MODULI OF SINGULAR SEXTIC CURVES VIA PERIODS OF $K3$ SURFACES

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Abstract. In this paper we realize the moduli spaces of singular sextic curves with specified symmetry type as arithmetic quotients of complex hyperbolic balls or type IV domains. We also identify their GIT compactifications with the Looijenga compactifications of the corresponding period domains, most of which are actually Baily-Borel compactifications. Some special cases were studied in [MSY92], [Laz09] and [GMGZ17]. This paper is a follow-up of [YZ18].

Contents

1. Introduction

A double cover of $\mathbb{P}^2$ branched along a smooth sextic curve is a smooth $K3$ surface with an ample class of self-intersection 2 (a degree two $K3$ surface, for short). The converse also holds for generic degree two $K3$ surfaces. Thanks to the global Torelli theorem for $K3$ surfaces (Pjateckiĭ-Šapiro and Šafarevič [PŠ71], Rappoport-Burns [BR75], Looijenga-Peters [LP81]), Shah realized the GIT compactification of moduli space of sextic curves as a modification of the Baily-Borel compactification of a type IV arithmetic quotient (see [Sha80]). This construction can also fit into a more general framework developed by Looijenga (see [Loo03a, Loo03b]). We will review this in section 2.

It is natural to consider reducible and nodal sextic curves. A double cover of $\mathbb{P}^2$ branched along a nodal sextic curve is a nodal $K3$ surface, with nodes at the preimage of singular points on that curve. After resolving the singularities, we obtain a $K3$ surface with natural lattice polarization. Along this direction, there are lots of works on moduli spaces of singular sextic curves, including [MSY92] about six lines on $\mathbb{P}^2$, [Laz09] about pairs consisting of a plane quintic curve and a line, [GMGZ17] about triples consisting of a plane quartic curve and two lines. In these works, the GIT compactifications are identified with Baily-Borel compactifications of the period domains. The main purpose of this paper is to give a uniform and systematic account of these results, hence provide more examples.

We briefly formulate our main theorem. Notice that there is a natural stratification on the moduli space of plane sextic curves induced by number of nodes. Let T be a singular type, which corresponds to an irreducible component of certain strata. See section 3.1 for detailed definition. Let $\mathcal{F}_T$ be the moduli space of sextic curves of type T and $\overline{\mathcal{F}}_T$ the GIT compactification, which are constructed via geometric invariant theory, see section 3.1. The period map of the corresponding lattice-poarized $K3$ surfaces gives a morphism $\mathscr{P}\colon \mathcal{F}_T \longrightarrow \Gamma\backslash\mathbb{D}$. Here $\Gamma\backslash\mathbb{D}$ is an arithemetic quotient of type IV domain. There is a Γ-invariant hyperplane arrangement $\mathcal{H}_*$ on $\mathbb{D}$. We have the Looijenga compactification $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ of $\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)$, see [Loo03b]. Our main theorem is:

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Theorem 1.1 (Main theorem). *For any singular type T, the period map $\mathscr{P}\colon \mathcal{F}_T \longrightarrow \Gamma\backslash\mathbb{D}$ is an algebraic open embedding and extends to an isomorphism between projective varieties $\mathscr{P}\colon \overline{\mathcal{F}}_T \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$.*

We will also give an criterian when $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ is Baily-Borel compactification.

Irreducible nodal sextic curves has very close relation with del Pezzo surfaces. We investigate this relation in section 4.1. When a generic sextic curve of singular type T is a union of smooth components, we identify $\overline{\mathcal{F}}_T$ with quotients of products of projective spaces. See equation (8) and table 1.

In our previous work [YZ18], we considered moduli spaces of symmetric cubic fourfolds. Similar construction can also be achieved sextic curves. Actually, in section 5, we will consider sextic curves with given singular type and symmetry type, and similar results hold.

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2. Review: Period of degree two K3 surface

In this section we review the characterization of moduli of sextic curves via period map of degree two $K3$ surfaces. See [Sha80], [Loo03b] (section 8, theorem 8.6), [Laz16] (section 1.2.3).

Let V be a 3-dimensional vector space over $\mathbb{C}$. We call a sextic curve together with an embedding into $\mathbb{P}(V)$ a plane sextic curve. The space of plane sextic curves is $\mathbb{P}\mathrm{Sym}^6(V^*)$. We call a homogeneous polynomial smooth if it defines a smooth hypersurface. Denote $\mathbb{P}\mathrm{Sym}^6(V^*)^{sm}$ to be the subset of $\mathbb{P}\mathrm{Sym}^6(V^*)$ consisting of smooth sextic polynomials. Consider the action of $\mathrm{SL}(V)$ on $\mathbb{P}\mathrm{Sym}^6(V^*)$, and define $\mathcal{M}$ to be the GIT quotient $\mathrm{SL}(V)\backslash\!\backslash\mathbb{P}\mathrm{Sym}^6(V^*)^{sm}$. Define $\overline{\mathcal{M}}$ to be the GIT compactification of $\mathcal{M}$, and $\mathcal{M}_1$ the moduli of sextic curves with at worst simple singularieties.

Consider the double cover $S_C \longrightarrow \mathbb{P}^2$ branched along a smooth plane sextic curve $C\subset\mathbb{P}^2$. Let $H\in H^2(S_C,\mathbb{Z})$ be the pull-back of the hyperplane class of $\mathbb{P}^2$. Then (S_C,H) is a polarized smooth $K3$ surface with $\varphi(H,H)=2$. Here φ is the topological intersection pairing on the second cohomology.

The isomorphism type of $(H^2(S_C,\mathbb{Z}),\varphi,H)$ does not depend on C. Let (Λ,φ,H) be a triple isomorphic to $(H^2(S_C,\mathbb{Z}),\varphi,H)$. Then $(\Lambda,\varphi)\cong U^3\oplus E_8(-1)^2$ is an even unimodular lattice of signature $(3,19)$. This is usually called the $K3$ lattice. We write Λ for (Λ,φ) for short. Let Λ_0 be the orthogonal complement of H in Λ. Let $\widehat{\mathbb{D}}$ be a component of $\mathbb{P}\{x\in\Lambda_0\otimes\mathbb{C}|\varphi(x,x)=0,\varphi(x,\overline{x})>0\}$.

The second cohomology of degree two $K3$ family over $\mathbb{P}\mathrm{Sym}^6(V^*)^{sm}$ gives a variation of Hodge structure on $\mathbb{P}\mathrm{Sym}^6(V^*)^{sm}$. This induces a period map $\mathscr{P}\colon \mathbb{P}\mathrm{Sym}^6(V^*)^{sm}\to\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$. Here $\widehat{\Gamma}\subset O(\Lambda,\varphi,H)$ is the index two subgroup leaving $\widehat{\mathbb{D}}$ stable. The quotient space $\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ is an arithmetic quotient of type IV domain, hence quasi-projective thanks to the Baily-Borel compactification (see [BB66]). Since $\mathrm{SL}(V)$ is a connected Lie group acting on the $K3$ family, the holomorphic map $\mathscr{P}$ descends to

$$\mathscr{P}\colon \mathrm{SL}(V)\backslash\!\backslash\mathbb{P}\mathrm{Sym}^6(V^*)^{sm}\to\widehat{\Gamma}\backslash\widehat{\mathbb{D}}. \tag{1}$$

We have two $\widehat{\Gamma}$-invariant hyperplane arrangements $\mathcal{H}_\Delta$ and $\mathcal{H}_\infty$ in $\widehat{\mathbb{D}}$, which correspond to two $O(\Lambda,\varphi,H)$-orbits of vectors in Λ_0 with self-intersection -2. The vectors defining $\mathcal{H}_\Delta$ have divisibility 1, and those defining $\mathcal{H}_\infty$ have divisibility 2. We have the Looijenga compactification of $\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty)$, denoted by $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$. See [Loo03b].

From [Sha80] and [Loo03b], we have

Theorem 2.1 (Shah, Looijenga)**.** *The period map* (1) *defined above is an algebraic open embedding with image the complement of* $\widehat{\Gamma}\backslash(\mathcal{H}_\Delta \cup \mathcal{H}_\infty)$*. Moreover* $\mathscr{P}$ *extends to an isomorphism* $\mathcal{M}_1 \cong \widehat{\Gamma}\backslash(\widehat{\mathbb{D}} - \mathcal{H}_\infty)$*, and further to* $\overline{\mathcal{M}} \cong \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$*.*

Remark 2.2. *The Looijenga compactification* $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$ *can be constructed from the Baily-Borel compactification* $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{bb}$ *in the following way. First we blow up some boundary components of* $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{bb}$ *to make the closure of each hypersurface in* $\widehat{\Gamma}\backslash\mathcal{H}_\infty$ *Cartier. Then we contract the strict transform of* $\overline{\widehat{\Gamma}\backslash\mathcal{H}_\infty}$ *to a point. The corresponding point in* $\overline{\mathcal{M}}$ *is a semi-stable sextic, represented by cube of quadric polynomials.*

3. Moduli of Singular Sextic Curves

In this section, we will first characterize the moduli spaces of singular sextic curves via GIT constructions (subsection 3.1). Using the Hodge structure of certain $K3$ surfaces naturally associated to singular sextic curve, we can realize those moduli spaces as arithmetic quotients of type IV domains (subsection 3.2).

3.1. **GIT construction of singular sextic curves.** Consider the subspace Σ^n of $\mathbb{P}\mathrm{Sym}^6(V^*)$ consisting of plane sextic curves with n different nodes. Let Σ be an irreducible component of Σ^n, with Zariski closure $\overline{\Sigma}$. A plane sextic curve in Σ is called to have singular type T_Σ. In the following discussion, we fix one singular type and write T for T_Σ. The group $\mathrm{SL}(V)$ acts on $\overline{\Sigma}$ with a natural linearization $\mathcal{O}(1)$. Let $\widetilde{\Sigma}$ be the normalization of $\overline{\Sigma}$, with a linearization given by the pull-back of $\mathcal{O}(1)$. This also admits an action of $\mathrm{SL}(V)$. By [Sev68], Σ is smooth. So it is isomorphic to its preimage in $\widetilde{\Sigma}$, which is also denoted by Σ.

By [Sha80], the points in Σ is stable under the action of $\mathrm{SL}(V)$. We consider the GIT quotient $\mathcal{F}_T = \mathrm{SL}(V)\backslash\!\backslash\Sigma$. There is a natural GIT compactification $\overline{\mathcal{F}}_T = \mathrm{SL}(V)\backslash\!\backslash\widetilde{\Sigma}^{ss}$ of $\mathcal{F}_T$. Here $\widetilde{\Sigma}^{ss}$ consists of semi-stable points in $\widetilde{\Sigma}$. We have a natural injective morphism $j\colon \mathcal{F}_T \to \overline{\mathcal{M}}$. Since the map between polarized varieties $(\widetilde{\Sigma}, \mathcal{O}(1)) \to (\mathbb{P}\mathrm{Sym}^6(V^*), \mathcal{O}(1))$ is $\mathrm{SL}(V)$-equivariant, we have a finite morphism $\overline{\mathcal{F}}_T \to \overline{\mathcal{M}}$, which extends j and is still denoted by j. In conclusion, we have

Proposition 3.1. *The morphism* $j\colon \overline{\mathcal{F}}_T \to \overline{\mathcal{M}}$ *is a normalizations of its image.*

3.2. **Period domain and period map.** Let $F \in \mathrm{Sym}^6(V^*)$ with $Z(F) \in \Sigma$. We denote S_F to be the double cover of $\mathbb{P}(V)$ branched along $Z(F)$. Then S_F is a $K3$ surface with n nodal singularities on the preimage of $Z(F)$. Let Q_F be the blowup of $\mathbb{P}(V)$ at the n nodes of $Z(F)$. Assume $Z(F)$ has l irreducible components. The proper transforms of those components in Q_F are l disjoint curves, denoted by $C_1, C_2, \cdots, C_l$. Let $\widetilde{S}_F$ be the double cover of Q_F branched along $\bigcup_{i=1}^l C_i$. Then $\widetilde{S}_F$ is a smooth $K3$ surface which resolves the nodal sigularities of S_F. There is an anti-symplectic involution ι on $\widetilde{S}_F$ induced by the double-cover construction. We use the same notation to denote the involution of $H^2(\widetilde{S}_F)$ induced by ι. Define

$$M_F \coloneqq \{x \in H^2(\widetilde{S}_F, \mathbb{Z}) | \iota x = x\}$$

to be a primitive sublattice of $H^2(\widetilde{S}_F, \mathbb{Z})$. Since the action of ι on $\widetilde{S}_F$ is anti-symmetric, the lattice M_F has signature $(1, n')$. Here n' is a non-negative integer and we will show $n' = n$ from the next lemma.

Lemma 3.2. *Let* $X \longrightarrow Y$ *be a branched cyclic cover between two closed manifolds with branch locus* B *a closed submanifold of* Y *with codimension* 2*. Suppose* ι *is a generator of the Deck transformation group, and denote* $H^*(X, \mathbb{Q})^\iota$ *the invariant subspace of* $H^*(X, \mathbb{Q})$ *under the induced action of* ι*. Then we have an isomorphism* $H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q})^\iota$*.*

Proof. The preimage of B in X is also denoted by B. Let $N_Y(B)$ be a tubular neighbourhood of B in Y, and $S_Y(B)$ be the boundary of $N_Y(B)$ which is a circle bundle over B. Take $Y^\circ \coloneqq Y - N_Y(B)$. Let $N_X(B), S_X(B)$ and X° be the preimage of $N_Y(B), S_Y(B)$ and Y° in X, respectively. The Mayer-Vietoris exact sequence gives the following commutative diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & H^k(Y) \arrow[r] \arrow[d] & H^k(N_Y(B)) \oplus H^k(Y^\circ) \arrow[r] \arrow[d] & H^k(S_Y(B)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^k(X) \arrow[r] & H^k(N_X(B)) \oplus H^k(X^\circ) \arrow[r] & H^k(S_X(B)) \arrow[r] & \cdots
\end{tikzcd} \tag{2}
$$

where all the cohomology groups have $\mathbb{Q}$-coefficients. The second long exact sequence in diagram (2) admits an induced action by ι. The image of the first exact sequence in diagram (2) is contained in the ι-invariant part. Since $N_X(B) \longrightarrow N_Y(B)$ is a homotopy equivalence, the morphism $H^k(N_Y(B)) \longrightarrow H^k(N_X(B))$ is an isomorphism. The maps $X^\circ \longrightarrow Y^\circ$ and $S_X(B) \longrightarrow S_Y(B)$ are regular covers, hence the maps $H^k(Y^\circ) \longrightarrow H^k(X^\circ)^\iota$ and $H^k(S_Y(B)) \longrightarrow H^k(S_X(B))^\iota$ are isomorphisms. Therefore, by five lemma, we have isomorphism $H^k(Y, \mathbb{Q}) \cong H^k(X, \mathbb{Q})^\iota$. □

Let H be the total transform of the hyperplane class in Q_F and E_i ($1 \le i \le n$) be the exceptional divisors in Q_F. These $n+1$ cycles form an integral basis of $H^2(Q_F, \mathbb{Z})$. By lemma 3.2, the rank of M is equal to $n+1$ and $n = n'$. Consider the embedding of lattice $H^2(Q_F, \mathbb{Z})(2)$ into $H^2(\widetilde{S}_F, \mathbb{Z})$, which has image contained in M. We use $[H]$ and $[E_i]$ to denote the corresponding cohomology classes in $H^2(\widetilde{S}_F, \mathbb{Z})$. The preimage of C_i in $\widetilde{S}_F$ is still denoted by C_i. The proper transform of $\mathcal{O}_{Q_F}(C_i)$ to $\widetilde{S}_F$ is $\mathcal{O}_{\widetilde{S}_F}(2C_i)$. We denote $[C_i]$ to be the first Chern class of $\mathcal{O}_{\widetilde{S}_F}(C_i)$. Let d_i be the degree of the image of C_i in $\mathbb{P}(V)$. Then

$$
[C_i] = \frac{d_i[H] - \sum_{j=1}^n a_{ij}[E_j]}{2}. \tag{3}
$$

Here $a_{ij} = 1$ if E_j is from blowing up of an intersection point between image of C_i in $\mathbb{P}(V)$ and another component of $Z(F)$, $a_{ij} = 2$ if E_j is from blowing up of a node on the image of C_i in $\mathbb{P}(V)$ and $a_{ij} = 0$ otherwise. We have the following description of M_F.

Proposition 3.3. *The primitive sublattice M_F is generated by $H^2(Q_F, \mathbb{Z})(2)$ and $[C_1], [C_2], \cdots, [C_l]$. Moreover*

$$
\frac{M_F}{H^2(Q_F, \mathbb{Z})(2)} \cong (\mathbb{Z}/2)^{l-1}.
$$

Proof. Take $M_1 = H^2(Q_F, \mathbb{Z})(2)$ and $\widetilde{M}$ the sublattice generated by M_1 and $[C_1], [C_2], \cdots, [C_l]$. We have $\widetilde{M} \subset M_F$ with the same rank. Since $H^2(Q_F, \mathbb{Z})$ is a unimodular lattice, we have $M_1^*/M_1 \cong (\mathbb{Z}/2)^{n+1}$. Since $M_1 \subset \widetilde{M} \subset M_F \subset M_1^*$, the quotients M_F/M_1 and $\widetilde{M}/M_1$ are both $\mathbb{Z}/2$-vector spaces. Tensoring the short exact sequence

$$
0 \to H^2(Q_F, \mathbb{Z}) \to H^2(\widetilde{S}_F, \mathbb{Z}) \to H^2(\widetilde{S}_F, \mathbb{Z})/M_1 \to 0
$$

with $\mathbb{Z}/2$, we obtain a long exact sequence

$$
0 \to \mathrm{Tor}(H^2(\widetilde{S}_F, \mathbb{Z})/M_1, \mathbb{Z}/2) \to H^2(Q_F, \mathbb{Z}/2) \to H^2(\widetilde{S}_F, \mathbb{Z}/2) \to (H^2(\widetilde{S}_F, \mathbb{Z})/M_1) \otimes \mathbb{Z}/2 \to 0. \tag{4}
$$

So the $\mathbb{Z}/2$-rank of the kernel of $H^2(Q_F, \mathbb{Z}/2) \to H^2(\widetilde{S}_F, \mathbb{Z}/2)$ is equal to the $\mathbb{Z}/2$-rank of the torsion part $(H^2(\widetilde{S}_F, \mathbb{Z})/M_1)_{\mathrm{tor}}$ of $H^2(\widetilde{S}_F, \mathbb{Z})/M_1$. On the other hand, according to theorem 1 in [LW95], we have the exact sequence

$$
0 \to H^1(Q_F, \bigcup_{i=1}^{l} C_i, \mathbb{Z}/2) \to H^2(Q_F, \mathbb{Z}/2) \to H^2(\widetilde{S}_F, \mathbb{Z}/2). \tag{5}
$$

The long exact sequence for relative cohomology

$$
0 \to H^0(Q_F, \mathbb{Z}/2) \to H^0(\bigcup_{i=1}^{l} C_i, \mathbb{Z}/2) \to H^1(Q_F, \bigcup_{i=1}^{l} C_i, \mathbb{Z}/2) \to 0 \tag{6}
$$

implies that the rank of $H^1(Q_F, \bigcup_{i=1}^l C_i, \mathbb{Z}/2)$ is $l-1$. Combining sequences (4), (5) and (6), we conclude that the $\mathbb{Z}/2$-rank of $(H^2(\widetilde{S}_F, \mathbb{Z})/M_1)_{\mathrm{tor}}$ is equal to $l-1$. Since $\widetilde{M}$ is the saturation of M_1, we have $M_F/M_1 \cong (\mathbb{Z}/2)^{l-1}$ as abelian groups.

We claim that $[C_1],[C_2],\dots,[C_{l-1}]\in \widetilde{M}/M_1$ are $\mathbb{Z}/2$-independent.

Apparently $l\le 6$. For $l=1$, the claim is clear. Suppose $l\ge 2$. For any $i\in\{1,2,\dots,l-1\}$, take $s(i)\in\{1,2,\dots,n\}$, such that $E_{s(i)}$ is from blowing up of an intersection point of images of C_i and C_l in $\mathbb{P}(V)$. Then $a_{is(i)}=1$. Thus the coefficient of $[E_{s(i)}]$ in the expression (3) of $[C_i]$ is equal to $-\frac{1}{2}$. Therefore, any nontrivial $\mathbb{Z}/2$-linear combination of $[C_1],[C_2],\dots,[C_{l-1}]$ can not vanish. We conclude the claim.

Since $\widetilde{M}/M_1\subset M_F/M_1$, we have $\widetilde{M}/M_1\cong(\mathbb{Z}/2)^{l-1}$ and hence $M_F=\widetilde{M}$. ◻

The isomorphism type of $(H^2(\widetilde{S}_F,\mathbb{Z}),M_F,H,\iota)$ does not depend on the choice of $F\in\Sigma$. Let (Λ,M,H,ι) be an abstract data isomorphic to $(H^2(\widetilde{S}_F,\mathbb{Z}),M_F,H,\iota)$. Let Λ_T be the orthogonal complement of M in Λ. We use the same notations for the corresponding elements of $[H],[E_i]$ in M. We have a family $p\colon\mathcal{S}\to\Sigma$ of smooth $K3$ surfaces with anti-symplectic involution ι. The Hodge structure of $K3$ surfaces gives rise to a variation of polarized Hodge structures on the local system $R^2p_*(\mathbb{Z})$ with induced involution ι. Let $\mathbb{H}$ be (-1)-eigensubsheaf of $R^2p_*(\mathbb{Z})$. We have a subvariation of Hodge structure on $\mathbb{H}$, which is naturally polarized by the topological intersection pairing φ.

Definition 3.4. *We define the period domain $\mathbb{D}=\mathbb{D}_T$ of type T to be the connected component of*

$$\mathbb{P}\{x\in\Lambda_T\otimes\mathbb{C}|\varphi(x,x)=0,\varphi(x,\overline{x})>0\}$$

contained in $\widehat{\mathbb{D}}$. The arithmetic group Γ is defined to be the centralizer of ι in $\widehat{\Gamma}$. Equivalently, $\Gamma=\{\sigma\in\widehat{\Gamma}|\sigma(M)=M\}$ can be viewed as group of automorphisms of the data (Λ,M,H,ι) with spinor norm 1.

The variation of Hodge structure $\mathbb{H}$ induces a period map $\mathscr{P}\colon\Sigma\longrightarrow\Gamma\backslash\mathbb{D}$. Since the group $\mathrm{SL}(V)$ acts equivariantly on the variation of Hodge structure $\mathbb{H}$, the period map is constant on each $\mathrm{SL}(V)$-orbit in Σ. Therefore, the map $\mathscr{P}$ descends to $\mathcal{F}_T\longrightarrow\Gamma\backslash\mathbb{D}$, still denoted by $\mathscr{P}$.

Define $\mathcal{H}_*$ to be the intersection of $\mathcal{H}_\infty$ with $\mathbb{D}$, which is a Γ-invariant hyperplane arrangement. From [YZ18] (appendix, theorem A.14), we have

Proposition 3.5. *There is a natural finite morphism $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}\longrightarrow\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$.*

We call an element in a lattice with self-intersection -2 a root. Let $H_M^\perp$ be the orthogonal complement of H in M. To show theorem 1.1, we need the following lemma:

Lemma 3.6. *The vectors $\pm E_i\in M$ are all the roots in $H_M^\perp$.*

Proof. Suppose not, let v be a root in $H_M^\perp$ which is distinct from $\pm E_1,\pm E_2,\dots,\pm E_n$. From proposition 3.3, M is generated by $[E_1],[E_2],\dots,[E_n],[C_1],[C_2],\dots,[C_l]$. We can write v as an integral linear combination of $[H],[E_1],[E_2],\dots,[E_n],[C_1],[C_2],\dots,[C_l]$. Suppose the coefficients of $[C_i]$ is ϵ_i. Let C' be the union of images of those C_i in $\mathbb{P}(V)$, such that ϵ_i is odd, and C'' be the union of images of other C_i in $\mathbb{P}(V)$.

There is unique expression of v as $\frac{1}{2}\mathbb{Z}$-linear combination of $[E_1],[E_2],\dots,[E_n]$. In this expression, the coefficient of $[E_j]$ is congruent to $\frac{1}{2}$ (mod $\mathbb{Z}$) if and only if E_j is from blowing up of certain intersection point of C' and C''. By the assumption on v, there is at least one $j\in\{1,2,\dots,n\}$ with the coefficient of $[E_j]$ congruent to $\frac{1}{2}$, hence both C' and C'' are nonempty. Since $\deg(C')+\deg(C'')=6$, there are at least 5 intersection points between C' with C''. Choose 5 intersection points, and let the corresponding exceptional rational curves be $E_{j_1},E_{j_2},E_{j_3},E_{j_4},E_{j_5}$. Notice that the orthogonal complement of $[H]$ in M is negative definite, we have

$$-2=v^2\le\left(\frac{[E_{j_1}]+[E_{j_2}]+[E_{j_3}]+[E_{j_4}]+[E_{j_5}]}{2}\right)^2=-\frac{5}{2},$$

contradiction! The lemma follows. ◻

Proof of theorem 1.1. We first show injectivity of $\mathscr{P}\colon \mathcal{F}_T \longrightarrow \Gamma\backslash\mathbb{D}$. Suppose $F_1, F_2 \in \Sigma$ satisfy $\mathscr{P}(F_1) = \mathscr{P}(F_2)$. Then there exists isomorphism κ between $(H^2(\widetilde{S}_{F_1}), M_{F_1}, H, \iota)$ and $(H^2(\widetilde{S}_{F_2}), M_{F_2}, H, \iota)$, sending $H^{2,0}(\widetilde{S}_{F_1})$ to $H^{2,0}(\widetilde{S}_{F_2})$. Notice that κ maps roots in $H^{\perp}_{M_{F_1}}$ to roots in $H^{\perp}_{M_{F_2}}$. By lemma 3.6, the roots $[E_i] \in M_{F_1}$ are sent to $\pm[E_j]$ in M_{F_2}. Denote r_i to be the reflection of $H^2(\widetilde{S}_F, \mathbb{Z})$ with respect to $[E_i]$, for any $F \in \Sigma$. Suppose κ sends $[E_i]$ to $-[E_j]$, then we use $r_j \circ \kappa$ instead of κ. After adjustments, we may assume κ sends exceptional divisor classes $[E_1], \cdots, [E_n]$ in $H^{\perp}_{M_{F_1}}$ to those in $H^{\perp}_{M_{F_2}}$. Furthermore, the class $16H - [E_1] - [E_2] - \cdots - [E_n]$ is ample on both $\widetilde{S}_{F_1}$ and $\widetilde{S}_{F_2}$. From global Torelli theorem, the adjusted κ is induced by an isomorphism $f\colon \widetilde{S}_{F_1} \to \widetilde{S}_{F_2}$, which commutes with ι. So f descends to an isomorphism $Q_{F_1} \longrightarrow Q_{F_2}$. This morphism preserves $\{E_1, \cdots, E_n\}$ on both sides. So it descends to a linear transformation $\mathbb{P}(V) \to \mathbb{P}(V)$ sending Z_{F_1} to Z_{F_2}. The injectivity follows.

It is a classical result ([Sev59], also see [Ser06], chapter 4.7) that $\dim \Sigma = 27 - n$, hence $\dim \mathcal{F} = \dim \Sigma - \dim \mathrm{PSL}(V) = 19 - n$. On the other hand, $\dim \mathbb{D} = \mathrm{rank}(\Lambda_T) - 2 = 20 - \mathrm{rank}(M) = 19 - n$. Thus $\mathcal{F}$ and $\Gamma\backslash\mathbb{D}$ are irreducible quasi-projective varieties with the same dimension. We conclude that $\mathscr{P}$ is a bimeromorphic morphism.

There is natural morphism $\pi\colon \Gamma\backslash\mathbb{D} \longrightarrow \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$. We claim this morphism is generically injective. Recall that Γ is the centralizer of ι in $\widehat{\Gamma}$. By [YZ18] (appendix, theorem A.14), the morphism π extends to a finite morphism $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*} \longrightarrow \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$, still denoted by π.

For any point $x \in \mathbb{D}$, denote $\mathrm{Pic}(x) = H^{1,1}_x \cap \Lambda$ to be the Picard lattice. Take two points $x, y \in \mathbb{D}$ with $\mathrm{Pic}(x) = M$. Suppose there exists $\sigma \in \widehat{\Gamma}$ with $\sigma(x) = y$. Then $\sigma^{-1}(\mathrm{Pic}(y)) \subset \mathrm{Pic}(x) = M$, hence $\mathrm{Pic}(y) = M$ and σ preserves M. Thus $\sigma \in \Gamma$. This implies the generic injectivity of π. We conclude that $\pi\colon \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$ is a normalization of its image.

We now have the following commutative diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathscr{P}"] \arrow[d, "j"] & \Gamma\backslash\mathbb{D} \arrow[d, "\pi"] \\ \overline{\mathcal{M}} \arrow[r, "\mathscr{P}"] & \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}. \end{tikzcd} \tag{7}$$

Here the commutativity follows from the standard construction of the period for nodal $K3$ surface. Since $\mathcal{F}$ and $\Gamma\backslash\mathbb{D}$ are bimeromorphic via $\mathscr{P}$, the Zariski closure of $j(\mathcal{F})$ in $\overline{\mathcal{M}}$ and that of $\pi(\Gamma\backslash\mathbb{D})$ in $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$ are identified via $\mathscr{P}$. By taking normalizations of the two closures, we obatin a unique isomorphism between $\overline{\mathcal{F}}$ and $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$, which extends the upper row morphism in diagram (7), and is still denoted by $\mathscr{P}$. The theorem follows. $\square$

In the remaining of this section, we give a criterion to determine whether $\overline{\mathcal{F}}$ is identified with the Baily-Borel compactification $\overline{\Gamma\backslash\mathbb{D}}^{bb}$. When $\mathcal{H}_*$ is empty, the Looijenga compactification $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ is actually the Baily-Borel compactification $\overline{\Gamma\backslash\mathbb{D}}^{bb}$ of $\Gamma\backslash\mathbb{D}$.

Proposition 3.7. *Let T be given a singular type. The hyperplane arrangement $\mathcal{H}_*$ is empty if and only if Σ_T does not contain cube of a quadric polynomial.*

Proof. This is direct from remark 2.2 and diagram (7). $\square$

4. Examples and Related Constructions

4.1. **Relation to del Pezzo surfaces.** Let T be a singular type. Suppose a generic sextic curve $Z(F)$ of type T is not irreducible, then it contains a line, a quadric curve or a cubic curve. Suppose $Z(F)$ contains a line, then there are at least 5 nodes on that line. Suppose $Z(F)$ contains a quadric curve, then there are at least 8 nodes on that quadric curve. Suppose $Z(F)$ contains a cubic curve, then there are at least 9 nodes on that cubic curve. In any cases, the nodes are not of general position.

Suppose a generic sextic curve of type T is irreducible. Let n be the number of nodes. Since a sextic curve has arithmetic genus 10, we have $n \leq 10$. It is well-known that for generically $n \leq 8$ points on $\mathbb{P}^2$, there exists an irreducible plane sextic curve containing those points as nodes. This implies the following relation between singular sextic curves and del Pezzo surfaces. For reader's convenience, we include a proof.

Proposition 4.1. *Suppose $n \leq 8$. Then for a generic sextic curve $Z(F)$ of type T, the surface Q_F is a del Pezzo surface of degree $9-n$, and the double cover $\widetilde{S}_F \longrightarrow Q_F$ is branched along a smooth irreducible curve of genus $10-n$.*

Proof. Consider the map
$$f\colon \Sigma_T \longrightarrow S_n \backslash ((\mathbb{P}^2)^n - \bigcup_{i\neq j} \Delta_{ij})$$
sending $C \in \Sigma_T$ to the set of its nodes. Here Δ_{ij} consists of points in $(\mathbb{P}^2)^n$ with the same i-th and j-th coordinates, and S_n is the permutation group of degree n. Take any $C \in \Sigma_T$. Let $p_1, \ldots, p_n$ be the n nodes of C. Let $\widetilde{C}$ be the normalization of C, and denote q_{2i-1}, q_{2i} to be the preimage of p_i. Let $D = p_1 + \cdots + p_n$ be a divisor on C, and $\widetilde{D} = q_1 + \cdots q_{2n}$ be a divisor on $\widetilde{C}$. By local calculation of variation of nodes, the tangent space of Σ_T at C is naturally identified with:
$$H^0(C, \mathcal{O}(6)(-D)) \cong H^0(\widetilde{C}, \pi^*\mathcal{O}(6)(-\widetilde{D})).$$
The tangent map df at C is:
$$df\big|_C\colon H^0(\widetilde{C}, \pi^*\mathcal{O}(6)(-\widetilde{D})) \longrightarrow \bigoplus_i \pi^*(\mathcal{O}(6))(-\widetilde{D})\big|_{q_i}$$
sending s to $(s(q_i))_i$. We claim that $df\big|_C$ is surjective. From the following exact sequence of coherent sheaves on $\widetilde{C}$:
$$0 \longrightarrow \pi^*\mathcal{O}(6)(-2\widetilde{D}) \longrightarrow \pi^*\mathcal{O}(6)(-\widetilde{D}) \longrightarrow \bigoplus_i \pi^*\mathcal{O}(6)(-\widetilde{D})\big|_{q_i} \longrightarrow 0$$
it suffices to show $H^1(\widetilde{C}, \pi^*\mathcal{O}(6)(-2\widetilde{D})) = 0$. The degree of $\pi^*\mathcal{O}(6)(-2\widetilde{D})$ equals to $36-4n$. The genus of $\widetilde{C}$ is $g(\widetilde{C}) = 10-n$. Since $n \leq 8$, we have $36-4n > 2g(\widetilde{C})-2$. Thus $H^1(\widetilde{C}, \pi^*\mathcal{O}(6)(-2\widetilde{D})) = 0$ and the claim follows.

It is now clear that a generic point $C = Z(F) \in \Sigma_T$ has n nodes of general position. Therefore, the surface Q_F is a del Pezzo surface of degree $9-n$. Moreover, the double cover $\widetilde{S}_F \longrightarrow Q_F$ is ramified along $\widetilde{C}$, which is a smooth irreducible curve of genus $10-n$. ☐

Remark 4.2. *A generic choice of 9 points on $\mathbb{P}^2$ can not be realized as 9 nodes of certain sextic curve. See* [Cay71].

4.2. **Union of smooth plane curves.** In this section, we apply theorem 1.1 to describe the moduli spaces of tuples of smooth plane curves with total degree 6. Special cases were studied in [Laz09], [MSY92], [GMGZ17]. Let T be the singular type such that a generic sextic curve in Σ_T is union of several smooth curves. Suppose the degrees of those smooth components are $m_1 \leq m_2 \leq \cdots \leq m_l$, then we also denote $T = (m_1, \ldots, m_l)$. We have 11 possibilities of such T. They are (6), $(1,5)$, $(2,4)$, $(3,3)$, $(1,1,4)$, $(1,2,3)$, $(2,2,2)$, $(1,1,1,3)$, $(1,1,2,2)$, $(1,1,1,1,2)$ and $(1,1,1,1,1,1)$.

Now we give an explicit description of the GIT quotient $\mathcal{F}_T$ for the types above. Let $n_1 \cdot 1 + n_2 \cdot 2 + \cdots + n_k \cdot k = 6$ be a partition of 6. Then we have a natural finite map
$$f\colon \prod_i \mathbb{P}(\mathrm{Sym}^i V^*)^{n_i} \longrightarrow \mathbb{P}(\mathrm{Sym}^6 V^*).$$

Taking quotient by the action of permutation group $\prod_i S_{n_i}$ on the left, we have a generically injective finite map

$$\prod_i S_{n_i} \backslash \prod_i \mathbb{P}(\mathrm{Sym}^i V^*)^{n_i} \longrightarrow \mathbb{P}(\mathrm{Sym}^6 V^*).$$

which is the normalization of $\overline{\Sigma}$. So we have

$$\widetilde{\Sigma} \cong \prod_i S_{n_i} \backslash \prod_i \mathbb{P}(\mathrm{Sym}^i V^*)^{n_i}.$$

Next we describe the polarization on $\widetilde{\Sigma}$. The pull-back of $\mathcal{O}(1)$ under f is

$$f^*(\mathcal{O}(1)) \cong \mathcal{O}(1)^{\boxtimes (n_1 + \cdots + n_k)}.$$

So the GIT construction has the following form

$$\mathrm{SL}(V) \times \prod_i S_{n_i} \backslash\!\!\backslash (\prod_i \mathbb{P}(\mathrm{Sym}^i V^*)^{n_i}, \mathcal{O}(1)^{\boxtimes (n_1 + \cdots + n_k)}). \tag{8}$$

In other words, the GIT quotients of the form (8) are isomorphic to Looijenga compactifications of arithmetic quotients of type IV domains. From proposition 3.7, we have:

Proposition 4.3. *For the following T: (6), $(1,5)$, $(3,3)$, $(1,1,4)$, $(1,2,3)$, $(1,1,1,3)$, $(1,1,2,2)$, $(1,1,1,1,2)$, $(1,1,1,1,1,1)$, we obatin identifications of $\overline{\mathcal{F}}_T$ with the Baily-Borel compactifications $\overline{\Gamma\backslash\mathbb{D}}^{bb}$.*

We put together the information in table 1.

Table 1. Information of singular types T when a generic member in Σ_T is the union of some smooth curves

Type	Number of Nodes	$Dim(\mathcal{F}_T)$	$Rank(M)$	A_M	Whether Baily-Borel
(6)	0	19	1	$(\mathbb{Z}/2\mathbb{Z})^1$	no
$(1,5)$	5	14	6	$(\mathbb{Z}/2\mathbb{Z})^4$	yes
$(2,4)$	8	11	9	$(\mathbb{Z}/2\mathbb{Z})^7$	no
$(3,3)$	9	10	10	$(\mathbb{Z}/2\mathbb{Z})^8$	yes
$(1,1,4)$	9	10	10	$(\mathbb{Z}/2\mathbb{Z})^6$	yes
$(1,2,3)$	11	8	12	$(\mathbb{Z}/2\mathbb{Z})^8$	yes
$(2,2,2)$	12	7	13	$(\mathbb{Z}/2\mathbb{Z})^9$	no
$(1,1,1,3)$	12	7	13	$(\mathbb{Z}/2\mathbb{Z})^7$	yes
$(1,1,2,2)$	13	6	14	$(\mathbb{Z}/2\mathbb{Z})^8$	yes
$(1,1,1,1,2)$	14	5	15	$(\mathbb{Z}/2\mathbb{Z})^7$	yes
$(1,1,1,1,1,1)$	15	4	16	$(\mathbb{Z}/2\mathbb{Z})^6$	yes

5. Moduli of Singular and Symmetric Sextic Curves

We follow [YZ18] to define the so-called symmetry type. Let A be a finite subgroup of $\mathrm{SL}(V)$ which contains the center $\mu_3 \subset \mathrm{SL}(V)$. For any $\zeta \in \mu_3$ and $F \in \mathrm{Sym}^6(V^*)$, we have $\zeta(F) = F \circ \zeta^{-1} = \zeta^{-6} F$. Let λ be a character of A such that the restriction to μ_3 sends ζ to ζ^{-6}. Let $\mathcal{V}_\lambda$ be the eigenspace of $\mathrm{Sym}^6(V^*)$ with respect to λ. Two pairs (A_1, λ_1) and (A_2, λ_2) are called equivalent if and only if there exists $g \in \mathrm{SL}(V)$ such that $gA_1g^{-1} = A_2$ and $\lambda_2(gag^{-1}) = \lambda_1(a)$ for all $a \in A_1$. In this case we write $(A_2, \lambda_2) = g(A_1, \lambda_1)$.

Definition 5.1. *A symmetry type for sextic curve is an equivalence class of pairs (A, λ), usually denoted by $T_s = [(A, \lambda)]$.*

Remark 5.2. *The space $\mathcal{V}_\lambda$ depends on the choice of the representative (A, λ). Suppose $g(A_1, \lambda_1) = (A_2, \lambda_2)$, then $g\mathcal{V}_{\lambda_1} = \mathcal{V}_{\lambda_2}$.*

Next we consider moduli of sextic curves with singular type T_d and symmetry type T_s. Recall Σ_{T_d} is the irreducible space of plane sextic curves of singular type T_d. Let n be the number of nodes. Choose a representative (A, λ) for the symmetry type T_s. Define $\Sigma_{T_d,T_s} \coloneqq \Sigma_{T_d} \cap \mathbb{P}\mathcal{V}_\lambda$. This may be reducible, see example 5.5. We denote Σ to be one of the irreducible components of Σ_{T_d,T_s}. Let $\overline{\Sigma}$ be the closure of Σ in $\mathbb{P}\mathrm{Sym}^6(V^*)$, and $\widetilde{\Sigma}$ the normalization of $\overline{\Sigma}$. Define

$$N \coloneqq \{g \in \mathrm{SL}(V) \big| gAg^{-1} = A, \lambda(a) = \lambda(gag^{-1}), \forall a \in A, g \in \mathrm{SL}(V)\}$$

which is a reductive subgroup of $\mathrm{SL}(V)$. Take $g \in N$ and $x \in \Sigma$, for any $a \in A$, we have:

$$a(gx) = g(g^{-1}ag)(x) = g\lambda(g^{-1}ag)x = \lambda(a)gx,$$

which implies that $gx \in \Sigma$. Thus N naturally acts on Σ, hence also on $\overline{\Sigma}$ and $\widetilde{\Sigma}$. The points in Σ are stable under the action of N. Define $\mathcal{F} \coloneqq N\backslash\!\!\backslash\Sigma$ and $\overline{\mathcal{F}} \coloneqq N\backslash\!\!\backslash\widetilde{\Sigma}^{ss}$. From [Lun75] and the argument in the proof of proposition 2.6 in [YZ18], we have:

Proposition 5.3. *There is a natural morphism $j\colon \mathcal{F} \longrightarrow \mathcal{F}_{T_d}$ which extends to a finite morphism $j\colon \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_{T_d}$.*

Next we define the period domain and period map corresponding to the type (T_d, T_s). Let (Λ, M, H, ι) be the data associated to T_d, see section 3. Take $F \in \mathrm{Sym}^6(V^*)$ with $[F] \in \Sigma$. Denote $\overline{A} = A/\mu_3 \subset \mathrm{PSL}(V)$. We have an action of $\overline{A}$ on $Z(F)$. This induces an action of $\overline{A}$ on $\widetilde{S}_F$ commuting with ι. The induced action of $\overline{A}$ on Λ preserves M and its orthogonal complement $M^\perp$ in Λ. Let η be the character of $\overline{A}$ associated to $H^{2,0}(\widetilde{S}_F)$. Let Λ_η be the eigenspace of $M^\perp_{\mathbb{C}}$ corresponding to η.

We consider the Hermitian form h on Λ_η defined by $h(x, y) \coloneqq \varphi(x, \overline{y})$ for any $x, y \in \Lambda_\eta$. When $\eta(\overline{A}) \subset \{\pm 1\}$ (we then say η is real), we define $\mathbb{D}$ to be one component of $\mathbb{P}\{x \in \Lambda_\eta \big| \varphi(x, x) = 0, \varphi(x, \overline{x}) > 0\}$, which is the type IV domain associated to Λ_η. When η is not real, we define $\mathbb{D} \coloneqq \mathbb{P}\{x \in \Lambda_\eta \big| \varphi(x, \overline{x}) > 0\}$, which is a complex hyperbolic ball.

Recall Γ_{T_d} is the centralizer of ι in $\widehat{\Gamma}$. We write r_i for the reflection in $[E_i] \in \Lambda$, for all $1 \le i \le n$. We have $r_i \in \Gamma_{T_d}$. We also have $\overline{A} \subset \Gamma_{T_d}$. Define Γ_{T_d,T_s} to be the normalizer of $\langle \overline{A}, r_1, \cdots, r_n\rangle$ in Γ_{T_d}. We write $\Gamma = \Gamma_{T_d,T_s}$ for short. Then Γ is an arithmetic group acting on $\mathbb{D}$. Take $F \in \mathrm{Sym}^6(V^*)$ with $[F] \in \Sigma$. Choose an isomorphism

$$\Phi\colon (H^2(\widetilde{S}_F), M_F, H, \iota, \overline{A}) \longrightarrow (\Lambda, M, H, \iota, \overline{A})$$

then $\Phi(H^{2,0}(\widetilde{S}_F)) \in \mathbb{D}$. Two choices of Φ differ by an element in the centralizer of $\overline{A}$ in Γ_{T_d}, which is contained in Γ. Thus we obatin an analytic morphism $\mathscr{P}\colon \Sigma \longrightarrow \Gamma\backslash\mathbb{D}$. Taking quotient by N on the left side, the map $\mathscr{P}$ descends to $\mathscr{P}\colon \mathcal{F} \longrightarrow \Gamma\backslash\mathbb{D}$. This morphism is called the period map for sextic curves of type (T_d, T_s). Let $\mathcal{H}_* \coloneqq \mathbb{D} \cap \mathcal{H}_\infty$ be a hyperplane arrangement in $\mathbb{D}$. It is clear that $\mathscr{P}(\mathcal{F}) \subset \Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$.

Theorem 5.4. *The period map $\mathscr{P}\colon \mathcal{F} \longrightarrow \Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$ is an algebraic open embedding, and extends to an isomorphism $\mathscr{P}\colon \overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$.*

Proof. Take $F_1, F_2 \in \mathrm{Sym}^6(V^*)$ such that $[F_1], [F_2] \in \Sigma$. Suppose $\mathscr{P}([F_1]) = \mathscr{P}([F_2])$. Then there exist markings Φ_1, Φ_2 of $\widetilde{S}_{F_1}, \widetilde{S}_{F_2}$ respectively, such that $\Phi_1(H^{2,0}(\widetilde{S}_{F_1})) = \Phi_2(H^{2,0}(\widetilde{S}_{F_2}))$. We have that $r_i \in \Gamma$ for any $1 \le i \le n$. Hence we can assume without loss of generality that $\Phi_2^{-1}\Phi_1$ sends effective roots in M_{F_1} to those in M_{F_2}. Then $\Phi_2^{-1}\Phi_1$ sends an ample class (say, $16H - [E_1] - \cdots - [E_n]$) to an ample class. By global Torelli of polarized $K3$ surface, there exists an isomorphism between $\widetilde{S}_{F_1}$ and $\widetilde{S}_{F_2}$ inducing $\Phi_2^{-1}\Phi_1\colon H^2(\widetilde{S}_{F_1}, \mathbb{Z}) \longrightarrow H^2(\widetilde{S}_{F_2}, \mathbb{Z})$. This isomorphism commutes with ι, hence is induced by a linear transform $g \in \mathrm{SL}(V)$. We have $gAg^{-1} = A$. Now take any $a \in A$. We have

$$\lambda(a)F_2 = aF_2 = agF_1 = gaF_1 = gag^{-1}F_2 = \lambda(gag^{-1})F_2,$$

hence $\lambda(gag^{-1}) = \lambda(a)$. This implies that $g \in N$. The injectivity of $\mathscr{P}$ follows.

Take $[F] \in \Sigma$ and Φ a marking of $\widetilde{S}_F$. Denote $x = \Phi(H^{2,0}(\widetilde{S}_F)) \in \mathbb{D} \subset \mathbb{D}_{T_d}$. By theorem 1.1, an open neighbourhood of x in $\mathbb{D}_T$ is induced by sextic curves of singular type T_d. In particular, there exists a contractible open neighbourhood U of x in $\mathbb{D}$, parametrizing a family of type T_d sextic curves $\mathscr{C} \longrightarrow U$. For every $x \in U$, let $Z(F_x)$ be the sextic curve over x. We have $\Phi(H^{2,0}(\widetilde{S}_{F_x})) = x$. Points in U are Hodge structures invariant under the action of $\overline{A}$. By theorem 1 in [BR75], the $K3$ surface $\widetilde{S}_{F_x}$ admits an action of $\overline{A}$ commuting with ι. Thus the family over U of $K3$ surfaces $\widetilde{S}_{F_x}$ admits an action of $\overline{A}$ commuting with ι. This action is induced from an action of $\overline{A}$ on $\mathscr{C} \longrightarrow U$. Therefore, The image of $\mathscr{P}\colon \mathcal{F} \longrightarrow \Gamma\backslash\mathbb{D}$ contains image of U in $\Gamma\backslash\mathbb{D}$. Thus $\mathscr{P}$ has open image in $\Gamma\backslash\mathbb{D}$. We conclude that $\mathscr{P}$ is a bimeromorphic morphism.

By [YZ18] (appendix, theorem A.14), there is a natural morphism $\Gamma\backslash\mathbb{D} \longrightarrow \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$, which extends to $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*} \longrightarrow \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$. Both two are finite.

We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathscr{P}"] \arrow[d, "j"] & \Gamma\backslash\mathbb{D} \arrow[d, "\pi"] \\ \overline{\mathcal{M}} \arrow[r, "\mathscr{P}"] & \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}. \end{tikzcd}$$

Since $\mathscr{P}(\mathcal{F})$ is open in $\Gamma\backslash\mathbb{D}$, the two closures $\overline{j(\mathcal{F})}$ and $\overline{\pi(\Gamma\backslash\mathbb{D})}$ are identified via $\mathscr{P}\colon \overline{\mathcal{M}} \cong \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$. By lemma 5.4 in [YZ18], there is an isomorphism $\overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$, which extends $\mathscr{P}\colon \mathcal{F} \longrightarrow \Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$. □

As an end, we give two examples of theorem 5.4.

Example 5.5. *An involution τ of $\mathbb{P}(V)$ must be represented by $\mathrm{diag}(1,1,-1)$ on V, with respect to certain choice of coordinate (x_1, x_2, x_3). Let $\overline{A} = \{id, \tau\}$ and A be the preimage of $\overline{A}$ in $\mathrm{SL}(V)$. Take λ to be the trivial character of A. We take symmetry type $T_s = [(A, \lambda)]$, and singular type $T_d = (1,1,1,1,1,1)$. We next describe Σ_{T_d,T_s}. A point in Σ_{T_d,T_s} corresponds to six lines on $\mathbb{P}(V)$ preserved by action of τ, such that any three of them do not intersect at one point. A line $l \subset \mathbb{P}(V)$ satisfies $\tau(l) = l$ if and only if $l = \{x_3 = 0\}$ or l passes through $[0:0:1]$. We call a pair of different lines τ-conjugate, if one line is mapped to the other via τ. The points in Σ_{T_d,T_s} belong to the following two irreducible components: one component Σ_1 with the corresponding six lines are union of three τ-conjugate pairs, the other component Σ_2 with the corresponding six lines are union of two different lines passing through $[0:0:1]$ and two τ-conjugate pairs. Denote $\mathbb{P}(V^*)$ to be the space of lines on $\mathbb{P}(V)$, and $\mathbb{P}(V^*)/\tau \cong \mathbb{P}(1,1,2)$ to be the space of τ-orbits in $\mathbb{P}(V^*)$. Let $N = \mathrm{SL}(\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^\times)$.*

Consider the morphism

$$j_1\colon S_3\backslash(\mathbb{P}(V^*)/\tau)^3 \longrightarrow S_6\backslash\mathbb{P}(V^*)^6$$

sending $[(l_1,l_2,l_3)]$ to $[(l_1,l_2,l_3,\tau(l_1),\tau(l_2),\tau(l_3))]$ for $l_1,l_2,l_3 \in \mathbb{P}(V^*)$. This morphism is a closed embedding with image $\overline{\Sigma}_1$. Hence $\widetilde{\Sigma}_1 = \overline{\Sigma}_1 = S_3\backslash(\mathbb{P}(V^*)/\tau)^3$. So the GIT *quotient is*

$$\overline{\mathcal{F}}_1 \cong (N \times S_3)\backslash\!\!\backslash((\mathbb{P}(V^*)/\tau)^3, \mathcal{O}(1)\boxtimes\mathcal{O}(1)\boxtimes\mathcal{O}(1)) \cong (N \times S_3)\backslash\!\!\backslash(\mathbb{P}(1,1,2)^3, \mathcal{O}(1)\boxtimes\mathcal{O}(1)\boxtimes\mathcal{O}(1))$$

and is isomorphic to the Baily-Borel compactification of an arithmetic quotient of a type IV *domain with dimension 2 by theorem 5.4.*

Let V_2 be the quotient of V by the line $\{x_1 = x_2 = 0\} \subset V$. Then $\mathbb{P}(V_2^*)$ parametrizes lines in $\mathbb{P}(V)$ passing through $[0:0:1]$. Consider the finite morphism

$$j_2\colon (S_2\backslash\mathbb{P}(V_2^*)^2) \times (S_2\backslash(\mathbb{P}(V^*)/\tau)^2) \longrightarrow S_6\backslash\mathbb{P}(V^*)^6$$

sending $[(l_1,l_2,l_3,l_4)]$ to $[(l_1,l_2,l_3,l_4,\tau(l_3),\tau(l_4))]$ for l_1,l_2 passing through $[0:0:1]$ and $l_3,l_4 \in \mathbb{P}(V^*)$. This morphism is generically injective with image $\overline{\Sigma}_1$. Hence $\widetilde{\Sigma}_1 = (S_2\backslash\mathbb{P}(V_2^*)^2) \times (S_2\backslash(\mathbb{P}(V^*)/\tau)^2)$. So the GIT-*quotient is $\overline{\mathcal{F}}_2 \cong N \times (S_2)^4\backslash\!\!\backslash(\mathbb{P}(V_2^*)^2 \times \mathbb{P}(V^*)^2, \mathcal{O}(1)^{\boxtimes 4})$, and is isomorphic to the Baily-Borel compactification of an arithmetic quotient of a type* IV *domain with dimension 2 by theorem 5.4.*

Example 5.6. *Let τ be an automorphism of $\mathbb{P}(V)$ represented by $\mathrm{diag}(1,1,\omega)$ with respect to certain choice of coordinate (x_1,x_2,x_3). Here $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$ is a third root of unity. Let $\overline{A} = \{1, \tau, \tau^2\}$ and A be the preimage* *of $\overline{A}$ in $\mathrm{SL}(V)$. Take λ to be the trivial character of A. We take symmetry type $T_s = [(A,\lambda)]$ and singular type $T_d = (1,5)$. In this case Σ_{T_d,T_s} is irreducible and $\Sigma = \Sigma_{T_d,T_s}$. The equation of a point in Σ can be arranged as*

$$(t_1x_1 + t_2x_2)(L_2(x_1,x_2)x_3^3 + L_5(x_1,x_2)),$$

where $t_1, t_2 \in \mathbb{C}$ and L_2, L_5 are polynomials of x_1, x_2 of degree $2, 5$ respectively. The line $(t_1x_1 + t_2x_2 = 0)$ passes through $[0:0:1]$, and such lines are parametrized by $\mathbb{P}(V_2^*)$ as defined in example 5.5. The quintics $Z(L_2(x_1,x_2)x_3^3 + L_5(x_1,x_2))$ are parametrized by $\mathbb{P}(\mathrm{Sym}^2(V_2^*) \oplus \mathrm{Sym}^5(V_2^*))$. Consider the natural finite morphism

$$j\colon \mathbb{P}(V_2^*) \times \mathbb{P}(\mathrm{Sym}^2(V_2^*) \oplus \mathrm{Sym}^5(V_2^*)) \longrightarrow \mathbb{P}(V^*) \times \mathbb{P}(\mathrm{Sym}^5(V^*)).$$

This morphism is generically injective with image $\overline{\Sigma}$. Thus $\widetilde{\Sigma} = \mathbb{P}(V_2^*) \times \mathbb{P}(\mathrm{Sym}^2(V_2^*) \oplus \mathrm{Sym}^5(V_2^*))$. The reductive group N is equal to $\mathrm{SL}(\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^\times)$. So the GIT-quotient is $\overline{\mathcal{F}} \cong N\backslash\!\!\backslash(\mathbb{P}(V_2^*) \times \mathbb{P}(\mathrm{Sym}^2(V_2^*) \oplus \mathrm{Sym}^5(V_2^*)), \mathcal{O}(1) \boxtimes \mathcal{O}(1))$. In this case, the character of μ_3 on $H^{2,0}(\widetilde{S}_F)$ is non-real. Thus $\overline{\mathcal{F}}$ is isomorphic to the Baily-Borel compactification of an arithmetic ball quotient with dimension 5 by theorem 5.4.

References

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

[BR75] D. Burns and M. Rapoport, *On the Torelli problem for Kählerian $K-3$ surfaces*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 2, 235–273.

[Cay71] A. Cayley, *A second memoir on quartic surfaces*, Proc. Lond. Math. Soc. (1869/71).

[GMGZ17] P. Gallardo, J. Martinez-Garcia, and Z. Zhang, *Compactifications of the moduli space of plane quartics and two lines*, arXiv preprint arXiv:1708.08420 (2017).

[Laz09] Radu Laza, *Deformations of singularities and variation of GIT quotients*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 2109–2161.

[Laz16] ———, *Perspectives on the construction and compactification of moduli spaces*, Compactifying moduli spaces, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2016, pp. 1–39.

[Loo03a] E. Looijenga, *Compactifications defined by arrangements. I. The ball quotient case*, Duke Math. J. **118** (2003), no. 1, 151–187.

[Loo03b] ———, *Compactifications defined by arrangements. II. Locally symmetric varieties of type IV*, Duke Math. J. **119** (2003), no. 3, 527–588.

[LP81] E. Looijenga and C. Peters, *Torelli theorems for Kähler $K3$ surfaces*, Compositio Math. **42** (1980/81), no. 2, 145–186.

[Lun75] D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.

[LW95] R. Lee and S. H. Weintraub, *On the homology of double branched covers*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 1263–1266.

[MSY92] K. Matsumoto, T. Sasaki, and M. Yoshida, *The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the hypergeometric function of type $(3,6)$*, Internat. J. Math. **3** (1992), no. 1, 164.

[PŠ71] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type $K3$*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.

[Sev59] F. Severi, *Geometria dei sistemi algebrici sopra una superficie e sopra una varietà algebrica. Vols. 2, 3*, Volumi secondo e terzo in continuazione del volume primo dello stesso autore che porta il titolo: Serie, sistemi d'equivalenza e corrispondenze algebriche sulle varietà algebriche, Edizioni Cremonese, Rome. Vol. 2, 1959.

[Sev68] ———, *Vorlesungen über algebraische Geometrie: Geometrie auf einer Kurve, Riemannsche Flächen, Abelsche Integrale*, Berechtigte Deutsche Übersetzung von Eugen Löffler. Mit einem Einführungswort von A. Brill. Begleitwort zum Neudruck von Beniamino Segre. Bibliotheca Mathematica Teubneriana, Band 32, Johnson Reprint Corp., New York-London, 1968.

[Sha80] J. Shah, *A complete moduli space for $K3$ surfaces of degree 2*, Ann. of Math. (2) **112** (1980), no. 3, 485–510.

[YZ18] C. Yu and Z. Zheng, *Moduli spaces of symmetric cubic fourfolds and locally symmetric varieties*, arXiv preprint arXiv:1806.04873v2 (2018).

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