Rapid Communications

Construction and classification of point-group symmetry-protected topological phases in two-dimensional interacting fermionic systems

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The construction and classification of symmetry-protected topological (SPT) phases in interacting bosonic and fermionic systems have been intensively studied in the past few years. Very recently, a complete classification and construction of space-group SPT phases were also proposed for interacting bosonic systems. In this Rapid Communication, we attempt to generalize this classification and construction scheme systematically into interacting fermion systems. In particular, we construct and classify point-group SPT phases for two-dimensional (2D) interacting fermion systems via lower-dimensional block-state decorations. We discover several intriguing fermionic SPT states that can only be realized in interacting fermion systems (i.e., not in free-fermion or bosonic SPT systems). Moreover, we also verify the recently conjectured crystalline equivalence principle for 2D interacting fermion systems. Finally, the potential experimental realization of these different classes of point-group SPT phases in 2D correlated superconductors is addressed.

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Introduction. In recent years, the concept of quantum entanglement patterns has played an essential role in constructing and classifying topological phases of quantum matter. At a very basic level, the ground state of a gapped quantum system can be classified as a long-range or short-range entangled state. In the presence of global symmetry, even short-range entangled states can be classified into numerous different phases, including the symmetry-protected topological (SPT) phases [1–4], in addition to the conventional symmetry-breaking phases. The simplest example of an SPT phase is a topological insulator, which is protected by time-reversal and charge-conservation symmetry [5,6]. Having a complete construction and classification of SPT phases is a crucial step towards understanding these peculiar quantum phases of matter. A general scheme of classifying bosonic SPT (BSPT) phases has been well established using group cohomology theory [3,4] and invertible topological quantum field theory (TQFT) [7–10]. An alternative strategy of classification was constructed in Ref. [11] by "gauging" the global symmetry and investigating the braiding statistics of gauge fluxes. A complete classification for fermionic SPT (FSPT) phases can also be obtained by the so-called general group supercohomology theory [12–14], spin cobordism theory [15,16], or by gauging the corresponding global symmetry [17–23].

Very recently, crystalline SPT phases have been intensively studied for free-fermion and interacting bosonic systems [24–37]. These states are not only of conceptual importance, but also provide great opportunities for experimental realization [38–41]. In particular, an explicit block-state construction scheme for crystalline SPT phases was established in Ref. [26]. Furthermore, it has been highlighted that the classification of space-group SPT phases is closely related to SPT phases with internal symmetry. In Ref. [28], a "crystalline equivalence principle" was proposed, i.e., crystalline topological phases with symmetry G are in one-to-one correspondence with topological phases protected by the same internal symmetry G, but acting in a twisted way, where if an element of G is a mirror reflection, it should be regarded as a time-reversal symmetry. Thus, the classification of crystalline SPT phases for interacting bosonic and free-fermion systems can be computed systematically. For interacting fermion systems, the strategy of classification schemes has also been discussed [28,33–35] and some simple examples have been studied [36,42], however, a detailed understanding of generic cases is still lacking.

In this Rapid Communication we systematically study the construction and classification of two-dimensional (2D) FSPT phases protected by point-group symmetry via a block-state decoration scheme. In particular, we discover several intriguing fermionic point-group SPT phases that cannot be realized in either free-fermion or in interacting boson systems.

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TABLE I. The classification of interacting 2D FSPT phases with point-group symmetry for spinless fermions and spin-1/2 fermions.

	Spin	
G_b	Spinless	Spin-1/2
C_{2m-1}	$\mathbb{Z}_{2m-1}$	$\mathbb{Z}_{2m-1}$
C_{2m}	$\mathbb{Z}_m$	$\begin{cases}\mathbb{Z}_2\times\mathbb{Z}_{4m}, & m\in\text{even}\\ \mathbb{Z}_{8m}, & m\in\text{odd}\end{cases}$
D_{2m-1}	$\mathbb{Z}_2$	$\mathbb{Z}_1$
D_{2m}	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$

Table I summarizes the classification results. We also compare these results with the classification of 2D FSPT phases with the corresponding internal symmetry. We conjecture a fermionic crystalline equivalence principle, stating that a mirror reflection symmetry action should be mapped onto a time-reversal symmetry action, and that spinless (spin-1/2) fermion systems should be mapped onto spin-1/2 (spinless) fermion systems. The possibility of experimental realization is also addressed.

A simple example with a D_4 symmetry. It is well known that there are ten point groups in 2D, classified as cyclic groups C_n and dihedral groups D_n ($n = 1, 2, 3, 4, 6$). As the C_n cases have already been discussed in Ref. [36], here we mainly focus on the D_n cases. Mathematically, each dihedral group is a semidirect product of a rotation and a reflection symmetry group $D_n = C_n \rtimes \mathbb{Z}_2^{\mathrm{M}}$. It eventuates that the most interesting and complicated cases arise for even numbers of n. Below, we will begin with the most intriguing case for spinless fermion systems, namely, the case protected by a D_4 symmetry (the simplest non-Abelian point group with even n).

Similar to Ref. [26], we begin with the "extended trivialization" of D_4. Suppose a $|\psi\rangle$ is an SPT state that cannot be trivialized by symmetric finite-depth local unitary transformations, we can still act with an alternative local unitary to *extensively* trivialize $|\psi\rangle$. First, we trivialize the region U (see Fig. 1), i.e., restrict O^{loc} to U as O_U^{loc} and act it on $|\psi\rangle$,

$$O_U^{\mathrm{loc}}|\psi\rangle = |T_U\rangle \otimes |\psi_{\bar{U}}\rangle, \tag{1}$$

where the system is in the product state $|T_U\rangle$ in region U and the remainder of the system $\bar{U}$ is in the state $|\psi_{\bar{U}}\rangle$. To trivialize the system symmetrically, we denote that $V_g O_U^{\mathrm{loc}} V_g^{-1}$ trivializes gU ($g \in D_4$, see Fig. 1). Therefore, we act on $|\psi\rangle$ with

$$O_R^{\mathrm{loc}} = \bigotimes_{g\in D_4} V_g O_U^{\mathrm{loc}} V_g^{-1}, \tag{2}$$

which results in

$$|\psi'\rangle = O_R^{\mathrm{loc}}|\psi\rangle = \bigotimes_{g\in D_4}|T_{gU}\rangle \otimes \bigotimes_{i=1,a}^{4,d}|\psi_i\rangle \otimes |\psi_{0\mathrm{D}}\rangle. \tag{3}$$

Now, all nontrivial properties of $|\psi\rangle$ are encoded in lower-dimensional block states $|\psi_i\rangle$ and $|\psi_{0\mathrm{D}}\rangle$.

Next, we consider the 1D block state $|\psi_i\rangle$. Let us divide 1D blocks into two categories, i.e., category I (1–4) and category II (a–d). As these two categories are independent under D_4 symmetry, we can discuss the decorations on these categories separately.

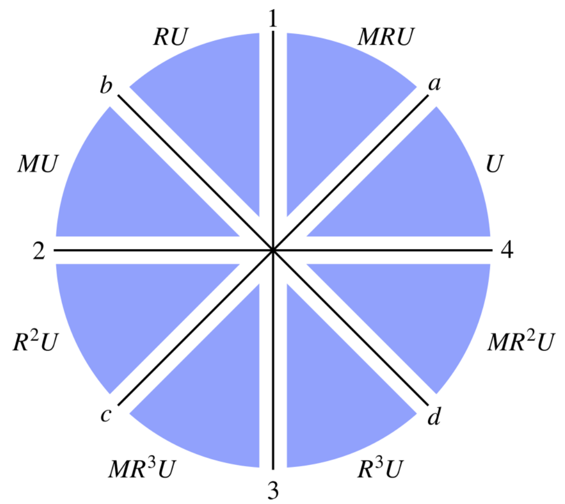

FIG. 1. Extended trivialization of 2D FSPT phases with dihedral group D_4. Here, all shadowed regimes are trivialized according to Eq. (3).

We investigate the decoration on category I ($\mathbb{Z}_2^f$ is the fermion parity, $\boldsymbol{R}$ and $\boldsymbol{M}$ are rotation and reflection generators of D_4). Decorations on $(1, 3)/(2, 4) : (1, 3)/(2, 4)$ are invariant under $\boldsymbol{M}/\boldsymbol{M}\boldsymbol{R}^2$, and the block state should consist of 1D FSPT phases with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ internal symmetry, and be compatible with all other space-group symmetries.

The classification of 1D-invertible topological order (ITO) with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry for interacting fermion systems is $\mathbb{Z}_2^2$ described by the following root phases [14]: (1) the Kitaev Majorana chain [43,44], and (2) an FSPT state that can be realized by two Majorana chains [12–14].

We note that all of these can be realized by fixed-point wave functions and exact-soluble parent Hamiltonians. First, we investigate the Majorana chain decoration. Considering four Majorana chains decorated on category I, there are four Majorana modes $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ located at the 0D block, with the local fermion parity symmetry $P_f = -\gamma_1\gamma_2\gamma_3\gamma_4$ which is odd under rotation: $\boldsymbol{R}P_f\boldsymbol{R}^{-1} = -\gamma_2\gamma_3\gamma_4\gamma_1 = -P_f$. Hence, these four Majorana modes form a projective representation of the group $C_4 \times \mathbb{Z}_2^f$ as a subgroup of $D_4 \times \mathbb{Z}_2^f$. Therefore, a nondegenerate ground state is forbidden. As a consequence, Majorana chain decoration on category I is forbidden by D_4 symmetry, and the above argument is also applicable for category II. Note that if we consider all 1D blocks together and decorate a Majorana chain on each, P_f commutes with rotation. Nevertheless, it is simple to verify that P_f anticommutes with reflection, thus Majorana chain decoration remains forbidden by D_4.

Subsequently, we investigate the decoration of 1D FSPT states on category I. Consider the geometry shown in Fig. 2, with eight Majorana modes located at the rotation center: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $(\gamma_1', \gamma_2', \gamma_3', \gamma_4')$, with rotation and reflection symmetry $\boldsymbol{R}^4 = 1$ and $\boldsymbol{M}^2 = 1$,

$$\boldsymbol{R}: \gamma_i \mapsto \gamma_{i+1}, \quad \gamma_i' \mapsto \gamma_{i+1}', \quad i = 1, 2, 3, 4, \tag{4}$$

$$\boldsymbol{M}: \gamma_i \mapsto -\gamma_{6-i}, \quad \gamma_i' \mapsto \gamma_{6-i}', \quad i = 1, 2, 3, 4. \tag{5}$$

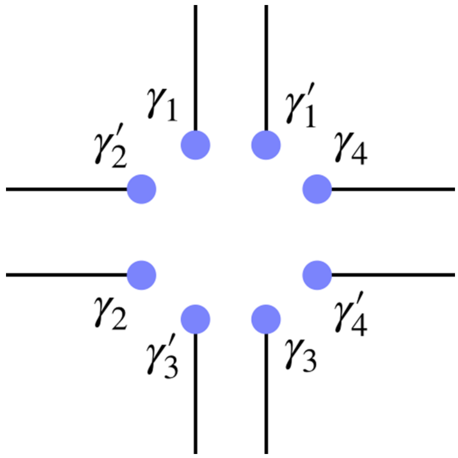

FIG. 2. 0D block of 2D FSPT with D_4 symmetry, corresponding to 1D FSPT phase decoration. Blue dots represent the Majorana zero modes on the edge of decorated root phases.

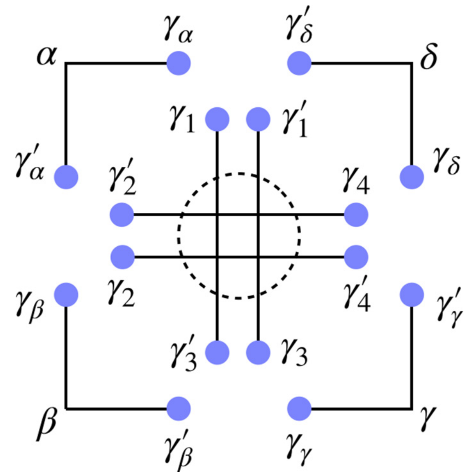

FIG. 3. Forbidden trivialization of 1D FSPT phase decoration on category I for spinless fermions. Again, blue dots represent the Majorana zero modes.

All subscripts take the values with modulo 4, e.g., $\gamma_5 \equiv \gamma_1$ and $\gamma_5' \equiv \gamma_1'$. Now we explain why we need to introduce the above symmetry properties for Majorana modes. Since the rotation properties are easier to understand, we mainly focus on the reflection properties below. For the vertical axis, $\boldsymbol{M}$ acts as an on-site $\mathbb{Z}_2$ symmetry, and the two Majorana modes at the edge of the decorated 1D FSPT state should anticommute with the fermion parity. Thus, for $i = 1, 3$, $\boldsymbol{M} : \gamma_i \mapsto -\gamma_i$, $\gamma_i' \mapsto \gamma_i'$. Similarly, for the horizontal axis, $\boldsymbol{MR}^2$ acts as an on-site $\mathbb{Z}_2$ symmetry, so for $j = 2, 4$, $\boldsymbol{MR}^2 : \gamma_j \mapsto -\gamma_j$, $\gamma_j' \mapsto \gamma_j'$. Together with rotational symmetry Eq. (4), it is easy to verify that for $j = 2, 4$, $\boldsymbol{M} : \gamma_j \mapsto -\gamma_{6-j}$, $\gamma_j' \mapsto \gamma_{6-j}'$.

Finally, we try to gap out these Majorana modes through interactions in a symmetric way. First, we consider the following interacting Hamiltonian,

$$\mathcal{H}_U = U[\gamma_1\gamma_1'\gamma_3\gamma_3' + \gamma_2\gamma_2'\gamma_4\gamma_4'], \quad U > 0. \tag{6}$$

For the ground state,

$$\gamma_1\gamma_1'\gamma_3\gamma_3' = \gamma_2\gamma_2'\gamma_4\gamma_4' = -1. \tag{7}$$

The ground state is fourfold degenerate from Eq. (6). To lift this degeneracy, we can further add a term,

$$\mathcal{H}_J = J(\gamma_1\gamma_2\gamma_1'\gamma_2' + \gamma_1\gamma_2\gamma_3'\gamma_4'), \quad J > 0. \tag{8}$$

Consider the total Hamiltonian $\mathcal{H} = \mathcal{H}_U + \mathcal{H}_J$ and take the limit $U \to \infty$, such that it leads to the constraint Eq. (7). Within the constraint subspace, Hamiltonian (8) is symmetric under D_4 symmetry. Then, because both terms in $\mathcal{H}_J$ commute with each other and have eigenvalues $\pm J$, $\mathcal{H}_J$ has a unique ground state with eigenvalue $-2J$. Thus, we can lift the degeneracy in a D_4 symmetric way and this decoration is compatible with D_4 symmetry.

Below, we argue that such a 1D block-state decoration cannot be trivialized. Considering a 2D system with an open boundary (see Fig. 3), we further place four additional Majorana chains $(\alpha, \beta, \gamma, \delta)$ on the boundary, adding eight additional more Majorana modes (γ_j, γ_j'), $j = \alpha, \beta, \gamma, \delta$. For any group of four Majorana modes, e.g., $(\gamma_1, \gamma_1', \gamma_\alpha, \gamma_\delta')$, at one side of the edge with the following reflection symmetry properties,

$$\boldsymbol{M} : (\gamma_1, \gamma_1', \gamma_\alpha, \gamma_\delta') \mapsto (-\gamma_1, \gamma_1', \gamma_\delta', \gamma_\alpha), \tag{9}$$

this group will be gapped without breaking the reflection symmetry due to the compatibility of local fermion parity $P_f = -\gamma_1\gamma_1'\gamma_\alpha\gamma_\delta'$ [45]. Nevertheless, it is remains unclear whether such a "trivialization scheme" is compatible with the full D_4 symmetry. Considering the Majorana chain labeled by α, the symmetry $\boldsymbol{M'} \equiv \boldsymbol{MR}^3 \in D_4$ acts on α as an effective reflection symmetry. However, because a single open Majorana chain is incompatible with reflection symmetry $\boldsymbol{M'}^2 = 1$ (anticommutes with the total fermion parity), this suggests that the boundary Majorana modes cannot be gapped out without breaking the full D_4 symmetry. As a result, we conclude that the 1D FSPT state decoration on category I must describe a nontrivial 2D FSPT state with D_4 symmetry. In particular, the 2D FSPT state that we have constructed here is an intrinsic interacting FSPT state that cannot be realized by free-fermion systems [46–49] or interacting bosonic systems.

Similar arguments hold for category II. Thus, the question naturally arises of whether or not these two cases describe independent FSPT states. Let us consider the geometry in Fig. 4, where we place eight different Majorana chains labeled by α–θ on the boundary of the system. It is simple to verify that this assignment respects D_4 symmetry. Then, as aforementioned, each Majorana mode on the boundary can be symmetrically gapped out without breaking the two reflection symmetries $\boldsymbol{M}$ and $\boldsymbol{M'}$. Therefore, this case indeed corresponds to a trivial bulk because a gapped, short-range entangled symmetric boundary termination is obtained. That is, the cases of decorating 1D FSPT phases on category I and category II are nonindependent and these two types of decorations give rise to only one nontrivial FSPT state.

Now we consider the 0D block state $|\psi_{0D}\rangle$. As D_4 acts on the 0D block as an internal symmetry, the full data of the 0D block state are [14]

$$\mathcal{H}^0(\mathbb{Z}_4 \rtimes \mathbb{Z}_2, \mathbb{Z}_2) \times \mathcal{H}^1[\mathbb{Z}_4 \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_2 \times \mathbb{Z}_2^2. \tag{10}$$

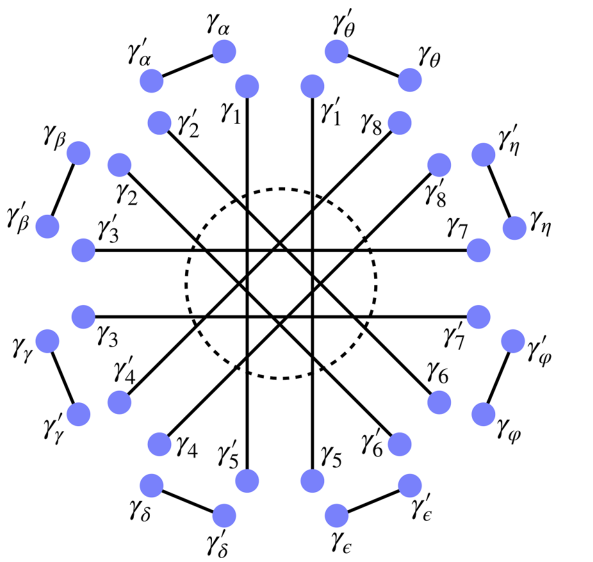

FIG. 4. Trivialization of decorated 1D FSPT state on both category I and category II 1D blocks for spinless fermion systems.

We first consider an atomically insulating state with four complex fermions,

$$|\phi\rangle_{0\mathrm{D}} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle, \tag{11}$$

with the following symmetry properties (all subscripts take the value of modulo 4),

$$\boldsymbol{R}: c_i^\dagger \mapsto c_{i+1}^\dagger, \quad \boldsymbol{M}: c_i^\dagger \mapsto c_{6-i}^\dagger, \quad i = 1, 2, 3, 4. \tag{12}$$

Again, all subscripts take values of modulo 4, and the above symmetry actions on Eq. (11) give rise to

$$\boldsymbol{R}|\psi\rangle_{0\mathrm{D}} = \boldsymbol{M}|\psi\rangle_{0\mathrm{D}} = -|\psi\rangle_{0\mathrm{D}}. \tag{13}$$

Thus, the eigenvalue -1 of rotation symmetry and reflection symmetry indeed corresponds to a topological trivial state. In Ref. [36], a closed Majorana chain surrounding the 0D block is introduced to trivialize the 0D block state with odd fermion parity. This construction can also be applied here and the 0D block state with odd fermion parity will also be trivialized (see Supplemental Material [45] for full details). Therefore, all 0D block states are trivialized and the classification of 2D FSPT phases protected by dihedral symmetry D_4 for spinless fermions is $\mathbb{Z}_2$ (see Table I). This classification coincides with the classification of the 2D FSPT protected by internal symmetry $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^\mathrm{T}$ (where $\mathbb{Z}_2^\mathrm{T}$ is time-reversal symmetry) for spin-1/2 fermions (see Table II).

Finally, we discuss systems with spin-1/2 fermions. Through similar block-state constructions, we obtain a $\mathbb{Z}_2^2$ classification: 1D block-state decorations do not contribute to any nontrivial FSPT phase, because for 1D systems with spin-1/2 fermions and $\mathbb{Z}_2$ symmetry (total symmetry group is $\mathbb{Z}_4^f$), there is no nontrivial SPT phase [14]. For the 0D block, the first $\mathbb{Z}_2$ of Eq. (10) is not allowed [14], and Eq. (13) has no nontrivial eigenvalue under rotation and reflection. [We note that for spin-1/2 fermions, there would be an extra i factor in Eq. (12) with $\boldsymbol{R}^2 = \boldsymbol{M}^2 = -1$, which cancels the -1 in Eq. (13).] As a result, there is no trivialization in this case. Thus, 0D block-state decorations contribute to a $\mathbb{Z}_2^2$ classification and the overall classification of 2D FSPT phases with D_4 symmetry for spin-1/2 fermions is $\mathbb{Z}_2^2$.

TABLE II. The interacting classification of 2D FSPT phases with internal symmetries, for spinless and spin-1/2 fermions, respectively. $\mathbb{Z}_2^\mathrm{T}$ is time-reversal symmetry.

	Spin	
G_b	Spinless	Spin-1/2
$\mathbb{Z}_{2m-1}$	$\mathbb{Z}_{2m-1}$	$\mathbb{Z}_{2m-1}$
$\mathbb{Z}_{2m}$	$\begin{cases}\mathbb{Z}_2\times\mathbb{Z}_{4m}, & m\in\text{even}\\ \mathbb{Z}_{8m}, & m\in\text{odd}\end{cases}$	$\mathbb{Z}_m$
$\mathbb{Z}_{2m-1}\rtimes\mathbb{Z}_2^\mathrm{T}$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$\mathbb{Z}_{2m}\rtimes\mathbb{Z}_2^\mathrm{T}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$

Point-group FSPT with general D_n symmetry. Spinless and spin-1/2 fermion systems with D_2 and D_6 point-group symmetry can also be constructed in a similar way and the classification results are exactly the same as the D_4 case. Moreover, a 2D FSPT state protected by D_n symmetry with odd n can also be constructed by similar block-state decorations as aforementioned. In fact, the essential contributions of nontrivial FSPT phases in these cases solely derive from the reflection subgroup. All results of classification are summarized in Table I, and the full details of the wave-function constructions can be found in the Supplemental Material [45].

Generalized crystalline equivalence principle. To this end, we would like to examine the generalized crystalline equivalence principle for 2D interacting fermion systems: We calculate the classification of 2D FSPT phases with internal symmetry $\mathbb{Z}_n$ and $\mathbb{Z}_n \rtimes \mathbb{Z}_2^\mathrm{T}$ using the so-called general group supercohomology theory [14]. The classification results are shown in Table II (see Supplemental Material [45] for detailed calculations). Comparing the results in Tables I and II, we conjecture that the crystalline equivalence principle can be generalized into 2D interacting fermion systems. We should map the *mirror reflection symmetry* onto an internal *time-reversal symmetry* and we should also map spinless (spin-1/2) fermions onto spin-1/2 (spinless) fermions. The twist on spinless and spin-1/2 fermions can be naturally interpreted as the spin rotation of fermions: A 2π rotation of a fermion around a specific axis results in a -1 phase factor (see Supplemental Material [45] for more details).

Conclusion and discussion. In this Rapid Communication, we systematically constructed and classified the 2D interacting FSPT phases with point-group symmetry using explicit block-state constructions (with D_4 symmetry as a concrete example). Our results also verify the generalized crystalline equivalence principle for 2D interacting fermionic systems. Experimentally, single-layer iron selenide (FeSe, an iron-based superconductor with space group $P4/nmm$ in 3D [50–53]) on a ferromagnetic substrate could be a natural candidate for realizing an intrinsic FSPT phase with D_4 symmetry [54]. (We note that the fermions in iron selenide are spin polarized due to the ferromagnetic proximity, and can thus be effectively treated as spinless.) According to our block-state construction, there are several dangling Majorana fermions located at the boundary of the system, and these gapless modes can be detected by experiments in principle. (These boundary Majorana modes might be related to the recent proposed "corner Majorana modes" in monolayer iron selenides [55,56].) Moreover, the bulk topological defects observed by scanning tunneling microscopy (STM) tomography might provide us another way to detect these SPT states. Finally, we stress that the method proposed here is also applicable to space-group symmetry, and that exploration of its higher-dimensional generalizations would be a very interesting future direction.

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[1] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[2] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[3] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[5] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[6] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[7] E. Plamadeala, M. Mulligan, and C. Nayak, Phys. Rev. B **88**, 045131 (2013).
[8] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[9] L. Kong and X.-G. Wen, arXiv:1405.5858
[10] D. S. Freed, arXiv:1406.7278.
[11] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[12] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **90**, 115141 (2014).
[13] Q.-R. Wang and Z.-C. Gu, Phys. Rev. X **8**, 011055 (2018).
[14] Q.-R. Wang and Z.-C. Gu, arXiv:1811.00536.
[15] A. Kapustin, arXiv:1403.1467.
[16] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, J. High Energy Phys. 12 (2015) 052.
[17] Z.-C. Gu and M. Levin, Phys. Rev. B **89**, 201113(R) (2014).
[18] M. Cheng and Z.-C. Gu, Phys. Rev. Lett. **112**, 141602 (2014).
[19] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Phys. Rev. B **100**, 115147 (2019).
[20] N. Tantivasadakarn, Phys. Rev. B **96**, 195101 (2017).
[21] C. Wang, C.-H. Lin, and Z.-C. Gu, Phys. Rev. B **95**, 195147 (2017).
[22] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, Phys. Rev. B **97**, 205109 (2018).
[23] M. Cheng, N. Tantivasadakarn, and C. Wang, Phys. Rev. X **8**, 011054 (2018).
[24] L. Fu, Phys. Rev. Lett. **106**, 106802 (2011).
[25] H. Isobe and L. Fu, Phys. Rev. B **92**, 081304(R) (2015).
[26] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Phys. Rev. X **7**, 011020 (2017).
[27] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, Phys. Rev. B **96**, 205106 (2017).
[28] R. Thorngren and D. V. Else, Phys. Rev. X **8**, 011040 (2018).
[29] L. Zou, Phys. Rev. B **97**, 045130 (2018).
[30] H. Song, C. Z. Xiong, and S.-J. Huang, arXiv:1811.06558.
[31] K. Shiozaki, M. Sato, and K. Gomi, arXiv:1802.06694.
[32] Z. Song, S.-J. Huang, Y. Qi, C. Fang, and M. Hermele, Sci. Adv. **5**, eaax2007 (2019).
[33] D. V. Else and R. Thorngren, Phys. Rev. B **99**, 115116 (2019).
[34] Z. Song, C. Fang, and Y. Qi, arXiv:1810.11013.
[35] K. Shiozaki, C. Z. Xiong, and K. Gomi, arXiv:1810.00801.
[36] M. Cheng and C. Wang, arXiv:1810.12308.
[37] A. Rasmussen and Y.-M. Lu, arXiv:1809.07325.
[38] Y. Tanaka, Z. Ren, T. Sato, K. Nakayama, S. Souma, T. Takahashi, K. Segawa, and Y. Ando, Nat. Phys. **8**, 800 (2012).
[39] P. Dziawa, B. J. Kowalski, K. Dybko, R. Buczko, A. Szczerbakow, M. Szot, E. Łusakowska, T. Balasubramanian, B. M. Wojek, M. H. Berntsen, O. Tjernbergm, and T. Story, Nat. Mater. **11**, 1023 (2012).
[40] Y. Okada, M. Serbyn, H. Lin, D. Walkup, W. Zhou, C. Dhital, M. Neupane, S. Xu, Y. J. Wang, R. Sankar, F. Chou, A. Bansil, M. Z. Hasan, S. D. Wilson, L. Fu, and V. Madhavan, Science **341**, 1496 (2013).
[41] J. Ma, C. Yi, B. Lv, Z. Wang, S. Nie, L. Wang, L. Kong, Y. Huang, P. Richard, P. Zhang, K. Yaji, K. Kuroda, S. Shin, H. Weng, B. A. Bernevig, Y. Shi, T. Qian, and Hong Ding, Sci. Adv. **3**, e1602415 (2017).
[42] A. Rasmussen and Y.-M. Lu, arXiv:1810.12317.
[43] A. Kitaev, Phys. Usp. **44**, 131 (2001).
[44] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).
[45] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.101.100501 for more detailed discussions containing the following: (a) An explanation of why there is a twist between spinless and spin-1/2 fermions when applying crystalline equivalence principle for 2D interacting fermion systems. (b) An explanation of why Eq. (8) is symmetric under D_4 symmetry. (c) How to introduce a mass term to gap out the boundary Majorana zero modes. (d) Some calculational details for 2D FSPT phases with odd-fold dihedral group. (e) A classification scheme for 2D FSPT phases with the corresponding internal symmetry.
[46] C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B **86**, 115112 (2012).
[47] R.-J. Slager, A. Mesaros, V. Juricic, and J. Zaanen, Nat. Phys. **9**, 98 (2013).
[48] L. Lu, C. Fang, L. Fu, S. G. Johnson, J. D. Joannopoulos, and M. Soljačić, Nat. Phys. **12**, 337 (2016).
[49] H. C. Po, A. Vishwanath, and H. Watanabe, Nat. Commun. **8**, 50 (2017).
[50] J. Paglione and R. J. Greeny, Nat. Phys. **6**, 645 (2010).
[51] S. He, J. He, W. Zhang, L. Zhao, D. Liu, X. Liu, D. Mou, Y.-B. Ou, Q.-Y. Wang, Z. Li, L. Wang, Y. Peng, Y. Liu, C. Chen,

L. Yu, G. Liu, X. Dong, J. Zhang, C. Chen, Z. Xu, X. Chen, X. Ma, Q. Xue, and X. J. Zhou, Nat. Mater. **12**, 605 (2013).

[52] S. Tan, Y. Zhang, M. Xia, Z. Ye, F. Chen, X. Xie, R. Peng, D. Xu, Q. Fan, H. Xu, J. Jiang, T Zhang, X. Lai, T. Xiang, J. Hu, B. Xie, and Donglai Feng, Nat. Mater. **12**, 634 (2013).

[53] J.-F. Ge, Z.-L. Liu, C. Liu, C.-L. Gao, D. Qian, Q.-K. Xue, Y. Liu, and J.-F. Jia, Nat. Mater. **14**, 285 (2015).

[54] The space group of 3D iron selenide is $P4/nmm$, and according to recent STM experiments, there is no nematicity for monolayer iron selenide. Some defect needs to be introduced to break translational symmetry to realize D_4 symmetry.

[55] R.-X. Zhang, W. S. Cole, X. Wu, and S. Das Sarma, Phys. Rev. Lett. **123**, 167001 (2019).

[56] X. Wu, X. Liu, R. Thomale, and C.-X. Liu, arXiv:1905.10648.