

by choosing small enough K and ϵ . From properties of Agmon distance ρ , we have

$$\tilde{\psi}(x) \geq \min_{z \in \text{supp}(\chi_S)} (f(z) + f(x) - f(z)) = f(x),$$

with equality holds only if $z \in V_S$ and there is a generalized gradient line joining z to x . Therefore, we have

$$\tilde{\psi}(x) + \rho(y, x) \geq f(x) + f(y) - f(x) = f(y),$$

with equality holds only if there is a generalized gradient line joining a point $z \in V_S$ to $x \in \text{supp}(\chi)$ and then to $y \in K$. This is impossible by our choices of χ and K . Hence we always have strict inequality and therefore we can find small ϵ by compactness argument. \square

We consider a closed neighborhood \tilde{W} of $\text{supp}(\chi)$ in W with smooth boundary. We let \tilde{G} to be the twisted Green's operator on \tilde{W} using Dirichlet boundary condition. We first argue that ζ_E can be replaced by $\tilde{\zeta}_E = d_f^* \tilde{G} \chi_S \zeta_S$.

Lemma 59. *There is a $\delta > 0$ such that*

$$\|e^{\lambda\psi_E} \nabla^j (\chi \zeta_E - \tilde{\zeta}_E)\|_{L^\infty(K)} \leq C_j e^{-\lambda\delta},$$

whenever $\text{supp}(\chi_S)$ and K are chosen small enough and $j \in \mathbb{Z}_+$.

Proof. We let $r_\lambda = \chi \zeta_E - \tilde{\zeta}_E$. First, r_λ satisfies the equation

$$(4.27) \quad \tilde{\Delta}_f r_\lambda = [\Delta, \chi] \zeta_E - \chi P_f d_f^* (\chi_S \zeta_S).$$

Therefore we have $r_\lambda = (\tilde{G}[\Delta, \chi]G - \tilde{G}\chi P_f) d_f^* (\chi_S \zeta_S)$. We consider it term by term to get estimate of r_λ . Making use of the lemma 16 and a similar statement for \tilde{G} , we have for any $\epsilon > 0$,

$$\tilde{G}[\Delta, \chi]G \sim \mathcal{O}_\epsilon \left(\exp(-\lambda(\min_{z \in \text{supp}(\nabla\chi)} (\rho(x, z) + \rho(z, y) - \epsilon))) \right).$$

Using lemma 58, we can show there is some $\delta_0 > 0$ such that

$$\tilde{G}[\Delta, \chi]G d_f^* (\chi_S \zeta_S) \sim \mathcal{O}(e^{-\lambda(\psi_E + \delta_0)})$$

in K , for λ small enough.

For the term $\tilde{G}\chi P_f$, we have

$$\tilde{G}\chi P_f \sim \mathcal{O}_\epsilon \left(\sum_{q \in C_f^l} \exp(-\lambda(\rho(x, q) + \rho(q, y) - \epsilon)) \right)$$

follows from lemma 17 and modified version of lemma 16 for \tilde{G} , where $l = \text{deg}(\zeta_S)$. Again, we can find a constant $\delta_1 > 0$ such that

$$\min_{x \in \text{supp}(\chi_S)} (\psi_S(x) + \rho(x, q) + \rho(q, y)) \geq \psi_E(y) + 2\delta_1,$$

for $y \in K$. Similarly we have

$$\tilde{G}\chi P_f d_f^* (\chi_S \zeta_S) \sim \mathcal{O}(e^{-\lambda(\psi_E + \delta_1)})$$

in K , for λ large enough. Notice that the constant $\delta = \min\{\delta_0, \delta_1\}$ can chosen to be the same if we shrink $\text{supp}(\chi_S)$ and K and keep \tilde{W} and χ fixed. \square

Next, we obtain estimates for $\tilde{\zeta}_E$ similar to those in lemma 57 for ζ_E .

Lemma 60. *For any $j \in \mathbb{Z}_+$, there exists $\lambda_{j,0} > 0$ such that if $\lambda > \lambda_{j,0}$, we have*

$$(4.28) \quad \|e^{\lambda\psi_E} \nabla^j \tilde{\zeta}_E\|_{L^\infty(\tilde{W})}^2 \leq C_j \lambda^{N_j},$$

where N_j is an positive integer depending on j .

Proof. We consider the equation

$$(4.29) \quad \Delta_f \tilde{\zeta}_E = d_f^*(\chi_S \zeta_S)$$

in \tilde{W} and divide the proof into steps:

Step 1: Without loss of generality, we assume there is a constant $C_0 > 0$ such that $C_0^{-1} \leq \psi_E \leq C_0$ and $C_0^{-1} \leq |df|^2 = |d\psi_E|^2 \leq C_0$ on \tilde{W} . We define the function

$$(4.30) \quad \Phi = \psi_E - \frac{C}{\lambda} \log(\lambda\psi_E),$$

with $C > 0$ to be chosen. Therefore we have

$$|df|^2 - |d\Phi|^2 \geq \frac{C|df|^2}{\lambda\psi_E} \geq \frac{C}{C_0^2\lambda}.$$

Using the equation (4.29) we get

$$\begin{aligned} \operatorname{Re}(\langle e^{2\lambda\Phi} d_f^*(\chi_S \zeta_S), \tilde{\zeta}_E \rangle) &= (\|d(e^{\lambda\Phi} \tilde{\zeta}_E)\|^2 + \|d^*(e^{\lambda\Phi} \tilde{\zeta}_E)\|^2) \\ &\quad + \langle (\lambda^2(|df|^2 - |d\Phi|^2) + \lambda M_f) e^{\lambda\Phi} \tilde{\zeta}_E, e^{\lambda\Phi} \tilde{\zeta}_E \rangle \end{aligned}$$

and if we choose $C > 0$ large enough to absorb the term $\langle \lambda M_f e^{\lambda\Phi} \tilde{\zeta}_E, e^{\lambda\Phi} \tilde{\zeta}_E \rangle$, we have

$$\begin{aligned} &(\|d(e^{\lambda\Phi} \tilde{\zeta}_E)\|^2 + \|d^*(e^{\lambda\Phi} \tilde{\zeta}_E)\|^2) + \frac{C\lambda}{2C_0^2} \|e^{\lambda\Phi} \tilde{\zeta}_E\|^2 \\ &\leq C_1 \|e^{\lambda\Phi} d_f^*(\chi_S \zeta_S)\|^2 \leq C_1 \left(\frac{C_0}{\lambda}\right)^{2C} \|e^{\lambda\psi_E} d_f^*(\chi_S \zeta_S)\|^2 \\ &\leq C_2 \left(\frac{C_0}{\lambda}\right)^{2C} \|e^{\lambda\psi_S} d_f^*(\chi_S \zeta_S)\|^2 \leq C_3 \lambda^{2-2C}. \end{aligned}$$

Therefore we get

$$(\|d(e^{\lambda\psi_E} \tilde{\zeta}_E)\|^2 + \|d^*(e^{\lambda\psi_E} \tilde{\zeta}_E)\|^2) + \lambda \|e^{\lambda\psi_E} \tilde{\zeta}_E\|^2 \leq C_3,$$

and obtained $\|e^{\lambda\psi_E} \tilde{\zeta}_E\|_{L^2(K)}^2 \leq C_4 \lambda^{-1}$, for $\lambda < \lambda_0$.

Step 2: We prove the L^2 estimate for derivatives of $\tilde{\zeta}_E$. We apply d_f and d_f^* to both sides of equation (4.29). We obtain

$$(4.31) \quad \Delta_f(d_f \tilde{\zeta}_E) = d_f d_f^*(\chi_S \zeta_S).$$

Applying the result in step 1 to $d_f \tilde{\zeta}_E$, we have

$$\|e^{\lambda\psi_E} d_f \tilde{\zeta}_E\|_{L^2(K)}^2 \leq C_4 \lambda^{-1}.$$

Since $d_f = d + \lambda df \wedge$, we have

$$\|e^{\lambda\psi_E} d\tilde{\zeta}_E\|_{L^2(K)}^2 \leq C_5 \lambda^1.$$

Corresponding result for $d^* \tilde{\zeta}_E$ can be obtained by a similar argument. These combine to obtain result for $\nabla \tilde{\zeta}_E$. By applying ∇ successively, we obtain all higher derivatives' estimates in a similar fashion.

Step 3: Finally, we improve the estimate to L^∞ norm. Since we have L^2 norm estimate for all the derivatives of $\tilde{\zeta}_E$. We use the Sobolev embedding on \tilde{W} to obtain the L^∞ norm estimate. Details are left to readers. \square

Lemma 57 follows from lemma 59 and lemma 60 directly.

4.7. WKB approximation. Next, we consider the WKB approximation of ζ_E . From the WKB approximation (4.1) of ζ_S , we can take d_f^* on both side and obtain a WKB approximation of $d_f^*(\chi_S \zeta_S)$

$$(4.32) \quad d_f^*(\chi_S \zeta_S) \sim e^{-\lambda\psi_S} (d^* + \lambda(\iota_{\nabla f} + \iota_{\nabla\psi_S})) (\chi_S \omega_{S,0} + \chi_S \omega_{S,1} \lambda^{-1/2} + \dots),$$

after grouping terms according to their orders of λ . We apply the iteration in the previous subsection 4.4 terms by terms to the above series and then group the terms according to orders of λ of their L^2 norms. As a result, we obtain a WKB expansion

$$(4.33) \quad \zeta_E \sim e^{-\lambda\psi_E} (\omega_{E,0}(\lambda) + \omega_{E,1}(\lambda) + \dots)$$

in W , where $\omega_{E,i}(\lambda)$'s are functions also depending on λ . Using Lemma 55 and Remark 56, we obtain that for every l and any compact subset $\tilde{K} \subset W$,

$$\|\omega_{E,l}(\lambda)\|_{L^2(\tilde{K})}^2 \leq C_{l,\tilde{K}} \lambda^{-l-1/2}$$

for those $\lambda < \lambda_{l,0}$, and also

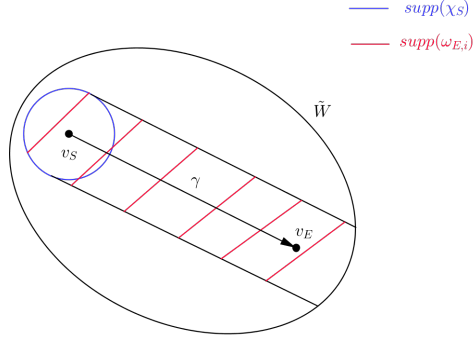
$$\|e^{\lambda\psi_E} (\Delta_f (e^{-\lambda\psi_E} \sum_{i=0}^N \omega_{E,i}(\lambda)) - d_f^* (e^{-\lambda\psi_S} \sum_{i=0}^N \omega_{S,i} \lambda^{-i/2}))\|_{L^2(\tilde{K})}^2 \leq C_{N,\tilde{K}} \lambda^{-N-1/2},$$

for $\lambda > \lambda_{N,0}$. We need to argue that it is a good approximation, which is the main theorem in this section.

Theorem 61. *For any $\text{supp}(\chi_S)$ and K small enough, and N large enough, there exists $\lambda_{j,N,0} > 0$ such that for $\lambda > \lambda_{j,N,0}$ we have*

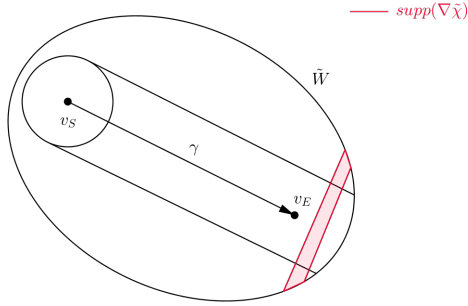
$$(4.34) \quad \|e^{\lambda\psi_E} \nabla^j \{\zeta_E - e^{-\lambda\psi_E} (\sum_{i=0}^N \omega_{E,i}(\lambda))\}\|_{L^2(K)}^2 \leq C_{j,N} \lambda^{-N+2j}.$$

Proof. Making use of lemma 59, we can again consider the equation 4.29. It suffices to show that the approximation works for $\tilde{\zeta}_E$ on some small enough pre-compact neighborhood K of the flow line γ . We divide the proof into several steps.


 FIGURE 7. Support of $\omega_{E,i}$'s

Step 1: As $\omega_{E,i}(\lambda)$'s do not vanish on boundary of \tilde{W} , we first need to cut them off suitably for applying integration by part. $\omega_{E,i}(\lambda)$'s, being defined by integrating along flow of τ , have support as shown in the following figure.

Suppose we have $\tau_{\tilde{T}}(v_S) = v_E$, then we can choose $\tilde{\chi}$ only depending on variable t (using coordinate defined by τ) such that $\tilde{\chi} \equiv 1$ for $t \leq \tilde{T}$. The support of $\nabla \tilde{\chi}$ is shown in the following figure.


 FIGURE 8. Support of $\nabla \tilde{\chi}$

By shrinking K and $\text{supp}(\chi_S)$ if necessary, we obtain some $\epsilon > 0$ such that

$$(4.35) \quad \psi_E(y) + \rho(y, x) \geq \psi_E(x) + \epsilon$$

for $x \in K$ and $y \in \text{supp}(\nabla \tilde{\chi})$. We define the function

$$(4.36) \quad \Phi_N = \min\{\Phi + N\lambda^{-1} \log(\lambda), \min_{y \in \text{supp}(\nabla \tilde{\chi})} (\Phi(y) + (1 - \epsilon)\rho(x, y))\},$$

where $\Phi := \psi_E - \frac{C}{\lambda} \log(\lambda\psi_E)$ is defined in (4.30), and the ϵ is chosen in lemma 58. We have

$$|df|^2 - |d\Phi_N|^2 \geq \frac{C|df|^2}{\lambda\psi_E} \geq \frac{C}{C_0^2\lambda},$$

for λ large enough. Notice that we have $\Phi_N = \Phi + N\lambda^{-1} \log(\lambda)$ in K for λ large enough, and $\Phi_N = \Phi$ in $\text{supp}(\nabla\tilde{\chi})$.

Step 2: Writing the reminder term as $r_k = \tilde{\chi}(\tilde{\zeta}_E - e^{-\lambda\psi_E}(\sum_{i=0}^{k-1} \omega_{E,i}(\lambda)))$, we get

$$\begin{aligned} & (\|d(e^{\lambda\Phi_N} r_k)\|_{L^2(K)}^2 + \|d^*(e^{\lambda\Phi_N} r_k)\|_{L^2(K)}^2) + \frac{C\lambda^1}{2C_0^2} \|e^{\lambda\Phi_N} r_k\|_{L^2(K)}^2 \\ \leq & D \|e^{\lambda\Phi_N} d_f^*(\chi_S \zeta_S - e^{-\lambda\psi_S} \sum_{i=0}^{k-1} \chi_S \omega_{S,i} \lambda^{-i/2})\|_{L^2(\tilde{W})}^2 \\ & + D \|e^{\lambda\Phi_N} (d_f^*(e^{-\lambda\psi_S} \sum_{i=0}^{k-1} \chi_S \omega_{S,i} \lambda^{-i/2}) - \Delta_f(e^{-\lambda\psi_E} \sum_{i=0}^{k-1} \omega_{E,i}(\lambda)))\|_{L^2(\tilde{W})}^2 \\ & + D (\|e^{\lambda\Phi} [\Delta, \tilde{\chi}] \tilde{\zeta}_E\|_{L^2(\tilde{W})}^2 + \|e^{\lambda\Phi} [\Delta, \tilde{\chi}] (e^{-\lambda\psi_E} \sum_{i=0}^{k-1} \omega_{E,i}(\lambda))\|_{L^2(\tilde{W})}^2). \end{aligned}$$

We handle the right hand side term by term. First, we have

$$\|e^{\lambda\Phi_N} d_f^*(\chi_S \zeta_S - e^{-\lambda\psi_S} \sum_{i=0}^{k-1} \chi_S \omega_{S,i} \lambda^{-i/2})\|_{L^2(\tilde{W})}^2 \leq C_k \lambda^{-2C+2N-k+2}.$$

Second, we have

$$\|e^{\lambda\Phi_N} (d_f^*(e^{-\lambda\psi_S} \sum_{i=0}^{k-1} \chi_S \omega_{S,i} \lambda^{-i/2}) - \Delta_f(e^{-\lambda\psi_E} \sum_{i=0}^{k-1} \omega_{E,i}(\lambda)))\|_{L^2(\tilde{W})}^2 \leq C_k \lambda^{-2C+2N-k+1}.$$

Third, we have

$$\|e^{\lambda\Phi} [\Delta, \tilde{\chi}] \tilde{\zeta}_E\|_{L^2(\tilde{W})}^2 \leq D_1 \lambda^{-2C+N_0},$$

where N_0 is the integer in lemma 57. Finally, we have

$$\|e^{\lambda\Phi} [\Delta, \tilde{\chi}] (e^{-\lambda\psi_E} \sum_{i=0}^{k-1} \omega_{E,i}(\lambda))\|_{L^2(\tilde{W})}^2 \leq C_k \lambda^{-2C+N_0},$$

by choosing a larger N_0 independent of k , if necessary. Combining the above, by choosing $N = N_0 + k$, we have

$$(\|d(e^{\lambda\psi_E} r_k)\|_{L^2(K)}^2 + \|d^*(e^{\lambda\psi_E} r_k)\|_{L^2(K)}^2) + \lambda \|e^{\lambda\psi_E} r_k\|_{L^2(K)}^2 \leq C_k \lambda^{-k+2},$$

which gives $\|e^{\lambda\psi_E} r_k\|_{L^2(K)}^2 \leq C_k \lambda^{-k+1}$, for those $\lambda < \lambda_{k,0}$.

Step 3: We obtain L^2 estimate for all derivatives of r_k . We repeat the above argument for $d_f r_k$ and $d_f^* r_k$. For any $j, N \in \mathbb{Z}_+$, we can find a $k_{j,N}$

large enough such that for any $k > k_{j,N}$, we have

$$\|e^{\lambda\psi_E} \nabla^j r_k\|_{L^2(K)}^2 \leq C_{j,K,N} \lambda^{-N},$$

for $\lambda > \lambda_{j,k,N,0}$.

Step 4: We apply interior Sobolev embedding to improve the statement in step 3 into L^∞ norm, by further shrinking K if necessary. As a result, we have for N large enough, there exists $\lambda_{j,N,0} > 0$ and M_N such that we have

$$(4.37) \quad \|e^{\lambda\psi_E} \nabla^j \{\tilde{\zeta}_E - e^{-\lambda\psi_E} (\sum_{i=0}^{M_N} \omega_{E,i}(\lambda))\}\|_{L^\infty(K)}^2 \leq C_{j,N} \lambda^{-N+2j}$$

for $\lambda < \lambda_{j,N,0}$. Finally, we observe that $\|\nabla^j \omega_{E,i}(\lambda)\|_{L^\infty(K)}^2 \leq C_{i,j} \lambda^{-i+j+\frac{1}{2}}$ and hence obtain the result by dropping redundant terms in the approximation series. \square

Finally, we restrict our attention to a small enough neighborhood W_E of v_E . Since the operator I is given by an integral with an exponential decay $e^{\lambda\Psi}$ along flow line, we can apply lemma 51 to obtain an expansion

$$\omega_{E,i}(\lambda) = \lambda^{-\frac{1}{2}} (\omega_{E,i,0} + \omega_{E,i,1} \lambda^{-1} + \omega_{E,i,2} \lambda^{-2} + \dots).$$

By regrouping terms according to their orders of λ , we obtain an expansion of the form given in equation (4.2).

4.8. Relation between $\omega_{S,0}$ and $\omega_{E,0}$. From section 4.4, we constructed a WKB approximation in W_E

$$\zeta_E = e^{-\lambda\psi_E} (\omega_{E,0}(\lambda) + \omega_{E,1}(\lambda) + \dots).$$

In particular, $\omega_{E,0}(\lambda)$ is given by

$$(4.38) \quad \omega_{E,0}(\lambda) = \frac{1}{2} \left(\int_{-\infty}^0 e^{\int_s^0 \frac{1}{2} \tau_\epsilon^*(M_{g_\epsilon}) d\epsilon} \tau_s^* (e^{\lambda\Psi} (\iota_{2\nabla f} + \iota_{\nabla g_s}) \chi_S \omega_{S,0}) ds \right).$$

In this section, we study the relation between integrals of $\omega_{S,0}$ and $\omega_{E,0}$ which is used in lemma 26. We begin by recalling lemma 25. Let M be a n -dimensional manifold and S be a k -dimensional submanifold in M , with a neighborhood B of S which can be identified as the normal bundle $\pi : NS \rightarrow S$. Suppose $\varphi : B \rightarrow \mathbb{R}_{\geq 0}$ is a Bott-Morse function with zero set S , we have

Lemma 62. *Let $\beta \in \Omega^*(B)$ which is vertically compact support along the fiber of π . Then, we have*

$$\pi_*(e^{-\lambda\varphi(x)} \beta) = \left(\frac{2\pi}{\lambda} \right)^{(n-k)/2} (\iota_{\text{vol}(\nabla^2\varphi)} \beta)|_V (1 + \mathcal{O}(\lambda^{-1})),$$

where π_* is the integration along fiber. Here $\text{vol}(\nabla^2\varphi)$ stands for the volume polyvector field defined for the positive symmetric tensor $\nabla^2\varphi$ along fibers of π .

We use the notations in section 4.1 and assume there is an identification of W_S and W_E with the normal bundle NV_S and NV_E of V_S and V_E respectively. We use π_S and π_E to stand for the bundle maps respectively. We have the following lemma which relates the integration of $\omega_{E,0}$ and $\omega_{S,0}$ along the fibers of π_E and π_S respectively.

Lemma 63. *Assume $\omega_{S,0} \in \wedge^{\text{top}} NV_S^*$ on V_S , then*

$$\lambda^{1/2} \pi_{E*}(e^{-\lambda g_E} \omega_{E,0}) = \varrho^* \pi_{S*}(e^{-\lambda g_S} \omega_{S,0})(1 + \mathcal{O}(\lambda^{-1/2})),$$

where $\varrho : V_E \rightarrow V_S$ is the projection map using the identification $V_E \equiv (V_S \times \mathbb{R}) \cap W_E$ given by τ (flow of $\nabla \psi_E$). Furthermore, we have $\omega_{E,0} \in \wedge^{\text{top}} NV_E^*$ on V_E .

Proof. We use the coordinates u_1, \dots, u_{n-1}, t for W , where u_1, \dots, u_{n-1} are coordinates of U_S . We further assume that $\{u_{s+1} = 0, \dots, u_{n-1} = 0\} = V_S$. From lemma 46, $\Psi \leq 0$ is a Bott-Morse function with zero set U_S . Applying lemma 62 to the equation (4.38), we have

$$\begin{aligned} & \omega_{E,0}(u, t) \\ \equiv & \left(\frac{\pi}{2\lambda}\right)^{1/2} \left(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0}\right)^{-1/2} \left(e^{\int_{-t}^0 \frac{1}{2} \tau_\epsilon^*(M_{g_E}) d\epsilon} \tau_{-t}^*((\iota_{2\nabla f} + \iota_{\nabla g_S}) \chi_S \omega_{S,0})\right), \end{aligned}$$

modulo terms of $\mathcal{O}(\lambda^{-1/2})$. From lemma 45, $g_E \geq 0$ is a Bott-Morse function with zero set V_E . Applying lemma 62 again, we get, modulo terms of $\mathcal{O}(\lambda^{-1/2})$,

$$\begin{aligned} & \pi_{E*}(e^{-\lambda g_E} \omega_{E,0})(u, t) \\ \equiv & \left(\frac{2\pi}{\lambda}\right)^{(n-s-1)/2} \iota_{\text{vol}(\nabla^2 g_E)}(\omega_{E,0}) \\ \equiv & \pi \left(\left(\frac{2\pi}{\lambda}\right)^{(n-s)/2} \iota_{\text{vol}(\nabla^2 g_E)} \left(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0} \right)^{-1/2} \left(e^{\int_{-t}^0 \frac{1}{2} \tau_\epsilon^*(M_{g_E}) d\epsilon} \tau_{-t}^*(\iota_{2\nabla f} \omega_{S,0}) \right) \right), \end{aligned}$$

for those $(u, t) \in V_E$. The term involving $\iota_{\nabla g_S}$ is dropped as $\tau_{-t}^*(dg_S)$ vanishes for $(u, t) \in V_E$. To make further simplifications, we need the following lemma.

Lemma 64. *Fixing a point $(u, t) \in V_E$, we have*

$$e^{\int_{-t}^0 \frac{1}{2} \tau_\epsilon^*(M_{g_E}) d\epsilon} = \left(\frac{\det(\nabla^2 g_E)(u, t)}{\det(\nabla^2 g_E)(u, 0)} \right)^{1/2}$$

as operators on $\wedge^{\text{top}} NV_E^*$, where the right hand side acts as multiplication. Here $\nabla^2 g_E$ is treated as an operator acting on NV_E using the metric tensor.

From the fact that $\omega_{S,0} \in \bigwedge^{\text{top}} NV_S^*$ upon restricting to V_S , we have $\tau_{-t}^*(\iota_{\nabla f} \omega_{S,0}) \in \bigwedge^{\text{top}} NV_E^*$ for those $(u, t) \in V_E$ and

$$\begin{aligned} & \pi_{E*}(e^{-\lambda g_E} \omega_{E,0})(u, t) \\ &= 2\pi \left(\frac{2\pi}{\lambda}\right)^{(n-s)/2} \left(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0}\right)^{-1/2} \left(\frac{\det(\nabla^2 g_E)(u, t)}{\det(\nabla^2 g_E)(u, 0)}\right)^{1/2} \iota_{\nabla f} \wedge \text{vol}(\nabla^2 g_E) \tau_{-t}^*(\omega_{S,0}). \end{aligned}$$

Notice that $\nabla f = \frac{\partial}{\partial t}$ when restricting on V_E , therefore we have

$$\left(\frac{\partial^2}{\partial t^2}(-\Psi)|_{t=0}\right)^{1/2} \nabla f = \text{vol}(\nabla_t^2(-\Psi)|_{t=0}),$$

where we view W as a \mathbb{R} -bundle over U_S and consider $\text{vol}(\nabla_t^2(-\Psi)|_{t=0})$ as the volume vector field along its fibers. Furthermore, we have the relation

$$d\tau_{-t}^* \left(\frac{\det(\nabla^2 g_E)(u, t)}{\det(\nabla^2 g_E)(u, 0)}\right)^{1/2} \text{vol}(\nabla^2 g_E)(u, t) = \text{vol}(\nabla^2 g_E)(u, 0).$$

Combining the above, we have

$$\begin{aligned} & \pi_{E*}(e^{-\lambda g_E} \omega_{E,0})(u, t) \\ &= (2\pi)^{(n-s)/2} \lambda^{-(n+s)/2} \left(\tau_{-t}^*(\iota_{\text{vol}(\nabla_t^2(-\Psi)|_{t=0})} \wedge \text{vol}(\nabla^2 g_E)|_{t=0} \omega_{S,0})\right). \end{aligned}$$

Finally, from the relation $\Psi = g_E - g_S$, we get

$$\text{vol}(\nabla_t^2(-\Psi)) \wedge \text{vol}(\nabla^2 g_E) = \text{vol}(\nabla^2 g_S)$$

on V_S , where $\text{vol}(\nabla^2 g_S)$ is the volume polyvector field along the fibers of π_S . Therefore, we have

$$\lambda^{1/2} \pi_{E*}(e^{-\lambda g_E} \omega_{E,0})(u, t) \equiv \tau_{-t}^*(\pi_{S*}(e^{-\lambda g_S} \omega_{S,0})(u, 0))$$

modulo terms of $\mathcal{O}(\lambda^{-1/2})$, for those $(u, t) \in V_E$. \square

Proof of Lemma 64. First of all, we have the equality

$$\frac{1}{2} M_{g_E} = \nabla^2 g_E - \frac{1}{2} \text{tr}(\nabla^2 g_E),$$

on the set $\{\nabla g_E = 0\}$. We can treat $\nabla^2 g_E$ as an operator acting on NV_E^* as g_E is Morse along V_S . Restricting to $\bigwedge^{\text{top}} NV_E^*$, it is just $\text{tr}(\nabla^2 g_E)$. Therefore we have

$$\frac{1}{2} M_{g_E} = \frac{1}{2} \text{tr}(\nabla^2 g_E),$$

acting on $\bigwedge^{\text{top}} NV_E^*$.

On V_E , we have

$$\begin{aligned} (4.39) \quad & \nabla_t \left(\int_0^t \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, \epsilon) d\epsilon - \frac{1}{2} \log(\det(\nabla_u^2 g_E)(u, t)) \right) \\ &= \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, t) - \frac{1}{2} \text{tr}((\nabla^2 g_E(u, t))^{-1} \nabla_t(\nabla^2 g_E(u, t))). \end{aligned}$$

We will show that the above expression vanish.

Restricting to the set $\{\nabla g_E = 0\}$, for any vector fields $X, Y \in TW$, we have

$$\begin{aligned}\nabla_t(\nabla_u^2 g_E)(X, Y) &= \nabla_t(\nabla^2 g_E(X, Y)) - \nabla^2 g_E(\nabla_t X, Y) - \nabla^2 g_E(X, \nabla_t Y) \\ &= \nabla_t \langle X, \nabla_Y \nabla g_E \rangle - \langle \nabla_t X, \nabla_Y \nabla g_E \rangle - \langle \nabla_X \nabla g_E, \nabla_t Y \rangle \\ &= \langle X, \nabla_t \nabla_Y \nabla g_E \rangle + \langle \nabla_X \nabla g_E, [\partial_t, Y] \rangle + \langle \nabla_X \nabla g_E, \nabla_Y \partial_t \rangle \\ &= \langle X, \nabla_Y \nabla_t \nabla g_E \rangle + \langle (\nabla^2 t \nabla^2 g_E) X, Y \rangle,\end{aligned}$$

and

$$\begin{aligned}\nabla^2(\nabla_t g_E)(X, Y) &= \langle \nabla_Y \nabla(\partial_t g_E), X \rangle \\ &= Y \langle \nabla(\partial_t g_E), X \rangle - \langle \nabla(\partial_t g_E), \nabla_Y X \rangle \\ &= Y \langle \nabla_X \nabla g_E, \partial_t \rangle + Y \langle \nabla g_E, \nabla_X \partial_t \rangle - \langle \nabla_{\nabla_Y X} \nabla g_E, \partial_t \rangle \\ &= Y \langle X, \nabla_t \nabla g_E \rangle + Y \langle \nabla g_E, \nabla_X \partial_t \rangle - \langle \nabla_Y X, \nabla_t \nabla g_E \rangle \\ &= \langle X, \nabla_Y \nabla_t \nabla g_E \rangle + (\nabla^2 g_E \nabla^2 t) X, Y).\end{aligned}$$

Therefore, we have

$$\nabla_t(\nabla^2 g_E) - \nabla^2(\nabla_t g_E) = [\nabla^2 t, \nabla^2 g_E],$$

where the Hessians are treated as endomorphisms of TM . Restricting the above equation to the subspace NV_E and multiplying by $(\nabla^2 g_E)^{-1}$, we have

$$\text{tr}((\nabla^2 g_E)^{-1}(\nabla_t(\nabla^2 g_E))) = \text{tr}((\nabla^2 g_E)^{-1} \nabla^2(\nabla_t g_E)).$$

Finally, from the equation $|\nabla \psi_E|^2 = |\nabla f|^2$, we obtain

$$\nabla_t g_E = \frac{1}{2} |\nabla g_E|^2.$$

Applying ∇^2 to both sides and restricting to V_E give

$$\nabla^2(\nabla_t g_E)(X, Y) = \langle \nabla^2 g_E(X), \nabla^2 g_E(Y) \rangle,$$

or simply

$$\nabla^2(\nabla_t g_E) = (\nabla^2 g_E)^2$$

if we treat both sides as operators on TM .

Substituting it back into equation (4.39), we find that the derivative in equation (4.39) vanish. Therefore we have

$$\begin{aligned}& \left(\int_0^t \frac{1}{2} \text{tr}(\nabla^2 g_E)(u, \epsilon) d\epsilon \right) \\ &= \frac{1}{2} \log(\det(\nabla^2 g_E)(u, t)) - \frac{1}{2} \log(\det(\nabla^2(g_E))(u, 0)),\end{aligned}$$

which is the equation we needed. \square

Therefore, we complete the proof of lemma 24 and 26 which are needed in the proof of our Main Theorem in section 3.

5. CONCLUSION

From the semi-classical analysis of the Witten twisted Green's operator in section 4, we obtain our main theorem 9 which can be viewed as an enhancement of the original Witten deformation of deRham complex, concerning cohomology of the manifold M , to one concerning its rational homotopy type by incorporating wedge product structures. In [6], Fukaya proposed a differential geometric approach to the Strominger-Yau-Zaslow (SYZ) by relating A-model holomorphic disks instantons of a Calabi-Yau manifold equipped with Lagrangian torus fibration, to certain Witten twisted differential constructed from the symplectic structure. Proving theorem 9 provides essential analytical technique for such an approach. For instance, the semi-classical analysis of Witten twisted Green's operator, can be applied to obtain a beautiful geometric interpretation of the complicated scattering diagram in [2].

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REFERENCES

1. M. Abouzaid, *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*, *Geom. Topol.* **10** (2006), 1097–1157.
2. K.-W. Chan, N. C. Leung, and Z. N. Ma, *Scattering diagrams from Witten-Morse theory*, preprint (2014).
3. M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.
4. Laudenbach F., *On the Thom-Smale complex in An Extension of a Theorem of Cheeger and Müller, by Bismut J.-M. and Zhang W.*, *Astérisque* (1992), no. No. 205.
5. K. Fukaya, *Morse homotopy, A^∞ -category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993), Lecture Notes Ser., vol. 18, Seoul Nat. Univ., Seoul, 1993, pp. 1–102. MR 1270931 (95e:57053)
6. ———, *Multivalued Morse theory, asymptotic analysis and mirror symmetry*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 205–278. MR 2131017 (2006a:53100)
7. K. Fukaya and Y.-G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, *Asian J. Math.* **1** (1997), no. 1, 96–180.
8. B. Helffer and F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, vol. 1862, Springer-Verlag, Berlin, 2005. MR 2130405 (2006a:58039)
9. B. Helffer and J. Sjöstrand, *Multiple wells in the semi-classical limit I*, *Comm. in PDE* **9** (1984), no. 4, 337–408.
10. ———, *Puits multiples en limite semi-classique II - Interaction moléculaire-Symétries-Perturbations*, *Annales de l'IHP(section Physique théorique)* **42** (1985), no. 2, 127–212.

11. ———, *Puits multiples en limite semi-classique IV - Etude du complexe de Witten*, Comm. in PDE **10** (1985), no. 3, 245–340.
12. M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263. MR 1882331 (2003c:32025)
13. D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969), 205–295.
14. D. Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l’IHÉS **47** (1977), no. 1, 269–331.
15. E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983). MR 683171 (84b:58111)
16. W. Zhang, *Lectures on chern-weil theory and witten deformations*, Nankai Tracts in Mathematics, vol. 4, World Scientific, 2001.

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