TROPICAL COUNTING FROM ASYMPTOTIC ANALYSIS ON MAURER-CARTAN EQUATIONS

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Abstract. Let $X = X_\Sigma$ be a toric surface and $(\check{X}, W)$ be its Landau-Ginzburg mirror where W is the Hori-Vafa potential [35]. In this paper, we study the extended deformation theory of $(\check{X}, W)$. We prove that the leading order terms in asymptotic expansions of the Fourier modes of Maurer-Cartan solutions with specific inputs are in bijective correspondences with tropical disks in X of Maslov index 0 or 2. The Maslov index 2 tropical disks give rise naturally to an n-th order perturbation W_n of W; for $X = \mathbb{P}^2$ this reproduces Gross' perturbed potential constructed in [27] which gives the universal unfolding of W in canonical coordinates. We also describe how the extended deformation theory dictates the jumping of W_n across the walls of a scattering diagram formed from the Maslov index 0 tropical disks.

Contents

1. Introduction

1.1. **Background.** The study of mirror symmetry for toric varieties goes back to Batyrev [5], Givental [24, 25, 26], Lian-Liu-Yau [42], Kontsevich [37] and Hori-Vafa [35]. Unlike the Calabi-Yau case, the mirror of a compact toric manifold X is given by a *Landau-Ginzburg (abbrev. LG) model* $(\check{X}, W)$ consisting of a noncompact Kähler manifold $\check{X}$ and a holomorphic function $W : \check{X} \to \mathbb{C}$ called the *potential* of the model [50, 51]. As a prototypical example, the mirror LG model of $X = \mathbb{P}^2$ is given by $\check{X} = \{(z^0, z^1, z^2) \in \mathbb{C}^3 \mid z^0 z^1 z^2 = 1\}$ together with the restriction of $W = z^0 + z^1 + z^2$ to $\check{X} \subset \mathbb{C}^3$.

At the genus 0 level, mirror symmetry can be understood as an isomorphism between Frobenius manifolds. For a large class of examples, the construction of the *B-model Frobenius manifold* from the LG model $(\check{X}, W)$ was carried out by Douai-Sabbah [13, 14] (see also the book [47]), generalizing the classic work of K. Saito [48]. Mirror symmetry then says that the B-model Frobenius manifold for $(\check{X}, W)$ is isomorphic (via a possibly nontrivial mirror map) to the *A-model Frobenius manifold* constructed from the genus 0 Gromov-Witten (abbrev. GW) theory or *big* quantum cohomology of X. In the case of projective spaces this was proved by Barannikov [3].

The geometry of this mirror symmetry can be understood using the Strominger-Yau-Zaslow (abbrev. SYZ) conjecture [49]. The moment map $p : X \to \mathbf{P}$ provides a natural Lagrangian torus fibration, and the mirror manifold $\check{X}$ can be constructed geometrically as a moduli space of A-branes, which are pairs (L, ∇) consisting of a Lagrangian torus fiber L of ρ and a flat $U(1)$-connection ∇ over it, or simply as the total space of the fiberwise dual of p restricted to the interior $\mathrm{Int}(\mathbf{P}) \subset \mathbf{P}$ [1]. The SYZ conjecture also suggests that mirror symmetry is a geometric Fourier transform; in this regard, the construction of $\check{X}$ from X may be regarded as describing the "0-th Fourier mode" of the mirror geometry.

The "higher Fourier modes" or "quantum corrections" come from the singular or degenerated fibers of p over the boundary $\partial\mathbf{P}$ and are captured by holomorphic disks in X with boundary on a Lagrangian torus fiber of p – this gives rise to the mirror LG potential W. Cho-Oh [11] were the first to prove, in the toric Fano case, that the so-called *Hori-Vafa potential* W can be expressed in terms of counts of Maslov index 2 holomorphic disks. This was later generalized by Fukaya-Oh-Ohta-Ono in [19] to all compact toric manifolds. The *Lagrangian Floer potential* W^{LF} is determined by Fukaya-Oh-Ohta-Ono's *obstruction cochain* $\mathfrak{m}_0$ in the Floer complex of a Lagrangian torus fiber of the moment map p [17, 18].

In explicit terms, coefficients of W^{LF} are counts of bubbled configurations, or more precisely, *genus 0 open Gromov-Witten invariants*, and W^{LF} should be viewed as a perturbation the Hori-Vafa potential W:

$$W^{\mathrm{LF}} = W + \text{correction terms}$$

as coefficients of the latter only encode counts of embedded holomorphic disks (which is why $W^{\mathrm{LF}} = W$ only when X is toric Fano). It is quite hard to compute W^{LF}, but explicit formulas are known in a few low-dimensional examples [2, 21, 6] and when X is semi-Fano [7, 8].

Using W^{LF}, one obtains an isomorphism of Frobenius algebras

$$QH^*(X) \cong Jac(W^{\mathrm{LF}}) \tag{1.1}$$

between the *small* quantum cohomology ring of X and the Jacobian ring of W, *without* going through a mirror map (or one can say that the mirror map is trivial). To upgrade this to an isomorphism between Frobenius manifolds, Fukaya-Oh-Ohta-Ono [20] introduced the *bulk-deformed potential* W_b^{LF} as a perturbation of W^{LF} by the ambient cycles in X. In [22] they proved that (1.1) can be enhanced to an isomorphism between the A-model Frobenius manifold of X and the B-model Frobenius manifold constructed from W_b^{LF}.

A closely related work and more relevant to our discussion is the paper [27] by Gross (see also his book [28]), in which he described another bulk-deformed potential mirror to $X = \mathbb{P}^2$ as an explicit perturbation W_n (for any fixed order n) of the Hori-Vafa potential $W = z^0 + z^1 + z^2$ by counts of Maslov index 2 *tropical disks* passing through n generic points $P_1, \dots, P_n$ in $\mathbb{R}^2$ (or the tropical projective plane $\mathbb{TP}^2$). He computed oscillatory integrals of his perturbed potential W_n to produce some beautiful tropical formulas for descendent GW invariants, thereby giving a very transparent proof of mirror symmetry for $\mathbb{P}^2$ via tropical geometry.

Gross' paper is in turn closely connected with the influential Gross-Siebert program [30, 31, 32, 33], where a key role is played by a combinatorial gadget called *scattering diagram* which was first introduced by Kontsevich-Soibelman in [39].

On the other hand, the precise correspondence between counting of tropical and holomorphic curves/disks has been studied in various cases, first by [44] in dimension 2, and later by [46, 45] in higher dimensions. These works show that tropical geometry is indeed sufficient in describing GW theory.

1.2. **Asymptotic behavior of Maurer-Cartan solutions.**

The main goal of this paper is to understand the relation between tropical disk counting on (the tropical counterpart of) a toric surface X and the (extended) deformation theory of its LG mirror $(\check{X}, W)$, building on the approach in [10]; here W is taken to be the Hori-Vafa potential.

Recall that in [10], an $\hbar \in \mathbb{R}_+$ parameter was introduced to twist the complex structure of the dual torus fibration $\check{p} : \check{X} \to \mathrm{Int}(\mathbf{P})$ which geometrically corresponds to shrinking of the torus fibers. We considered the differential-geometric deformation theory of $\check{X}$ captured by the Kodaira-Spencer dgLa

$$KS^*_{\check{X}} := \Omega^{0,*}(\check{X}, T^{1,0})$$

and the associated Maurer-Cartan (abbrev. MC) equation

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0. \tag{1.2}$$

Following a proposal put forward by Kontsevich-Soibelman [38] and Fukaya [16], we study the Fourier expansions of the MC solutions along fibers of $\check{p} : \check{X} \to \mathrm{Int}(\mathbf{P})$. One of the main results of [10] says that the leading order terms (or the semiclassical limit) of the Fourier modes of a solution (of a specific type) of the MC equation (1.2) naturally gives rise to a consistent scattering diagram $\mathcal{D}$ as $\hbar \to 0$, and such MC solutions can be constructed as a sum of terms with support concentrated along the walls in $\mathcal{D}$.

In this paper, we consider the extension of $KS^*_{\check{X}}$ given by taking the full exterior power $\bigwedge^* T^{1,0}$ of polyvector fields:

$$PV^{*,*}(\check{X}) := \Omega^{0,*}\left(\check{X}, \wedge^* T^{1,0}\right),$$

which is equipped with a naturally extended Lie-bracket $[\cdot,\cdot]$ and a BV operator Δ, producing *a differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra*. As pointed out in [41], this dgBV algebra of polyvector fields is closely related to the B-model Frobenius manifold, especially in view of the approach in [4, 3].

For a LG model $(\check{X}, W)$, the dgBV algebra $PV^{*,*}$ should be equipped with the Dolbeault differential twisted by W:

$$\bar{\partial}_W := \bar{\partial} + [W, \cdot].$$

It is therefore natural to consider the following *extended* Maurer-Cartan equation:

$$\bar{\partial}_W \varphi + \frac{1}{2}[\varphi, \varphi] = 0 \tag{1.3}$$

for $\varphi \in PV^{*,*}(\check{X})$ (see Section 3.1). We will apply the machinery developed in [10] to analyze the solutions of (1.3).

We will restrict our attention to the 2-dimensional case so that X is a toric surface (implicitly equipped with its toric anticanonical divisor $D = D_\infty$), and study the equation (1.3) associated to the mirror LG model ($\check{X} \cong (\mathbb{C}^*)^2, W$) where W is the Hori-Vafa potential. Following Gross [27], we consider n points $P_1, \ldots, P_n \in \mathbb{R}^2$ in generic position and define the n-pointed perturbed LG potential $W_n(Q)$ by counting Maslov index 2 tropical disks with interior marked points possibly passing through $P_1, \ldots, P_n$ and with stop at a fixed point $Q \in \mathbb{R}^2$; see Section 2 for the precise definitions.

The n-pointed potential $W_n(Q)$ depends on Q, and Gross [27] proved that the dependence is dictated by wall-crossing across walls of a scattering diagram $\mathscr{D}$ constructed from Maslov index 0 tropical disks with interior marked points possibly passing through $P_1, \ldots, P_n$. Our main results describe how solutions to the extended MC equation (1.3) give rise to the perturbed potential W_n and explain the wall-crossing formulas by gauge equivalence in the extended deformation theory of $(\check{X}, W)$.

More precisely, we will only be concerned with the leading order behavior of the Fourier modes of the MC solutions as $\hbar \to 0$ and will therefore restrict ourselves to the quotient $(\mathcal{G}/\mathcal{I})^{*,*}$ of a subalgebra $\mathcal{G}^{*,*} \leq PV^{*,*}$ consisting of terms with growth control as $\hbar \to 0$ by the ideal $\mathcal{I}^{*,*}$ generated by error terms in $\hbar$ as $\hbar \to 0$ (see Section 3.3.1); here we will employ the key notion of *asymptotic support* introduced in [10] (see Section 3.2).

As in [10], a solution Φ to the extended MC equation 1.3 of $(\mathcal{G}/\mathcal{I})^{*,*}$ will be constructed using Kuranishi's method [40], namely, by writing it as a sum over directed ribbon weighted d-pointed k-trees (see Notation 3.28) with a specific input Π chosen to be of the form

$$\Pi = \sum_i u_i \delta_{P_i} (\partial_1 \wedge \partial_2), \tag{1.4}$$

where u_i is a formal variable in the ring

$$R = R_n := \frac{\mathbb{C}[u_1, \ldots, u_n]}{(u_i^2 \mid 1 \leq i \leq n)}$$

(equipped with the maximal ideal $\mathbf{m} = \mathbf{m}_n := (u_1, \ldots, u_n)$) corresponding to the point P_i, $\partial_1 \wedge \partial_2$ is the canonical holomorphic bi-vector field on $(\mathbb{C}^*)^2$ and δ_{P_i} is a Dolbeault $(0,2)$-form with support concentrated at P_i as $\hbar \to 0$. The MC solution Φ can be decomposed as

$$\Phi = \Pi + \Xi^{0,0} + \Xi^{1,1},$$

with $\Xi^{i,i} \in (\mathcal{G}/\mathcal{I})^{i,i}$, and the behavior of each term as $\hbar \to 0$ is described by our main result:

Theorem 1.1 (=Theorem 4.12)**.** *Each of the terms $\Xi^{0,0}, \Xi^{1,1}$ of the Maurer-Cartan solution Φ can be expressed as a sum over tropical disks Γ whose moduli space $\overline{\mathfrak{M}}^{\Gamma}$ is non-empty of codimension $1 - \frac{MI(\Gamma)}{2}$ in $B_0 = \mathbb{R}^2$ (where MI denotes the Maslov index):*

$$\Xi^{0,0} = \sum_{MI(\Gamma)=2} \alpha_\Gamma \mathit{Mono}(\Gamma),$$
$$\Xi^{1,1} = \sum_{MI(\Gamma)=0} \alpha_\Gamma \mathit{Log}(\Theta_\Gamma),$$

where $\mathit{Mono}(\Gamma)$ is a holomorphic function, $\mathit{Log}(\Theta_\Gamma)$ is a holomorphic vector field defined explicitly for the tropical disk Γ, and α_Γ is a Dolbeault $(0, 1 - \frac{MI(\Gamma)}{2})$-form with support concentrated along the $(1 + \frac{MI(\Gamma)}{2})$-dimensional tropical polyhedral subset $Q_\Gamma \subset B_0$ traced out by the stop Q of the tropical disks in the moduli space $\overline{\mathfrak{M}}^{\Gamma}$ (see Definition 2.8).

Furthermore, the following properties hold:

$$\lim_{\hbar \to 0} \alpha_\Gamma|_x = 1 \qquad \text{for any } x \text{ in the interior } \mathit{Int}(Q_\Gamma) \text{ when } MI(\Gamma) = 2,$$
$$\lim_{\hbar \to 0} \int_\varrho \alpha_\Gamma = -1 \quad \text{for any } \varrho \pitchfork \mathit{Int}_{re}(Q_\Gamma) \text{ positively when } MI(\Gamma) = 0,$$

where ϱ is any affine line intersecting Q_Γ positively and transversally in its relative interior $\mathit{Int}_{re}(Q_\Gamma)$.

In particular, this theorem gives a bijective correspondence between tropical disks Γ with $MI(\Gamma) = 0$ and $\overline{\mathfrak{M}}^{\Gamma} \neq \emptyset$, walls in $\mathcal{D}$, and leading order terms of $\Xi^{1,1}$ as $\hbar \to 0$:

$$\left\{ \begin{array}{c} \text{terms in the} \\ \text{expression of } \Xi^{1,1} \end{array} \right\} \overset{\text{Theorem 1.1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Tropical disks } \Gamma \\ \text{with } MI(\Gamma) = 0 \end{array} \right\} \overset{[27]}{\longleftrightarrow} \left\{ \begin{array}{c} \text{walls } \mathbf{w}_\Gamma = (m_\Gamma, Q_\Gamma, \Theta_\Gamma) \\ \text{in the diagram } \mathcal{D} \end{array} \right\}.$$

If we fix the stop Q, or equivalently a chamber in $\mathbb{R}^2$, the above theorem gives a bijective correspondence between tropical disks Γ with $MI(\Gamma) = 2$ and $\overline{\mathfrak{M}}^{\Gamma} \neq \emptyset$ and leading order terms of $\Xi^{0,0}$ as $\hbar \to 0$:

$$\left\{ \begin{array}{c} \text{terms in the} \\ \text{expression of } \Xi^{0,0} \end{array} \right\} \overset{\text{Theorem 1.1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Tropical disks } \Gamma \\ \text{with } MI(\Gamma) = 2 \end{array} \right\} \overset{[27]}{\longleftrightarrow} \left\{ \begin{array}{c} \text{terms in the} \\ \text{perturbation } W_n(Q) \end{array} \right\}.$$

In the case of $X = \mathbb{P}^2$, the tropical disks in these correspondences are exactly those considered by Gross [27]. Indeed, by Theorem 4.12, we have

$$\lim_{\hbar \to 0} \Xi^{0,0}(Q) = \sum_{MI(\Gamma)=2} \mathrm{Mono}(\Gamma),$$

which coincides with Gross' definition of the n-pointed potential $W_n(Q)$.

Underlying the above bijective correspondences is an interplay between the differential-geometric properties of the dgBV algebra $(\mathcal{G}/\mathcal{I})^{*,*}$ and the combinatorial properties of tropical counts (which also play an important role in [10]). We will see in Section 3.3.1 that his leads to an extended version of the tropical Lie algebra which we call the *tropical dgLa*.

1.3. **Wall-crossing from Maurer-Cartan solutions.** Besides giving enumerative meanings to the MC solution Φ, Theorem 1.1, together with the main results in [10], have an interesting corollary which can be viewed as providing an alternative proof of the wall-crossing formula ([27, Theorem 4.12]) for Gross' perturbed potential W_n.

By definition the scattering diagram $\mathscr{D} = \mathscr{D}(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ consists of walls $\mathbf{w}_\Gamma = (m_\Gamma, Q_\Gamma, \Theta_\Gamma)$, the support of each is the tropical polyhedral subset Q_Γ traced out by the stop of a tropical disk Γ (see Section 2.3.3). The intersection of these Q_Γ's is the set $\mathrm{Sing}(\mathscr{D}) \setminus \{P_1, \ldots, P_n\}$, where $\mathrm{Sing}(\mathscr{D})$ denotes the singular set of $\mathscr{D}$, and a point $\mathfrak{j} \in \mathrm{Sing}(\mathscr{D}) \setminus \{P_1, \ldots, P_n\}$ is called a *joint* in the Gross-Siebert program [33].

Restricting to a contractible open neighborhood $U = U_\mathfrak{j} \subset \mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}$ containing a single joint $\mathfrak{j}$, we have
$$\Phi|_U = \mathit{\Xi}^{0,0} + \mathit{\Xi}^{1,1}$$
because $\delta_{P_i}|_U = 0$ and hence $\Pi|_U = 0$ in $(\mathcal{G}/\mathcal{I})^{*,*}$. By degree reasons, we see that $\mathit{\Xi}^{1,1}|_U$ is itself a solution to the non-extended Maurer-Cartan equation (1.2), namely,
$$\bar{\partial}\mathit{\Xi}^{1,1} + \frac{1}{2}[\mathit{\Xi}^{1,1}, \mathit{\Xi}^{1,1}] = 0.$$

Now [10, Theorems 1.5 and 1.6], the proofs of which were by asymptotic analysis and completely different from that of [27], imply the following statement which originally appeared in Gross [27]:

Corollary 1.2 (Proposition 4.7 in [27])**.** *For any point $\mathfrak{j} \in \mathrm{Sing}(\mathscr{D}) \setminus \{P_1, \ldots, P_n\}$, we have*
$$\Theta_{\gamma_\mathfrak{j}, \mathscr{D}} = Id$$
for any loop $\gamma_\mathfrak{j}$ around $\mathfrak{j}$ in a sufficiently small contractible neighborhood $U_\mathfrak{j}$ of $\mathfrak{j}$.

Next we would like to work locally near a wall Q_Γ of the scattering diagram $\mathscr{D}$. So we consider a contractible open subset $U \subset M_\mathbb{R} \setminus (\{P_1, \ldots, P_n\} \cup \mathrm{Sing}(\mathscr{D}))$, which is separated into two chambers U_+ and U_- by the wall $Q_\Gamma \cap U$ as shown in following Figure 1 (here $U_\pm$ are chosen according to the orientation of the ray Q_Γ).

Results from [10, Section 4] imply that
$$\varphi := \begin{cases} \mathrm{Log}(\Theta_\Gamma) & \text{on } U_+ \\ 0 & \text{on } U_- \end{cases} \tag{1.5}$$
is the unique gauge solving the equation
$$e^{\mathrm{ad}_\varphi}\bar{\partial}e^{-\mathrm{ad}_\varphi} = \bar{\partial} - \left[\left(\frac{e^{\mathrm{ad}_\varphi} - \mathrm{Id}}{\mathrm{ad}_\varphi}\right)\bar{\partial}(\varphi), \cdot\right] = \bar{\partial} + [\mathit{\Xi}^{1,1}, \cdot] \tag{1.6}$$
and satisfying the condition that $\varphi|_{U_-} = 0$. The Maurer-Cartan solution $\Xi^{1,1}$ behaves like a delta-function supported on the wall Q_Γ, while the local gauge φ in U behaves like a step-function jump across the wall Q_Γ as shown in the following Figure 1.

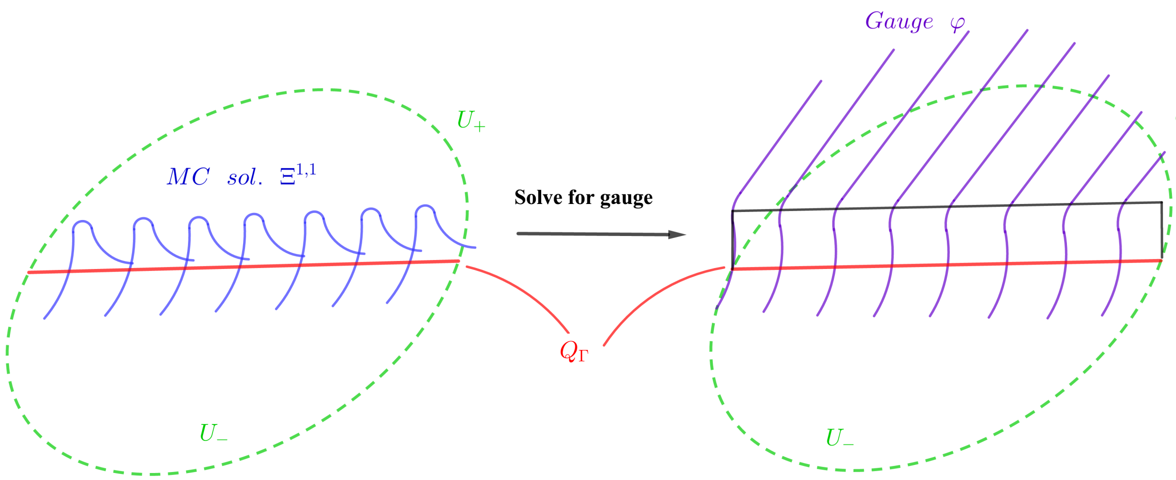

FIGURE 1. The gauge φ as step-function

When the extended MC equation (3.3) is restricted to U, we have $\Phi|_U = \varXi^{0,0} + \varXi^{1,1}$ and it decomposes into two equations:

$$\bar{\partial}\varXi^{1,1} + \frac{1}{2}[\varXi^{1,1}, \varXi^{1,1}] = 0,$$
$$\bar{\partial}\varXi^{0,0} + [W, \varXi^{1,1}] + [\varXi^{0,0}, \varXi^{1,1}] = 0.$$

The second equation can be rewritten as

$$\left(\bar{\partial} + [\varXi^{1,1}, \cdot]\right)\left(W + \varXi^{0,0}\right) = 0,$$

saying precisely that $W + \varXi^{0,0}$ is a holomorphic function with respect to the complex structure defined by $\bar{\partial} + [\varXi^{1,1}, \cdot]$. Since $e^{\mathrm{ad}_\varphi}\bar{\partial}e^{-\mathrm{ad}_\varphi} = \bar{\partial} + [\varXi^{1,1}, \cdot]$, this is equivalent to saying that the function $e^{-\mathrm{ad}_\varphi}(W + \varXi^{0,0})$, which is globally defined on U, is holomorphic with respect to the *original* Dolbeault operator $\bar{\partial}$.

Now letting $W_{n,\pm} := (W + \varXi^{0,0})\,|_{U_\pm}$ on $U_\pm$ respectively, we have

$$e^{-\mathrm{ad}_\varphi}(W + \varXi^{0,0}) = \begin{cases} \Theta_\Gamma^{-1}(W_{n,+}) & \text{on } U_+, \\ W_{n,-} & \text{on } U_-. \end{cases}$$

So the fact that $e^{-\mathrm{ad}_\varphi}(W + \varXi^{0,0})$ is a globally defined function on U implies that

$$\Theta_\Gamma(W_{n,-}) = W_{n,+}.$$

Applying this to a finite number of walls, we obtain the following wall-crossing formula which originally appeared in Gross [27]:

Corollary 1.3 (Theorem 4.12 in Gross [27])**.** *If $Q, Q' \in \mathbb{R}^2$ are not lying on any walls in the scattering diagram $\mathcal{D}$, then we have*

$$W_n(Q') = \Theta_{\gamma,\mathcal{D}}(W_n(Q)), \tag{1.7}$$

for any path $\gamma \subset M_\mathbb{R} \setminus Sing(\mathcal{D})$ joining Q to Q'.

1.4. **Remarks.** We end this introduction by a couple remarks.

(1) Just as in Gross [27], the tropical curves which appear here (see Section 2) correspond to holomorphic curves in the toric variety X which are transversal to the toric divisor D_∞, so the tropical counts would give the correct Gromov-Witten invariants only when X is a product of projective spaces (i.e. when $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ in the 2-dimensional case). One way to get the correct Gromov-Witten invariants in general is to apply the technique of *tropical modification* which deforms the ambient space so that the hidden curves (i.e. curves lying inside the toric divisor D_∞) can be seen; see e.g. [36] for an introduction. We expect that our results would hold if we use the corrected tropical counts and replace the Hori-Vafa potential W by the Lagrangian Floer potential W^{LF}.

(2) To generalize our results to higher dimensions, one again needs a proper definition of tropical counts. In that case, we shall allow interior insertions of tropical cycles of different codimensions, instead of just points in generic position. It should be straightforward to generalize our results concerning only points insertions. For other tropical cycles, what is missing is a description of such cycles by means of elements in the tropical dgLa as in equation (1.4).

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2. Tropical counting in dimension 2

We fix once and for all a rank 2 lattice M together with its dual lattice N, and write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ for the real vector spaces spanned by them respectively. We will use m to denote an element in M and n to denote an element in N. We let $\Sigma \subset M_{\mathbb{R}}$ be a complete rational polyhedral fan and X_Σ be the associated toric surface and take $K := \mathcal{K}_{X_\Sigma} \cap H^2(X_\Sigma, \mathbb{Z})$ to be the monoid of integral Kähler forms, where $\mathcal{K}_{X_\Sigma}$ is the Kähler cone of X_Σ. We also use $\Sigma(1)$ to denote the set of 1-dimensional cones in Σ and D_ρ to be the toric divisor corresponding to $\rho \in \Sigma(1)$. We will define the counting of tropical disks following [44, 27], with slight modifications.

2.1. **Tropical disks.**

Notations 2.1. *If we fix a Lagrangian torus fiber L of the moment map $p : X_\Sigma \to \mathbf{P}$, then $\pi_2(X, L) \cong \sum_{\rho\in\Sigma(1)} \mathbb{Z}\cdot m_\rho$ is freely generated by the classes m_ρ's of Maslov index 2 holomorphic disks in (X, L), where m_ρ is the disk class intersecting the toric divisor D_ρ exactly once.*

We let $\mathcal{P}$ to be the nonnegative cone $\pi_2(X, L)_{\geq 0} \subset \pi_2(X, L)$ generated by those classes $\beta \in \pi_2(X, L)$ satisfying $\int_\beta \omega \geq 0$ for all $\omega \in K$, and $Q := \pi_2(X) \cap \pi_2(X, L)_{\geq 0}$ to be the effective cone (or the Mori cone) of X_Σ. We will have $\mathcal{P}^{gp} = \pi_2(X, L)$ and $Q^{gp} = \pi_2(X)$, where $\mathcal{P}^{gp}$ and Q^{gp} are the abelian groups associated to $\mathcal{P}$ and Q respectively. Then we have the following exact sequence of monoids

$$0 \longrightarrow Q \longrightarrow \mathcal{P} \xrightarrow{\theta} M \longrightarrow 0 \tag{2.1}$$

where M is identified with $\pi_1(L)$ and $\theta(\beta) = \partial\beta$ defined by taking boundary of the class $\beta \in \mathcal{P}$.

We will use m to denote an element of $\mathcal{P}$ and $\bar{m}$ to denote its image $\theta(m)$ in M in the rest of this paper. Notice that $\theta : \mathcal{P} \to M$ maps the standard basis $\{m_\rho\}$ (of $\mathbb{Z}^{|\Sigma(1)|}$) to the generators of the 1-dimensional cones ρ.

Example 2.2. *We consider a fan Σ in $M_\mathbb{R} \cong \mathbb{Z}^2$, with its 1-dimensional cones given by $\rho_i = \mathbb{R}\cdot \mathsf{m}_i$ where we have $\mathsf{m}_1 = (0, 1)$, $\mathsf{m}_2 = (-1, 0)$, $\mathsf{m}_3 = (0, -1)$ and $\mathsf{m}_4 = (1, 1)$. The corresponding toric variety X_Σ is the surface F_1.*

We write $\pi_2(X_\Sigma, L) = \bigoplus_{i=1}^4 \mathbb{Z}\cdot m_i$ with m_i being the unique disk class support Maslov 2 holomorphic disk intersecting exactly once with the toric boundary D_{ρ_i}. In this case Q is the monoid of integral points in the cone generated by $-m_1 + m_2 + m_4$ and $m_1 + m_3$, while $\mathcal{P}$ is the monoid of integral points in the cone generated by m_i's together with $-m_1 + m_2 + m_4$.

Before defining tropical disks, we first introduce the combinatorial notion of k-trees:

Definition 2.3. *A* (directed) k-tree T *consists of the following data:*

- *a finite set of vertices $\bar{T}^{[0]}$ together with a decomposition*
$$\bar{T}^{[0]} = T^{[0]}_{in} \sqcup T^{[0]} \sqcup \{v_o\},$$
where $T^{[0]}_{in}$, called the set of incoming vertices, is a set of size k and v_o is called the outgoing vertex (we also write $T^{[0]}_\infty := T^{[0]}_{in} \sqcup \{v_o\}$),
- *a finite set of edges $\bar{T}^{[1]}$, and*
- *two boundary maps $\partial_{in}, \partial_o : \bar{T}^{[1]} \to \bar{T}^{[0]}$ (here ∂_{in} stands for incoming and ∂_o stands for outgoing)*

satisfying the following conditions:

1. *Every vertex $v \in T^{[0]}$ is trivalent, and satisfies $\#\partial_o^{-1}(v) = 2$ and $\#\partial_{in}^{-1}(v) = 1$.*
2. *Every vertex $v \in T^{[0]}_{in}$ has valency one, and satisfies $\#\partial_o^{-1}(v) = 0$ and $\#\partial_{in}^{-1}(v) = 1$; we let $T^{[1]} := \bar{T}^{[1]} \setminus \partial_{in}^{-1}(T^{[0]}_{in})$.*
3. *For the outgoing vertex v_o, we have $\#\partial_o^{-1}(v_o) = 1$ and $\#\partial_{in}^{-1}(v_o) = 0$; we let $e_o := \partial_o^{-1}(v_o)$ be the outgoing edge and denote by $v_r \in T^{[0]}_{in} \sqcup T^{[0]}$ the unique vertex (which we call the root vertex) with $e_o = \partial_{in}^{-1}(v_r)$.*

(4) *The* topological realization $|\bar{T}| := \left(\coprod_{e\in\bar{T}^{[1]}}[0,1]\right)/\sim$ *of the tree* T *is connected and simply connected; here* $\sim$ *is the equivalence relation defined by identifying boundary points of edges if their images in* $T^{[0]}$ *are the same.*

Two k*-trees* T_1 *and* T_2 *are* isomorphic *if there are bijections* $\bar{T}_1^{[0]} \cong \bar{T}_2^{[0]}$ *and* $\bar{T}_1^{[1]} \cong \bar{T}_2^{[1]}$ *preserving the decomposition* $\bar{T}_i^{[0]} = T_{i,in}^{[0]} \sqcup T_i^{[0]} \sqcup \{v_{i,o}\}$ *and boundary maps* $\partial_{i,in}$ *and* $\partial_{i,o}$*. The set of isomorphism classes of* k*-trees will be denoted by* $\mathbb{T}_k$*. For a* k*-tree* T*, we will abuse notations and use* T *(instead of* $[T]$*) to denote its isomorphism class.*

Now we consider n points $P_1,\ldots,P_n \in M_{\mathbb{R}}$.

Notations 2.4. *As in* [27, 28]*, we introduce the following ring :*

$$R = R_n := \frac{\mathbb{C}[u_1,\ldots,u_n]}{(u_i^2 \mid 1 \le i \le n)}, \tag{2.2}$$

where we associate to each point P_i *the variable* u_i*. This ring has the maximal ideal* $\mathbf{m} = \mathbf{m}_n := (u_1,\ldots,u_n)$*.*

Definition 2.5. *A* weighted d-pointed k-tree *is a* $(k+d)$*-tree* Γ *together with*

- *an injective map* $p : \{1,\ldots,d\} \hookrightarrow \partial_{in}^{-1}(\Gamma_{in}^{[0]})$ *(we will write* p_j *to stand for the image* $p(j) \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$*),*
- *a weight* $m : \bar{\Gamma}^{[1]} \to \mathcal{P}$ *(we will write* m_e *to stand for the image* $m(e) \in \mathcal{P}$*) and*
- *a map* $u : \bar{\Gamma}^{[1]} \to R_n$ *(we will write* u_e *to stand for the image* $u(e) \in R_n$*)*

satisfying the following conditions:

(1) *For every* $e \in \bar{\Gamma}^{[1]}$*,* $u_e \in R_n$ *is a monomial. Furthermore, for each* $j = 1,\ldots,d$*, we have* $u_{p_j} = u_{i_j}$ *for some* $i_j \in \{1,\ldots,n\}$ *such that* $1 \le i_1 < i_2 < \cdots < i_d \le n$*, and for each* $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1,\ldots,p_d\}$*, we have* $u_e = 1$*.*
(2) *For every trivalent vertex* $v \in \Gamma^{[0]}$ *attached with two incoming edges* e_1, e_2 *and an outgoing edge* e_3*,*
   (a) *at least one of* e_1, e_2 *do not belong to* $\partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1,\ldots,p_d\}$*,*
   (b) $u_{e_3} = u_{e_1} \cdot u_{e_2}$ *and*
   (c) $m_{e_3} = m_{e_1} + m_{e_2}$*.*
(3) $m_e = 0$ *if and only if* $e \in \{p_1,\ldots,p_d\}$*.*
(4) *For every* $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1,\ldots,p_d\}$*, we have* $m_e = m_\rho$ *for some* $\rho \in \Sigma(1)$*.*

Two weighted d*-pointed* k*-trees* Γ_1 *and* Γ_2 *are said to be* isomorphic *if they are isomorphic as* k*-trees and the isomorphism preserves the marked points* p_i*'s and the weight functions* m_e*'s. The set of isomorphism classes of weighted* d*-pointed* k*-trees will be denoted by* $\mathtt{WPT}_{k,d}$*. For a weighted* d*-pointed* k*-tree* Γ*, we will again abuse notations and use* Γ *(instead of* $[\Gamma]$*) to stand for its isomorphism class.*

Notations 2.6. *Given a weighted* d*-pointed* k*-tree* Γ *(*$d \ge 0$ *and* $k \ge 1$*), we will write* $\Gamma_{in}^{[1]} := \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1,\ldots,p_d\}$ *for the set of incoming edges excluding those which correspond to the marked points.*

Given any $e \in \bar{\Gamma}^{[1]} \setminus \{p_1,\ldots,p_d\}$*, we let* $k_e = 0$ *if* $\bar{m}_e = 0$*, and when* $\bar{m}_e \neq 0$*, we let* $k_e \in \mathbb{Z}_{\ge 0}$ *be the unique positive integer such that* $\bar{m}_e = k_e \hat{m}_e$ *where* $\hat{m}_e \in M$ *is the primitive element.*

The integers $\{k_e\}_{e\in\bar{\Gamma}[1]\backslash\{p_1,\dots,p_d\}}$ *define the* weight function $w_\Gamma : \bar{\Gamma}[1]\backslash\{p_1,\dots,p_d\} \to \mathbb{Z}_{\geq 0}$ *(hence the name "weighted k-tree") and the formula* $\bar{m}_{e_2} = \bar{m}_{e_1} + \bar{m}_{e_0}$ *corresponds to the* balancing condition, *both of which appear in the original definition of tropical curves in* [44, 27].

We will also write m_Γ *(or* k_Γ*) and* u_Γ *for the weight and monomial, respectively, associated to the unique outgoing edge* e_o *attached to the unique outgoing vertex* v_o *of* Γ.

Definition 2.7. *Given a d-pointed weighted k-tree* Γ, *we define the* multiplicity at a trivalent vertex $v \in \Gamma_0^{[0]} := \Gamma^{[0]} \setminus \partial_o(\{p_1,\dots,p_d\})$ *by*

$$Mult_v(\Gamma) := |\det(\bar{m}_{e_1}, \bar{m}_{e_2})| = |\det(\bar{m}_{e_1}, \bar{m}_{e_3})| = |\det(\bar{m}_{e_2}, \bar{m}_{e_3})|,$$

where e_1, e_2 *are the incoming edges and* e_3 *the outgoing edge attached to* v. *Then we define the* multiplicity $Mult(\Gamma)$ *of* Γ *by*

$$Mult(\Gamma) := \prod_{v\in\Gamma_0^{[0]}} Mult_v(\Gamma).$$

Note that at a trivalent vertex $v \in \Gamma_0^{[0]}$ with incoming edges e_1, e_2, the multiplicity $\mathrm{Mult}_v(\Gamma) \neq 0$ if and only if $\bar{m}_{e_1}, \bar{m}_{e_2}$ are linearly independent in $M_\mathbb{R}$.

Given a weighted d-pointed k-tree Γ, a *realization* of Γ is defined as

$$|\Gamma_{\vec{s}}| := \left(\left(\coprod_{e\in\partial_{in}^{-1}(\Gamma_{in}^{[0]})} (\mathbb{R}_{\leq 0})_e\right) \sqcup \left(\coprod_{e\in\Gamma^{[1]}} [s_e, 0]\right)\right) / \sim,$$

for a set of parameters $\vec{s} := (s_e)_{e\in\Gamma^{[1]}} \in (\mathbb{R}_{<0})^{|\Gamma^{[1]}|}$; here $(\mathbb{R}_{\leq 0})_e$ is just a copy of $\mathbb{R}_{\leq 0}$ and $\sim$ is the equivalence relation defined by identifying boundary points of edges if their images in $\Gamma^{[0]}$ are the same. The set of realizations of Γ is parametrized by $\vec{s} \in (\mathbb{R}_{<0})^{|\Gamma^{[1]}|}$.

Starting from now on, we will fix $n+1$ points $P_1,\dots,P_n,Q \in M_\mathbb{R}$. Here comes the notion of a tropical disk in $(\mathcal{P},\Sigma,P_1,\dots,P_n;Q)$.

Definition 2.8. *A* d-pointed tropical disk ς in $(\mathcal{P},\Sigma,P_1,\dots,P_n;Q)$ *consists of the following data*

- *a weighted d-pointed k-tree* Γ *with* $u_\Gamma \neq 0$,
- *a set of parameters* $\vec{s} = (s_e)_{e\in\Gamma^{[1]}} \in (\mathbb{R}_{<0})^{|\Gamma^{[1]}|}$, *and*
- *a proper map* $\varsigma : |\Gamma_{\vec{s}}| \to M_\mathbb{R}$ *from the realization* $|\Gamma_{\vec{s}}|$ *of* Γ *to* $M_\mathbb{R}$

satisfying the following conditions:

1. $\varsigma|_{(\mathbb{R}_{\leq 0})_{p_j}} \equiv P_{i_j}$ *if the monomial assigned to* p_j *is* u_{i_j}*; in particular* $\varsigma|_{(\mathbb{R}_{\leq 0})_{p_j}}$ *a constant map (playing the role of a marked point).*
2. *For each incoming edge* $e \in \Gamma_{in}^{[1]}$, *we have* $\varsigma|_{(\mathbb{R}_{\leq 0})_e}(s) = \varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) + s(-\bar{m}_e)$ *for all* $s \in \mathbb{R}_{\leq 0}$.
3. *For each* $e \in \Gamma^{[1]}$, *we have* $\varsigma|_{[s_e,0]}(s) = \varsigma|_{[s_e,0]}(0) + s(-\bar{m}_e)$ *for all* $s \in [s_e, 0]$ *(so that the image* $Im(\varsigma|_{[s_e,0]})$ *is an affine line segment with slope* $-\bar{m}_e$*).*
4. *The point* $\varsigma(v_o) := \varsigma|_{[s_{e_0},0]}(0) \in M_\mathbb{R}$ *is called the* stop *of the tropical disk* ς *and we require that* $\varsigma(v_o) = Q$.

The multiplicity $Mult(\varsigma)$ *of a tropical disk* ς *is defined as the multiplicity* $Mult(\Gamma)$ *of the underlying weighted* d*-pointed* k*-tree* Γ*. Note that* $Mult(\varsigma) \neq 0$ *if and only if the images of the two incoming edges at any trivalent vertex are intersecting transversally.*

The underlying tree Γ *is said to be the* combinatorial type *of the tropical disk* ς*. We will use* $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n; Q)$ *to denote the moduli space of tropical disks in* $(\mathcal{P}, \Sigma, P_1, \ldots, P_n; Q)$ *with a fixed combinatorial type* Γ*.*

Similarly, we define a tropical disk ς in $(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ *by allowing the stop* Q *to vary or by dropping condition (4) above, and we denote by* $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ *the moduli space of tropical disks in* $(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ *with a fixed combinatorial type* Γ*. In other words,*

$$\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n) = \bigcup_Q \mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n; Q).$$

Notice that there is a natural $\mathbb{R}_+$ *action on* $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ *given by translating the stop* $\varsigma(v_o) = Q$ *along the direction* $-\bar{m}_{e_o}$*, so we have a well-defined quotient*

$$\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n)/\mathbb{R}_+,$$

which can be regarded as the moduli space of tropical disks as the stop Q *goes to infinity along the direction* $-\bar{m}_{e_o}$*; see* [27, 28]*.*

We further define a tropical disk ς in $(\mathcal{P}, \Sigma)$ *with a fixed combinatorial type* Γ *by dropping condition (4) above and replacing condition (1) by only requiring that* $\varsigma|_{(\mathbb{R}_{\leq 0})_{p_j}}$ *is a constant map for each* $j = 1, \ldots, d$

The reader may ask why all the internal vertices $\Gamma^{[0]}$ are required to be trivalent. Indeed we have only defined *generic* tropical disks and the above moduli spaces are all noncompact. We use this approach because this suffices for the purpose of tropical counting. To compactify these moduli spaces, we need to allow the intervals $[s_e, 0]$'s corresponding to the internal edges $e \in \Gamma^{[1]}$ to shrink to zero lengths (i.e. by allowing $s_e = 0$), so that some internal vertices are allowed to be of higher valencies.

We use $\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ to denote the moduli space of tropical disks Γ in $(\mathcal{P}, \Sigma)$ with a fixed combinatorial type thus obtained, which gives a compactification of the union of the moduli spaces $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ as $P_1, \ldots, P_n$ vary. We will use the notation $\partial\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ to stand for the set of tropical disks with at least one degenerated internal edges (i.e. $s_e = 0$ for some $e \in \Gamma^{[1]}$).

It is not hard to see that

$$\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \cong (\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|} \times M_\mathbb{R},$$

where the first component $\vec{s} \in (\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|}$ parametrizes the realization $|\Gamma_{\vec{s}}|$ of Γ and the second component $M_\mathbb{R}$ parametrizes the stop $\varsigma(v_o)$. Its dimension is given by

$$\dim_\mathbb{R}(\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)) = |\Delta(\Gamma)| + d + 1, \tag{2.3}$$

where $|\Delta(\Gamma)| := k$ for a d-pointed weighted k-tree Γ. This moduli space has a natural stratification coming from the one on $(\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|}$ given naturally by the coordinate hyperplanes $s_e = 0$.

We will also need to consider the partial compactification

$$\hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) := \left(\overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \setminus \{\varsigma \mid s_{e_o} = 0\}\right)/\mathbb{R}_+,$$

and denote by $\partial\hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma)$ the set of tropical disks with $s_e = 0$ for some $e \in \Gamma^{[1]} \setminus \{e_o\}$.

Definition 2.9. *We define the evaluation maps*

$$ev_* : \overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \to M_{\mathbb{R}},$$

where $* \in \{1,\ldots,d\} \cup \{o\}$, to be the evaluation at a marked point $ev_*(\varsigma) = \varsigma(p_*)$ when $* \in \{1,\ldots,d\}$, and the evaluation at the outgoing vertex $ev_*(\varsigma) = \varsigma(v_o)$ if $* = o$. We put these evaluation maps together to obtain the map

$$\vec{ev} = (ev_1,\ldots,ev_d,ev_o) : \overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \to M^{d+1}_{\mathbb{R}}.$$

Similarly, we have the evaluation map

$$\hat{ev} = (ev_1,\ldots,ev_d) : \hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \to M^{d}_{\mathbb{R}}.$$

Definition 2.10. *We say that n distinct points $P_1,\ldots,P_n$ are in* generic position *if for any $d \leq n$, any d-tuple $(P_{i_1},\ldots,P_{i_d})$ is not lying in the image $\hat{ev}(S)$ of a stratum $S \subset \hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma)$ over which the evaluation map $\hat{ev}$ is degenerated, meaning that the differential $D(\hat{ev}|_S)$ is not surjective (notice that $\hat{ev}|_S$ is an affine map and hence $D(\hat{ev}|_S)$ is a well-defined constant linear map), and this holds for any combinatorial type Γ.*

We say that $n+1$ distinct points $P_1,\ldots,P_n,Q$ are in generic position *if the n points $P_1,\ldots,P_n$ are in generic position, and for any $d \leq n$ and any d-tuple $(P_{i_1},\ldots,P_{i_d})$, the $(d+1)$-tuple $(P_{i_1},\ldots,P_{i_d},Q)$ is not lying in the image $\vec{ev}(S)$ of a stratum $S \subset (\overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma))$ over which the evaluation map $\vec{ev}$ is degenerated and this holds for any combinatorial type Γ.*

Definition 2.11. *We define the* Maslov index $MI(\Gamma)$ *of a weighted d-pointed k-tree Γ by*

$$MI(\Gamma) = 2(k-d),$$

and define the Maslov index $MI(\varsigma)$ of a tropical disk ς to be that of its combinatorial type Γ.

We have the following lemma from [27].

Lemma 2.12 (Lemma 2.6 in [27])**.** *If $P_1,\ldots,P_n,Q$ are in generic position and $MI(\Gamma) = 2r$, then $\mathfrak{M}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n;Q)$ is an $(r-1)$-dimensional (over $\mathbb{R}$) affine linear subspace of $\overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \setminus \partial\overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma)$; in particular, $\mathfrak{M}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n;Q) = \emptyset$ when $r \leq 0$.*

If $P_1,\ldots,P_n$ are in generic position and $MI(\Gamma) = 2r$, then $\mathfrak{M}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n)/\mathbb{R}_+$ is an r-dimensional (over $\mathbb{R}$) affine linear subspace of $\hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma) \setminus \partial\hat{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma)$; in particular, $\mathfrak{M}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n)/\mathbb{R}_+ = \emptyset$ when $r < 0$.

2.2. **Perturbing the Landau-Ginzburg potential.** Following [28], we will define a monomial $\mathrm{Mono}(\varsigma)$ associated to each Maslov index 2 tropical disk ς for defining the n-pointed Landau-Ginzburg (abbrev. LG) potential as a perturbation of the Hori-Vafa mirror family $(\check{\mathcal{X}},W)$ [35] (here $W : \check{\mathcal{X}} \to \mathbb{C}$ is the LG potential) of X_Σ as follows.

We let $\mathbb{C}[\mathcal{P}] := \mathbb{C}[z^m \mid m \in \mathcal{P}]$ be the ring of polynomial functions on the affine toric variety $\check{\mathcal{X}} := \mathrm{Spec}(\mathbb{C}[\mathcal{P}])$. The universal piecewise linear function $\psi : |\Sigma| = M_{\mathbb{R}} \to Q^{gp}$ is strictly convex and therefore we have a unique primitive element $m_\rho \in \mathcal{P}$ for each $\rho \in \Sigma(1)$ as mentioned in Notations 2.1. We further let $W = \sum_{\rho\in\Sigma(1)} z^{m_\rho}$ be the Hori-Vafa Landau-Ginzburg potential on $\check{\mathcal{X}}$.

Furthermore, we let $\mathcal{S} = \mathrm{Spec}(\mathbb{C}[Q])$ be the affine toric variety whose ring of polynomial functions is given by $\mathbb{C}[Q]$. There is a natural morphism $\pi : \check{\mathcal{X}} \to \mathcal{S}$ induced by the map $Q \to \mathcal{P}$ which appears in the exact sequence (2.1).

As a result we obtain a toric degeneration of the Landau-Ginzburg model as in [28]:

$$\begin{array}{ccc} \check{\mathcal{X}} & \xrightarrow{W} & \mathbb{C} \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{S} & & \end{array} \tag{2.4}$$

We define Mono(ς) as an element in $\mathbb{C}[\mathcal{P}]$ as follows.

Definition 2.13. *Given a tropical disk $\varsigma \in \mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \ldots, P_n; Q)$ with $MI(\varsigma) = 2$, we define*

$$Mono(\varsigma) := Mult(\Gamma) z^{m_\Gamma} u_\Gamma$$

as the monomial in $\mathbb{C}[\mathcal{P}]$ associated to ς, where $m_\Gamma \in \mathcal{P}$ is the weight and u_Γ is the monomial associated to the unique outgoing edge e_o of ς as in Definition 2.5.

Definition 2.14. *Fixing the points $P_1, \ldots, P_n, Q$ in generic position, we follow* [28] *to define the n-*pointed Landau-Ginzburg (LG) potential *as*

$$W_n(Q) := \sum_{\varsigma} Mono(\varsigma),$$

where the sum is over all Maslov index 2 tropical disks ς in $(P_1, \ldots, P_n; Q)$.

We notice that the 0-pointed LG potential $W_0(Q) = W$ is nothing but the Hori-Vafa potential. Therefore the n-pointed potential $W_n(Q)$ should be viewed as a higher order deformation (in formal variables u_i's) of the Hori-Vafa potential W by tropical disks.

2.3. **Scattering diagram from the Maslov index 0 disks.** According to [27, 28], the dependence of the n-pointed LG potential $W_n(Q)$ on Q is governed by a scattering diagram constructed from the Maslov index 0 tropical disks. Here we recall the definition of scattering diagrams from [10, Section 3] with slight modifications; the original definition was due to Kontsevich-Soibelman [39] and can be found in [29].

2.3.1. *Tropical vertex group.* We consider $\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N$, whose general elements are finite linear combinations of elements of the form $z^m \otimes \check{\partial}_n$ (here $\check{\partial}_n$ is a holomorphic vector field associated to $n \in N$ to be defined in (3.7)). We also define the Lie-bracket $[\cdot,\cdot]$ on $\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N$ via the following formula from [29]:

$$\left[z^m \otimes \check{\partial}_n, z^{m'} \otimes \check{\partial}_{n'}\right] = z^{m+m'} \check{\partial}_{(\bar{m}',n)n' - (\bar{m},n')n}; \tag{2.5}$$

here $(\cdot,\cdot)$ is the natural pairing between M and N. We consider the Lie algebra

$$\mathfrak{g} := (\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N) \otimes_{\mathbb{C}} R_n,$$

where R_n is the formal power series ring R_n in (2.2) equipped with its maximal ideal $\mathbf{m}$.

Definition 2.15. *The* tropical Lie-algebra *over $R = R_n$ is defined to be the nilpotent Lie subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$ given explicitly by*

$$\left(\bigoplus_{m \in \mathcal{P} \setminus \{0\}} \mathbb{C} \cdot z^m \otimes_{\mathbb{Z}} (m^{\perp}) \right) \otimes_{\mathbb{C}} R \to \mathfrak{g}.$$

The tropical vertex group *is defined as the exponential group of $\mathfrak{h}$.*

Definition 2.16. *Given $m \in \mathcal{P} \setminus \{0\}$ and $n \in m^{\perp}$, we let*

$$\mathfrak{h}_{m,n} := (\mathbb{C}[z^m] \cdot z^m) \check{\partial}_n \otimes_{\mathbb{C}} \mathbf{m} \hookrightarrow \mathfrak{h},$$

whose general elements are of the form

$$\sum_{k \geq 1} \sum_{I} a_{k,I} z^{km} \check{\partial}_n u_I,$$

where $I \subset \{1, \ldots, n\}$. This defines an abelian Lie subalgebra of $\mathfrak{h}$ by (2.5).

Definition 2.17. *A* wall $\mathbf{w}$ *over R is a triple (m, Q, Θ), where*

- *$m \in \mathcal{P} \setminus \{0\}$,*
- *Q, called the* support *of $\mathbf{w}$, is a connected oriented codimension one convex tropical polyhedral subset of $M_{\mathbb{R}}$, meaning that it is a connected convex subset locally defined by affine linear equations and inequalities defined over $\mathbb{Q}$,*
- *$\Theta \in \exp(\mathfrak{h}_{m,n_Q})$, where $n_Q \in N$ is the unique primitive element satisfying $n_Q \in (TQ)^{\perp}$ and $(\nu_Q, n) < 0$, and $\nu_Q \in M_{\mathbb{R}}$ here is a vector normal to Q such that the orientation of $TQ \oplus \mathbb{R} \cdot \nu_Q$ agrees with that of $M_{\mathbb{R}}$.*

Definition 2.18. *A* scattering diagram $\mathcal{D}$ *over $R = R_n$ is a finite set of walls $\{(m_\alpha, Q_\alpha, \Theta_\alpha)\}_\alpha$.*

Notations 2.19. *For the scattering diagram $\mathcal{D}$, we will define the* support *of $\mathcal{D}$ to be*

$$supp(\mathcal{D}) = \bigcup_{\mathbf{w} \in \mathcal{D}} Q_{\mathbf{w}},$$

and the singular set *of $\mathcal{D}$ to be*

$$Sing(\mathcal{D}) = \bigcup_{\mathbf{w} \in \mathcal{D}} \partial Q_{\mathbf{w}} \cup \bigcup_{\mathbf{w}_1 \pitchfork \mathbf{w}_2} (Q_{\mathbf{w}_1} \cap Q_{\mathbf{w}_2}),$$

where $\mathbf{w}_1 \pitchfork \mathbf{w}_2$ means transversally intersecting walls.

2.3.2. *Path ordered products.* An embedded path

$$\gamma : [0, 1] \to B_0 \setminus \mathrm{Sing}(\mathcal{D})$$

is said to be *intersecting* $\mathcal{D}$ *generically* if $\gamma(0), \gamma(1) \notin \mathrm{supp}(\mathcal{D})$, $\mathrm{Im}(\gamma) \cap \mathrm{Sing}(\mathcal{D}) = \emptyset$ and it intersects all the walls in $\mathcal{D}$ transversally. Given such an embedded path γ, we define the *path ordered product* along γ, denoted by

$$\Theta_{\gamma,\mathcal{D}} = \prod_{\mathbf{w} \in \mathcal{D}}^{\gamma} \Theta_{\mathbf{w}} \in \exp(\mathfrak{h} \otimes_R \mathbf{m})$$

following [29].

Explicitly, we will have a sequence of real numbers

$$0 = t_0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = 1$$

such that $\{\gamma(t_1), \ldots, \gamma(t_s)\} = \gamma \cap \mathrm{supp}(\mathscr{D})$.

For each $1 \leq i \leq s$, there are walls $\mathbf{w}_{i,1}, \ldots, \mathbf{w}_{i,l_i}$ in $\mathscr{D}$ such that $\gamma(t_i) \in P_{i,j} := \mathrm{supp}(\mathbf{w}_{i,j})$ for all $j = 1, \ldots, l_i$. Since γ does not hit $\mathrm{Sing}(\mathscr{D})$, we have $\mathrm{codim}(\mathrm{supp}(\mathbf{w}_{i,j_1}) \cap \mathrm{supp}(\mathbf{w}_{i,j_2})) = 1$ for any j_1, j_2, i.e. the walls $\mathbf{w}_{i,1}, \ldots, \mathbf{w}_{i,l_i}$ are overlapping with each other and contained in a common tropical hyperplane. Then we have an element

$$\Theta_{\gamma(t_i)} := \prod_{j=1}^{k} \Theta_{\mathbf{w}_{i,j}}^{\sigma_j},$$

where

$$\sigma_j = \begin{cases} 1 & \text{if orientation of } P_{i,j} \oplus \mathbb{R} \cdot \gamma'(t_i) \text{ agree with that of } M_{\mathbb{R}}, \\ -1 & \text{if orientation of } P_{i,j} \oplus \mathbb{R} \cdot \gamma'(t_i) \text{ does not agree with that of } M_{\mathbb{R}}; \end{cases}$$

this element is well defined without prescribing the order of the product since the elements $\Theta_{\mathbf{w}_{i,j}}$'s are commuting with each other.

Finally, we take the ordered product along the path γ as

$$\Theta^k_{\gamma,\mathscr{D}} := \Theta_{\gamma(t_s)} \cdots \Theta_{\gamma(t_i)} \cdots \Theta_{\gamma(t_1)}.$$

Definition 2.20. *A scattering diagram $\mathscr{D}$ is said to be* consistent *if we have*

$$\Theta_{\gamma,\mathscr{D}} = Id,$$

for any embedded loop γ intersecting $\mathscr{D}$ generically.

Two scattering diagrams $\mathscr{D}$ and $\tilde{\mathscr{D}}$ are said to be equivalent *if*

$$\Theta_{\gamma,\mathscr{D}} = \Theta_{\gamma,\tilde{\mathscr{D}}}$$

for any embedded path γ intersecting both $\mathscr{D}$ and $\tilde{\mathscr{D}}$ generically.

2.3.3. *Maslov index 0 tropical disks.*

Definition 2.21. *We define $\mathscr{D}(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ to be the scattering diagram consisting of walls $\mathbf{w}_\Gamma = (m_\Gamma, Q_\Gamma, \Theta_\Gamma)$ for each weighted d-pointed k-tree Γ with $MI(\Gamma) = 0$ and $\mathfrak{M}^\Gamma_d(\mathcal{P}, \Sigma, P_1, \ldots, P_n)/\mathbb{R}_+ \neq \emptyset$, where*

(1) the ray $Q_\Gamma \subset M_\mathbb{R}$ is given by the closure of the image of the evaluation map $ev_o : \mathfrak{M}^\Gamma_d(\mathcal{P}, \Sigma, P_1, \ldots, P_n) \to M_\mathbb{R}$ at the outgoing vertex v_o (i.e. the locus of the stop of a tropical disk ς),

(2) the Fourier mode $m_\Gamma = m_\varsigma$ is the weight associated to the outgoing edge e_o attached to the unique outgoing vertex v_o, and

(3) the wall-crossing automorphism Θ_Γ is given by the formula

$$Log(\Theta_\Gamma) = k_\Gamma Mult(\Gamma) z^{m_\Gamma} \check{\partial}_{n_\Gamma} u_\Gamma,$$

where k_Γ is introduced in Notation 2.6, u_Γ is defined as in Definition 2.13, and $n_\Gamma \in N$ is the clockwise primitive normal to Q_Γ.

We end this section by stating two of the main results in [27] describing how the perturbed Landau-Ginzburg potential $W_n(Q)$ jumps across the walls in the scattering diagram $\mathscr{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$:

Theorem 2.22 (Proposition 4.7 and Theorem 4.12 in [27])**.** *For any point $\mathfrak{j} \in Sing(\mathscr{D}) \setminus \{P_1, \dots, P_n\}$ and any loop $\gamma_{\mathfrak{j}}$ around $\mathfrak{j}$ in a sufficiently small contractible neighborhood $U_{\mathfrak{j}}$ of $\mathfrak{j}$, we have*

$$\Theta_{\gamma_{\mathfrak{j}},\mathscr{D}} = Id.$$

Furthermore, if $Q, Q' \in M_{\mathbb{R}}$ are not lying on any walls in $\mathscr{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$, then we have

$$W_n(Q') = \Theta_{\gamma,\mathscr{D}}(W_n(Q)), \tag{2.6}$$

for any path $\gamma \subset M_{\mathbb{R}} \setminus Sing(\mathscr{D})$ joining Q to Q'.

The main result Theorem 1.1 of this paper together with the main results of [10] can be used to give alternative proofs of these results, as we have seen in the introduction.

3. Extended deformation theory of the LG model

In this section, we will investigate the Maurer-Cartan equation governing the extended deformation theory of the Landau-Ginzburg model $(\check{\mathcal{X}}, W)$, and the asymptotic behavior of the Maurer-Cartan solutions when the torus fibers of the fibration $\check{p} : \check{X} \to \mathrm{Int}(\mathbf{P})$ shrink, by using the techniques developed in [10].

3.1. The dgBV algebra coming from polyvector fields.

Given a Landau-Ginzburg model $(\check{X}, W)$ equipped with a holomorphic volume form $\check{\Omega}$, one can construct a natural differential graded Batalin-Vilkovisky (dgBV) algebra on the Dolbeault resolution of the sheaf of polyvector fields on $\check{X}$:

$$PV^{i,j}(\check{X}) := \Omega^{0,j}(\check{X}, \wedge^i T^{1,0}_{\check{X}}),$$

where the degree on $PV^{i,j}(\check{X})$ is taken to be $j - i$. We briefly review this construction; see, e.g. [41].

Notations 3.1. *Given local holomorphic coordinates $u^1, \dots, u^n$ on $\check{X}$ and an ordered subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, we set*

$$d\bar{u}^I := d\bar{u}^{i_1} \wedge \cdots \wedge d\bar{u}^{i_k},\ \partial_I := \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_k}},$$

and similarly for du^I and $\bar{\partial}_I$.

First of all, the space of smooth sections of $\bigwedge^* T^{1,0}_{\check{X}}$ is equipped with a natural wedge product $\wedge$. We make use of the holomorphic volume form $\check{\Omega} = e^f du^1 \cdots du^n$ to define the BV differential Δ. Given a polyvector field of the form ∂_I where $I = \{i_1, \dots, i_k\}$, we define

$$\partial_I \lrcorner \check{\Omega} := \iota_{\frac{\partial}{\partial u^{i_1}}} \cdots \iota_{\frac{\partial}{\partial u^{i_k}}} \check{\Omega}.$$

Definition 3.2. *We define the* BV differential $\Delta_{\check{\Omega}}$ *(depending on* $\check{\Omega}$*) by*

$$\Delta_{\check{\Omega}}\alpha \lrcorner \check{\Omega} := \partial(\alpha \lrcorner \check{\Omega}). \tag{3.1}$$

We will suppress the dependence of the BV differential on $\check{\Omega}$ *when there is no danger of confusion.*

The operation $\delta_v : \bigwedge^* T^{1,0} \to \bigwedge^{*-1} T^{1,0}$ defined by

$$\delta_v(w) := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^k v \wedge \Delta(w) \tag{3.2}$$

is a derivation of degree $k+1$.

Definition 3.3. *We define the* bracket operation $[\cdot,\cdot] : V \otimes V \to V$ *by*

$$[v,w] = (-1)^{|v|+1}\delta_v(w),$$

where $|v|$ *stands for the degree of a homogeneous element* v*.*

The well-known *Schouten-Nijenhuis Lie bracket* $[\cdot,\cdot]$ on smooth section of $\bigwedge^* T^{1,0}$ can then be expressed as

$$[v_1 \wedge \cdots \wedge v_k, \mathsf{v}_1 \wedge \cdots \wedge \mathsf{v}_{k'}] = \sum_{\substack{1\leq i\leq k\\1\leq j\leq k'}} (-1)^{i+j}[v_i, \mathsf{v}_j] \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \wedge \ldots \widehat{\mathsf{v}_j} \wedge \cdots \wedge \mathsf{v}_{k'},$$

$$[v_1 \wedge \cdots \wedge v_k, f] = \sum_i (-1)^{k-i} v_i(f) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \ldots v_k.$$

These structures can be extended to the Dolbeault resolution $PV^{*,*}(\check{X})$ of $\bigwedge^* T^{1,0}$ equipped with $\bar{\partial}_W = \bar{\partial} + [W,]$ and the graded commutative wedge product $\wedge$. In the local holomorphic coordinates $u^1,\ldots,u^n$, writing $\alpha = \alpha^I_J d\bar{u}^J \wedge \partial_I$ (with $|I| = i$ and $|J| = j$) and $\beta = \beta^K_L d\bar{u}^L \wedge \partial_K$ (with $|K| = k$ and $|L| = l$), we have

$$\begin{aligned}
\bar{\partial}(\alpha) &= \bar{\partial}(\alpha^I_J) d\bar{u}^J \wedge \partial_I,\\
\Delta(\alpha) &= (-1)^j d\bar{u}^J \wedge \Delta(\alpha^I_J \partial_I),\\
\alpha \wedge \beta &= (-1)^{il} \alpha^I_J \beta^K_L d\bar{u}^J \wedge d\bar{u}^L \wedge \partial_I \wedge \partial_K,\\
[\alpha,\beta] &= (-1)^{(i+1)l} d\bar{u}^J d\bar{u}^L [\alpha^I_J \partial_I, \beta^K_L \partial_K].
\end{aligned}$$

These operations satisfy

$$\begin{aligned}
[v,w] &= -(-1)^{(|v|+1)(|w|+1)}[w,v],\\
[v,[w,u]] &= [[v,w],u] + (-1)^{(|v|+1)(|w|+1)}[w,[v,u]],\\
\Delta[v,w] &= [\Delta(v),w] + (-1)^{|v|+1}[v,\Delta(w)],\\
\bar{\partial}_W[v,w] &= [\bar{\partial}_W v, w] + (-1)^{|v|+1}[v, \bar{\partial}_W w].
\end{aligned}$$

From these we obtain the differential graded Lie algebra (dgLa) $(PV^{*,*}[1], [\cdot,\cdot], \bar{\partial}_W)$ (here [1] is a degree shift). We will be considering the asymptotic behavior of solutions of the Maurer-Cartan equation

$$\bar{\partial}_W \varphi + \frac{1}{2}[\varphi,\varphi] = 0, \tag{3.3}$$

for degree 0 elements φ in $PV^{*,*} \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}} \otimes_{\mathbb{C}} \mathbf{m}_n$. (Recall that $\mathbf{m}_n$ is the maximal ideal introduced in Notations 2.4.)

Going back to our situation, by passing the exact sequence of monoids (2.1) to the associated abelian groups, we obtain the exact sequence

$$0 \longrightarrow Q^{gp} \longrightarrow \mathcal{P}^{gp} \xrightarrow{\theta} M \longrightarrow 0\,, \tag{3.4}$$

which is usually called the *fan sequence* in toric geometry [12, 23]. We have $\mathcal{P}^{gp} \cong \mathbb{Z}^{|\Sigma(1)|} \cong M \times Q^{gp}$, giving a trivialization of the family over $\mathrm{Spec}(\mathbb{C}[Q^{gp}]) \subset \mathrm{Spec}(\mathbb{C}[Q])$ as

$$\check{\mathcal{X}} \times_{\mathrm{Spec}(\mathbb{C}[Q])} \mathrm{Spec}(\mathbb{C}[Q^{gp}]) \cong T_N \times \mathrm{Spec}(\mathbb{C}[Q^{gp}]), \tag{3.5}$$

where $T_N := (N \otimes_{\mathbb{Z}} \mathbb{C})/N$ is a 2-dimensional algebraic torus.

Since Q is a strictly convex polyhedral cone, there is a natural maximal ideal $\mathbf{m} = \mathbf{m}_Q := \langle z^m \mid m \in Q \setminus \{0\}\rangle$ in $\mathbb{C}[Q]$. We consider the completion $\widehat{\mathbb{C}[Q]} := \varprojlim_k \mathbb{C}[Q]/\mathbf{m}^k$ and its localization $\widehat{\mathbb{C}[Q]}_{\mathcal{S}}$ at the multiplicative system $\mathcal{S} = \{z^m \mid m \in Q \setminus \{0\}\}$. By taking the tensor product $\mathbb{C}[\mathcal{P}^{gp}] \otimes_{\mathbb{C}[Q^{gp}]} \widehat{\mathbb{C}[Q]}_{\mathcal{S}} = \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}}$, we can treat $W \in \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}}$ as a family of Landau-Ginzburg potentials parametrized by $\widehat{\mathbb{C}[Q]}_{\mathcal{S}}$ on the (fixed) algebraic torus T_N.

For the Landau-Ginzburg model $(\check{X}, W) := (T_N, W)$, we fix once and for all the local holomorphic coordinates as follows:

Notations 3.4. *We fix once and for all a $\mathbb{Z}$-basis e_1, e_2 for M and identify $\mathsf{m} = \mathsf{m}_1 e_1 + \mathsf{m}_2 e_2$ with $(\mathsf{m}_1, \mathsf{m}_2) \in \mathbb{Z}^2$. We also use $w^{\mathsf{m}} = (w^1)^{\mathsf{m}_1}(w^2)^{\mathsf{m}_2}$, for $\mathsf{m} = (\mathsf{m}_1, \mathsf{m}_2) \in M$, to denote a monomial on $\check{X}$.*

Notice that every $\mathsf{m} \in M$ naturally gives a $(1,0)$-form $d\log(\mathsf{m}) := d\log(w^{\mathsf{m}})$; similarly, every $n \in N$ naturally gives a vector field $\check{\partial}_n$ satisfying

$$\check{\partial}_n(w^{\mathsf{m}}) = (n, \mathsf{m}) w^{\mathsf{m}},$$

where $(\cdot,\cdot)$ is the natural pairing between M and N.

We can define a natural holomorphic volume form $\check{\Omega}$ on $\check{X}$ by

$$\check{\Omega} := d\log w^1 \wedge d\log w^2, \tag{3.6}$$

so we obtain the triple $(\check{X}, W, \check{\Omega})$, and hence a dgBV algebra by the above discussion.

3.1.1. *$\hbar$-family of SYZ fibrations.* Following a proposal by Kontsevich-Soibelman [38] and Fukaya [16], we are going to construct an $\hbar$-family of SYZ fibrations which corresponds to a large structure limit, so that we can apply asymptotic analysis as in the previous work [10].

We consider the log map $\mathrm{Log} : T_N \cong (N_{\mathbb{C}}/N) \to \sqrt{-1}N_{\mathbb{R}}$ which is naturally a torus fibration. We fix a symplectic structure ω_0 on the toric surface X_Σ and consider the associated moment polytope $\mathbf{P} \subset \sqrt{-1}N_{\mathbb{R}}$. From the SYZ viewpoint [49, 9], the mirror manifold is obtained by dualizing the moment map on X_Σ, so we choose the base of the SYZ fibration to be $\check{B}_0 := \mathrm{Int}(\mathbf{P})$ and take $\check{X} = \check{p}^{-1}(\check{B}_0)$ instead of the whole algebraic torus T_N.

Let $\{e^1, e^2\}$ be the $\mathbb{Z}$-basis of N dual to the chosen basis $\{e_1, e_2\}$ of M. We then let (x^1, x^2) be the oriented affine coordinates of $\check{B}_0$ with respect to the basis $\{e^1, e^2\}$ and (y^1, y^2) be the affine coordinates on the torus fibers of $\check{p} : \check{X} \to \check{B}_0$.

Associated to the symplectic structure ω_0, there is a symplectic potential $\check{\phi}$ in the action-angle coordinates as in [34]. We take $\check{\phi}$ and apply the Legendre transform

$$\check{L}_{\check{\phi}} : \mathrm{Int}(\mathbf{P}) \to M_{\mathbb{R}}$$

to obtain the dual integral affine manifold B_0 equipped with affine coordinates

$$x_1 := \frac{\partial \check{\phi}}{\partial x^1},\ x_2 := \frac{\partial \check{\phi}}{\partial x^2}.$$

We prefer to work with the affine manifold B_0 because then we can deal with tropical trees instead of Morse trees as mentioned in [33] (see also [10]).

We introduce a small $\hbar > 0$ parameter to rescale the affine coordinates on $\check{B}_0$ as $x^j \mapsto \hbar^{-1} x^j$, and obtain the ($\hbar$-dependent) holomorphic coordinates $w^j = \exp(-2\pi i(y^j + i\hbar^{-1}x^j))$ (cf. [10, Section 2]). Under these $\hbar$-twisted coordinates, the holomorphic vector field $\check{\partial}_j$ is explicitly given by

$$\begin{aligned} n = (n^j) \mapsto \check{\partial}_n &:= \sum_j n^j \check{\partial}_j = \sum_j n^j \frac{\partial}{\partial \log w^j} = \frac{i}{4\pi} \sum_j n^j \left(\frac{\partial}{\partial y^j} - i\hbar \frac{\partial}{\partial x^j} \right) \\ &= \frac{i}{4\pi} \sum_j n^j \left(\frac{\partial}{\partial y^j} - i\hbar \sum_k g_{jk} \frac{\partial}{\partial x_k} \right). \end{aligned} \tag{3.7}$$

We take the corresponding $\hbar$-dependent dgLa of polyvector fields, which will be denoted by $PV^{*,*}_{\hbar}$. We will always be considering differential forms on B_0 depending on $\hbar$ and hence we introduce the following notation.

Notations 3.5. *We use $\Omega^*_{\hbar}(B_0)$ (similarly for $\Omega^*_{\hbar}(U)$ for any open subset $U \subset B_0$) to denote $\Gamma(B_0 \times \mathbb{R}_{>0}, \bigwedge^* T^* B_0)$, where the extra $\mathbb{R}_{>0}$ direction is parametrized by $\hbar$.*

3.1.2. *Fourier expansions of polyvector fields.* We recall the Fourier transform $\hat{\mathcal{F}}$ introduced in [10, Section 2], which gives us an inclusion of dg Lie subalgebras (it is an inclusion since we restrict ourselves to Fourier modes m in $\mathcal{P}$, and take finite sums instead of infinite Fourier series):

$$\hat{\mathcal{F}} : \mathbf{G}^{*,*}_n := \left(\bigoplus_{m \in \mathcal{P}} \Omega^*_{\hbar}(B_0) z^m \right) \otimes_{\mathbb{Z}} \bigwedge^* N \otimes_{\mathbb{C}} R_n \hookrightarrow PV^{*,*}_{\hbar} \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}} \otimes_{\mathbb{C}} R_n \tag{3.8}$$

by

1. identifying the Fourier modes $z^m \in \mathbb{C}[\mathcal{P}] \hookrightarrow \mathbb{C}[\mathcal{P}^{gp}]$ as $z^m = w^{\bar{m}} \otimes q^{\hat{m}} \in \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}}$ through the isomorphism $\mathbb{C}[\mathcal{P}^{gp}] \cong \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}}$,
2. pulling back smooth functions $f(x, \hbar)$ on B_0 to $\check{X}$ via the equation $\hat{\mathcal{F}}(f(x, \hbar)) = \check{p}^{-1}(f(x, \hbar))$ and using the torus fibration $\check{p} : \check{X} \to B_0$,
3. identifying the 1-form $dx^j = \sum_k \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_k$ (where $\phi : B_0 \to \mathbb{R}$ is the Legendre dual to $\check{\phi}$) on B_0 with the $(0,1)$-form $\hat{\mathcal{F}}(dx^j) = \frac{\hbar}{4\pi} d \log \bar{w}^j$ on $\check{X}$ for $j = 1, 2$,

(4) identifying $n \in N$ with the holomorphic vector field $\check{\partial}_n$ on $\check{X}$ by (3.7), and
(5) extending the map skew-symmetrically.

The Dolbeaut differential $\bar{\partial}$ is identified with the deRham differential d acting on each summand $\Omega^*_\hbar(B_0)$ via $\hat{\mathcal{F}}$. The action of a vector field $\check{\partial}_n = (n^1, n^2)$ on $f(x,\hbar)$ by differentiation is identified as

$$\check{\partial}_n(f) = \frac{\hbar}{4\pi}\sum_{j,k} n^j \frac{\partial^2 \check{\phi}}{\partial x^j \partial x^k}\frac{\partial}{\partial x_k}(f) \tag{3.9}$$

via $\hat{\mathcal{F}}$ (recall that x^1, x^2 are affine coordinates on $\check{B}_0 \cong \mathrm{Int}(\mathbf{P})$ while x_1, x_2 are affine coordinates on $B_0 = M_{\mathbb{R}}$).

3.2. **Differential forms with asymptotic support.** We will be working with a dgLa constructed as a suitable quotient of a subalgebra of $\mathbf{G}^{*,*}_n$ (defined above in (3.8)), which turns out to be closely related to the tropical counting defined in Section 2. In order to do so, we need to recall the notion of *asymptotic support on a closed codimension k tropical polyhedral subset $P \subset U$* for some *convex $U \subset B_0$* which describes the behavior of differential forms $\alpha \in \Omega^*_\hbar(B_0)$ as $\hbar \to 0$ and also some of its basic properties from [10].

First of all, by a tropical polyhedral subset in U we mean a connected convex subset which is defined by *finitely many* affine linear equations or inequalities over $\mathbb{Q}$. In this paper, we will only be considering 3 cases:

(1) P is a point whence $\dim_{\mathbb{R}}(P) = 0$;
(2) P is a ray or a line whence $\dim_{\mathbb{R}}(P) = 1$;
(3) P is a polyhedral domain whence $\dim_{\mathbb{R}}(P) = 2$.

We begin with recalling the definitions and lemmas from [10] for a convex open subset $U \subset B_0$ in a general (oriented) affine manifold B_0.

Definition 3.6. *We define $\mathcal{W}^{-\infty}_k(U) \subset \Omega^k_\hbar(U)$ to be the set of differential k-forms $\alpha \in \Omega^k_\hbar(U)$ such that for each point $q \in U$, there exists a neighborhood V of q where we have*

$$\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} e^{-c_V/\hbar}$$

for some constants c_V and $D_{j,V}$. The association $U \mapsto \mathcal{W}^{-\infty}_k(U)$ defines a sheaf over B_0 which we denote by $\mathcal{W}^{-\infty}_k$.

We will also consider differential forms which only blow up at polynomial orders in $\hbar^{-1}$:

Definition 3.7. *We define $\mathcal{W}^{\infty}_k(U) \subset \Omega^k_\hbar(U)$ to be the set of differential k-forms $\alpha \in \Omega^k_\hbar(U)$ such that for each point $q \in U$, there exists a neighborhood V of q where we have*

$$\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} \hbar^{-N_{j,V}}$$

for some constants $D_{j,V}$ and $N_{j,V} \in \mathbb{Z}_{>0}$. The association $U \mapsto \mathcal{W}^{\infty}_k(U)$ defines a sheaf over B_0 which we denote by $\mathcal{W}^{\infty}_k$.

Notice that the sheaves $\mathcal{W}^{\pm\infty}_k$ in Definitions 3.6 and 3.7 are closed under the actions of $\nabla_{\frac{\partial}{\partial x}}$, the deRham differential d and the wedge product of differential forms. We also observe the fact that $\mathcal{W}^{-\infty}_k$ is a differential graded ideal of $\mathcal{W}^{\infty}_k$. In particular, we can consider the sheaf of differential graded algebras $\mathcal{W}^{\infty}_*/\mathcal{W}^{-\infty}_*$, equipped with the deRham differential.

Notations 3.8. *Let $P \subset U$ be a closed codimension k tropical polyhedral subset.*

(1) *There is a natural foliation $\{P_q\}_{q\in N}$ in U obtained by parallel transporting the tangent space of P (at some interior point in P) to every point in U by the affine connection ∇ on B_0.*

We let $\nu_P \in \Gamma(U, \bigwedge^k(N_P^*))$ be a top covariant constant form (i.e. $\nabla(\nu_P) = 0$) in the conormal bundle N_P^* of P (which is unique up to scaling by constants); we regard ν_P as a volume form on space of leaves N if it admits a smooth structure. We also let $\nu_P^\vee \in \wedge^k N_P$ be a volume element dual to ν_P, and choose a lifting of $\nu_P^\vee$ as an element in $\wedge^k TU$ (which will again be denoted by $\nu_P^\vee$ by abusing notations).

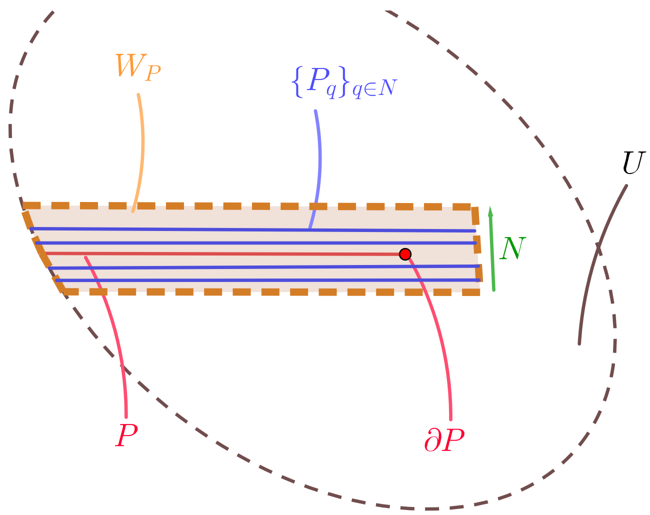

FIGURE 2. The foliation near P

(2) *For any point $p \in P$, we choose a sufficiently small convex neighborhood $V \subset U$ containing p so that there exists a slice $N_V \subset V$ transversal to the foliation $\{P_q \cap V\}$ given by intersection of $\{P_q\}_{q\in N}$ with V, i.e. a dimension k affine subspace which is transversal to all the leaves in $\{P_q \cap V\}$; we denote this foliation on V by $\{(P_{V,q})\}_{q\in N_V}$, using N_V as the parameter space. See Figure 2 for an illustration.*

In V, we take local affine coordinates $x = (x_1, \dots x_n)$ such that $x' := (x_1, \dots, x_k)$ parametrizes N_V with $x' = 0$ corresponding to the unique leaf containing P. Using these coordinates, we can write $\nu_P = dx_1 \wedge \cdots \wedge dx_k$ and $\nu_P^\vee = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_k}$.

Definition 3.9. *A differential k-form $\alpha \in \mathcal{W}_k^\infty(U)$ is said to have* asymptotic support on a closed codimension k tropical polyhedral subset $P \subset U$ with weight s, *denoted by $\alpha \in \mathcal{W}_P^s$ if the following conditions are satisfied:*

(1) *For any $p \in U \setminus P$, there is a neighborhood $V \subset U \setminus P$ of p such that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$ on V.*

(2) *There exists a neighborhood W_P of P in U such that we can write*

$$\alpha = h(x, \hbar)\nu_P + \eta,$$

where ν_P is the volume form Notations 3.8(1), $h(x,\hbar) \in C^\infty(W_P \times \mathbb{R}_{>0})$ and η is an error term satisfying $\eta \in \mathcal{W}_k^{-\infty}(W_P)$ on W_P.

(3) *For any $p \in P$, there exists a sufficiently small convex neighborhood V containing p such that using the coordinate system chosen in Notations 3.8(2) and considering the foliation $\{(P_{V,x'})\}_{x' \in N_V}$ in V, we have, for all $j \in \mathbb{Z}_{\geq 0}$ and multi-index $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k_{\geq 0}$, the estimate*

$$\int_{x' \in N_V} (x')^\beta \left(\sup_{P_{V,x'}} |\nabla^j (\iota_{\nu_P^\vee} \alpha)| \right) \nu_P \leq D_{j,V,\beta} \hbar^{-\frac{j+s-|\beta|-k}{2}}, \tag{3.10}$$

for some constant $D_{j,V,\beta}$ and some $s \in \mathbb{Z}$, where $|\beta| = \sum_l \beta_l$ is the vanishing order of the monomial $(x')^\beta = x_1^{\beta_1} \cdots x_k^{\beta_k}$ along $P_{x'=0}$.

Remark 3.10. *Note that condition (3) in Definition 3.9 is independent of the choice of the convex neighborhood V, the transversal slice N_V and the choice of the local affine coordinates $x = (x_1, \ldots x_n)$ (although the constant $D_{j,V,\beta}$ may depends these choices). Therefore this condition can be checked by choosing a sufficiently nice neighborhood V at every point $p \in P$.*

By definition, we have

$$\nabla_{\frac{\partial}{\partial x_l}} \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s+1}(U)$$

for any $l = 1, \ldots, n$, and

$$(x')^\beta \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s-|\beta|}(U)$$

for any affine monomial $(x')^\beta$ with vanishing order $|\beta|$ along P, so we have the nice property that

$$(x')^\beta \nabla_{\frac{\partial}{\partial x_{l_1}}} \cdots \nabla_{\frac{\partial}{\partial x_{l_j}}} \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s+j-|\beta|}(U). \tag{3.11}$$

The weight s in the Definition 3.9 defines the following filtration (we will drop the U dependence whenever it is clear from the context)[1]

$$\mathcal{W}_k^{-\infty} \cdots \subset \mathcal{W}_P^{-s} \subset \cdots \mathcal{W}_P^{-1} \subset \mathcal{W}_P^0 \subset \mathcal{W}_P^1 \subset \mathcal{W}_P^2 \subset \cdots \subset \mathcal{W}_P^s \subset \cdots \subset \mathcal{W}_k^\infty \subset \Omega_\hbar^k(U). \tag{3.12}$$

This filtration keeps track of the polynomial orders of $\hbar$ for differential k-forms with asymptotic support on P, and it provides a convenient tool for us to prove and express our results in asymptotic analysis.

3.2.1. *Behavior under d and $\wedge$.* We have the following lemma describing how compatible the filtration is with the wedge product:

Lemma 3.11 (Lemma 4.24 in [10]). *For two closed tropical polyhedral subsets $P_1, P_2 \subset U$ of codimension k_1, k_2 respectively, we have $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_P^{r+s}(U)$ for any codimension $k_1 + k_2$ polyhedral subset P containing $P_1 \cap P_2$ normal to $\nu_{P_1} \wedge \nu_{P_2}$ if they intersect transversally (in particular if $codim_{\mathbb{R}}(P_1 \cap P_2) = k_1 + k_2$ we can take $P = P_1 \cap P_2$), and $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_{k_1+k_2}^{-\infty}(U)$ if their intersection is not transversal.*

Let us clarify that when we say two closed tropical polyhedral subsets $P_1, P_2 \subset U$ of codimension k_1, k_2 are *intersecting transversally*, we mean the affine subspaces containing

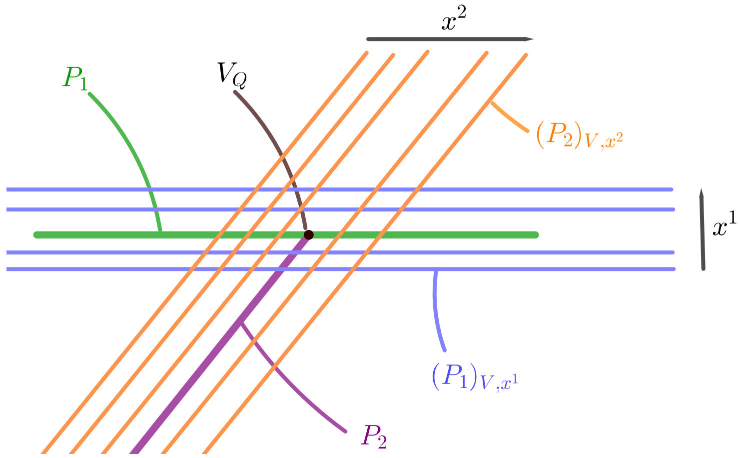

FIGURE 3. Foliation in the neighborhood V

P_1, P_2 and of codimension k_1, k_2 respectively are intersecting transversally; this applies even to the case when $\partial P_i \neq \emptyset$, as shown in Figure 3.

Definition 3.12. *A differential k-form α is in $\tilde{\mathcal{W}}_k^s(U)$ if it there exist finitely many polyhedral subsets $P_1, \dots, P_l$ of codimension k such that $\alpha \in \bigoplus_{j=1}^l \mathcal{W}_{P_j}^s(U)$. If we further have $d\alpha \in \tilde{\mathcal{W}}_{k+1}^{s+1}(U)$, then we say α is in $\mathcal{W}_k^s(U)$.*

We have the following lemma by applying Lemma 3.11.

Lemma 3.13. *We have $\mathcal{W}_{k_1}^{s_1}(U) \wedge \mathcal{W}_{k_2}^{s_2}(U) \subset \mathcal{W}_{k_1+k_2}^{s_1+s_2}(U)$.*

Proof. From Lemma 3.11, we notice that we will always have some polyhedral subset P of codimension $k = k_1 + k_2$ such that $\mathcal{W}_{P_1}^{s_1}(U) \wedge \mathcal{W}_{P_2}^{s_2}(U) \subset \mathcal{W}_P^{s_1+s_2}(U)$, whenever P_1 and P_2 are polyhedral subsets of codimensions k_1 and k_2 respectively. Therefore, we conclude that $\tilde{\mathcal{W}}_{k_1}^{s_1}(U) \wedge \tilde{\mathcal{W}}_{k_2}^{s_2}(U) \subset \tilde{\mathcal{W}}_{k_1+k_2}^{s_1+s_2}(U)$.

Now, suppose $\alpha_i \in \mathcal{W}_{k_i}^{s_i}(U)$. Then we will have $d\alpha_i \in \tilde{\mathcal{W}}_{k_i+1}^{s_i+1}(U)$ and therefore $(d\alpha_1) \wedge \alpha_2 \in \tilde{\mathcal{W}}_{k_1+k_2+1}^{s_1+s_2+1}(U)$, and similar statement holds for $\alpha \wedge (d\alpha_2)$. $\square$

Definition 3.14. *We let $\mathcal{W}_*^s(U) := \bigoplus_k \mathcal{W}_k^{s+k}(U)$ for every $s \in \mathbb{Z}$.*

Proposition 3.15. *$\mathcal{W}_*^0(U) \subset \mathcal{W}_*^\infty(U)$ is a dg subalgebra and $\mathcal{W}_*^{-1}(U) \subset \mathcal{W}_*^0(U)$ is a dg ideal of $\mathcal{W}_*^0(U)$, under the operations d and $\wedge$.*

Proof. From Definition 3.12, we see that $\mathcal{W}_*^s(U)$ is closed under the deRham differential for any $s \in \mathbb{Z}$. Furthermore, Proposition 3.13 says that $\mathcal{W}_{k_1}^{k_1}(U) \wedge \mathcal{W}_{k_2}^{k_2}(U) \subset \mathcal{W}_{k_1+k_2}^{k_1+k_2}(U)$ and $\mathcal{W}_{k_1}^{k_1-1}(U) \wedge \mathcal{W}_{k_2}^{k_2}(U) \subset \mathcal{W}_{k_1+k_2}^{k_1+k_2-1}(U)$, which are exactly the properties we need to show. $\square$

[1]Note that the degree k of the differential forms has to be equal to the codimension of P. Also note that the sets $\mathcal{W}_k^{\pm\infty}(U)$ are independent of the choice of P.

3.2.2. *Behavior under integral operators.* For the rest of this Section 3.2, we will study the behavior of $\mathcal{W}_P^s(U)$ under the application of a homotopy type operator I. For a given closed tropical polyhedral subset $P \subset U$, we choose a reference tropical hyperplane $R \subset U$ which divide the domain U into $U \setminus R = U_+ \cup U_-$, together with an affine vector field v (meaning $\nabla v = 0$) transversal to R pointing into U_+.

By shrinking U if necessary, we assume that for any point $p \in U$, the unique flow line of v in U passing through p intersects R uniquely at a point $x \in R$. Then the time-t flow along v defines a diffeomorphism

$$\tau : W \to U, \ (t, x) \mapsto \tau(t, x),$$

where $W \subset \mathbb{R} \times R$ is the maximal domain of definition of τ (namely, for any $x \in R$, there is a maximal time interval $I_x \subset \mathbb{R}$ so that the flow line through x has its image lying inside U). For any point $x \in R$, we denote by $\tau_x(t) := \tau(t, x)$ the flow line of v passing through x. Figure 4 illustrates the situation.

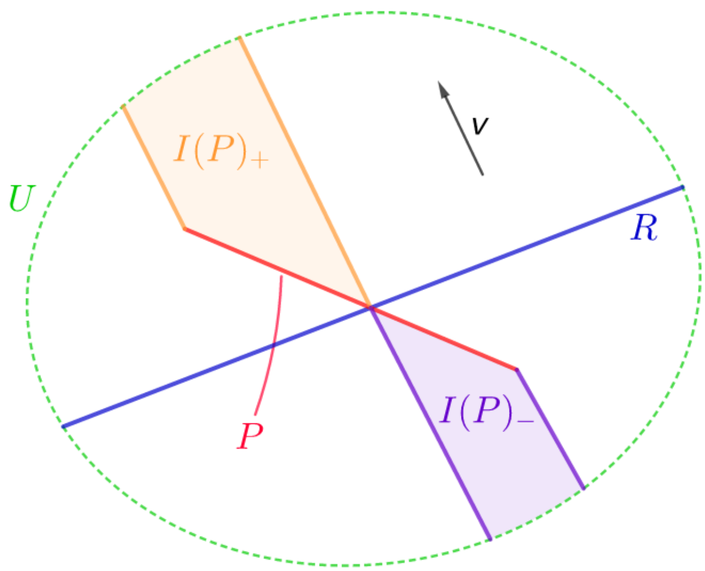

FIGURE 4. The flow along v and $I(P)$

We let $P_\pm = P \cap \overline{U}_\pm$ and define

$$\begin{aligned} I(P)_+ &:= (P_+ + \mathbb{R}_{\geq 0} \cdot v) \cap U, \\ I(P)_- &:= (P_- + \mathbb{R}_{\leq 0} \cdot v) \cap U, \end{aligned} \tag{3.13}$$

as shown in the above picture 4, and we will write $I(P) = I(P)_+ \cup I(P)_-$.

We now define an integral operator I as

$$I(\alpha)(t, x) := \int_0^t \iota_{\frac{\partial}{\partial s}}(\tau^*(\alpha))(s, x) ds. \tag{3.14}$$

Note that I depends on the choice of the tropical hyperplane R. We have the following lemma, which is a modification of [10, Lemma 4.25]:

Lemma 3.16 (cf. Lemma 4.25 in [10]). *For $\alpha \in \mathcal{W}^s_P(U)$, we have $I(\alpha) \in \mathcal{W}^{-\infty}_{k-1}(U)$ if v is not transversal to P, and $I(\alpha) \in \mathcal{W}^{s-1}_{I(P)_+}(U) + \mathcal{W}^{s-1}_{I(P)_-}(U)$ if v is transversal to P, where $I(P)_\pm$ is defined in (3.13). Moveover for $\alpha \in \tilde{\mathcal{W}}^s_k(U)$, we will have $I(\alpha) \in \tilde{\mathcal{W}}^{s-1}_{k-1}(U)$.*

Before proving Lemma 3.16, we introduce a decomposition $\alpha = \alpha_+ + \alpha_-$ of α, where the components α_+ and α_- have asymptotic support of the same weight on P_+ and P_- respectively, using cut-offs as follows. We first consider the functions depending only on the t-coordinate given by

$$\chi_+(t) := \left(\frac{1}{\hbar\pi}\right)^{\frac{1}{2}} \int_{-\infty}^t e^{-\frac{s^2}{\hbar}} ds$$
$$\chi_-(t) := 1 - \chi_+(t) = \left(\frac{1}{\hbar\pi}\right)^{\frac{1}{2}} \int_t^{\infty} e^{-\frac{s^2}{\hbar}} ds,$$

which have asymptotic support on $U_+ = \{t \geq 0\} \cap U$ and $U_- = \{t \leq 0\} \cap U$ respectively with weight 0. Proposition 3.13 implies that the cut-offs

$$\alpha_\pm := \chi_\pm \alpha$$

have asymptotic support on $U_\pm \cap P$ with the same weight s respectively.

Proof of Lemma 3.16. We begin by assuming $\alpha \in \mathcal{W}^s_P(U)$ with $P \subset \overline{U}_+$ and we simply write $I(P)$ to stand for $I(P)_+$. In order to simplify notations in this proof, we will omit τ^* in the definition (3.14) of I by treating $\tau : W \to U$ as an affine coordinate chart.

Suppose that v is not transversal to P. By condition (2) of Definition 3.9, we have a neighborhood $W_P \subset U$ such that $\alpha = h\nu_P + \eta$. For each point $x \in R$, the path $\tau_x(t)$ is tangent to the foliation $\{P_q\}_{q \in N}$ in W_P whenever $\tau_x(t) \in W_P$ by the non-transversal assumption. This means $\iota_{\frac{\partial}{\partial t}}(\nu_P) = 0$ in $\tau_x^{-1}(W_P)$ and hence we have

$$I(\alpha)(t,x) = \int_{[0,t]} \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds = \int_{[0,t]\cap\tau_x^{-1}(U \setminus W_P)} \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds + \int_{[0,t]\cap\tau_x^{-1}(W_P)} \iota_{\frac{\partial}{\partial s}} \eta(s,x) ds.$$

So we have $I(\alpha) \in \mathcal{W}^{-\infty}_{k-1}(U)$ by conditions (1) and (2) of Definition 3.9.

Now suppose that v is transversal to P. Let

$$I(W_P) := \bigcup_{t \geq 0} (W_P + t \cdot v) \cap U,$$

which gives an open neighborhood of $I(P)$. Concerning condition (1) in Definition 3.9, we take $\tau(t_0, x_0) \in U \setminus I(P)$, and then a neighborhood V of $\tau(t_0, x_0)$ in $U \setminus I(P)$ and a neighborhood $W'_P \subset W_P$ of P, such that, for any point $\tau(t,x) \in V$, the flow line joining $\tau(t,x)$ to R does not hit $\overline{W'_P}$. This implies that $I(\alpha)|_V \in \mathcal{W}^{-\infty}_{k-1}(V)$ since we have $\alpha|_{U \setminus \overline{W'_P}} \in \mathcal{W}^{-\infty}_k(U \setminus \overline{W'_P})$ and

$$I(\alpha)(t,x) = \int_0^t \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds = \int_{[0,t]\cap\tau_x^{-1}(U \setminus \overline{W'_P})} \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds.$$

So condition (1) in Definition 3.9 holds for $I(\alpha)$.

Concerning condition (2) in Definition 3.9, we first note that $v = \frac{\partial}{\partial t}$ is tangent to $I(P)$, so by parallel transporting the form $\iota_{\frac{\partial}{\partial t}}\nu_P$ to the neighborhood $I(W_P)$, we obtain a volume element in the normal bundle of $I(P)$, which we denote by $\nu_{I(P)}$. For a point $q \in I(W_P)$, we take a small neighborhood V near q, and for $\tau(t,x) \in V$, we write

$$\begin{aligned} I(\alpha)(t,x) &= \int_0^t \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds \\ &= \int_{[0,t]\cap \tau_x^{-1}(U\setminus W_P)} \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds + \int_{[0,t]\cap\tau_x^{-1}(W_P)} \iota_{\frac{\partial}{\partial s}} (h\nu_P + \eta)(s,x) ds \\ &= \left(\int_{[0,t]\cap\tau_x^{-1}(W_P)} h(s,x) ds \right) \nu_{I(P)} + \int_{[0,t]\cap\tau_x^{-1}(W_P)} \iota_{\frac{\partial}{\partial s}} \eta(s,x) ds \\ &\quad + \int_{[0,t]\cap\tau_x^{-1}(U\setminus W_P)} \iota_{\frac{\partial}{\partial s}} \alpha(s,x) ds, \end{aligned}$$

where the last two terms are in $\mathcal{W}_{k-1}^{-\infty}(V)$, and condition (2) in Definition 3.9 holds for $I(\alpha)$.

Concerning condition (3) in Definition 3.9, we fix a point $p = \tau(b,x) \in I(P)$ and let $p' = \tau(a,x) \in P$ be the unique point such that p, p' lie on the same flow line τ_x. We take local affine coordinates $x = (x_1, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}) \in (-\delta,\delta)^{n-1}$ of R centered at p' (meaning that $p' = \tau(a,0)$) such that $x' = (x_1,\ldots,x_{k-1})$ are normal to the tropical polyhedral subset $p_R(\tau^{-1}(P)) \subset R$, where $p_R : W(\subset \mathbb{R}\times R) \to R$ is the natural projection.

By taking δ small enough, we have $\tau : (a-\delta, b+\delta) \times (-\delta,\delta)^{n-1} \to U$ mapping diffeomorphically onto its image, such that it contains the part the flow line $\tau_0|_{[a,b]}$ joining p' to p. We can also take $V = \tau((b-\delta,b+\delta)\times(-\delta,\delta)^{n-1})$ with $\tau(b,0) = p$, and arrange that $V' = \tau((a-\delta,a+\delta)\times(-\delta,\delta)^{n-1}) \subset W_P$ with $\tau(a,0) = p'$. Notice that there is a possibility that $p = p' \in P$ and therefore $a = b$ in the above description which means $V = V'$. Figure 5 illustrates the situation.

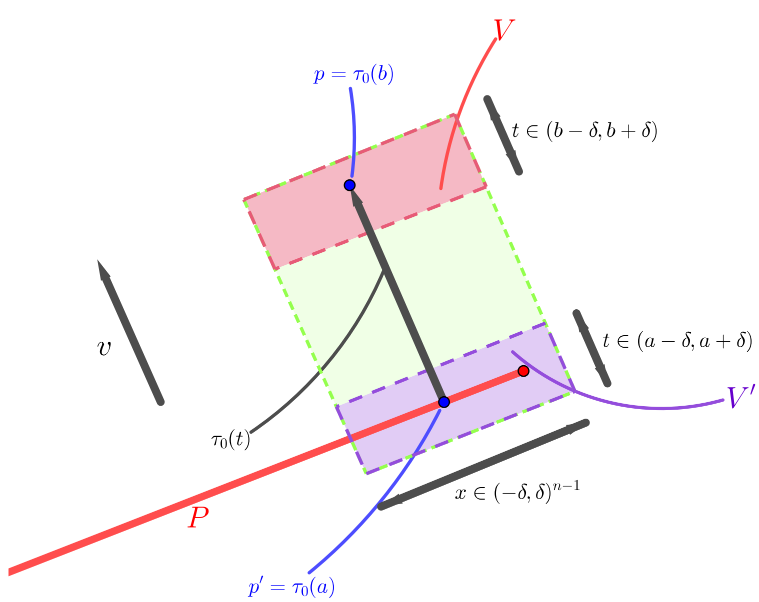

FIGURE 5. Neighborhood along the flow line $\tau_0(t)$

Recall that there is a foliation $\{P_q\}_{q\in N}$ codimension k affine subspaces parallel to P. Then the induced foliation $\{P_{t,x'}\}_{(t,x')\in N_{V'}}$ of the neighborhood V' can be parametrized by $N_{V'} := (a-\delta, a+\delta)\times(-\delta,\delta)^{k-1}$. Therefore the foliation of V induced by $I(P)$ is parametrized as $\{I(P)_{x'}\}_{x'\in N_V}$, where

$$I(P)_{x'} = \bigcup_{t\in(b-\delta,b+\delta)} (P_{0,x'} + tv)$$

and $N_V = (-\delta,\delta)^{k-1}$.

For $\alpha \in \mathcal{W}_P^s$, we consider $I(\alpha) = \int_0^t \iota_{\frac{\partial}{\partial s}} \alpha(s,x)ds$ in the neighborhood V, and what we need to estimate is the term

$$\int_{x'} (x')^\beta \sup_{I(P)_{x'}} |\nabla^j \iota_{\nu^\vee_{I(P)}} I(\alpha)| \nu_{I(P)}.$$

Writing $\nabla^j = \nabla_\perp^{j_1}\nabla^{j_2}_{\frac{\partial}{\partial t}}$, where $\nabla_\perp(t) = 0$, we have two cases depending on whether $j_2 = 0$ or $j_2 > 0$:

Case 1: $j_2 = 0$. Then we have

$$|\nabla^j_\perp \iota_{\nu^\vee_{I(P)}}(I(\alpha))| \leq \int_0^{a-\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds + \int_{a-\delta}^{a+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds + \int_{a+\delta}^{b+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds,$$

in the case that $p' \notin R\cap P$ (equivalent to $a > 0$ in the local coordinates given by τ), or

$$|\nabla^j_\perp \iota_{\nu^\vee_{I(P)}}(I(\alpha))| \leq \int_{a-\delta}^{a+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds + \int_{a+\delta}^{b+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds,$$

in the case that $p' \in R\cap P$. In both cases, the latter term can be dropped because the domain $\int_{a+\delta}^{b+\delta}$ misses the support of P, so it lies in $\mathcal{W}_{k-1}^{-\infty}$. For the case $p' \notin R\cap P$, we notice that the first integral $\int_0^{a-\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds$ can also be dropped as the domain $\int_0^{a-\delta}$ misses the support of P as well, contributing to a term in $\mathcal{W}_{k-1}^{-\infty}$.

As a conclusion, we only have to consider the term $\int_{a-\delta}^{a+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds$ which we treat as a function of (t,x) on V that is constant along the t-direction. Therefore we estimate

$$\begin{aligned}
&\int_{x'} (x')^\beta \sup_{I(P)_{x'}} \left(\int_{a-\delta}^{a+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds\right) \nu_{I(P)}\\
=&\int_{x'} (x')^\beta \sup_{P_{0,x'}+bv} \left(\int_{a-\delta}^{a+\delta} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds\right) \nu_{I(P)}\\
\leq&\int_{x'} \sup_{P_{0,x'}+bv} \left(\int_{a-\delta}^{a+\delta} (x')^\beta \sup_{P_{s,x'}} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds\right) \nu_{I(P)}\\
=&\int_{x'} \left(\int_{a-\delta}^{a+\delta} (x')^\beta \sup_{P_{s,x'}} |\nabla^j_\perp(\iota_{\nu^\vee_P}\alpha)|ds\right) \nu_{I(P)}\\
\leq& C_{j,V',\beta}\hbar^{-\frac{j+s-|\beta|-k}{2}},
\end{aligned}$$

where the first inequality follows from the inequality

$$\int_{a-\delta}^{a+\delta} |\nabla^j_{\perp}(\iota_{\nu_P^\vee}\alpha)|ds \leq \int_{a-\delta}^{a+\delta} \sup_{P_{t,x'}} |\nabla^j_{\perp}(\iota_{\nu_P^\vee}\alpha)|ds,$$

and the second equality is due to the fact that $\int_{a-\delta}^{a+\delta} \sup_{P_{t,x'}} |\nabla^j_{\perp}(\iota_{\nu_P^\vee}\alpha)|ds$, treated as a function on V, is constant along the leaf $P_{0,x'} + bv$. Writing $j + s - |\beta| - k = j + (s-1) - |\beta| - (k-1)$, we obtain the desired estimate so that $\alpha \in \mathcal{W}^{s-1}_{I(P)}(U)$.

Case 2: $j_2 > 0$. Then we have $\nabla^{j_2}_{\frac{\partial}{\partial t}} \iota_{\nu^\vee_{I(P)}}(I_{a-\delta}(\alpha)) = \nabla^{j_2-1}_{\frac{\partial}{\partial t}}(\iota_{\nu_P^\vee}\alpha)$. We can rewrite it as

$$\nabla^{j_1}_{\perp}\nabla^{j_2}_{\frac{\partial}{\partial t}} \iota_{\nu^\vee_{I(P)}}(I_{a-\delta}(\alpha))(t,x) = \int_{a-\delta}^{t} \nabla^j(\iota_{\nu_P^\vee}\alpha)(s,x)ds + \left(\nabla^{j_2-1}_{\frac{\partial}{\partial t}}\nabla^{j_1}_{\perp}(\iota_{\nu_P^\vee}\alpha)\right)(a-\delta,x),$$

where the latter term lies in $\mathcal{W}^{-\infty}_{k-1}$ because it misses the support P of α, and the first term is bounded by

$$|\int_{a-\delta}^{t} \nabla^j(\iota_{\nu_P^\vee}\alpha)(s,x)ds| \leq \int_{a-\delta}^{a+\delta} |\nabla^j(\iota_{\nu_P^\vee}\alpha)|(s,x)ds + \int_{a+\delta}^{b+\delta} |\nabla^j(\iota_{\nu_P^\vee}\alpha)|(s,x)ds.$$

The same argument as in Case 1 can then be applied to get the desired estimate.

Finally, the last statement that I is a map from $\tilde{\mathcal{W}}^s_k(U)$ to $\tilde{\mathcal{W}}^{s-1}_{k-1}(U)$ follows immediately from the first statement. □

In order to obtain the corresponding statement about $\mathcal{W}^s_k(U)$, we need the following Lemmas 3.17 and 3.18 which describe the behavior of $\mathcal{W}^s_P(U)$ under pullbacks. Following the notations in Lemma 3.16, we consider the tropical hypersurface $\mathbf{i} : R \subset U$ with an affine projection $\mathbf{p} : U \to R$ (which are explicitly given by the $\mathbf{i}(x) = (0,x)$ and $\mathbf{p}(t,x) = x$ using the affine coordinates given by τ).

Lemma 3.17. *For $\alpha \in \mathcal{W}^s_P(U)$, we have $\mathbf{i}^*(\alpha) \in \mathcal{W}^s_Q(R)$ if P intersects R transversally and Q is any polyhedral subset of R of codimension k (here $k = codim_{\mathbb{R}}(P \subset U)$) containing $P \cap R$ and normal to $\mathbf{i}^*(\nu_P)$, and $\mathbf{i}^*(\alpha) \in \mathcal{W}^{-\infty}_k(U)$ if the intersection of P and R is not transversal. Moveover, the pull back gives a map $\mathbf{i}^* : \mathcal{W}^s_k(U) \to \mathcal{W}^s_k(R)$.*

Proof. We begin by showing the corresponding statement for $\mathcal{W}^s_P(U)$. First, we will show condition (1) of Definition 3.9. Suppose that $p \in R \setminus P$, then we can find a neighborhood V of p in $U \setminus P$ such that $\alpha|_V \in \mathcal{W}^{-\infty}_k(V)$ from the assumption that $\alpha \in \mathcal{W}^s_P(U)$. Therefore $\alpha|_{V\cap R} \in \mathcal{W}^{-\infty}_k(V \cap R)$.

For condition (2) of Definition 3.9, we first assume that P and R are not intersecting transversally. We notice that there is a neighborhood W_P of P such that α can be written as $h(x,\hbar)\nu_P + \eta$ in W_P from the assumption that $\alpha \in \mathcal{W}^s_P(U)$. Therefore we will have $\mathbf{i}^*(\nu_P) = 0$ if the intersection is not transversal, and so $\mathbf{i}^*(\alpha) \in \mathcal{W}^{-\infty}_k(R)$. Suppose P and R intersect transversally, then we can take $W_{P\cap R} := \mathbf{i}^{-1}(W_P)$, and we will have

$$\mathbf{i}^*(\alpha)|_{W_{P\cap R}} = \mathbf{i}^*(h)\mathbf{i}^*(\nu_P) + \mathbf{i}^*(\eta)$$

in $W_{P\cap R}$ with $\mathbf{i}^*(\nu_P)$ being the volume form of normal bundle of $\mathbf{i}^{-1}(P)$ as desired for condition (2). Notice that if $Q \neq R \cap P$ and for any point $p \notin R \cap P$, there is a neighborhood V of p

such that $\alpha|_{V\cap R} \in \mathcal{W}_k^{-\infty}(V\cap R)$ from our earlier discussion, and therefore condition (2) still holds for arbitrary such Q.

For condition (3), we consider a point $p \in R\cap P$ with affine coordinates

$$(x_1,\dots,x_k,x_{k+1},\dots,x_n) \in (-\delta,\delta)^n$$

in $V \subset U$ such that $R\cap V = \{x_n = 0\}$, and $x' = (x_1,\dots,x_k)$ are parametrizing the parallel foliation $\{P_{V,x'}\}_{x'\in(-\delta,\delta)^k}$ to P in V. Then $\{P_{V,x'}\cap R\}_{x'\in(-\delta,\delta)^k}$ is the foliation parallel to $P\cap R$ in $V\cap R$. Using the fact that $\sup_{P_{V,x'}\cap R}|\nabla^j(\iota_{\nu_P^\vee}\alpha)| \le \sup_{P_{V,x'}}|\nabla^j(\iota_{\nu_P^\vee}\alpha)|$, we have

$$\int_{x'\in N_V}(x')^\beta\left(\sup_{P_{V,x'}\cap R}|\nabla^j(\iota_{\nu_P^\vee}\alpha)|\right)\nu_P \le \int_{x'\in N_V}(x')^\beta\left(\sup_{P_{V,x'}}|\nabla^j(\iota_{\nu_P^\vee}\alpha)|\right)\nu_P \le D_{j,V,\beta}\hbar^{-\frac{j+s-|\beta|-k}{2}},$$

which is the desired estimate for condition (3) of Definition 3.9.

Finally, the second statement that $\mathbf{i}^*$ is a map from $\mathcal{W}_k^s(U)$ to $\mathcal{W}_k^s(R)$ immediately follows from the first statement. □

Lemma 3.18. *For $\alpha \in \mathcal{W}_P^s(R)$, we will have $\mathbf{p}^*(\alpha) \in \mathcal{W}^s_{\mathbf{p}^{-1}(P)}(U)$. Moreover, the pull back gives a map $\mathbf{p}^* : \mathcal{W}_k^s(R) \to \mathcal{W}_k^s(U)$.*

Proof. For condition (1) of Definition 3.9, suppose we take $x \in U\setminus \mathbf{p}^{-1}(P)$, we will have an open subset $V \subset R\setminus P$ containing $\mathbf{p}(x)$. Therefore from the fact that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$ (here $k = \mathrm{codim}_{\mathbb{R}}(\mathbf{p}^{-1}(P)))$ we get $\mathbf{p}^*(\alpha)|_{\mathbf{p}^{-1}(V)} \in \mathcal{W}_k^{-\infty}(\mathbf{p}^{-1}(V))$.

For condition (2) of Definition 3.9, we take a neighborhood W_P of P in R such that we can write α as $h\nu_P + \eta$ with $\eta \in \mathcal{W}_k^{-\infty}(R)$ and ν_P is the normal of P in R. We let $W_{\mathbf{p}^{-1}(P)} = \mathbf{p}^{-1}(W_P)$, and observe that $\mathbf{p}^*(\alpha) = \mathbf{p}^*(h)\mathbf{p}^*(\nu_P) + \mathbf{p}^*(\eta)$ with $\mathbf{p}^*(\nu_P)$ being normal of $\mathbf{p}^{-1}(P)$ in U which is the desired decomposition.

For condition (3), we consider a point $p \in \mathbf{p}^{-1}(P)$ with affine coordinates

$$(x_1,\dots,x_k,x_{k+1},\dots,x_{n-1}) \in (-\delta,\delta)^{n-1}$$

around $q := \mathbf{p}(p)$ in $V \subset R$ such that $x' = (x_1,\dots,x_k)$ are parametrizing the foliation $\{P_{V,x'}\}_{x'\in(-\delta,\delta)^k}$ parallel to P in V. Therefore, we can extend the affine coordinates as $(x_1,\dots,x_k,x_{k+1},\dots,x_n)$ of $\mathbf{p}^{-1}(V)$ such that

$$\mathbf{p}(x_1,\dots,x_k,x_{k+1},\dots,x_n) = (x_1,\dots,x_k,x_{k+1},\dots,x_{n-1})$$

in these coordinates. We notice that $\{\mathbf{p}^{-1}(P_{V,x'})\}_{x'\in(-\delta,\delta)^k}$ is the foliation parallel to $\mathbf{p}^{-1}(P)$ in $\mathbf{p}^{-1}(V)$ and we also have

$$\sup_{P_{V,x'}\cap R}|\nabla^j(\iota_{\nu_P^\vee}\alpha)| = \sup_{\mathbf{p}^{-1}(P_{V,x'})}|\nabla^j(\iota_{\mathbf{p}^*(\nu_P)^\vee}\mathbf{p}^*(\alpha))|.$$

Therefore we conclude that

$$\begin{aligned}&\int_{x'\in N_{\mathbf{p}^{-1}(V)}}(x')^\beta\left(\sup_{\mathbf{p}^{-1}(P_{V,x'})}|\nabla^j(\iota_{\mathbf{p}^*(\nu_P)^\vee}\mathbf{p}^*(\alpha))|\right)\mathbf{p}^*(\nu_P)\\ =&\int_{x'\in N_V}(x')^\beta\left(\sup_{P_{V,x'}}|\nabla^j(\iota_{\nu_P^\vee}\alpha)|\right)\nu_P \le D_{j,V,\beta}\hbar^{-\frac{j+s-|\beta|-k}{2}},\end{aligned}$$

which is the desired estimate.

Finally, the second statement that $\mathbf{p}^*$ is a map from $\mathcal{W}_k^s(R)$ to $\mathcal{W}_k^s(U)$ follows immediately from the first statement. □

Lemma 3.19. *For $\alpha \in \mathcal{W}_k^s(U)$, we have $I(\alpha) \in \mathcal{W}_{k-1}^{s-1}(U)$.*

Proof. With the same notations as in Lemma 3.16, we note that the integral operator I satisfies the equation $dI + Id = \mathrm{Id} - \mathbf{p}^* \circ \mathbf{i}^*$. For a given $\alpha \in \mathcal{W}_k^s(U)$, we have $I(\alpha) \in \tilde{\mathcal{W}}_{k-1}^{s-1}(U)$ by Lemma 3.16. Making use of Lemmas 3.17 and 3.18, we have

$$d(I(\alpha)) = -I(d(\alpha)) + \alpha - \mathbf{p}^* \circ \mathbf{i}^*(\alpha) \in \tilde{\mathcal{W}}_k^s(U),$$

which implies $I(\alpha) \in \mathcal{W}_{k-1}^{s-1}(U)$. □

Finally, we extend the definition of the integral operator which retracts U to a point q_0 instead of the hypersurface R. We consider the chain of affine subspace $\{q_0\} = U_0 \leq U_1 \cdots \leq U_n = U$ with $\dim_{\mathbb{R}}(U_j) = j$, equipped with the natural inclusions $\mathbf{i}_j : U_j \to U_{j+1}$ and affine projections $\mathbf{p}_j : U_{j+1} \to U_j$ such that the fiber of $\mathbf{p}_j$ is tangent to a constant affine vector field v_j on U_{j+1}. Composition of the inclusion operators gives $\mathbf{i}_{i,j} : U_i \to U_j$, and similarly for the projection operator $\mathbf{p}_{i,j} : U_j \to U_i$ for $i < j$. We let $I_j : \mathcal{W}_k^s(U_{j+1}) \to \mathcal{W}_{k-1}^{s-1}(U_{j+1})$ be the integral operator defined on U_{j+1} using the vector field v_j as in the beginning of this subsection (Section 3.2.2).

We will further choose q_0 to be an irrational point in U_1 (strictly speaking it is not a tropical polyhedral subset of U_1) for later applications in Section 3.4 when we solve the Maurer-Cartan equation. The definitions of $\mathbf{p}_{0,j}^*$'s are still valid if they are treated as inclusions of constant functions. The operator I_0 will define a map $\mathcal{W}_k^s(U_1) \to \mathcal{W}_{k-1}^{s-1}(U_1)$ regardless of the fact that q_0 is irrational, because every $\alpha \in \mathcal{W}_1^s(U_1)$ is a finite sum of $\sum_l \alpha_l$ with $\alpha_l \in \tilde{\mathcal{W}}_{P_l}^s(U_l)$ for some *rational* points P_l's on U_1 which in particular miss q_0 and therefore $I_0(P_l)$ will still be a tropical subspace of U_1.

We define the integral operator by

$$I = \mathbf{p}_{1,n}^* I_0 \mathbf{i}_{1,n}^* + \cdots + \mathbf{p}_{n-1,n}^* I_{n-2} \mathbf{i}_{n-1,n}^* + I_{n-1}, \tag{3.15}$$

which is defined as $\mathcal{W}_*^s(U) \to \mathcal{W}_{*-1}^{s-1}(U)$, with the corresponding operator $\mathbf{i}^* := \mathbf{i}_{0,n}^*$ being the evaluation at q_0 and the operator $\mathbf{p}^* := \mathbf{p}_{0,n}^*$.

Proposition 3.20. *We have the identity*

$$dI + Id = Id - \mathbf{p}^* \circ \mathbf{i}^*,$$

meaning that I is contracting the cohomology of U to that of the point q_0.

Proof. We first notice that

$$\mathbf{p}_{j+1,n}^*(dI_j + I_j d)\mathbf{i}_{j+1,n}^* = \mathbf{p}_{j+1,n}^*(id_{U_{j+1}} - \mathbf{p}_{j,j+1}^* \mathbf{i}_{j,j+1}^*)\mathbf{i}_{j+1,n}^*,$$

which gives

$$d(\mathbf{p}_{j+1,n}^* I_j \mathbf{i}_{j+1,n}^*) + (\mathbf{p}_{j+1,n}^* I_j \mathbf{i}_{j+1,n}^*)d = \mathbf{p}_{j+1,n}^* \mathbf{i}_{j+1,n}^* - \mathbf{p}_{j,n}^* \mathbf{i}_{j,n}^*.$$

Taking summation over $j = 0, \ldots, n-1$ gives the desired equation. □

3.3. **The tropical dgLa and its homotopy operator.** Starting from now on, we will restrict ourselves to the case that $B_0 = M_{\mathbb{R}}$ with $U \subset M_{\mathbb{R}}$.

3.3.1. *The tropical dgLa and the extended tropical vertex group.* As an analogue of dgLa $\mathbf{G}_n^{*,*}$ introduced in (3.8), we further impose the requirement of the asymptotic behavior as $\hbar \to 0$ and replace $\Omega_{\hbar}^{*}(U)$ by the dg subalgebra $\mathcal{W}_*^0(U)$.

Definition 3.21. *For every convex open subset $U \subset B_0$, we define a dg Lie subalgebra of $PV_{\hbar}^{*,*}|_U \otimes_{\mathbb{C}} R_n$ by*

$$\mathcal{G}_n^{*,*}(U) := \hat{\mathcal{F}}\Big[\left(\bigoplus_{m\in\mathcal{P}} \mathcal{W}_*^0(U)z^m\right) \otimes_{\mathbb{Z}} \bigwedge^{*} N \otimes_{\mathbb{C}} R_n\Big],$$

making use of the Fourier transform (3.8)*. By abusing of notations, we will drop the identification via Fourier transform in* (3.8) *and simply write*

$$\mathcal{G}_n^{*,*}(U) = \left(\bigoplus_{m\in\mathcal{P}} \mathcal{W}_*^0(U)z^m\right) \otimes_{\mathbb{Z}} \bigwedge^{*} N \otimes_{\mathbb{C}} R_n.$$

Then we take the quotient by the dg Lie ideal

$$\mathcal{I}_n^{*,*}(U) := \left(\bigoplus_{m\in\mathcal{P}} \mathcal{W}_*^{-1}(U)z^m\right) \otimes_{\mathbb{Z}} \bigwedge^{*} N \otimes_{\mathbb{C}} R_n$$

to obtain

$$(\mathcal{G}/\mathcal{I})_n^{*,*}(U) = \left(\bigoplus_{m\in\mathcal{P}} \left(\mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)\right) z^m\right) \otimes_{\mathbb{Z}} \bigwedge^{*} N \otimes_{\mathbb{C}} R_n \tag{3.16}$$

which defines a dgLa (since $\mathcal{W}_*^{-1}(U)$ is a dg ideal of $\mathcal{W}_*^0(U)$).

A general element of $\mathcal{G}_n^{i,j}$, $\mathcal{E}_n^{i,j}$ and $(\mathcal{G}/\mathcal{I})_n^{i,j}$ is a finite sum of the form

$$\sum_I \sum_{m,n_1,\dots,n_j} \alpha_{m,I}^{n_1,\dots,n_j} z^m \check{\partial}_{n_1} \wedge \cdots \wedge \check{\partial}_{n_j} u_I,$$

where $I \subset \{1,\dots,n\}$ and $u_I = \prod_{i\in I} u_i$, with $\alpha_{m,I}^{n_1,\dots,n_j} \in \mathcal{W}_i^i(U)$, $\alpha_{m,I}^{n_1,\dots,n_j} \in \mathcal{W}_i^{i-1}(U)$ and $\alpha_{m,I}^{n_1,\dots,n_j} \in \mathcal{W}_i^i(U)/\mathcal{W}_i^{i-1}(U)$ respectively. We will be concerned with the Maurer-Cartan equation (3.3) of the dgLa $(\mathcal{G}/\mathcal{I})_n^{*,*}(U)$ instead of $PV_{\hbar}^{*,*}|_U \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}} \otimes_{\mathbb{C}} R_n$, because the former is more closely related to tropical counting defined in Section 2.

Making use of the holomorphic volume form (3.6) on $\check{X}$, we obtain a BV operator Δ acting on $PV_{\hbar}^{*,*}$ as in Section 3.1. The BV operator can be carried to $\mathcal{G}_n^{*,*}$ and naturally to $(\mathcal{G}/\mathcal{I})_n^{*,*}$, equipping them with dgBV structures. The BV operator is explicitly given by

$$\Delta(\alpha z^m \check{\partial}_{n_1} \wedge \cdots \wedge \check{\partial}_{n_j}) = \sum_j (-1)^{|\alpha|+r-1} \check{\partial}_{n_r}(\alpha z^m) \check{\partial}_{n_1} \wedge \cdots \widehat{\check{\partial}_{n_r}} \cdots \wedge \check{\partial}_{n_j}$$

in $\mathcal{G}_n^{*,*}$, which is further reduced to

$$\Delta(\alpha z^m \check{\partial}_{n_1} \wedge \cdots \wedge \check{\partial}_{n_j}) = \sum_j (-1)^{|\alpha|+r-1} (n_r, \bar{m}) \alpha z^m \check{\partial}_{n_1} \wedge \cdots \widehat{\check{\partial}_{n_r}} \cdots \wedge \check{\partial}_{n_j}$$

in $(\mathcal{G}/\mathcal{I})^{*,*}_n$. This is because of the extra $\hbar$ in the formula (3.9) giving $\check{\partial}_n(\alpha) \in \mathcal{I}^{*,*}_n$. As a consequence, the Lie bracket $[\cdot,\cdot]$ in $(\mathcal{G}/\mathcal{I})^{*,*}_n$ is given by

$$[\alpha z^m \check{\partial}_{n_I}, \beta z^{m'} \check{\partial}_{n_J}] = (-1)^{(|I|+1)|\beta|} \alpha\beta [z^m \check{\partial}_{n_I}, z^{m'} \check{\partial}_{n_J}],$$

where $\check{\partial}_{n_I} = \check{\partial}_{n_{i_1}} \wedge \cdots \wedge \check{\partial}_{n_{i_l}}$.

Definition 3.22. *We call the dg Lie subalgebra $\mathcal{H}^{*,*}_n \leq (\mathcal{G}/\mathcal{I})^{*,*}_n$ defined by $\mathcal{H}^{*,*}_n := ker(\Delta)$, equipped with the differential $\bar{\partial}_W$ and Lie-bracket $[\cdot,\cdot]$, the* tropical dgLa.

We call $\mathfrak{h}^*_n := \mathcal{H}^{*,0}_n \cap ker(\bar{\partial})$ the extended tropical Lie-algebra*. The corresponding exponential group* $\exp(\mathfrak{h}^*_n)$ *is called the* extended tropical vertex group.

Explicitly, we have

$$\begin{aligned}
\mathfrak{h}^0_n &= \left(\bigoplus_{m \in \mathcal{P}} \mathbb{C} \cdot z^m \right) \otimes_{\mathbb{C}} R_n, \\
\mathfrak{h}^1_n &= \left(\bigoplus_{m \in \mathcal{P}} \mathbb{C} \cdot z^m \right) \otimes_{\mathbb{Z}} m^{\perp} \otimes_{\mathbb{C}} R_n, \\
\mathfrak{h}^2_n &= \mathbb{C} \cdot \check{\partial}_1 \wedge \check{\partial}_2 \otimes_{\mathbb{C}} R_n,
\end{aligned}$$

and $\mathcal{H}^{*,*}_n$ can be thought of as the Dolbeault resolution of $\mathfrak{h}^*_n$. We will see that solving the Maurer-Cartan equation (3.3) in $\mathcal{H}^{*,*}_n$ is intimately related to tropical counting.

3.3.2. *The homotopy operator.* In order to solve the Maurer-Cartan equation (3.3), we need a homotopy operator H (also called a *propagator*) to fix the gauge and apply Kuranishi's method [40] to construct the solution as a sum over trivalent trees. We begin with the construction of a homotopy operator H using the operator I introduced in (3.15). We will take $U = B_0 = M_{\mathbb{R}}$ in the rest of this subsection, and drop the dependence on U in notations whenever there is no confusion.

Notations 3.23. *For each $m \in \mathcal{P}$ with the associated $\bar{m} \in M$, $\bar{m}$ naturally gives an affine vector field on $B_0 = M_{\mathbb{R}}$ which, by abuse of notations, will also be denoted as $\bar{m}$. We fix an affine linear metric g_0 on $M_{\mathbb{R}}$. We choose a chain of affine subspaces $\{pt\} = U^m_0 \leq U^m_1 \leq U^m_2 = M_{\mathbb{R}}$ (we choose U^m_0 to be irrational point on U^m_1 as in Section 3.2), together with the projections $\mathbf{p}^m_j$'s (determined by the vector field v^m_j's) for each $m \in \mathcal{P}$, as in the definition of I in* (3.15) *by requiring $v^m_1 = -\bar{m}$ and $U^m_1 = \{x \mid g_0(-\bar{m}, x) = -R\}$ such that the half space $U^m_{1,+} = \{x \mid g_0(-\bar{m}, x) \geq -R\}$ containing $Sing(\mathfrak{D})$ for some large enough R to be fixed later in Lemma 4.4, for $\bar{m} \neq 0$.*

Such a choice will give us a homotopy operator $H_m : \mathcal{W}^0_*(U) \to \mathcal{W}^0_{*-1}(U)$ using the construction in (3.15) *(which was denoted by I there).*

Definition 3.24. *For each $m \in \mathcal{P}$, we define the homotopy operator*

$$H_m : \mathcal{W}^0_*(U) z^m \to \mathcal{W}^0_{*-1}(U) z^m$$

on the direct summand for each Fourier mode z^m by simply taking $H_m(\alpha z^m) := H_m(\alpha) z^m$.

We also define the projection

$$\mathsf{P}_m : \mathcal{W}^0_*(U) z^m \to \mathcal{W}^0_0(U^m_0) z^m$$

by $\mathsf{P}_m(\alpha z^m) := (\alpha|_{x^0_m})z^m$*, where* $\alpha|_{x^0_m}$ *is evaluation of* α *at the point* $\{x^0_m\} = U^m_0$*, and the operator*

$$\iota_m : \mathcal{W}^0_0(U^m_0)z^m \to \mathcal{W}^0_*(U)z^m$$

by $\iota_m(\alpha z^m) := \iota_m(\alpha)z^m$ *by setting* $\iota_m : \mathcal{W}^0_0(U^m_0) \hookrightarrow \mathcal{W}^0_*(U)$ *to be the embedding of constant functions over* $M_{\mathbb{R}}$*.*

We will abuse notations by treating H_m*,* P_m *and* ι_m *as acting on the spaces* $\mathcal{W}^0_*(U)$ *and* $\mathcal{W}^0_0(U^m_0)$ *respectively.*

As in [10], these operators satisfy the following identity of homotopy retracting $\mathcal{W}^0_*(U)z^m$ onto its cohomology $\mathcal{W}^0_0(U^m_0)z^m = H^*(\mathcal{W}^0_*(U), d)z^m$, i.e. we have

$$\mathrm{Id} - \iota_m \mathsf{P}_m = dH_m + H_m d. \tag{3.17}$$

Moreover, these operators can be descended to $(\mathcal{W}^0_*(U)/\mathcal{W}^{-1}_*(U))z^m$ contracting to its cohomology $\mathbb{C}\cdot z^m \cong \big(\mathcal{W}^0_0(U^m_0)/\mathcal{W}^{-1}_0(U^m_0)\big)z^m$.

Definition 3.25. *We define the operators*

$$H := \bigoplus H_m,\ \mathsf{P} := \bigoplus \mathsf{P}_m \text{ and } \iota := \bigoplus \iota_m$$

acting on the direct sum $\bigoplus_m \mathcal{W}^0_*(U)z^m$ *and its cohomology. These operators extend naturally to the tensor product* $\mathcal{G}^{*,*}_n(U) = \big(\bigoplus_{m\in\mathcal{P}} \mathcal{W}^0_*(U)z^m\big)\otimes_{\mathbb{Z}}\bigwedge^* N\otimes_{\mathbb{C}} R_n$*, and descend to the quotient* $(\mathcal{G}/\mathcal{I})^{*,*}_n$*. Moveover, these operators preserve* $\mathcal{H}^{*,*}_n$ *and hence can also be defined on* $\mathcal{H}^{*,*}_n$*. All of the above operators will be denoted by the same notations.*

3.4. **Solving the Maurer-Cartan equation.** Recall that we have fixed n points $P_1, \dots, P_n$ in generic position, with each P_i corresponding to a formal variable $u_i \in R_n$. To each P_i, we associate an input term of the form

$$\Pi^{(i)} = u_i \delta_{P_i}(\check{\partial}_1 \wedge \check{\partial}_2),$$

where $\check{\partial}_1, \check{\partial}_1$ are the holomorphic vector fields corresponding to the basis $\{e^1, e^2\}$ of N (fixed at the beginning of Section 3.1.1), and

$$\delta_{P_i} = \frac{1}{\pi\hbar} e^{-(\eta_{i,1}^2+\eta_{i,2}^2)/\hbar} d\eta_{i,1}\wedge d\eta_{i,2} \in \mathcal{W}^\infty_2 \tag{3.18}$$

is an $\hbar$-dependent smoothing of the delta-function at P_i, for some affine coordinates $(\eta_{i,1}, \eta_{i,2})$ on $M_{\mathbb{R}}$ taking the values $(0,0)$ at P_i. We will be interested in the Maurer-Cartan solution in $\mathcal{H}^{*,*}_n$ constructed by a sum over trees formula with the input $\sum_{i=1}^n \Pi^{(i)}$.

Notice that we have $\delta_{P_i} \in \mathcal{W}^2_{P_i}$ because because we can apply Lemma 3.11 to the expression

$$\delta_{P_i} = \left(\left(\frac{1}{\pi\hbar}\right)^{1/2} e^{-(\eta_{i,1}^2)/\hbar} d\eta_{i,1}\right) \wedge \left(\left(\frac{1}{\pi\hbar}\right)^{1/2} e^{-(\eta_{i,2}^2)/\hbar} d\eta_{i,2}\right),$$

and we have the following lemma from [10]:

Lemma 3.26 (Lemma 4.14 in [10])**.** *For any affine linear function* η *on* U*, the 1-form* $\left(\frac{1}{\pi\hbar}\right)^{1/2} e^{-(\eta^2)/\hbar} d\eta$ *has asymptotic support on the line* $L := \{\eta = 0\}$ *with weight* 1*.*

Instead of solving the Maurer-Cartan equation directly, we will try to solve the equation (3.19):

$$\Phi = \Pi - H([W, \Phi] + \frac{1}{2}[\Phi, \Phi]), \tag{3.19}$$

where Φ is a degree 0 element in $\mathcal{H}^{*,*}_n$, with the input

$$\Pi := \sum_{i=1}^{n} \Pi^{(i)}. \tag{3.20}$$

This originates from a method of Kuranishi [40] in solving the Maurer-Cartan equation of the classical Kodaira-Spencer dgLa. His method can be generalized to our current situation as follows (see e.g. [43])

Proposition 3.27. *Suppose that Φ satisfies the equation* (3.19)*. Then Φ satisfies the Maurer-Cartan equation* (3.3) *if and only if* $\mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$.

Proof. Applying $\bar{\partial}$ to both sides of (3.19) (recall that $\bar{\partial}$ is identified with the de Rham differential d using the Fourier transform $\mathcal{F}$ (3.8)) and using $\bar{\partial}\Pi = 0$, we obtain

$$\bar{\partial}\Phi + [W, \Phi] + \frac{1}{2}[\Phi, \Phi] = H([\bar{\partial}\Phi, W + \Phi]) + \iota \circ \mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]).$$

Suppose that Φ satisfies the MC equation (3.3). Then we see that $[\bar{\partial}\Phi, W + \Phi] = -[[W, \Phi] + \frac{1}{2}[\Phi, \Phi], W + \Phi] = 0$ and hence $\mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$.

For the converse, we let $\delta = \bar{\partial}\Phi + [W, \Phi] + \frac{1}{2}[\Phi, \Phi]$. It follows from the assumption $\mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$ that

$$\delta = H[W + \Phi, \delta] = (H \circ ad_{W+\Phi})^m(\delta)$$

for any $m \in \mathbb{Z}_+$. Then by the fact that $\Phi \in \mathcal{H}^{*,*}_n \otimes \mathbf{m}$, and the fact that ad_W is an operator of degree $(1, 0)$, we have $\delta = 0$ by taking m large enough. ☐

We notice that $\mathsf{P}\alpha \neq 0$ only if $\alpha \in \mathcal{H}^{i,0}_n$ by its construction. When we write $\Phi = \sum_{i=0}^{2} \Phi^{i,i}$ with $\Phi^{i,i} \in \mathcal{H}^{i,i}_n$, and consider the term $\mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$, we notice that

$$\mathsf{P}([W, \Phi^{1,1} + \Phi^{2,2}] + \frac{1}{2}[\Phi^{1,1} + \Phi^{2,2}, \Phi^{1,1} + \Phi^{2,2}] + [\Phi^{0,0}, \Phi^{1,1} + \Phi^{2,2}]) = 0$$

by degree reason. Furthermore, we have $[W, \Phi^{0,0}] = 0 = [\Phi^{0,0}, \Phi^{0,0}]$, and therefore we have $\mathsf{P}([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$. As a result, it suffices to solve the equation (3.19).

Now we look at the equation (3.19). Letting $\Xi = \Phi - \Pi$, we solve

$$\Xi + H([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$$

iteratively in $\mathcal{H}^{*,*}_n \otimes_R (R/\mathbf{m}^k)$ by increasing the power in $\mathbf{m}^k$. We write $\Xi = \sum_{i=1} \Xi_i$ and $\Phi = \sum_i \Phi_i$ with $\Xi_i, \Phi_i \in \mathcal{H}^{*,*}_n \otimes (\mathbf{m}^i/\mathbf{m}^{i+1})$. We further decompose each Ξ_i and Φ_i by its degree and write $\Xi_i = \sum_{j=0}^{2} \Xi_i^{j,j}$ with $\Xi_i^{j,j} \in \mathcal{H}^{j,j}_n \otimes \mathbf{m}^i/\mathbf{m}^{i+1}$ and similarly $\Phi_i = \sum_{j=0}^{2} \Phi_i^{j,j}$.

The first order terms are simply given by $\varXi_1^{1,1} = -H[W,\Pi]$ and $\varXi_1^{0,0} = -H[W,\varXi_1^{1,1}]$. In general, the k-th equation is given by

$$\varXi_k + H[W,\Phi_k] + \sum_{j+l=k} \frac{1}{2} H[\Phi_j,\Phi_l] = 0, \tag{3.21}$$

and $\varXi_k^{j,j}$ is uniquely determined by $\varXi_i$ with $i<k$ and $\varXi_k^{r,r}$ with $r>j$. In this way, the solution $\varXi$ to (3.19) is uniquely determined.

There is a beautiful way to express the unique solution $\varXi$ as a sum of terms involving the input Π over directed trees (reminiscent of a Feynman sum). To this end, we will introduce the notions of *weighted d-pointed k-tree with ribbon structure*, following [10] whose definitions are originated from [15].

Definition 3.28. *A* weighted ribbon d-pointed k-tree *is a weighted d-pointed k-tree Γ equipped with a ribbon structure on it. That is a cyclic ordering of $\partial_{in}^{-1}(v) \sqcup \partial_o^{-1}(v)$ for each trivalent vertex $v \in \Gamma^{[0]}$ such that if e_1 and e_2 are the incoming edges of v with outgoing edge e_3 with e_1, e_2, e_3 in cyclic ordering, then only e_1 can possibly be an edge from $\Gamma_{in}^{[0]} = \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1,\dots,p_d\}$ (recall that only one of e_1 or e_2 is allow to be edges from $\Gamma_{in}^{[0]}$ from Definition 2.5).*

Equivalently, it can be regarded as an embedding $|\bar{\Gamma}| \hookrightarrow D$ of $|\bar{\Gamma}|$ (the topological space associated to a tree $\bar{\Gamma}$) into the unit disc $D \subset \mathbb{R}^2$ mapping $\Gamma_{\infty}^{[0]}$ to ∂D, up to orientation preserving automorphism of the disk D, from which the cyclic ordering is induced by the clockwise orientation on D.

Isomorphisms between these trees are defined as isomorphisms between weighted d-pointed k-trees which preserve the relevant structures. The set of isomorphism classes of weighted ribbon d-pointed k-tree is denoted by $\mathtt{WRT}_{k,d}$. We will use the notation $\mathcal{T}$ to denote a weighted ribbon d-pointed k-tree as well as its isomorphism class.

Definition 3.29. *Given a weighted ribbon d-pointed k-tree $\mathcal{T} \in \mathtt{WRT}_{k,d}$, we align the marked points $p_1,\dots,p_d$ (recall that marked points is itself an edge in $\partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})$) by $p_{i_1},\dots,p_{i_d}$ according to its cyclic ordering (or the clockwise orientation on D if we use the embedding $|\mathcal{T}| \hookrightarrow D$). We define the graded operator*

$$\mathfrak{l}_{\mathcal{T}} : \mathcal{H}^{*,*}[2]^{\otimes d} \to \mathcal{H}^{*,*}[2]$$

for input $\zeta_1,\dots,\zeta_d \in \mathcal{H}^{*,*}[2]$ by

(1) *writing $\zeta_j = \sum_{I\subset\{1,\dots,n\}} \alpha_{j,I} u_I$, and extracting the term $\alpha_{j,i_j} u_{i_j}$ in ζ_j and aligning it as the input at p_{i_j}, where $u_{i_j} \in R_n$ is the monomial associated to the marked point p_{i_j} in Definition 2.5,*
(2) *aligning the term z^{m_e} at each incoming edge in $e \in \mathcal{T}_{in}^{[1]} = \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]}) \setminus \{p_1,\dots,p_d\}$,*
(3) *applying $[\cdot,\cdot]$ at each vertex in $\mathcal{T}^{[0]}$ according to the ordering of the ribbon structure,*
(4) *applying the homotopy operator $-H$ to each edge in $\mathcal{T}^{[1]}$.*

We then define $\mathfrak{l}_{k,d} : \mathcal{H}^{*,*}[2]^{\otimes d} \to \mathcal{H}^{*,*}[2]$ by

$$\mathfrak{l}_{k,d} := \sum_{\mathcal{T}\in\mathtt{WRT}_{k,d}} \frac{1}{2^{d-1}} \mathfrak{l}_{\mathcal{T}}.$$

Setting

$$\Phi := \Pi + \varXi = \sum_{k,d\geq 1} \mathfrak{l}_{k,d}(\Pi,\dots,\Pi) \tag{3.22}$$

gives the unique solution to the equation (3.19) which is obtained by recursively solving (3.21). Note that the sum above is finite because the ideal $\mathbf{m}_n$ is nilpotent.

4. Proof of Theorem 1.1 by asymptotic analysis

In this section, we give a proof of our main result Theorem 1.1 by using asymptotic analysis to relate the Maurer-Cartan solution $\Phi \in \mathcal{H}^{*,*}_n$ constructed via the sum over tree formula in (3.22) with the specific input Π (3.20) to the counts of tropical disks we introduced in Section 2.

We begin by fixing n points $P_1,\dots,P_n \in M_{\mathbb{R}}$ in generic position.

Notations 4.1. *We use* $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P},\Sigma,P_1,\dots,P_n)$ *to denote the space* $\vec{ev}^{-1}((P_{i_1},\dots,P_{i_d})\times M_{\mathbb{R}})$ *(here* $\vec{ev}$ *is the evaluation map defined in Definition 2.9,* P_{i_j} *is the point such that the monomial weight at the marked point* p_j *is* u_{i_j}*, and note that the subset* $\{i_1,\dots,i_d\}\subset\{1,\dots,n\}$ *is determined by the weight of* $\mathcal{T}$*) which is a compactification of* $\mathfrak{M}^{\mathcal{T}}_d(\mathcal{P},\Sigma,P_1,\dots,P_n)$ *for any weighted ribbon tree* $\mathcal{T}$*.*

Definition 4.2. *Given a weighted ribbon d-pointed k-tree* $\mathcal{T}\in \mathtt{WRT}_{k,d}$ *with* $u_{\mathcal{T}}\neq 0$*, we associate to each of its edges* $e\in\bar{\mathcal{T}}$ *a tropical polyhedral subset* $Q_e\subset M_{\mathbb{R}}$ *as follows: For each incoming edge* $e\in\mathcal{T}^{[1]}_{in}$*, we assign* $Q_e = M_{\mathbb{R}}$*, and for each marked point* p_j *we assign* $Q_{p_j}=P_{i_j}$ *where the monomial weight at* p_j *is* u_{i_j}*. We then inductively assign a (possibly empty) tropical polyhedral subset* Q_e *to each edge* $e\in\mathcal{T}^{[1]}$ *by the following rule:*

If e_1 *and* e_2 *are two incoming edges meeting at a vertex* v *with an outgoing edge* e_3 *for which* Q_{e_1} *and* Q_{e_2} *are defined beforehand, we set*

$$Q_{e_3} := (Q - \mathbb{R}_{\geq 0}\bar{m}_{e_3})$$

if both Q_{e_1} *and* Q_{e_1} *are non-empty and they intersect transversally at* $Q := Q_{e_1}\cap Q_{e_2}$ *with* $Mult_v(\mathcal{T})\neq 0$*, and*

$$Q_{e_3} := \emptyset$$

otherwise (recall that transversal intersection *between two closed tropical polyhedral subsets, including the case when they have nonempty boundaries, was defined right after Lemma 3.11).*

We denote the tropical polyhedral subset associated to the unique outgoing edge e_o *by* $Q_{\mathcal{T}}$*.*

We begin with the following combinatorial lemma concerning the tropical polyhedral subset $Q_{\mathcal{T}}$.

Lemma 4.3. *If* $MI(\mathcal{T})<0$*, then both* $Q_{\mathcal{T}}$ *and* $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P},\Sigma,P_1,\dots,P_n)$ *are empty.*

For $MI(\mathcal{T})=0$ *or* 2*, we have* $Q_{\mathcal{T}}\neq\emptyset$ *only when* $Mult(\mathcal{T})\neq 0$*, and in that case,* ev_o *is a diffeomorphism onto its image and we have* $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P},\Sigma,P_1,\dots,P_n))$ *which is of dimension* $\frac{MI(\mathcal{T})}{2}+1$*.*

Proof. We prove by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$, i.e. when there are no trivalent vertices. Then the only possible trees are the ones with a unique edge e. In this case we have $MI(\mathcal{T}) = 2$ and $Q_{\mathcal{T}} = M_{\mathbb{R}}$, and the lemma holds automatically.

For the induction step, suppose we have a tree $\mathcal{T}$ with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2. We split $\mathcal{T}$ at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points respectively. Then we have the decomposition

$$\left(\overline{\mathfrak{M}}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)_{ev_o} \times_{ev_o} \overline{\mathfrak{M}}^{\mathcal{T}_2}_{d_2}(\mathcal{P}, \Sigma, P_1, \dots, P_n)\right) \times \mathbb{R}_{\geq 0} \cdot (-\bar{m}_{\mathcal{T}}) = \overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n),$$

which is compatible with the inductive definition of $Q_{\mathcal{T}}$. There are two cases to consider.

The first case is when one of the incoming edges, say e_2, is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. In this case $\mathcal{T}_2$ is not a weighted tree in the sense of Definition 2.5, but we can still take $Q_{\mathcal{T}_2}$ to be the point P_{e_2} associated to e_2.

If $MI(\mathcal{T}_1) \leq 0$, then by the induction hypothesis and the generic assumption (Definition 2.10), $Q_{\mathcal{T}_1}$ cannot intersect $Q_{\mathcal{T}_2}$ transversally and hence $Q_{\mathcal{T}} = \emptyset$. On the other hand we will have $MI(\mathcal{T}) < 0$, so $\mathfrak{M}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$.

If $MI(\mathcal{T}_1) = 2$, then $Q_{\mathcal{T}_1}$ intersect $Q_{\mathcal{T}_2}$ transversally at $Q_{\mathcal{T}_2}$ automatically if $Q_{\mathcal{T}_2}$ lies on $Q_{\mathcal{T}_1}$. In this case $MI(\mathcal{T}) = 0$, $\mathrm{Mult}(\mathcal{T}) = \mathrm{Mult}(\mathcal{T}_1)$ and $m_{\mathcal{T}} = m_{\mathcal{T}_1}$. Assuming $\mathrm{Mult}(\mathcal{T}_1) \neq 0$, we have $Q_{\mathcal{T}_1} = (ev_o)_*(\overline{\mathfrak{M}}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ by the induction hypothesis and the above decomposition becomes

$$\left(\overline{\mathfrak{M}}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \cap (ev_{\mathcal{T}_1,o})^{-1}(Q_{\mathcal{T}_2})\right) \times \mathbb{R}_{\geq 0} \cdot (-\bar{m}_{\mathcal{T}_1}) = \overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n),$$

implying that $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$. In the case when $Q_{\mathcal{T}}$ is nonempty, i.e. when $(ev_{\mathcal{T}_1,o})^{-1}(Q_{\mathcal{T}_2}) \in \overline{\mathfrak{M}}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$, we indeed have $(ev_{\mathcal{T}_1,o})^{-1}(Q_{\mathcal{T}_2}) \in \mathfrak{M}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ due to the generic assumption on the points $P_1, \dots, P_n$, and hence the dimension of $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ is exactly given by $\frac{MI(\mathcal{T})}{2} + 1$.

The second case is when both $\mathcal{T}_1$ and $\mathcal{T}_2$ have $k_1, k_2 \geq 1$. In this case we have $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$ and the two moduli spaces $\overline{\mathfrak{M}}^{\mathcal{T}_i}_{d_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ have dimensions $MI(\mathcal{T}_i)/2 + 1$ respectively. We further have the decomposition

$$\left(\overline{\mathfrak{M}}^{\mathcal{T}_1}_{d_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)_{ev_o} \times_{ev_o} \overline{\mathfrak{M}}^{\mathcal{T}_2}_{d_2}(\mathcal{P}, \Sigma, P_1, \dots, P_n)\right) \times \mathbb{R}_{\geq 0} \cdot (-\bar{m}_{\mathcal{T}}) = \overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n).$$

Notice that if $\overline{\mathfrak{M}}^{\mathcal{T}_i}_{d_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$ for $i = 1$ or 2, then $\overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$. So $\overline{\mathfrak{M}}^{\mathcal{T}}_{d}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$ if $MI(\mathcal{T}) < 0$. Therefore it remains to consider the cases when $MI(\mathcal{T}_i) = 0, 2$ and $MI(\mathcal{T}) = 0, 2$.

From the induction hypothesis we have $Q_{\mathcal{T}_i} = ev_o(\overline{\mathfrak{M}}^{\mathcal{T}_i}_{d_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ for $i = 1, 2$. If $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ are intersecting non-transversally, by dimension reasons, it is only possible if $MI(\mathcal{T}_1) = MI(\mathcal{T}_2) = 0$ so that $MI(\mathcal{T}) = 0$. In this case, $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ are parallel rays or lines, so $\mathrm{Mult}_{v_r}(\mathcal{T}) = 0$. Therefore, if $\mathrm{Mult}(\mathcal{T}) \neq 0$, we will have $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ intersecting transversally and $Q_{\mathcal{T}} = (ev_o)_*(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \ldots, P_n))$. Finally, by the generic assumption on $P_1, \ldots, P_n$, $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \ldots, P_n))$ has dimension $\frac{MI(\mathcal{T})}{2}+1$ whenever it is nonempty. □

Lemma 4.4. *There exist a large enough $R > 0$ such that the tropical polyhedral subset $Q_{\mathcal{T}}$ is a subset of the half space $U^m_{1,+}$ in Notations 3.23 for any $\mathcal{T}$ with $MI(\mathcal{T}) = 0, 2$, Mult$(\mathcal{T}) \neq 0$ and $u_{\mathcal{T}} \neq 0$ with at least one marked point such that $m_{\mathcal{T}} = m$.*

Proof. The existence of a fixed R depends on the finiteness of the total number of weighted ribbon trees $\mathcal{T}$ (for arbitrary number of marked points and $k = |\mathcal{T}^{[0]}_{in}|$) with $MI(\mathcal{T}) = 0, 2$, $\mathrm{Mult}(\mathcal{T}) \neq 0$ and $u_{\mathcal{T}} \neq 0$.

We prove by induction on the number N of vertices in $\mathcal{T}^{[0]}$ the existence of $R_N > 0$ satisfying the lemma for all $\mathcal{T}$ with $|\mathcal{T}^{[0]}| \leq N$.

The initial case concerns the tree $\mathcal{T}$ with an unique internal vertex v_r, with two incoming edges e_1 and e_2, and one outgoing edge e_o in clockwise orientation. Furthermore, we have $e_1 \in \mathcal{T}^{[1]}_{in}$ and e_2 is an edge corresponding to a marked point with monomial weight u_{e_2}. In this case we have $MI(\mathcal{T}) = 0$ and $Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ which is lying in $U^m_{1,+}$ as we required $\mathrm{Sing}(\mathfrak{D}) \subset U^m_{1,+}$ when we chose $U^m_{1,+}$ in Notations 3.23.

For the induction step, suppose we have a tree $\mathcal{T}$ with $|\mathcal{T}^{[0]}| = N + 1$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2. We split $\mathcal{T}$ at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points respectively. There are two cases to consider (as in the proof of Lemma 4.3).

This first case is when one of the incoming edges, say e_2, is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. We let $Q_{\mathcal{T}_2} = P_{e_2}$ to be the corresponding marked point. From the proof of Lemma 4.3, we know that we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$ for $Q_{\mathcal{T}} \neq \emptyset$. In this case $Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ and we have $Q_{\mathcal{T}} \subset U^m_{1,+}$ by the same reason as in the initial step.

In the second case we have both $\mathcal{T}_1$ and $\mathcal{T}_2$ having $k_1, k_2 \geq 1$, and we have $MI(\mathcal{T}) = MI(\mathcal{T}_1)+MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of the $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$ is a ray or a line, and we assume that it is $Q_{\mathcal{T}_1}$, with $MI(\mathcal{T}_2) = 0, 2$. Therefore for any point $x \in Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}$ we have the relations $g_0(-\bar{m}_{\mathcal{T}_1}, x) \geq -R_N$ and $g_0(-\bar{m}_{\mathcal{T}_2}, x) \geq -R_N$, and hence $g_0(-\bar{m}_{\mathcal{T}}, x) \geq -2R_N$. Therefore by taking $R_{N+1} = 2R_N$, we have $g_0(-\bar{m}_{\mathcal{T}}, x) \geq -R_{N+1}$ and hence $Q_{\mathcal{T}} = Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}} \subset U^m_{1,+}$ for $\bar{m} \neq 0$ as desired. □

We are now ready to prove the key lemma (Lemma 4.7 relating our Maurer-Cartan solution and the locus $Q_{\mathcal{T}}$ traced out by the moduli space of tropical disks introduced in Definition 4.2.

Notations 4.5. *Given a weighted ribbon d-pointed k-tree $\mathcal{T}$, we define a differential form $\alpha_{\mathcal{T}} \in \mathcal{W}^0_*$ (cf. Definition 3.29). We align the marked points $p_1, \ldots, p_d$ (recall that a marked point is itself an edge in $\partial^{-1}_{in}(\mathcal{T}^{[0]}_{in})$) by $p_{i_1}, \ldots, p_{i_d}$ according to its cyclic ordering. We define $\alpha_{\mathcal{T}}$ as the output of the operation given by*

(1) aligning $\delta_{P_{i_j}}$ as the input at the edge corresponding to the marked point p_{i_j}, if the monomial weight associated to p_{i_j} is u_{i_j},
(2) aligning the constant 1 at each incoming edge in $e \in \mathcal{T}^{[1]}_{in} = \partial^{-1}_{in}(\mathcal{T}^{[0]}_{in}) \setminus \{p_1, \ldots, p_d\}$,
(3) applying the wedge product $\wedge$ at each vertex in $\mathcal{T}^{[0]}$ according to the ordering of the ribbon structure,
(4) applying the homotopy operator $-H$ to each edge in $\mathcal{T}^{[1]}$.

Definition 4.6. *Given a weighted ribbon d-pointed k-tree $\mathcal{T} \in \mathtt{WRT}_{k,d}$ with Mult$(\mathcal{T}) \neq 0$, we set*

$$(-1)^{\chi(\mathcal{T})} := \prod_{v \in \mathcal{T}^{[0]}} (-1)^{\chi(\mathcal{T},v)},$$

where $(-1)^{\chi(\mathcal{T},v)}$ is defined as follows (with the convention that $(-1)^{\chi(\mathcal{T})} = 1$ if $\mathcal{T}^{[0]} = \emptyset$):

- *if v is connected to a marked point, we set $\chi(\mathcal{T}, v) = 0$, and*
- *$(-1)^{\chi(\mathcal{T},v)}$ is defined for each trivalent vertex v not connecting to any marked point p_i's (attached to two incoming edges e_1, e_2 and one outgoing edge e_3 so that e_1, e_2, e_3 are arranged in the clockwise orientation) by comparing the orientation of the ordered basis $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ with that of B_0.*

Lemma 4.7. *Let $\mathcal{T} \in \mathtt{WRT}_{k,d}$ be a weighted ribbon d-pointed k-tree. Then we have*

$$\mathfrak{l}_{\mathcal{T}}(\Pi, \ldots, \Pi) = \begin{cases} 0 & \text{if } MI(\mathcal{T}) \neq 0, 2 \text{ or } Q_{\mathcal{T}} = \emptyset, \\ (-1)^{\chi(\mathcal{T})} \alpha_{\mathcal{T}} Mult(\mathcal{T}) z^{m_{\mathcal{T}}} u_{\mathcal{T}} & \text{if } MI(\mathcal{T}) = 2 \text{ and } Q_{\mathcal{T}} \neq \emptyset, \\ (-1)^{\chi(\mathcal{T})} \alpha_{\mathcal{T}} k_{\mathcal{T}} Mult(\mathcal{T}) z^{m_{\mathcal{T}}} \check{\partial}_{n_{Q_{\mathcal{T}}}} u_{\mathcal{T}} & \text{if } MI(\mathcal{T}) = 0 \text{ and } Q_{\mathcal{T}} \neq \emptyset \end{cases}$$

in $\mathcal{H}^{*,*}_n$, where $\alpha_{\mathcal{T}} \in \mathcal{W}^{s_{\mathcal{T}}}_{Q_{\mathcal{T}}}$ in which $s_{\mathcal{T}} := 1 - \frac{MI(\mathcal{T})}{2}$ and $n_{Q_{\mathcal{T}}}$ is the clockwise oriented normal to the ray or line $Q_{\mathcal{T}}$ when $Q_{\mathcal{T}} \neq \emptyset$.

Proof. First of all, from Notations 4.5 we can see that the degree of the form $\alpha_{\mathcal{T}}$ is exactly given by $s_{\mathcal{T}}$ which can only be 0 or 1 since the operator associated to the outgoing edge is a homotopy operator and it decreases the degree by 1. Therefore we notice that $\alpha_{\mathcal{T}} \neq 0$ except when $MI(\mathcal{T}) = 0$ or 2.

Once again, we prove by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$ and the only possible trees the ones with a unique edge e. In this case, we have $MI(\mathcal{T}) = 2$, $Q_{\mathcal{T}} = M_{\mathbb{R}}$ and $\mathfrak{l}_{\mathcal{T}}(\Pi, \ldots, \Pi) = z^{m_{\mathcal{T}}}$, so the lemma holds.

For the induction step, suppose we have a tree $\mathcal{T}$ with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2. We split $\mathcal{T}$ at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points respectively. As before, there are two possible scenarios.

This first case is when one of the incoming edges, say e_2, is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. In this case we let $P_{e_2} = Q_{\mathcal{T}_2}$ to be the marked point associated to e_2. The proof of Lemma 4.3 shows that we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$ in order to have $Q_{\mathcal{T}} \neq \emptyset$. By the induction hypothesis we have $\mathfrak{l}_{\mathcal{T}_1}(\Pi, \ldots, \Pi) = (-1)^{\chi(\mathcal{T}_1)} \alpha_{\mathcal{T}_1} \mathrm{Mult}(\mathcal{T}_1) z^{m_{\mathcal{T}_1}} u_{\mathcal{T}_1}$ with $\alpha_1 \in \mathcal{W}^0_{Q_{\mathcal{T}_1}}$. Therefore we have

$$\mathfrak{l}_{\mathcal{T}}(\Pi, \ldots, \Pi) = -(-1)^{\chi(\mathcal{T}_1)} \mathrm{Mult}(\mathcal{T}_1) H(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}})[z^{m_{\mathcal{T}_1}}, \check{\partial}_1 \wedge \check{\partial}_2] u_{\mathcal{T}_1} u_{e_2},$$

where $u_{\mathcal{T}_1} u_{e_2} = u_{\mathcal{T}}$.

Now Lemma 3.11 implies that $\alpha_1 \wedge \delta_{P_{e_2}} \in \mathcal{W}^2_{P_{e_2}}$. By our choice we have $P_{e_2} \in U^{m_{\mathcal{T}}}_{1,+}$ and hence applying Lemma 3.19, we get $\alpha_{\mathcal{T}} = -H(\alpha_1 \wedge \delta_{P_{e_2}}) \in \mathcal{W}^1_{Q_{\mathcal{T}}}$, where $Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ as in Definition 4.2. Furthermore, we have $[z^{m_{\mathcal{T}_1}}, \check{\partial}_1 \wedge \check{\partial}_2] = (e_2^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_2 - (e_1^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_1$, where e_1^*, e_2^* is the dual basis to e_1, e_2 introduced in Notations 3.4. As in Notations 2.6, we can write $k_{\mathcal{T}}\hat{m}_{\mathcal{T}} = \bar{m}_{\mathcal{T}}$ for some primitive $\hat{m}_{\mathcal{T}} \in M$. Since we have $m_{\mathcal{T}_1} = m_{\mathcal{T}}$, we find that $(e_2^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_2 - (e_1^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_1 = k_{\mathcal{T}} n_{Q_{\mathcal{T}}}$. Together with the fact that $\chi(\mathcal{T}) = \chi(\mathcal{T}_1)$ and $\mathrm{Mult}(\mathcal{T}) = \mathrm{Mult}(\mathcal{T}_1)$, we obtain the desired identity in this case.

In the second case we have both $\mathcal{T}_1$ and $\mathcal{T}_2$ having $k_1, k_2 \geq 1$, and $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$, say $Q_{\mathcal{T}_1}$, is a ray or a line. There are two subcases depending on whether $MI(\mathcal{T}_2) = 0$ or 2.

We first assume that $MI(\mathcal{T}_2) = 0$. Then we can write

$$\mathfrak{l}_{\mathcal{T}_i}(\Pi, \dots, \Pi) = (-1)^{\chi(\mathcal{T}_i)} \mathrm{Mult}(\mathcal{T}_i) \alpha_{\mathcal{T}_i} z^{m_{\mathcal{T}_i}} \check{\partial}_{n_i} u_{\mathcal{T}_i},$$

where we abbreviate $n_i = k_{\mathcal{T}_i} n_{Q_{\mathcal{T}_i}}$ and $n_{Q_{\mathcal{T}_i}}$ is the primitive clockwise oriented normal to $Q_{\mathcal{T}_i}$. Therefore we have

$$\mathfrak{l}_{\mathcal{T}}(\Pi, \dots, \Pi) = -(-1)^{\chi(\mathcal{T}_1)+\chi(\mathcal{T}_2)} \mathrm{Mult}(\mathcal{T}_1)\mathrm{Mult}(\mathcal{T}_2) H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})[z^{m_{\mathcal{T}_1}}\check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}}\check{\partial}_{n_2}] u_{\mathcal{T}_1} u_{\mathcal{T}_2}.$$

Using Lemma 3.11 we see that $\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2} \in \mathcal{W}^2_{Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}}$ in the case that $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ are intersecting transversally (otherwise the product is $0 \in \mathcal{H}^{0,*}_n$). Applying Lemma 4.4 together with Lemma 3.19, we get $\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \in \mathcal{W}^1_{Q_{\mathcal{T}}}$. Furthermore, we have

$$[z^{m_{\mathcal{T}_1}}\check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}}\check{\partial}_{n_2}] = z^{m_{\mathcal{T}}}\big((\bar{m}_{\mathcal{T}_2}, n_{\mathcal{T}_1})\check{\partial}_{n_2} - (\bar{m}_{\mathcal{T}_1}, n_{\mathcal{T}_2})\check{\partial}_{n_1}\big),$$
$$(\bar{m}_{\mathcal{T}_2}, n_{\mathcal{T}_1})\check{\partial}_{n_2} - (\bar{m}_{\mathcal{T}_1}, n_{\mathcal{T}_2})\check{\partial}_{n_1} = \det(\bar{m}_{e_1}, \bar{m}_{e_2})(n_{\mathcal{T}_1} + n_{\mathcal{T}_2}).$$

If $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ is positively oriented, then $\det(\bar{m}_{e_1}, \bar{m}_{e_2}) > 0$ and $n_{\mathcal{T}_1} + n_{\mathcal{T}_2} = k_{\mathcal{T}} n_{Q_{\mathcal{T}}}$, where $k_{\mathcal{T}}$ is introduced in Notations 2.6, and $\det(\bar{m}_{e_1}, \bar{m}_{e_2}) = \mathrm{Mult}_{v_r}(\mathcal{T})$. Notice that switching to the assumption that $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ is negatively oriented will result in a minus sign in $\det(\bar{m}_{e_1}, \bar{m}_{e_2})$ and hence contribute an extra $(-1)^{\chi(\mathcal{T}, v_r)}$ in the formula (i.e. in this case $\chi(\mathcal{T}, v_r) = 1$). Combining with the fact that $\mathrm{Mult}(\mathcal{T}) = \mathrm{Mult}(\mathcal{T}_1)\mathrm{Mult}_{v_r}(\mathcal{T})$, $(-1)^{\chi(\mathcal{T})} = (-1)^{\chi(\mathcal{T}_1)}(-1)^{\chi(\mathcal{T}, v_r)}$, we obtain the desired formula.

In the second subcase we assume that $MI(\mathcal{T}_2) = 2$, so by the induction hypothesis we have $\mathfrak{l}_{\mathcal{T}_2}(\Pi, \dots, \Pi) = (-1)^{\chi(\mathcal{T}_2)} \alpha_{\mathcal{T}_2} \mathrm{Mult}(\mathcal{T}_2) z^{m_{\mathcal{T}_2}} u_{\mathcal{T}_2}$. Therefore we have

$$\mathfrak{l}_{\mathcal{T}}(\Pi, \dots, \Pi) = -(-1)^{\chi(\mathcal{T}_1)+\chi(\mathcal{T}_2)} \mathrm{Mult}(\mathcal{T}_1)\mathrm{Mult}(\mathcal{T}_2) H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})[z^{m_{\mathcal{T}_1}}\check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}}] u_{\mathcal{T}_1} u_{\mathcal{T}_2},$$

where we absorb the $k_{\mathcal{T}_1}$ into $n_1 = k_{\mathcal{T}_1} n_{Q_{\mathcal{T}_1}}$ again. Applying Lemma 4.4, 3.11 and 3.19 as in the previous subcase, we obtain that $\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \in \mathcal{W}^0_{Q_{\mathcal{T}}}$. Furthermore, we have $[z^{m_{\mathcal{T}_1}}\check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}}] = det(\bar{m}_{\mathcal{T}_1}, \bar{m}_{\mathcal{T}_2}) z^{m_{\mathcal{T}}} = (-1)^{\chi(\mathcal{T}, v_r)} \mathrm{Mult}_{v_r}(\mathcal{T})$ which gives us the desired identity. □

Next we would like to take a closer look at the differential form $\alpha_{\mathcal{T}}$ defined in Notations 4.5.

Definition 4.8 (cf. Definition 5.29 in [10]). *We attach a differential form ν_e on $\mathbb{R}^{|\mathcal{T}^{[1]}|}_{\leq 0}$ to each $e \in \bar{\mathcal{T}}^{[1]}$ recursively as follows:*

- $\nu_e := 1$ *for each incoming edge $e \in \partial_{in}^{-1}(\mathcal{T}^{[0]}_{in})$;*

- $\nu_{e_3} = (-1)^{|\nu_{e_1}||\nu_{e_2}|}\nu_{e_1} \wedge \nu_{e_2} \wedge ds_{e_3}$ *(here* $|\nu_{e_2}|$ *is the cohomological degree of* ν_{e_2}*) if* v *is an internal vertex with incoming edges* $e_1, e_2 \in \mathcal{T}_0$ *and outgoing edge* e_3 *such that* e_1, e_2, e_3 *is clockwise oriented.*

We let $\nu_{\mathcal{T}}$ *be the differential form attached to the unique outgoing edge* $e_o \in \mathcal{T}^{[1]}$*, which defines a volume form or orientation on* $\mathbb{R}^{|\mathcal{T}^{[1]}|}_{\leq 0}$*.*

Given a weighted ribbon d-pointed k-tree $\mathcal{T}$ with $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$, which is either a ray or a line, we let $\eta_{\mathcal{T}}$ be the unique affine function on $M_{\mathbb{R}}$ such that $\eta_{\mathcal{T}} = 0$ on $Q_{\mathcal{T}}$ and $\eta_{\mathcal{T}}$ takes positive values on the anti-clockwise oriented normal to $Q_{\mathcal{T}}$.

Lemma 4.9. *For a weighted ribbon d-pointed k-tree* $\mathcal{T}$ *with* $MI(\mathcal{T}) = 0, 2$ *and* $Q_{\mathcal{T}} \neq \emptyset$*, there exists some* $c > 0$ *such that*

$$(ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2) = \begin{cases} (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 2, \\ (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 0, \end{cases}$$

where ε *satisfies* $\iota_{\nu_{\mathcal{T}}^\vee}\varepsilon = 0$ *(here* $\nu_{\mathcal{T}}^\vee$ *is a top polyvector field dual to* $\nu_{\mathcal{T}}$ *over the component* $\mathbb{R}^{|\mathcal{T}^{[1]}|}_{\leq 0}$*) and* η_1, η_2 *are the affine coordinates on* $M_{\mathbb{R}}$ *with respect to the oriented basis* e_1, e_2 *introduced in Notations 3.4.*

Proof. First of all, notice that both $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma)$ and $M^d_{\mathbb{R}}$ are affine manifolds and $\vec{ev}$ is affine linear. So all the differential forms appearing in this lemmma are affine differential forms. Therefore it suffices to check the equality at a point in $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma)$. Also since $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma) \cong \mathbb{R}^{|\mathcal{T}^{[1]}|}_{\leq 0} \times M_{\mathbb{R}}$, we can always write

$$(ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2) = \begin{cases} c'\nu_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 2, \\ c'\nu_{\mathcal{T}} \wedge \alpha + \varepsilon & \text{if } MI(\mathcal{T}) = 0 \end{cases}$$

for some $c' \in \mathbb{R}$, and some 1-form $\alpha \in \Omega^1(M_{\mathbb{R}})$ with $\iota_{\nu_{\mathcal{T}}^\vee}\varepsilon = 0$. We need to show that α is a constant multiple of $d\eta_{\mathcal{T}}$ and the constant $c' = (-1)^{\chi(\mathcal{T})}c$ for some $c > 0$.

In the case $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$, the moduli space $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma, P_1, \ldots, P_n)$ is a 1-dimensional affine subspace of $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma)$. We take any path ς lying inside $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma, P_1, \ldots, P_n) \subset \overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma)$. Since $ev_{\mathcal{T},i_j} \circ \varsigma$ is a constant map for any $j = 1, \ldots, d$, we have

$$\iota_{\varsigma'}((ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2)) = 0,$$

where ς' is the affine vector field on $\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma)$ induced by ς. On the other hand, $(ev_o)_*(\varrho')$ is tangent to $Q_{\mathcal{T}} = ev_o\left(\overline{\mathfrak{M}}^{\mathcal{T}}_d(\mathcal{P}, \Sigma, P_1, \ldots, P_n)\right)$. So α must be a constant multiple of $d\eta_{\mathcal{T}}$ and we can write

$$(ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2) = c'\nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}} + \varepsilon,$$

for some constant c'.

We now prove that c' is of the form $(-1)^{\chi(\mathcal{T})}c$ for some $c > 0$ by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$ and the only possible trees are those with a unique edge e. Since there are no evaluation maps, we adopt the convention that the left hand side of the equality in the lemma is equal to 1 to make the statement true in this case.

For the induction step, suppose we have a tree $\mathcal{T}$ with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2. We split $\mathcal{T}$ at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points respectively. There are two possible cases.

This first case is when one of the incoming edges, say e_2, is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. As in the proof of Lemma 4.3, we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$. We use the identification

$$\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \ {}_{ev_{\mathcal{T}_1,o}} \times_{\tau_{e_o}} (\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma),$$

under which the evaluation map $ev_{\mathcal{T},o} : \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \to M_{\mathbb{R}}$ is identified as the projection to the last coordinate of the product on the left hand side, and the evaluation at the marked point e_2 is identified as the projection τ_{e_o} to the second factor of $\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}$. We have

$$(ev_{\mathcal{T}_1,i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_1,i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_1)} c\nu_{\mathcal{T}_1} + \varepsilon_{\mathcal{T}_1}$$

for some $c > 0$ by the induction hypothesis. Since $MI(\mathcal{T}) = 0$, $Q_{\mathcal{T}}$ is a ray or a line. We take an affine path ϱ in $M_{\mathbb{R}}$ transversal to $Q_{\mathcal{T}}$ parametrized by the affine coordinate $\eta_{\mathcal{T}}$. Then restricting to $\mathbb{R}_{\leq 0} \times \varrho$, we have

$$ev_{\mathcal{T},i_d}^*(d\eta_1 d\eta_2) = \tau_{e_o}^*(d\eta_1 d\eta_2) = ds_{e_o} \wedge d\eta_{\mathcal{T}},$$

where s_{e_o} is the coordinate on $\mathbb{R}_{\leq 0}$ associated to the outgoing edge e_o. Putting these together we have

$$(-1)^{\chi(\mathcal{T}_1)} c\nu_{\mathcal{T}_1} \wedge ds_{e_o} \wedge d\eta_{\mathcal{T}} = (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}}.$$

In the second case we have both $\mathcal{T}_1$ and $\mathcal{T}_2$ having $k_1, k_2 \geq 1$, and we have $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$, say $Q_{\mathcal{T}_1}$, must be a ray or a line. There are two subcases depending on whether $MI(\mathcal{T}_2) = 0$ or $MI(\mathcal{T}_2) = 2$.

We first assume that $MI(\mathcal{T}_2) = 0$. In this case we have $MI(\mathcal{T}) = 0$, and both $|\mathcal{T}_1^{[1]}|, |\mathcal{T}_2^{[1]}|$ are odd, and hence so is $|\mathcal{T}_1^{[1]}||\mathcal{T}_2^{[1]}|$. Similar to the previous case, we use the identification

$$\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \ {}_{ev_{\mathcal{T}_1,o}} \times_{ev_{\mathcal{T}_2,o}} \overline{\mathfrak{M}}_{d_2}^{\mathcal{T}_2}(\mathcal{P}, \Sigma) \ {}_{ev_{\mathcal{T}_2,o}} \times_{\tau_{e_o}} (\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma).$$

From the induction hypothesis we have

$$(ev_{\mathcal{T}_a,i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_a,i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_a)} c_a \nu_{\mathcal{T}_a} \tau_{e_o}^*(d\eta_{\mathcal{T}_a}) + \varepsilon_{\mathcal{T}_a}$$

with $c_a > 0$ and $\iota_{\nu_{\mathcal{T}_a}^{\vee}} \varepsilon_{\mathcal{T}_a} = 0$ for $a = 1, 2$. Taking their product we get

$$(ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_1) + \chi(\mathcal{T}_2) + |\mathcal{T}_1^{[1]}||\mathcal{T}_2^{[1]}|} \nu_{\mathcal{T}_1} \wedge \nu_{\mathcal{T}_2} \tau_{e_o}^*(d\eta_{\mathcal{T}_1} \wedge d\eta_{\mathcal{T}_2}) + \varepsilon,$$

where $c := c_1 c_2 > 0$ and $\iota_{\nu_{\mathcal{T}}^{\vee}} \varepsilon = 0$. Furthermore, we have

$$\tau_{e_o}^*(d\eta_{\mathcal{T}_1} \wedge d\eta_{\mathcal{T}_2}) = (-1)^{\chi(\mathcal{T}, v)} ds_{e_o} \wedge d\eta_{\mathcal{T}},$$

where s_{e_o} is the coordinate on $\mathbb{R}_{\leq 0}$ associated to the outgoing edge e_o. Putting these together we obtain the desired identity.

Now assuming $MI(\mathcal{T}_2) = 2$, we have

$$(ev_{\mathcal{T}_2,i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_2,i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_2)} c_2 \nu_{\mathcal{T}_2} + \varepsilon_{\mathcal{T}_2}$$

instead. In this case, $|\mathcal{T}_1^{[1]}||\mathcal{T}_2^{[1]}|$ is even and $\nu_{\mathcal{T}_2}$ is an even degree differential form. Therefore we obtain

$$\begin{aligned}(ev_{\mathcal{T},i_1})^*(d\eta_1 d\eta_2)\cdots(ev_{\mathcal{T},i_d})^*(d\eta_1 d\eta_2) &= (-1)^{\chi(\mathcal{T}_1)+\chi(\mathcal{T}_2)} c\nu_{\mathcal{T}_1}\wedge\nu_{\mathcal{T}_2}\tau_{e_o}^*(d\eta_{\mathcal{T}_1})+\varepsilon\\ &= (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}_1}\wedge\nu_{\mathcal{T}_2}\wedge ds_{e_o}+\varepsilon'\\ &= (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}}+\varepsilon'.\end{aligned}$$

Notice that switching the roles of $\mathcal{T}_1$ and $\mathcal{T}_2$ will also give the same result. This completes the proof of the lemma. □

Lemma 4.10. *For $MI(\mathcal{T})=0,2$ with $Q_{\mathcal{T}}\neq\emptyset$, we have the identity*

$$\alpha_{\mathcal{T}}=(-1)^{k+d-1}(ev_o)_*\left((ev_{i_1})^*(\delta_{P_{k_1}})\cdots(ev_{i_d})^*(\delta_{P_{k_d}})\right),$$

where $ev_*:\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P},\Sigma)\to M_{\mathbb{R}}$'s are the evaluation maps introduced in Definition 2.9, and the orientation on fibers of ev_o is defined similarly as in Definition 4.8 (notice that $(-1)^{k+d-1}=(-1)^{MI(\mathcal{T})/2-1}$).

Proof. We prove by using induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]}=\emptyset$ and the only possible trees the ones with a unique edge e, for which the statement is trivial.

For the induction step, suppose we have a tree $\mathcal{T}$ with $\mathcal{T}^{[0]}\neq\emptyset$ with the unique root vertex $v_r\in\mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2. We split $\mathcal{T}$ at v_r to obtain two trees $\mathcal{T}_1,\mathcal{T}_2$ with outgoing edges e_1,e_2 and k_1,k_2 incoming edges, d_1 and d_2 marked points respectively. There are two possible cases.

This first case is when one of the incoming edges, say e_2, is an edge corresponding to a marked point so that $k_2=0$ and $d_2=1$. As in the proof of Lemma 4.3, we must have $MI(\mathcal{T}_1)=2$ and $MI(\mathcal{T})=0$. In this case we let $P_{e_2}=Q_{\mathcal{T}_2}$ be the marked point associated to e_2. The induction hypothesis says that

$$\alpha_{\mathcal{T}_1}=(-1)^{k+d-2}(ev_{\mathcal{T}_1,o})_*\left((ev_{\mathcal{T}_1,i_1})^*(\delta_{P_{k_1}})\cdots(ev_{\mathcal{T}_1,i_{d_1}})^*(\delta_{P_{k_{d_1}}})\right),$$

which is a function with asymptotic support on $Q_{\mathcal{T}_1}$. Then we have

$$\alpha_{\mathcal{T}}=-H(\alpha_{\mathcal{T}_1}\wedge\delta_{P_{e_2}})=-\int_{-\infty}^0\tau_{e_o}^*(\alpha_{\mathcal{T}_1}\wedge\delta_{P_{e_2}})$$

in $\mathcal{W}_*^0/\mathcal{W}_*^{-1}$, where $\tau_{e_o}:\mathbb{R}\times M_{\mathbb{R}}\to M_{\mathbb{R}}$ is the flow associated to $-\bar{m}_{\mathcal{T}}$. This equality holds because we have $P_{e_2}\in U_{1,+}^{m_{\mathcal{T}}}$ and hence any integral over a domain not intersecting P_{e_2} gives 0 in $\mathcal{W}_*^0/\mathcal{W}_*^{-1}$.

Writing

$$\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P},\Sigma)\ {}_{ev_{\mathcal{T}_1,o}}\times_{\tau_{e_o}}(\mathbb{R}_{\leq 0}\times M_{\mathbb{R}})=\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P},\Sigma),$$

where the evaluation map $ev_{\mathcal{T},o}:\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P},\Sigma)\to M_{\mathbb{R}}$ is identified as the projection to the last factor in the product on the left hand side, and the evaluation at the marked point e_2 is identified as τ_{e_o} on $\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}$. Then we have

$$\begin{aligned}
& -\int_{-\infty}^{0} \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}}) \\
=& (-1)^{k+d-1} \int_{-\infty}^{0} \tau_{e_o}^* \left((ev_{\mathcal{T}_1,o})_*((ev_{\mathcal{T}_1,i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1,i_{d_1}})^*(\delta_{P_{k_{d_1}}})) \wedge \delta_{P_{e_2}} \right) \\
=& (-1)^{k+d-1} \int_{-\infty}^{0} \left(\int_{\mathbb{R}_{\leq 0}^{|\mathcal{T}_1|}} (ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_{d-1}})^*(\delta_{P_{k_{d-1}}}) \right) \wedge ev_{i_d}^*(\delta_{P_{e_2}}) \\
=& (-1)^{k+d-1} (ev_{\mathcal{T},o})_* \left((ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_d})^*(\delta_{P_{e_2}}) \right).
\end{aligned}$$

In the second case we have both $\mathcal{T}_1$ and $\mathcal{T}_2$ having $k_1, k_2 \geq 1$ and $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Making use of Lemma 4.4 again, we notice that by comparing the domain of integration intersecting $Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}$ we have

$$\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) = -\int_{-\infty}^{0} \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}),$$

where τ_{e_o} is the flow of $-\bar{m}_{\mathcal{T}}$.

Notice that we have

$$\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \ {}_{ev_{\mathcal{T}_1,o}} \times_{ev_{\mathcal{T}_2,o}} \overline{\mathfrak{M}}_{d_2}^{\mathcal{T}_2}(\mathcal{P}, \Sigma) \ {}_{ev_{\mathcal{T}_2,o}} \times_{\tau_{e_o}} (\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_{d}^{\mathcal{T}}(\mathcal{P}, \Sigma),$$

and therefore we obtain

$$\begin{aligned}
& -\int_{-\infty}^{0} \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \\
=& (-1)^{k+d-1} \int_{-\infty}^{0} \tau_{e_o}^* \big((ev_{\mathcal{T}_1,o})_*((ev_{\mathcal{T}_1,i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1,i_{d_1}})^*(\delta_{P_{k_{d_1}}})) \\
& \qquad \wedge (ev_{\mathcal{T}_2,o})_*((ev_{\mathcal{T}_2,i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1,i_{d_2}})^*(\delta_{P_{k_{d_2}}})) \big) \\
=& (-1)^{k+d-1} \int_{-\infty}^{0} (-1)^{|\mathcal{T}_1^{[1]}||\mathcal{T}_2^{[1]}|} \int_{\mathbb{R}_{\leq 0}^{|\mathcal{T}_1|+|\mathcal{T}_2|}} \left((ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_d})^*(\delta_{P_{k_d}}) \right) \\
=& (-1)^{k+d-1} (ev_{\mathcal{T},o})_* \left((ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_d})^*(\delta_{P_{k_d}}) \right).
\end{aligned}$$

□

Lemma 4.10 allows us to compute the contribution of $\alpha_{\mathcal{T}}$ explicitly as follows:

Lemma 4.11. *For $MI(\mathcal{T}) = 2$ with $Q_{\mathcal{T}} \neq \emptyset$ and for any point x in the interior Int$(Q_{\mathcal{T}})$, we have*

$$\lim_{\hbar \to 0} \alpha_{\mathcal{T}}|_x = (-1)^{\chi(\mathcal{T})}.$$

For $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$ and for an arbitrary embedded path $\varrho : (a, b) \to M_{\mathbb{R}}$ intersecting the relative interior Int$_{re}(Q_{\mathcal{T}})$ transversally and positively (here positive means the orientation of $\{-\bar{m}_{\mathcal{T}}, \varrho'\}$ agrees with that of $M_{\mathbb{R}}$), we have

$$\lim_{\hbar \to 0} \int_{\varrho} \alpha_{\mathcal{T}} = (-1)^{\chi(\mathcal{T})+1}.$$

Proof. We begin with $MI(\mathcal{T}) = 2$. In this case, $k + d - 1$ is even so we have $\alpha_{\mathcal{T}} = (ev_o)_*\Big((ev_{i_1})^*(\delta_{P_{k_1}})\cdots(ev_{i_d})^*(\delta_{P_{k_d}})\Big)$. Fixing a point $x \in \mathrm{Int}(Q_{\mathcal{T}})$, we consider the evaluation map $\hat{ev}_x : \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P},\Sigma) \cap ev_o^{-1}(x) \to M_{\mathbb{R}}^d$ which pulls back the volume form $\prod^d d\eta_1 \wedge d\eta_2$ to $(-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}}$, and in particular $\hat{ev}_x$ is a diffeomorphism onto its image (notice that $\hat{ev}_x$ is affine linear). We let $C_x := \mathrm{Im}(\hat{ev}_x) \subset M_{\mathbb{R}}^d$. Then we have

$$(ev_o)_*\big((ev_{i_1})^*(\delta_{P_{k_1}})\cdots(ev_{i_d})^*(\delta_{P_{k_d}})\big)|_x = (-1)^{\chi(\mathcal{T})}\int_{C_x}\delta_{P_{k_1}}\wedge\cdots\wedge\delta_{P_{k_d}}.$$

Using the fact that $x \in \mathrm{Int}(Q_{\mathcal{T}})$ and the assumption that $P_1,\ldots,P_n$ are in generic position (Definition 2.10), we see that $(P_{k_1},\ldots,P_{k_d}) \in \mathrm{Int}(C_x)$. Together with the explicit form of δ_{P_i}'s in (3.18), we have $\lim_{\hbar\to 0}\int_{C_x}\delta_{P_{k_1}}\wedge\cdots\wedge\delta_{P_{k_d}} = 1$.

For $MI(\mathcal{T}) = 0$, $k + d - 1$ is odd. We consider $\mathcal{I}_\varrho := \bigcup_{t\in(a,b)}\mathcal{I}_{\varrho(t)}$, where we write $\mathcal{I}_x := \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|} \times \{x\} \cong \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$ and treat $\nu_{\mathcal{T}}$ as a volume element on each $\mathcal{I}_x$. Similar to the previous case we consider $\hat{ev}_\varrho : \mathcal{I}_\varrho \to M_{\mathbb{R}}^d$ which gives $\hat{ev}_\varrho^*(\prod^d d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}}$. Therefore we have

$$\int_\varrho \alpha_{\mathcal{T}} = (-1)\int_{\mathcal{I}_\varrho}(ev_{i_1})^*(\delta_{P_{k_1}})\cdots(ev_{i_d})^*(\delta_{P_{k_d}}) = (-1)^{\chi(\mathcal{T})+1}\int_{C_\varrho}\delta_{P_{k_1}}\wedge\cdots\wedge\delta_{P_{k_d}}.$$

Again using the generic assumption on the points $P_1,\ldots,P_n$, we get $(P_{k_1},\ldots,P_{k_d}) \in \mathrm{Int}(C_\varrho)$ and therefore $\lim_{\hbar\to 0}\int_{C_\varrho}\delta_{P_{k_1}}\wedge\cdots\wedge\delta_{P_{k_d}} = 1$. □

For a a weighted d-pointed k-tree Γ with $MI(\Gamma) = 0, 2$ and $Q_\Gamma \neq \emptyset$ (notice that the definition of the polyhedral subset Q_Γ does not depend on the ribbon structure), since the monomial weights u_{k_j}'s at the marked points p_{i_j}'s are all distinct, there are exactly 2^{d-1} ribbon structures (up to isomorphisms) on Γ. Furthermore, $\mathfrak{l}_{\mathcal{T}}(\Pi,\ldots,\Pi)$ does not depend on the ribbon structure as well since $\Pi \in \mathcal{H}_n^{2,2}$ and Π commutes with even elements in $\mathcal{H}_n^{*,*}$ (one can also see from Lemmas 4.7 and 4.11 that the terms $(-1)^{\chi(\mathcal{T})}$ which depend on the ribbon structure indeed cancel).

Therefore for each weighted d-pointed k-tree Γ, we can fix an arbitrary ribbon tree $\mathcal{T}$ as representative whose underlying tree $\underline{\mathcal{T}}$ is Γ, and write

$$\mathfrak{l}_{k,d}(\Pi,\ldots,\Pi) := \sum_{\Gamma\in\mathtt{WPT}_{k,d}}\mathfrak{l}_{\mathcal{T}}(\Pi,\ldots,\Pi).$$

By setting $\alpha_\Gamma := (-1)^{\chi(\mathcal{T})}\alpha_{\mathcal{T}}$ and combining Lemmas 4.7 and 4.11 which dose not depends on the ribbon structure, we obtain our main theorem:

Theorem 4.12. *The Maurer-Cartan solution $\Phi \in \mathcal{H}_n^{*,*}$ constructed in* (3.22) *is of the form*

$$\Phi = \Pi + \Xi^{0,0} + \Xi^{1,1},$$

with $\varXi^{i,i} \in \mathcal{H}^{i,i}_n$ for $i = 0, 1, 2$, and both $\varXi^{0,0}, \varXi^{1,1}$ can be expressed as a sum over tropical disks:

$$\varXi^{0,0} = \sum_{k,d} \sum_{\substack{\Gamma \in \mathtt{WPT}_{k,d},\ MI(\Gamma)=2 \\ \overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n) \neq \emptyset}} \alpha_\Gamma Mult(\Gamma) z^{m_\Gamma} u_\Gamma,$$

$$\varXi^{1,1} = \sum_{k,d} \sum_{\substack{\Gamma \in \mathtt{WPT}_{k,d},\ MI(\Gamma)=0 \\ \overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P},\Sigma,P_1,\ldots,P_n) \neq \emptyset}} \alpha_\Gamma k_\Gamma Mult(\Gamma) z^{m_\Gamma} \check{\partial}_{n_{Q_\Gamma}} u_\Gamma,$$

where $\mathtt{WPT}_{k,d}$ is the set of isomorphism classes of weighted d-pointed k-trees introduced in Definition 2.5. Furthermore, in the above expressions we have

$$\alpha_\Gamma \in \mathcal{W}^{s_\Gamma}_{Q_\Gamma},$$

where $Q_\Gamma = ev_o(\overline{\mathfrak{M}}^{\Gamma}_{d}(\mathcal{P}, \Sigma, P_1, \ldots, P_n))$ is of codimension $s_\Gamma := 1 - \frac{MI(\Gamma)}{2}$ in $M_\mathbb{R}$, and

$$\lim_{\hbar \to 0} \alpha_\Gamma|_x = 1 \qquad \text{for any } x \in Int(Q_\Gamma) \text{ when } MI(\Gamma) = 2,$$
$$\lim_{\hbar \to 0} \int_\varrho \alpha_\Gamma = -1 \quad \text{for any } \varrho \pitchfork Int_{re}(Q_\Gamma) \text{ positively when } MI(\Gamma) = 0.$$

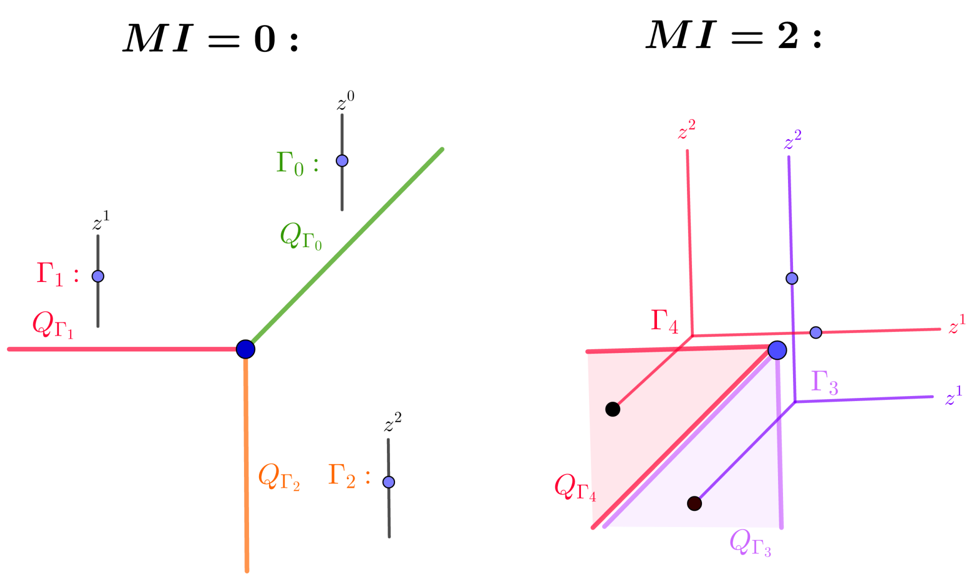

FIGURE 6. Tropical disks and their moduli spaces for $n = 1$

Example 4.13. *We give an example on the the locus Q_Γ traced out by weighted 1-pointed k-trees Γ in the case $n = 1$, i.e. there is only 1 marked point. For a tree Γ with $MI(\Gamma) = 0$, the only possibility is that $k = 1$ and there are precisely 3 such trees $\Gamma_0, \Gamma_1, \Gamma_2$ as shown in Figure 6 together with the corresponding 1-dimensional loci $Q_{\Gamma_0}, Q_{\Gamma_1}, Q_{\Gamma_2}$.*

For the case $MI(\Gamma) = 2$, we have $k = 2$, and there are 6 such trees. Two of them, which we call Γ_3 and Γ_4, with the same attached monomial $Mono(\Gamma) = z^1 z^2$ are shown in Figure 6. Note that the boundary between Q_{Γ_3} and Q_{Γ_4} is not a wall in $\mathcal{D}$ although the moduli space jumps across it. This is because the attached monomial $Mono(\Gamma)$ does not jump across the boundary, and this agrees with the fact that Φ is simply a holomorphic function outside the support $Supp(\mathcal{D})$.

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