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Hybrid models for homological projective duals and noncommutative resolutions

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Abstract

We study hybrid models arising as homological projective duals (HPD) of certain projective embeddings $f : X \to \mathbb{P}(V)$ of Fano manifolds X. More precisely, the category of B-branes of such hybrid models corresponds to the HPD category of the embedding f. B-branes on these hybrid models can be seen as global matrix factorizations over some compact space B or, equivalently, as the derived category of the sheaf of $\mathcal{A}$-modules on B, where $\mathcal{A}$ is an A_∞ algebra. This latter interpretation corresponds to a noncommutative resolution of B. We compute explicitly the algebra $\mathcal{A}$ by several methods, for some specific class of hybrid models, and find that in general it takes the form of a smash product of an A_∞-algebra with a cyclic group. Then we apply our results to the HPD of f corresponding to a Veronese embedding of projective space and the projective embedding of Fano complete intersections in $\mathbb{P}^n$.

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1 Introduction

It is known that hybrid models provide realizations of a series of two-dimensional superconformal field theories which can be obtained from certain phases of gauged linear sigma models (GLSM) [1]. Roughly speaking, a hybrid model is a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric field theory whose target space is of the form $Y = \mathrm{Tot}(\mathcal{E} \to B)$ for some holomorphic vector bundle (or an orbibundle, in general) $\mathcal{E}$ over the base space B where the fields interact via a superpotential $W \in H^0(\mathcal{O}_Y)$ which is a holomorphic function on the total space. The hybrid model can be viewed as a family of Landau-Ginzburg (LG) models fibred over the base space. Further sufficient conditions (but not necessary) in Y and W guarantees that these models RG flows to SCFTs and also makes them tractable as quantum field theories (QFT), see for instance [2].

Recently, it is found that homological projective dual (HPD) [3] of certain projective embeddings can be described by hybrid models. This was found in mathematics [4, 5] and a physics formulation is presented in[1] [7]: if a GLSM $\mathcal{T}_X$ for a projective morphism $f : X \to \mathbb{P}(V)$ is known, one can build up an extended GLSM $\mathcal{T}_{\mathcal{X}}$ such that the Higgs branch of one of its phases gives rise to the HPD of $f : X \to \mathbb{P}(V)$. In the abelian cases, this Higgs branch is a hybrid model.

For a LG model with quadratic superpotential W_{LG}, it was shown in [8] that the category of B-branes (homotopy category of matrix factorizations) $MF(W_{LG})$ is equivalent to the

[1] The first appeareance of HPD in the context of dynamics of GLSMs can be found in [6].

derived category of finite dimensional Clifford modules, where the Clifford algebra is defined by the Hessian of the superpotential W_{LG}. In addition, if a $\mathbb{Z}_2$ orbifold is present that leaves W_{LG} invariant, then the category of B-branes of this LG orbifold $MF(W_{LG}, \mathbb{Z}_2)$ is equivalent to the derived category of finite dimensional modules of the even subalgebra of the corresponding Clifford algebra. Consequently, the category of matrix factorizations of a $\mathbb{Z}_2$-orbifold hybrid model with superpotential quadratic along the fibre coordinates is equivalent to the derived category of the sheaf of modules of the sheaf of even parts of a Clifford algebra. This is the case of the HPD category of the degree 2 Veronese embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V)$ [9]. Thus, the hybrid model orbifold can be viewed as a noncommutative resolution of the base space.

In this work, we generalize the idea and construct an explicit correspondence between hybrid models and noncommutative spaces. Locally, at a generic point in the base $p \in B$ of a hybrid model, we can model its dynamics by a LG orbifold. Denote the category of B-branes of this LG orbifold as $MF(W, G)$, where G is the orbifold group. We first study the A_∞-algebra $\mathcal{A}_{D0} = \mathrm{End}(\mathcal{B}_{D0})$ associated with the endomorphism algebra of a $D0$-brane $\mathcal{B}_{D0} \in MF(W)$ of the LG model. The algebra $\mathcal{A}_{D0}$ takes the form of a A_∞ algebra with a finite number of generators (as an algebra) ψ_i that satisfy the higher products relations given in (4.65) and (4.66) (for a homogeneous W) and for general elements (4.69) and (4.70)[2]. It then sets up the equivalence between matrix factorizations and A_∞-modules of $\mathcal{A}_{D0}\sharp G$, where $\sharp$ denotes the smash product (the mathematical approach toward this equivalence can be found in [10]), more precisely

$$MF(W, G) \cong D(Mod - \mathcal{A}_{D0}\sharp G) \tag{1.1}$$

where the appeareance of the derived category is a consequence that $MF(W, G)$ is taken to be the homotopy category. We then use this equivalence to relate a hybrid model to a noncommutative resolution

$$D(Y, W) \cong D(B, \mathcal{A}_{D0}\sharp G), \tag{1.2}$$

i.e. the derived category of sheaf of $\mathcal{A}_{D0}\sharp G$-modules.

This result can be used to study HPD of several spaces. As mentioned above, the GLSM construction realizes the HPDs as hybrid model orbifolds, which can be identified with noncommutative resolutions as the equivalence suggests. Therefore, given a projective embedding engineered by an abelian GLSM, the hybrid model describing the HPD can be read off following [7]. One can then use the correspondence discussed in this paper to give a noncommutative geometric decription of the HPD.

This paper is organized as follows. We review the basic facts about GLSMs for HPD and A_∞-algebras in sections 2 and 3 respectively. In section 4, we set up the relationship between matrix factorizations and A_∞-modules. We first find the structure of the A_∞-algebra $\mathcal{A}_{D0}$ by various means (deformation theory of $\mathcal{B}_{D0}$ and effective superpotential, A_∞-homomorphism), then we propose a functor realizing the equivalence between the category of matrix factorizations and the derived category of A_∞-modules of $\mathcal{A}_{D0}$ and sketch its generalization to the orbifold case $\mathcal{A}_{D0}\sharp G$. We provide checks of this proposal in appendix A. We then apply this correspondence in section 5 to describe the HPD of degree d Veronese embedding of projective space, Fano hypersurfaces and complete intersections in projective spaces as noncommutative spaces with the structure sheaf given by the corresponding sheaf of A_∞-algebras. The same result was obtained for Veronese embeddings by summing over the ribbon trees in [11], we review this method in appendix B.

[2]We also consider the case of inhomogenous W. The higher products are given in (B.2) and (B.3).

2 Lightning review of GLSMs for HPD

In this section, we review the construction of homological projective duals (HPD) of projective morphisms $f : X \to \mathbb{P}(S)$ (where $S \cong \mathbb{C}^{n+1}$) proposed in [7]. We refer the reader to [7] for the details of definitions and notations. The construction of [7] assumes that we have a gauged linear sigma model (GLSM) construction for $f : X \to \mathbb{P}(S)$, i.e. a GLSM having a geometric phase corresponding to a Higgs branch[3] that RG flows to a nonlinear sigma model (NLSM) whose target space is the image of f in $\mathbb{P}(S)$. Denote that GLSM by $\mathcal{T}_X = (G, \rho_m : G \to GL(V), W, t_{\mathrm{ren}}, R)$. Then, there exist a distinguished $U(1)_{\mathcal{L}} \subset G$ (with associated FI-theta parameter $t_{\mathcal{L}} = \zeta_{\mathcal{L}} - i\theta_{\mathcal{L}}$) characterizing the morphism f. The components of f have homogeneous weight under $U(1)_{\mathcal{L}}$ and corresponds to sections of the line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(S)}(1)$ over X. In the following we assume that the aforementioned geometric phase is a pure Higgs phase[4] and its category of B-branes will be denoted by $D(X_{\zeta_{\mathcal{L}} \gg 1})$, if this phase is located at $\zeta_{\mathcal{L}} \gg 1$. As we vary the parameter $\zeta_{\mathcal{L}}$ we find, in general that the phase at $\zeta_{\mathcal{L}} \ll -1$ has a Higgs branch whose category of B-branes we denote[5] $D(Y_{\zeta_{\mathcal{L}} \ll -1}, W_{\zeta_{\mathcal{L}} \ll -1})$ and a mixed Coulomb-higgs branch that splits into several isolated vacua, whose categories of B-branes we denote as $E_1, \ldots, E_k$. Both categories of B-branes at the different values of $\zeta_{\mathcal{L}}$ are related by

$$\langle D(Y_{\zeta_{\mathcal{L}} \ll -1}, W_{\zeta_{\mathcal{L}} \ll -1}), E_1, \ldots, E_k \rangle \cong D(X_{\zeta_{\mathcal{L}} \gg 1}) \tag{2.1}$$

The equivalence (2.1) is realized at the level of the GLSM via the so called window categories. They are defined entirely via the UV datum (i.e. GLSM datum). Defining a B-brane $\mathcal{B}$ in the GLSM requires to specify a representation $\rho_M : G \to GL(M)$. If we denote $q^{\mathcal{L}}$ the weight of ρ_M restricted to $U(1)_{\mathcal{L}}$, then we define two conditions on the weights $q^{\mathcal{L}}$:

$$\begin{aligned} &\textbf{Small window}: \quad |\theta_{\mathcal{L}} + 2\pi q^{\mathcal{L}}| < \pi \min(N_{\mathcal{L},\pm}) \\ &\textbf{Big window}: \quad |\theta_{\mathcal{L}} + 2\pi q^{\mathcal{L}}| < \pi \max(N_{\mathcal{L},\pm}) \end{aligned} \tag{2.2}$$

where $N_{\mathcal{L},\pm} := \sum_a (Q_a^{\mathcal{L}})^{\pm}$, $(x)^{\pm} := (|x| \pm x)/2$ and $Q_a^{\mathcal{L}}$ are the weights of ρ_m restricted to $U(1)_{\mathcal{L}}$. Therefore we have the definition of the window subcategories by the constraints (2.2): $\mathcal{W}^{\mathcal{L}}_{+,b}$ (resp. $\mathcal{W}^{\mathcal{L}}_{-,b}$) corresponds to the objects $\mathcal{B}$ such that the weights $q^{\mathcal{L}}$ of ρ_M satisfy the big (resp. small) window constraint for $b = \lfloor \frac{\theta_{\mathcal{L}}}{2\pi} \rfloor$. The we have

$$D(Y_{\zeta_{\mathcal{L}} \ll -1}, W_{\zeta_{\mathcal{L}} \ll -1}) \cong \mathcal{W}^{\mathcal{L}}_{-,b} \hookrightarrow D(X_{\zeta_{\mathcal{L}} \gg 1}) \cong \mathcal{W}^{\mathcal{L}}_{+,b} \tag{2.3}$$

for any $b \in \mathbb{Z}$. Starting from $\mathcal{T}_X$, we define an extension $\mathcal{T}_{\mathcal{X}}$ of $\mathcal{T}_X$ given by

$$\mathcal{T}_{\mathcal{X}} = (\widehat{G} = G \times U(1)_{s+1}, \hat{\rho}_m : \widehat{G} \to GL(V \oplus V'), \widehat{W}, \widehat{R}), \tag{2.4}$$

where V' is a representation of $U(1)_{s+1} \times U(1)_{\mathcal{L}} \subseteq \widehat{G}$ with weights $(-1, -Q) \oplus (1, 0)^{\oplus(n+1)}$. Denoting the coordinates of V' as $(p, s_0, \ldots, s_n)$, the superpotential $\widehat{W}$ is given by

$$\widehat{W} = W + p \sum_{j=0}^{n} s_j f_j(x), \tag{2.5}$$

[3]We do not need to assume the GLSM is nonanomalous.

[4]This is always possible in the case X is Fano or Calabi-Yau (CY), wich are essentially the cases we will cover in this work. The generalization is straightforward.

[5]The space $D(Y_{\zeta_{\mathcal{L}} \ll -1}, W_{\zeta_{\mathcal{L}} \ll -1})$ denotes the category of B-branes of a hybrid Landau-Ginzburg (LG) model with target space $Y_{\zeta_{\mathcal{L}} \ll -1}$ and superpotential $W_{\zeta_{\mathcal{L}} \ll -1}$. The details on how such category arises in the current context can be found in [7].

where $f_j(x)$ are the components of the image of the map f. The GLSM $\mathcal{T}_{\mathcal{X}}$ is identified with the GLSM of the universal hyperplane section[6] $\mathcal{X}$ of X: Its Higgs branch deep in the first quadrant of the FI parameter of $U(1)_{s+1} \times U(1)_{\mathcal{L}}$ corresponds to a NLSM with target space $\mathcal{X}$. Keeping $\zeta_{s+1} \gg 1$ and varying $\zeta_{\mathcal{L}}$ leads to the following equivalence of categories

$$\mathcal{C} = D(\widehat{Y}_{\zeta_{\mathcal{L}} \ll -1}, \widehat{W}_{\zeta_{\mathcal{L}} \ll -1}) \cong \widehat{\mathcal{W}}^{\mathcal{L}}_{-,b} \hookrightarrow D(\mathcal{X}_{\zeta_{\mathcal{L}} \gg 1}) \cong \widehat{\mathcal{W}}^{\mathcal{L}}_{+,b}, \tag{2.6}$$

where the categories $\widehat{\mathcal{W}}^{\mathcal{L}}_{\pm,b}$ are defined analogously to $\mathcal{W}^{\mathcal{L}}_{\pm,b}$. The category $\mathcal{C}$ is identified with the HPD category of X. Everything can be carried over when taking linear sections of X, but in this work we will be mainly interested in the HPD of X.

Using this proposal, we can express $\mathcal{C}$ as the category of B-branes on a Higgs branch that we can describe as a fibered LG model i.e. a hybrid model [2].

3 Lightning review of A_∞ algebras and their relation to open topological strings

In this section we present the useful definitions and results that relate A_∞ to the relevant physical systems we are going to need in the subsequent sections. Let us start with the definition of A_∞ algebra (our main reference is [12] but other useful sources are [13–15]).

Definition. An A_∞ algebra over a field $\mathbb{K}$ Consist of a $\mathbb{Z}$-graded $\mathbb{K}$-vector space A

$$A = \bigoplus_{p \in \mathbb{Z}} A^p \tag{3.1}$$

endowed with homogeneous $\mathbb{K}$-linear maps[7]

$$m_n : A^{\otimes n} \to A \qquad n \geq 1 \tag{3.2}$$

of degree $2-n$ satisfying the relations

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0 \qquad n \geq 1 \tag{3.3}$$

where $u = r+t+1$ and $s \geq 1$, $r,t \geq 0$.

Let us make a few important remarks. First, note that (3.3) implies $m_1 \circ m_1 = 0$ hence, m_1 is a differential. Second, the maps in the tensor products, such as in (3.3) are subjected to the Koszul sign rule:

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b) \tag{3.4}$$

where we assume a is an homogeneous element of degree denoted by $|a|$ and $|g|$ denotes the degree of the map g. We define next a morphism of A_∞ algebras

[6]We recall the reader that $\mathcal{X}$ is defined as the fiber product $X \times_{\mathbb{P}(S)} \mathcal{H} \subset X \times \mathbb{P}(S^\vee)$ where $\mathcal{H} = \{(u,v) \in \mathbb{P}(S) \times \mathbb{P}(S^\vee) | v(u) = 0\} \subset \mathbb{P}(S) \times \mathbb{P}(S^\vee)$ is the incidence divisor.

[7]The grading of $A^{\otimes n}$ is given by $(A^{\otimes n})^p = \bigoplus_{i_1+\ldots+i_n=p} A^{i_1} \otimes \cdots \otimes A^{i_n}$

Definition. A morphism $f : A \to B$ between A_∞ algebras consist of a family of maps

$$f_n : A^{\otimes n} \to B \tag{3.5}$$

of degree $1-n$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} f_u(\mathbf{1}^{\otimes r} \otimes m_s^A \otimes \mathbf{1}^{\otimes t}) = \sum_{l=1}^{n} \sum_{I=n} (-1)^{\epsilon(l)} m_l^B (f_{i_1} \otimes \cdots \otimes f_{i_l}) \qquad n \geq 1 \tag{3.6}$$

where $u = r + t + 1$, $s \geq 1$, $r, t \geq 0$ and the second sum over $I = n$ means sum over all decompositions $i_1 + \ldots + i_l = n$ (with $i_k \geq 1$). The sign $\epsilon(l)$ is given by

$$\epsilon(l) = (l-1)(i_1 - 1) + (l-2)(i_2 - 1) + \ldots + 2(i_{l-2} - 1) + (i_{l-1} - 1) \tag{3.7}$$

finally, we denoted $m_n^{A,B}$ the maps of A and B, respectively.

Note that the map f_1 induces a map $f_{1,*}$ between the cohomologies

$$f_{1,*} : H(A) \to H(B) \tag{3.8}$$

where $H(A)$ ($H(B)$) denotes the homology of the differential m_1^A (m_1^B). Then, a morphism is called quasi-isomorphism if $f_{1,*}$ is an isomorphism and is called strict if $f_i = 0$ for all $i \neq 1$.

Definition. An A_∞-module over A is given by a $\mathbb{Z}$-graded vector space endowed with maps

$$m_n^M : M \otimes A^{\otimes n-1} \to M \qquad n \geq 1 \tag{3.9}$$

of degree $2-n$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_u^M (\mathbf{1}^{\otimes r} \otimes \tilde{m}_s \otimes \mathbf{1}^{\otimes t}) = 0 \qquad n \geq 1 \tag{3.10}$$

where $u = r + t + 1$, $s \geq 1$, $r, t \geq 0$ and

$$m_u^M (\mathbf{1}^{\otimes r} \otimes \tilde{m}_s \otimes \mathbf{1}^{\otimes t}) = \begin{cases} m_u^M (\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}), & \text{if } r > 0 \\ m_u^M (m_s^M \otimes \mathbf{1}^{\otimes t}), & \text{if } r = 0 \end{cases} \tag{3.11}$$

There is an alternative construction of the A_∞-algebra know as the bar construction. Consider a $\mathbb{Z}$-graded $\mathbb{K}$-vector space V and the tensor algebra

$$T^\bullet V := \bigoplus_{n \geq 1} V^{\otimes n} \tag{3.12}$$

Then, any coderivation $b : T^\bullet V \to T^\bullet V$ can be written in terms of degree 1 maps $b_n : V^{\otimes} \to V$. Explicitly, by denoting $b_{n,u}$ component of b mapping $V^{\otimes n} \to V^{\otimes u}$, we can write

$$b_{n,u} = \sum_{r+s+t=n, r+t+1=u} \mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t} \qquad r, t \geq 1, s \geq 1 \tag{3.13}$$

Imposing $b^2 = 0$ is equivalent to the conditions

$$\sum_{r+s+t=n} b_u(\mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}) = 0 \qquad n \geq 1 \tag{3.14}$$

where $u = r + t + 1$ and $s \geq 1$, $r, t \geq 0$. Then, if we identify $V = A[1]$, where $A[1]$ is the grading shift $(A[1])^p = A^{p+1}$ and we denote the natural degree -1 map $s : A \to A[1]$, then if we write

$$m_n = s^{-1} \circ b_n \circ s^{\otimes n} \tag{3.15}$$

or equivalently[8]

$$b_n = (-1)^{\frac{n(n-1)}{2}} s \circ m_n \circ (s^{-1})^{\otimes n} \tag{3.16}$$

the relations (3.14) are equivalent to (3.3). An A_∞-algebra A is called minimal if $m_1 \equiv 0$ and is called strictly unital if there exist a degree 0 element $1_A \in A^0$ satisfying

$$\begin{aligned} &m_1(1_A) = 0 \\ &m_2(1_A \otimes a) = m_2(a \otimes 1_A) = a \\ &m_i(a_1 \otimes \cdots \otimes a_i) = 0 \ \text{ if any } a_k = 1_A \qquad i > 2 \end{aligned} \tag{3.17}$$

for all $a, a_1, \ldots, a_i \in A$. Moreover, if A is equipped with a bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{C}$, then A is called cyclic (w.r.t. $\langle \cdot, \cdot \rangle$) if it satisfies

$$\langle a_0, b_n(a_1 \otimes \cdots \otimes a_n)) \rangle = (-1)^{(|a_0|+1)(|a_1|+\ldots+|a_n|+n)} \langle a_1, b_n(a_2 \otimes \cdots \otimes a_0)) \rangle \tag{3.18}$$

where $a_i \in A$ are homogeneous elements.
We have the following important theorem [16, 17]

Theorem. Any A_∞-algebra (A, m_n) is A_∞-quasi-isomorphic to a minimal A_∞-algebra called a minimal model for A. Moreover this minimal model can be taken to be $(H(A), m_n^H)$ which is unique up to A_∞-isomorphism and satisfies

1. The map $f_1 : H(A) \to A$ is given by the inclusion map.
2. The map m_2^H is given by the map induced by m_2.

Then, this theorem plus the conditions (3.6) for A_∞ morphisms applied to the inclusion map $\iota : H(A) \to A$ give us a way to recursively determine the products m_n^H from the knowledge of (A, m_n). Let us write some of these relations to ilustrate this point (recall that $m_1^H \equiv 0$):

$$\begin{aligned} \iota \circ m_2^H &= m_2(\iota \otimes \iota) + m_1 \circ f_2 \\ \iota \circ m_3^H &= f_2(m_2^H \otimes \mathbf{1}) - f_2(\mathbf{1} \otimes m_2^H) + m_2(\iota \otimes f_2) - m_2(f_2 \otimes \iota) + m_1 \circ f_3 \\ &\vdots \end{aligned} \tag{3.19}$$

so, the maps $f_n : H(A)^{\otimes n} \to A$ and the higher products m_n^H can be determined recursively (see for example [18]).

[8] Here we used that the inverse of $s^{\otimes n}$ is $(-1)^{\frac{n(n-1)}{2}}(s^{-1})^{\otimes n}$. We remak that there are different ways to define b_n in terms of m_n, leading to different sign conventions.

In the case of topological strings we will be interested in A_∞-categories, which are defined as follows

Definition. A A_∞-category $\mathcal{A}$ with objects $\mathrm{Ob}(\mathcal{A})$ consists of the datum

1. For all $A, B \in \mathrm{Ob}(\mathcal{A})$ the space $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is a $\mathbb{Z}$-graded vector space.
2. For all $n \geq 1$ and any set of objects $A_0, \ldots, A_n \in \mathrm{Ob}(\mathcal{A})$ there exists a degree $2-n$ map

   $m_n : \mathrm{Hom}_{\mathcal{A}}(A_{n-1}, A_n) \otimes \mathrm{Hom}_{\mathcal{A}}(A_{n-2}, A_{n-1}) \otimes \cdots \mathrm{Hom}_{\mathcal{A}}(A_0, A_1) \to \mathrm{Hom}_{\mathcal{A}}(A_0, A_n)$

   satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0 \tag{3.20}$$

we also define

Definition. An A_∞-functor between A_∞-categories $\mathcal{A}_1$ and $\mathcal{A}_2$ consists of the datum

1. A map $\mathcal{F} : \mathrm{Ob}(\mathcal{A}_1) \to \mathrm{Ob}(\mathcal{A}_2)$.
2. For all $n \geq 1$ and any set of objects $A_0, \ldots, A_n \in \mathrm{Ob}(\mathcal{A}_1)$ there exists a degree $1-n$ map

   $\mathcal{F}_n : \mathrm{Hom}_{\mathcal{A}_1}(A_{n-1}, A_n) \otimes \mathrm{Hom}_{\mathcal{A}_1}(A_{n-2}, A_{n-1}) \otimes \cdots \mathrm{Hom}_{\mathcal{A}_1}(A_0, A_1) \to \mathrm{Hom}_{\mathcal{A}_2}(\mathcal{F}(A_0), \mathcal{F}(A_n))$

   satisfying conditions analogous to (3.6)

More precisely, in topological string theory we encounter cyclic, unital and minimal A_∞-categories[9] and we take the field $\mathbb{K} = \mathbb{C}$ from now on. It is easy to see that the A_∞-category of a single objectis equivalent to an A_∞-algebra. Next we move on to explain how these structures arise in topological strings. For simplicty we consider a worldsheet with disk topology and boundary conditions characterized by a single D-brane $\mathcal{D}$. Upon topological twist, this configuration has a single scalar nilpotent supercharge $\mathbf{Q}$. The 'off-shell' space of open strings stretching from $\mathcal{D}$ to itself is given by a graded vector space, which we denote $V_{\mathcal{D}}$ and there is an action of $\mathbf{Q}$ in this vector space, hence we can take the cohomology

$$\mathrm{End}(\mathcal{D}) := H_{\mathbf{Q}}(V_{\mathcal{D}}) \tag{3.21}$$

which is the space of physical states of the topological strings stretching between $\mathcal{D}$ and itself. If we denote ψ_a the elements of $\mathrm{End}(\mathcal{D})$, their disc correlators encode the Stasheff conditions (3.3). More precisely, the disk correlator of two elements (the boundary topological metric), denoted $\langle \psi_a, \psi_b \rangle$ equips $\mathrm{End}(\mathcal{D})$ with an (nondegenerate) inner product. In [19,20] it is found that the relation between the disk correlators with more than two insertions and the maps b_k

$$B_{i_0 i_1 \cdots i_k} := (-1)^{|a_1|+\ldots+|a_{k-1}|+k-1} \left\langle \psi_{i_0} \psi_{i_1} P \int \psi_{i_2}^{(1)} \cdots \int \psi_{i_{k-1}}^{(1)} \psi_{i_k} \right\rangle = \langle \psi_{i_0}, b_k(\psi_{i_1}, \cdots, \psi_{i_k}) \rangle, \tag{3.22}$$

[9]We say an A_∞-category has strict identities if, for every $A \in \mathrm{Ob}(\mathcal{A})$ there is a degree 0 element $1_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ satisfying the conditions (3.17), whenever it can be consistently inserted in a map m_n as defined in (3.20).

where $\psi_a^{(1)}$ denote the 1-form descendants of ψ_a. The correlators (3.22) are defined using an appropriate regulator [20] and they satisfy a cyclicity condition:

$$B_{i_0 i_1 \cdots i_k} = (-1)^{(|a_m|+1)(|a_0|+\ldots+|a_{k-1}|+k)} B_{i_k i_0 \cdots i_{k-1}}. \tag{3.23}$$

It is important to remark that on the right hand side of (3.22), the operators ψ_a should be considered in the space End($\mathcal{D}$)[1]. In other words, the graded space A is identified with End($\mathcal{D}$). Hence, up to a sign that, in general depends on the degree of the insertions, we can identify

$$B_{i_0 i_1 \cdots i_k} \sim \langle m_k(\psi_{i_0}, \cdots, \psi_{i_{k-1}}), \psi_{i_k} \rangle, \tag{3.24}$$

In general, for a SCFT we can define a trace function[10]

$$\gamma : A \to \mathbb{C} \tag{3.25}$$

of degree $-\hat{c} = -\frac{c}{3}$, where c is the central charge of the SCFT. Then the inner product can be written as

$$\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{C} \qquad (\psi_a, \psi_b) \mapsto \gamma(m_2(\psi_a, \psi_b)) \tag{3.26}$$

then, we simply write the relation

$$B_{i_0 i_1 \cdots i_k} = \gamma\left(m_2(m_k(\psi_{i_0}, \cdots, \psi_{i_{k-1}}), \psi_{i_k}) \right), \tag{3.27}$$

where the sign is hidden in γ. When considering multiple branes, this structure becomes an A_∞-category. For instance, in the case of a SCFT defined by the NLSM with a CY target space X, the category of B-branes (topological open strings in the B-model) is equivalent to $DCoh(X)$, the derived category of coherent sheaves on X [22–25] and an A_∞ structure on this category as been derived from physics and mathematical point of view [26–29]. Analogous results also exist in the case of G-equivariant categories of matrix factorizations $MF_G(W)$, when G is a finite abelian group and W is a quasi-homogeneous polynomial [20, 30, 31]

4 A_∞-algebras associated with Landau-Ginzburg models

In this section we will apply the results reviewed in section 3 to the specific case of LG orbifolds. We begin by reviewing the physics approach of categories of matrix factorizations, arising as B-branes on LG orbifolds. Fix a vector space $\mathbb{V}$ of rank N with coordinates denoted by x_i $i = 1, \ldots, N$. We specify a left R-symmetry given by a $\mathbb{C}_L^*$ action on $\mathbb{V}$ with weights $q_i \in \mathbb{Q} \cap (0, 1)$. The *orbifold group* will be specified by a finite abelian group G and a representation $\rho_{orb} : G \to GL(\mathbb{V})$. We specify a *superpotential*, that is a holomorphic, G-invariant function $W : \mathbb{C}^N \to \mathbb{C}$, $W \in \mathbb{C}[x_1, \ldots, x_N]$. As an $\mathcal{N} = (2, 2)$ theory, the LG orbifold is specified by the data

$$(W, G, \rho_{orb}, \mathbb{C}_L^*) \tag{4.1}$$

but we impose some extra requirement on (4.1). In order for the vector R-symmetry to be nonanomalous, we require W to be quasi-homogeneous, of weight 1 under a $\mathbb{C}_L^*$ i.e. $W(\lambda^{q_i}\phi_i) = \lambda W(\phi_i)$ [32] (this implies that W has charge 2 under the vector R-symmetry). Moreover, W being quasi-homogeneous implies $dW^{-1}(0) = \{0\}$ i.e. W is compact, in the sense that defines

[10] This is just the (twisted) correlator on the sphere. See for example the review [21].

a compact SCFT in the IR. Quasi-homogeneity of W guarantees that we always have the symmetry $x_j \to e^{2i\pi q_i} x_j$. If d denotes the lowest nonzero integer such that $dq_i \in \mathbb{Z}$ for all i, then this specifies a $\mathbb{Z}_d$ action generated by $J = \mathrm{diag}(e^{2i\pi q_1}, \ldots, e^{2i\pi q_N})$. Denote by $\mathrm{Aut}(W)$ the group of diagonal automorphisms of W, i.e.

$$\mathrm{Aut}(W) = \left\{ \mathrm{diag}(e^{2\pi i\lambda_1}, \ldots, e^{2\pi i\lambda_N}) \in U(1)^N : W(e^{2\pi i\lambda_i} x_i) = W(x_i) \right\}. \tag{4.2}$$

we then say an orbifold group G is *admissible* if it satisfies

$$\langle J \rangle \subseteq G \subseteq \mathrm{Aut}(W), \tag{4.3}$$

and we will require this condition. B-type D-branes $\mathcal{B}$ in LG orbifolds are characterized in terms of matrix factorizations of W [8,33]. More precisely, $\mathcal{B}$ consists of the data

$$\mathcal{B} = (M, \sigma, Q, R_M, \rho_M) \tag{4.4}$$

where M (the Chan-Paton space) is a free $\mathbb{C}[x_1, \ldots, x_N]$-module, σ is an involution on M, inducing a $\mathbb{Z}_2$-grading (so we can write $M = M_0 \oplus M_1$, with $\sigma M_i = (-1)^i M_i$) and $Q(x)$ is a $\mathbb{Z}_2$-odd endomorphism on M satisfying

$$Q^2 = W \cdot \mathrm{id}_M. \tag{4.5}$$

Under the vector R-charge, W has charge 2: $W(\lambda^{2q_i} x_i) = \lambda^2 W(x_i)$ with the charges q_i of the left R-symmetry. Therefore, by (4.5), Q must have vector R-charge 1. This defines a compatible representation $R_M : U(1)_V \to GL(M)$ of the vector R-symmetry, satisfying:

$$R_M(\lambda) Q(\lambda^{2q_i} x_i) R_M^{-1}(\lambda) = \lambda Q(x_i), \tag{4.6}$$

as well as another compatible representation of G, $\rho_M : G \to GL(M)$ satisfying

$$\rho_M(g)^{-1} Q(\rho_{orb}(g) \cdot x_j) \rho_M(g) = Q(x_j). \tag{4.7}$$

Given a pair of B-branes $\mathcal{B}_i = (M^{(i)}, \sigma_i, Q_i, R_M^{(i)}, \rho_M^{(i)})$, $i = 1, 2$ we can define the space of morphisms betwen them, $Hom(\mathcal{B}_1, \mathcal{B}_2)$ as graded morphisms

$$\Psi : M^{(1)} \to M^{(2)} \tag{4.8}$$

i.e. $\Psi \in V_{r_1,r_2} := Mat_{r_1,r_2}(\mathbb{C}[x_1, \ldots, x_N])$, the space of $r_1 \times r_2$ matrices with coefficients in $\mathbb{C}[x_1, \ldots, x_N]$, where $r_i = \mathrm{rk}(M^{(i)})$ satisfying

$$D_{12} \circ \Psi := Q_2 \Psi - \sigma_2 \Psi \sigma_1 Q_1 = 0 \tag{4.9}$$

modulo D_{12}-exact morphisms. The differential D_{12} can be identified with the conserved supercharge $\mathbf{Q}$ of the worldsheet theory on the open string stretching between $\mathcal{B}_1$ and $\mathcal{B}_2$. Therefore we can denote

$$Hom(\mathcal{B}_1, \mathcal{B}_2) = H_{D_{12}}(V_{r_1,r_2}). \tag{4.10}$$

The space $Hom(\mathcal{B}_1, \mathcal{B}_2)$ is $\mathbb{Z}_2$-graded and we denote its homogeneous components, and elements, as

$$Hom(\mathcal{B}_1, \mathcal{B}_2) = H^0(\mathcal{B}_1, \mathcal{B}_2) \oplus H^1(\mathcal{B}_1, \mathcal{B}_2) \qquad \phi_i \in H^0(\mathcal{B}_1, \mathcal{B}_2) \qquad \psi_i \in H^1(\mathcal{B}_1, \mathcal{B}_2) \tag{4.11}$$

The category $MF(W,G)$ of objects $\mathcal{B}$ with morphisms defined as

$$Hom_{MF(W,G)}(\mathcal{B}_1,\mathcal{B}_2) := Hom(\mathcal{B}_1,\mathcal{B}_2)^G \tag{4.12}$$

i.e., $\Psi \in Hom_{MF(W,G)}(\mathcal{B}_1,\mathcal{B}_2)$ satisfies

$$\rho_{M^{(2)}}(g)^{-1}\Psi(\rho_{orb}(g)\cdot x_i)\rho_{M^{(1)}}(g) = \Psi(x_i), \tag{4.13}$$

will be referred as the category of B-branes on the LG orbifold. This category also has a grading that we will review next.

Gradings

The category $MF(W,G)$ defined above has a natural $\mathbb{Q}$-grading given by the R-charge. More precisely, it is the fact that the superpotential W is quasi-homogeneous that guarantees the existence of this $\mathbb{Q}$-grading (because then the vector R-charge is conserved) [34]. The orbifold by G satisfying (4.3) guarantees that the physical states will have integer R-charges [35] and hence, we can put an integer grading on open string states. For a reduced and irreducible matrix factorization $\mathcal{B} \in MF(W,G)$, the map ρ_M satisfies [34]

$$\rho_M(J) = \sigma \circ R_M(e^{i\pi})e^{-i\pi\varphi} \tag{4.14}$$

for some $\varphi \in \frac{2}{d}\mathbb{Z}$. The morphism $\Psi \in Hom_{MF(W,G)}(\mathcal{B}_1,\mathcal{B}_2)$ has R-charge $q_\Psi \in \mathbb{Q}$ defined by

$$R_{M^{(2)}}(\lambda)\Psi(\lambda^{2q_i}x_i)R_{M^{(1)}}(\lambda)^{-1} = \lambda^{q_\Psi}\Psi(x_i), \tag{4.15}$$

Then, a $\mathbb{Z}$-grading on Ψ is defined by

$$\deg(\Psi) = \varphi_2 - \varphi_1 + q_\Psi \tag{4.16}$$

The category $MF(W,G)$ with this additional grading is known in the mathematics literature as the category of graded, G-equivariant matrix factorizations [36].

4.1 Effective superpotential, deformations and A_∞ structures

The category $MF(W,G)$ can be given an A_∞ structure [37], and the higher products can be read off from the computation of the unobstructed deformations of the objects $\mathcal{B} \in MF(W,G)$, as we will explain in this section, and will become useful later. However, it is very convenient to use a description of $MF(W,G)$ that follows very closely [8]. Consider first the case of a trivial orbifold

$$G = \mathbf{1}, \tag{4.17}$$

then we denote the category just as $MF(W)$. Then, in the case $dW^{-1}(0) = \{0\}$, this category has a single generator [38] given by the matrix factorization $\mathcal{B}_{D0} = (M,\sigma,Q_{D0},R_M)$

$$Q_{D0} = \sum_{i=1}^{N}\left(x_i\overline{\eta}_i + q_i\frac{\partial W}{\partial x_i}\eta_i\right), \tag{4.18}$$

where the subscript $D0$ is because this matrix factorization is reminiscent to the $D0$-brane in [8]. The objects $\overline{\eta}_i$, η_i, $i=1,\ldots,N$ are generators of a Clifford algebra of rank $2N$, namely they satisfy the relations

$$\{\overline{\eta}_i,\eta_j\} = \delta_{i,j}\mathbf{1} \qquad \{\overline{\eta}_i,\overline{\eta}_j\} = \{\eta_i,\eta_j\} = 0. \tag{4.19}$$

Then, we can consider

$$\mathcal{A}_{D0} := Hom_{MF(W)}(\mathcal{B}_{D0}, \mathcal{B}_{D0}), \tag{4.20}$$

which has the structure of a A_∞-algebra[11] [38] and moreover we have the equivalence

$$MF(W) \cong D(\text{Mod} - \mathcal{A}_{D0}) \tag{4.21}$$

where $D(\text{Mod} - \mathcal{A}_{D0})$ stands for the derived category of A_∞-modules over $\mathcal{A}_{D0}$. Given an object $\mathcal{B} \in MF(W)$, the module associated to $\mathcal{B}$ is given by $M_\mathcal{B} := Hom_{MF(W)}(\mathcal{B}_{D0}, \mathcal{B})$ where the maps

$$m_n^\mathcal{B} : M_\mathcal{B} \otimes \mathcal{A}_{D0}^{\otimes n-1} \to M_\mathcal{B} \tag{4.22}$$

come from the A_∞ structure of the category $MF(W)$, in particular

$$m_2^\mathcal{B} : M_\mathcal{B} \otimes \mathcal{A}_{D0} \to M_\mathcal{B} \qquad m_2^\mathcal{B}(\Psi^\mathcal{B}, \Psi) = \Psi^\mathcal{B} \circ \Psi. \tag{4.23}$$

When we add an orbifold, we expect the following equivalence

$$MF(W, G) \cong D(\text{Mod} - \mathcal{A}_{D0} \sharp G) \tag{4.24}$$

where $\mathcal{A}_{D0}\sharp G$ is the smash product between $\mathcal{A}_{D0}$ and the group algebra $\mathbb{C}[G]$, the product in $\mathcal{A}_{D0}\sharp G$ is given by [10]

$$(a\sharp g_1) \cdot (b\sharp g_2) = (a \cdot g_1 b g_1^{-1})\sharp g_1 g_2 \tag{4.25}$$

hence, studying the algebra $\mathcal{A}_{D0}$ is crucial. The higher products $m_n : \mathcal{A}_{D0}^{\otimes n} \to \mathcal{A}_{D0}$ can be read off from the effective superpotential $\mathcal{W}_{eff}$ defined by

$$\mathcal{W}_{\text{eff}} = \text{Tr}\left(\sum_{k=2}^{\infty} \sum_{i_0, i_1, \cdots, i_k} \frac{B_{i_0 \cdots i_k}}{k+1} Z_{i_0} Z_{i_1} \cdots Z_{i_k}\right), \tag{4.26}$$

the function $\mathcal{W}_{eff}$ encodes obstructions to the boundary deformations of the SCFT, and can be computed as follows. Consider the matrix factorization Q_{D0}, then our objective is to find a deformed matrix factorization

$$Q_{D0}^{\text{def}} = Q_{D0} + \sum_{\vec{m} \in B} \alpha_{\vec{m}} u^{\vec{m}} \qquad u^{\vec{m}} := \prod_{i=1}^{n} u_i^{m_i} \tag{4.27}$$

where $B \subset \mathbb{N}^n$, $n = \text{dim}H^1(\mathcal{B}_{D0}, \mathcal{B}_{D0})$, $\alpha_{\vec{m}}$ are fermionic operators and u_i, $i = 1, \dots, n$ are commutative parameters. The matrix factorization satisfies

$$(Q_{D0}^{\text{def}})^2 = W \cdot \text{id}_M + \sum_{i=1}^{n} f_i(u)\phi_i \tag{4.28}$$

where, using the same notation as (4.11), $\phi_i \in H^0(\mathcal{B}_{D0}, \mathcal{B}_{D0})$. Then, the critical locus of $\mathcal{W}_{\text{eff}}$ coincides, as a set, with $f_1 = f_2 = \cdots = f_n = 0$. More precisely, if we identify the variables Z_i with the parameters u_i, $Z_i \equiv u_i$ in $\mathcal{W}_{\text{eff}}$, then $d\mathcal{W}_{\text{eff}}^{-1}(0)$ coincides with the solutions to the equations $f_1 = f_2 = \cdots = f_n = 0$ i.e. we can integrate the equations

$$\frac{\partial \widetilde{\mathcal{W}_{\text{eff}}}}{\partial u_i} = f_i(u) \tag{4.29}$$

[11] In the particular case that W is homogeneous of degree 2, then $\mathcal{A}_{D0}$ becomes simply the (complex) Clifford algebra $Cl(q)$ associated with the quadratic form $q_{ij} := \partial_i \partial_j W$ [8, 39].

and $\widetilde{\mathcal{W}}_{\text{eff}}$ coincides with $\mathcal{W}_{\text{eff}}$ up to a nonlinear redefinition of the parameters u_i. The operators $\alpha_{\vec{m}}$ are computed iteratively. We can summarize this process as follows. Define $|\vec{m}| := \sum_i m_i$. We start by defining

$$\alpha_{e_i} := \psi_i \qquad i = 1, \ldots, n \tag{4.30}$$

with e_i, $i = 1, \ldots, n$ the cannonical basis of $\mathbb{N}^n$. Then, in the first step we write

$$Q_{D0}^{\text{def},(1)} = Q_{D0} + \sum_{i=1}^{n} u_i \alpha_{e_i} \tag{4.31}$$

and we look at the terms of order $|\vec{m}| = 2$ in $(Q_{D0}^{\text{def},(1)})^2$, denote them $\sum_{|\vec{m}|=2} y_{\vec{m}} u^{\vec{m}}$. Then, if $y_{\vec{m}}$ is Q_{D0}-exact, then we can define an operator $\alpha_{\vec{m}}$ (with $|\vec{m}| = 2$) such that

$$\beta_{\vec{m}} := -y_{\vec{m}} = [Q_{D0}, \alpha_{\vec{m}}]. \tag{4.32}$$

Then, if we denote B_2 the set of all vectors $\vec{m}$ with $|\vec{m}| = 2$ such that $y_{\vec{m}}$ is exact, we can write

$$Q_{D0}^{\text{def},(2)} = Q_{D0} + \sum_{i=1}^{n} u_i \alpha_{e_i} + \sum_{\vec{m} \in B_2} \alpha_{\vec{m}} u^{\vec{m}} \tag{4.33}$$

and repeat the process to find the operators $\alpha_{\vec{m}}$ with $|m| > 2$. The process ends when none of the $y_{\vec{m}}$ are Q_{D0}-exact.

4.2 LG model with homogeneous superpotential

In this section we consider the case of a LG model with chiral superfields $x_i, i = 1, \cdots, n$. The superpotential W is a homogeneous polynomial in x_i with degree $d \geq 2$. We will set the orbifold G to be trivial in this section, hence the relevant category of B-branes will be $MF(W)$. The $D0$-brane of this model is the matrix factorization $\mathcal{B}_{D0}$ described in the previous subsection (4.18), therefore

$$Q_{D0} = \sum_{i=1}^{n} \left(x_i \overline{\eta}_i + \frac{1}{d} \frac{\partial W}{\partial x_i} \eta_i \right). \tag{4.34}$$

We want to deduce the multiplication rule of the A_∞-algebra

$$\mathcal{A}_{D0} = \text{End}(\mathcal{B}_{D0}). \tag{4.35}$$

The generators of the ring $H^1(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ are straightforward to compute and given by[12]

$$\psi_i = i\sqrt{\frac{d(d-1)}{2}} \overline{\eta}_i - \frac{i}{\sqrt{2d(d-1)}} \sum_{j=1}^{n} \frac{\partial^2 W}{\partial x_i \partial x_j} \eta_j, \; 1 \leq i \leq n. \tag{4.36}$$

which satisfy $\{Q_{D0}, \psi_i\} = 0$. Note that

$$\{\psi_i, \psi_j\} = \frac{\partial^2 W}{\partial x_i \partial x_j} \mathbf{1} \tag{4.37}$$

[12]Equation (4.36) denotes a particular representative of the Q_{D0}-class of ψ_i. All the computations where the explicit matrix form of ψ_i is used do not depend on the choice of representative.

where $\mathbf{1} \in H^0(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ is the identity operator and $\{\psi_i, \psi_j\} \simeq 0$, for $d > 2$, i.e. (4.37) says that $\{\psi_i, \psi_j\}$ is Q_{D0}-exact, because $\{Q_{D0}, \eta_i\} = x_i$. Hence, any monomial in x_i with positive degree is Q_{D0}-exact. We remark that the operators (4.36) generate the whole $H^1(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ as a ring (not necessarily as a vector space). Indeed they generate the whole space $\mathrm{End}(\mathcal{B}_{D0})$. This can be shown, for instance, using the explicit form of Q_{D0} and the fact that the Dirac matrices $\eta_i, \bar{\eta}_i$, $i = 1 \ldots, n$ plus the identify $\mathbf{1}$ generate the off-shell dg algebra.

Next we propose an explicit expression for the functor from $MF(W)$ to the category of modules of $\mathcal{A}_{D0}$, for the case at hand. For any matrix factorization $\mathcal{B} = (M, \sigma_M, Q_M, R_M)$, the corresponding A_∞-module is given by $Hom(\mathcal{B}_{D0}, \mathcal{B})$. The A_∞-module structure is given by the A_∞-multiplications of the A_∞-category $MF(W)$ as described in (4.22). Conversely, given any $\mathbb{Z}_2$-graded A_∞-module $\mathbf{N}$ of $\mathcal{A}_{D0}$, we propose that the corresponding matrix factorization is given by $M = \mathbf{N} \otimes \mathbb{C}[x_1, \cdots, x_n]$ and

$$Q_M(\phi) = \sum_{k=1}^{d-1} \sum_{i_1, i_2, \cdots, i_k} m_{k+1}^{\mathbf{N}}(\phi, \psi_{i_1}, \cdots, \psi_{i_k}) x_{i_1} x_{i_2} \cdots x_{i_k} \tag{4.38}$$

for $\phi \in M$. We provide several consistency checks for (4.38) in Appendix A

The A_∞ structure of $\mathcal{A}_{D0}$ was constructed explicitly in [11], for the case W homogeneous of degree d. The A_∞-algebra relations were found to be given by (A.1) when $d = 2$ or (A.2) and (A.3) when $d > 2$. In [11], this A_∞ structure was proved by summing over the ribbon trees. In the remainder of this subsection, we give some alternative derivation of (A.1), (A.2) and (A.3) and we give explicit expressions for the higher products m_d, when acting on arbitrary elements of $\mathcal{A}_{D0}$. For this purpose we analyze separately the case $d = 2$ and $d > 2$. In the following we write the map

$$\iota : \mathcal{A}_{D0} = \mathrm{End}(\mathcal{B}_{D0}) \to V_{M_{D0}} \tag{4.39}$$

where $V_{M_{D0}}$ is the space of $\mathrm{rk}(M)$-square matrices with values in $\mathbb{C}[x_1, \ldots, x_n]$ i.e., the space of endomorphisms of $\mathcal{B}_{D0}$ without taking the homology. We can always give to the algebra $V_{M_{D0}}$ a dg algebra structure and will not spoil the A_∞-relations of $\mathcal{A}_{D0}$. In other words, $V_{M_{D0}}$ is the off-shell algebra of open strings and we can always find a dg algebra that is A_∞-quasi-isomprphic to it (this fact is true for any A_∞-algebra [15]). We will denote the image under ι of ψ_i in $V_{M_{D0}}$ by v_i:

$$\iota(\psi_i) = v_i. \tag{4.40}$$

In the following we will also make use of the open disk one-point function correlators on the disk, for LG models. This was computed in [40] (see [41, 42] for a mathematical treatment) and is given by

$$\langle \Phi \rangle = \frac{1}{(2\pi i)^n} \oint_{x_i=0} \frac{\mathrm{Str}\left(\frac{\partial \mathrm{Q_{D0}}}{\partial \mathrm{x_1}} \wedge \cdots \wedge \frac{\partial \mathrm{Q_{D0}}}{\partial \mathrm{x_n}} \Phi\right)}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_n}} dx_1 \cdots dx_n. \tag{4.41}$$

This corresponds to the (B-model) correlator in D^2 with a single boundary insertion $\Phi \in \mathcal{A}_{D0}$ and boundary conditions defined by the brane $\mathcal{B}_{D0}$.

4.2.1 $\mathcal{A}_{D0}$ for $d = 2$

From the relations (3.6) and (3.19) (i.e. $f_2 : \mathcal{A}_{D0}^{\otimes 2} \to V_{M_{D0}}$):

$$\iota(m_2(\psi_i, \psi_j)) = v_i v_j + \{Q_{D0}, f_2(\psi_i, \psi_j)\}. \tag{4.42}$$

When $d = 2$, $v_i v_j$ has no Q_{D0}-exact terms, therefore we can choose $f_2(\psi_i, \psi_j) = 0$. Consequently,

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}, \tag{4.43}$$

i.e. under the multiplication m_2, $\mathcal{A}_{D0}$ is the same as the Clifford algebra $Cl(n, \mathbb{C})$ with the quadratic form given by the Hessian of W. (4.43) can also be obtained by computing the correlation functions, using (4.41). We illustrate this case with a simple example:

Example: $W = x_1 x_2$

The D0-brane is

$$Q = x_1 \overline{\eta}_1 + x_2 \overline{\eta}_2 + \frac{x_2}{2}\eta_1 + \frac{x_1}{2}\eta_2. \tag{4.44}$$

The fermionic open string states are

$$\psi_1 = \overline{\eta}_1 - \frac{1}{2}\eta_2, \quad \psi_2 = \overline{\eta}_2 - \frac{1}{2}\eta_1. \tag{4.45}$$

The bosonic open string states can be taken to be[13] $e = 1$ and ϕ such that $\langle e, \phi \rangle = 1$ and $\langle e, e \rangle = \langle \phi, \phi \rangle = 0$. One can compute

$$\gamma(m_2(\psi_1, \psi_2)) = \langle \psi_1 \psi_2 \rangle = \frac{1}{(2\pi i)^2} \oint_{x_1 = x_2 = 0} \frac{\mathrm{Str}\left(\frac{\partial Q}{\partial x_1}\frac{\partial Q}{\partial x_2}\psi_1\psi_2\right)}{\frac{\partial W}{\partial x_1}\frac{\partial W}{\partial x_2}} = 1, \tag{4.46}$$

from which we also deduce

$$\gamma(m_2(m_2(\psi_1, \psi_2), e)) = 1.$$

Furthermore,

$$\gamma(m_2(m_2(\psi_1, \psi_2), \psi_1\psi_2)) = \langle \psi_1\psi_2\psi_1\psi_2 \rangle = \frac{1}{(2\pi i)^2} \oint_{x_1 = x_2 = 0} \frac{\mathrm{Str}\left(\frac{\partial Q}{\partial x_1}\frac{\partial Q}{\partial x_2}\psi_1\psi_2\psi_1\psi_2\right)}{\frac{\partial W}{\partial x_1}\frac{\partial W}{\partial x_2}} = 1,$$

thus

$$m_2(\psi_1, \psi_2) = e + \phi.$$

The same computation shows

$$m_2(\psi_1, \psi_1) = m_2(\psi_2, \psi_2) = 0.$$

4.2.2 $d > 2$

Now assume the degree of W is greater than 2. Because $\{v_i, v_j\} = \frac{\partial^2 W}{\partial x_i \partial x_j}$ is Q_{D0}-exact, we can take f_2 such that

$$\iota(m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i)) = \{v_i, v_j\} + \{Q_{D0}, f_2(\psi_i, \psi_j) + f_2(\psi_j, \psi_i)\} = 0,$$

and therefore

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0, \tag{4.47}$$

[13] A representative for ϕ can be taken to be, for instance, $\bar{\eta}_1\bar{\eta}_2 - \frac{1}{2}\bar{\eta}_1\eta_1 - \frac{1}{2}\bar{\eta}_2\eta_2 + \frac{1}{4}\eta_1\eta_2$

which means that $\mathcal{A}_{D0}$ is the exterior algebra $\wedge^\bullet \mathbb{C}^n$ under the multiplication m_2.

Alternatively, $m_2(\psi_i, \psi_j)$ can be determined by the correlation functions (4.41). First note that the one-point correlator

$$\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \quad \text{if } i_s = i_t \tag{4.48}$$

for any pair of indices i_s and i_t. This is simply because $\{\psi_i, \psi_j\}$ and ψ_i^2 are Q_{D0}-exact and as a consequence the correlation function can be rewritten as a sum of correlation functions, each involving a Q_{D0}-exact operator. A corollary of this observation is that

$$\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \quad \text{if } m > n. \tag{4.49}$$

Now assume that $m \leq n$. The formula (4.41) implies

$$\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = \frac{1}{(2\pi i)^n} \oint_{x_i = 0} \frac{\mathrm{Str}\left(\frac{\partial \mathrm{Q_{D0}}}{\partial \mathrm{x_1}} \wedge \cdots \wedge \frac{\partial \mathrm{Q_{D0}}}{\partial \mathrm{x_n}} \psi_{\mathrm{i_1}} \cdots \psi_{\mathrm{i_m}} \right)}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_n}} dx_1 \cdots dx_n. \tag{4.50}$$

The degree of the denominator of the integrand is $nd - n$. In order to have a nonzero result, the numerator of the integrand must have degree $nd - 2n$, which can only result from the term in $\frac{\partial Q_{D0}}{\partial x_1} \wedge \cdots \wedge \frac{\partial Q_{D0}}{\partial x_n} \psi_{i_1} \cdots \psi_{i_m}$ proportional to $\eta_1 \cdots \eta_n$. But in order to make nonzero contribution to the supertrace, there should also be n $\overline{\eta}$'s to contract with the $\eta' s$, this is possible only when $m = n$. In conclusion,

$$\langle \psi_{i_1} \cdots \psi_{i_m} \rangle = 0 \quad \text{if } m < n, \tag{4.51}$$

and it is straightforward to compute from (4.41) that

$$\langle \psi_1 \psi_2 \cdots \psi_n \rangle = 1 \tag{4.52}$$

up to a normalization factor, which implies $\psi_{i_1} \cdots \psi_{i_m}$ is dual to $\pm \prod_{j \neq i_s, 1 \leq s \leq m} \psi_j$. Combining (4.48), (4.49) and (4.51), we see that for a monomial f in ψ_i

$$\gamma(m_2(m_2(\psi_i, \psi_j), f(\psi_1, \cdots, \psi_n))) = \pm \langle \psi_i \psi_j f(\psi_1, \cdots, \psi_n) \rangle \tag{4.53}$$

vanishes unless $f(\psi_1, \cdots, \psi_n) = \pm \prod_{k \neq i,j} \psi_k$. This allows us to conclude that $\iota(m_2(\psi_i, \psi_j))$ only contains the term dual to $\prod_{k \neq i,j} v_k$, i.e.

$$m_2(\psi_i, \psi_j) = \frac{1}{2} \pi([v_i, v_j]). \tag{4.54}$$

where $\pi : V_{M_{D0}} \to \mathcal{A}_{D0}$ is the projection onto Q_{D0}-classes.

Now we compute the higher order multiplications $m_k, k > 2$. In principle, this can be done by performing the algorithm described in section 3 using (3.6). Here we take the physical perspective and determine the multiplications by studying the deformations of Q_{D0} as reviewed in section 4.1.

Assume that we deform Q_{D0} using the fermionic generators ψ_i

$$Q_{D0}^{\mathrm{def}} = Q_{D0} + \sum_{\vec{m}: |\vec{m}| > 0} \alpha_{\vec{m}} u^{\vec{m}}, \tag{4.55}$$

where $\alpha_{e_i} = \psi_i/(\sqrt{d(d-1)})$. In principle, we can consider further deformations by other elements of $H^1(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ (and even elements of $H^0(\mathcal{B}_{D0}, \mathcal{B}_{D0})$), but if we are interested in extracting the higher products involving only ψ_i operators, this suffices. Indeed, if the most general first order deformation (i.e. $|m| = 1$) has the form

$$\sum_i u_i \psi_i + \sum_\mu u_\mu \Lambda_\mu \tag{4.56}$$

where Λ_μ denote the operators in $H^1(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ that are not ψ_i's, then if we set $u_\mu = 0$, after running the algorithm outlined in section 4.1 we will get that $(Q_{D0}^{\text{def}})^2$ has the form

$$(Q_{D0}^{\text{def}})^2 = W \cdot \mathbf{1} + \sum_a f_a(u_i) \phi_a \tag{4.57}$$

where f_a's will have the following interpretation

$$f_a(u_i) = \frac{\partial \widetilde{W}_{eff}}{\partial u_a}\Big|_{u_\mu = 0} = \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \langle m_k(\psi_{i_1}, \ldots, \psi_{i_k}), \Lambda_a^D \rangle u_{i_1} \cdots u_{i_k} \tag{4.58}$$

where a runs over all operators in $\mathcal{A}_{D0}$ and Λ_a^D denotes the operator dual to $\Lambda_a \in \{\psi_i, \Lambda_\mu\}$. Hence, (4.57) will contain all the information we need about the higher products $m_k(\psi_{i_1}, \ldots, \psi_{i_k})$, when we set $u_\mu = 0$.

Define $\alpha_{i_1 \cdots i_s} := \alpha_{e_{i_1} + \cdots + e_{i_s}}$, $\beta_{i_1 \cdots i_s} := \beta_{e_{i_1} + \cdots + e_{i_s}}$ and $W_{i_1 \ldots i_s} := \partial_{i_1} \cdots \partial_{i_s} W$. Then we have

$$\beta_{ij} = \{\alpha_i, \alpha_j\} = \frac{1}{d(d-1)} W_{ij}.$$

As $d > 2$, β_{ij} is Q_{D0}-exact: $d(d-1)\beta_{ij} = W_{ij} = \{Q_{D0}, \sum_k W_{ijk}\eta_k\}/(d-2)$. Then we can take $\alpha_{ij} = -1 \sum_k W_{ijk}\eta_k/(d(d-1)(d-2))$ to cancel β_{ij} in $(Q_{D0}^{\text{def}})^2$. Then at degree 3, one computes $\beta_{ijk} = -W_{ijk}/(d(d-1)(d-2))$ and $\alpha_{ijk} = \sum_l W_{ijkl}\eta_l/(d(d-1)(d-2))$. This process continues and at degree m we have

$$\beta_{i_1 \cdots i_m} = (-1)^m \frac{W_{i_1 \cdots i_m}}{d(d-1) \cdots (d-m+1)} \tag{4.59}$$

and

$$\alpha_{i_1 \cdots i_m} = (-1)^{m-1} \frac{\sum_j W_{i_1 \cdots i_m j} \eta_j}{d(d-1) \cdots (d-m+1)} \tag{4.60}$$

for $2 \leq m \leq d$. In particular, $\beta_{i_1 \cdots i_d}$ is not Q_{D0}-exact and cannot be cancelled by a choice of $\alpha_{i_1 \cdots i_d}$. As a result,

$$(Q_{D0}^{\text{def}})^2 = Q_{D0}^2 + \frac{(-1)^d}{d!} \sum_{r_1 + \cdots + r_n = d} \frac{\partial^d W}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} u_1^{r_1} u_2^{r_2} \cdots u_n^{r_n} \mathbf{1}. \tag{4.61}$$

This means the obstruction to the deformation is given by the identity operator. Therefore $\widetilde{\mathcal{W}}_{eff}$ takes the form

$$\widetilde{\mathcal{W}}_{eff} = \sum_{i_1, \ldots, i_d} \langle m_d(\psi_{i_1}, \ldots, \psi_{i_d}), \Lambda \rangle u_{i_1} \cdots u_{i_d} u_0 + \mathcal{O}(u_0^2) \qquad \Lambda = \psi_1 \cdots \psi_n \tag{4.62}$$

where Λ is the dual to the identity operator. Due to the correlation function (4.52) and (4.61) together with (4.62) we conclude that

$$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) + \text{cyclic permutations} = \frac{(-1)^d}{d!} W_{i_1 \cdots i_d}, \tag{4.63}$$

and

$$m_k(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k}) + \text{cyclic permutations} = 0 \qquad k \neq d \tag{4.64}$$

where the i_l's are not necessarily distinct. Using directly the algorithm outlined in 3, and (3.6)[14] we can further determine exactly all the higher products m_k. This computation ends up determining the A_∞ relations as

- If $d = 2$, $\mathcal{A}_{D0}$ is a family of Clifford algebras:

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}. \tag{4.65}$$

- If $d > 2$, $\mathcal{A}_{D0}$ is an A_∞-algebra, where m_2 is the wedge product, $m_k = 0$ for $k = 1, 3, 4, \cdots, d-1$ and

$$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \frac{1}{d!} \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}. \tag{4.66}$$

We illustrate with an example the computation of m_k, exactly, for $d = 3$ (using the relations (3.6) and (3.19)):

$$\{v_i, v_j\} = W_{ij} = \{Q, \sum_k W_{ijk} \eta_k\}.$$

Because

$$v_i v_j = \frac{1}{2}\{v_i, v_j\} + \frac{1}{2}[v_i, v_j] = \frac{1}{2} W_{ij} + \frac{1}{2}[v_i, v_j],$$

we have

$$m_2(\psi_i, \psi_j) = \frac{1}{2}\pi\left([v_i, v_j]\right), \quad f_2(\psi_i, \psi_j) = -\frac{1}{2}\sum_k W_{ijk}\eta_k,$$

Thus $m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0$.

$$v_j v_i v_k = \{v_i, v_j\} v_k - v_i v_j v_k = W_{ij} v_k - W_{jk} v_i + v_i v_k v_j$$

yields

$$v_i [v_j, v_k] - [v_i, v_j] v_k = \{Q, -\sum_l W_{jkl} \eta_l v_i - \sum_l W_{ijl} v_k \eta_l\}.$$

Therefore

$$f_2(\psi_i, m_2(\psi_j, \psi_k)) - f_2(m_2(\psi_i, \psi_j), \psi_k) = \frac{1}{2}(\sum_l W_{ijl} v_k \eta_l + \sum_l W_{jkl} \eta_l v_i).$$

$$\iota(m_3(\psi_i, \psi_j, \psi_k)) = f_2(\psi_i, m_2(\psi_j, \psi_k)) - f_2(m_2(\psi_i, \psi_j), \psi_k) - v_i f_2(\psi_j, \psi_k) - f_2(\psi_i, \psi_j) v_k$$
$$= \frac{1}{2}\sum_l (W_{ijl} v_k \eta_l + W_{jkl} \eta_l v_i + W_{jkl} v_i \eta_l + W_{ijl} \eta_l v_k) = \frac{1}{2}(W_{ijl}\delta_{lk} + W_{jkl}\delta_{li}) = W_{ijk}.$$

[14] [11] used a different approach, namely summing the ribbon trees. We review this idea in the appendix B

In conclusion, $m_3(\psi_i, \psi_j, \psi_k) = W_{ijk}$.

Therefore, in general, we can say that all the elements of $\mathcal{A}_{D0}$ can be written as linear combinations of the form ($d > 2$)

$$\Lambda_i = \psi_1 \wedge \cdots \wedge \psi_{i_r} \qquad \deg(\Lambda_i) := r \tag{4.67}$$

where we defined $\deg(\Lambda_i)$ for later convenience and $\wedge$ denotes the usual skew-symmetric wedge product. The fact that all the elements can be write as in the formula (4.67) is just a consequence of (4.47). Now, we can determine $m_d(\Lambda_1, \Lambda_2, \cdots, \Lambda_d)$ for $\Lambda_i = \psi_{i_1} \wedge \cdots \wedge \psi_{i_{\deg(\Lambda_i)}}$. Since $m_k = 0$ for $k \neq 2, d$, the relation (3.3) can be solved by the rule[15]

$$\begin{aligned}
&m_k(\Lambda_1, \Lambda_2, \cdots, m_2(\Lambda_i, \Lambda_{i+1}), \cdots, \Lambda_{k+1}) \\
=&(-1)^{\deg(\Lambda_i)(\deg(\Lambda_{i+1})+\cdots+\deg(\Lambda_{k+1}))} m_2(m_k(\Lambda_1, \cdots, \Lambda_{i-1}, \Lambda_{i+1}, \cdots, \Lambda_{k+1}), \Lambda_i) \\
&+ (-1)^{\deg(\Lambda_{i+1})(\deg(\Lambda_{i+2})+\cdots+\deg(\Lambda_{k+1}))} m_2(m_k(\Lambda_1, \cdots, \Lambda_i, \Lambda_{i+2}, \cdots, \Lambda_{k+1}), \Lambda_{i+1}).
\end{aligned} \tag{4.68}$$

By repeated use of (4.68), we are lead to the conclusion that

$$m_d(\psi_{i_0} \wedge \cdots \wedge \psi_{i_{t_1}}, \psi_{j_0} \wedge \cdots \wedge \psi_{j_{t_2}}, \cdots, \psi_{k_0} \wedge \cdots \wedge \psi_{k_{t_d}}), \tag{4.69}$$

is equal to the sum

$$\begin{aligned}
&\sum_{a_1=0}^{t_1} \sum_{a_2=0}^{t_2} \cdots \sum_{a_d=0}^{t_d} m_d(\psi_{i_{a_1}}, \psi_{j_{a_2}}, \cdots, \psi_{k_{a_d}})(-1)^{a_1+\cdots+a_d}. \\
&\psi_{i_0} \wedge \cdots \hat{\psi}_{i_{a_1}} \cdots \wedge \psi_{i_{t_1}} \wedge \psi_{j_0} \wedge \cdots \hat{\psi}_{j_{a_2}} \cdots \wedge \psi_{j_{t_2}} \wedge \cdots \wedge \psi_{k_0} \wedge \cdots \hat{\psi}_{k_{a_d}} \cdots \wedge \psi_{k_{t_d}}
\end{aligned} \tag{4.70}$$

up to an overall sign.

We finish this section with a very simple example to illustrate the consistency between the $\widetilde{\mathcal{W}}_{eff}$ computation and the relations (4.65) and (4.66):

Example: $W = x^d$

Let's consider the LG model with a single chiral superfield x and a superpotential $W = x^d$, $d \geq 2$. The D0-brane is gven by

$$Q = x\overline{\eta} + x^{d-1}\eta. \tag{4.71}$$

The bosonic open string state is $e = 1$ and the fermionic open string state is $\psi = \overline{\eta} - \eta$. One can use the Kapustin-Li formula to compute the three-point correlation function

$$\langle \psi\psi\psi \rangle = \frac{1}{2\pi i} \oint_{x=0} \frac{\mathrm{Str}\left(\frac{\mathrm{dQ}}{\mathrm{dx}} \psi\psi\psi\right)}{\frac{dW}{dx}} = \begin{cases} 1, & d = 2, \\ 0, & d > 2. \end{cases}$$

From the relation

$$\langle \psi\psi\psi \rangle = \gamma\left(m_2(m_2(\psi, \psi), \psi)\right)$$

and

$$\gamma(e) = 0, \quad \gamma(\psi) = 1,$$

[15] Notice that $\Lambda_1 m_k(\Lambda_2, \cdots, \Lambda_{k+1}) = (-1)^{\deg(\Lambda_1)(\deg(\Lambda_2)+\cdots+\deg(\Lambda_{k+1})+k)} m_k(\Lambda_2, \cdots, \Lambda_{k+1})\Lambda_1$.

we see

$$m_2(\psi, \psi) = \begin{cases} e, & d = 2, \\ 0, & d > 2. \end{cases}$$

It was shown in [43] that the effective superpotential, or disk partition function, of the LG model with Dirichlet boundary condition, which is equivalent to (4.71), is

$$\mathcal{W}_{\text{eff}} = \text{Tr}\left(\frac{Z^{d+1}}{d+1}\right)$$

up to a rescaling, where Z is the world volume field dual to ψ. From this effective superpotential we conclude

$$\gamma\left(m_2(m_s(\psi^{\otimes s}), \psi)\right) = \begin{cases} 1, & s = d, \\ 0, & s \neq d. \end{cases}$$

or equivalently

$$m_s(\psi^{\otimes s}) = \begin{cases} e, & s = d, \\ 0, & s \neq d. \end{cases}$$

Finally, we remark that in principle we can study the correspondence between $MF(W)$ and $D(\text{Mod} - \mathcal{A}_{D0})$ from the point of view of tensor products of minimal models. A homogeneous superpotential $W \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d at a special point in complex structure moduli can be seen as the tensor product of n A_{d-1} minimal models. This relates to the well known structure of tensor products in matrix factorization categories. It will be interesting to study further how this tensor product structure translates to the category $D(\text{Mod} - \mathcal{A}_{D0})$ as is well known that tensor products of A_∞-algebras is rather nontrivial [44–46].

4.3 Inhomogeneous superpotential

Consider now a quasi-homogeneous superpotential $W \in \mathbb{C}[x_1, \ldots, x_N]$. Then, we write the superpotential as a sum of homogeneous polynomial-degree terms:

$$W(x) = \sum_{l=2}^{d} W^{(l)}(x), \tag{4.72}$$

where each term $W^{(l)}(x)$ has polynomial-degree l i.e., where we assign degree 1 to each variable x_i. The brane $\mathcal{B}_{D0}$ and the fermionic generators of the open string states are still given by (4.34) and (4.36) respectively. If we turn on deformations as in (4.55), we can use the same argument[16] to deduce that the obstruction is given by

$$\frac{(-1)^l}{l!} \sum_{i_1+\cdots+i_n=l} \frac{\partial^l W^{(l)}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n} \tag{4.73}$$

at degree l. Therefore we have the following multiplications

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W^{(2)}}{\partial x_i \partial x_j} \tag{4.74}$$

and

$$m_l(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_l}) = \frac{1}{l!} \frac{\partial^l W^{(l)}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}} \tag{4.75}$$

for $3 \leq l \leq d$ and $m_k = 0$ for $k > d$.

[16] Also one can apply an argument based on ribbon trees as reviewed in Appendix B.

4.4 Landau-Ginzburg Orbifold

So far we have considered LG models with trivial orbifold gorup. For a LG orbifold, there is a finite abelian group G acting on the field space, and $W \in \mathbb{C}[x_1, \ldots, x_n]^G$. As such, the open string states are those invariant under the action of G. Let $\psi_1, \cdots, \psi_n$ be the degree-one fermionic generators (4.36) in $\mathrm{End}_{MF(W)}(\mathcal{B}_{D0})$ of the LG model without orbifolding. Note that each ψ_1 transforms in a definite representation, when we take the G action into account. When we incorporate the orbifold the brane defined by the matrix factorization Q_{D0} in (4.34) requires the specification of a representation ρ_M, compatible with ρ_{orb}. It is easy to see that $\rho_M : G \to GL(M)$ is almost completely fixed by its action, via conjugation, over $\eta_j, \bar{\eta}_j$, in the definition of Q_{D0}. Then ρ_M is fixed up to its action on the Clifford vacuum $|0\rangle$ (defined by $\eta_j|0\rangle = 0$ for all j). Hence, we can label $\mathcal{B}_{D0}^{(a)}$ in the category $MF(W, G)$ by a single (one-dimensional) irreducible representation[17]: the representation of $|0\rangle$. Then, if $G = \mathbb{Z}_d$, we have $a = 0, \ldots, d-1$ and we denote

$$\mathcal{B}_{orb} := \bigoplus_{a=0}^{d-1} \mathcal{B}_{D0}^{(a)} \in MF(W, G), \tag{4.76}$$

where the set of branes $\mathcal{B}_{D0}^{(a)}$, $a = 0, \ldots, d-1$ form a set of generators of $MF(G, W)$ [10] and the algebra $\mathcal{A}_{D0}$ must be replaced by

$$\mathcal{A}_{orb} = \mathrm{End}_{MF(W,G)}(\mathcal{B}_{orb}). \tag{4.77}$$

The correspondence $MF(W, G) \cong D(Mod - \mathcal{A}_{orb})$ was studied in [47], for homogeneous potentials. Moreover, the results in [10] implies the following isomorphism of A_∞ algebras:

$$\mathcal{A}_{orb} \cong \mathcal{A}_{D0} \sharp G \tag{4.78}$$

where $\mathcal{A}_{D0}\sharp G$, the smash product of $\mathcal{A}_{D0}$ and $\mathbb{C}[G]$, is regarded as an A_∞-algebra with $m_2(g_1, g_2) = g_1 \cdot g_2$ and $m_k = 0$ for $k \neq 2$. Then $\mathcal{A}_{orb}$ can be regarded as the product of two A_∞-algebras. We can still use the construction introduced in section 4.2 to set up the correspondence between objects of $MF(W, G)$ and the A_∞-modules over $\mathcal{A}_{D0}\sharp G$. The difference is that the module is not only an A_∞-module of $\mathcal{A}_{D0}$, but also a $\mathbb{C}G$-module, this corresponds to the fact that the Chan-Paton spaces of the matrix factorizations of LG orbifold all carries a G-representation.

Specifically for the case $G = \mathbb{Z}_d$. Because the multiplication $m_2^{\mathcal{A}_{D0}\sharp\mathbb{Z}_d}$ of $\mathcal{A}_{D0}\sharp\mathbb{Z}_d$ satisfies

$$m_2^{\mathcal{A}_{D0}\sharp\mathbb{Z}_d}(a\sharp g_1, b\sharp g_2) = m_2^{\mathcal{A}_{D0}}(a, g_1 b g_1^{-1})\sharp(g_1 \cdot g_2), \tag{4.79}$$

an $\mathcal{A}_{D0}\sharp\mathbb{Z}_d$-module is of the form $\oplus_{i=0}^{d-1} M_i \otimes \rho_{\lambda+i}$, where $\oplus_{i=0}^{d-1} M_i$ is a $\mathbb{Z}_d$-graded $\mathcal{A}_{D0}$-module and ρ_l denotes the one-dimensional representation of $\mathbb{Z}_d$ with weight $\exp(2\pi i l/d)$.

For example, when $d = 2$, $\mathcal{A}_{D0}$ is a Clifford algebra $Cl(n, \mathbb{C})$, hence if $G = \mathbb{Z}_2$ we have

$$MF(W, \mathbb{Z}_2) \cong D(Mod - \mathcal{A}_{D0}\sharp\mathbb{Z}_2). \tag{4.80}$$

The category $MF(W, \mathbb{Z}_2)$ is very similar to the graded category $MF(W)$ but its morphisms are diferent. Because of the $\mathbb{Z}_2$ orbifold all the morphisms between irreducible objects are

[17] These are sometimes called the orbit branes related to $\mathcal{B}_{D0}$ [47, 48].

either even or odd, but not both. This is exactly the category studied in [49] and so, we can use the results in [49] to conclude

$$MF(W, \mathbb{Z}_2) \cong D(Mod_{\mathbb{Z}_2} - Cl(n, \mathbb{C})) \tag{4.81}$$

where $D(Mod_{\mathbb{Z}_2} - Cl(n, \mathbb{C}))$ denotes the derived category of graded modules over $Cl(n, \mathbb{C})$. A classical result of Atiyah-Bott-Shapiro [50] (see also [49]) establish

$$MF(W, \mathbb{Z}_2) \cong D(Mod_{\mathbb{Z}_2} - Cl(n, \mathbb{C})) \cong D(Mod - Cl_0(n, \mathbb{C})) \tag{4.82}$$

where $Cl_0(n, \mathbb{C})$ denotes the even part of the Clifford algebra $Cl(n, \mathbb{C})$. We remark that the category considered in [8] is the category $MF(W)$ where the morphisms are odd and even, and hence is equivalent to $D(Mod - Cl(n, \mathbb{C}))$, where the modules are not graded. Finally, we illustrate the correspondence (4.80) with an example. Set $W = \sum_{i=1}^{2m} x_i^2$ and each x_i is $\mathbb{Z}_2$-odd. Let S_+ and S_- be the spinor representation with left and right chirality respectively. The $\mathcal{A}_{D0}\sharp\mathbb{Z}_2$ module $M := S_+ \otimes \rho_0 \oplus S_- \otimes \rho_1$ corresponds to the matrix factorization with $Q_M = \sum_{i=1}^{m}(x_{2i-1} + ix_{2i})\overline{\eta}_i + \sum_{i=1}^{m}(x_{2i-1} - ix_{2i})\eta_i$ and the vacuum being $\mathbb{Z}_2$-even. The $\mathcal{A}_{D0}\sharp\mathbb{Z}_2$ module $S_+ \otimes \rho_1 \oplus S_- \otimes \rho_0$ corresponds to the matrix factorization with the same Q_M but the vacuum being $\mathbb{Z}_2$-odd ($\overline{\eta}_i$ and η_i are $\mathbb{Z}_2$-odd in both cases).

4.5 Hybrid Model

Finally, we consider a hybrid model (for details on the precise definition of hybrid models see [2]), defined on a space of the form

$$Y := \mathrm{Tot}\left(\mathcal{V} \to B\right)/G \tag{4.83}$$

with superpotential $W(x, p) \in H^0(\mathcal{O}_Y)$, where $\mathcal{V}$ is a G-equivariant vector bundle over the base space B, x and p are fibre and base coordinates respectively. Assume W is quasi-homogeneous with degree d in x. As discussed above, at each fixed point $p_0 \in B$, the matrix factorizations of the LG model on the fiber $\mathcal{V}|_{p_0}$ with superpotential $W(x, p_0)$ is in one-to-one correspondence with the A_∞-modules of the A_∞-algebra $\mathcal{A}_{p_0}\sharp G$, where $\mathcal{A}_{p_0}$ has A_∞-products given by (4.65) and (4.66) with $W = W(x, p_0)$. Each $\mathcal{A}_{p_0}$ can be viewed as the stalk at p_0 of a sheaf of A_∞-algebra $\mathcal{A}$ over B. Therefore, we have the equivalence between the category of matrix factorizations of the hybrid model and the derived category of the noncommutative space:

$$MF_{(\mathcal{V},B)}(W, G) := D(Y, W) = D(B, \mathcal{A}\sharp G), \tag{4.84}$$

where the latter is the derived category of A_∞-modules of the sheaf of A_∞-algebra $\mathcal{A}\sharp G$. In the next section, we apply this result to homological projective duality.

5 Examples of categories of B-branes on HPD phases

In this section we apply the results from the previous section to HPD constructed from GLSMs from [7]. The Higgs branch category $\mathcal{C}$ defined in (2.6) takes the form

$$\mathcal{C} = D(\widehat{Y}_{\zeta_{\mathcal{L}} \ll -1}, \widehat{W}_{\zeta_{\mathcal{L}} \ll -1}) \tag{5.1}$$

which, generically corresponds to a hybrid model as the ones reviewed in section 4.5. We analyze the following examples in detail:

- Degree d Veronese embeddings.

- Fano complete intersections in $\mathbb{P}^n$.

and we will particularly focus on the case of Fano hypersurfaces.

5.1 HPD of Veronese embedding

As reviewed in section 2, the HPD category (2.6) of degree-d Veronese embedding of $\mathbb{P}^n = \mathbb{P}(V)$ ($\dim V = n+1$), can be described by the category of B-branes on hybrid model with target space[18] [7]

$$\mathrm{Tot}\left(\mathcal{O}\left(-\frac{1}{d}\right)^{\oplus(n+1)} \to \mathbb{P}^{\binom{n+d}{d}-1}\right)/\mathbb{Z}_d \tag{5.2}$$

with superpotential

$$W = \sum_{a=1}^{\binom{n+d}{d}} s_a f_a(x), \tag{5.3}$$

where f_a's are the degree d monomials in x_i, $i = 0, \cdots, n$ and s_a are homogeneous coordinates in $\mathbb{P}^{\binom{n+d}{d}-1}$. One can interpret the category of matrix factorizations of this LG model as the derived category of a noncommutative space $D(\mathbb{P}^{\binom{n+d}{d}-1}, \mathcal{A}_{D0}\sharp\mathbb{Z}_d)$, i.e. the category of sheaves of modules over the sheaf of A_∞-algebras $\mathcal{A}_{D0}\sharp\mathbb{Z}_d$. So, we want to determine $\mathcal{A}_{D0}$. Let us start by denoting by Q_{D0} the matrix factorization corresponding to the $D0$-brane $\mathcal{B}_{D0}^{(0)}$ given by

$$Q_{D0} = \sum_i \left(x_i\overline{\eta}_i + \frac{1}{d}\frac{\partial W}{\partial x_i}\eta_i\right), \tag{5.4}$$

where we have chosen the trivial $\mathbb{Z}_d$ representation for the Clifford vaccum. Then, at a generic point $p \in \mathbb{P}^{\binom{n+d}{d}-1}$, $\mathcal{A}_{D0,p}$ is given by the A_∞-algebra with relations (4.65) and (4.66).

For completeness, let us write the generators of $D(\mathbb{P}^{\binom{n+d}{d}-1}, \mathcal{A}_{D0}\sharp\mathbb{Z}_d)$ as $\mathcal{A}_{D0}\sharp\mathbb{Z}_d$-modules. The B-brane $\mathcal{B}_{D0}^{(0)}$ can be represented as the curved complex

$$\mathcal{O} \underset{\frac{\partial W}{\partial x}\eta}{\overset{x\bar{\eta}}{\rightleftarrows}} \mathcal{O}(-\tfrac{1}{d})\otimes V \underset{\frac{\partial W}{\partial x}\eta}{\overset{x\bar{\eta}}{\rightleftarrows}} \mathcal{O}(-\tfrac{2}{d})\otimes \wedge^2 V \underset{\frac{\partial W}{\partial x}\eta}{\overset{x\bar{\eta}}{\rightleftarrows}} \cdots \underset{\frac{\partial W}{\partial x}\eta}{\overset{x\bar{\eta}}{\rightleftarrows}} \mathcal{O}(-\tfrac{n}{d})\otimes \wedge^n V \underset{\frac{\partial W}{\partial X}\eta}{\overset{x\bar{\eta}}{\rightleftarrows}} \mathcal{O}(-\tfrac{n+1}{d})\otimes \wedge^{n+1} V. \tag{5.5}$$

where $\mathcal{O}(m)$ denotes orbibundles over $\mathbb{P}^{\binom{n+d}{d}-1}$. Define $\mathcal{B}_{D0}^{(l)}$ to be the matrix factorization (5.5) twisted by $\mathcal{O}(l/d)$ for $l \in \mathbb{Z}$ and therefore the sheaf of A_∞ modules corresponding to $\mathcal{B}_{D0}^{(l)}$ is given by[19]

$$\mathcal{A}^{(l)} := Hom\left(\bigoplus_{a=0}^{d-1} \mathcal{B}_{D0}^{(a)}, \mathcal{B}_{D0}^{(l)}\right) = \bigoplus_{a=0}^{d-1}\bigoplus_{k\geq 0} \mathcal{O}\left(\frac{l-a}{d} - k\right)\otimes \wedge^{kd+a} V, \tag{5.6}$$

[18] The notation $\mathcal{O}(m)$ with $m \in \mathbb{Q}$ denotes an orbibundle over $\mathbb{P}^{\binom{n+d}{d}-1}$. See [7] for a short review or [51] for details.

[19] Note that the Hom's are taken over the orbifold category of the hybrid model defined by (5.2), (5.3).

where the sum over k is such that $kd + a \leq n+1$ then, we can simplify (5.6) to

$$\mathcal{A}^{(l)} = \bigoplus_{i=0}^{n+1} \mathcal{O}\left(\frac{l-i}{d}\right) \otimes \wedge^i V \tag{5.7}$$

We can therefore write

$$D(\mathbb{P}^{\binom{n+d}{d}-1}, \mathcal{A}_{D0} \sharp \mathbb{Z}_d) = \langle \mathcal{A}^{(1-C_{d,n})}, \cdots, \mathcal{A}^{(-1)}, \mathcal{A}^{(0)} \cong \mathcal{A}_{D0} \rangle, \tag{5.8}$$

where $C_{d,n} = d\binom{n+d}{d} - (n+1)$ is the expected number of factors obtained from the analysis of the pure Coulomb vacua in [7].

5.2 HPD of Fano hypersurface in projective space

A degree $d \leq n$ Fano hypersurface in $\mathbb{P}^n$, denoted $\mathbb{P}^n[d]$, can be described by a GLSM with $U(1)$ gauge group, $n+1$ chiral multiplets x_i with gauge charge 1, one chiral multiplet p with gauge charge $-d$ and a superpotential

$$W_{\text{Fano}} = p_0 F_d(x), \tag{5.9}$$

where the polynomial $F_d(x)$, of degree d, is the defining equation of the hypersurface (which we assume to be smooth). We consider x_i to be the coordinates on a complex vector space V ($\dim V = n+1$), hence $\mathbb{P}^n = \mathbb{P}(V)$. In this case, the equivalence (2.1) takes the form

$$D^b Coh(\mathbb{P}^n[d]) = \langle MF(F_d, \mathbb{Z}_d), \mathcal{O}_{\mathbb{P}^n[d]}, \mathcal{O}_{\mathbb{P}^n[d]}(1), \cdots, \mathcal{O}_{\mathbb{P}^n[d]}(n-d) \rangle. \tag{5.10}$$

The associated Lefschetz decomposition is

$$D^b(\mathbb{P}^n[d]) = \langle \mathcal{A}_0, \mathcal{A}_1, \cdots, \mathcal{A}_{n-d} \rangle, \tag{5.11}$$

where $\mathcal{A}_0 := \langle MF(F_d, \mathbb{Z}_d), \mathcal{O}_{\mathbb{P}^n[d]} \rangle$, $\mathcal{A}_i := \langle \mathcal{O}_{\mathbb{P}^n[d]} \rangle$, $i > 0$. The small window category $\mathcal{W}^{\mathcal{L}}_{-,b}$ of this GLSM, defined in (2.2) consists of B-branes with charges q satisfying

$$\left| q + \frac{\theta}{2\pi} \right| < \frac{d}{2}. \tag{5.12}$$

We choose the theta angle (i.e. the integer b) as $\theta = \pi d - \varepsilon$, with $0 < \varepsilon \ll 1$, so q to take the values $-(d-1), -(d-2), \cdots, -2, -1, 0$.

As shown in [7], the universal hyperplane section can be described by the geometric phase (with FI parameters lying in the first quadrant) of the GLSM $\mathcal{T}_{\mathcal{X}}$, this corresponds to a GLSM with gauge group $U(1)_{\mathcal{L}} \times U(1)$ and with matter content

	x_0	x_1	$\cdots$	x_n	p_0	p	$\cdots$	y_0	$\cdots$	y_n
$U(1)_{\mathcal{L}}$	1	1	$\cdots$	1	$-d$	-1	$\cdots$	0	$\cdots$	0
$U(1)_1$	0	0	$\cdots$	0	0	-1	$\cdots$	1	$\cdots$	1

and superpotential

$$\widehat{W} = p_0 F_d(x) + p \sum_{i=0}^{n} x_i y_i.$$

The HPD of $\mathbb{P}^n[d]$ with Lefschetz decomposition given by (5.11) can be described by the Higgs branch of the phase of $\mathcal{T}_{\mathcal{X}}$ corresponding to the FI parameters lying in the second quadrant: $(\zeta_{\mathcal{L}} \ll -1, \zeta_1 \gg 1)$. This is a hybrid model with target space

$$Y := \mathrm{Tot}\left(\mathcal{O}_{\check{\mathbb{P}}^n}^{\oplus(n+1)} \oplus \mathcal{O}_{\check{\mathbb{P}}^n}(-1) \to \check{\mathbb{P}}^n\right)/\mathbb{Z}_d$$

and superpotential

$$W = F_d(x) + p\sum_{i=0}^{n} x_i y_i, \tag{5.13}$$

where $\check{\mathbb{P}}^n := \mathbb{P}(V^\vee)$, x_i's are fibre coordinates of $\mathcal{O}_{\check{\mathbb{P}}^n}^{\oplus(n+1)}$, p is the fibre coordinate of $\mathcal{O}_{\check{\mathbb{P}}^n}(-1)$ and the y_i's are homogeneous coordinates on the base $\check{\mathbb{P}}^n$. The $\mathbb{Z}_d$ orbifold acts with weight 1 on the x_i's and -1 on p. Denote the category of B-branes of this hybrid model as $D(Y, W)$. Then $D(Y, W)$ has a (dual) Lefschetz decomposition that takes the form (as proposed in [7]):

$$D(Y, W) = \langle \mathcal{B}_{n-1}(1-n), \mathcal{B}_{n-2}(2-n), \cdots, \mathcal{B}_2(-2), \mathcal{B}_1(-1), \mathcal{B}_0\rangle, \tag{5.14}$$

where we have the equivalence of categories $\mathcal{B}_0 \cong \mathcal{A}_0$. Denote the functor implementing this equivalence by $\mathcal{F}$:

$$\mathcal{F} : \mathcal{A}_0 \to \mathcal{B}_0.$$

Then $\mathcal{B}_i = \langle \mathcal{F}(MF(F_d, \mathbb{Z}_d)), \mathcal{F}(\mathcal{O})\rangle$ for $0 \leq i \leq d$, $\mathcal{B}_j = \langle \mathcal{F}(MF(F_d, \mathbb{Z}_d))\rangle$ for $d+1 \leq j \leq n-1$. The relationship between the Lefschetz decomposition and its dual decomposition is illustrated by figure 1. Next we describe the functor $\mathcal{F}$ explicitly. Define the matrix factorization Q' of

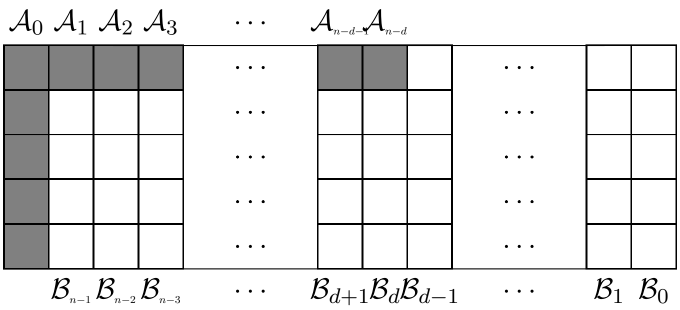

Figure 1: Lefschetz decomposition of the hypersurface $\mathbb{P}^n[d]$ and the dual Lefschetz decomposition of the HPD.

$p\sum_{i=0}^n x_i y_i$ as

$$\mathcal{O}_{\check{\mathbb{P}}^n}(1) \underset{\sum_{i=0}^n x_i y_i}{\overset{p}{\rightleftarrows}} \mathcal{O}_{\check{\mathbb{P}}^n}.$$

Any matrix factorization $\mathcal{M} \in MF(F_d, \mathbb{Z}_d)$ can be lifted to a GLSM B-brane with $U(1)_{\mathcal{L}}$ charges in the small window (5.12). On the other hand the category $\widehat{\mathcal{W}}^{\mathcal{L}}_{-,b} \cong D(Y, W)$ in (2.6) can be chosen (by adjusting b) such that the $U(1)_{\mathcal{L}}$ charges q' in the $\mathcal{T}_{\mathcal{X}}$ model satisfy $q' \in \{-(d-1), -(d-2), \cdots, -1, 0, 1\}$. Then, the tensor product $\mathcal{M} \otimes Q'$ has $U(1)_{\mathcal{L}}$ charges belonging to $\widehat{\mathcal{W}}^{\mathcal{L}}_{-,b}$.The same is true for the B-brane[20] $\mathcal{O}_{\mathbb{P}^n[d]} \otimes Q'$. We conclude that the

[20]Here we should think of $\mathcal{O}_{\mathbb{P}^n[d]}$ as its lift to a matrix factorization for the GLSM $\mathcal{T}_X$, with theta angle $\theta = \pi d - \varepsilon$ in whose case it has $U(1)_{\mathcal{L}}$ charges $q \in \{-d, 0\}$. So, it does not belong to $\mathcal{W}^{\mathcal{L}}_{-,b}$, but it does belong to $\widehat{\mathcal{W}}^{\mathcal{L}}_{-,b}$ upon tensoring with Q'.

functor is given by

$$\mathcal{F}(\mathcal{M}) = \mathcal{M} \otimes Q'.$$

For every fixed point on the base, the superpotential along the fibre is given by (5.13) with fixed y_i. The $D0$-brane $\mathcal{B}_{D0}^{(0)}$ can be written as

$$Q_{D0} = \sum_{i=0}^{n} (x_i \overline{\eta}_i) + p\overline{\eta}_{n+1} + \sum_{i=0}^{n} \left(\frac{1}{d} \frac{\partial F_d}{\partial x_i} + \frac{1}{2} p y_i \right) \eta_i + \frac{1}{2} \left(\sum_{i=0}^{n} x_i y_i \right) \eta_{n+1}.$$

where the trivial represenation of $\mathbb{Z}_d$ is chosen for the Clifford vacuum. Because the superpotential (5.13) has a quadratic term $p \sum_{i=0}^{n} x_i y_i$ and a degree-d term $F_d(x)$, the structure of the $\mathcal{A}_{D0}$ factor in the sheaf of A_∞-algebras $\mathcal{A}_{D0} \sharp \mathbb{Z}_d$ is determined by the relations[21]

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \left. \frac{\partial^2 W}{\partial x_i \partial x_j} \right|_{x_i=0}, \tag{5.15}$$

$$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \left. \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}} \right|_{x_i=0} \tag{5.16}$$

at each point of the base, where we have identified p with x_{n+1}.

As in the case of Veronese embedding, the global sheaf structure of $\mathcal{A}_{D0} \sharp \mathbb{Z}_d$ can also be read off from the global behavior of x_i and p, we present two examples for illustration.

Example: Quadrics

In the case $d = 2$, at each point of the base, the superpotential is quadratic in the fibre coordinates. Therefore, the sheaf of algebra is the Clifford algebra associated with the quadratic form given by $\frac{\partial^2 W}{\partial x_i \partial x_j}$, $i = 0, \ldots, n+1$, $x_{n+1} := p$ at fixed y_i. We can take the $D0$-brane $\mathcal{B}_{D0}^{(0)}$ to be given by the matrix factorization

$$Q_{D0} = \sum_{i=0}^{n} (x_i \overline{\eta}_i) + p\overline{\eta}_{n+1} + \sum_{i=0}^{n} \left(\frac{1}{2} \frac{\partial F_2}{\partial x_i} + \frac{1}{2} p y_i \right) \eta_i + \frac{1}{2} \left(\sum_{i=0}^{n} y_i x_i \right) \eta_{n+1},$$

hence, the curved complex associated to $\mathcal{B}_{D0}^{(0)}$ is given by

$$\mathcal{O}_+ \rightleftarrows \begin{matrix} \mathcal{O}_- \otimes V \\ \oplus \\ \mathcal{O}_-(-1) \end{matrix} \rightleftarrows \cdots \rightleftarrows \begin{matrix} \mathcal{O}_{(-)^{n-1}} \otimes \wedge^{n+1} V \\ \oplus \\ \mathcal{O}_{(-)^{n-1}}(-1) \otimes \wedge^n V \end{matrix} \rightleftarrows \mathcal{O}_{(-)^n}(-1) \otimes \wedge^{n+1} V, \tag{5.17}$$

where all the sheaves $\mathcal{O}_{(-)^i}(a)$ in (5.17) denote sheaves over $\mathbb{P}^n$ and the subindex $\pm$ indicates the weight the sheaf carries under the action of the $\mathbb{Z}_2$ orbifold. A similar computation as the one in section 5.1 let us conclude that

$$\mathcal{A}_{D0} \cong \left(\bigoplus_{i=0}^{n+1} \mathcal{O}_{(-)^i} \otimes \wedge^i V \right) \oplus \left(\bigoplus_{i=0}^{n+1} \mathcal{O}_{(-)^{i+1}}(-1) \otimes \wedge^i V \right) \tag{5.18}$$

[21] At a generic point $y \in \mathbb{P}^n$ the superpotential $W|_y$ satisfies $dW|_y^{-1}(0) = \{0\}$, hence we satisfy the condition on the potential of a LG orbifold we assumed.

globally. So, we can write

$$D(Y,W) \cong D(\mathbb{P}^n, \mathcal{A}_{D0}\sharp\mathbb{Z}_2) \tag{5.19}$$

i.e., the hybrid model B-brane category $D(Y,W)$ is equivalent to the derived category of sheaves of $\mathcal{A}_{D0}\sharp\mathbb{Z}_2$ A_∞-modules.

Example: Cubic hypersurfaces

The $\mathcal{B}^{(0)}_{D0}$-brane is given by

$$Q_{D0} = \sum_{i=0}^{n}(x_i\overline{\eta}_i) + p\overline{\eta}_{n+1} + \sum_{i=0}^{n}\left(\frac{1}{3}\frac{\partial F_3}{\partial x_i} + \frac{1}{2}py_i\right)\eta_i + \frac{1}{2}\left(\sum_{i=0}^{n} x_i y_i\right)\eta_{n+1},$$

and its associated curved complex

$$\mathcal{O}_0 \rightleftarrows \begin{matrix}\mathcal{O}_1\otimes V\\ \oplus \\ \mathcal{O}_{-1}(-1)\end{matrix} \rightleftarrows \cdots \rightleftarrows \begin{matrix}\mathcal{O}_{n+1}\otimes \wedge^{n+1}V\\ \oplus \\ \mathcal{O}_{n-1}\otimes\wedge^n V\end{matrix} \rightleftarrows \mathcal{O}_n\otimes\wedge^{n+1}V, \tag{5.20}$$

where the subscripts of the line bundles are the $\mathbb{Z}_3$-weights. At each point of the base, the A_∞-algebra is given by

$$m_2(\psi_i,\psi_{n+1}) + m_2(\psi_{n+1},\psi_i) = y_i,$$
$$m_3(\psi_i,\psi_j,\psi_k) = \frac{\partial^3 F_3}{\partial x_i\partial x_j\partial x_k}.$$

Globally, the sheaf of A_∞-algebra $\mathcal{A}_{D0}$ is

$$\mathcal{A}_{D0} = \left(\bigoplus_{i=0}^{n+1}\mathcal{O}_i\otimes\wedge^i V\right)\oplus\left(\bigoplus_{i=-1}^{n-1}\mathcal{O}_i(-1)\otimes\wedge^{i+1}V\right)\oplus\left(\mathcal{O}_n\otimes\wedge^{n+1}V\right).$$

Therefore, the HPD of $\mathbb{P}^n[3]$ is the noncommutative space $(\check{\mathbb{P}}^n, \mathcal{A}_{D0}\sharp\mathbb{Z}_3)$.

5.3 HPD of complete intersections

We finally make some remarks on the case of HPD of the Fano complete intersection $\mathbb{P}^n[d_1,d_2,\cdots,d_k]$, $\sum_{\alpha=1}^k d_\alpha < n+1$. The HPD is described by the hybrid model on

$$Y = \text{Tot}\left(\mathcal{O}(-1,0)^{\oplus(n+1)}\oplus\mathcal{O}(1,-1)\to \mathrm{W}\mathbb{P}[d_1,\cdots,d_k]\times\check{\mathbb{P}}^n\right) \tag{5.21}$$

with superpotential

$$W = \sum_{\alpha=1}^{k} p_\alpha F_{d_\alpha}(x) + p\sum_{i=0}^{n} x_i y_i, \tag{5.22}$$

where p_α are homogeneous coordinates of the weighted projective space $\mathrm{W}\mathbb{P}[d_1,\cdots,d_k]$, y_i are homogeneous coordinates of $\check{\mathbb{P}}^n$, x_i and p are coordinates along the fibers of $\mathcal{O}(-1,0)^{\oplus(n+1)}$ and $\mathcal{O}(1,-1)$ respectively.

As in the case of hypersurfaces, $\mathcal{A}_{D0}$ is spanned by $\psi_0, \ldots, \psi_{n+1}$ and the A_∞-products of $\mathcal{A}_{D0}$ are determined by

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \left.\frac{\partial^2 W}{\partial x_i \partial x_j}\right|_{x_i=0},$$

$$m_{d_\alpha}(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \left.\frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}\right|_{x_i=0}$$

at each point of the base $\mathbb{WP}[d_1, \cdots, d_k] \times \check{\mathbb{P}}^n$, where we have identified $x_{n+1} := p$. However, this is only valid if we ignore the orbifold singularity coming from the affine patches of $\mathbb{WP}[d_1, \cdots, d_k]$ and, for generic d_α, we cannot write the space Y as a global orbifold. Exceptions are, for instance if $d_\alpha = d$ for all α, then we have a noncommutative resolution of $\mathbb{P}^{k-1} \times \check{\mathbb{P}}^n$. Otherwise we have to deal with a sheaf of algebras over a singular space. We hope to return to this problem in a sequel.

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A Checkings on the $(Mod - \mathcal{A}_{D0}) \to MF(W)$ functor

Here we provide sevral checks for the proposed functor (4.38).

Check 1. $d = 2$ **Case.** When $d = 2$, we have $\mathcal{A}_{D0} = Cl(n, \mathbb{C})$, the Clifford algebra defined by the quadratic form $\frac{\partial^2 W}{\partial x_i \partial x_j}$. The correspondence between the matrix factorization for quadratic superpotentials and Clifford modules is given in [8,49], which matches (4.38).

Check 2. $\mathbf{N} = \mathcal{A}_{D0}$**.** It is straightforward to check (4.38) for the case the module is $\mathcal{A}_{D0}$ itself. Then (4.38) corresponds to Q_{D0}. In fact, as shown in [11], the fermionic generators $\psi_i, i = 1, \cdots, n$, satisfy

- If $d = 2$, $\mathcal{A}_{D0}$ is the Clifford algebra:

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}. \tag{A.1}$$

- If $d > 2$, $\mathcal{A}_{D0}$ is an A_∞-algebra, where m_2 satisfies

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0, \tag{A.2}$$

$m_k = 0$ for $k = 1, \cdots, d-1$ and

$$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \frac{1}{d!} \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}. \tag{A.3}$$

Thus when $d > 2$, if we identify $m_2(\cdot, v_i)$ with $\{\overline{\eta}_i, \cdot\}$ according to (A.2), then (A.3) tells us that $m_d(\cdot, \psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_{d-1}})$ should be identified with

$$\frac{1}{d!}\sum_{i=1}^{n} \frac{\partial^d W}{\partial x_i \partial x_{i_1} \cdots \partial x_{i_{d-1}}} \{\eta_i, \cdot\}.$$

Because W is homogeneous with degree d, we see that Q_M defined by (4.38) is exactly Q_{D0} defined by (4.34) in this case.

Check 3. $Q_M^2 = W \cdot \mathrm{id}$**.**. Here we will show that the object Q_M^2 is indeed a matrix factorization of W. We will make the assumption that $m_s^{\mathbf{N}} = 0$ for $s > d$ (which can be shown below to be true for the case $n = 1$), it can be shown that $Q_M^2 = W \cdot \mathrm{id}$. For example, when $d = 3$

$$Q_M(\phi) = \sum_{ij} m_3^{\mathbf{N}}(\phi, \psi_i, \psi_j) x_i x_j + \sum_i m_2^{\mathbf{N}}(\phi, \psi_i) x_i.$$

Therefore

$$\begin{aligned} Q_M^2(\phi) = &\sum_{ijkl} m_3^{\mathbf{N}}(m_3^{\mathbf{N}}(\phi, \psi_i, \psi_j), \psi_k, \psi_l) x_i x_j x_k x_l + \sum_{ijk} m_3^{\mathbf{N}}(m_2^{\mathbf{N}}(\phi, \psi_i), \psi_j, \psi_k) x_i x_j x_k \\ &- \sum_{ijk} m_2^{\mathbf{N}}(m_3^{\mathbf{N}}(\phi, \psi_i, \psi_j), \psi_k) x_i x_j x_k + \sum_{ij} m_2^{\mathbf{N}}(m_2^{\mathbf{N}}(\phi, \psi_i), \psi_j) x_i x_j. \end{aligned} \tag{A.4}$$

From (3.3), we get

$$m_2^{\mathbf{N}}(m_2^{\mathbf{N}}(\phi, \psi_i), \psi_j) = m_2^{\mathbf{N}}(\phi, m_2(\psi_i, \psi_j)),$$

then the last term of (A.4) vanishes due to (A.2). The first term of (A.4) also vanishes because of (3.3), (A.3) and $m_k(\cdots, 1, \cdots) = 0$ for $k > 2$. Also from (3.3) and (A.3), the second and third terms of (A.4) yield

$$\begin{aligned} &\sum_{ijk} \left(m_3^{\mathbf{N}}((m_2^{\mathbf{N}}((\phi, \psi_i), \psi_j, \psi_k) - m_2^{\mathbf{N}}((m^{\mathbf{N}}(_3(\phi, \psi_i, \psi_j), v_k)\right) x_i x_j x_k \\ &= \sum_{ijk} m_2^{\mathbf{N}}((\phi, m_3(\psi_i, \psi_j, \psi_k)) x_i x_j x_k = \frac{1}{3!}\phi \sum_{ijk} \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} x_i x_j x_k = W \cdot \phi, \end{aligned}$$

which shows $Q_M^2 = W \cdot \mathrm{id}$.

Check 4. $n = 1$ **case.** Finally, we show that the functor reproduces the matrix factorizations for the case $n = 1$, i.e. $W = x^d$. In this case, the $D0$-brane is given by the matrix factorization

$$Q_{D0} = x\overline{\eta} + x^{d-1}\eta. \tag{A.5}$$

The fermionic generator of $\mathcal{A}_{D0} = Hom(\mathcal{B}_{D0}, \mathcal{B}_{D0})$ is $\psi = \overline{\eta} - x^{d-2}\eta$. Let $\mathcal{B}_l$ be the matrix factorization with

$$Q_M = x^l\overline{\pi} + x^{d-l}\pi \tag{A.6}$$

where $1 < l < d$ and $\{\pi, \pi\} = \{\overline{\pi}, \overline{\pi}\} = 0$. Next, we will show that (4.38) recovers Q_M. Start by considering the bosonic state $\phi_0 \in Hom_0(\mathcal{B}_{D0}, \mathcal{B}_l)$ and the fermionic state $\phi_1 \in Hom_1(\mathcal{B}_{D0}, \mathcal{B}_l)$. If the vacuum state of M is denoted as $|\Omega\rangle$, then

$$\phi_0|0\rangle = |\Omega\rangle, \qquad \phi_0\overline{\eta}|0\rangle = x^{l-1}\overline{\pi}|\Omega\rangle,$$

and

$$\phi_1|0\rangle = \overline{\eta}|\Omega\rangle, \qquad \phi_1\overline{\eta}|0\rangle = -x^{d-l-1}|\Omega\rangle.$$

In matrix form,

$$\psi = \begin{pmatrix} 0 & -x^{d-2} \\ 1 & 0 \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & x^{l-1} \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 & -x^{d-l-1} \\ 1 & 0 \end{pmatrix}.$$

Using the algorithm reviewed in section 3, one can compute ($\iota\psi = v$)[22]

$$m_k(\psi^{\otimes k}) = 0, \quad f_k(\psi^{\otimes k}) = (-1)^k x^{d-k-1}\eta, \quad 1 < k < d,$$

and

$$m_d(\psi^{\otimes d}) = 1, \quad f_d(\psi^{\otimes d}) = 0.$$

By composing the homomorphisms, one gets

$$\phi_0 \circ \psi = x^{l-1}\phi_1, \quad \phi_1 \circ \psi = -x^{d-l-1}\phi_0.$$

Therefore,

$$\phi_0 \circ \psi = \mathrm{d}\tilde{\phi}_0^{(1)} = \tilde{\phi}_0^{(1)} Q_{D0} - Q_M \tilde{\phi}_0^{(1)}, \quad \phi_1 \circ \psi = -\mathrm{d}\tilde{\phi}_1^{(d-l-1)} = -\tilde{\phi}_1^{(d-l-1)} Q_{D0} - Q_M \tilde{\phi}_1^{(d-l-1)},$$

where

$$\tilde{\phi}_0^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & x^{l-2} \end{pmatrix}, \quad \tilde{\phi}_1^{(d-l-1)} = \begin{pmatrix} 0 & x^{d-l-2} \\ 0 & 0 \end{pmatrix},$$

from which one can deduce that[23]

$$m_2^{\mathbf{N}}(\phi_0, v) = 0, \quad f_2^{\mathbf{N}}(\phi_0, v) = -\tilde{\phi}_0^{(1)},$$

and

$$m_2^{\mathbf{N}}(\phi_1, \psi) = 0, \quad f_2^{\mathbf{N}}(\phi_1, \psi) = \tilde{\phi}_1^{(d-l-1)}.$$

It can be shown by induction that

$$m_{k+1}^{\mathbf{N}}(\phi_0, \psi^{\otimes k}) = 0, \quad f_{k+1}^{\mathbf{N}}(\phi_0, \psi^{\otimes k}) = -\tilde{\phi}_0^{(k)}, \quad 1 < k < l,$$

where

$$\tilde{\phi}_0^{(k)} = \begin{pmatrix} 0 & 0 \\ 0 & x^{l-k-1} \end{pmatrix}.$$

Similarly,

$$m_{k+1}^{\mathbf{N}}(\phi_1, \psi^{\otimes k}) = 0, \quad f_{k+1}^{\mathbf{N}}(\phi_1, \psi^{\otimes k}) = \tilde{\phi}_1^{(d-l-k)}, \quad 1 < k < d-l,$$

where

$$\tilde{\phi}_1^{(k)} = \begin{pmatrix} 0 & x^{k-1} \\ 0 & 0 \end{pmatrix}.$$

Now, one can compute

$$\iota m_{l+1}^{\mathbf{N}}(\phi_0, \psi^{\otimes l}) = -f_l^{\mathbf{N}}(\phi_0, \psi^{\otimes(l-1)}) \circ \psi - \phi_0 \circ f_l(\psi^{\otimes l}) = \phi_1,$$

[22]Another computation for this result can be found in the example at the end of section 4.2.

[23]We use ϕ_i to denote the cohomology class and the representative, the meaning should be clear from the context.

similarly $\iota m^N_{d-l+1}(\phi_1, \psi^{\otimes(d-l)}) = \phi_0$, and all the higher order multiplications vanish. Therefore, in the basis $\{\phi_0, \phi_1\}$, (4.38) yields

$$Q_M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x^l + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^{d-l} = \begin{pmatrix} 0 & x^{d-l} \\ x^l & 0 \end{pmatrix},$$

which is exactly the matrix factorization (A.6) we started with.

B A_∞-algebras defined by ribbon trees

The structure of the A_∞-algebra $\mathcal{A} = \mathrm{End}(\mathcal{B}_{D0})$ corresponding to a Landau-Ginzburg model with homogeneous superpotential was derived in [11] using the method of summing over ribbon trees. In this appendix, we review the idea of [11] and generalize it to LG models with inhomogeneous superpotentials.

Let ι be an embedding of $H(\mathcal{A}) := H_{m_1^{\mathcal{A}}}(\mathcal{A})$ into $\mathcal{A}$. If we define the projection π : $\mathcal{A} \to H(\mathcal{A})$ such that $\pi \circ \iota = 1$ and there is a map $h : \mathcal{A} \to \mathcal{A}$ of degree -1 such that $1 - \iota\circ\pi = m_1^{\mathcal{A}} \circ h + h \circ m_1^{\mathcal{A}}$ and $h^2 = \pi h = h\iota = 0$, then the A_∞ products $m_k : H(\mathcal{A})^{\otimes k} \to H(\mathcal{A})$, $k \geq 2$ can be defined by summing over the contributions from ribbon trees [52]:

$$m_k = \sum_T m_{k,T}. \tag{B.1}$$

For a LG model with degree-d superpotential, the ribbon trees contributing to the sum have one root and d leaves such that the valency of any vertex is 2 or 3 [11]. (B.1) is a solution to the defining relations (3.3).

In our convention, $\iota(\psi_i) = v_i$ defined by (4.36), consequently h can be defined to be $h = \sum_i \eta_i \frac{\partial}{\partial x_i}$ where η_i acts by multiplication in the Clifford algebra.

By definition, a ribbon tree is a tree T with a collection of vertices, external edges and internal edges such that: (a) Each external edge is incident to a single vertex. (b) Each internal edge is incident to exactly two vertices. (c) One of the external edge is the root, the other external edges are the leaves. Every ribbon tree T with one root and k leaves determines a term $m_{k,T}$ in (B.1).

Given a tree T, to compute $m_{k,T}(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k})$ we put $\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k}$ on the leaves from left to right and then act on them a series of maps as follows:

- Each leaf gives a map ι;
- Each bivalent vertex gives a map f;
- Each internal edge gives a map h;
- Each trivalent vertex corresponds to the multiplication in $\mathcal{A}$;
- The root gives the map π

while reading from the top to the bottom. Here f is defined by

$$\left\{ \sum_i \frac{\partial W}{\partial x_i} \eta_i, \cdot \right] - \sum_i \frac{\partial W}{\partial x_i} \eta_i,$$

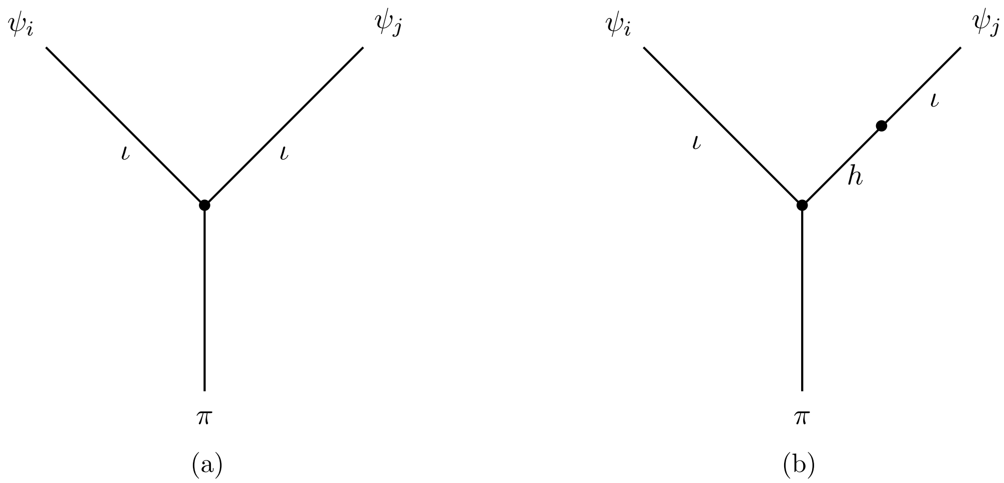

Figure 2: (a) Ribbon tree contributing to m_2. (b) Another contribution to m_2 when $d = 2$.

where {} denotes commutator/anticommutator depending on whether the second argument is of even/odd degree and the second term is the usual multiplication of the Clifford algebra.

It is shown in [11] that there is always a tree given by Figure.2a making a nontrivial contribution to m_2. For $d > 2$, this is the only contribution and it makes m_2 to satisfy $m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = 0$. When $d = 2$, there is another nontrivial contribution from the tree given by Figure.2b. The effect of Figure.2b is to modify m_2 such that $m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}$. In general, other than the tree in Figure.2a, the only ribbon tree that can make a nonzero contribution is the one in Figure.3. If the input of the tree is $\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k}$, then before hitting the root, the image of the set of maps encoded in the tree is $\frac{1}{k!}\frac{\partial^k W}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}$ plus some Q-exact terms. When $k \neq d$, this image is Q-exact and annihilated by the projection π, so the output of the tree is zero. When $k = d$, the output is $\frac{1}{k!}\frac{\partial^k W}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}$ because it is not Q-exact. In summary, we have

- If $d = 2$, $\mathcal{A}$ is a Clifford algebra given by:

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}.$$

- If $d > 2$, m_2 is the wedge product, $m_k = 0$ for $k \neq 2$ and $k \neq d$.

$$m_d(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_d}) = \frac{\partial^d W}{\partial x_{i_1} \cdots \partial x_{i_d}}.$$

Now assume we have a inhomogeneous superpotential of the form

$$W = \sum_{l=2}^{d} W^{(l)},$$

where $\deg W^{(l)} = l$. Because now every nonvanishing derivative $\frac{\partial^l W^{(l)}}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_l}}$ is not Q-exact, we can use the same argument to deduce that

$$m_2(\psi_i, \psi_j) + m_2(\psi_j, \psi_i) = \frac{\partial^2 W^{(2)}}{\partial x_i \partial x_j} \tag{B.2}$$

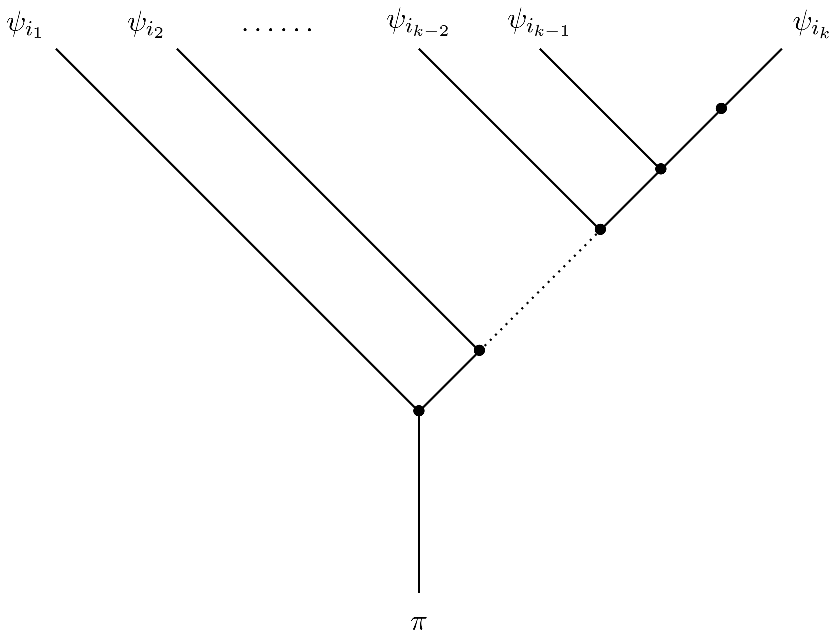

Figure 3: Ribbon tree contributing to m_k.

and

$$m_l(\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_l}) = \frac{1}{l!} \frac{\partial^l W^{(l)}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}} \tag{B.3}$$

for $3 \leq l \leq d$.

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