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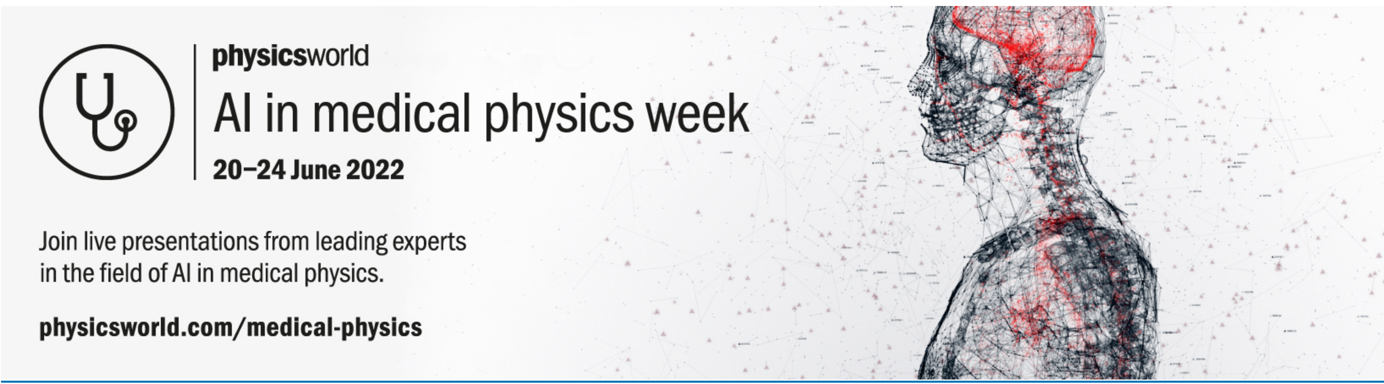

Computing Classification of Interacting Fermionic Symmetry-Protected Topological Phases Using Topological Invariants

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In recent years, great success has been achieved on the classification of symmetry-protected topological (SPT) phases for interacting fermion systems by using generalized cohomology theory. However, the explicit calculation of generalized cohomology theory is extremely hard due to the difficulty of computing obstruction functions. Based on the physical picture of topological invariants and mathematical techniques in homotopy algebra, we develop an algorithm to resolve this hard problem. It is well known that cochains in the cohomology of the symmetry group, which are used to enumerate the SPT phases, can be expressed equivalently in different linear bases, known as the resolutions. By expressing the cochains in a reduced resolution containing much fewer basis than the choice commonly used in previous studies, the computational cost is drastically reduced. In particular, it reduces the computational cost for infinite discrete symmetry groups, like the wallpaper groups and space groups, from infinity to finity. As examples, we compute the classification of two-dimensional interacting fermionic SPT phases, for all 17 wallpaper symmetry groups.

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Topological phases[1,2] are quantum states of matter beyond the Landau paradigm of classifying phases through spontaneous symmetry breaking. Among them, the so-called symmetry-breaking topological (SPT) phases[3−6] are distinguished from topologically trivial phases through symmetry-protected gapless edge states (or more generally, symmetry anomalies on the edge) and nontrivial responses to the insertion of symmetry fluxes, while the bulk is short-range entangled and lacks of fractionalized excitations. Examples of SPT phases include topological insulators (TI),[7,8] topological superconductors (TSC), topological crystalline insulators (TCI)[9] and one-dimensional(1D) Haldane chain,[10] which all have been realized in solid-state materials.

The classification of interacting fermion SPT (fSPT) states is still not fully resolved for the most general symmetry groups, despite of rapid progress in recent years.[11−20] Physically, the ground state of fSPT phases can be understood in terms of decorated fluctuation domain walls. Mathematically, the fSPT classification is described by generalized cohomology theory which is a combination of layers of cochains in the cohomology of the symmetry group. These cochains must satisfy certain conditions, which are expressed by obstruction functions mapping cochains to another cocycle.[20−22] A central task in the classification scheme is to compute such obstruction functions. For realistic symmetry groups of physical interests, the computational cost of such functions can be high or even prohibitive. This not only seriously limits the application of the fSPT-classification results to realistic systems, but also inconveniences theoretical studies of fSPT classification by causing difficulties in constructing and studying examples.

In this Letter, we develop an efficient algorithm to make it possible to evaluate obstruction functions for general discrete symmetry groups, based on the physical concept of topological invariants and mathematical techniques in homology algebra. In particular, when the outcome of the cocycle function is a cocycle in the bosonic SPT layer (the details of the layers will be reviewed below), the algorithm has an intuitive interpretation as evaluating partition functions on representative space-time manifolds with suitable symmetry-flux insertions.[23] Previously, these partition functions or equivalent quantities like braiding statistics[24−28] were used as topological invariants in the process of computing the outcome of cocycle functions. So far, such invariants have only been

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constructed for special symmetry groups, e.g., finite Abelian groups. Our algorithm not only provides a way to efficiently construct and compute these invariants automatically for a large class of symmetry groups, even including infinite ones, but also generalizes them to more tasks, e.g., including the cases where the outcoming cocycle is in other layers of an underlying generalized cohomology theory. For example, our algorithm can be easily generalized for solving similar obstruction functions in the classification of symmetry enriched topological (SET) phases.[29–33]

Since the techniques of homology algebra we use to develop this algorithm may be unfamiliar to physicists, we include a brief review of related concepts and notations, as well as implementation details of algorithm stated with standard mathematical notations, in the Supplementary Material (SM), while avoiding the jargons in the main text. The SM also includes a short introduction to the software package SptSet,[34] which is written in GAP[35] by one of the authors. It implements our algorithm and can produce the results listed in Tables 1 and 2.

The fSPT Classification. We begin with a brief review for the classification of fSPT states we want to compute. One fruitful scheme for computing the classification is to construct the SPT states using domain-wall decoration.[15,20] In this scheme, a $(d+1)$-dimensional SPT state is divided into different layers, where $(p+1)$-dimensional invertible topological orders (iTOs) are decorated onto the $(d-p)$-dimensional symmetry domain walls, respectively. Here, iTOs are states that do not have any fractionalized excitations, but are still nontrivial even when there is no symmetry to protect it. For interacting-fermion systems, the invertible topological orders include complex-fermion modes in (0+1)D, Kitaev chains in (1+1)D and $p+ip$-wave topological superconductors in (2+1)D. Denoting the classification of $(d+1)$-dimensional iTOs by iTO^{d+1}, we have $\mathrm{iTO}^1=\mathbb{Z}_2$, $\mathrm{iTO}^2=\mathbb{Z}_2$, $\mathrm{iTO}^3=\mathbb{Z}$, respectively.[36] Using this notation, the decoration of $(p+1)$-dimensional iTOs on $(d-p)$-dimensional symmetry domain walls is classified by a cocycle $n_{d-p}\in H^{d-p}(G_\mathrm{b},\mathrm{iTO}^{p+1})$, where $0\le p<d$ and G_b denotes the group of bosonic symmetries, which is the quotient group $G_\mathrm{f}/\mathbb{Z}_2^\mathrm{f}$, where G_f and $\mathbb{Z}_2^\mathrm{f}$ denotes the total symmetry group of the fermionic system and the fermion-parity symmetry, respectively. Finally, there is another layer of a bosonic SPT, described by $\nu_{d+1}\in H^{d+1}[G_\mathrm{b},U(1)_T]$.

However, the classification of fSPTs is not simply a direct sum of the aforementioned cohomology classes. In particular, a decoration n_p can be anomalous: it cannot be realized in a purely d-dimensional system, and can only be realized on the boundary of a system in one-higher dimension, with a decoration in a higher layer $n_{p'}$ or ν_{d+2}.[37] Mathematically, such a bulk-boundary relation is described by an obstruction function:[15,20]

$$n_{p'}=O_{p'}[n_p]\quad\text{or}\quad\nu_{d+2}=O_{d+2}[n_p].\qquad(1)$$

Physically, this means that the n_p decoration can and must be realized on the surface of the $O_{p'}[n_p]$ decoration. The application of these obstruction functions is two-fold:[37] On the one hand, if $O_{d+2}[n_p]$ does not vanish, it signals that n_p does not describe a valid SPT state and should be eliminated. On the other hand, the corresponding $O_{p'}$ describes a trivial SPT state in one higher dimension, because its surface can be gapped out without symmetry-breaking or fractionalization. Hence, the computation of the obstruction functions plays a central role in classifying fSPTs. After the obstruction-free and nontrivial cocycles are obtained, another subtlety in determining the group structure of fSPT classes is the group-extension problem: when adding two decorations n_p and n'_p, if the result is a trivial cocycle in this layer, the physical result may be a nontrivial SPT state in a higher layer.

The obstruction functions and the group-extension functions are both functions mapping one or two p-cocycles to a p'-cocycle. We shall use the evaluation of obstruction functions, in particular O_{d+2}, as an example in the main part of this work, although the algorithm can be readily applied to other tasks. In previous works, the obstruction functions are derived in terms of a special form of cocycles, known as the homogeneous or the inhomogeneous cocycles.[20] In particular, an n-cocycle is a function $\alpha(g_1,\ldots,g_n)$, mapping a combination of n group elements (which will be denoted by $[g_1|\cdots|g_n]$), to a complex number in $U(1)$. Mathematically, it can be viewed as cocycle on a simplicial-complex realization of the classifying space of the symmetry group. Although this form of cocycles is convenient for theoretical derivation, it is cumbersome for computation. In particular, the computational complexity of determining the cohomology class of a cocycle computed from an obstruction function scales as $(|G|-1)^{3n}$, where $|G|$ is the order of the group and n is the order of the cocycle. This complexity quickly becomes prohibitive for complex symmetry groups of physical interests.

The Algorithm. Using mathematical tools in homology algebra, we construct an algorithm to accelerate this and other similar tasks in the evaluation of the cocycle functions. Intuitively, this is carried out by mapping the cocycles into a different basis, which can equivalently express all cohomology classes in the group-cohomology theory but is much smaller.

In group-cohomology theory, it is well known that the cohomology of a group is related to the cohomology of the classifying space of that group, and therefore can be computed using the chain complex of the classifying space, known as the (free) resolution associated with the group.[38,39] The resolution provides the basis for writing down the cocycles. For a given group, there are different choices of realizations of the classifying space, resulting in different resolutions, and all choices are homotopically equivalent to each other, meaning that the resulting cohomology group is independent of the choice. In particular, the well-known inhomogeneous cocycles correspond to a particular simplicial construction of the classifying space, with the resolution known as the bar resolution in mathematics. However, not all resolutions are created equally: some resolutions are smaller than others, meaning that there are fewer basis for expressing the cocycles in each dimension, and therefore require much less computational resources in practice. The bar resolution, on the other hand, is one of the biggest resolutions. Unfortunately, some of the obstruction functions are only known, to the best of our knowledge, in terms of the inhomogeneous cocycles or the bar resolution, which is indeed convenient for theoretically deriving the obstruction functions due to the simplicial structure of the corresponding classifying space. Hence, we propose an algorithm for accelerating the evaluation of the cocycle functions, by first converting the input cocycle to an inhomogeneous cycle, computing the resulting cocycle using the inhomogeneous-cocycle formula, then converting the result back to a cocycle in a smaller resolution. Since the most time-consuming step is to find the cohomology class of a cocycle, this step becomes much faster in the smaller resolution, providing an overall acceleration to the whole process.

In particular, in a resolution with only a few bases, the task is simplified to evaluating a few topological invariants using entries in the inhomogeneous cocycle. Physically, when the coefficient of the cohomology is $U(1)$, these invariants can be viewed as partition functions on space-time manifolds with nontrivial symmetry fluxes. Such invariants have been constructed case-by-case for simple cases before.[23,26,27] Our algorithm generalizes and automates the construction of these invariants to a large class of groups. We note that similar techniques have been used in mathematics in computing properties of higher groups.[40]

To be more concrete, for a discrete group G, we construct two sets of basis (known as resolutions) representing the same set of cohomology classes: the bar resolution whose bases are $[g_1|\cdots|g_n]$, and a simplified solution whose bases are denoted abstractly by $e_1^n,\ldots,e_{r_n}^n$. The core task of our algorithm is to construct two maps between the two sets of bases: f maps each basis in the simplified resolution to the bar resolution,

$$f(e_i^n)=\sum_{g_1,\ldots,g_n}\phi_i(g_1,\ldots,g_n)[g_1|\cdots|g_n];\qquad(2)$$

g maps each basis in the bar resolution to the simplified resolution,

$$g([g_1|\cdots|g_n])=\sum_i\gamma_i(g_1,\ldots,g_n)e_i^n.\qquad(3)$$

Here, for certain i and $[g_1|\cdots|g_n]$, the coefficients ϕ and γ actually belong to the integral group ring $\mathbb{Z}G$ reviewed in the SM.

Using the maps f and g, we can compute the obstruction function O_{d+2} for cocycles in the simplified basis as the following. Given a p-cocycle in the simplified basis, represented as $\alpha(e_i^p)$, we first convert it to an inhomogeneous cocycle, denoted as $\bar{\alpha}$, using the map g:

$$\bar{\alpha}(g_1,\ldots,g_n)=\alpha[g([g_1|\cdots|g_n])].$$

The obstruction function O_{d+2} is then computed using the formula for inhomogeneous cocycles. The result of this formula is an inhomogeneous $(d+2)$-cocycle, denoted by $\bar{\beta}=O_{d+2}[\bar{\alpha}]$; $\bar{\beta}$ is then converted to the simplified basis using the map f, as $\beta(e_i^{d+2})=\bar{\beta}[f(e_i^{d+2})]$. We note that, to convert cocycles in the simplified basis (bar-resolution basis) to ones in the bar-resolution basis (simplified basis), we use the map g (f), respectively. This is because the cocycles, analogous to linear functions in linear spaces, are contravariant under the changes of the basis. Mathematically, the conversion between two types of cocycles are known as pullback maps induced by the chain maps g and f.

Construction of the Chain Maps. The maps f and g are known as chain maps between the two resolutions, and can be constructed using standard homology-algebra techniques, which are outlined here. Details of the algorithm and reviews of standard notation in homology can be found in the SM. Without losing generality, we consider constructing a chain map f between two resolutions F and F', whose bases in dimension n are denoted by e_i^n and $e_i^{n\prime}$, respectively. The construction is iterative: We start with the lowest dimension $n=0$, where both resolutions contain only one basis and the construction of the map is obvious. Then, assuming the map is constructed for dimension $n-1$, we now proceed to dimension n and compute $f(e_i^n)$ for each basis e_i^n. In order to preserve the algebraic structure, it is required that the chain map f commutes with the boundary operators ∂ and ∂' of the two resolutions F and F' (the boundary can be viewed as the dual operation of the coboundary operator on the cochains). Hence, we require that

$$\partial' f(e_i^n)=f(\partial e_i^n).$$

Notice that the right hand side of the equation can be computed using the map constructed for dimension $n-1$. A proper choice of $f(e_i^n)$ can then be computed using a contracting homotopy s' of F':

$$f(e_i^n)=s'[f(\partial e_i^n)].$$

Intuitively, a contracting homotopy can be viewed as an "inverse" of the boundary operator ∂', and it is essential to our algorithm. For the inhomogeneous cocycles or the bar resolution, a standard choice of contracting homotopy is reviewed in the SM. For the simplified resolutions, a contracting homotopy must be constructed along with the resolution.

Construction of the Resolution. For a large class of groups, including all finite groups and the space groups, a simplified resolution suitable for our algorithm, accompanied by a contracting homotopy, can be constructed using the procedures introduced in Ref. [41], which is implemented by the HAP package[42] in the GAP software.[35] This includes all 17 2D wallpaper groups and most 3D space groups.

Moreover, when a group G is expressed as an extension of Q by N, a resolution of G can be constructed using resolutions of Q and N.[43] Compared to the construction in Ref. [41], this method is easier to implement. This method can also be applied to all 2D wallpaper groups. It is well known that a 2D wallpaper group can be viewed as an extension of a point group P by the translation-symmetry group $T = \mathbb{Z}^2$: $P = G/T$. Hence, we can construct a resolution over G from resolutions over P and $\mathbb{Z}^2$, using Wall's construction.[43] $\mathbb{Z}^2$ has a simple resolution because $B\mathbb{Z}^2$ is simply the 2D torus T^2. For the point group P, we recall that there are eight possible nontrivial point groups in 2D: $C_{2,3,4,6}$ and $D_{2,3,4,6}$. For the cyclic groups, simple resolutions over $C_n = \mathbb{Z}_n$ can be reviewed in the SM. For the dihedral groups, they can in turn be expressed as split extensions $D_n = C_n \rtimes \mathbb{Z}_2$. Hence, a simple free resolution can be constructed again using Ref. [43]. Therefore, finite-rank free resolutions over 2D wallpaper groups can be constructed by combining simple resolutions over $\mathbb{Z}_n$ and $\mathbb{Z}$ using Ref. [43]. We note that this approach also works for 3D space groups.

Example of the $\mathbb{Z}_{2n}$ Symmetry Group. Here, we sketch a simple example for SPT phases protected by the $G = \mathbb{Z}_{2n}$ group. For concreteness, we demonstrate how to simplify the computation of O_4 obstruction for a 3D SPT state with a Majorana-chain decoration, denoted by n_2. This obstruction represents the failure of finding a complex-fermion decoration, and its formula is given by Eq. (38) in Ref. [16], for the simple case where the unitary $\mathbb{Z}_{2n}$ symmetry extends trivially over the fermion-parity symmetry.

As explained in the SM, a simplified resolution for $G = \mathbb{Z}_{2n}$ consists of only a single basis in each dimension k, denoted by e_k. Using this resolution, the single nontrivial choice of n_2 is given by $n_2(e_2) = 1$. The obstruction cocycle, $\alpha = O_4[n_2]$, is also represented by a single entry $\alpha(e_4)$. To compute this, we first express α by an inhomogeneous cocycle using the mapping f in Eq. (2), which maps e^4 to $f(e^4) = \sum_{i,j=1}^{2n-1}[a^i|a|a^j|a]$ [see Eq. (21) in the SM.] This expresses $\alpha(e^4)$ as a linear combination of inhomogeneous cocycles $\alpha(e^4) = \sum_{i,j=1}^{2n-1}\bar{\alpha}(a^i,a,a^j,a)$. Using the equation for O_4 obstruction in inhomogeneous cocycles, $\bar{\alpha} = O_4[n_2] = \bar{n}_2 \cup \bar{n}_2$, we obtain

$$\alpha(e^4) = \sum_{i,j=1}^{2n-1} \bar{n}_2(a^i,a)\bar{n}_2(a^j,a). \tag{4}$$

Next, we compute the entries of the inhomogeneous cocycle $\bar{n}_2$ from n_2, using the chain map g in Eq. (3). Using the form of g in Eq. (24) of the SM, we can reach $\bar{n}_2(a^i,a) = \delta_{i,2n-1}$. Plug this into Eq. (4), we obtain $\alpha(e^4) = 1$, indicating that the O_4 obstruction is nontrivial. Therefore, such a Majorana-chain decoration is obstructed.

Using the simplified resolution, determining $\alpha = O_4$ only needs to compute the $(2n-1)^2$ entries in Eq. (4). Therefore, the computational cost increases with n as $O[(2n-1)^2]$. On the other hand, in traditional methods, not only do we need to compute all $(2n-1)^4$ entries in $\bar{\alpha}$, but we also need to solve a set of linear equations (with integral coefficients) with dimension $(2n-1)^4$-by-$(2n-1)^4$, and the associated computational cost is $O[(2n-1)^{12}]$. Therefore, the algorithm presented here greatly reduces the computational cost.

Solving a Twisted Cocycle Equation. This is another computationally heavy task that needs to be and can be accelerated using a simplified resolution. In the computation of fSPT classification, when an obstruction function in Eq. (1) is a trivial but nonvanishing cocycle, the cochain representing the decoration $n_{p'-1}$ or ν_{d+1} must satisfy the following twisted cocycle equation[16,20]

$$dn_{p'-1} = O_{p'}[n_p] \quad \text{or} \quad d\nu_{d+1} = O_{d+2}[n_p]. \tag{5}$$

The fact that the right hand side of this equation trivially guarantees that the equation has solutions, and the task is to seek one particular solution. Using the inhomogeneous cochains or the bar resolution, this task is also time-consuming as the matrix form of Eq. (5) has dimension $(|G|-1)^{p'-1} \times (|G|-1)^{p'}$ or $(|G|-1)^{d+1} \times (|G|-1)^{d+2}$. Here, we briefly sketch the idea of accelerating this using a simplified resolution, and the details of the implementation can be found in the SM.

To simplify the notations, we consider the generic problem of finding one solution of the cocycle equation in terms of inhomogeneous cochains,

$$d\bar{\beta} = \bar{\alpha}. \tag{6}$$

Naively, one may try to solve Eq. (6) by mapping $\bar{\alpha}$ to a cochain in the simplified resolution using the map f as α, find a solution $\beta' = d\alpha$ there, and map it back to an inhomogeneous cocycle using the map g, which we denote by $\bar{\beta}'$. However, the inhomogeneous cocycle $\bar{\beta}'$ constructed by this way is not a solution to Eq. (6), because the two chain maps f and g are not the inverse of each other. Instead, fg is only homotopic to the identity map, meaning that it can be related to identity using a homotopy $h : fg \sim 1$. In fact, the homotopy h can be used to construct a "correction" cochain we denote by $h^*(\bar{\alpha})$: $h^*(\bar{\alpha})(g_1,\ldots,g_n) = \bar{\alpha}[h([g_1|\cdots|g_n])]$, and one solution to Eq. (6) is then given by $\bar{\beta} = \bar{\beta}' - h^*(\bar{\alpha})$. As described in details in the SM, the homotopy h can be viewed as an analogue of the chain maps f and g (actually, it is a degree-1 map from the bar resolution to itself), and can be computed using a similar iterative procedure.

Wallpaper-Group SPT Classification. To demonstrate the power of our algorithm, we compute the classification of 2D fSPTs protected by the 17 2D wallpaper groups.[44] This task is impossible using the inhomogeneous cocycles, because the wallpaper groups are infinite. However, using our algorithm, it becomes a finite problem and can be solved using a computer program.

Table 1. Classification of 2D fSPTs protected by 2D wallpaper groups, where fermions are spinless (spin-1/2 if the symmetry group is treated as spatial symmetries). The answer is listed in terms of the Majorana-chain (MC), complex-fermion (CF), and bosonic (B) layers. The total number of phases combining all three layers is also listed in the last column. The overall group structure is not computed in this study.

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	$3\mathbb{Z}_2$	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	2048
p1m1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p2mm	0	0	$8\mathbb{Z}_2$	256
p2mg	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$3\mathbb{Z}_2$	256
p2gg	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	64
c2mm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$5\mathbb{Z}_2$	128
p4	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_4$	1024
p4mm	0	0	$6\mathbb{Z}_2$	64
p4gm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_4$	64
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	4
p31m	0	$\mathbb{Z}_2$	$\mathbb{Z}_6$	12
p6	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_6$	288
p6mm	0	0	$4\mathbb{Z}_2$	16

The problem we consider is the 2D fSPTs protected by an onsite symmetry group G, which have the same group structure as one of the 17 wallpaper groups. We assume that the proper and improper operations in G act as unitary and antiunitary operations, respectively. Furthermore, we assume that the total symmetry group is a direct product of the wallpaper group G and the fermion-parity symmetry group $\mathbb{Z}_2^{\mathrm{f}}$: $G_{\mathrm{f}} = G \times \mathbb{Z}_2^{\mathrm{f}}$. According to the crystalline equivalence principle,[45,46] this fSPT classification is the same as the classification of topological crystalline states protected by the wallpaper group G, formed by fermions transforming projectively under G as the physical spin-$\frac{1}{2}$ electrons perform. Therefore, the classification we compute here can also guide the search of such topological crystalline states on 2D lattices. The results we obtain are listed in Table 1. Moreover, we also compute the classification for spin-$\frac{1}{2}$ fermions, which corresponds to topological crystalline states formed by spinless fermions, and the results are listed in Table 2. We note that they agree with recent results obtained by real-space constructions of the corresponding topological crystalline states.[47]

Table 2. Classification of 2D fSPTs protected by 2D wallpaper groups, where fermions carry spin-1/2 (spinless if the symmetry group is treated as spatial symmetries). The answer is listed in terms of the Majorana-chain (MC), complex-fermion (CF), and bosonic (B) layers. The total number of phases combining all three layers is also listed in the last column. The overall group structure is not computed in this study.

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	0	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p1m1	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	64
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p2mm	0	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	256
p2mg	$\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	32
p2gg	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
c2mm	0	$3\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p4	0	$2\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	32
p4mm	0	$4\mathbb{Z}_2$	$3\mathbb{Z}_2$	128
p4gm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p31m	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	24
p6	0	$\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	36
p6mm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16

In summary, we have given an algorithm to accelerate the computation of certain maps between different cohomology classes of a symmetry group, which is a common and essential task in classifying fSPTs. Using the fact that the same cohomology classes can be obtained from different choices of classifying space of the group, or algebraically different resolutions, the algorithm constructs chain-maps between the standard choice of resolution, where the formula of the desiring maps are known, and a simplified choice of resolution where the computation is much easier. Such chain-maps then allow us to convert cocycles between two choices of resolutions, and to simplify the computation of the maps between cohomology classes.

Our algorithm not only reproduces some known results on finite groups with a faster speed, but also works for infinite discrete groups, like the 2D wallpaper groups and 3D space groups. Hence, it can be used to compute examples for the study of fSPT classification, and to compute fSPT classification for symmetry groups relevant to materials, which usually include space-group symmetries. Furthermore, recent progresses on the classification of 2D symmetry-enriched topological (SET) states[29–32] and 3D U(1) quantum spin liquids[33] also involve obstruction functions that map between cohomology classes of the symmetry group. The computation of these obstruction functions can also be accelerated by our algorithm.

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