FUKAYA'S CONJECTURE ON S^1-EQUIVARIANT DE RHAM COMPLEX

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ABSTRACT. Getzler-Jones-Petrack [7] introduced A_∞ structures on the equivariant complex for manifold M with smooth $\mathbb{S}^1$ action, motivated by geometry of loop spaces. Applying Witten's deformation by Morse functions followed by homological perturbation we obtained a new set of A_∞ structures. We extend and prove Fukaya's conjecture [6] relating this Witten's deformed equivariant de Rham complexes, to a new Morse theoretical A_∞ complexes defined by counting gradient trees with jumping which are closely related to the $\mathbb{S}^1$ equivariant symplectic cohomology proposed by Siedel [15].

1. INTRODUCTION

In the influential paper [17] by Witten, harmonic forms on a compact oriented Riemannian manifold (M, g) are related to the Morse complex $CM_f^* := \bigoplus_{p\in \mathrm{Crit}(f)} \mathbb{C}\cdot p$ on M with a Morse function f [1]. More precisely, Witten introduced the twisted Laplacian $\Delta_{f,\lambda} := d^*_{f,\lambda} \circ d + d \circ d^*_{f,\lambda}$ [2] with a large real parameter λ, and an isomorphism

$$\phi : (CM_f^*, \delta) \to (\Omega^*_{f,<1}(M), d) \tag{1.1}$$

where $\Omega^*_{f,<1}(M)$ refers to the small eigensubspace of $\Delta_{f,\lambda}$ (see Section 2.2). The detailed analysis of ϕ is later carried out in [9, 11, 10, 12] and readers may also see [18] for this correspondence.

In [6], Fukaya conjectured that Witten's isomorphism (1.1) can be enhanced to an isomorphism of A_∞ algebras (or categories), a generalization of differential graded algebras (abbrev. dga), encoding rational homotopy type by work of Quillen [14] and Sullivan [16]. The A_∞ structures $m_k(\lambda)$'s on $\Omega^*_{f,<1}(M)$ are obtained by pulling back the structures of the de Rham dga $(\Omega^*(M), d, \wedge)$ using the homological perturbation lemma (see e.g. [13]) with homotopy operator $H_{f,\lambda} = d^*_{f,\lambda} G_{f,\lambda}$. The Morse A_∞ structures m_k^{Morse}'s are defined via counting gradient flow trees of Morse functions as in [5]. Fukaya conjectured that they are related by

$$\lim_{\lambda\to\infty} m_k(\lambda) = m_k^{Morse} \tag{1.2}$$

via the Witten's isomorphism (1.1). This conjectured is proven in [3] by extending the analytic technique in [12] to incorporate the homotopy operator $H_{f,\lambda}$.

When M is equipped with a smooth $\mathbb{S}^1$ action, motivated by the geometry of loop space $\mathbb{S}^1 \curvearrowright \mathcal{L}X$ for some X, Getzler-Jones-Petrack [7] introduced an enhancement of the equivariant de Rham complex on M. They defined new A_∞ algebra structures consisting of

$$\tilde{m}_k : \left(\Omega^*(M)[[u]]\right)^{\otimes k} \to \Omega^*(M)[[u]] \tag{1.3}$$

by adding higher order (in u) operations $u\mathcal{P}_k$'s (see Section 2.1) to ordinary de Rham dga structures. Witten's deformed A_∞ structures $m_k(\lambda)$'s are constructed from $\tilde{m}_k$'s in (1.3) using the technique of homological perturbation as in original Fukaya's conjecture.

[1] Here $\mathrm{Crit}(f)$ refers to set of critical points of f, and the differential δ is given by counting gradient flow lines.

[2] We let $d^*_{f,\lambda}$ to be the adjoint of d, and $G_{f,\lambda}$ to be Witten's Green function of $\Delta_{f,\lambda}$ w.r.t. volume form $e^{-2\lambda f}\,\mathrm{vol}_M$.

Inspired by Fukaya's correspondence, we define new Morse theoretic type counting structures m_k^{eMorse}'s (where m_1^{eMorse} is known before in [2]) associated to $\mathbb{S}^1 \curvearrowright M$, counting of Morse flow trees with jumpings coming from the $\mathbb{S}^1$ action (see the following Section 1.1). We prove the generalization of (1.2) for $\mathbb{S}^1 \curvearrowright M$ relating these two structures.

Theorem 1.1 (=Theorem 2.11)**.** *We have*

$$\lim_{\lambda\to\infty} m_k(\lambda) = m_k^{eMorse}.$$

1.1. **The operation m_k^{eMorse}'s.** To describe m_k^{eMorse}'s, we fix a generic sequence (see Definition 2.8) of functions $(f_0, \ldots, f_k)$ such that their differences $f_{ij} := f_j - f_i$ are assumed to be Morse-Smale as in Definition 2.5. The Morse theoretical A_∞ product m_k^{eMorse}'s take the form

$$m_k^{\text{eMorse}} := \sum_T m_{k,T}^{\text{eMorse}} : CM^*_{f_{(k-1)k}}[[u]] \otimes \cdots \otimes CM^*_{f_{01}}[[u]] \to CM^*_{f_{0k}}[[u]]$$

which is a summation over directed labeled ribbon k-tree T with k-incoming edges and 1 outgoing edge, where internal vertices are either labeled by 1 or by u. For example (see Section 2.3 for details), if we take the tree T to be the one with two incoming edges e_{12} and e_{01} joining the vertex v_r connected to the outgoing edge e_{02}, with v_r being labeled by u. The gradient flow trees with type T will be consisting of gradient flow lines of f_{12}, f_{01} and f_{02} which ending at critical points q_{12}, q_{01} and q_{02} respectively, that can be joined together at a point $x_{v_r} \in M$ with further help of the $\mathbb{S}^1$ action $\sigma_t : M \to M$ (for some t) as shown in the Figure 1. As a consequence of the above Theorem 1.1, the Morse (pre)-category (here pre-category means this operation only defined for generic sequence $(f_0, \ldots, f_k)$) on $\mathbb{S}^1 \curvearrowright M$ is an A_∞ (pre)-category.

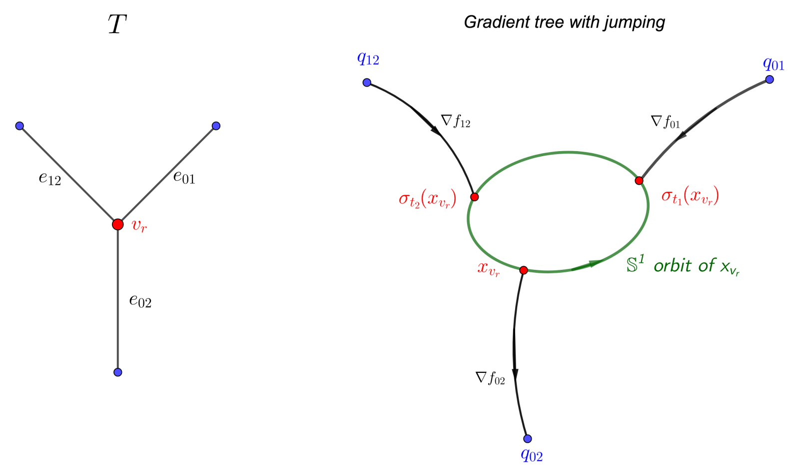

FIGURE 1. Gradient tree with jumping of type T

Corollary 1.2. *The operations m_k^{eMorse}'s satisfy the A_∞ relation for generic sequences of functions.*

Remark 1.3. *In* [15, Section 8b], *Seidel proposed the A_∞ operators m_k^{Floer} on the symplectic cochain complex for a Liouville domain X, which corresponds to m_k^{eMorse}'s if we think of M as a finite dimensional analogue of $\mathcal{L}X$. The corresponding m_1^{Floer} operation is studied in details in* [19]*. The above Theorem 1.1 suggest how Witten deformation can provide a linkage between the Getzler-Jones-Petrack's operation $\tilde{m}_k$ on $\mathcal{L}X$ and the Floer theoretical operations introduced by Seidel through the investigation of the corresponding finite dimensional situation.*

This paper consists of three parts. In Section 2 we set up the Witten deformation of Getzler-Jones-Petrack's A_∞ operations $\tilde{m}_k$'s, the definition of counting gradient flow trees with jumping, and state our Main Theorem 2.11. In Section 3.1, we recall the necessary analytic result by following [3]. The rest of Section 3 will be a proof of Theorem 2.11 by figuring out the exact relations between the operations $m_{k,T}(\lambda)$ and counting of gradient trees.

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2. Witten's deformation of S^1-equivariant de Rham complex

We always let (M, g) to be an n-dimensional compact oriented Riemannian manifold, and denote it volume form by vol_M (or simply vol). We assume there is an smooth $\mathbb{S}^1$ action $\sigma : \mathbb{S}^1 \times M \to M$ on M preserving (g, vol). We should write $\sigma_t : M \to M$ to be the action for a fixed $t \in \mathbb{S}^1$.

2.1. S^1-equivariant de Rham complex and category.

We begin with recalling the Definition of S^1-equivariant de Rham A_∞ algebra introduced in [7], which is reformulated to be A_∞ category as follows for the convenient of presentation of this paper.

Definition 2.1. *The S^1-equivariant de Rham A_∞ category $dR(M)$ consisting of object being smooth functions $f : M \to \mathbb{R}$, with morphism* $\mathrm{Hom}(f, g) := \Omega^*(M)[[u]]$ *where u is a formal variable. The A_∞ operations* $\tilde{m}_k : \mathrm{Hom}(f_{k-1}, f_k) \otimes \cdots \otimes \mathrm{Hom}(f_0, f_1) \cong (\Omega^*(M)[[u]])^{\otimes k} \to \mathrm{Hom}(f_0, f_k) \cong \Omega^*(M)[[u]]$ *is defined by* $\tilde{m}_1(\alpha_{01}) = d(\alpha_{01}) + u\mathcal{P}_1(\alpha_{01})$, $\tilde{m}_2(\alpha_{12}, \alpha_{01}) = (-1)^{|\alpha_{12}|+1}\alpha_{12} \wedge \alpha_{01} + u\mathcal{P}_2(\alpha_{12}, \alpha_{01})$ *and* $\tilde{m}_k(\alpha_{(k-1)k}, \dots, \alpha_{01}) = u\mathcal{P}_k(\alpha_{(k-1)k}, \dots, \alpha_{01})$ *for* $\alpha_{ij} \in \mathrm{Hom}(f_i, f_j)$.

Here the operator $\mathcal{P}_k$ is defined by the action $\mathcal{P}_1(\alpha_{ij}) = \int_{\mathbb{S}^1}(\iota_{\frac{\partial}{\partial t}}\sigma^*(\alpha_{ij}))dt$, *and for $k \geq 2$ we use*

$$\mathcal{P}_k(\alpha_{(k-1)k}, \dots, \alpha_{01}) := \int_{0 \leq t_k \leq \cdots \leq t_1 \leq 1} \left(\iota_{\frac{\partial}{\partial t_k}}(\sigma^*(\alpha_{(k-1)k})) \wedge \cdots \wedge \iota_{\frac{\partial}{\partial t_1}}(\sigma^*(\alpha_{01})) \right) dt_k \cdots dt_1.$$

The fact that the about operations $\tilde{m}_k$'s form an A_∞ category is proven in [7, Theorem 1.7].

2.2. Homological perturbation via Witten's deformation.

We follow [3, Section 2.2.] to introduced the Witten deformation with a real parameter $\lambda > 0$, which is orignated from [17]. For each f_i and f_j, we twist the volume form vol by $f_{ij} := f_j - f_i$ as $\mathrm{vol}_{ij} = e^{-2\lambda f_{ij}}\mathrm{vol}$, and let $d^*_{ij} := e^{2\lambda f_{ij}} d^* e^{-2\lambda f_{ij}} = d^* + 2\lambda\iota_{\nabla f_{ij}}$ to be the adjoint of d with respect to the volume form vol_{ij}. The Witten Laplacian is defined by $\Delta_{ij} := dd^*_{ij} + d^*_{ij}d$, acting on the complex $\Omega^*(M)[[u]]$ [3]. We denote the span of eigenspaces with eigenvalues contained in $[0, 1)$ by $\Omega^*_{ij,<1}(M)[[u]]$, or simply $\Omega^*_{ij,<1}[[u]]$. We use construction in [3] originated from [6] using homological perturbation lemma [13], which obtain a new A_∞ structure from m_k's as follows.

Definition 2.2. *A* (directed) k-tree *labeled T consists of a finite set of vertices $\bar{T}^{[0]}$ together with a decomposition $\bar{T}^{[0]} = T^{[0]}_{in} \sqcup T^{[0]} \sqcup \{v_o\}$, where $T^{[0]}_{in}$, called the set of incoming vertices, is a set of size k and v_o is called the outgoing vertex (we also write $T^{[0]}_\infty := T^{[0]}_{in} \sqcup \{v_o\}$ and $T^{[0]}_{ni} := T^{[0]} \cup \{v_o\}$), a finite set of edges $\bar{T}^{[1]}$, two boundary maps $\partial_{in}, \partial_o : \bar{T}^{[1]} \to \bar{T}^{[0]}$ (here ∂_{in} stands for incoming and*

[3]Stictly speaking, the differential forms here depend on the real parameter λ while we prefer to subpress the dependence in our notation.

∂_o stands for outgoing), and a labeling of every internal vertices $T^{[0]}$ by either 1 or u, satisfying the following conditions:

(1) Every vertex $v \in T^{[0]}_{in}$ has valency one, and satisfies $\#\partial_o^{-1}(v) = 0$ and $\#\partial_{in}^{-1}(v) = 1$; we let $T^{[1]} := \bar{T}^{[1]} \setminus \partial_{in}^{-1}(T^{[0]}_{in})$.
(2) Every vertex $v \in T^{[0]}$ has an unique edge $e_{v,o} \in \bar{T}^{[1]}$ such that $\partial_{in}(e_{v,o}) = v$, and only trivalent vertices in $T^{[0]}$ can be labeled with 1.
(3) For the outgoing vertex v_o, we have $\#\partial_o^{-1}(v_o) = 1$ and $\#\partial_{in}^{-1}(v_o) = 0$; we let $e_o := \partial_o^{-1}(v_o)$ be the outgoing edge and denote by $v_r \in T^{[0]}_{in} \sqcup T^{[0]}$ the unique vertex (which we call the root vertex) with $e_o = \partial_{in}^{-1}(v_r)$.
(4) The topological realization *$|\bar{T}| := \left(\coprod_{e \in \bar{T}^{[1]}} [0,1]\right) / \sim$ of the tree T is connected and simply connected; here $\sim$ is the equivalence relation defined by identifying boundary points of edges if their images in $T^{[0]}$ are the same.*

By convention we also allow the unique labeled 1-tree with $T^{[0]} = \emptyset$. Two labeled k-trees T_1 and T_2 are isomorphic *if there are bijections $\bar{T}_1^{[0]} \cong \bar{T}_2^{[0]}$ and $\bar{T}_1^{[1]} \cong \bar{T}_2^{[1]}$ preserving the decomposition $\bar{T}_i^{[0]} = T^{[0]}_{i,in} \sqcup T^{[0]}_i \sqcup \{v_{i,o}\}$ and boundary maps $\partial_{i,in}$ and $\partial_{i,o}$ and the labelling of $T^{[0]}$. The set of isomorphism classes of labeled k-trees will be denoted by $\mathbb{T}_k$. For a labeled k-tree T, we will abuse notations and use T (instead of $[T]$) to denote its isomorphism class.*

A labeled ribbon k-tree is a k-tree T with a cyclic ordering of $\partial_{in}^{-1}(v) \sqcup \partial_o^{-1}(v)$ for each trivalent vertex $v \in T^{[0]}$, and isomorphism of labeled ribbon k-trees are further required to preserve this ordering. A labeled ribbon k-tree can have its topological realization $|\bar{T}|$ being embedded into the unit disc D, with $T^{[0]}_\infty$ lying on the boundary ∂D such that the cyclic ordering of $\partial_{in}^{-1}(v) \sqcup \partial_o^{-1}(v)$ agree with the anti-clockwise orientation of D. The set of isomorphism classes of labeled ribbon k-trees will be denoted by $\mathbb{LT}_k$.

Notations 2.3. *For each $T \in \mathbb{LT}_k$, we can associated to each edge $e \in \bar{T}^{[1]}$ a numbering by pair of integer ij using the embedding $|\bar{T}| \to D$ by the rules: there are $k+1$ connected components of $D \setminus |\bar{T}|$, and we assign each component by integers $0, \ldots, k$; each (directed) edge $e \in \bar{T}^{[1]}$ with region numbered by i on its left and region numbered by j on its right is numbered by ij; the incoming edges numbered by $e_{(k-1)k}, \ldots, e_{01}$ and the outgoing edge e_{0k} are in clockwise ordering of ∂D.*

A pair of $v \in T^{[0]} \cup \{v_o\}$ attached to an edge $e \in \bar{T}^{[1]}$ is called a flag, and we will let $F(T)$ to be the set of all flags. For every flag (e, v), we let $T_{e,v}$ to be the unique subtree with outgoing vertex being v if $\partial_o(e) = v$, and we let $T_{e,v}$ to be the unique subtree with outgoing edge being e if $\partial_{in}(e) = v$.

Definition 2.4. *Given a labeled ribbon k-tree $T \in \mathbb{LT}_k$ with an embedding $|\bar{T}| \to D$, we assoicate to it an operation $m_{k,T}(\lambda) : \Omega^*_{(k-1)k,<1}[[u]] \otimes \cdots \otimes \Omega^*_{01,<1}[[u]] \to \Omega^*_{0k,<1}[[u]]$ by the following rules :*

(1) aligning the inputs $\varphi_{(k-1)k}, \cdots, \varphi_{01}$ at the incoming vertices $T^{[0]}_{in}$ according to the clockwise ordering induced from D;
(2) if a vertex $v \in T^{[0]}$ has incoming edges $e_{v,1}, \ldots, e_{v,l}$ and outgoing edge $e_{v,o}$ attached to it such that $e_{v,l}, \ldots, e_{v,1}, e_{v,o}$ is in clockwise orientation, we apply the operation $\wedge$ if v is labeled with 1 (and hence trivalent) and the operation $\mathcal{P}_l$ if v is labeled with u;
(3) for an edge $e \in T^{[0]}$ which is numbered by ij, we apply the homotopy operator $H_{ij} := d^*_{ij} G_{ij}$ where G_{ij} is the Witten's twisted Green operator associated to the Witten Laplacian Δ_{ij};
(4) for the unique outgoing edge e_o, we apply the operator P_{0k} which is the orthogonal projection $P_{0k} : \Omega^*[[u]] \to \Omega^*_{0k,<1}[[u]]$ with respect to the twisted L_2-norm obtained from the volume form vol_{0k}.

By convention, we define $m_{1,T}(\lambda)$ for the unique tree with $T^{[0]} = \emptyset$ to be the restriction of d on $\Omega^*_{ij,<1}[[u]]$. For each labeled ribbon k-tree T, we assign n_T to be the number of vertices in $T^{[0]}$ labeled with u, and we let $m_k(\lambda) := \sum_{T\in\mathbb{LT}_k} u^{n_T} m_{k,T}(\lambda)$ to be the homological perturbed A_∞ strucutre.

It is well-known that (see e.g. [1, Chapter 8]) the perturbed A_∞ structure $m_k(\lambda)$'s satisfy the A_∞ relation. And we obtain a new category $\mathrm{dR}_{<1}(M)$ via Witten deformation.

2.3. **Relation with S^1-equivariant Morse flow trees.** In [12, 17, 18], a relation between the Morse complex $\mathrm{CM}_{f_{ij}}$ and $\Omega^*_{ij,<1}$ is established when f_{ij} is a Morse-Smale function in following Definition 2.5. Following [18], it is an isomorphism

$$\Phi_{ij} : \Omega^*_{ij,<1} \to \mathrm{CM}_{f_{ij}}; \quad \Phi_{ij}(\alpha) := \sum_{p\in\mathrm{Crit}(f_{ij})} \int_{V_p^-} \alpha, \tag{2.1}$$

where $\mathrm{Crit}(f_{ij})$ is the finite set of critical points of f_{ij} (with Morse index of p given by number of negative eigenvalues of $\nabla^2 f_{ij}(p)$), and V_p^- (Notice that we further choose an orientation of V_p^- by choosing a volume element of the normal bundle NV_p^+) is the unstable submanifold associated to p which is the union of all gradient flow lines $\gamma(s)$ of ∇f_{ij} which limit toward p as $s\to\infty$. Furthermore, the de Rham differential is identified with the Morse differential δ_1 defined via counting Morse flow lines.

Definition 2.5. *A Morse function f_{ij} is said to satisfy the Morse-Smale condition if V_p^+ and V_q^- intersecting transversally for any two critical points $p\neq q$ of f_{ij}.*

We illustrate how the technique in [3] can be used to establish a relation between $\lambda\to\infty$ limit of the operation $m_k^T(\lambda)$ with a new Morse-theoretical counting for $\mathbb{S}^1\to M$ defined as follows.

Notations 2.6. *A metric labeled k-tree (ribbon) $\mathcal{T}$ is a labeled (ribbon) k-tree together with a length function $l : T^{[1]}\setminus\{e_o\}\to(0,+\infty)$. For each $e\in\bar{T}^{[1]}$, we let $\mathcal{I}_e = (-\infty, 0]$ if $e\in T^{[1]}_{in}$, $\mathcal{I}_e = [0, l(e)]$ for $e\in T^{[1]}\setminus\{e_o\}$ and $\mathcal{I}_{e_o} = [0,\infty)$. The space of metric structure on T, denoted by $\mathcal{S}(T)$, is a copy of $(0,+\infty)^{|T^{[1]}|-1}$. The space $\mathcal{S}(T)$ can be partially compactified to a manifold with corners $(0,+\infty]^{|T^{[1]}|-1}$, by allowing the length of internal edges going to be infinity. In particular, it has codimension-1 boundary $\partial\overline{\mathcal{S}(T)} = \coprod_{T=T'\sqcup T''}\mathcal{S}(T')\times\mathcal{S}(T'')$.*

For every vertex $v\in\bar{T}$, we use $\nu(v)+1$ to denote the valency of v. We write $\blacktriangle_l := \{(t_l,\dots,t_1)\in[0,1]^l \mid 0\le t_l\le\cdots\le t_1\le 1\}$ for $l>1$, and $\blacktriangle_1 = \mathbb{S}^1$ [4], and attach to each vertex v labeled with u a simplex $\blacktriangle_{\nu(v)}$. Writing $LT^{[0]}$ to be the collection of all vertices with label u, we let $\mathbf{S}(T) := \prod_{v\in LT^{[0]}}\blacktriangle_{\nu(v)}\times\mathcal{S}(T)$.

Definition 2.7. *Given a sequence $\vec{f} = (f_0,\dots,f_k)$ such that all the difference f_{ij}'s are Morse, with a sequence of points $\vec{q} = (q_{(k-1)k},\dots,q_{01},q_{0k})$ such that q_{ij} is a critical point of f_{ij}, and a metric labeled ribbon k-tree $\mathcal{T}$, a gradient flow tree (with jumping) Γ (readers may see Figure 1 for an example) of type $(T,\vec{f},\vec{q})$ consisting of a gradient flow line $\gamma_{ij} : \mathcal{I}_{e_{ij}}\to M$ of the Morse function f_{ij} for each edge $e_{ij}\in\bar{T}^{[1]}$ numbered by ij, and a point $\mathbf{t}_v = (t_{v,\nu(v)},\dots,t_{v,1})\in\blacktriangle_{\nu(v)}$ for every $v\in LT^{[0]}$ satisfying:*

1. *(1) $\lim_{s\to-\infty}\gamma_{e_{i(i+1)}}(s) = q_{i(i+1)}$ for the incoming edges $e_{i(i+1)}\in T^{[1]}_{in}$, and $\lim_{s\to\infty}\gamma_{e_{0k}}(s) = q_{0k}$ for the unique outgoing edge e_o;*
2. *(2) for a trivalent vertex $v\in T^{[0]}$ labeled by 1 with two incoming edges e_{jl}, e_{ij} and outgoing edge e_{il}, we require that $\gamma_{ij}(l(e_{ij})) = \gamma_{jl}(l(e_{jl})) = \gamma_{il}(0)$;*

[4] This is not the 1-simplex, but we would like to unify our notation in this way.

(3) *for a vertex* $v \in LT^{[0]}$ *with incoming edges* $e_{i_{l-1}i_l}, \ldots, e_{i_0 i_1}$ *and outgoing edge* $e_{i_0 i_l}$*, we require that* $\sigma(-t_{v,l}, \gamma_{i_{l-1}i_l}(l(e_{i_{l-1}i_l}))) = \cdots = \sigma(-t_{v,1}, \gamma_{i_0 i_1}(l(e_{i_0 i_1}))) = \gamma_{i_0 i_l}(0)$*, where* $l = \nu(v)$ *and* σ *is the* $\mathbb{S}^1$ *action map in the beginning of Section 2.*

We will let $\mathcal{M}_T(\vec{f}, \vec{q})$ *to denote the moduli space (as a set) of gradient flow lines of type* T*. For the unique tree with* $T^{[0]} = \emptyset$*, we let* $\mathcal{M}_T(\vec{f}, \vec{q})$ *to be the moduli space of gradient flow lines quotient by the extra* $\mathbb{R}$ *symmetry by convention.*

Similar to the moduli space of gradient flow trees without $\mathbb{S}^1$ action (see e.g. [3, Section 2.1.]), we can describe $\mathcal{M}_T(\vec{f}, \vec{q})$ as intersection of stable and unstable submanifolds.

Definition 2.8. *Given the sequence* $\vec{f}$ *and* $\vec{q}$ *as in the above Definition 2.7, we define a smooth map* $\mathbf{f}_{T,i(i+1)} : V^+_{q_{i(i+1)}} \times \mathbf{S}(T) \to M$ *for each* $i = 0, \ldots, k-1$ *as follows. Given a incoming edge* $e_{i(i+1)}$*, there is a unique sequence of edges* $e_{i_0 j_0} = e_{i(i+1)}, e_{i_1 j_1}, \ldots, e_{i_m j_m}, e_{i_{m+1} j_{m+1}} = e_o$ *with* $v_d := \partial_o(e_{i_d j_d})$ *forming a path from the incoming vertex* $v_{i(i+1)}$ *to the outgoing vertex* v_o*. Fixing a point* $x_0 \in V^+_{q_{i(i+1)}}$ *and a point* $((\mathbf{t}_v)_{v \in LT^{[0]}}, (l(e))_{e \in T^{[1]} \setminus \{e_o\}}) \in \mathbf{S}(T)$*, we determind a point* $x_d \in M$ *inductively for* $0 \leq d \leq m+1$ *by the rules:*

(1) *if* v_d *is labeled with* 1*, we simply take* x_{d+1} *to be the image of* x_d *under* $l(e_{i_{d+1} j_{d+1}})$ *time flow of* $\nabla f_{i_{d+1} j_{d+1}}$ *for* $d < m$*, and* $x_{d+1} = x_d$ *for* $d = m$*;*
(2) *and if* v_d *is labeled with* u*, we take* x_{d+1} *to be the image of* $\sigma(-t_{v_d,l}, x_d)$ *under the* $l(e_{i_{d+1} j_{d+1}})$ *time flow of* $\nabla f_{i_{d+1} j_{d+1}}$ *if* $d < m$*, and* $x_{d+1} = \sigma(-t_{v_d,l}, x_d)$ *for* $d = m$*, where* $e_{i_d j_d}$ *is the* l*-th incoming edge attached to* v_d *in the* anti-clockwise *orientation.*

These map can be put together as $\mathbf{f}_T : V^-_{q_{0k}} \times V^+_{q_{(k-1)k}} \times \cdots \times V^+_{q_{01}} \times \mathbf{S}(T) \to M^k$ *using the natural embedding* $V^-_{q_{0k}} \hookrightarrow M$ *for the first component. Therefore we see that* $\mathcal{M}_T(\vec{f}, \vec{q}) = \mathbf{f}_T^{-1}(\mathbf{D})$ *where* $\mathbf{D} = M \hookrightarrow M^{k+1}$ *is the diagonal.*

We say a sequence of function $\vec{f}$ *generic if for any sequence of critical points* $\vec{q}$*, any labeled tree* T *the associated intersection* $\mathbf{f}_T$ *with* $\mathbf{D}$ *is transversal with expected dimension (meaning that it is empty when expected negative dimensional intersection), and the same hold when restricting* $\mathbf{f}_T$ *on any boundary strata of* $V^-_{q_{0k}} \times V^+_{q_{(k-1)k}} \times \cdots \times V^+_{q_{01}} \times \mathbf{S}(T)$ *(the stratification coming from that of* $\blacktriangle_{\nu(v)}$*) and for any subsequence of* $\vec{f}$*.*

Suppose we are given a generic sequence $\vec{f}$ with $\vec{q}$ and T as in the above Definition 2.8, then we can compute the dimension of the moduli space as

$$\dim(\mathcal{M}_T(\vec{f}, \vec{q})) = \deg(q_{0k}) - \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) + \sum_{v \in LT^{[0]}} \nu(v) + |T^{[1]}| - 1. \tag{2.2}$$

Definition 2.9. *Given generic* $\vec{f}$*,* $\vec{q}$ *and* T *as in the above Definition 2.8 such that* $\dim(\mathcal{M}_T(\vec{f}, \vec{q})) = 0$*, with a flow tree* $\Gamma \in \mathcal{M}_T(\vec{f}, \vec{q})$*, we assign a sign* $(-1)^{\chi(\Gamma)}$ *by assigning a differential form* $\mathrm{vol}_{e,v} \in \bigwedge^n T^* M_{\gamma_e(v)}$ *(Here we abuse the notation to use* v *to stand for the corresponding point in* $\mathcal{I}_e$*) for each flag* $(e, v) \in \digamma(T)$*, inductively along the tree* T *as follows:*

(1) *for an incoming edge* $e_{i(i+1)}$ *with* $v = \partial_o(e_{i(i+1)})$*, we let* $\mathrm{vol}_{e_{i(i+1)},v}$ *to be the restriction of the volume form of the normal bundle* $NV^+_{q_{i(i+1)}}$ *onto* $\gamma_{e_{i(i+1)}}(v)$*;*
(2) *for a vertex* $v \in T^{[0]}$ *with incoming edges* $e_{i_{l-1}i_l}, \ldots, e_{i_0 i_1}$ *and outgoing edge* $e_{i_0 i_l}$ *arranged in clockwise orientation with* $\mathrm{vol}_{e_{i_{d-1}i_d},v}$ *defined, we let* $\mathrm{vol}_{e_{i_0 i_2},v} := (-1)^{|\mathrm{vol}_{e_{i_2 i_1},v}|+1} \mathrm{vol}_{e_{i_2 i_1},v} \wedge \mathrm{vol}_{e_{i_0 i_1},v}$

when v is labeled with 1 [5], *and we let* $\mathrm{vol}_{e_{i_0i_l},v} := \sigma^*_{t_{v,l}}(\iota_{\sigma_*(\frac{\partial}{\partial t_l})}\mathrm{vol}_{e_{i_{l-1}i_l},v})\wedge\cdots\wedge\sigma^*_{t_{v,1}}(\iota_{\sigma_*(\frac{\partial}{\partial t_1})}\mathrm{vol}_{e_{i_0i_1},v})$ *when v is labeled with u;*

(3) *for an edge e_{ij} with incoming vertex $v_0 = \partial_{in}(e_{ij})$ and outgoing vertex $v_1 = \partial_o(e_{ij})$, we let* $\mathrm{vol}_{e_{ij},v_1} = (\tau_{l(e_{ij})})_*(\mathrm{vol}_{e_{ij},v_0})$ *where $\tau_{l(e_{ij})}$ is the gradient flow of ∇f_{ij} for time $l(e_{ij})$.*

Therefore, for the outgoing edge e_{0k} starting at the root vertex v_r and ending at the outgoing vertex v_o, we obtain a differential form $\mathrm{vol}_{e_{0k},v_r}$ from the above construction, and we determine the sign $(-1)^{\chi(\Gamma)}$ by $(-1)^{\chi(\Gamma)}\mathrm{vol}_{e_{0k},v_r}\wedge *\mathrm{vol}_{q_{0k}} = \mathrm{vol}_M$ where $\mathrm{vol}_{q_{0k}}$ is the chosen volume element in $NV^+_{q_{0k}}$ for the critical point q_{0k}. (For the case $T^{[0]} = \emptyset$, we define by convention that $(-1)^{\chi(\Gamma)}\Gamma'\wedge\mathrm{vol}_p\wedge *\mathrm{vol}_q = \mathrm{vol}_M$ for a gradient flow line Γ from p to q.)

Definition 2.10. *Given a generic sequence of functions $\vec{f} = (f_0,\dots,f_k)$, with a sequence of critical points $(q_{(k-1)k},\dots,q_{01})$ we define the operation $m_k^{eMorse}(q_{(k-1)k},\dots,q_{01}) \in CM^*_{f_{0k}}[[u]]$ by extending linearly the formula*

$$m_{k,T}^{eMorse}(q_{(k-1)k},\dots,q_{01}) := \begin{cases} \sum_{q_{0k}\in Crit(f_{0k})}\Big(\sum_{\Gamma\in\mathcal{M}_T(\vec{f},\vec{q})}(-1)^{\chi(\Gamma)}\Big)q_{0k} & \text{if } \dim(\mathcal{M}_T(\vec{f},\vec{q})) = 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $\vec{q} = ((q_{(k-1)k},\dots,q_{01},q_{0k})$. We further let $m_k^{eMorse} = \sum_{T\in\mathbb{LT}_k} u^{n_T} m_{k,T}^{eMorse}$ where $n_T = |LT^{[0]}|$.

We have the following Theorem 2.11 which is the main result for this paper.

Theorem 2.11. *Given a generic sequence of functions $\vec{f} = (f_0,\dots,f_k)$, with a sequence of critical points $\vec{q} = (q_{(k-1)k},\dots,q_{01},q_{0k})$, then we have*

$$\lim_{\lambda\to\infty}\Phi\big(m_{k,T}(\lambda)(\phi(q_{(k-1)k}),\dots,\phi(q_{01}))\big) = m_{k,T}^{eMorse}(q_{(k-1)k},\dots,q_{01}),$$

where $\phi := \Phi^{-1}$ [6] *is the inverse of the isomorphism in equation* (2.1).

As a consequence, the Morse product m_k^{eMorse}'s satisfy the A_∞-relation whenever we consider a generic sequence of functions such that every operation appearing in the formula is well-defined.

3. Proof of Theorem 2.11

3.1. **Analytic results.** For the proof of Theorem 2.11, we assume $T^{[0]} \neq \emptyset$ since this is exactly the case carried out by [12]. We begin with recalling the necessary analytic results from [12, 18, 3].

3.1.1. *Results for a single Morse function.* We will assume that the function f_{ij} we are dealing with satisfy the Morse-Smale assumption 2.5. Due to difference in convention, $e^{-\lambda f_{ij}}\Delta_{ij}e^{\lambda f_{ij}}$ is called the Witten's Laplacian in [3], and result stated in this Section is obtain by the corresponding statements in [3] by conjugating $e^{\lambda f_{ij}}$.

Theorem 3.1 ([12, 18])**.** *For each f_{ij}, there is $\lambda_0 > 0$ and constants $c, C > 0$ such that we have* $\mathrm{Spec}(\Delta_{ij})\cap[ce^{-c\lambda}, C\lambda^{1/2}) = \emptyset$, *for $\lambda > \lambda_0$. The map $\Phi = \Phi_{ij} : \Omega^*_{ij,<1} \to CM^*_{f_{ij}}$ in equation* (2.1) *is a chain isomorphism for λ large enough. We will denote the inverse by $\phi = \phi_{ij}$.*

We will the asymptotic behaviour of $\phi(q)$ for a critical point q of f_{ij}, and we will need the following Agmon distance d_{ij} for this purpose.

[5] Hence we have valency of v being 3.

[6] We omit the numbering ij from our notation here.

Definition 3.2. *For a Morse function f_{ij}, the Agmon distance d_{ij}* [7]*, or simply denoted by d, is the distance function with respect to the degenerated Riemannian metric $\langle\cdot,\cdot\rangle_{f_{ij}} = |df_{ij}|^2\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the background metric. We will also write $\rho_{ij}(x,y) := d_{ij}(x,y) - f_{ij}(y) + f_{ij}(x)$.*

Lemma 3.3. *We have $\rho_{ij}(x,y) \geq 0$ with equality holds if and only if x is connected to y via a generalized flow line $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Here a generalized flow line means that γ is continuous, and there is a partition $0 = t_0 < t_1 < \cdots < t_l = 1$ such that $\gamma|_{(t_r,t_{r+1})}$ is a reparameterization of a gradient flow line of f_{ij} and $\gamma(t_r) \in Crit(f_{ij})$ for $0 < r < l$.*

Lemma 3.4. *Let $\gamma \subset \mathbb{C}$ to be a subset whose distance from $Spec(\Delta_{ij})$ is bounded below by a constant. For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon} > 0$ such that $\|\nabla^j((z-\Delta_{ij})^{-1}u)\|_{C^0(V)} \leq C_{j,\epsilon}e^{-\lambda(\rho_{ij}(x_0,y_0)-\epsilon)}\|u\|_{W^{k_j,2}(U)}$, for all $\lambda > \lambda_0$ and $u \in C^0_c(U)$, where $W^{k,p}$ refers to the Sobolev norm.*

We will also need modified version of the resolvent estimate for G_{ij}, which can be obtained by applying the original resolvent estimate to the the formula

$$G_{ij}(u) = \oint_\gamma z^{-1}(z-\Delta_{ij})^{-1}u. \tag{3.1}$$

Lemma 3.5. *For any $j \in \mathbb{Z}_+$ and $\epsilon > 0$, there is $k_j \in \mathbb{Z}_+$ and $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for any two points $x_0, y_0 \in M$, there exist neighborhoods V and U (depending on ϵ) of x_0 and y_0 respectively, and $C_{j,\epsilon} > 0$ such that $\|\nabla^j(G_{ij}u)\|_{C^0(V)} \leq C_{j,\epsilon}e^{-\lambda(\rho_{ij}(x_0,y_0)-\epsilon)}\|u\|_{W^{k_j,2}(U)}$, for all $\lambda < \lambda_0$ and $u \in C^0_c(U)$, where $W^{k,p}$ refers to the Sobolev norm.*

For a critical point q of f_{ij}, $\phi(q)$, has certain exponential decay measured by the Agmon distance from the critical point q.

Lemma 3.6. *For any ϵ, there exists $\lambda_0 = \lambda_0(\epsilon) > 0$ such that for $\lambda > \lambda_0$, we have $\phi(q) = \mathcal{O}_\epsilon(e^{-\lambda(g^+_q(x)-\epsilon)})$, and same estimate holds for the derivatives of $\phi_{ij}(q)$ as well. Here $\mathcal{O}_\epsilon$ refers to the dependence of the constant on ϵ and $g^+_q(x) = \rho_{ij}(q,x) = d_{ij}(q,x) + f_{ij}(q) - f_{ij}(x)$.*

Remark 3.7. *We notice that g^+_q is a nonnegative function with zero set V^+_q that is smooth and Bott-Morse in a neighborhood W of $V^+_q \cup V^-_q$. Similarly, if we write $g^-_q = d_{ij}(q,x) + f_{ij}(x) - f_{ij}(q)$ which is a nonnegative function with zero set V^-_q and is smooth and Bott-Morse in W, and we have $*_{ij}\phi(q)/\|\phi(q)e^{-\lambda f_{ij}}\|^2 = \mathcal{O}_\epsilon(e^{-\lambda(g^-_q-\epsilon)})$ where $*_{ij} = *e^{-2\lambda f_{ij}}$ comparing to the usual star operator $*$.*

Lemma 3.8. *The normalized basis $\phi(q)/\|\phi(q)\|$'s are almost orthonormal basis with respect to the twisted inner product $\langle\cdot,\cdot\rangle e^{-2\lambda f_{ij}}$. More precisely, there is a $C, c > 0$ and λ_0 such that when $\lambda > \lambda_0$, we will have $\int_M \langle \frac{\phi(p)}{\|\phi(p)\|}, \frac{\phi(q)}{\|\phi(q)\|}\rangle \mathrm{vol}_{ij} = \delta_{pq} + Ce^{-c\lambda}$.*

Restricting our attention to a small enough neighborhood W containing $V^+_q \cup V^-_q$, the above decay estimate of $\phi(q)$ from [12] can be improved from an error of order $\mathcal{O}_\epsilon(e^{\epsilon\lambda})$ to $\mathcal{O}(\lambda^{-\infty})$.

Lemma 3.9. *There is a WKB approximation of the $\phi(q)$ as $\phi(q) \sim \lambda^{\frac{\deg(q)}{2}}e^{-\lambda g^+_q}(\omega_{q,0} + \omega_{q,1}\lambda^{-1/2} + \dots)$*[8]*, which is an approximation in any precompact open subset $K \subset W_q$ of the form*

$$\|e^{\lambda g^+_q}\nabla^j\big(\lambda^{-\deg(q)/2}\phi(q) - e^{-\lambda g^+_q}\sum_{l=0}^{N}\omega_{q,j}\lambda^{-l/2}\big)\|^2_{L^\infty(K)} \leq C_{j,K,N}\lambda^{-N-1+2j}$$

[7] Readers may see [8] for its basic properties.

[8] Notice that we indeed have $\omega_{q,2j+1} = 0$ in this case while we prefer to write it in this form to unify our notations.

for any $j, N \in \mathbb{Z}_+$, where $W_q \supset V_q^+ \cup V_q^-$ is an open neighborhood of $V_q^+ \cup V_q^-$.

Furthermore, the integral of the leading order term $\omega_{q,0}$ in the normal direction to the stable submanifold V_q^+ is computed in [12].

Lemma 3.10. *Fixing any point $x \in V_q^+$ and $\chi \equiv 1$ around x compactly supported in W, we take any closed submanifold (possibly with boundary) $NV_{q,x}^+$ of W intersecting transversally with V_q^+ at x. We have*

$$\lambda^{\frac{\deg(q)}{2}} \int_{NV_{q,x}^+} e^{-\lambda g_q^+} \chi \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1}); \quad \frac{\lambda^{\frac{\deg(q)}{2}}}{\|e^{-\lambda f_{ij}} \phi_{ij}(q)\|^2} \int_{NV_{q,x}^-} e^{-\lambda g_q^-} \chi * \omega_{q,0} = 1 + \mathcal{O}(\lambda^{-1}),$$

for any point $x \in V_q^-$, with $NV_{q,x}^-$ intersecting transversally with V_q^-.

3.1.2. *WKB for homotopy operator.* We recall the key estimate for the homotopy operator H_{ij} proven in [3, Section 4]. Let $\gamma(t)$ be a flow line of $\nabla f_{ij}/|\nabla f_{ij}|_{d_{ij}}$ starts at $\gamma(0) = x_S$ and $\gamma(T) = x_E$ for a fixed $T > 0$ as shown in the following figure 2. We consider an input form ζ_S defined in a

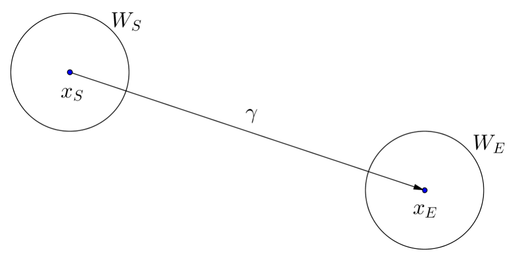

FIGURE 2. gradient flow line γ

neighborhood W_S of x_S. Suppose we are given a WKB approximation of ζ_S in W_S, which is an approximation of ζ_S according to order of λ of the form

$$\zeta_S \sim e^{-\lambda g_S}(\omega_{S,0} + \omega_{S,1}\lambda^{-1/2} + \omega_{S,2}\lambda^{-1} + \dots) \tag{3.2}$$

which means we have $\lambda_{j,0} > 0$ such that when $\lambda > \lambda_{j,N,0}$ we have

$$\|e^{\lambda g_S} \nabla^j (\zeta_S - e^{-\lambda g_S}(\sum_{i=0}^{N} \omega_{S,i}\lambda^{-i/2}))\|^2_{L^\infty(W_S)} \leq C_{j,N}\lambda^{-N-1+2j},$$

for any $j, N \in \mathbb{Z}_+$. We further assume that g_S is a nonnegative Bott-Morse function in W_S with zero set V_S such that γ is not tangent to V_S at x_S. We consider the equation

$$\Delta_{ij}\zeta_E = (I - P_{ij})d_{ij}^*(\chi_S\zeta_S), \tag{3.3}$$

where χ_S is a cutoff function compactly supported in W_S, $P_{ij} : \Omega^*(M) \to \Omega^*_{ij,<1}$ is the projection. We want to have a WKB approximation of $\zeta_E = H_{ij}(\chi_S\zeta_S)$

Lemma 3.11. *For* $\operatorname{supp}(\chi_S)$ *small enough (the size only depends on g_S and f_{ij}), there is a WKB approximation of ζ_E in a small enough neighborhood W_E of x_E, of the form $\zeta_E \sim e^{-\lambda g_E}\lambda^{-1/2}(\omega_{E,0} + \omega_{E,1}\lambda^{-1/2} + \dots)$ in the sense that we have $\lambda_{j,0} > 0$ such that when $\lambda > \lambda_{j,N,0}$ we have*

$$\|e^{\lambda g_E} \nabla^j \{\zeta_E - e^{-\lambda g_E}(\sum_{i=0}^{N} \omega_{E,i}\lambda^{-(i+1)/2})\}\|^2_{L^\infty(W_E)} \leq C_{j,N}\lambda^{-N+2j}.$$

Furthermore, the function g_E (only depending on g_S and f_{ij}) is a nonnegative function which is Bott-Morse in W_E with zero set $V_E = (\bigcup_{-\infty<t<+\infty} \varsigma_t(V_S)) \cap W_E$ which is a closed submanifold in W_E, where ς_t is the t-time $\nabla f_{ij}/|\nabla f_{ij}|^2$.

Finally, we have the following Lemma 3.12 from [3] relating the integrals of $\omega_{S,0}$ and $\omega_{E,0}$.

Lemma 3.12. *Using same notations in lemma 3.11 and suppose χ_S and χ_E are cutoff functions supported in W_S and W_E respectively, then we have*

$$\lambda^{-\frac{1}{2}} \int_{N_{x_E}} e^{-\lambda g_E} \chi_E \omega_{E,0} = (\int_{N_{x_S}} e^{-\lambda g_S} \chi_S \omega_{S,0})(1 + \mathcal{O}(\lambda^{-1})). \tag{3.4}$$

Furthermore, suppose $\omega_{S,0}(x_S) \in \bigwedge^{top} N(V_S)^*_{x_S}$, we have $\omega_{E,0}(x_E) \in \bigwedge^{top} N(V_E)^*_{x_E}$. Here $\bigwedge^{top} E$ refers to $\bigwedge^r E$ for a rank r vector bundle E. Here N_{v_S} and N_{v_E} are any closed submanifold of W_S and W_E intersecting V_S and V_E transversally at x_S and x_E respectively.

3.2. **Apriori Estimate.**

Notations 3.13. *From now on, we will consider a fixed generic sequence $\vec{f} = (f_0, \dots, f_k)$ with corresponding sequence of critical points $\vec{q} = (q_{(k-1)k}, \dots, q_{01}, q_{0k})$ and a fixed labeled ribbon k-tree T such that $\dim(\mathcal{M}_T(\vec{f}, \vec{q})) = 0$ (the dimension is given by formula (2.2)). We use q_{ij} to denote a fixed critical point of f_{ij}. $\phi(q_{ij})$ associated to q_{ij} is abbreviated by ϕ_{ij}.*

Notations 3.14. *For $T \in \mathbb{T}_k$ or $\mathbb{LT}_k$ with $\vec{q}$, we let $\blacktriangle_T := \prod_{v \in LT^{[0]}} \blacktriangle_{\nu(v)}$ of dimension $\nu(T) := \sum_{v \in LT^{[0]}} \nu(v)$, and we also let $\deg(T) := \sum_{i=0}^{k-1} \deg(q_{i(i+1)}) - |T^{[1]}| - \nu(T)$. We inductively define a volume form ν_T on $\blacktriangle_T$ for labeled ribbon tree $T \in \mathbb{LT}_k$ by: letting $\nu_l = dt_l \wedge \dots \wedge dt_1$ on the $\blacktriangle_l$; and for v_r labeled with 1 we split T at v_r into T_2 and T_1 such that T_2, T_1, e_o is clockwisely oriented, then we take $\nu_T = \nu_{T_2} \wedge \nu_{T_1}$; and for v_r labeled with u we split T at v_r into $T_l, \dots, T_1$ clockwisely, and we take $\nu_T = \nu_{T_l} \wedge \dots \wedge \nu_{T_1} \wedge \nu_l$. We should also write $\nu_T^\vee$ to be the polyvector field dual to ν_T.*

Definition 3.15. *Given a labeled ribbon k-tree T with $\vec{f}$ and $\vec{q}$ as above, we associate to it a length function $\hat{\rho}_T$ on $\mathfrak{M}(T) := \blacktriangle_T \times M^{|T^{[0]}_{ni}|} \to \mathbb{R}_+$ [9] with coordinates $(\vec{\mathbf{t}}_T, \hat{x}_T)$ (where $\vec{\mathbf{t}}_T = (\mathbf{t}_v)_{v \in LT^{[0]}}$ and $\hat{x}_T = (x_v)_{v \in T^{[0]}_{ni}}$) inductively along the tree by the rules:*

1. *for the unique tree with one edge e numbered by ij, we take $\hat{\rho}_T(x_{v_o}) := \rho_{ij}(q_{ij}, x_{v_o})$;*
2. *when v_r is labeled with 1, we split T at the root vertex v_r into T_2, T_1. We notice that $\mathfrak{M}(T) = \mathfrak{M}(T_2) \times_M \mathfrak{M}(T_1) \times M_{v_o}$ (with coordinates $\vec{\mathbf{t}}_T = (\vec{\mathbf{t}}_{T_2}, \vec{\mathbf{t}}_{T_1})$, and $\hat{x}_T = (\hat{x}_{T_2}, \hat{x}_{T_1}, x_{v_o})$ such that $x_{T_2,v_r} = x_{T_1,v_r} = x_{v_r}$ in M) and we let*
$$\hat{\rho}_T(\vec{\mathbf{t}}_T, \hat{x}_{F(T)}) = \hat{\rho}_{ij}(x_{v_r}, x_{v_o}) + \sum_{j=1}^{2} \hat{\rho}_{T_j}(\vec{\mathbf{t}}_{T_j}, \hat{x}_{T_j})$$
if the numbering on e_o is ij;
3. *when v_r is labeled with u, we split T at v_r into $T_l, \dots, T_1$ and we can write $\mathfrak{M}(T) = \mathfrak{M}_{T_l} \times_M \dots \times_M \mathfrak{M}(T_1) \times_M (\blacktriangle_l \times M_{v_r}) \times M_{v_o}$ where $l = \nu(v_r)$. By writing coordinates $(\vec{\mathbf{t}}_{T_j}, \hat{x}_{T_j})$ for $\mathfrak{M}(T_j)$, $\mathbf{t}_{v_r} = (t_{v_r,l}, \dots, t_{v_r,1})$ for $\blacktriangle_l$, x_{v_r} for M_{v_r} and x_{v_o} for M_{v_o} satisfying $x_{T_l,v_r} = \sigma_{t_{v_r,l}}(x_{v_r}), \cdots, x_{T_1,v_r} = \sigma_{t_{v_r,1}}(x_{v_r})$, we let*
$$\hat{\rho}_T(\vec{\mathbf{t}}_T, \hat{x}_T) := \hat{\rho}_{ij}(x_{v_r}, x_{v_o}) + \sum_{j=1}^{l} \hat{\rho}_{T_j}(\vec{\mathbf{t}}_{T_j}, \hat{x}_{T_j})$$
if the numbering on e_o is ij.

[9] Here $T^{[0]}_{ni}$ is the set of all vertices besides incoming edges introduced in Definition 2.2

Fixing the outgoing point $x_{v_o} = q_{0k}$ giving coordinates $\vec{x}_T = (x_v)_{v \in T^{[0]}}$ for $M^{|T^{[0]}|}$, we let $\rho_T(\vec{\mathbf{t}}_T, \vec{x}_T) := \hat{\rho}_T(\vec{\mathbf{t}}_T, \vec{x}_T, q_{0k})$.

Example 3.16. *Suppose that T is the labeled ribbon 2-tree with two incoming vertices v_2 and v_1 joining to v labeled with u by e_{12} and e_{01}, and v is joining to the root vertex v_r labeled with u via e. Then we have $\blacktriangle_T \times M^{|T^{[0]}_{ni}|} = \blacktriangle_2 \times \mathbb{S}^1 \times M^3$ and $\hat{\rho}_T(t_{v,2}, t_{v,1}, t_{v_r}, x_v, x_{v_r}, x_{v_o}) = \rho_{02}(x_{v_r}, x_{v_o}) + \rho_{02}(x_v, \sigma_{t_{v_r}}(x_{v_r})) + \rho_{12}(q_{12}, \sigma_{t_{v,2}}(x_v)) + \rho_{01}(q_{01}, \sigma_{t_{v,1}}(x_v))$. The following Figure 3 shows the tree T and its associated $\hat{\rho}_T$.*

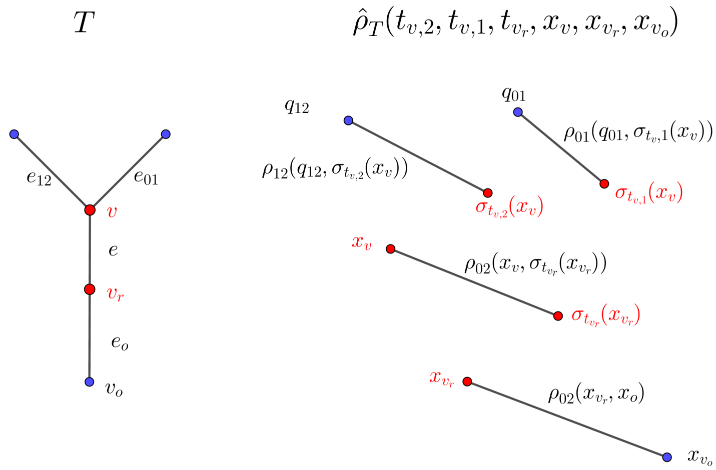

FIGURE 3. Distance function associated to T

From its construction and Lemma 3.3, we notice that $\rho_T(\vec{\mathbf{t}}_T, \vec{x}_T) \geq 0$ and equality holds if and only if for each edge e numbered by ij with $\partial_{in}(e) = v_1$ and $\partial_o(e) = v_2$, there is a generalized flow line of ∇f_{ij} joining x_{v_1} to $\tilde{x}_{v_2}$, where $\tilde{x}_{v_2} = x_{v_2}$ when v_2 is labeled by 1; and $\tilde{x}_{v_2} = \sigma_{t_{v_2,j}}(x_{v_2})$ if v_2 is labeled by u with and e is the j^{th} incoming edges of v_2 in the anti-clockwise orientation. Therefore, we have a generalized flow tree (with jumping) of type $(T, \vec{f}, \vec{q})$ (which is a generalization of flow tree in Definition 2.7 by allow broken flow lines as in Definition 3.3). With the condition that $\dim(\mathcal{M}(\vec{f}, \vec{q})) = 0$ as mentioned in Notation 3.13, we notice that every such generalized flow line is an actual flow line from the generic assumption 2.8 for $\vec{f}$, because the expected dimension for flow tree with broken flow line is negative.

Notations 3.17. *We let $\Gamma_1, \ldots, \Gamma_d$ be the gradient flow tree of type $(T, \vec{f}, \vec{q})$, such that each Γ_i is associated with a point $\mathbf{t}_{\Gamma_i,v} \in \blacktriangle_{\nu(v)}$ (for $v \in LT^{[0]}$) and $x_{\Gamma_i,v} \in M$ (for $v \in T^{[0]}$) such that*

(1) $x_{\Gamma_i,v}$ is the starting point of a gradient flow line γ_e associated to edge e if $\partial_{in}(e) = v$, and we write $x_{\Gamma_i,e,v} = x_{\Gamma_i,v}$ in this case;
(2) $x_{\Gamma_i,v}$ is the end point of the gradient flow line γ_e if v is labeled by 1 if $\partial_o(e) = v$, and we write $x_{\Gamma_i,e,v} = x_{\Gamma_i,v}$ in this case;
(3) and $\sigma_{t_{\Gamma_i,v,j}}(x_{\Gamma_i,v})$ is the end point of a gradient flow line γ_e associated to j^{th}-edge e clockwisely if v is labeled by u and $\partial_o(e) = v$, and we write $x_{\Gamma_i,e,v} = \sigma_{t_{\Gamma_i,v,j}}(x_{\Gamma_i,v})$ in this case.

We consider a sequence of cut off functions $\vec{\chi} := (\chi_v)_{v \in T^{[0]}}$ such that χ_v compactly supported in a ball $U_v := B(x_v, r/2)$ of radius r centered at a fixed point $x_v \in M$, and $(\vec{\varkappa}_v)_{v \in LT^{[0]}}$ with $\varkappa_v$

compactly support in a small neighborhood $\mathbf{C}_v$ containing a fixed $\mathbf{t}_v = (t_{v,\nu(v)}, \dots, t_{v,1}) \in \blacktriangle_{\nu(v)}$ such that the Riemannian distance between $\sigma_{t_j}(x)$ and $\sigma_{t'_j}(x)$ is strictly less than $r/2$ for any j and any $x \in M$ and any $\mathbf{t}$ and $\mathbf{t}'$ in $\mathbf{C}_v$.

Definition 3.18. *With $\vec{\chi}$ and $\vec{\varkappa}$ as above, we define $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}} \in \Omega^*(\blacktriangle_{T_{e,v}} \times M)$ [10] for each flag $(e,v) \in F(T)$ inductively along T by letting:*

1. *(1) for the incoming edge e_{ij} with $\partial_o(e_{ij}) = v$, we take $\mathfrak{m}^{(e_{ij},v)}_{\vec{\chi},\vec{\varkappa}} = \phi_{ij}$;*
2. *(2) when we have (e,v) with $\partial_{in}(e) = v$ with v is labeled with 1 with, we let T_2, T_1 to be subtrees with outgoing edges e_2, e_1 ending at v such that e_2, e_1, e clockwisely oriented. With coordinates $\vec{\mathbf{t}}_{T_{e,v}} = (\vec{\mathbf{t}}_{T_2}, \vec{\mathbf{t}}_{T_1})$ for $\blacktriangle_T = \blacktriangle_{T_2} \times \blacktriangle_{T_1}$, we let*
$$\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_{e,v}}, x) = (-1)^{\varepsilon} \nu_{T_{e,v}} \chi_{v_r}(x) \big(\iota_{\nu^\vee_{T_2}} \mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_2}, x)\big) \wedge \big(\iota_{\nu^\vee_{T_1}} \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_1}, x)\big),$$
where $\varepsilon = \deg\big(\iota_{\nu^\vee_{T_2}} \mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_2}, x)\big) + 1$;
3. *(3) when we have v labeled with u, we let $T_l, \dots, T_1$ be subtrees with outgoing edges $e_l, \dots, e_1$ ending at v with $e_l, \dots, e_1, e$ clockwisely oriented. We let*
$$\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_{e,v}}, x) = \nu_{T_{e,v}} \chi_v(x) \varkappa_v(\mathbf{t}_v) \sigma^*_{t_{v,l}}\big(\iota_{w_{v,l} \wedge \nu^\vee_{T_l}} \mathfrak{m}^{(e_l,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_l}, x)\big) \wedge \dots \wedge \sigma^*_{t_{v,1}}\big(\iota_{w_{v,1} \wedge \nu^\vee_{T_1}} \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}(\vec{\mathbf{t}}_{T_1}, x)\big),$$
where $t_{v,l}, \dots, t_{v,1}$ is the coordinates for $\blacktriangle_{\nu(v)}$ and $\vec{\mathbf{t}}_{T_{e,v}} = (\vec{\mathbf{t}}_{T_l}, \dots, \vec{\mathbf{t}}_{T_1}, t_{v,l}, \dots, t_{v,1})$, and $w_{v,j} = \sigma_*(\frac{\partial}{\partial t_{v,j}})$;
4. *(4) for an edge e numbered by ij with $\partial_{in}(e) = v_0$ and $\partial_o(e) = v_1$ with v_1 not being the outgoing vertex v_o, we let $\mathfrak{m}^{(e,v_1)}_{\vec{\chi},\vec{\varkappa}} = d^*_{ij} G_{ij}(\mathfrak{m}^{(e,v_0)}_{\vec{\chi},\vec{\varkappa}})$ where G_{ij} is introduced in Definition 2.4;*
5. *(5) for the outgoing edge e_o with $\partial_{in}(e_o) = v_r$ and $\partial_o(e_o) = v_o$, we take $\mathfrak{m}^T_{\vec{\chi},\vec{\varkappa}} = \mathfrak{m}^{(e_o,v_o)}_{\vec{\chi},\vec{\varkappa}} = \mathfrak{m}^{(e_o,v_r)}_{\vec{\chi},\vec{\varkappa}}$.*

Example 3.19. *We the tree T described in the previous Example 3.16, we have $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}(t_{v,2}, t_{v,1}, x_v) = \chi_v(x_v) \varkappa_v(t_{v,2}, t_{v,1}) dt_{v,2} dt_{v,1} \sigma^*_{t_{v_2}}(\iota_{w_{v,2}} \phi_{02})(x_v) \wedge \sigma^*_{t_{v_1}}(\iota_{w_{v,1}} \phi_{01})(x_v)$, $\mathfrak{m}^{(e,v_r)}_{\vec{\chi},\vec{\varkappa}} = d^*_{02} G_{02}(\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}})$ ($d^*_{02} G_{02}$ only acting on the component M) and*
$$\mathfrak{m}^{(e_o,v_r)}_{\vec{\chi},\vec{\varkappa}}(t_{v,2}, t_{v,1}, t_{v_r}, x_{v_r}) = \chi_{v_r}(x_{v_r}) \varkappa(t_{v_r}) dt_{v,2} dt_{v,1} dt_{v_r} \sigma^*_{t_{v_r}}(\iota_{w_{v_r} \wedge \frac{\partial}{\partial t_{v,1}} \wedge \frac{\partial}{\partial t_{v,2}}} \mathfrak{m}^{(e,v_r)}_{\vec{\chi},\vec{\varkappa}})(x_{v_r}),$$
and finally we have $\mathfrak{m}^T_{\vec{\chi},\vec{\varkappa}} = \mathfrak{m}^{(e_o,v_r)}_{\vec{\chi},\vec{\varkappa}}$.

We take a collection $\{\vec{\chi}_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{I}}$ and $\{\vec{\varkappa}_{\mathbf{j}}\}_{\mathbf{j} \in \mathcal{J}}$ such that $\vec{\chi}_{\mathbf{i}} = (\chi_{i,v})_{\substack{i \in \mathcal{I}_v \\ v \in T^{[0]}}}$ and $\vec{\varkappa}_{\mathbf{j}} = (\varkappa_{j,v})_{\substack{j \in \mathcal{J}_v \\ v \in LT^{[0]}}}$ and such that every collection $\{\chi_{i,v}\}_{i \in \mathcal{I}_v}$ and $\{\varkappa_{j,v}\}_{j \in \mathcal{J}_v}$ is a partition of unity for M_v and $\blacktriangle_{\nu(v)}$ respectively (Here we use the notation $\mathcal{I} = \prod_{v \in T^{[0]}} \mathcal{I}_v$ and $\mathcal{J} = \prod_{v \in T^{[0]}} \mathcal{J}_v$). With the cut off construction in Definition 3.18 and the Definition 2.4, we have

$$\int_M m_{k,T}(\lambda)(\phi_{(k-1)k}, \dots, \phi_{01}) \wedge \frac{*e^{-2\lambda f_{0k}} \phi_{0k}}{\|e^{-\lambda f_{0k}} \phi_{0k}\|^2} = \sum_{\mathbf{i} \in \mathcal{I}} \sum_{\mathbf{j} \in \mathcal{J}} \int_{\blacktriangle_T \times M} \mathfrak{m}^T_{\vec{\chi}_{\mathbf{i}}, \vec{\varkappa}_{\mathbf{j}}} \wedge \frac{*e^{-2\lambda f_{0k}} \phi_{0k}}{\|e^{-\lambda f_{0k}} \phi_{0k}\|^2}. \tag{3.5}$$

Lemma 3.20. *We fix a point $(\vec{\mathbf{t}}_T, \vec{x}_T)$ in $\mathfrak{M}(T)$ with the cut off functions $\vec{\chi}$ and $\vec{\varkappa}$ and $\mathfrak{m}^T_{\vec{\chi},\vec{\varkappa}}$ as before Definition 3.18, for any $\epsilon > 0$ we have $\lambda_0(\epsilon)$ and small enough radius $r = r(\epsilon)$ of cut off functions (which is described before Definition 3.18) such that when $\lambda > \lambda_0$ we have the norm estimate*
$$\|\mathfrak{m}^T_{\vec{\chi},\vec{\varkappa}} \wedge \frac{*e^{-2\lambda f_{0k}} \phi_{0k}}{\|e^{-\lambda f_{0k}} \phi_{0k}\|^2}\|_{C^j(\blacktriangle_T \times M)} \leq C_{j,\epsilon} e^{-\lambda(\rho_T(\vec{\mathbf{t}}_T, \vec{x}_T) - b_T \epsilon)},$$

[10] recall that $T_{e,v}$ is introduced in Notation 2.3

for any $j \in \mathbb{Z}_+$ (Here we fix an arbitrary metric on the simplices $\blacktriangle_l$'s), where b_T is a constant depending the combinatorics of T.

Proof. We prove by induction along the tree T that for each flag (e,v) with $\partial_o(e) = v \neq v_o$ we have

$$\|\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}\|_{C^j(\blacktriangle_{T_{e,v}}\times U_v)} \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda(\hat{\rho}_{T_{e,v}}(\vec{\mathbf{t}}_{T_{e,v}}, \hat{x}_{T_{e,v}}) - b_{T_{e,v}}\epsilon)\right),$$

where $U_v = B(x_v, r/2)$, for any points $\vec{\mathbf{t}}_T \in \blacktriangle_T$, $\hat{x}_T \in M^{|T^{[0]}_{ni}|}$ with the assoicated cut off functions $\vec{\varkappa}$ and $\vec{\chi}$ with small enough r. The initial case follows from the estimate in Lemma 3.6. For induction we consider an edge e with $\partial_{in}(e) = v$ and $\partial_o(e) = \tilde{v}$. We take subtrees (of T) $T_l, \dots, T_1$ with edges $e_l, \dots, e_1$ attached to v such that $e_l, \dots, e_1, e$ is clockwisely oriented. There are two cases.

The first case is when v is labeled with 1 and we have $l = 2$. In this case we have the estimate

$$\|\mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}} \wedge \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}\|_{C^j(\blacktriangle_{T_{e_2,v}}\times\blacktriangle_{T_{e_1,v}}\times U_v)} \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda\big(\hat{\rho}_{T_2}(\vec{\mathbf{t}}_{T_2}, \hat{x}_{T_2}) + \hat{\rho}_{T_1}(\vec{\mathbf{t}}_{T_1}, \hat{x}_{T_1}) - b_{T_{e,v}}\epsilon\big)\right)$$

by choosing $b_{T_{e,v}} \geq b_{T_1} + b_{T_2}$, where we require $x_{T_1,v} = x_{T_2,v} = x_v$ in the R.H.S. of the above equation. Assuming that e is numbered by ij, and we apply the Lemma 3.5 to the term $\mathfrak{m}^{(e,\tilde{v})}_{\vec{\chi},\vec{\varkappa}} = d^*_{ij}G_{ij}\big(\chi_v \mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}} \wedge \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}\big)$ (we choose smaller r if necessary) we obtain the estimate

$$\|d^*_{ij}G_{ij}\big(\chi_v \mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}} \wedge \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}\big)\|_{C^j(\blacktriangle_{T_{e,\tilde{v}}}\times U_{\tilde{v}})} \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda\big(\hat{\rho}_{T_{e,\tilde{v}}}(\vec{\mathbf{t}}_{T_{e,\tilde{v}}}, \hat{x}_{T_{e,\tilde{v}}}) - b_{T_{e,\tilde{v}}}\epsilon\big)\right),$$

by taking $b_{T_{e,\tilde{v}}} \geq b_{T_{e,v}} + 1$ which is the desired estimate.

The second case is when v is labeled with u, and we have the estimate

$$\begin{aligned}&\|\sigma^*_{t_l}\big(\iota_{w_{v,l}\wedge\nu^\vee_{T_l}} \mathfrak{m}^{(e_l,v)}_{\vec{\chi},\vec{\varkappa}}\big) \wedge \cdots \wedge \sigma^*_{t_1}\big(\iota_{w_{v,1}\wedge\nu^\vee_{T_1}} \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}\big)\|_{C^j(\prod_{j=1}^l \blacktriangle_{T_j}\times \mathbf{C}_v\times U_v)} \\ &\qquad \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda\Big(\sum_{j=1}^l \hat{\rho}_{T_j}(\vec{\mathbf{t}}_{T_j}, \hat{x}_{T_j}) - b_{T_{e,v}}\epsilon\Big)\right),\end{aligned}$$

using the induction hypothesis and by taking $b_{T_{e,v}} \geq l + \sum_{j=1}^l b_{T_j}$, for $(t_l, \dots, t_1)$ varying in small enough neighborhood $\mathbf{C}_v$ of $(t_{v,l}, \dots, t_{v,1})$ ($\mathbf{C}_v$ introduced in the paragraph before Definition 3.18), where we require that the identity $x_{T_j,v} = \sigma_{t_{v,j}}(x_v)$ on the R.H.S. as in the Definition 3.15. By applying $d^*_{ij}G_{ij}$ (if e is numbered by ij) to the term $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}} = \nu_{T_{e,v}} \chi_v \varkappa_v \sigma^*_{t_l}\big(\iota_{w_{v,l}\wedge\nu^\vee_{T_l}} \mathfrak{m}^{(e_l,v)}_{\vec{\chi},\vec{\varkappa}}\big) \wedge \cdots \wedge \sigma^*_{t_1}\big(\iota_{w_{v,1}\wedge\nu^\vee_{T_1}} \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}}\big)$ as in Definition 3.18, and using Lemma 3.5 again we have the desired estimate

$$\|\mathfrak{m}^{(e,\tilde{v})}_{\vec{\chi},\vec{\varkappa}}\|_{C^j(\blacktriangle_{T_{e,\tilde{v}}}\times U_{\tilde{v}})} \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda\big(\hat{\rho}_{T_{e,\tilde{v}}}(\vec{\mathbf{t}}_{T_{e,\tilde{v}}}, \hat{x}_{T_{e,\tilde{v}}}) - b_{T_{e,\tilde{v}}}\epsilon\big)\right),$$

where we take $b_{T_{e,\tilde{v}}} \geq b_{T_{e,v}} + 1$.

To obtain the statement of the Lemma, we observe that if $T_l, \cdots, T_1$ are the incoming trees joining to the root vertex we have

$$\|\mathfrak{m}^{(e_o,v_o)}_{\vec{\chi},\vec{\varkappa}}\|_{C^j(\blacktriangle_T\times U_{v_r})} \leq C_{j,\epsilon,\vec{\chi},\vec{\varkappa}} \exp\left(-\lambda\Big(\sum_{j=1}^l \hat{\rho}_{T_j}(\vec{\mathbf{t}}_{T_j}, \hat{x}_{T_j}) - b_{T_{e_o,v_o}}\epsilon\Big)\right)$$

in a small enough neighborhood U_{v_r} of x_{v_r}, where we have $l = 2$ and $x_{T_2,v_r} = x_{T_1,v_r} = x_{v_r}$ in R.H.S. as in the first case with v_r labeled with 1, and $x_{T_j,v_r} = \sigma_{t_{v_r,j}}(x_{v_r})$ in R.H.S. as in the second case that v_r is labeled with u. The Lemma follows from the estimate for $\mathfrak{m}^{(e_o,v_o)}_{\vec{\chi},\vec{\varkappa}}$ and that for $\frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2}$ in Remark 3.7. ∎

The above Lemma allows us to estimate the terms $\mathfrak{m}^T_{\vec{\chi},\vec{\varkappa}}$ appearing in the R.H.S., and from the discussion after Example 3.16 we notice that it is closely related to gradient flow tree of type T. With the gradient flow trees Γ_i's as in Notation 3.17, we assume there are open neighborhoods $D_{\Gamma_i,v}$ and $W_{\Gamma_i,v}$ of $x_{\Gamma_i,v}$ for $v \in T^{[0]}$ such that $\overline{D_{\Gamma_i,v}} \subset W_{\Gamma_i,v}$ together with $\chi_{\Gamma_i,v} \equiv 1$ on $\overline{D_{\Gamma_i,v}}$ which is compactly supported in $W_{\Gamma_i,v}$ giving $\vec{\chi}_{\Gamma_i} = (\chi_{\Gamma_i,v})_{v\in T^{[0]}}$. Similarly, we also assume there are open neighborhoods $\mathbf{C}_{\Gamma_i,v}$ and $\mathbf{E}_{\Gamma_i,v}$ of $\mathbf{t}_{\Gamma_i,v}$ in $\blacktriangle_{\nu(v)}$ satisfying $\overline{\mathbf{C}_{\Gamma_i,v}} \subset \mathbf{E}_{\Gamma_i,v}$ together with $\varkappa_{\Gamma_i,v} \equiv 1$ on $\overline{\mathbf{C}_{\Gamma_i,v}}$ which is compactly supported in $\mathbf{E}_{\Gamma_i,v}$ giving $\vec{\varkappa}_{\Gamma_i} = (\varkappa_{\Gamma_i,v})_{v\in LT^{[0]}}$. We should further prescribe the size of these neighborhood $W_{\Gamma_i,v}$'s and $\mathbf{E}_{\Gamma_i,v}$ in the upcoming Section 3.3 which is defined along the gradient tree Γ_i's together with the WKB approximation [11]. By writing $\overline{\vec{D}_{\Gamma_i}} = \prod_{v\in T^{[0]}} \overline{D_{\Gamma_i,v}}$ and $\overline{\vec{\mathbf{C}}_{\Gamma_i}} = \prod_{v\in LT^{[0]}} \overline{\mathbf{C}_{\Gamma_i,v}}$, we have $\rho_T \geq c > 0$ for some constant c outside $\bigcup_{i=1}^d \overline{\vec{\mathbf{C}}_{\Gamma_i}} \times \overline{\vec{D}_{\Gamma_i}}$ by continuity of ρ_T and the discussion after Example 3.16. As a result, we can fix a small enough ϵ (and the associated $r(\epsilon)$) such that $b_T\epsilon < c/2$. The following Figure 4 show the situation for these open subsets $W_{\Gamma_i,v}$'s and $\mathbf{E}_{\Gamma_i,v}$'s for the tree in Example 3.16.

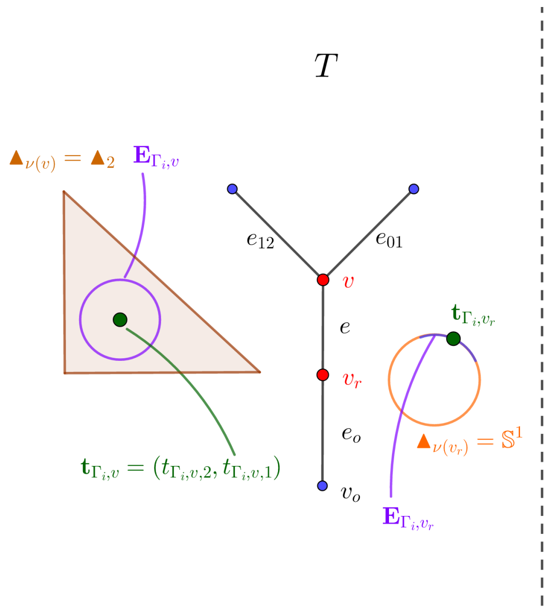

FIGURE 4. Open subsets near gradient tree Γ_i

We can take a finite collection $\{\vec{\chi}_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}}$ and $\{\vec{\varkappa}_{\mathbf{j}}\}_{\mathbf{j}\in\mathcal{J}}$ in the paragraph before Lemma 3.20 such that $\{\vec{\chi}_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}} \cup \{\vec{\chi}_{\Gamma_1},\dots,\vec{\chi}_{\Gamma_d}\}$ forms a partition of unity of $M^{|T^{[0]}|}$ and finite collection $\{\vec{\varkappa}_{\mathbf{j}}\}_{\mathbf{j}\in\mathcal{J}} \cup \{\vec{\varkappa}_{\Gamma_1},\dots,\vec{\varkappa}_{\Gamma_d}\}$ forms a partition of unity of $\blacktriangle_T$ respectively, further satisfying $\big(\mathrm{Supp}(\vec{\chi}_{\mathbf{i}})\times\mathrm{Supp}(\vec{\varkappa}_{\mathbf{j}})\big)\cap \overline{\vec{\mathbf{C}}_{\Gamma_i}} \times \overline{\vec{D}_{\Gamma_i}} = \emptyset$ for each flow tree Γ_i and any $\mathbf{i},\mathbf{j}$. Therefore we have the estimate $\|\mathfrak{m}^T_{\vec{\chi}_{\mathbf{i}},\vec{\varkappa}_{\mathbf{j}}} \wedge \frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2}\|_{C^0(\blacktriangle_T\times M)} \leq C_{\epsilon,\vec{\chi}_{\mathbf{i}},\vec{\varkappa}_{\mathbf{j}}} e^{-\lambda c/2}$. As a conclusion of this Section 3.2, we have

(3.6)

$$\int_M m_{k,T}(\lambda)(\phi_{(k-1)k},\dots,\phi_{01}) \wedge \frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} = \sum_{i=1}^d \int_{\blacktriangle_T\times M} \mathfrak{m}^T_{\vec{\chi}_{\Gamma_i},\vec{\varkappa}_{\Gamma_i}} \wedge \frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} + \mathcal{O}(e^{-\lambda c/2}),$$

[11] Roughly speaking, these are the open subsets that WKB approximation for $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}$ can be constructed. These open subsets does not depend on $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}$ but rather depend on the geometry of gradient flow tree Γ_i's when applying Lemma 3.9 and Lemmma 3.11 along Γ_i's.

where $\mathcal{O}(e^{-\lambda c/2})$ refers to function in λ bounded by $Ce^{-\lambda c/2}$ for some C. This cut off the contribution to integral near the gradient flow trees Γ_i's.

3.3. WKB approximation method.

3.3.1. *WKB expansion for* $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}$. We fix a particular gradient flow tree $\Gamma = \Gamma_i$ (we omit i in our notations for the rest of this paper) and compute the contribution from the integral $\int_{\blacktriangle_T \times M} \mathfrak{m}^T_{\vec{\chi}_\Gamma,\vec{\varkappa}_\Gamma} \wedge \frac{e^{-2\lambda f_{ij}} * \phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2}$ in the above equation 3.6 using techniques from [3, Section 3].

We inductively define the open subset $W_{e,v} \subset M$ and $\mathbf{E}_v$ of $\mathbf{t}_v$ along the tree Γ, together with a WKB expansion of $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}}$ in $\vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v} = \prod_{v \in LT^{[0]}_{e,v}} \mathbf{E}_v \times W_{e,v}$ [12] for each flag (e,v) of T

$$\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}} \sim \lambda^{r_{e,v}} e^{-\lambda g_{e,v}} \big(\omega_{(e,v),0} + \omega_{(e,v),1}\lambda^{-\frac{1}{2}} + \cdots\big), \tag{3.7}$$

which is a norm estimate (here we fix arbitrary metric on $\blacktriangle_l$ as before) in the sense of Lemma 3.11, where $g_{e,v} \in \mathcal{C}^\infty(\vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v})$ is non-negative Bott-Morse function with zero set $V_{e,v} \subset \vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v}$ and $\omega_{(e,v),i} \in \Omega^*(\vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v})$ as follows:

(1) for the incoming edges e_{ij} with $\partial_o(e_{ij}) = v$, we define $W_{e_{ij},v}$ to be a open subset of $x_{\Gamma,e_{ij},v}$ (We use the notation as in Notation 3.17) together with the WKB expansion for ϕ_{ij} in $W_{e_{ij},v}$ from Lemma 3.9, with $r_{e_{ij},v} = \frac{\deg(q_{ij})}{2}$ and $g_{e_{ij},v} = g^+_{q_{ij}}$. In this case we have $V_{e_{ij},v} = V^+_{q_{ij}} \cap W_{e_{ij},v}$ being the stable submanifold;

(2) for (e,v) with $\partial_{in}(e) = v$ with v is labeled with 1, we let T_2, T_1 to be subtrees with outgoing edges e_2, e_1 ending at v such that e_2, e_1, e clockwisely oriented, we let $\vec{\mathbf{E}}_{T_{e,v}} = \vec{\mathbf{E}}_{T_2} \times \vec{\mathbf{E}}_{T_1}$ and $W_{e,v} = W_{e_2,v} \cap W_{e_1,v}$, with the product WKB expansion as
$$(-1)^{\varepsilon}\chi_v \mathfrak{m}^{(e_2,v)}_{\vec{\chi},\vec{\varkappa}} \wedge \mathfrak{m}^{(e_1,v)}_{\vec{\chi},\vec{\varkappa}} \sim \lambda^{r_{e,v}} e^{-\lambda g_{e,v}} \big(\omega_{(e,v),0} + \omega_{(e,v),1}\lambda^{-\frac{1}{2}} + \cdots\big)$$
by taking $\lambda^{r_{e,v}} = \lambda^{r_{e_2,v}+r_{e_1,v}}$, $g_{e,v} = g_{e_2,v} + g_{e_1,v}$ and $\omega_{(e,v),l} = \sum_{i+j=l} \chi_v \omega_{(e_2,v),i} \wedge \omega_{(e_1,v),j}$ (Here ε is given (2) in Definition 3.18). In this case we have $g_{e,v}$ being a non-negative Bott-Morse function in $\vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v}$ with zero set $V_{e,v} = (V_{e_2,v} \times \vec{\mathbf{E}}_{T_1}) \cap (V_{e_1,v} \times \vec{\mathbf{E}}_{T_1})$;

(3) when we have v labeled with u, we let $T_l, \ldots, T_1$ be subtrees with outgoing edges $e_l, \ldots, e_1$ ending at v with $e_l, \ldots, e_1, e$ clockwisely oriented, we let $\vec{\mathbf{E}}_{T_{e,v}} = \prod_{j=1}^l \vec{\mathbf{E}}_{T_j} \times \mathbf{C}_v$ and take $W_{e,v}$ (Here $\mathbf{C}_v$ is neighborhood of $\mathbf{t}_{\Gamma,v}$, and $W_{e,v}$ is a neighborhood of $x_{\Gamma,v} = x_{\Gamma,e,v}$) such that $\sigma_{t_j}(W_{e,v}) \subset W_{e_j,v}$ for each $j = 1, \ldots, l$ for $(t_l, \ldots, t_1) \in \mathbf{C}_v$. Therefore we have the WKB expansion $\mathfrak{m}^{(e,v)}_{\vec{\chi},\vec{\varkappa}} \sim \lambda^{r_{e,v}} e^{-\lambda g_{e,v}} \big(\omega_{(e,v),0} + \omega_{(e,v),1}\lambda^{-\frac{1}{2}} + \cdots\big)$ by taking $r_{e,v} = \sum_{j=1}^l r_{e_j,v}$, $g_{e,v} = \sum_{j=1}^l \tau_j^*(g_{e_j,v})$ and
$$\omega_{(e,v),m} = \sum_{i_l+\cdots+i_1=m} \nu_{T_{e,v}}\chi_v\varkappa_v\big(\iota_{\frac{\partial}{\partial t_{v,l}} \wedge \nu^\vee_{T_l}} \tau_l^*(\omega_{(e_l,v),i_l})\big) \wedge \cdots \wedge \big(\iota_{\frac{\partial}{\partial t_{v,1}} \wedge \nu^\vee_{T_1}} \tau_1^*(\omega_{(e_1,v),i_1})\big),$$
where $\tau_j : \prod_{j=1}^l \vec{\mathbf{E}}_{T_j} \times \blacktriangle_{\nu(v)} \times W_{e,v} \to \vec{\mathbf{E}}_{T_j} \times W_{e_j,v}$ is induced by taking product of the projection $\prod_{j=1}^l \vec{\mathbf{E}}_{T_j} \to \vec{\mathbf{E}}_{T_j}$ with $\tau_j : \blacktriangle_{\nu(v)} \times W_{e,v} \to W_{e_j,v}$ (here we abuse the notation) given by $\tau_j(t_{v,l}, \cdots, t_{v,1}, x) = \sigma_{t_{v,j}}(x)$. In this case we have $V_{e,v} = \bigcap_{j=1}^l \tau_j^{-1}(V_{e_j,v})$;

(4) for an edge e numbered by ij with $\partial_{in}(e) = v_0$ and $\partial_o(e) = v_1$ with v_1 not being the outgoing vertex v_o, we apply the Lemma 3.11 by taking $\zeta_S = \mathfrak{m}^{(e,v_0)}_{\vec{\chi},\vec{\varkappa}}$ (and shrinking W_{e,v_0} if necessary) together with its WKB approximation, therefore we obtain the WKB approximation for

[12] Here $T_{e,v}$ is the combinatorial subtree of T as in Notation 2.3.

$\zeta_E = \mathfrak{m}^{(e,v_1)}_{\vec{\chi},\vec{\varkappa}}$ in a neighborhood $\vec{\mathbf{E}}_{T_{e,v_1}} \times W_{e,v_1}$ for some small neighborhood W_{e,v_1} of x_{Γ,e,v_1}. In this case we have $V_{e,v_1} = \bigcup_{t\in\mathbb{R}} \varsigma_t(V_{e,v_0}) \cap \big(\vec{\mathbf{E}}_{T_{e,v_1}} \times W_{e,v_1}\big)$ where ς_t here is t-time flow of $\nabla f_{ij}/|\nabla f_{ij}|^2$ extended to $\vec{\mathbf{E}}_{T_{e,v_1}} \times (M \setminus \mathrm{Crit}(f_{ij}))$ by taking product with $\vec{\mathbf{E}}_{T_{e,v_1}}$;

(5) for the outgoing edge e_o with outgoing vertex v_o, we simply take the WKB expansion of $\mathfrak{m}^{(e_o,v_o)}_{\vec{\chi},\vec{\varkappa}}$ to be that of $\mathfrak{m}^{(e_o,v_r)}_{\vec{\chi},\vec{\varkappa}}$. In this case we have $V_{e_o,v_o} = V_{e_o,v_r}$.

Having the WKB approximation of $\mathfrak{m}^{(e_o,v_o)}_{\vec{\chi},\vec{\varkappa}}$, together with that for

$$\frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} \sim \frac{\lambda^{\deg(q_{0k})/2}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} e^{-\lambda g^-_{0k}}\big(*\omega_{0k,0} + *\omega_{0k,1}\lambda^{-\frac{1}{2}} + \cdots\big)$$

from Lemma 3.9 (here we abbreviated $g^-_{q_{0k}}$ and $\omega_{q_{0k},i}$'s by g^-_{0k} and $\omega_{0k,i}$'s respectively), we obtain

(3.8)
$$\int_{\blacktriangle_T\times M} \mathfrak{m}^T_{\vec{\chi}_\Gamma,\vec{\varkappa}_\Gamma} \wedge \frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} = \frac{\lambda^{r_{e_o,v_o}+\deg(q_{0k})/2}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2}\int_{\blacktriangle_T\times M} e^{-\lambda(g_{e_o,v_o}+g^-_{0k})}\omega_{(e_o,v_o),0}\wedge *\omega_{0k,0} + \mathcal{O}(\lambda^{-\frac{1}{2}}).$$

3.3.2. *Explicit computation of the integral.* From the generic assumption of $\vec{f}$ in Definition 2.8, we notice that all the points $\mathbf{t}_{\Gamma,v} \in \mathrm{int}(\blacktriangle_{\nu(v)})$. In the above WKB construction, by shrinking $\mathbf{E}_v$'s and $W_{e,v}$'s if necessary, we may always assume that $\pi_{e,v} : \vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v} \to V_{e,v}$ being identified with a neighborhood of zero section in the normal bundle $NV_{e,v}$ in $\vec{\mathbf{E}}_{T_{e,v}} \times W_{e,v}$. We notice that the element $\nu_{T_{e,v}} \wedge \mathrm{vol}_{e,v}$ (Here $\mathrm{vol}_{e,v}$ is introduced in Definition 2.9 as element in $\bigwedge^* T^*M_{x_{\Gamma,e,v}}$) is a top degree element in $\bigwedge^* NV^*_{e,v}$, serves as an orientation in the normal direction (by extending to whole $V_{e,v}$).

We show inductively along gradient tree Γ that the integration along fiber

$$(\pi_{e,v})_*\big(\lambda^{r_{e,v}} e^{-\lambda g_{e,v}}\omega_{(e,v),0}\big) = 1 + \mathcal{O}(\lambda^{-\frac{1}{2}})$$

at the point $(\vec{\mathbf{t}}_{\Gamma_{e,v}}, x_{\Gamma,e,v})$ (here $x_{\Gamma,e,v}$ is introduced in Notation 3.17) in $V_{e,v}$ (Here $(\pi_{e,v})_*$ refers integration along fibers of $\pi_{e,v}$ with respect to orientation $\nu_{T_{e,v}} \wedge \mathrm{vol}_{e,v}$) using techniques from [3, Section 3]. Since $g_{e,v}$ is non-negative Bott-Morse function with zero set $V_{e,v}$, using the well known stationary phase expansion (see e.g. [4] or [3, Lemma 58]) we notice the leading order in $\lambda^{-\frac{1}{2}}$ in above integral only depend on the values of $\omega_{(e,v),0}$ at $(\vec{\mathbf{t}}_{\Gamma_{e,v}}, x_{\Gamma,e,v})$, and can be computed inductively as follows (we use the same notations as in the inductive WKB construction in earlier Section 3.3):

(1) for the incoming edges e_{ij} with $\partial_o(e_{ij}) = v$, this is exactly Lemma 3.10;

(2) for (e,v) with $\partial_{in}(e) = v$ with v is labeled with 1, with subtree T_2, T_1 and outgoing edges e_2, e_1 ending at v, we have $V_{e,v} = (V_{e_2,v} \times \vec{\mathbf{E}}_{T_1}) \cap (V_{e_1,v} \times \vec{\mathbf{E}}_{T_1})$ and we can compute

$$(\pi_{e,v})_*(\lambda^{r_{e,v}}e^{-\lambda g_{e,v}}\omega_{(e,v),0}) = (-1)^{\varepsilon}(\pi_{e_2,v})_*(\lambda^{r_{e_2,v}}e^{-\lambda g_{e_2,v}}\omega_{(e_2,v),0})(\pi_{e_1,v})_*(\lambda^{r_{e_1,v}}e^{-\lambda g_{e_1,v}}\omega_{(e_1,v),0}) = 1$$

at the point $(\vec{\mathbf{t}}_{\Gamma_{e,v}}, x_{\Gamma,e,v})$ in $V_{e,v}$ modulo error $\mathcal{O}(\lambda^{-\frac{1}{2}})$ (ε as in (2) Definition 3.18);

(3) when we have v labeled with u, we let $T_l, \dots, T_1$ be subtrees with outgoing edges $e_l, \dots, e_1$ ending at v with $e_l, \dots, e_1, e$ clockwisely oriented, we notice that $V_{e,v} = \bigcap_{j=1}^l \tau_j^{-1}(V_{e_j,v})$ from WKB construction in previous Section 3.3. From the induction, we can compute the integral $(\pi_{e_j,v})_*\big(\lambda^{r_{e_j,v}}e^{-\lambda\tau_j^*(g_{e_j,v})}\tau_j^*(\omega_{(e_j,v),0})\big) = 1 + \mathcal{O}(\lambda^{-1})$ as function on $\tau_j^{-1}((\vec{\mathbf{t}}_{\Gamma_{e_j,v}}, x_{\Gamma,e_j,v}))$ if we identify a neighborhood $\tau_j^{-1}(\vec{\mathbf{E}}_{T_j} \times W_{e_j,v})$ of $\tau_j^{-1}(V_{e_j,v})$ with a neighborhood of zero section in the pull back normal bundle $\tau_j^{-1}(NV_{e_j,v})$ as treat $\pi_{e_j,v} : \tau_j^{-1}(NV_{e_j,v}) \to \tau_j^{-1}(V_{e_j,v})$ as

integration along fibers. We obtain the identity

$$(\pi_{e,v})_*(\lambda^{r_{e,v}} e^{-\lambda g_{e,v}} \omega_{(e,v),0}) = \prod_{j=1}^{l} (\pi_{e_j,v})_* \big(\lambda^{r_{e_j,v}} e^{-\lambda \tau_j^*(g_{e_j,v})} \tau_j^*(\omega_{(e_j,v),0})\big) = 1,$$

at $(\vec{\mathbf{t}}_{\Gamma_{e,v}}, x_{\Gamma,e,v})$ modulo error $\mathcal{O}(\lambda^{-\frac{1}{2}})$;

(4) for an edge e numbered by ij with $\partial_{in}(e) = v_0$ and $\partial_o(e) = v_1$ with v_1 not being the outgoing vertex v_o, we can compute $(\pi_{e,v_1})_*(\lambda^{r_{e,v_1}} e^{-\lambda g_{e,v_1}} \omega_{(e,v_1),0}) = 1 + \mathcal{O}(\lambda^{-\frac{1}{2}})$ at the point $(\vec{\mathbf{t}}_{\Gamma_{e,v_1}}, x_{\Gamma,e,v_1})$ using the fact that $(\pi_{e,v_0})_*(\lambda^{r_{e,v_0}} e^{-\lambda g_{e,v_0}} \omega_{(e,v_0),0}) = 1 + \mathcal{O}(\lambda^{-\frac{1}{2}})$ at the point $(\vec{\mathbf{t}}_{\Gamma_{e,v_0}}, x_{\Gamma,e,v_0})$ by applying Lemma 3.12 with $x_S = x_{\Gamma,e,v_0}$ an $x_E = x_{\Gamma,e,v_1}$ (notice that $\vec{\mathbf{t}}_{\Gamma_{e,v_0}} = \vec{\mathbf{t}}_{\Gamma_{e,v_1}}$);

(5) for the outgoing edge e_o with outgoing vertex v_o, since we have V_{e_o,v_o} and $\vec{\mathbf{E}}_T \times V_{0k}^-$ intersecting transversally at $(\vec{\mathbf{t}}_\Gamma, x_{\Gamma,e_o,x_r})$, we can compute

$$\begin{aligned}
&\frac{\lambda^{r_{e_o,v_o}+\deg(q_{0k})/2}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} \int_{\blacktriangle_T \times M} e^{-\lambda(g_{e_o,v_o}+g_{0k}^-)} \omega_{(e_o,v_o),0} \wedge *\omega_{0k,0} \\
&= \pm\, (\pi_{e_o,v_o})_*(\lambda^{r_{e_o,v_o}} e^{-\lambda g_{e_o,v_o}} \omega_{(e_o,v_o),0}) \Big(\frac{\lambda^{\frac{\deg(q_{0k})}{2}}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} \int_{NV^-_{x_{\Gamma,e_o,x_r}}} e^{-\lambda g_{0k}^-} * \omega_{0k,0} \Big) + \mathcal{O}(\lambda^{-\frac{1}{2}}) \\
&= \pm\, 1 + \mathcal{O}(\lambda^{-\frac{1}{2}})
\end{aligned}$$

where the $\pm$ sign depending on whether the sign of gradient flow tree Γ obtained by comparing $\mathrm{vol}_{e_o,v_r} \wedge * \mathrm{vol}_{q_{0k}}$ with vol_M as described in Definition 2.9.

As a conclusion, we have proven that

$$\int_M m_{k,T}(\lambda)(\phi_{(k-1)k}, \ldots, \phi_{01}) \wedge \frac{*e^{-2\lambda f_{0k}}\phi_{0k}}{\|e^{-\lambda f_{0k}}\phi_{0k}\|^2} = \sum_{i=1}^{d} (-1)^{\chi(\Gamma_i)} + \mathcal{O}(\lambda^{-\frac{1}{2}})$$

and hence Theorem 2.11.

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