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Nonarchimedean components of non-endoscopic automorphic representations for quasisplit $Sp(N)$ and $O(N)$

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Abstract
Arthur classified the discrete automorphic representations of symplectic and orthogonal groups over a number field by that of the general linear groups. In this classification, those that are not from endoscopic lifting correspond to pairs (ϕ, b), where ϕ is an irreducible unitary cuspidal automorphic representation of some general linear group and b is an integer. In this paper, we study the local components of these automorphic representations at a nonarchimedean place, and we give a complete description of them in terms of their Langlands parameters.

Keywords Symplectic and orthogonal group · Arthur packet · Jacquet functor

Mathematics Subject Classification Primary 22E50; Secondary 11F70

1 Introduction

Let G be a split symplectic or special odd orthogonal group over a number field k. Arthur [1] proved the automorphic representations of $G(\mathbb{A}_k)$ can be parametrized by the global Arthur parameters, which are isobaric sums

$$\psi = \boxplus_i (\phi_i \boxtimes \nu_{b_i}),$$

where ϕ_i is certain irreducible unitary cuspidal automorphic representation of a general linear group and ν_{b_i} is the $(b_i - 1)$-th symmetric power representation of $SL(2, \mathbb{C})$. For any such ψ, Arthur attached a global Arthur packet Π_ψ, which is a multi-set of isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_k)$. This packet admits a restricted tensor

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product decomposition

$$\Pi_{\psi} := \otimes_v \, \Pi_{\psi_v}$$

where we denote by ψ_v the local component of ψ at each place v, and Π_{ψ_v} is a multi-set of isomorphism classes of irreducible admissible representations of $G(k_v)$, called local Arthur packet. By the local Langlands correspondence for general linear groups [2–4,8], we can associate $\phi_{i,v}$ with a representation of the Weil–Deligne group $WD_{k_v} := W_{k_v} \times SL(2, \mathbb{C})$ at the nonarchimedean places (resp. W_{k_v} at the archimedean places), which will still be denoted by $\phi_{i,v}$. Then ψ_v can be viewed as a representation of $WD_{k_v} \times SL(2, \mathbb{C})$ at the nonarchimedean places (resp. $W_{k_v} \times SL(2, \mathbb{C})$ at the archimedean places). In particular, Arthur showed that it factors through the Langlands dual group of $G(k_v)$. We will call ψ_v a local Arthur parameter for $G(k_v)$. In this paper, we would like to describe the Langlands parameters of the elements inside Π_{ψ_v}, when ψ consists of a single term, i.e.,

$$\psi = \phi \boxtimes \nu_b \tag{1.1}$$

and v is a nonarchimedean place. It follows from Arthur's theory [1] that the representations in such Π_{ψ} do not come from endoscopic lifting, so this justifies our title.

From now on, we will let G be a split symplectic or special odd orthogonal group over a p-adic field F. Let $\widehat{G}$ be the complex dual group of G. We recall an Arthur parameter for $G(F)$ is a $\widehat{G}$-conjugacy class of admissible homomorphisms

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow \widehat{G}$$

with the property that $\psi(W_F)$ is bounded. By composing with the standard representation of $\widehat{G}$, we can view ψ as a representation of $W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. It decomposes as

$$\psi = \oplus_{i=1}^{n} \rho_i \otimes \nu_{a_i} \otimes \nu_{b_i} \tag{1.2}$$

where ρ_i is an irreducible unitary representation of W_F and $a_i, b_i \in \mathbb{Z}$. To describe the associated packet Π_{ψ}, we will take ρ_i to be the corresponding irreducible supercuspidal representation of $GL(d_{\rho_i}, F)$ through the local Langlands correspondence. Then we can construct a self-dual representation of $GL(N, F)$ by

$$\pi_{\psi}^{GL} := \times_{i=1}^{n} \; Sp(St(\rho_i, a_i), b_i),$$

which is an induction of the Speh representations. Recall the Steinberg representation $St(\rho_i, a_i)$ is the unique irreducible subrepresentation of the induction

$$\rho_i||^{(a_i-1)/2} \times \rho_i||^{(a_i-3)/2} \cdots \times \rho_i||^{-(a_i-1)/2}$$

and the Speh representation $Sp(St(\rho_i, a_i), b_i)$ is the unique irreducible subrepresentation of

$$St(\rho_i, a_i)||^{-(b_i-1)/2} \times St(\rho_i, a_i)||^{-(b_i-3)/2} \cdots \times St(\rho_i, a_i)||^{(b_i-1)/2} \tag{1.3}$$

We will also denote the Steinberg representation by

$$\langle (a_i - 1)/2, \ldots, -(a_i - 1)/2 \rangle$$

and the Speh representation by a matrix

$$\begin{pmatrix} (a_i - b_i)/2 & \cdots & 1 - (a_i + b_i)/2 \\ \vdots & & \vdots \\ (a_i + b_i)/2 - 1 & \cdots & -(a_i - b_i)/2 \end{pmatrix} \tag{1.4}$$

where each row corresponds to the exponents of the shifted Steinberg representations in (1.3). Since π_{ψ}^{GL} is self-dual, one can consider its twisted character. Arthur [1] proved that there exists a stable finite linear combination of characters on $G(F)$, whose twisted endoscopic transfer is this twisted character. By the linear independence of characters, this determines Π_{ψ} as a finite subset of isomorphism classes of irreducible admissible representations of $G(F)$. (Moeglin [7] proved the Arthur packet is always multiplicity free in this case.) However, this does not tell us explicitly which representations are contained in it. To answer this question, we need a parametrization of the set $\mathrm{Irr}(G(F))$ of isomorphism classes of irreducible admissible representations of $G(F)$. This is given by the local Langlands correspondence for $G(F)$. Arthur [1] proved that there is a canonical bijection (after fixing a Whittaker datum)

$$\mathrm{Irr}(G(F)) \cong \{(\phi, \epsilon) | \phi \in \Phi(G(F)), \epsilon \in \mathrm{Irr}(\mathcal{S}_{\phi})\},$$

where $\Phi(G(F))$ is the set of Langlands parameters, which are $\widehat{G}$-conjugacy classes of admissible homomorphisms

$$\phi : W_F \times SL(2, \mathbb{C}) \to \widehat{G},$$

and

$$\mathcal{S}_{\phi} := \pi_0(Z_{\widehat{G}}(\phi)/Z(\widehat{G})),$$

where $Z_{\widehat{G}}(\phi)$ is the stabilizer of ϕ in $\widehat{G}$ and $Z(\widehat{G})$ is the center of $\widehat{G}$. We will call the pair (ϕ, ϵ) complete Langlands parameter of $G(F)$, and denote the corresponding representation by $\pi(\phi, \epsilon)$.

Back to the Arthur parameter (1.2), let us write $A_i = (a_i + b_i)/2 - 1$, $B_i = |a_i - b_i|/2$ and $\zeta_i = \mathrm{sgn}(a_i - b_i)$. When $a_i = b_i$, we may choose ζ_i arbitrarily. We will call (ρ, a_i, b_i) or $(\rho, A_i, B_i, \zeta_i)$ Jordan blocks, and denote the set of Jordan blocks by $Jord(\psi)$. For simplicity, we will assume that

$$\rho_i = \rho \text{ for some fixed } \rho, \text{ and } (\rho, a_i, b_i) \text{ all have the same parity as } \widehat{G}. \tag{1.5}$$

Since we want to study the local component of a global Arthur parameter of the type (1.1), we can assume all b_i are equal and denote it by b. So we may rewrite (1.2) as

$$\psi = \oplus_{i=1}^{n} \rho \otimes \nu_{a_i} \otimes \nu_b \tag{1.6}$$

Under our assumptions, all a_i will have the same parity. The simplest case is when ψ consists of a single term, i.e.,

$$\psi = \rho \otimes \nu_a \otimes \nu_b$$

and $a \geq b$. In this case, we have the following result due to Mœglin [6, Theorem 4.2]. Firstly, there is a bijection

$$\Pi_{\psi} \to \{(l, \eta) \in \mathbb{Z} \times \{\pm 1\} \,|\, 0 \leq l \leq [(A - B + 1)/2] \text{ and } \epsilon_{l,\eta} = 1\}/\sim$$

where

$$\epsilon_{l,\eta} := \eta^{A-B+1}(-1)^{[(A-B+1)/2]+l} \tag{1.7}$$

and the equivalence relation $\sim$ only identifies those (l, η) and (l', η') for $l = l' = (A - B + 1)/2$. Secondly, the representation $\pi(\psi, l, \eta)$ parametrized by (l, η) satisfies

$$\pi(\psi, l, \eta) \hookrightarrow \begin{pmatrix} B & \cdots & -A \\ \vdots & & \vdots \\ B + l - 1 & \cdots & -(A - l + 1) \end{pmatrix} \rtimes \pi(\phi', \epsilon') \tag{1.8}$$

as the unique irreducible subrepresentation. Here the matrix represents a shifted Speh representation (cf. (1.4)) and $\pi(\phi', \epsilon')$ is a discrete series representation of $G'(F)$, which is of the same type as $G(F)$. The complete Langlands parameter ϕ_l of the shifted Speh representation factors through that of the inducing representation (cf. (1.3)), i.e.,

$$\phi_l = \oplus_{i=0}^{l-1} (\rho||^{i-(A-B)/2} \otimes \nu_{A+B+1})$$

and

$$\phi' = \oplus_{C=B+l}^{A-l} \rho \otimes \nu_{2C+1}.$$

We can view ϕ' as an Arthur parameter, where the second $SL(2, \mathbb{C})$ maps trivially. Then its Jordan blocks are $(\rho, C, C, +)$ for $B + l \leq C \leq A - l$. The character ϵ' can be represented by a sign function over this set of Jordan blocks. In this way, we have

$$\epsilon'(\rho, C, C, +) = (-1)^{C-(B+l)}\eta.$$

The sign condition $\epsilon_{l,\eta} = 1$ guarantees that

$$\prod_{C=B+l}^{A-l} \epsilon'(\rho, C, C, +) = 1,$$

which is the necessary condition for ϵ' to define a character of $\mathcal{S}_{\phi'}$. One can also describe the complete Langlands parameter (ϕ, ϵ) for $\pi(\psi, l, \eta)$ from the embedding (1.8). Indeed,

$$\phi = \phi_l \oplus \phi' \oplus \phi_l^\vee.$$

Moreover, ϵ corresponds to ϵ' under the natural isomorphisms $\mathcal{S}_\phi \cong \mathcal{S}_{\phi'}$. By Mœglin's result, one can also view $\pi(\phi', \epsilon')$ as an element in $\Pi_{\psi'}$, where ψ' is the Arthur parameter of $G'(F)$ consisting of only one Jordan block $(\rho, A - l, B + l, +)$. In particular,

$$\pi(\phi', \epsilon') = \pi(\psi', l', \eta') \tag{1.9}$$

for $l' = 0$ and $\eta' = \epsilon'(\rho, B + l, B + l, +)$. To save notations, we will write

$$\pi(\psi', l', \eta') = \pi((\rho, A - l, B + l, 0, \eta', +)).$$

In general, we can divide the Jordan blocks in (1.6) into two classes.

- $a_i \geq b$, i.e., $\zeta_i = +$: $A_i - B_i = b - 1$, $A_i + B_i = a_i - 1$. So all intervals $[B_i, A_i]$ have the same length and are centered beyond $(b - 1)/2$.
- $a_i < b$, i.e., $\zeta_i = -$: $A_i - B_i = a_i - 1$, $A_i + B_i = b - 1$. So all intervals are centered at $(b - 1)/2$.

We reorder the Jordan blocks such that $A_i \geq A_{i-1}$. Then there exists an integer m such that $(\rho, A_i, B_i, \zeta_i)$ is in the first class if $i > m$ and the second class if $i \leq m$. Now we can state our main results.

1.1 Reductions

Theorem 1.1 *Suppose $A_i, B_i \in \mathbb{Z}$. Let ψ' be obtained from ψ by replacing all $(\rho, A_i, B_i, \zeta_i)$ by $(\rho, A'_i, B'_i, \zeta'_i)$ such that: $\zeta'_i = +$ and*

- *$A'_i = A_i$, $B'_i = B_i$ for $i > m$;*
- *$A'_i = A_i - B_i$, $B'_i = 0$ for $i \leq m$.*

Then there is a bijection

$$\Pi_\psi \to \Pi_{\psi'}, \quad \pi \mapsto \pi'$$

such that any representation $\pi \in \Pi_\psi$ *is given as the unique irreducible subrepresentation of*

$$\pi \hookrightarrow \times_{i \le m} \begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1 & \cdots & -(A_i - B_i + 1) \end{pmatrix} \rtimes \pi'$$

for the corresponding $\pi' \in \Pi_{\psi'}$*. Moreover, if* (ϕ', ϵ') *is the complete Langlands parameter of* π'*, then the complete Langlands parameter* (ϕ, ϵ) *of* π *is given as follows:*

$$\phi = (\oplus_{i \le m} \phi_i) \oplus \phi' \oplus (\oplus_{i \le m} \phi_i^\vee),$$

where ϕ_i *is the Langlands parameter of the corresponding shifted Speh representation and* ϵ *corresponds to* ϵ' *under the natural isomorphism* $\mathcal{S}_\phi \cong \mathcal{S}_{\phi'}$*.*

Theorem 1.2 *Suppose* $A_i, B_i \notin \mathbb{Z}$*. We consider the maximal sequence of integers*

$$0 = s_0 < s_1 < \cdots < s_l = m$$

such that $A_{s_j} - B_{s_j} \neq A_{s_j+1} - B_{s_j+1}$*. For any* $0 \le k \le l$*, we get a new parameter* ψ'_k *by replacing all* $(\rho, A_i, B_i, \zeta_i)$ *by* $(\rho, A'_i, B'_i, \zeta'_i)$ *such that:* $\zeta'_i = +$ *and*

- $A'_i = A_i, B'_i = B_i$ *for* $i > m$*;*
- $A'_i = A_i - B_i - 1/2$, $B'_i = 1/2$ *for* $i \le m$ *and* $i \neq s_k$*;*
- $A'_i = A_i - B_i + 1/2$, $B'_i = 1/2$ *for* $i = s_k$*.*

Then we can divide Π_ψ *into* $l+1$ *classes, i.e.,*

$$\Pi_\psi = \sqcup_{k=0}^{l} \Pi_\psi(k),$$

and for any $0 \le k \le l$*, we can get an injection*

$$\Pi_\psi(k) \hookrightarrow \Pi_{\psi'_k}, \quad \pi \mapsto \pi(\psi'_k, \underline{l}', \underline{\eta}'),$$

such that

$$\pi \hookrightarrow \times_{s_k \neq i \le m} \begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i - B_i + 1/2) \end{pmatrix}$$
$$\times \begin{pmatrix} -B_{s_k} & \cdots & -A_{s_k} \\ \vdots & & \vdots \\ -3/2 & \cdots & -(A_{s_k} - B_{s_k} + 3/2) \end{pmatrix} \rtimes \pi(\psi'_k, \underline{l}', \underline{\eta}')$$

as the unique irreducible subrepresentation. Here we have parametrized the elements of $\Pi_{\psi'_k}$ *by* $(\underline{l}', \underline{\eta}')$ *as explained in Sect. 1.2 below. The image is characterized by the condition that for all* $i \le s_k$*,*

- $l'_i = 0$*;*
- $\eta'_i = -\prod_{j<i} (-1)^{A_j - B_j + 1}$*.*

When $k \neq 0$, the second condition can also be simplified as $\eta'_1 = -1$. Moreover, if (ϕ', ϵ') is the complete Langlands parameter of $\pi(\psi'_k, \underline{l}', \underline{\eta}')$, then the complete Langlands parameter (ϕ, ϵ) of π is given as follows:

$$\phi = (\oplus_{i \le m} \phi_i) \oplus \phi' \oplus (\oplus_{i \le m} \phi_i^{\vee}),$$

where ϕ_i is the Langlands parameter of the corresponding shifted Speh representation and ϵ corresponds to ϵ' under the natural isomorphism $\mathcal{S}_\phi \cong \mathcal{S}_{\phi'}$.

1.2 A special case

The previous two theorems reduce our problem to the following special case (cf. ψ', ψ'_k):

$$\psi = \oplus_{i=1}^{n} (\rho \otimes \nu_{a_i} \otimes \nu_{b_i}),$$

where $A_i \ge A_{i-1}$, $B_i \ge B_{i-1}$ and $\zeta_i = +$. In this case, we have the following result.

Theorem 1.3 *Suppose we are in the special case described above.*

(1) There is a bijection

$$\Pi_\psi \to \{(\underline{l}, \underline{\eta}) \in \mathbb{Z}^n \times \{\pm 1\}^n \,|\, 0 \le l_i \le [(A_i - B_i + 1)/2], \text{(1.10) and (1.11) are satisfied}\}/\sim$$

where $\underline{l} = (l_i)$, $\underline{\eta} = (\eta_i)$, and

$$\begin{cases} \eta_{i+1} = (-1)^{A_i - B_i} \eta_i & \Rightarrow A_{i+1} - l_{i+1} \ge A_i - l_i, \quad B_{i+1} + l_{i+1} \ge B_i + l_i, \\ \eta_{i+1} \neq (-1)^{A_i - B_i} \eta_i & \Rightarrow B_{i+1} + l_{i+1} > A_i - l_i. \end{cases} \tag{1.10}$$

and

$$\prod_{i=1}^{n} \epsilon_{l_i, \eta_i} = 1 \tag{1.11}$$

where ϵ_{l_i, η_i} is defined as in (1.7). We have identified $(\underline{l}, \underline{\eta}) \sim (\underline{l}', \underline{\eta}')$, whenever

$$\begin{cases} \underline{l} = \underline{l}' \\ \eta_i = \eta'_i \text{ unless } l_i = (A - B + 1)/2 \end{cases}$$

(2) Let $\pi(\psi, \underline{l}, \underline{\eta})$ be the representation parametrized by $(\underline{l}, \underline{\eta})$. Consider the maximal sequence of integers

$$0 = k_0 < \cdots < k_r = n$$

such that $A_{k_j} - l_{k_j} < B_{k_j + 1} + l_{k_j + 1}$. When $A_i - l_i \ge B_{i+1} + l_{i+1}$, we take

$$t_i = \frac{(A_i - l_i) + (B_{i+1} + l_{i+1})}{2}$$

and

$$\delta_i = \begin{cases} 1 & \text{if } t_i - A_i \in \mathbb{Z} \\ 1/2 & \text{if } t_i - A_i \notin \mathbb{Z} \end{cases}$$

Then we have

$$\pi(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=1}^{n} \underbrace{\begin{pmatrix} B_i & \cdots & -A_i \\ \vdots & & \vdots \\ B_i + l_i - 1 & \cdots & -(A_i - l_i + 1) \end{pmatrix}}_{I_i}$$

$$\times_{i=1}^{r} \times_{k_{i-1} < j < k_i} \underbrace{\begin{pmatrix} B_{j+1} + l_{j+1} & \cdots & -(A_j - l_j) \\ \vdots & & \vdots \\ t_j - \delta_j & \cdots & -(t_j + \delta_j) \end{pmatrix}}_{\tilde{I}_j} \rtimes \pi'$$

as the unique irreducible subrepresentation, where

$$\pi' = \pi\Big(\cup_i \Big\{ \cdots \Big\} \Big)$$

with

$$\Big\{ \cdots \Big\} = \Big\{ (\rho, A_{k_i} - l_{k_i}, B_{k_i} + l_{k_i}, 0, \eta_{k_i}, +) \Big\}$$

if $k_i - k_{i-1} = 1$, *and*

$$\begin{aligned} \Big\{ \cdots \Big\} = \Big\{ &\Big(\rho, A_{k_i} - l_{k_i}, t_{k_i-1} - \delta_{k_i-1} + 1, 0, (-1)^{t_{k_i-1} - \delta_{k_i-1} + 1 - (B_{k_i} + l_{k_i})} \eta_{k_i}, +\Big), \\ &\cup_{k_{i-1}+1 < j < k_i} \Big(\rho, t_j + \delta_j - 1, t_{j-1} - \delta_{j-1} + 1, 0, (-1)^{t_{j-1} - \delta_{j-1} + 1 - (B_j + l_j)} \eta_j, +\Big), \\ &\Big(\rho, t_{k_{i-1}+1} + \delta_{k_{i-1}+1} - 1, B_{k_{i-1}+1} + l_{k_{i-1}+1}, 0, \eta_{k_{i-1}+1}, +\Big) \Big\} \end{aligned}$$

otherwise. Here π' *is a tempered representation of a group* $G'(F)$ *of the same type as* $G(F)$, *and its complete Langlands parameter* (ϕ', ϵ') *can be described as in* (1.9). *Moreover, the complete Langlands parameter* (ϕ, ϵ) *of* $\pi(\psi, \underline{l}, \underline{\eta})$ *is given as follows:*

$$\phi = (\oplus_j \tilde{\phi}_j) \oplus (\oplus_i \phi_i) \oplus \phi' \oplus (\oplus_i \phi_i^\vee) \oplus (\oplus_j \tilde{\phi}_j^\vee)$$

where ϕ_i *(resp.* $\tilde{\phi}_j$*) is the Langlands parameter of the corresponding shifted Speh representation* I_i *(resp.* $\tilde{I}_j$*) and* ϵ *corresponds to* ϵ' *under the natural isomorphism* $\mathcal{S}_\phi \cong \mathcal{S}_{\phi'}$.

Remark 1.4 When the intervals $[B_i, A_i]$ are disjoint, the condition (1.10) becomes void. In that case, the result is due to Mœglin [6, Theorem 4.2].

1.3 Even orthogonal groups

Let G be a quasisplit special even orthogonal group over F, split over a quadratic extension E/F. Let θ_0 be an outer automorphism of G over F, induced from the conjugate action of the even orthogonal group. Let $\Sigma_0 = \langle \theta_0 \rangle$ and $G^{\Sigma_0} = G \rtimes \Sigma_0$, which is isomorphic to the even orthogonal group. Let $\hat{\theta}_0$ be the dual automorphism on $\hat{G}$, which commutes with the action of $\mathrm{Gal}(E/F)$. The local Langlands correspondence for $G^{\Sigma_0}(F)$ takes the following form: there is a canonical bijection (after fixing a θ_0-stable Whittaker datum)

$$\mathrm{Irr}(G^{\Sigma_0}(F)) \cong \{(\phi, \epsilon) | \phi \in \bar{\Phi}(G(F)), \epsilon \in \mathrm{Irr}(\mathcal{S}_\phi^{\Sigma_0})\},$$

where $\bar{\Phi}(G(F))$ is the set of $\hat{\theta}_0$-orbits of Langlands parameters of $G(F)$, which are $\widehat{G} \rtimes \langle\hat{\theta}_0\rangle$-conjugacy classes of admissible homomorphisms

$$\phi : W_F \times SL(2, \mathbb{C}) \to \widehat{G} \rtimes \mathrm{Gal}(E/F)$$

and

$$\mathcal{S}_{\phi}^{\Sigma_0} := \pi_0(Z_{\widehat{G} \rtimes \langle\hat{\theta}_0\rangle}(\phi)/Z(\widehat{G})^{\mathrm{Gal}(E/F)}).$$

This follows from Arthur's results on the local Langlands correspondence for $G(F)$ and the θ_0-twisted endoscopic character relations (cf. [1] and [9, Theorem 4.3]). We will call the pair (ϕ, ϵ) complete Langlands parameter of $G^{\Sigma_0}(F)$, and denote the corresponding representation by $\pi^{\Sigma_0}(\phi, \epsilon)$.

For any Arthur parameter ψ of $G(F)$, Arthur [1] has associated it with a finite multi-set $\bar{\Pi}_{\psi}$ of Σ_0-orbits in $\mathrm{Irr}(G(F))$, in the same way as we have described for symplectic and special odd orthogonal groups. This is also multiplicity free due to Moeglin [7]. In [10] we define the Arthur packet $\Pi_{\psi}^{\Sigma_0}$ for $G^{\Sigma_0}(F)$ to be the subset of isomorphism classes of irreducible representations of $G^{\Sigma_0}(F)$, whose restriction to $G(F)$ have irreducible constituents in $\bar{\Pi}_{\psi}$. In this paper, we will also prove the analogues of Theorem 1.1, 1.2, 1.3 for $\Pi_{\psi}^{\Sigma_0}$.

2 Review of Moeglin's parametrization

From now on, we will let G be a quasisplit symplectic or special orthogonal group over a p-adic field F. In order to get a uniform description, we will also take $\Sigma_0 = 1$ and $G^{\Sigma_0} = G$, when G is not special even orthogonal. Let ψ be an Arthur parameter of $G(F)$. We will review Moeglin's parametrization of elements in $\Pi_{\psi}^{\Sigma_0}$. The reader is referred to [9,10] for more details.

Let ψ_p be the parameter consisting of Jordan blocks of ψ that has the same parity as $\widehat{G}$, and $>_{\psi}$ be an admissible order on $Jord(\psi_p)$. The admissibility condition requires that for any $(\rho, A, B, \zeta), (\rho, A', B', \zeta') \in Jord(\psi_p)$ satisfying $A > A'$, $B > B'$ and $\zeta = \zeta'$, we have $(\rho, A, B, \zeta) >_{\psi} (\rho, A', B', \zeta')$. Then Mœglin showed that there is an injection depending on $>_{\psi}$

$$\begin{aligned} \Pi_{\psi}^{\Sigma_0} \hookrightarrow \Big\{ (\underline{l}, \underline{\eta}) \in \mathbb{Z}^{Jord(\psi_p)} \times \{\pm 1\}^{Jord(\psi_p)} \,|\, \underline{l}(\rho, A, B, \zeta) \\ \in [0, (A - B + 1)/2], \text{(2.2) is satisfied} \Big\} /\!\sim_{\Sigma_0}, \end{aligned} \tag{2.1}$$

where

$$\prod_{(\rho, A, B, \zeta) \in Jord(\psi_p)} \epsilon_{\underline{l}, \underline{\eta}}(\rho, A, B, \zeta) = 1 \tag{2.2}$$

and

$$\epsilon_{\underline{l}, \underline{\eta}}(\rho, A, B, \zeta) := \underline{\eta}(\rho, A, B, \zeta)^{A-B+1}(-1)^{[(A-B+1)/2] + \underline{l}(\rho, A, B, \zeta)}.$$

Here we say $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$ if and only if

$$\begin{cases} \underline{l} = \underline{l}' \\ (\underline{\eta}/\underline{\eta}')(\rho, A, B, \zeta) = 1 \text{ unless } \underline{l}(\rho, A, B, \zeta) = (A - B + 1)/2. \end{cases}$$

This is the parametrization appearing in Sect. 1.2, where we have implicitly chosen the order $>_\psi$ to be that of the indexes. For any $(\underline{l}, \underline{\eta})$ in (2.1), we let $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ be the associated representation if $(\underline{l}, \underline{\eta})$ is in the image, or zero otherwise. Mœglin also expressed the nonvanishing of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \eta)$ in terms of the nonvanishing of certain Jacquet module (cf. (2.3)). Following this description, we have developed a procedure in [10] to determine the image explicitly. As an application, we give the formula (1.10) for characterizing the image in the special case (cf. Sect. 1.2). The proof will be given in Appendix A.

What turns out crucial to this procedure [10] is to understand how the injection (2.1) changes when one changes the order $>_\psi$. This is also one of the main results in [10] and we will recall it here. Suppose we have two adjacent Jordan blocks $(\rho, A_i, B_i, \zeta_i)$ $(i = 1, 2)$ with respect to the admissible order $>_\psi$, and

$$(\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

Suppose the new order $>'_\psi$ obtained by switching the two is still admissible. Then by definition, either $\zeta_1 \neq \zeta_2$ or one of $\{[B_i, A_i]\}_{i=1,2}$ is included in the other. Let us define ψ_- by

$$Jord(\psi_-) = Jord(\psi)\backslash\{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}.$$

Suppose

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0,$$

then the restrictions of $(\underline{l}, \underline{\eta})$ and $(\underline{l}', \underline{\eta}')$ to $Jord(\psi_-)$ are equivalent with respect to $(\sim_{\Sigma_0})$ and the following conditions are satisfied.

(1) If $\zeta_1 = \zeta_2$, it suffices to consider the case $[B_2, A_2] \supseteq [B_1, A_1]$. Then we are in one of the following situations.

(a) If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ and $\eta'_1 = (-1)^{A_2-B_2}\eta'_2$, then

$$\begin{cases} l_1 = l'_1 \\ l_2 - l'_2 = (A_1 - B_1 - 2l_1) + 1 \\ \eta'_1 = (-1)^{A_2-B_2}\eta_1 \end{cases}$$

(b) If $\eta_2 = (-1)^{A_1-B_1}\eta_1$ and $\eta'_1 \neq (-1)^{A_2-B_2}\eta'_2$, then

$$\begin{cases} l_1 = l'_1 \\ l'_2 - l_2 = (A_1 - B_1 - 2l_1) + 1 \\ \eta'_1 = (-1)^{A_2-B_2}\eta_1 \end{cases}$$

(c) If $\eta_2 = (-1)^{A_1-B_1}\eta_1$ and $\eta'_1 = (-1)^{A_2-B_2}\eta'_2$, then

$$\begin{cases} l_1 = l'_1 \\ (l'_2 - l'_1) + (l_2 - l_1) = (A_2 - B_2) - (A_1 - B_1) \\ \eta'_1 = (-1)^{A_2-B_2}\eta_1 \end{cases}$$

(2) If $\zeta_1 \neq \zeta_2$, then

$$\begin{cases} l_2' = l_2 \\ l_1' = l_1 \\ \eta_2 = (-1)^{A_1-B_1+1}\eta_2' \\ \eta_1 = (-1)^{A_2-B_2+1}\eta_1' \end{cases}$$

This formula suggests that for $(\rho, A, B, \zeta) \in Jord(\psi_p)$ with $B = 0$, the choice of sign ζ will affect the parametrization (cf. [10, Proposition 7.5]).

2.1 Terminology

We recall a few terminologies from [9] [10]. Let ψ be an Arthur parameter of $G(F)$ such that $\psi = \psi_p$. Let ρ be an irreducible unitary supercuspidal representation of $GL(d_\rho, F)$. We denote by $Jord_\rho(\psi)$ the subset of $Jord(\psi)$ containing ρ. A subset J of $Jord_\rho(\psi)$ is said to have **discrete diagonal restriction** if the intervals $[B, A]$, $[B', A']$ do not intersect for any $(\rho, A, B, \zeta), (\rho, A', B', \zeta') \in Jord_\rho(\psi)$. We say ψ has discrete diagonal restriction if $Jord_\rho(\psi)$ has discrete diagonal restriction for all ρ. A Jordan block (ρ, A, B, ζ) is said to be **far away** from a subset J of $Jord_\rho(\psi)$ if

$$B > 2^{|J|} \cdot \Big(\sum_{(\rho,A',B',\zeta')\in J} A' + |J| \sum_{(\rho,A',B',\zeta')\in Jord_\rho(\psi)} (A' - B' + 1) \Big)$$

and we will write $(\rho, A, B, \zeta) \gg J$ (cf. [10, Section 2]).

Let $>_\psi$ be an admissible order on $Jord(\psi)$ and we index $Jord_\rho(\psi)$ for each ρ so that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

A new parameter $\psi_\gg$ is said to **dominate** ψ with respect to $>_\psi$ if $Jord_\rho(\psi_\gg)$ is obtained by shifting $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ with $T_i \geq 0$ for each ρ, and $>_\psi$ induces an admissible order on $Jord(\psi_\gg)$. In this case, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ and $\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta})$ are related as follows:

$$\begin{aligned} \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) := {} & \circ_{\{\rho: Jord_\rho(\psi)\neq\emptyset\}} \circ_{(\rho,A_i,B_i,\zeta_i)\in Jord_\rho(\psi)} \\ & \mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}), \end{aligned} \tag{2.3}$$

where i is decreasing (cf. [9, Remark 8.4]). If we further assume both of them are nonzero, then we have

$$\begin{aligned} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow {} & \times_{\{\rho: Jord_\rho(\psi)\neq\emptyset\}} \\ & \times_{(\rho,A_i,B_i,\zeta_i)\in Jord_\rho(\psi)} \begin{pmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

where i is increasing (cf. [9, Proposition 8.5]).

At last, we will say a few words about the operators used in (2.3). Let $M = GL(d_\rho) \times G_-$ be the Levi component of a standard maximal parabolic subgroup P of G. For any finite-length smooth representation π^{Σ_0} of $G^{\Sigma_0}(F)$, we can decompose the semisimplification of its Jacquet module as follows

$$s.s.\,\mathrm{Jac}_P(\pi^{\Sigma_0}) = \bigoplus_i \tau_i \otimes \sigma_i,$$

where τ_i (resp. σ_i) are irreducible representations of $GL(d_\rho, F)$ (resp. $G_-^{\Sigma_0}(F)$). We define $\mathrm{Jac}_x \pi^{\Sigma_0}$ for any real number x to be

$$\mathrm{Jac}_x(\pi^{\Sigma_0}) = \bigoplus_{\tau_i = \rho||^x} \sigma_i.$$

We also define

$$\mathrm{Jac}_{x_1,\ldots,x_s}\pi^{\Sigma_0} = \mathrm{Jac}_{x_s} \circ \cdots \circ \mathrm{Jac}_{x_1}\pi^{\Sigma_0}$$

for any ordered sequence of real numbers $\{x_1, \ldots, x_s\}$. Let

$$X_i = \begin{bmatrix} \zeta_i(B_i + T_i) & \cdots & \zeta_i(B_i + 1) \\ \vdots & & \vdots \\ \zeta_i(A_i + T_i) & \cdots & \zeta_i(A_i + 1) \end{bmatrix}$$

with respect to $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ in the previous paragraph. Then $\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$ is defined to be $\mathrm{Jac}_{X_i} := \circ_{x\in X_i} \mathrm{Jac}_x$, where x ranges over X_i from top to bottom and left to right.

3 Step one

In the next three sections, we will give the proofs of the main results stated in the introduction. The Arthur parameters (cf. (1.2)) considered in these proofs are always under the Assumption (1.5). Later we will make some comments on the general case (cf. Sect. 6).

In step one, we consider a subclass of representations in the special case (cf. Sect. 1.2). In the special case, we have

$$\psi = \oplus_{i=1}^n (\rho \otimes \nu_{a_i} \otimes \nu_{b_i})$$

where $A_i \geq A_{i-1}$, $B_i \geq B_{i-1}$ and $\zeta_i = +$. We fix the order so that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Now let us consider $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$, where $\underline{l} = 0$. In this case, we can reinterpret the nonvanishing condition (1.10) as follows.

- If $A_i \geq B_{i+1}$, then $\eta_{i+1} = (-1)^{A_i - B_i}\eta_i$. Let $t_i = (A_i + B_{i+1})/2$ and

$$\delta_i = \begin{cases} 1 & \text{if } t_i - A_i \in \mathbb{Z} \\ 1/2 & \text{if } t_i - A_i \notin \mathbb{Z} \end{cases}$$

- If $A_i < B_{i+1}$, then there is no condition on η_{i+1}.

Consider the maximal sequence of integers

$$0 = k_0 < \cdots < k_r = n$$

such that $A_{k_j} < B_{k_j+1}$. We would like to show

Theorem 3.1

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \hookrightarrow \times_{i=1}^{r} \times_{k_{i-1}<j<k_i} \underbrace{\begin{pmatrix} B_{j+1} & \cdots & -A_j \\ \vdots & & \vdots \\ t_j-\delta_j & \cdots & -(t_j+\delta_j) \end{pmatrix}}_{\tilde{I}_j} \rtimes \sigma^{\Sigma_0} \tag{3.1}$$

as the unique irreducible subrepresentation, where

$$\sigma^{\Sigma_0} := \pi^{\Sigma_0}_{M,>_\psi}\left(\cup_i \left\{\cdots\right\}\right)$$

is a tempered representation with

$$\left\{\cdots\right\} = \left\{(\rho, A_{k_i}, B_{k_i}, 0, \eta_{k_i}, +)\right\}$$

if $k_i - k_{i-1} = 1$, *and*

$$\begin{aligned}\left\{\cdots\right\} = \Big\{&\Big(\rho, A_{k_i}, t_{k_i-1}-\delta_{k_i-1}+1, 0, (-1)^{t_{k_i-1}-\delta_{k_i-1}+1-B_{k_i}}\eta_{k_i}, +\Big),\\ &\cup_{k_{i-1}+1<j<k_i}\Big(\rho, t_j+\delta_j-1, t_{j-1}-\delta_{j-1}+1, 0, (-1)^{t_{j-1}-\delta_{j-1}+1-B_j}\eta_j, +\Big),\\ &\Big(\rho, t_{k_{i-1}+1}+\delta_{k_{i-1}+1}-1, B_{k_{i-1}+1}, 0, \eta_{k_{i-1}+1}, +\Big)\Big\}\end{aligned}$$

otherwise. Moreover, the induced representation $\mathcal{I}$ in (3.1) is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from rows of $\tilde{I}_j$ together with σ^{Σ_0}.

The following corollary is an immediate consequence of the theorem.

Corollary 3.2 *In the notations of Theorem 3.1, the complete Langlands parameter (ϕ,ϵ) of $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ is given as follows:*

$$\phi = (\oplus_j \tilde{\phi}_j) \oplus \phi' \oplus (\oplus_j \tilde{\phi}_j^\vee)$$

where $\tilde{\phi}_j$ is the Langlands parameter of $\tilde{I}_j$, (ϕ',ϵ') is the complete Langlands parameter of σ^{Σ_0}, and ϵ corresponds to ϵ' under the natural isomorphism $\mathcal{S}_\phi^{\Sigma_0} \cong \mathcal{S}_{\phi'}^{\Sigma_0}$.

We will prove Theorem 3.1 by induction on

$$\sum_{i=1}^{n-1} \max\{A_i - B_{i+1}, 0\}.$$

Suppose it is not zero. Let us choose the maximal integer $s < n$ such that $A_s - B_{s+1} \geq A_i - B_{i+1}$ for all $1 \leq i < n$. By maximality of s, we have $B_{s+2} > B_{s+1}$ or $s = n-1$. Moreover, there exists $l \leq s+1$ such that

$$A_s = A_{s-1} = \cdots = A_{l-1} \text{ and } B_{s+1} = B_s = \cdots = B_l$$

and $A_{l-1} > A_{l-2}$ or $l = 2$.

Lemma 3.3

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times^{s}_{j=l-1}\langle B_{j+1}, \ldots, -A_j\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi', \underline{l}, \underline{\eta}'),$$

where $A'_j = A_j - 1$, $B'_{j+1} = B_{j+1} + 1$ *and* $\eta'_{j+1} = -\eta_{j+1}$ *for* $l - 1 \leq j \leq s$.

Proof Since $\underline{l} = 0$, we can reorganize the Jordan blocks for $l - 1 \leq j \leq s + 1$ as

$$\begin{aligned}&(\rho, A_{s+1}, B_{s+1} + 1, 0, -\eta_{s+1}, +) > (\rho, B_{s+1}, B_{s+1}, 0, \eta_{s+1}, +) > (\rho, A_s, B_{s+1}, 0, \eta_s, +)\\ &\quad > \cdots > (\rho, A_l, B_{l+1}, 0, \eta_l, +) > (\rho, A_{l-1}, B_l, 0, (-1)^{B_l - B_{l-1}}\eta_{l-1}, +)\\ &\quad > (\rho, B_l - 1, B_{l-1}, 0, \eta_{l-1}, +),\end{aligned}$$

where we have splitted the first and last ones. Note the last Jordan block above disappear when $B_l = B_{l-1}$. Then we can move $(\rho, B_{s+1}, B_{s+1}, 0, \eta_{s+1}, +)$ to the second last position (or last when $B_l = B_{l-1}$) above. By the change of order formula, it can be combined with the last term. So we get

$$\begin{aligned}&(\rho, A_{s+1}, B_{s+1} + 1, 0, -\eta_{s+1}, +) > (\rho, A_s, B_{s+1}, 1, -\eta_s, +)\\ &\quad > \cdots > (\rho, A_l, B_{l+1}, 1, -\eta_l, +) > (\rho, A_{l-1}, B_l, 1, -(-1)^{B_l - B_{l-1}}\eta_{l-1}, +)\\ &\quad > (\rho, B_l, B_{l-1}, 0, \eta_{l-1}, +)\end{aligned}$$

Since $A_{l-1} > A_{l-2}$ and $B_{s+2} > B_{s+1}$, we get by applying Lemma 4.3

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times^{s}_{j=l-1}\langle B_{j+1}, \ldots, -A_j\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi', \underline{l}, \underline{\eta}').$$

□

By Lemma 3.3 and the induction assumption, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times^{s}_{j=l-1}\langle B_{j+1}, \ldots, -A_j\rangle \rtimes \mathcal{I}' \tag{3.2}$$

and

$$\mathcal{I}' := \times^{r}_{i=1} \times_{k_{i-1}<j<k_i} \underbrace{\begin{pmatrix} B'_{j+1} & \cdots & -A'_j \\ \vdots & & \vdots \\ t_j - \delta_j & \cdots & -(t_j + \delta_j) \end{pmatrix}}_{\tilde{I}_j} \rtimes \sigma^{\Sigma_0}$$

Moreover, $\mathcal{I}'$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from rows of $\tilde{I}_j$ together with σ^{Σ_0}. Combined with the maximality of $A_s - B_{s+1}$, we see the induction in (3.2) is a subrepresentation of the costandard representation as we want. It also follows that the induction in (3.2) has a unique irreducible subrepresentation. Since $\tilde{I}_j$ are interchangeable with each other (cf. [10, Corollary 4.3]), one can combine $\langle B_{j+1}, \ldots, -A_j\rangle$ with $\tilde{I}_j$ for $l - 1 \leq j \leq s$, and this gives (3.1).

4 Step two

We will settle the special case in this step, hence complete the proof of Theorem 1.3. Let

$$\psi = \oplus^{n}_{i=1}(\rho \otimes \nu_{a_i} \otimes \nu_{b_i})$$

where $A_i \geq A_{i-1}$, $B_i \geq B_{i-1}$ and $\zeta_i = +$. We fix the order so that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

By Theorem A.1, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if (1.10) is satisfied. We would like to prove the following theorem.

Theorem 4.1

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_i \underbrace{\begin{pmatrix} B_i & \cdots & -A_i \\ \vdots & & \vdots \\ B_i + l_i - 1 & \cdots & -(A_i - l_i + 1) \end{pmatrix}}_{I_i}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}\Big(\cup_i (\rho, A_i - l_i, B_i + l_i, 0, \eta_i, \zeta_i)\Big) \tag{4.1}$$

as the unique irreducible subrepresentation. Moreover, after applying (3.1) *to*

$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\cup_i (\rho, A_i - l_i, B_i + l_i, 0, \eta_i, \zeta_i)\Big), \tag{4.2}$$

we can embed the right hand side of (4.1) *into an induced representation $\mathcal{I}$. Then $\mathcal{I}$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with a tempered representation σ^{Σ_0} as in Theorem 3.1.*

The following corollary is an immediate consequence of the theorem.

Corollary 4.2 *In the notations of Theorem 4.1, the complete Langlands parameter (ϕ, ϵ) of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ is given as follows:*

$$\phi = (\oplus_i \phi_i) \oplus \phi' \oplus (\oplus_i \phi_i^\vee)$$

where ϕ_i is the Langlands parameter of I_i, (ϕ', ϵ') is the complete Langlands parameter of (4.2)*, and ϵ corresponds to ϵ' under the natural isomorphism $\mathcal{S}^{\Sigma_0}_\phi \cong \mathcal{S}^{\Sigma_0}_{\phi'}$.*

We will prove Theorem 4.1 by induction on $\sum_{i=1} l_i$. Among all i such that $l_i \neq 0$, let us choose maximal s for the property that $A_s - B_s \geq A_i - B_i$ for any such i. By the maximality of s, we have $B_{s+1} > B_s$ or $s = n$. Moreover, there exists $l \leq s$ such that

$$A_s = \cdots = A_l \text{ and } B_s = \cdots = B_l$$

and $A_l > A_{l-1}$ or $l = 1$.

Lemma 4.3

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=l}^s \langle B_i, \ldots, -A_i \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi', \underline{l}', \underline{\eta}),$$

where $A'_i = A_i - 1$, $B'_i = B_i + 1$, and $l'_i = l_i - 1$ for $l \leq i \leq s$.

Proof Let $\psi^{(l)}_\gg$ be a dominating parameter of ψ such that the Jordan blocks for $i < l$ remains the same, and the Jordan blocks for $i \geq l$ are shifted by T_i, so that they are disjoint and far away from $\cup_{i<l}\{(\rho, A_i, B_i, \zeta_i)\}$. Similarly, we can define $\psi^{(s+1)}_\gg$ (resp. $\psi'^{(s+1)}_\gg$).

$$
\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(l)}_{\gg},\underline{l},\underline{\eta}) &\hookrightarrow \times_{i=l}^{s}\langle B_i+T_i,\ldots,-(A_i+T_i)\rangle\times \\
&\quad \times_{i=l}^{s}\begin{pmatrix} B_i+T_i+1 & \cdots & B_i+2 \\ \vdots & & \vdots \\ A_i+T_i-1 & \cdots & A_i \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi'^{(s+1)}_{\gg},\underline{l}',\underline{\eta}) \\
&\hookrightarrow \times_{i=l}^{s}\Big(\langle B_i+T_i,\ldots,-A_i\rangle\times \underbrace{\langle -(A_i+1),\ldots,-(A_i+T_i)\rangle}_{I_i}\Big)\times \\
&\quad \times_{i=l}^{s}\underbrace{\begin{pmatrix} B_i+T_i+1 & \cdots & B_i+2 \\ \vdots & & \vdots \\ A_i+T_i-1 & \cdots & A_i \end{pmatrix}}_{II_i} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi'^{(s+1)}_{\gg},\underline{l}',\underline{\eta})
\end{aligned}
$$

We can switch I_i with II_j (cf. [10, Corollary 4.3]). Since $A_l > A_{l-1}$, we can then take the dual of I_i (cf. [10, Proposition 4.6]). Moreover, we can combine II_i with $I_i^\vee$, for otherwise, $\mathrm{Jac}_{A_i+T_i}\,\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(l)}_{\gg},\underline{l},\underline{\eta}) \neq 0$, which is impossible. Therefore, we get

$$
\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(l)}_{\gg},\underline{l},\underline{\eta}) &\hookrightarrow \times_{i=l}^{s}\langle B_i+T_i,\ldots,-A_i\rangle\times \\
&\quad \times_{i=l}^{s}\begin{pmatrix} B_i+T_i+1 & \cdots & B_i+2 \\ \vdots & & \vdots \\ A_i+T_i & \cdots & A_i+1 \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi'^{(s+1)}_{\gg},\underline{l}',\underline{\eta}) \\
&\hookrightarrow \times_{i=l}^{s}\left(\langle B_i+T_i,\ldots,-A_i\rangle\times \begin{pmatrix} B_i+T_i+1 & \cdots & B_i+2 \\ \vdots & & \vdots \\ A_i+T_i & \cdots & A_i+1 \end{pmatrix}\right) \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi'^{(s+1)}_{\gg},\underline{l}',\underline{\eta})
\end{aligned}
$$

It follows

$$
\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(s+1)}_{\gg},\underline{l},\underline{\eta}) &\hookrightarrow \times_{i=l}^{s}\langle B_i,\ldots,-A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi'^{(s+1)}_{\gg},\underline{l}',\underline{\eta}) \\
&\hookrightarrow \times_{i=l}^{s}\langle B_i,\ldots,-A_i\rangle\times\Big(\times_{i>s}\underbrace{\begin{pmatrix} B_i+T_i & \cdots & B_i+1 \\ \vdots & & \vdots \\ A_i+T_i & \cdots & A_i+1 \end{pmatrix}}_{III_i}\Big) \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi',\underline{l}',\underline{\eta}).
\end{aligned}
$$

Since $B_{s+1} > B_s$, we can switch $\langle B_i,\ldots,-A_i\rangle$ with III_j (cf. [10, Corollary 4.3]). Hence,

$$
\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \hookrightarrow \times_{i=l}^{s}\langle B_i,\ldots,-A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi',\underline{l}',\underline{\eta}).
$$

□

By Lemma 4.3 and the induction assumption, we have

$$
\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \hookrightarrow \times_{i=l}^{s}\langle B_i,\ldots,-A_i\rangle \rtimes \mathcal{I}' \tag{4.3}
$$

and

$$\mathcal{I}':=\times_{i=l}^{s}\underbrace{\begin{pmatrix} B_i+1 & \cdots & -(A_i-1) \\ \vdots & & \vdots \\ B_i+l_i-1 & \cdots & -(A_i-l_i+1) \end{pmatrix}}_{II_i}\times\times_{i<l \text{ or } i>s}\underbrace{\begin{pmatrix} B_i & \cdots & -A_i \\ \vdots & & \vdots \\ B_i+l_i-1 & \cdots & -(A_i-l_i+1) \end{pmatrix}}_{II_i}$$

$$\times\times_{\text{some } j}\underbrace{\begin{pmatrix} B_{j+1}+l_{j+1} & \cdots & -(A_j-l_j) \\ \vdots & & \vdots \\ t_j-\delta_j & \cdots & -(t_j+\delta_j) \end{pmatrix}}_{\tilde{I}_j}\rtimes\sigma^{\Sigma_0}$$

Moreover, $\mathcal{I}'$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with σ^{Σ_0}. We claim the induction in (4.3) is a subrepresentation of the costandard representation as we want.

To prove the claim, we need to show any shifted Steinberg representation above, whose shift is less than that of $\langle B_s,\ldots,-A_s\rangle$, can be moved to the front. By our choice of s, it suffices to consider $\langle x,\ldots,-y\rangle$ from rows of $\tilde{I}_j$. Moreover, it is necessary that $l_{j+1}=l_j=0$. There are two cases.

(1) If $A_s\leq A_j$, then $y\geq A_s$.
(2) If $B_s\geq B_{j+1}$, then $x\leq B_s$.

In either case, we see $\langle x,\ldots,-y\rangle$ and $\langle B_s,\ldots,-A_s\rangle$ are interchangeable (cf. [10, Corollary 4.3]). This finishes the proof of our claim. As a consequence, the induction in (4.3) has a unique irreducible subrepresentation. So we can combine $\langle B_i,\ldots,-A_i\rangle$ with II_i for $l\leq i\leq s$, and this gives (4.1).

5 Step three

In this step we will prove Theorems 1.1 and 1.2, which reduce our problem to the special case settled in the previous step. In order to apply the induction argument, we need to generalize our problem (cf. (1.6)) to the following case

$$\psi=\oplus_{i=1}^{n}(\rho\otimes\nu_{a_i}\otimes\nu_{b_i})$$

where $A_i\geq A_{i-1}$ and there exists $m\leq n$ such that

- if $i>m$, then $\zeta_i=+$, $B_{i+1}\geq B_i$;
- if $i\leq m$, then $\zeta_i=-$, and $B_i\leq B_{i-1}$.

We choose the order so that

$$(\rho,A_i,B_i,\zeta_i)>_\psi(\rho,A_{i-1},B_{i-1},\zeta_{i-1}).$$

Among all $i\leq m$ such that $B_i\neq 0$ (resp. $1/2$), we choose $s\leq m$ maximal for the property that

- $A_s\geq A_i$ for all such i;
- $B_s\geq B_i$, if $A_s=A_i$ for any such i

By the maximality of s, we have $B_s > B_{s+1}$ or $s = m$. Moreover, there exists $l \le s$ such that

$$A_s = \cdots = A_l \text{ and } B_s = \cdots = B_l$$

and $A_l > A_{l-1}$ or $l = 1$.

Lemma 5.1 *There is a bijection between*

$$\Pi_{\psi}^{\Sigma_0} \to \Pi_{\psi^s}^{\Sigma_0}$$
$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \mapsto \pi_{M,>_\psi}^{\Sigma_0}(\psi^s, \underline{l}, \underline{\eta})$$

such that

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=l}^{s} \langle -B_i, \cdots, -A_i \rangle \rtimes \pi_{M,>_\psi}^{\Sigma_0}(\psi^s, \underline{l}, \underline{\eta}),$$

where ψ^s is obtained from ψ by changing $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i - 1, B_i - 1, \zeta_i)$ for $l \le i \le s$.

Proof We first show

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0 \Leftrightarrow \pi_{M,>_\psi}^{\Sigma_0}(\psi^s, \underline{l}, \underline{\eta}) \neq 0.$$

Following the procedure in [10, Section 8], we can first reduce to the case that all $(\rho, A_i, B_i, \zeta_i)$ for $i > m$ are far away from $\cup_{i \le m}\{(\rho, A_i, B_i, \zeta_i)\}$, except for one $(\rho, A_j, B_j, \zeta_j)$. This is done by the operations of "pull" and "expand" (cf. [10, Section 7.1, 7.2]). Since $A_j \ge A_i$ for all $i \le m$, then we can "expand" $[B_j, A_j]$ and change the sign ζ_j to negative (cf. [10, Section 7.3]). In this way, we can further reduce to the case that all $(\rho, A_i, B_i, \zeta_i)$ for $i > m$ are far away from $\cup_{i \le m}\{(\rho, A_i, B_i, \zeta_i)\}$. Since $A_l > A_{l-1}$ and $B_s > B_{s+1}$, the inclusion relations of intervals are not changed after shifting $[B_i, A_i]$ to $[B_i - 1, A_i - 1]$ for $l \le i \le s$. Then it is not hard to see from the procedure in [10, Section 8] again that the nonvanishing condition is not changed.

Next we impose a new order $>'_\psi$ by moving $\{(\rho, A_i, B_i, \zeta_i)\}_{i=l}^{s}$ to the front. Suppose

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{M,>'_\psi}^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}'),$$

then

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi^s, \underline{l}, \underline{\eta}) = \pi_{M,>'_\psi}^{\Sigma_0}(\psi^s, \underline{l}', \underline{\eta}')$$

by the change of order formula. So it suffices to prove the lemma under this new order. Let $\psi_{\gg}$ be the parameter obtained by shifting $[B_i, A_i]$ to $[B_i + T_i, A_i + T_i]$ for $l \le i \le s$, which are disjoint and far away from the rest. Then

$$\pi_{M,>'_\psi}^{\Sigma_0}(\psi_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \times_{i=l}^{s} \begin{pmatrix} -(B_i + T_i) & \cdots & -(A_i + T_i) \\ \vdots & & \vdots \\ -B_i & \cdots & -A_i \end{pmatrix} \rtimes \pi_{M,>'_\psi}^{\Sigma_0}(\psi^s, \underline{l}', \underline{\eta}'),$$

where i increases. It follows

$$\begin{aligned}\pi^{\Sigma_0}_{M,>'_\psi}(\psi_\gg,\underline{l}',\underline{\eta}') &\hookrightarrow \begin{pmatrix}-(B_l+T_l) & \cdots & -(A_l+T_l)\\ \vdots & & \vdots\\ -(B_l+1) & \cdots & -(A_l+1)\end{pmatrix} \times \langle -B_l,\ldots,-A_l\rangle \times\\ &\quad \times_{i=l+1}^{s} \begin{pmatrix}-(B_i+T_i) & \cdots & -(A_i+T_i)\\ \vdots & & \vdots\\ -B_i & \cdots & -A_i\end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi^s,\underline{l}',\underline{\eta}')\\ &\hookrightarrow \begin{pmatrix}-(B_l+T_l) & \cdots & -(A_l+T_l)\\ \vdots & & \vdots\\ -(B_l+1) & \cdots & -(A_l+1)\end{pmatrix} \times\\ &\quad \times_{i=l+1}^{s} \begin{pmatrix}-(B_i+T_i) & \cdots & -(A_i+T_i)\\ \vdots & & \vdots\\ -B_i & \cdots & -A_i\end{pmatrix}\\ &\quad \times \langle -B_l,\ldots,-A_l\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi^s,\underline{l}',\underline{\eta}')\end{aligned}$$

Continuing this way, we should get

$$\begin{aligned}\pi^{\Sigma_0}_{M,>'_\psi}(\psi_\gg,\underline{l}',\underline{\eta}') \hookrightarrow \times_{i=l}^{s} \begin{pmatrix}-(B_i+T_i) & \cdots & -(A_i+T_i)\\ \vdots & & \vdots\\ -(B_i+1) & \cdots & -(A_i+1)\end{pmatrix} \times\\ \times_{i=l}^{s} \langle -B_i,\ldots,-A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi^s,\underline{l}',\underline{\eta}')\end{aligned}$$

It follows

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}') \hookrightarrow \times_{i=l}^{s}\langle -B_i,\ldots,-A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi^s,\underline{l}',\underline{\eta}').$$

This finishes the proof. $\square$

Remark 5.2 In this lemma, it is critical to have $A_j \geq A_i$ for $\zeta_j = +, \zeta_i = -$. Here we give a counter-example when this condition is not satisfied. Suppose

$$Jord(\psi) = \{(\rho, A_1, B_1, \zeta_1), (\rho, A_2, B_2, \zeta_2)\}$$

with $A_i, B_i \in \mathbb{Z}$, $\zeta_1 = -$, $\zeta_2 = +$, and $A_1 > B_1 + B_2 > A_2 > A_1 - B_1$. Let

$$Jord(\psi') = \{(\rho, A_1 - B_1, 0, \zeta_1), (\rho, A_2, B_2, \zeta_2)\}.$$

We claim $|\Pi^{\Sigma_0}_{\psi}| > |\Pi^{\Sigma_0}_{\psi'}|$, which will result in a contradiction to Lemma 5.1. After changing ζ_1 to positive, we can apply Theorem 1.3 to ψ'. Then there is a bijection

$$\Pi^{\Sigma_0}_{\psi'} \to \{(\underline{l},\underline{\eta}) \in \mathbb{Z}^2 \times \{\pm 1\}^2 \,|\, 0 \leq l_i \leq [(A_i - B_i + 1)/2], \text{(1.10) and (1.11) are satisfied}\}/\sim_{\Sigma_0}$$

where $\underline{l} = (l_i)$, $\underline{\eta} = (\eta_i)$. By our assumption, $[0, A_1 - B_1]$ intersects with $[A_2, B_2]$, so the condition (1.10) is not void. On the other hand, let

$$Jord(\psi'') = \{(\rho, A_1, B_1, \zeta_1), (\rho, A_2 - B_2, 0, \zeta_2)\}$$

By applying $\mathrm{Jac}_{(\rho,A_2,B_2,\zeta_2)\mapsto(\rho,A_2-B_2,0,\zeta_2)}$, we get a surjection

$$\Pi_{\psi}^{\Sigma_0} \to \Pi_{\psi''}^{\Sigma_0} \tag{5.1}$$

By our assumption, $[0, A_2 - B_2]$ does not intersect with $[A_1, B_1]$. So by [6, Theorem 4.2], there is a bijection

$$\Pi_{\psi''}^{\Sigma_0} \to \{(\underline{l}, \underline{\eta}) \in \mathbb{Z}^2 \times \{\pm 1\}^2 \,|\, 0 \le l_i \le [(A_i - B_i + 1)/2], \text{(1.11) is satisfied}\}/\sim_{\Sigma_0}$$

Hence, $|\Pi_{\psi}^{\Sigma_0}| \ge |\Pi_{\psi''}^{\Sigma_0}| > |\Pi_{\psi'}^{\Sigma_0}|$. From (2.1), we also see (5.1) is actually a bijection.

5.1 Integral case

We assume $A_i, B_i \in \mathbb{Z}$. Recall $\zeta_i = +$ for $i > m$ and $\zeta_i = -$ for $i \le m$. We get a new parameter ψ' by replacing all $(\rho, A_i, B_i, \zeta_i)$ by $(\rho, A_i', B_i', \zeta_i')$ such that: $\zeta_i' = \zeta_i$ and

- $A_i' = A_i$, $B_i' = B_i$ for $i > m$;
- $A_i' = A_i - B_i$, $B_i' = 0$ for $i \le m$.

Theorem 5.3 *There is a bijection*

$$\begin{gathered}\Pi_{\psi}^{\Sigma_0} \to \Pi_{\psi'}^{\Sigma_0}\\ \pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \mapsto \pi_{M,>_\psi}^{\Sigma_0}(\psi', \underline{l}, \underline{\eta})\end{gathered}$$

such that

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i \le m} \underbrace{\begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1 & \cdots & -(A_i - B_i + 1) \end{pmatrix}}_{I_i} \rtimes \pi_{M,>_\psi}^{\Sigma_0}(\psi', \underline{l}, \underline{\eta}) \tag{5.2}$$

as the unique irreducible subrepresentation.

Let us assume

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi', \underline{l}, \underline{\eta}) = \pi_{M,>_\psi}^{\Sigma_0}(\psi', \underline{l}', \underline{\eta}').$$

after changing ζ_i to positive for $i \le m$. By applying (4.1) and (3.1) to $\pi_{M,>_\psi}^{\Sigma_0}(\psi', \underline{l}', \underline{\eta}')$, we can embed the right hand side of (5.2) into an induced representation $\mathcal{I}$. Then $\mathcal{I}$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with a tempered representation σ^{Σ_0} as in Theorem 3.1.

Proof We can prove this by induction on $\sum_{i \le m} B_i$. By Lemma 5.1 and the induction assumption, we have

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=l}^{s} \langle -B_i, \ldots, -A_i \rangle \rtimes \mathcal{I}^s \tag{5.3}$$

and

$$
\mathcal{I}^s := \times_{i=l}^{s} \underbrace{\begin{pmatrix} -(B_i-1) & \cdots & -(A_i-1) \\ \vdots & & \vdots \\ -1 & \cdots & -(A_i-B_i+1) \end{pmatrix}}_{III_i} \times \times_{i<l \text{ or } m\geq i>s} \underbrace{\begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1 & \cdots & -(A_i-B_i+1) \end{pmatrix}}_{III_i} \times
$$

$$
\times_{i=1}^{n} \underbrace{\begin{pmatrix} B_i' & \cdots & -A_i' \\ \vdots & & \vdots \\ B_i'+l_i'-1 & \cdots & -(A_i'-l_i'+1) \end{pmatrix}}_{II_i} \times \times_{\text{some } j} \underbrace{\begin{pmatrix} B_{j+1}'+l_{j+1}' & \cdots & -(A_j'-l_j') \\ \vdots & & \vdots \\ t_j'-\delta_j & \cdots & -(t_j'+\delta_j) \end{pmatrix}}_{\tilde{I}_j} \rtimes \sigma^{\Sigma_0}
$$

Moreover, $\mathcal{I}^s$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with σ^{Σ_0}. We claim the induced representation in (5.3) is a subrepresentation of the costandard representation as we want.

To prove the claim, we need to show any shifted Steinberg representation above, whose shift is less than that of $\langle -B_s, \ldots, -A_s \rangle$, can be moved to the front. There are two cases.

(1) If it is in the form $\langle -x, \ldots, -y \rangle$ from III_i, then by our choice of s, we have $A_s \geq y$ and $x \geq B_s$. Hence, $\langle -x, \ldots, -y \rangle$ and $\langle -B_s, \ldots, -A_s \rangle$ are interchangeable.
(2) If it is in the form $\langle x, \ldots, -y \rangle$ from II_i or $\tilde{I}_j$, then we have $y \geq A_s$. Otherwise,

$$
\frac{x-y}{2} \geq -\frac{y}{2} > -\frac{A_s}{2} \geq -\frac{A_s+B_s}{2},
$$

which contradicts to our assumption about the shifts. Hence, $\langle x, \ldots, -y \rangle$ and $\langle -B_s, \ldots, -A_s \rangle$ are interchangeable (cf. [10, Corollary 4.3]).

This finishes the proof of our claim. As a consequence, the induction in (5.3) has a unique irreducible subrepresentation. So we can combine $\langle -B_i, \ldots, -A_i \rangle$ with III_i for $l \leq i \leq s$, and this gives (5.2). □

The following corollary is an immediate consequence of Theorem 5.3.

Corollary 5.4 *In the notations of Theorem 5.3, the complete Langlands parameter (ϕ, ϵ) of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ is given as follows:*

$$
\phi = (\oplus_i \phi_i) \oplus \phi' \oplus (\oplus_i \phi_i^\vee)
$$

where ϕ_i is the Langlands parameter of I_i, (ϕ', ϵ') is the complete Langlands parameter of $\pi^{\Sigma_0}_{M,>_\psi}(\psi', \underline{l}, \underline{\eta})$, and ϵ corresponds to ϵ' under the natural isomorphism $\mathcal{S}^{\Sigma_0}_{\phi} \cong \mathcal{S}^{\Sigma_0}_{\phi'}$.

5.2 Half-integral case

We assume $A_i, B_i \notin \mathbb{Z}$. Recall $\zeta_i = +$ for $i > m$ and $\zeta_i = -$ for $i \leq m$.

Theorem 5.5 *Consider the maximal sequence of integers*

$$
0 = s_0 < s_1 < \cdots < s_l = m
$$

such that $A_{s_j} - B_{s_j} \neq A_{s_j+1} - B_{s_j+1}$. For any $0 \leq k \leq l$, we get a new parameter ψ_k' by replacing all $(\rho, A_i, B_i, \zeta_i)$ by $(\rho, A_i', B_i', \zeta_i')$ such that: $\zeta_i' = +$ and

- $A'_i = A_i, B'_i = B_i$ *for* $i > m$*;*
- $A'_i = A_i - B_i - 1/2, B'_i = 1/2$ *for* $i \leq m$ *and* $i \neq s_k$*;*
- $A'_i = A_i - B_i + 1/2, B'_i = 1/2$ *for* $i = s_k$.

Then we can divide $\Pi_\psi^{\Sigma_0}$ *into* $l+1$ *classes, i.e.,*

$$\Pi_\psi^{\Sigma_0} = \sqcup_{k=0}^{l} \Pi_\psi^{\Sigma_0}(k).$$

For any $0 \leq k \leq l$*, we put an order* $>_{\psi'_k}$ *on* $Jord(\psi'_k)$ *so that*

$$(\rho, A'_i, B'_i, \zeta'_i) >_{\psi'_k} (\rho, A'_{i-1}, B'_{i-1}, \zeta'_{i-1}).$$

Then we can get an injection

$$\Pi_\psi^{\Sigma_0}(k) \hookrightarrow \Pi_{\psi'_k}^{\Sigma_0}, \quad \pi^{\Sigma_0} \mapsto \pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}'),$$

such that

$$\pi^{\Sigma_0} \hookrightarrow \times_{s_k \neq i \leq m} \underbrace{\begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i - B_i + 1/2) \end{pmatrix}}_{I_i} \times \underbrace{\begin{pmatrix} -B_{s_k} & \cdots & -A_{s_k} \\ \vdots & & \vdots \\ -3/2 & \cdots & -(A_{s_k} - B_{s_k} + 3/2) \end{pmatrix}}_{I_{s_k}}$$
$$\rtimes \pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}') \tag{5.4}$$

as the unique irreducible subrepresentation. The image is characterized by the condition that for all $i \leq s_k$,

- $l'_i = 0$*;*
- $\eta'_i = -\prod_{j<i}(-1)^{A_j - B_j + 1}$.

When $k \neq 0$*, the second condition can also be simplified as* $\eta'_1 = -1$.

After applying (4.1) *and* (3.1) *to* $\pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}')$*, we can embed the right hand side of* (5.4) *into an induced representation* $\mathcal{I}$*. Then* $\mathcal{I}$ *is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with* σ^{Σ_0} *as in Theorem 3.1.*

The following corollary is an immediate consequence of the theorem.

Corollary 5.6 *In the notations of Theorem 5.5, the complete Langlands parameter* (ϕ, ϵ) *of* π^{Σ_0} *is given as follows*

$$\phi = (\oplus_i \phi_i) \oplus \phi' \oplus (\oplus_i \phi_i^\vee)$$

where ϕ_i *is the Langlands parameter of* I_i*,* (ϕ', ϵ') *is the complete Langlands parameter of* $\pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}')$*, and* ϵ *corresponds to* ϵ' *under the natural isomorphism* $\mathcal{S}_\phi^{\Sigma_0} \cong \mathcal{S}_{\phi'}^{\Sigma_0}$.

We will prove Theorem 5.5 by induction on $\sum_{i \leq m}(B_i - 1/2)$.

5.2.1 Change sign

Suppose

$$\sum_{i \leq m} (B_i - 1/2) = 0, \tag{5.5}$$

i.e., $B_i = 1/2$ for $i \leq m$. We change the order $>_\psi$ so that

$$(\rho, A_i, 1/2, \zeta_i) >_\psi (\rho, A_{i+1}, 1/2, \zeta_{i+1})$$

for $1 \leq i \leq m-1$. Then there are two cases.

First case: $l_m \neq 0$ or $\eta_m = 1$. There exists $l \leq m$ such that

$$A_m = A_{m-1} = \cdots = A_l \text{ with } A_l > A_{l-1} \text{ or } l = 1.$$

Let ψ'' be obtained from ψ by replacing all $(\rho, A_i, B_i, \zeta_i)$ by $(\rho, A_i - 1, 1/2, -\zeta_i)$ for $l \leq i \leq m$. Note ψ'' also falls into the general case that we consider in the beginning of Sect. 5. We also change $>_{\psi''}$ by reversing the order for the Jordan blocks with negative ζ. It can be obtained by moving $(\rho, A_i - 1, 1/2, -\zeta_i)$ to the front of the last m Jordan blocks, one by one as i goes from l to m. In particular, it satisfies

$$\begin{aligned}(\rho, A_i, B_i, \zeta_i) &>_{\psi''} (\rho, A_m - 1, 1/2, -\zeta_m) >_{\psi''} \cdots >_{\psi''} (\rho, A_l - 1, 1/2, -\zeta_l)\\ &>_{\psi''} (\rho, A_j, 1/2, \zeta_j)\end{aligned}$$

for any $i > m$ and $j < l$.

Lemma 5.7 *There is a bijection*

$$\left\{\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \in \Pi^{\Sigma_0}_\psi \,|\, l_m \neq 0 \text{ or } \eta_m = 1\right\} \to \Pi^{\Sigma_0}_{\psi''}$$

such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=l}^m \langle -1/2, \cdots, -A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'', \underline{l}'', \underline{\eta}''),$$

where $(\underline{l}'', \underline{\eta}'')$ *only differs from* $(\underline{l}, \underline{\eta})$ *for* $1 \leq i \leq m$. *We will set* $\eta_m = -1$ *if* $l_m = (A_m + \frac{1}{2})/2$. *Then*

$$(i < l) \quad \begin{cases} l''_i = l_i \\ \eta''_i = \eta_i(-1)^{(m-l+1)(A_i - \frac{1}{2})} \end{cases}$$

$$(l \leq i < m) \quad \begin{cases} \eta''_l = -\eta_m(-1)^{(l-1)(A_i + \frac{1}{2})} \\ \eta''_{i+1} = (-1)^{A_i - \frac{3}{2}}\eta''_i \end{cases}$$

$$(l \leq i \leq m) \quad l''_i = \begin{cases} l_m - 1 & \text{if } \eta_m = -1 \\ l_m & \text{if } \eta_m = 1 \end{cases}$$

Proof If $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then it is necessary that

$$l_i = l_m \text{ and } \eta_i = (-1)^{A_{i+1} - \frac{1}{2}}\eta_{i+1} \quad \text{for } l \leq i < m.$$

(cf. [10, Lemma 5.5]) Similarly, if $\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'', \underline{l}'', \underline{\eta}'') \neq 0$, then it is necessary that

$$l''_i = l''_m \text{ and } \eta''_{i+1} = (-1)^{A_i - \frac{3}{2}}\eta''_i \quad \text{for } l \leq i < m.$$

So for the bijection, it suffices to show

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \neq 0 \Leftrightarrow \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'',\underline{l}'',\underline{\eta}'') \neq 0$$

Let $\psi_{\gg}$ (resp. $\psi''_{\gg}$) be dominating parameters obtained from ψ (resp. ψ'') by changing $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ for $i > m$, far away from the remaining ones. By Proposition B.1 together with the change of order formulas, we know

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \neq 0 \Leftrightarrow \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{\gg},\underline{l}'',\underline{\eta}'') \neq 0.$$

Moreover, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \times_{i=l}^{m}\langle -1/2,\ldots,-A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{\gg},\underline{l}'',\underline{\eta}''). \tag{5.6}$$

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \neq 0$, then if we apply $\circ_{i>m}\mathrm{Jac}_{X_i}$ (i decreasing) with

$$X_i = \begin{bmatrix} B_i + T_i & \cdots & B_i + 1 \\ \vdots & & \vdots \\ A_i + T_i & \cdots & A_i + 1 \end{bmatrix}$$

to the right hand side of (5.6), we should also get something nonzero. We claim the result must be

$$\times_{i=l}^{m}\langle -1/2,\ldots,-A_i\rangle \rtimes \circ_{i>m}\mathrm{Jac}_{X_i}\,\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{\gg},\underline{l}'',\underline{\eta}''),$$

which shows the nonvanishing of $\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'',\underline{l}'',\underline{\eta}'')$. Otherwise, since $A_i + 1 > A_m$ for $i > m$, there exist $j > m$ and $t \geq 1$ such that

$$\mathrm{Jac}_{X_{j,t}} \circ \circ_{j>i>m}\mathrm{Jac}_{X_i}(\mathrm{RHS}.(5.6))$$

contains a term

$$\begin{aligned} &\times_{i=l}^{m-1}\langle -1/2,\ldots,-A_i\rangle \times \langle -1/2,\ldots,-(A_m-1)\rangle \\ &\times \mathrm{Jac}_{X^-_{j,t}} \circ \circ_{j>i>m}\mathrm{Jac}_{X_i}\,\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{\gg},\underline{l}'',\underline{\eta}'') \neq 0 \end{aligned}$$

where

$$X_{j,t} := \begin{bmatrix} B_j + T_j & \cdots & B_j + t \\ \vdots & & \vdots \\ A_j + T_j & \cdots & A_j + t \end{bmatrix}$$

and $X^-_{j,t}$ means that we take away the entry A_m from the last column of $X_{j,t}$. For $X^-_{j,t}$ to be well-defined, we must have

$$B_j + 1 \leq B_j + t \leq A_m < A_j + 1 \leq A_j + t$$

Let $\psi''_{>}$ be obtained from $\psi''_{\gg}$ by changing $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $j > i > m$ and $(\rho, A_j + T_j, B_j + T_j, \zeta_i)$ to $(\rho, A_j + t, B_j + t, \zeta_j)$. Then we can conclude

$$\mathrm{Jac}_{B_j+t,\ldots,A_m-1,A_m+1,\ldots,A_j+t}\,\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{>},\underline{l}'',\underline{\eta}'') \neq 0.$$

It follows $\mathrm{Jac}_{A_m+1}\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{>},\underline{l}'',\underline{\eta}'') \neq 0$. But this is impossible, since $B_i + T_i > A_m + 1$ for $i > j$, and

$$B_i \leq B_j < B_j + t \leq A_m < A_m + 1$$

for $j > i > m$.

Conversely, if $\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'', \underline{l}'', \underline{\eta}'') \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_{\psi''}}(\psi''_{\gg}, \underline{l}'', \underline{\eta}'') \hookrightarrow \times_{i>m} \mathcal{C}_{X_i} \rtimes \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'', \underline{l}'', \underline{\eta}''),$$

where

$$\mathcal{C}_{X_i} := \begin{pmatrix} B_i + T_i & \cdots & B_i + 1 \\ \vdots & & \vdots \\ A_i + T_i & \cdots & A_i + 1 \end{pmatrix}.$$

Since $\mathcal{C}_{X_i}$ and $\langle -1/2, \ldots, -A_m \rangle$ are interchangeable for $i > m$, it follows from (5.6) that

$$\pi^{\Sigma_0}_{M,>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i=l}^{m} \langle -1/2, \ldots, -A_i \rangle \rtimes \pi^{\Sigma_0}_{M,>_{\psi''}}(\psi'', \underline{l}'', \underline{\eta}'').$$

Hence $\pi^{\Sigma_0}_{M,>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \neq 0$. In the meantime, we have also shown the inclusion relation as well. □

Second case: $l_m = 0$ and $\eta_m = -1$. By [10, Lemma 5.5], it is necessary that $l_i = 0$ for $i < m$. Therefore,

$$\eta_i = (-1)^{A_{i+1} - \frac{1}{2}} \eta_{i+1} \quad \text{for } i < m.$$

We get a new parameter ψ' by changing $(\rho, A_m, 1/2, \zeta_m)$ to $(\rho, A_m, 1/2, -\zeta_m)$, and $(\rho, A_i, 1/2, \zeta_i)$ to $(\rho, A_i - 1, 1/2, -\zeta_i)$ for $i < m$. After imposing the usual order $>_{\psi'}$ on the Jordan blocks of ψ', i.e.,

$$(\rho, A_m, 1/2, -\zeta_m) >_{\psi'} (\rho, A_{m-1} - 1, 1/2, -\zeta_i) >_{\psi'} \cdots >_{\psi'} (\rho, A_1 - 1, 1/2, -\zeta_1),$$

we would get $(\underline{l}', \underline{\eta}')$ from $(\underline{l}, \underline{\eta})$ by requiring $l'_i = 0$ for $i \leq m$ and

$$\eta'_1 = -1 \text{ and } \eta'_{i+1} = (-1)^{A_i - \frac{3}{2}} \eta'_i \quad \text{for } i < m.$$

Lemma 5.8

$$\pi^{\Sigma_0}_{M,>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<m} \langle -1/2, \ldots, -A_i \rangle \rtimes \pi^{\Sigma_0}_{M,>_{\psi'}}(\psi', \underline{l}', \underline{\eta}'),$$

Proof Let $\psi^{(k)}$ be the parameter by changing $(\rho, A_m, 1/2, \zeta_m)$ to $(\rho, A_m, 1/2, -\zeta_m)$, and $(\rho, A_i, 1/2, \zeta_i)$ to $(\rho, A_i - 1, 1/2, -\zeta_i)$ for $m - k < i < m$. We will also change the order to $>_{\psi^{(k)}}$ by moving these Jordan blocks to the front of the last m Jordan blocks as i goes from $m - k + 1$ to m. The representations inside the corresponding packets are parametrized by $(\underline{l}^{(k)}, \underline{\eta}^{(k)})$ with respect to $>_{\psi^{(k)}}$.

Since $l_m = 0$ and $\eta_m = -1$,

$$\pi^{\Sigma_0}_{M,>_{\psi}}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_{\psi^{(1)}}}(\psi^{(1)}, \underline{l}^{(1)}, \underline{\eta}^{(1)})$$

where $(\underline{l}^{(1)}, \underline{\eta}^{(1)})$ satisfies

$$\eta_i^{(1)} = (-1)^{A_m + \frac{1}{2}} \eta_i \quad \text{for } i < m.$$

Since $l_i^{(1)} = 0$ for $i < m$, we also have

$$\eta_i^{(1)} = (-1)^{A_{i+1} - \frac{1}{2}} \eta_{i+1}^{(1)} \quad \text{for } i < m - 1.$$

We compute

$$\eta^{(1)}_{m-1} = (-1)^{A_m+\frac{1}{2}}\eta_{m-1} = (-1)^{A_m+\frac{1}{2}}(-1)^{A_m-\frac{1}{2}}\eta_m = 1.$$

So we can apply Lemma 5.7. Choose $l \le m-1$ such that

$$A_{m-1} = \cdots = A_l \text{ with } A_l > A_{l-1} \text{ or } l = 1.$$

Then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_{\psi^{(1)}}}(\psi^{(1)}, \underline{l}^{(1)}, \underline{\eta}^{(1)}) &\hookrightarrow \times_{i=l}^{m-1}\langle -1/2, \ldots, -A_i\rangle \\ &\rtimes \pi^{\Sigma_0}_{M,>_{\psi^{(m-l+1)}}}(\psi^{(m-l+1)}, \underline{l}^{(m-l+1)}, \underline{\eta}^{(m-l+1)}),\end{aligned}$$

To go further, we need to compute

$$\eta^{(m-l+1)}_{l-1} = \eta^{(1)}_{l-1}\prod_{i=l}^{m-1}(-1)^{A_i-\frac{1}{2}} = \prod_{i=l}^{m-1}(-1)^{A_i-\frac{1}{2}}\prod_{i=l}^{m-1}(-1)^{A_i-\frac{1}{2}} = 1$$

This means we can repeat the previous process. It is not hard to see that one gets eventually

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_{\psi^{(m-l+1)}}}(\psi^{(m-l+1)}, \underline{l}^{(m-l+1)}, \underline{\eta}^{(m-l+1)}) &\hookrightarrow \times_{i<l}\langle -1/2, \ldots, -A_i\rangle \\ \rtimes \pi^{\Sigma_0}_{M,>_{\psi^{(m)}}}(\psi^{(m)}, \underline{l}^{(m)}, \underline{\eta}^{(m)})\end{aligned}$$

As a result, we get

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<m}\langle -1/2, \ldots, -A_i\rangle \rtimes \pi^{\Sigma_0}_{M,>_{\psi^{(m)}}}(\psi^{(m)}, \underline{l}^{(m)}, \underline{\eta}^{(m)})$$

By definition $\psi^{(m)} = \psi'$. So it remains to show $(\underline{l}^{(m)}, \underline{\eta}^{(m)}) = (\underline{l}', \underline{\eta}')$. It is clear that $l^{(m)}_i = 0$ for $i \le m$. Hence,

$$\eta^{(m)}_{i+1} = (-1)^{A_i-\frac{3}{2}}\eta^{(m)}_i \quad \text{for } i < m.$$

By Lemma 5.7,

$$\eta^{(m)}_1 = -\eta^{(m-k)}_k$$

for maximal $k < m$ such that $A_1 = A_i$ for all $i \le k$. From the above discussion, we see $\eta^{(m-k)}_k = 1$. So $\eta^{(m)}_1 = -1$. Hence, $\underline{\eta}^{(m)} = \underline{\eta}'$. □

Combining the two cases, we can describe $\Pi^{\Sigma_0}_\psi$ under the assumption (5.5) as follows. Consider the maximal sequence of integers

$$0 = s_0 < s_1 < \cdots < s_l = m$$

such that $A_{s_j} \ne A_{s_j+1}$. For any $0 \le k \le l$, we get a new parameter ψ'_k by replacing all $(\rho, A_i, B_i, \zeta_i)$ by $(\rho, A'_i, B'_i, \zeta'_i)$ such that: $\zeta'_i = +$ and

- $A'_i = A_i$, $B'_i = B_i$ for $i > m$;
- $A'_i = A_i - 1$, $B'_i = 1/2$ for $i \le m$ and $i \ne s_k$;
- $A'_i = A_i$, $B'_i = 1/2$ for $i = s_k$.

Then we can divide $\Pi_{\psi}^{\Sigma_0}$ into $l+1$ classes, i.e.,

$$\Pi_{\psi}^{\Sigma_0} = \sqcup_{k=0}^{l} \Pi_{\psi}^{\Sigma_0}(k).$$

For any $0 \leq k \leq l$, we can get an injection

$$\Pi_{\psi}^{\Sigma_0}(k) \hookrightarrow \Pi_{\psi'_k}^{\Sigma_0}, \quad \pi^{\Sigma_0} \mapsto \pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}'),$$

such that

$$\pi^{\Sigma_0} \hookrightarrow \times_{i \leq m \text{ and } i \neq s_k} \langle -1/2, \ldots, -A_i \rangle \rtimes \pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}'). \tag{5.7}$$

The image is characterized by the condition that for all $i \leq s_k$,

- $l'_i = 0$;
- $\eta'_i = -\prod_{j<i}(-1)^{A_j + \frac{1}{2}}$.

Because of the first condition, the second condition can be simplified as $\eta'_1 = -1$ when $k \neq 0$.

After applying (4.1) and (3.1) to $\pi_{M,>_{\psi'_k}}^{\Sigma_0}(\psi'_k, \underline{l}', \underline{\eta}')$, we get

$$\pi^{\Sigma_0} \hookrightarrow \times_{i \leq m \text{ and } i \neq s_k} \langle -1/2, \ldots, -A_i \rangle \rtimes \mathcal{I}' \tag{5.8}$$

and

$$\mathcal{I}' := \times_{i=1}^{n} \underbrace{\begin{pmatrix} B'_i & \cdots & -A'_i \\ \vdots & & \vdots \\ B'_i + l'_i - 1 & \cdots & -(A'_i - l'_i + 1) \end{pmatrix}}_{II_i}$$
$$\times \times_{\text{some } j} \underbrace{\begin{pmatrix} B'_{j+1} + l'_{j+1} & \cdots & -(A'_j - l'_j) \\ \vdots & & \vdots \\ t'_j - \delta_j & \cdots & -(t'_j + \delta_j) \end{pmatrix}}_{\tilde{I}_j} \rtimes \sigma^{\Sigma_0} \tag{5.9}$$

Note if $\langle x, \ldots, -y \rangle$ from II_i or $\tilde{I}_j$ has shift less than that of $\langle -1/2, \ldots, -A_i \rangle$, then it is necessary that $y \geq A_i$. So they are interchangeable (cf. [10, Corollary 4.3]). This shows the induced representation in (5.8) is a subrepresentation of the costandard representation as we want. As a consequence, the induced representation in (5.8) has a unique irreducible subrepresentation. Therefore, the same is true for that of (5.7).

5.2.2 Resolution

Now we can complete the proof of Theorem 5.5. By Lemma 5.1 and induction assumption, we have

$$\pi^{\Sigma_0} \hookrightarrow \times_{i=l}^{s} \langle -B_i, \ldots, -A_i \rangle \rtimes \mathcal{I}^s \tag{5.10}$$

and

$$\mathcal{I}^s := \times_{i=l}^{s} \underbrace{\begin{pmatrix} -(B_i-1) & \cdots & -(A_i-1) \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i-B_i+1/2) \end{pmatrix}}_{III_i}$$

$$\times \times_{\{i<l \text{ or } m\geq i>s\}\setminus\{s_k\}} \underbrace{\begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i-B_i+1/2) \end{pmatrix}}_{III_i}$$

$$\times \underbrace{\begin{pmatrix} -B_{s_k} & \cdots & -A_{s_k} \\ \vdots & & \vdots \\ -3/2 & \cdots & -(A_{s_k}-B_{s_k}+3/2) \end{pmatrix}}_{III_{s_k}} \rtimes \mathcal{I}'$$

if $s \neq s_k$, and

$$\mathcal{I}^s := \times_{i=l}^{s-1} \underbrace{\begin{pmatrix} -(B_i-1) & \cdots & -(A_i-1) \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i-B_i+1/2) \end{pmatrix}}_{III_i}$$

$$\times \times_{i<l \text{ or } m\geq i>s} \underbrace{\begin{pmatrix} -B_i & \cdots & -A_i \\ \vdots & & \vdots \\ -1/2 & \cdots & -(A_i-B_i+1/2) \end{pmatrix}}_{III_i}$$

$$\times \underbrace{\begin{pmatrix} -(B_{s_k}-1) & \cdots & -(A_{s_k}-1) \\ \vdots & & \vdots \\ -3/2 & \cdots & -(A_{s_k}-B_{s_k}+3/2) \end{pmatrix}}_{III_{s_k}} \rtimes \mathcal{I}'$$

if $s = s_k$. Here $\mathcal{I}'$ is defined as in (5.9). Moreover, $\mathcal{I}^s$ is a subrepresentation of the costandard representation, obtained by taking induction of the shifted Steinberg representations from the shifted Speh representations with σ^{Σ_0}. We claim the induced representation in (5.10) is a subrepresentation of the costandard representation as we want.

To prove the claim, we need to show any shifted Steinberg representation above, whose shift is less than that of $\langle -B_s, \ldots, -A_s\rangle$, can be moved to the front. There are two cases.

(1) If it is in the form $\langle -x, \ldots, -y\rangle$ from III_i, then by our choice of s,

$$\begin{cases} x \geq B_s & \text{if } y \leq A_s, \\ x = 1/2 < B_s & \text{if } y > A_s. \end{cases}$$

In either case, $\langle -x, \ldots, -y\rangle$ and $\langle -B_s, \ldots, -A_s\rangle$ are interchangeable.

(2) If it is in the form $\langle x, \ldots, -y\rangle$ from II_i or $\tilde{I}_j$ in (5.9), then we have $y \geq A_s$. Hence, $\langle x, \ldots, -y\rangle$ and $\langle -B_s, \ldots, -A_s\rangle$ are interchangeable (cf. [10, Corollary 4.3]).

This finishes the proof of our claim. As a consequence, the induction in (5.10) has a unique irreducible subrepresentation. So we can combine $\langle B_i, \ldots, -A_i \rangle$ with III_i for $l \leq i \leq s$, and this gives (5.4).

6 Comments on the general case

Let ψ be an Arthur parameter of $G(F)$ (cf. (1.2)) with the assumption that all $b_i = b$. Note we do not assume (1.5) here. Let ψ_{np} be any representation of $W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ such that

$$\psi = \psi_{np} \oplus \psi_p \oplus \psi_{np}^\vee,$$

where $\psi_{np}^\vee$ is the dual of ψ_{np}. Mœglin [5, Theorem 6] proved that there is a bijection

$$\Pi_\psi^{\Sigma_0} \to \Pi_{\psi_p}^{\Sigma_0}, \quad \pi^{\Sigma_0} \mapsto \tau^{\Sigma_0}.$$

such that

$$\pi^{\Sigma_0} = \Big(\times_{(\rho_i, a_i, b_i) \in Jord(\psi_{np})} Sp(St(\rho_i, a_i), b_i) \Big) \rtimes \tau^{\Sigma_0}$$

We can embed τ^{Σ_0} into a costandard representation, which is an induction of shifted Steinberg representations and a tempered representation of a group of the same type as $G(F)$. Note these shifted Steinberg representations are interchangeable with the ones from $Sp(St(\rho_i, a_i), b_i)$ for $(\rho_i, a_i, b_i) \in Jord(\psi_{np})$ by the parity condition. So the complete Langlands parameter (ϕ, ϵ) of π^{Σ_0} will be given as

$$\phi = (\oplus_i \phi_i) \oplus \phi' \oplus (\oplus_i \phi_i^\vee)$$

where ϕ_i is the Langlands parameter of $Sp(St(\rho_i, a_i), b_i)$ for $(\rho_i, a_i, b_i) \in Jord(\psi_{np})$, (ϕ', ϵ') is the complete Langlands parameter of τ^{Σ_0} and ϵ corresponds to ϵ' under the canonical isomorphism $\mathcal{S}_\phi^{\Sigma_0} \cong \mathcal{S}_{\phi'}^{\Sigma_0}$.

At last, we can extend our main results (Theorem 1.1, 1.2, 1.3) to ψ_p by applying them to each ρ appearing in $Jord(\psi_p)$. For the proofs, it suffices to modify the induction assumptions in the proofs of Theorems 3.1, 4.1, 5.3, 5.5 by considering all Jordan blocks of ψ_p, and apply Theorem A.3 for the nonvanishing result in the special case (cf. Sect. 3, 4).

Appendix A: A nonvanishing result

In this appendix, we will prove the following nonvanishing result. Let ψ be an Arthur parameter of $G(F)$ (cf. (1.2)) under the Assumption (1.5). Let $>_\psi$ be an admissible order and we index the Jordan blocks in $Jord(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Let

$$J := \cup_{i=1}^n \{(\rho, A_i, B_i, \zeta_i)\} \subseteq Jord(\psi)$$

Suppose

$$A_{i+1} \geq A_i, \quad B_{i+1} \geq B_i, \quad \zeta_{i+1} = \zeta_i \quad \text{for } i < n$$

and

$$J^c \gg J, \quad J^c \text{ has discrete diagonal restriction,}$$

where $J^c := Jord(\psi) \backslash J$. Then we have the following theorem.

Theorem A.1 $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ *if and only if the following condition are satisfied for all* $i < n$*:*

$$\begin{cases} \eta_{i+1} = (-1)^{A_i - B_i}\eta_i & \Rightarrow A_{i+1} - l_{i+1} \geq A_i - l_i, \quad B_{i+1} + l_{i+1} \geq B_i + l_i, \\ \eta_{i+1} \neq (-1)^{A_i - B_i}\eta_i & \Rightarrow B_{i+1} + l_{i+1} > A_i - l_i \end{cases} \tag{A.1}$$

Proof The necessity of the condition follows from [10, Lemma 5.5]. So it remains to prove its sufficiency. We will proceed by induction on $|J|$. If $|J| = 2$, this has been proved in [10, Proposition 5.2].

Suppose $|J| = m+1$. We first "expand" $[B_{m+1}, A_{m+1}]$ to $[B^*_{m+1}, A^*_{m+1}]$ (cf. [10, Section 7.2]), so that $B^*_{m+1} = B_m$. By [10, Proposition 7.4], we know $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A^*_{m+1}, B^*_{m+1}, l^*_{m+1}, \eta_{m+1}, \zeta_{m+1})\Big) \neq 0 \tag{A.2}$$

where ψ_- is defined by

$$Jord(\psi_-) = Jord(\psi) \backslash \{(\rho, A_{m+1}, B_{m+1}, \zeta_{m+1})\}$$

and

$$l^*_{m+1} = l_{m+1} + (B_{m+1} - B_m).$$

It is easy to check that the condition (A.1) holds for $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ if and only if it holds for the representation in (A.2). So we will assume $B_{m+1} = B_m$ from now on.

Next we can "pull" $[B_{m+1}, A_{m+1}], [B_m, A_m]$ (cf. [10, 7.1]), so that they are far away from $\cup_{i<m}\{(\rho, A_i, B_i, \zeta_i)\}$. By [10, Proposition 7.1, 7.3], we know $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if the following representations are all nonzero. So it suffices to show each of them is nonzero by our induction assumption. Let ψ_- be defined by

$$Jord(\psi_-) = Jord(\psi) \backslash \{(\rho, A_{m+1}, B_{m+1}, \zeta_{m+1}), (\rho, A_m, B_m, \zeta_m)\}.$$

(1) Show

$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{m+1} + T, B_{m+1} + T, l_{m+1}, \eta_{m+1}, \zeta_{m+1}), (\rho, A_m + T, B_m + T, l_m, \eta_m, \zeta_m)\Big) \neq 0 \tag{A.3}$$

for some T. Let $J_- = Jord(\psi_-)$. Then we will choose T so that $J^c_- \gg J_-$. To make J^c_- having discrete diagonal restriction, we will shift $[B_{m+1} + T, A_{m+1} + T]$ further to $[B_{m+1} + T', A_{m+1} + T']$ such that $B_{m+1} + T' > A_m + T$. Then by our induction assumption,

$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{m+1} + T', B_{m+1} + T', l_{m+1}, \eta_{m+1}, \zeta_{m+1}), (\rho, A_m + T, B_m + T, l_m, \eta_m, \zeta_m)\Big) \neq 0$$

Let $\psi_{\gg}$ be the dominating parameter with discrete diagonal restriction, obtained by shifting $[B_i, A_i]$ to $[B_i + T_i, A_i + T_i]$ with $A_i + T_i < B_m + T$ for all $1 \leq i \leq m-1$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<m} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-;$$
$$(\rho, A_{m+1}+T', B_{m+1}+T', l_{m+1}, \eta_{m+1}, \zeta_{m+1}), (\rho, A_m+T, B_m+T, l_m, \eta_m, \zeta_m)\Big)$$

By [10, Proposition 5.2],

$$\mathrm{Jac}_{(\rho, A_{m+1}+T', B_{m+1}+T', \zeta_{m+1}) \mapsto (\rho, A_{m+1}+T, B_{m+1}+T, \zeta_{m+1})} \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

So after we apply the same Jacquet functor to the full induced representation above, we should get something nonzero. Since $B_{m+1} + T + 1 > A_i + T_i$ for $i < m$, the result is

$$\times_{i<m} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix}$$
$$\rtimes \mathrm{Jac}_{(\rho, A_{m+1}+T', B_{m+1}+T', \zeta_{m+1}) \mapsto (\rho, A_{m+1}+T, B_{m+1}+T, \zeta_{m+1})}$$
$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{m+1}+T', B_{m+1}+T', l_{m+1}, \eta_{m+1}, \zeta_{m+1}),$$
$$(\rho, A_m+T, B_m+T, l_m, \eta_m, \zeta_m)\Big) \neq 0$$

This shows (A.3).

(2) Show

$$\pi^{\Sigma_0}_{M,>_\psi}\Big(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{m+1}+T, B_{m+1}+T, l_{m+1}, \eta_{m+1}, \zeta_{m+1}), (\rho, A_m, B_m, l_m, \eta_m, \zeta_m)\Big) \neq 0 \tag{A.4}$$

for some T. Let $J_- = Jord(\psi_-) \sqcup \{(\rho, A_m, B_m, \zeta_m)\}$. We can choose T so that $J^c_- \gg J_-$. Then the statement follows from our induction assumption immediately.

(3) Show

$$\pi^{\Sigma_0}_{M,>'_\psi}\Big(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_{m+1}, B_{m+1}, l'_{m+1}, \eta'_{m+1}, \zeta_{m+1}),$$
$$(\rho, A_m+T, B_m+T, l'_m, \eta'_m, \zeta_{n-1})\Big) \neq 0 \tag{A.5}$$

for some T, where $>'_\psi$ is obtained by switching $(\rho, A_{m+1}, B_{m+1}, \zeta_{m+1})$ with $(\rho, A_m, B_m, \zeta_m)$, and $(\underline{l}', \underline{\eta}') = S^+_{m+1}(\underline{l}, \underline{\eta})$ (cf. [10, Section 6.1]) given by the change of order formula. Let $J_- = Jord(\psi_-) \sqcup \{(\rho, A_{m+1}, B_{m+1}, \zeta_{m+1})\}$. We can choose T so that $J^c_- \gg J_-$. Then the statement follows from our induction assumption again, provided we can verify the representation in (A.5) satisfies (A.1). Indeed, we only need to show

$$\begin{cases} \eta'_{m+1} = (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1} & \Rightarrow A_{m+1} - l'_{m+1} \geq A_{m-1} - l_{m-1}, \quad B_{m+1} + l'_{m+1} \geq B_{m-1} + l_{m-1}, \\ \eta'_{m+1} \neq (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1} & \Rightarrow B_{m+1} + l'_{m+1} > A_{m-1} - l_{m-1}. \end{cases} \tag{A.6}$$

We leave it to the next lemma. □

Lemma A.2 (A.6) *holds.*

Proof We divide into three cases according to the change of order formula.

(1) If $\eta_{m+1} \neq (-1)^{A_m - B_m}\eta_m$, then

$$\begin{cases} \eta'_{m+1} = \eta_m \\ l'_{m+1} = (B_m + l_m) - (A_m - l_m) + l_{m+1} - 1 \end{cases}$$

We get

$$\begin{aligned} B_{m+1} + l'_{m+1} &= (B_{m+1} + l_{m+1}) + (B_m + l_m) - (A_m - l_m) - 1 \\ A_{m+1} - l'_{m+1} &= (A_{m+1} - l_{m+1}) + (A_m - l_m) - (B_m + l_m) + 1 \end{aligned}$$

By (A.1), we have

$$B_{m+1} + l_{m+1} > A_m - l_m.$$

(a) When $\eta_m \neq (-1)^{A_{m-1} - B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} \neq (-1)^{A_{m-1} - B_{m-1}}\eta_{m-1}$. We need to show

$$B_{m+1} + l'_{m+1} > A_{m-1} - l_{m-1}.$$

By (A.1), we have

$$B_m + l_m > A_{m-1} - l_{m-1}.$$

Then

$$B_{m+1} + l'_{m+1} \geq B_m + l_m > A_{m-1} - l_{m-1}.$$

(b) When $\eta_m = (-1)^{A_{m-1} - B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} = (-1)^{A_{m-1} - B_{m-1}}\eta_{m-1}$. We need to show

$$\begin{cases} B_{m+1} + l'_{m+1} \geq B_{m-1} + l_{m-1} \\ A_{m+1} - l'_{m+1} \geq A_{m-1} - l_{m-1} \end{cases}$$

By (A.1), we have

$$\begin{cases} B_m + l_m \geq B_{m-1} + l_{m-1} \\ A_m - l_m \geq A_{m-1} - l_{m-1} \end{cases}$$

Then

$$\begin{aligned} B_{m+1} + l'_{m+1} &\geq B_m + l_m \geq B_{m-1} + l_{m-1} \\ A_{m+1} - l'_{m+1} &\geq A_{m+1} - l_{m+1} \geq A_m - l_m \geq A_{m-1} - l_{m-1} \end{aligned}$$

(2) If $\eta_{m+1} = (-1)^{A_m - B_m}\eta_m$ and

$$l_{m+1} - l_m < (A_{m+1} - B_{m+1})/2 - (A_m - B_m) + l_m,$$

then

$$\begin{cases} \eta'_{m+1} \neq \eta_m \\ l'_{m+1} = (A_m - l_m) - (B_m + l_m) + l_{m+1} - 1 \end{cases}$$

We get

$$B_{m+1} + l'_{m+1} = (B_{m+1} + l_{m+1}) - (B_m + l_m) + (A_m - l_m) + 1$$
$$A_{m+1} - l'_{m+1} = (A_{m+1} - l_{m+1}) - (A_m - l_m) + (B_m + l_m) - 1$$

By (A.1), we have

$$\begin{cases} B_{m+1} + l_{m+1} \geq B_m + l_m \\ A_{m+1} - l_{m+1} \geq A_m - l_m \end{cases}$$

(a) When $\eta_m \neq (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} = (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$. We need to show

$$\begin{cases} B_{m+1} + l'_{m+1} \geq B_{m-1} + l_{m-1} \\ A_{m+1} - l'_{m+1} \geq A_{m-1} - l_{m-1} \end{cases}$$

By (A.1), we have

$$B_m + l_m > A_{m-1} - l_{m-1}.$$

Then

$$B_{m+1} + l'_{m+1} \geq (A_m - l_m) + 1 \geq (A_{m-1} - l_{m-1}) + 1 \geq B_{m-1} + l_{m-1}$$
$$A_{m+1} - l'_{m+1} \geq (B_m + l_m) - 1 \geq A_{m-1} - l_{m-1}$$

(b) When $\eta_m = (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} \neq (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$. We need to show

$$B_{m+1} + l'_{m+1} > A_{m-1} - l_{m-1}$$

By (A.1), we have

$$\begin{cases} B_m + l_m \geq B_{m-1} + l_{m-1} \\ A_m - l_m \geq A_{m-1} - l_{m-1} \end{cases}$$

Then

$$B_{m+1} + l'_{m+1} \geq (A_m - l_m) + 1 \geq (A_{m-1} - l_{m-1}) + 1 > A_{m-1} - l_{m-1}$$

(3) If $\eta_{m+1} = (-1)^{A_m - B_m}\eta_m$ and

$$l_{m+1} - l_m \geq (A_{m+1} - B_{m+1})/2 - (A_m - B_m) + l_m,$$

then

$$\begin{cases} \eta'_{m+1} = \eta_m \\ l'_{m+1} = (A_{m+1} - B_{m+1}) - l_{m+1} - (A_m - l_m) + (B_m + l_m) \end{cases}$$

We get

$$B_{m+1} + l'_{m+1} = (A_{m+1} - l_{m+1}) - (A_m - l_m) + (B_m + l_m)$$
$$A_{m+1} - l'_{m+1} = (B_{m+1} + l_{m+1}) - (B_m + l_m) + (A_m - l_m)$$

By (A.1), we have

$$\begin{cases} B_{m+1} + l_{m+1} \geq B_m + l_m \\ A_{m+1} - l_{m+1} \geq A_m - l_m \end{cases}$$

(a) When $\eta_m \neq (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} \neq (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$. We need to show

$$B_{m+1} + l'_{m+1} > A_{m-1} - l_{m-1}$$

By (A.1), we have

$$B_m + l_m > A_{m-1} - l_{m-1}.$$

Then

$$B_{m+1} + l'_{m+1} \geq B_m + l_m > A_{m-1} - l_{m-1}$$

(b) When $\eta_m = (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$, then $\eta'_{m+1} = (-1)^{A_{m-1}-B_{m-1}}\eta_{m-1}$. We need to show

$$\begin{cases} B_{m+1} + l'_{m+1} \geq B_{m-1} + l_{m-1} \\ A_{m+1} - l'_{m+1} \geq A_{m-1} - l_{m-1} \end{cases}$$

By (A.1), we have

$$\begin{cases} B_m + l_m \geq B_{m-1} + l_{m-1} \\ A_m - l_m \geq A_{m-1} - l_{m-1} \end{cases}$$

Then

$$B_{m+1} + l'_{m+1} \geq B_m + l_m \geq B_{m-1} + l_{m-1}$$
$$A_{m+1} - l'_{m+1} \geq A_m - l_m \geq A_{m-1} - l_{m-1}$$

□

More generally, we can drop the Assumption (1.5), but only assume $\psi = \psi_p$. Suppose for each ρ appearing in $Jord(\psi)$, we have the same setup as in Theorem A.1. Then we have

Theorem A.3 $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ *if and only if the condition* (A.1) *is satisfied for each* ρ.

Proof We can apply the arguments of the proof of Theorem A.1 to each ρ one by one, which reduces it to the case that $|J| = 2$ for each ρ. Then this case follows from [10, Proposition 5.3]. □

Appendix B. Change sign

In this appendix, we would like to extend [10, Proposition 7.6] as follows. Let ψ be an Arthur parameter of $G(F)$ such that $\psi = \psi_p$. We choose an admissible order $>_\psi$ and fix an irreducible unitary supercuspidal representation ρ of $GL(d_\rho, F)$. Let us index the Jordan blocks in $Jord_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \cup_{j=1}^n \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover, there exists $1 \leq m \leq n$ such that

$$A_m = \cdots = A_1 \geq A_i, \quad B_m = \cdots = B_1 = 1/2, \quad \zeta_m = \cdots = \zeta_1 \neq \zeta_i$$

for $m < i \leq n$. We introduce another parameter ψ^* by changing $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i + 1, B_i, -\zeta_i)$ for $i \leq m$. For any $(\underline{l}, \underline{\eta})$, such that

$$l_{i+1} = l_i, \quad \eta_{i+1} = (-1)^{A_i - \frac{1}{2}} \eta_i \quad \text{for } i < m, \tag{B.1}$$

we can associate it with $(\underline{l}^*, \underline{\eta}^*)$, defined as follows. For $i > m$,

$$l_i^* = l_i, \quad \eta_i^* = \eta_i.$$

For $i < m$,

$$l_{i+1}^* = l_i^*, \quad \eta_{i+1}^* = (-1)^{A_i + \frac{1}{2}} \eta_i^*. \tag{B.2}$$

Then it remains to specify l_1^*, η_1^*, which are given by

$$\eta_1^* = -\eta_1, \quad l_1^* = \begin{cases} l_1 + 1 & \text{if } \eta_1 = 1 \\ l_1 & \text{if } \eta_1 = -1 \end{cases}$$

In case $l_1 = (A_1 + \frac{1}{2})/2$, we fix $\eta_1 = -1$.

Proposition B.1 $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ *if and only if* $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$. *Moreover,*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \times_{i=1}^{m} \langle -\zeta_i 1/2, \ldots, -\zeta_i (A_i + 1) \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \tag{B.3}$$

Remark B.2 [10, Proposition 7.6] settles the case when $m = 1$.

Proof As in the proof of [10, Proposition 7.6], we can reduce it to the case that

$$m = n \text{ and } Jord(\psi) \backslash \cup_{i=1}^{n} \{(\rho, A_i, B_i, \zeta_i)\} \text{ has discrete diagonal restriction.}$$

Because of the conditions (B.1) and (B.2), we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0 \text{ and } \pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$$

by Theorem A.3. So we only need to show (B.3), and we will proceed by induction on n.

Let $\psi^*_>$ be obtained from ψ^* by changing $(\rho, A_n + 1, 1/2, -\zeta_n)$ to $(\rho, A_n + 1 + T_n, 1/2 + T_n, -\zeta_n)$ for T_n sufficiently large. Then by our induction assumption, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_>, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \times_{i=1}^{n-1} \langle -\zeta_i 1/2, \ldots, -\zeta_i (A_i + 1) \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}_>, \underline{l}^{(n)}, \underline{\eta}^{(n)})$$

where $\psi^{(n)}_>$ is obtained from $\psi^*_>$ by changing $(\rho, A_i + 1, 1/2, -\zeta_i)$ back to $(\rho, A_i, 1/2, \zeta_i)$ for $1 \leq i < n$. Moreover,

$$l_i^{(n)} = l_i, \quad \eta_i^{(n)} = \eta_i \quad \text{for } i < n,$$

and

$$l_i^{(n)} = l_i^*, \quad \eta_i^{(n)} = \eta_i^* \quad \text{for } i \geq n.$$

Then we claim

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}_>, \underline{l}^{(n)}, \underline{\eta}^{(n)}) \hookrightarrow \underbrace{\begin{pmatrix} -\zeta_n(1/2 + T_n) & \cdots & -\zeta_n 1/2 \\ \vdots & & \vdots \\ -\zeta_n(A_n + 1 + T_n) & \cdots & -\zeta_n(A_n + 1) \end{pmatrix}}_{\mathcal{C}_{X_n}} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}). \tag{B.4}$$

where

$$X_n := \begin{bmatrix} -\zeta_n(1/2+T_n) & \cdots & -\zeta_n 1/2 \\ \vdots & & \vdots \\ -\zeta_n(A_n+1+T_n) & \cdots & -\zeta_n(A_n+1) \end{bmatrix}$$

If this is the case, then

$$\begin{aligned} \pi^{\Sigma_0}_{>_\psi}(\psi^*_>, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \times_{i=l}^{n-1} \langle -\zeta_i 1/2, \ldots, -\zeta_i(A_i+1)\rangle \times \mathcal{C}_{X_n} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \\ &\cong \mathcal{C}_{X_n} \times \times_{i=l}^{n-1} \langle -\zeta_i 1/2, \ldots, -\zeta_i(A_i+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}), \end{aligned}$$

from which (B.3) follows.

We still need to show the claim (B.4). Let $\psi^{(n)}$ be obtained from $\psi^{(n)}_>$ by moving $(\rho, A_n + 1 + T_n, 1/2 + T_n, -\zeta_n)$ back to $(\rho, A_n + 1, 1/2, -\zeta_n)$. Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}, \underline{l}^{(n)}, \underline{\eta}^{(n)}) \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}_>, \underline{l}^{(n)}, \underline{\eta}^{(n)}) \hookrightarrow \begin{pmatrix} -\zeta_n(1/2+T_n) & \cdots & -\zeta_n 3/2 \\ \vdots & & \vdots \\ -\zeta_n(A_n+1+T_n) & \cdots & -\zeta_n(A_n+2) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}, \underline{l}^{(n)}, \underline{\eta}^{(n)}). \tag{B.5}$$

To show the nonvanishing of $\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}, \underline{l}^{(n)}, \underline{\eta}^{(n)})$, we need to switch to a new order $>'_\psi$ by moving $(\rho, A_n + 1, 1/2, -\zeta_n)$ to the last position. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}, \underline{l}^{(n)}, \underline{\eta}^{(n)}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(n)}, \underline{l}'^{(n)}, \underline{\eta}'^{(n)}),$$

where

$$l_i'^{(n)} = l_i^{(n)}, \quad \eta_i'^{(n)} = \eta_i^{(n)} \quad \text{for } i > n,$$

and

$$l_i'^{(n)} = l_i^{(n)}, \quad \eta_i'^{(n)} = (-1)^{A_n - 1/2}\eta_i^{(n)} \quad \text{for } i < n,$$

and

$$l_n'^{(n)} = l_1^*, \quad \eta_n'^{(n)} = \eta_1^*.$$

Let $\psi^{(n)}_\gg$ be a dominating parameter for $\psi^{(n)}$ with respect to $>'_\psi$, obtained by changing $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ for $i < n$. We also require that $\psi^{(n)}_\gg$ has discrete diagonal restriction. Then by [10, Proposition 7.6],

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(n)}_\gg, \underline{l}'^{(n)}, \underline{\eta}'^{(n)}) \hookrightarrow \langle -\zeta_n 1/2, \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_\gg, \underline{l}', \underline{\eta}'),$$

where $\psi_\gg$ is obtained from $\psi^{(n)}_\gg$ by changing $(\rho, A_n + 1, 1/2, -\zeta_n)$ back to $(\rho, A_n, 1/2, \zeta_n)$. Note

$$l_i' = l_i'^{(n)}, \quad \eta_i' = \eta_i'^{(n)} \quad \text{for } i \neq n,$$

and

$$l_n' = l_1, \quad \eta_n' = \eta_1.$$

It is easy to check by the change of order formula that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

In particular, the right hand side is nonzero. Therefore,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \times_{i=1}^{n-1} \begin{pmatrix} \zeta_i(1/2+T_i) & \cdots & \zeta_i 3/2 \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$$

Combined with the previous inclusion, we get

$$\begin{aligned} \pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(n)}_{\gg}, \underline{l}'^{(n)}, \underline{\eta}'^{(n)}) \hookrightarrow & \langle -\zeta_n 1/2, \ldots, -\zeta_n(A_n+1)\rangle \times \\ & \times_{i=1}^{n-1} \begin{pmatrix} \zeta_i(1/2+T_i) & \cdots & \zeta_i 3/2 \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \\ & \cong \times_{i=1}^{n-1} \begin{pmatrix} \zeta_i(1/2+T_i) & \cdots & \zeta_i 3/2 \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \\ & \times \langle -\zeta_n 1/2, \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \end{aligned}$$

Consequently, $\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(n)}, \underline{l}'^{(n)}, \underline{\eta}'^{(n)}) \neq 0$ and

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(n)}, \underline{l}'^{(n)}, \underline{\eta}'^{(n)}) \hookrightarrow \langle -\zeta_n 1/2, \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

Substitute the above expression into (B.5), we obtain

$$\begin{aligned} \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}_{>}, \underline{l}^{(n)}, \underline{\eta}^{(n)}) \hookrightarrow & \begin{pmatrix} -\zeta_n(1/2+T_n) & \cdots & -\zeta_n 3/2 \\ \vdots & & \vdots \\ -\zeta_n(A_n+1+T_n) & \cdots & -\zeta_n(A_n+2) \end{pmatrix} \\ & \times \langle -\zeta_n 1/2, \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}). \end{aligned}$$

Note the Jordan blocks in $Jord_\rho(\psi)$ satisfies $A_i < A_n + 1$ for $i \leq n$, and $B_i > A_n + 1 + T_n$ for $i > n$. If we apply Jac_{X_n} to the right hand side of the above expression, we can only get $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$. This means the left hand side is the unique irreducible subrepresentation of the right hand side. Therefore,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n)}_{>}, \underline{l}^{(n)}, \underline{\eta}^{(n)}) \hookrightarrow \mathcal{C}_{X_n} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}),$$

which is exactly (B.4). This finishes our proof. □

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