Math. Res. Lett.
Volume 28, Number 2, 415–434, 2021

Period integrals of vector bundle sections and tautological systems

An Huang, Bong Lian, Shing-Tung Yau, and Chenglong Yu

Tautological systems developed in [8, 9] are Picard-Fuchs type systems to study period integrals of complete intersections in Fano varieties. We generalize tautological systems to zero loci of global sections of vector bundles. In particular, we obtain similar criterion as in [8, 9] about holonomicity and regularity of the systems. We also prove solution rank formulas and geometric realizations of solutions following the work on hypersurfaces in homogeneous varieties [4].

1. Introduction

Computing period integrals has a long history in geometry. One important approach is to find enough linear differential operators that annihilate period integrals and study the corresponding Picard-Fuchs systems. Following this idea, tautological systems are introduced in [8, 9]. It is a generalization of GKZ-hypergeometric systems introduced by Gel'fand, Kapranov and Zelevinski [3]. The new systems have been used to study period integrals in much more general situations, and also to reveal new information regarding the GKZ systems in certain important cases [4, 5]. Tautological systems have also recently been applied to prove the *hyperplane conjecture* for $\mathbb{CP}^n$ in mirror symmetry [10].

Let X^n be a smooth n-dimensional complex Fano variety and E be a vector bundle of rank r on X. Denote the dual space of global sections by $V = H^0(X, E)^\vee$. Assume that a generic section $s \in V^\vee$ defines a smooth subvariety $Y_s = \{s = 0\}$ in X with codimension r. (When E is very ample, the zero locus of a generic section is either empty or smooth due to a Bertini-type theorem for vector bundles followed by Cayley's trick. For example, see Lemma 1.6 in [13]. When it is empty, we can consider the quotient bundle of E by the trivial line bundle.) The dimension of Y_s is denoted by $d = n - r$. Consider the smooth family of varieties formed by zero loci of sections in $V^\vee$, denoted by $\pi\colon \mathcal{Y} \to B = V^\vee - D$, where D is the discriminant locus. If we further assume $\det E \cong K_X^{-1}$, the adjunction formula implies that

$$K_{Y_s} \cong K_X \otimes \det E|_{Y_s} \cong \mathcal{O}_{Y_s}. \tag{1.1}$$

Then $\pi\colon \mathcal{Y} \to B$ is a family of Calabi-Yau varieties together with a global trivialization Ω_s of $\pi_* K_{\mathcal{Y}/B}$ corresponding to the constant section 1 of $\mathcal{O}_{Y_s}$. Consider the period integrals

$$\int_\gamma \Omega_s, \quad \gamma \in H_n(Y_s),$$

where γ is a local flat section with respect to the Gauss-Manin connection. The goal of this paper is to find explicit Picard-Fuchs differential systems consisting of linear differential operators for these period integrals, study their solution rank formulas and the geometric realizations of all the solutions as chain integrals, generalizing [1, 4, 5].

When E is a line bundle or splits as a direct sum of line bundles, the global trivialization Ω_s is constructed by a global residue formula in [9], see Theorem 6.6 in [9] for the definition of Poincaré residue. In the nonsplitting case, the idea is that we apply the residue formula in the splitting case locally and glue it together to obtain a global residue formula. We realize this idea explicitly in the present paper. We generalize the *Calabi-Yau bundle* construction in [8, 9] to carry this out.

Now we state the main results. We construct the differential systems in Theorem 3.2. We prove that the differential systems are regular holonomic when the vector bundle admits sufficiently large symmetry in Theorem 3.3. In section 6, we give different solution rank formulas in terms of Lie algebra homology, Corollary 6.3 and in terms of geometric formula, Theorem 6.4. In the special case when Y_s is complete intersection in homogeneous variety, the solutions are given by chain integrals in Theorem 6.10. This generalizes the result about hypersurfaces in [4]. In particular, when Y_s is the intersection of prime divisors $Y_1, \dots, Y_r$, we conjecture that the solutions are given by Euler type integrals along the middle dimensional cycles in the complement of $\cup Y_i$. This has been recently proved by Lee, Lian and Zhang [7]. The main contributions of this paper are in two aspects. One is the generalization of tautological systems introduced in [8, 9] for complete intersections to vector bundle sections. The other is the study of Riemann-Hilbert type problems for both complete intersections and noncomplete intersections, generalizing the work of hypersurfaces in [4, 5].

The main techniques in this paper are the following. We use Cayley's trick to identify vector bundle sections of E with line bundle sections on the projectivization $\mathbb{P}(E^\vee)$, and also the corresponding Hodge structures on the zero loci. The key step in solution rank formula is to obtain a motivic formula of the tautological system, Theorem 6.4. This is similar to the formula obtained in the Calabi-Yau hypersurface case [4]. It turns out that for complete intersections in homogeneous varieties, the behavior of the tautological system has the flavor of both the GKZ system and the system for hypersurfaces in homogeneous varieties.

2. Calabi-Yau bundles and adjunction formulas

Motivated by the residue formulas for projective spaces and toric varieties, the notion of Calabi-Yau bundles is introduced in [9] and used to write down an adjunction formula on principal bundles. The canonical sections of holomorphic top forms used in period integrals are given by this construction. First we recall the definition of Calabi-Yau bundles in [9] and adapted it to the local complete intersections.

Definition 2.1 (Calabi-Yau bundle). *Denote by H a complex Lie group. Let $p\colon P \to X$ be a principal H-bundle. A Calabi-Yau bundle structure on (X, H) says that the canonical bundle of X is the associated line bundle with character $\chi\colon H \to \mathbb{C}^*$. Then the following short exact sequence*

$$0 \to \operatorname{Ker} p_* \to TP \to p^*TX \to 0$$

induces an isomorphism

$$K_P \cong p^*(K_X \otimes \det(P \times_{ad(H)} \mathfrak{h}^\vee)). \tag{2.1}$$

Fixing an isomorphism $P \times_H \mathbb{C}_\chi \cong K_X$, the isomorphism (2.1) implies that K_P is a trivial bundle on P and has a trivializing section ν which is the tensor product of nonzero elements in $\mathbb{C}_\chi$ and $\det \mathfrak{h}^\vee$. This holomorphic top form satisfies that

$$h^*(\nu) = \chi(h)\chi_{\mathfrak{h}}^{-1}(h)\nu, \quad \forall h \in H, \tag{2.2}$$

where $\chi_{\mathfrak{h}}$ is the character of H on $\det \mathfrak{h}$ by adjoint action. The tuple (P, H, ν, χ) satisfying (2.2) is called a Calabi-Yau bundle. In the following, we also consider Calabi-Yau bundle with a group action. Let G be a complex Lie group. We say (P, H, ν, χ) is G-equivariant if the principal H-bundle P has a left G-equivariant action. In other words, there is a left action of G

on P which commutes with the H-action, hence also induces an action of G on X.

Conversely, any section ν satisfying (2.2) determines an isomorphism $P \times_H \mathbb{C}_\chi \cong K_X$. Since the only line bundle automorphism of $K_X \to X$ fixing X is rescaling when X is compact, such ν is determined up to rescaling (Theorem 3.12 in [9]). So the equivariant action of G on $P \to X$ changes ν according to some character ψ^{-1} of G. We say the Calabi-Yau bundle is (G, ψ^{-1})-equivariant.

Example 2.2. *Let X be $\mathbb{CP}^{d+1}$ and P be $\mathbb{C}^{d+2} \setminus \{0\}$ with natural actions of $G = GL(d+2, \mathbb{C})$ and $H = \mathbb{C}^*$. The volume form is $\nu = dx_0 \wedge \cdots \wedge dx_{d+1}$. The character $\beta^{-1} = \det g$ for any $g \in G$.*

When E is the line bundle K_X^{-1}, the following is the residue formula for Calabi-Yau bundles:

Theorem 2.3 ([9], Theorem 4.1). *If (P, H, ν, χ) is a Calabi-Yau bundle over a Fano manifold X, the subsheaf $\pi_* K_{X/B}$ in the Hodge filtration of $R^d \pi_*(\underline{\mathbb{C}}) \otimes \mathcal{O}_B$ associated with the family $\pi\colon \mathcal{Y} \to B$ of Calabi-Yau hypersurfaces has a canonical section of the form*

$$\omega = Res \frac{\iota_{\xi_1} \cdots \iota_{\xi_m} \nu}{f}. \tag{2.3}$$

Here $\xi_1, \ldots, \xi_m$ are independent vector fields generating the distribution of H-action on P, and $f\colon B \times P \to \mathbb{C}$ is the function representing the universal section of $P \times_H \mathbb{C}_{\chi^{-1}} \cong K_X^{-1}$.

When E is a direct sum of line bundles associated to characters of H, the residue formula is similar to (2.3) by induction.

Definition 2.4 (Residue formula for vector bundles). *When E is a vector bundle associated with representation $\rho\colon H \to GL(W)$, we can also construct a residue formula as follows. Under the assumption that $E = P \times_H W$, a section of E is an (H, ρ)-equivariant map $f\colon P \to W$, i.e. $f(p \cdot h) = h \cdot f(p)$. Choose a basis $e_1, \ldots, e_r$ for W, then $f = f^1 e_1 + \cdots + f^r e_r$. Assume f defines a smooth Calabi-Yau subvariety Y_f with codimension r in X. We have the following residue formula:*

$$\omega = \iota_{\xi_1} \cdots \iota_{\xi_m} Res \frac{\nu}{f^1 \cdots f^r} \tag{2.4}$$

The residue defines a holomorphic top form on the zero locus of $f^i = 0$ on P, which is the restriction of the principal bundle on the Calabi-Yau subvariety. After contracting with $\xi_1, \dots, \xi_m$, the holomorphic d-form w is invariant under the action of H and vanishes for the vertical distribution, hence it defines a d-form on Y_f.

The vector bundle E can be associated with different principal bundles. The residue construction is canonical in the following sense.

Proposition 2.5. *For different choices of principal bundles P or basis $e_1, \dots, e_r$, the residue form ω is unique up to rescaling.*

Proof. Firstly, for different choices of basis of W, the functions f_i are changed by a linear transformation. Hence the denominator of the residue formula is changed by the determinant of the linear transformation along the common zero locus of f_i. So the residue is changed by a scalar.

Secondly, we prove the independence on the choice of principal bundles. Let P' be the frame bundle of E. We only need to compare any P with P'. The frame bundle can be constructed by $P' = P \times_H GL(W)$. So we have a quotient map from $P \times GL(W)$ to P'. Especially we have a principal bundle map

$$c\colon P \to P',$$

which is equivariant under the actions of H and $GL(W)$ on both sides related by $\rho\colon H \to GL(W)$. Then we have a morphism between the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} p_* & \longrightarrow & TP & \longrightarrow & p^*TX & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} {p'}_* & \longrightarrow & TP' & \longrightarrow & {p'}^*TX & \longrightarrow & 0 \end{array}$$

Notice that one exact sequence gives an isomorphism

$$p^*K_X \cong K_P \otimes \det(P \times_{ad(H)} \mathfrak{h})$$

and induces the form ν and $\iota_{\xi_1} \cdots \iota_{\xi_m} \nu$. So we have

$$\iota_{\xi_1} \cdots \iota_{\xi_m} \nu = c^*(\iota_{\xi'_1} \cdots \iota_{\xi'_r} \nu').$$

Furthermore, the functions f_i defined on P are also pull back of the corresponding functions on P'. So we have the same residue formula. $\square$

With the canonical choice of ω, the period integral for the family is defined to be

$$\Pi_\gamma = \int_\gamma \omega.$$

Here γ is a local horizontal section of the d-th homology group of the family. The period integrals are local holomorphic functions on the base B and generates a subsheaf of $\mathcal{O}_B$ called period sheaf.

3. Tautological systems

In order to study the period sheaf, we look for differential operators which annihilate the period integrals. In [8] and [9], tautological systems are then introduced and the solution sheaves contain the period sheaves. When H is the complex torus and X toric variety, tautological systems are the GKZ systems and extended GKZ systems. When X is homogeneous variety, tautological systems provide new interesting $\mathcal{D}$-modules. The notion of tautological system also provides convenient ways to apply $\mathcal{D}$-module theory to study the solution sheaves and period sheaves. The regularity and holonomicity are discussed in [8], [9]. The Riemann-Hilbert problems and geometric realizations are discussed in [1], [4], [5].

The differential operators in tautological systems come from two sources: one from symmetry group G called symmetry operators and the other from the defining ideal of X in the linear system $|K_X^{-1}|$ called geometric constraints. In this section we have similar constructions. First we fix a basis $a_1, \ldots, a_m$ of V and dual basis $a_1^\vee, \ldots, a_m^\vee$ of $V^\vee$. Viewing a_i as coordinates on $V^\vee$, the universal section of E is denoted by $f = a_1 a_1^\vee + \cdots + a_m a_m^\vee$. According to the discussion in last section, the section $a_i^\vee$ corresponds to a map $f_i \colon P \to W$ and a tuple of functions $f_i = (f_i^1, \ldots, f_i^r)$. Then the residue formula has the following form:

$$\omega = \iota_{\xi_1} \cdots \iota_{\xi_m} Res \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)}.$$

Considering the action of G on V, we have a Lie algebra representation

$$Z \colon \mathfrak{g} \to \operatorname{End} V$$

For any $x \in \mathfrak{g}$, let $Z(x) = \sum_{i,j} x_{ij} a_i a_j^\vee$ and the dual representation $Z^\vee(x) = \sum_{i,j} -x_{ij} a_j^\vee a_i$.

From Proposition 2.5, the uniqueness of residue form, we know that the G-action changes the period integral according to a character of G. So the first order differential operators

$$\sum_{ij} x_{ij} a_i \frac{\partial}{\partial a_j} + \beta(x)$$

annihilate the period integral, where $\beta \coloneqq d\psi\colon \mathfrak{g} \to \mathbb{C}$ is the corresponding character on Lie algebra. More specifically, consider the action of the one parameter group $\exp(tx)$ on the period integral. From Cartan's formula, we have

$$\begin{aligned} L_x \iota_{\xi_1} \cdots \iota_{\xi_m} & \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} \\ &= d\left(\iota_x \iota_{\xi_1} \cdots \iota_{\xi_m} \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} \right). \end{aligned}$$

So integration along γ gives

$$\int_\gamma \iota_{\xi_1} \cdots \iota_{\xi_m} Res \left(L_x \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} \right) = 0.$$

The Lie derivative is

$$\begin{aligned} L_x & \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} \\ &= -\beta(x) \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} \\ &\quad + \sum_k \frac{\sum_{i,j} a_i x_{ij} f_j^k \nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^k)^2 \cdots (\sum_i a_i f_i^r)} \\ &= \left(-\beta(x) - \sum_{ij} x_{ij} a_i \frac{\partial}{\partial a_j} \right) \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)}. \end{aligned}$$

So we have

$$\left(\beta(x) + \sum_{ij} x_{ij} a_i \frac{\partial}{\partial a_j} \right) \Pi_\gamma = 0.$$

The geometric constraints arise from the following observation. We consider the first order differential operators:

$$\frac{\partial}{\partial a_j} \frac{\nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^r)} = -\sum_k \frac{f_j^k \nu}{(\sum_i a_i f_i^1) \cdots (\sum_i a_i f_i^k)^2 \cdots (\sum_i a_i f_i^r)};$$

and the second order differential operators:

$$\frac{\partial^2}{\partial a_l \partial a_j} \frac{\nu}{f^1 \cdots f^r} = \sum_{a,b} Q_{ab} f_l^a f_j^b \nu. \tag{3.1}$$

Here Q_{ab} are rational functions of $f_1, \ldots, f_r$, not depending on j, l. By induction, we have similar formulas for higher order differential operators $\partial_{i_1,\ldots,i_s}$ and the coefficients Q are independent of the multi-index $i_1, \ldots, i_s$. Notice that we can switch the order of l and j. So we have $Q_{ab} = Q_{ba}$. On the other hand, consider the product of $a_l^\vee$ and $a_j^\vee$ in $H^0(X, \mathrm{Sym}^2 E)$. The symmetric product $\mathrm{Sym}^2 E$ is associated with the symmetric product of the representation of ρ. Hence $a_l^\vee \cdot a_k^\vee$ can be viewed as a map from the principal bundle $P \to \mathrm{Sym}^2 W$ given by

$$f_l \cdot f_j = \left(\sum_a f_l^a e_a \right) \cdot \left(\sum_b f_j^a e_b \right) = \sum_a f_l^a f_j^a e_a^2 + \sum_{a<b} (f_l^a f_j^b + f_j^a f_l^b) e_a e_b.$$

Consider the elements in the kernel of the map

$$H^0(X, E) \otimes H^0(X, E) \to H^0(X, \mathrm{Sym}^2 E).$$

The Fourier transform of these elements annihilate the period integral. For example if $(a_1^\vee)^2 - a_2^\vee a_3^\vee = 0$, then

$$\begin{aligned} &\left(\frac{\partial^2}{\partial {a_1}^2} - \frac{\partial^2}{\partial a_2 \partial a_3} \right) \frac{\nu}{f^1 \cdots f^r} \\ &\quad = \left(\sum_a Q_{aa}((f_1^a)^2 - f_2^a f_3^a) + \sum_{a<b} Q_{ab}(2 f_1^a f_1^b + f_2^a f_3^b + f_3^a f_2^b) \right) \nu = 0. \end{aligned}$$

This is because the terms of f_i^a are the coefficients of $(a_1^\vee)^2 - a_2^\vee a_3^\vee$ written under the basis $e_a^2, e_a e_b$.

In order to describe the geometric origin of the differential operators above, we need the well-known fact relating the vector bundle E and hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E^\vee)$ derived from Leray spectral sequence.

Proposition 3.1. *Assume $E \to X$ is a holomorphic vector bundle on complex manifold X and $\mathcal{O}(1) \to \mathbb{P}(E^\vee)$ is the hyperplane bundle on the projectivization of $E^\vee$. There is a canonical ring isomorphism*

$$\oplus_k H^0(X, \mathrm{Sym}^k(E)) \cong \oplus_k H^0(\mathbb{P}(E^\vee), \mathcal{O}(k)). \tag{3.2}$$

The above identification of $V^\vee$ with $H^0(\mathbb{P}(E^\vee), \mathcal{O}(1))$ gives a map $\mathbb{P}(E^\vee) \to \mathbb{P}(V)$ by $\mathcal{O}(1)$ when $|\mathcal{O}(1)|$ is base-point free. Consider the ideal determined by image of this map $I(\mathbb{P}(E^\vee), V)$, which is the kernel of the map

$$\oplus_k \operatorname{Sym}^k(V^\vee) \to \oplus_k H^0(X, \operatorname{Sym}^k(E))$$

With the discussion above, we collect all the differential operators in the following theorem.

Theorem 3.2. *The period integral Π_γ satisfies the following system of differential equations:*

$$Q(\partial_a)\Pi_\gamma = 0 \qquad (Q \in I(\mathbb{P}(E^\vee), V)) \tag{3.3}$$

$$(Z_x + \beta(x))\Pi_\gamma = 0 \qquad (x \in \mathfrak{g}) \tag{3.4}$$

$$\left(\sum_i a_i \frac{\partial}{\partial a_i} + r\right)\Pi_\gamma = 0. \tag{3.5}$$

The last operator is called Euler operator and comes from ω being homogeneous of degree $-r$ with respect to a_i. We can also view Euler operator as symmetry operator. Consider the frame bundle of E with structure group $H = GL(r)$. It has a symmetry $G = \mathbb{C}^*$ acting as the center of H. The symmetry operator of G is the Euler operator.

We call the differential system in Theorem 3.2 tautological system for (X, E, H, G). It is the same as the cyclic D-module $\tau(G, \mathbb{P}(E^\vee), \mathcal{O}(-1), \hat{\beta})$ defined in [8] [9] by

$$\tau = D_{V^\vee}/D_{V^\vee}(J(\mathbb{P}(E^\vee)) + Z(x) + \hat{\beta}(x), x \in \hat{\mathfrak{g}}).$$

Here $J(\mathbb{P}(E^\vee)) = \{Q(\partial_a) | Q \in I(\mathbb{P}(E^\vee))\}$, $\hat{G} = G \times \mathbb{C}^*$ with Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e$ and $\hat{\beta} = (\beta, r)$.

We can apply the holonomicity criterion for tautological system in [8, 9].

Theorem 3.3. *If the induced action of G on $\mathbb{P}(E^\vee)$ has finitely many orbits, the corresponding tautological system τ is holonomic. Moreover, when G is reductive and $\beta = 0$, the system is also regular.*

4. Examples

In this section, we give some examples of the tautological system introduced in Theorem 3.2. In particular, the example 4.1 shows that this system is a generalization of the tautological system introduced in [9] for complete intersections.

Example 4.1 (Complete intersections). *When $E = \oplus_1^r L_i$ is a direct sum of very ample line bundles, the above system recovers the tautological system for complete intersections in [9]. This case is equivalent to say that the structure group of E is reduced to the complex torus $(\mathbb{C}^*)^r$. So we have symmetry group $(\mathbb{C}^*)^r$ acting on the fibers of E. This gives the usual Euler operators in [9]. Let $\hat{X}_i$ be the cone of X inside $V_i = H^0(X, L_i)^\vee$ under the linear system of L_i. The cone of $\mathbb{P}(E^\vee)$ inside $V = \oplus_{i=1}^r V_i$ is fibered product $\hat{X}_i$ over X. So the geometric constraints are the same as [9]. Assume X is a G-variety consisting of finitely many G-orbits and L_i are G-equivariant line bundles. Then $\mathbb{P}(E^\vee)$ admits an action of $\tilde{G} = G \times (\mathbb{C}^*)^{r-1}$ with finitely many orbits. This is the same holonomicity criterion as [9] for complete intersections.*

Example 4.2. *[Homogeneous vectore bundles] Let G be a semisimple complex Lie group and $X = G/R$ is a generalized flag variety given by a parabolic subgroup R of G. This forms a principal R-bundle over X. We assume E to be a homogeneous vector bundle from a representation of R and the action of G on $\mathbb{P}(E^\vee)$ is transitive. Then the projectivization of $\mathbb{P}(E^\vee)$ is also a generalized flag variety for another parabolic subgroup $R' \subset R$. If $\mathcal{O}(1)$ on $G/R' = \mathbb{P}(E^\vee)$ corresponds to a dominant weight of $\mathfrak{g}$, the defining ideal of $\mathbb{P}(E^\vee)$ in $\mathbb{P}(V)$ is generated by the Kostant-Lichtenstein quadratic relations [6, 11], which are analogues of Plüker relations. In other words, the differential operators 3.3 are given by second order differential operators. Furthermore, any character of $\mathfrak{g}$ is trivial, hence β is zero. So the differential system is regular holonomic and explicitly given in this case.*

For instance, let $X = G(k, l)$ be Grassmannian of k-dimensional subspaces in $\mathbb{C}^l$ and F be the tautological bundle of rank k. Then $E \cong F^\vee \otimes \mathcal{O}(\frac{l-1}{k})$ is an ample vector bundle with $\det E \cong K_X^{-1}$. The corresponding $\mathbb{P}$ is homogeneous under the action of $SL(l)$.

Example 4.3. *In this example, we work out explicitly the differential operators in a specific case of Example 4.2. Let $X = G(2,3) \cong \mathbb{P}^2$ and $E \cong F^\vee \otimes \mathcal{O}(1)$, where F is the tautological rank-2 vector bundle on X. Then $\det(E) \cong \mathcal{O}(3) \cong K_X^{-1}$. The projectivization $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}(F)$, the complete flag variety $SL(3)/B$, where B is the set of lower triangular matrices with determinant 1. We follow the notation of representation theory of semi-simple Lie algebra. Let s_1 and s_2 be two simple roots of the root system of $\mathfrak{sl}_3$. The polarization $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}(E)$ is a homogeneous line bundle,* *and the associated character of B corresponds to weight $s_1 + s_2$. According to Borel-Weil-Bott theorem, the space of global sections $H^0(\mathbb{P}(E), \mathcal{O}(1)) \cong H^0(X, E)$ is an irreducible $\mathfrak{sl}_3$-representation with highest weight $s_1 + s_2$. The dual space V is also an $\mathfrak{sl}_3$-representation with highest weight $s_1 + s_2$, which is isomorphic to $\mathfrak{sl}_3$ with adjoint action. Denote by e_{ij} the matrix with 1 at i-th row, j-th column and 0 elsewhere. Under the identification $V \cong \mathfrak{sl}_3$, let $a_1 = e_{12}, a_2 = e_{23}, a_3 = e_{13}, a_4 = e_{11} - e_{22}, a_5 = e_{22} - e_{33}, a_6 = e_{21}, a_7 = e_{32}, a_8 = e_{31}$ be the basis of $\mathfrak{sl}_3$ with highest weight vector a_3. Then the symmetric product $a_3 \cdot a_3 \in \operatorname{Sym}^2 V$ generates an irreducible $\mathfrak{sl}_3$ sub-representation $V^{2s_1+2s_2}$ with highest weight $2s_1 + 2s_2$ in $\operatorname{Sym}^2 V$. The dual of the quotient $W = \operatorname{Sym}^2 V / V^{2s_1+2s_2}$ gives $W^\vee = \{x \in \operatorname{Sym}^2 V^\vee | x(v) = 0, \forall v \in V^{2s_1+2s_2}\}$. By Theorem 1.1 in [6] due to Kostant or Theorem in [11], the elements in $W^\vee$ are generators of the ideal defining X in $\mathbb{P}(V)$. A direct calculation gives the basis of $W^\vee$*

$$\begin{gathered}
a_1^\vee a_2^\vee + a_3^\vee a_4^\vee - a_3^\vee a_5^\vee, \quad a_1^\vee a_5^\vee + a_3^\vee a_7^\vee, \quad a_2^\vee a_4^\vee - a_3^\vee a_6^\vee, \\
a_1^\vee a_6^\vee + a_3^\vee a_8^\vee + 2a_4^\vee a_4^\vee, \quad a_2^\vee a_7^\vee + a_3^\vee a_8^\vee + 2a_5^\vee a_5^\vee, \quad a_3^\vee a_8^\vee + a_4^\vee a_5^\vee, \\
a_6^\vee a_7^\vee + a_8^\vee a_4^\vee - a_8^\vee a_5^\vee, \quad a_6^\vee a_5^\vee + a_8^\vee a_2^\vee, \quad a_7^\vee a_4^\vee - a_8^\vee a_1^\vee.
\end{gathered}$$

So the geometric operators (3.3) *in this case are*

$$\begin{gathered}
\frac{\partial^2}{\partial a_1 \partial a_2} + \frac{\partial^2}{\partial a_3 \partial a_4} - \frac{\partial^2}{\partial a_3 \partial a_5}, \quad \frac{\partial^2}{\partial a_1 \partial a_5} + \frac{\partial^2}{\partial a_3 \partial a_7}, \\
\frac{\partial^2}{\partial a_2 \partial a_4} - \frac{\partial^2}{\partial a_3 \partial a_6}, \quad \frac{\partial^2}{\partial a_1 \partial a_6} + \frac{\partial^2}{\partial a_3 \partial a_8} + 2\frac{\partial^2}{\partial a_4^2}, \\
\frac{\partial^2}{\partial a_2 \partial a_7} + \frac{\partial^2}{\partial a_3 \partial a_8} + 2\frac{\partial^2}{\partial a_5^2}, \quad \frac{\partial^2}{\partial a_3 \partial a_8} + \frac{\partial^2}{\partial a_4 \partial a_5}, \\
\frac{\partial^2}{\partial a_6 \partial a_7} + \frac{\partial^2}{\partial a_8 \partial a_4} - \frac{\partial^2}{\partial a_8 \partial a_5}, \quad \frac{\partial^2}{\partial a_6 \partial a_5} + \frac{\partial^2}{\partial a_8 \partial a_2}, \\
\frac{\partial^2}{\partial a_7 \partial a_4} - \frac{\partial^2}{\partial a_8 \partial a_1}.
\end{gathered}$$

There are 8 first order differential operators generating the symmetry operators (3.4)

$$\begin{gathered}
a_3 \frac{\partial}{\partial a_2} - 2a_1 \frac{\partial}{\partial a_4} + a_1 \frac{\partial}{\partial a_5} + a_4 \frac{\partial}{\partial a_6} - a_7 \frac{\partial}{\partial a_8}, \\
- a_3 \frac{\partial}{\partial a_1} - 2a_2 \frac{\partial}{\partial a_5} + a_2 \frac{\partial}{\partial a_4} + a_5 \frac{\partial}{\partial a_7} + a_6 \frac{\partial}{\partial a_8},
\end{gathered}$$

$$
\begin{aligned}
&a_6\frac{\partial}{\partial a_7} - 2a_6\frac{\partial}{\partial a_4} + a_6\frac{\partial}{\partial a_5} + a_4\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_3},\\
&a_7\frac{\partial}{\partial a_6} - 2a_7\frac{\partial}{\partial a_5} + a_7\frac{\partial}{\partial a_4} + a_5\frac{\partial}{\partial a_2} + a_1\frac{\partial}{\partial a_3},\\
&2a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2} - 2a_6\frac{\partial}{\partial a_6} + a_7\frac{\partial}{\partial a_7} + a_3\frac{\partial}{\partial a_3} - a_8\frac{\partial}{\partial a_8},\\
&2a_2\frac{\partial}{\partial a_2} - a_1\frac{\partial}{\partial a_1} - 2a_7\frac{\partial}{\partial a_7} + a_6\frac{\partial}{\partial a_6} + a_3\frac{\partial}{\partial a_3} - a_8\frac{\partial}{\partial a_8},\\
&-a_3\frac{\partial}{\partial a_4} - a_3\frac{\partial}{\partial a_5} - a_2\frac{\partial}{\partial a_6} + a_1\frac{\partial}{\partial a_7} + a_4\frac{\partial}{\partial a_8} + a_5\frac{\partial}{\partial a_8},\\
&-a_8\frac{\partial}{\partial a_4} - a_8\frac{\partial}{\partial a_5} - a_7\frac{\partial}{\partial a_1} + a_6\frac{\partial}{\partial a_2} + a_4\frac{\partial}{\partial a_3} + a_5\frac{\partial}{\partial a_3}.
\end{aligned}
$$

Together with Euler operator $(\sum_i a_i \frac{\partial}{\partial a_i}) + 1$, *these operators form the tautological system in this case.*

5. Global residue on projectivization of vector bundle

The residue formula in Definition 2.4 is motivated by residue formulas for hypersurfaces. It is locally the same as complete intersections. In this section, we introduce another approach more directly related to the geometry of $\mathbb{P}(E^\vee)$ by Cayley's trick.

First we fix the notation. Let $\mathbb{P} = \mathbb{P}(E^\vee)$ and $\mathcal{O}(1)$ for the hyperplane section bundle. The projection map is denoted by $\pi\colon \mathbb{P} \to X$. Any section $f \in H^0(X, E)$ is identified with a section $f \in H^0(\mathbb{P}, \mathcal{O}(1))$. The zero locus of $f \in H^0(\mathbb{P}, \mathcal{O}(1))$ is denoted by $\widetilde{Y}_f$.

We collect the propositions relating the geometry of X and $\mathbb{P}$ in the following.

Proposition 5.1. 1) *Hypersurface* $\widetilde{Y}_f$ *is smooth if and only if* Y_f *is smooth with codimension* r *or empty.*

2) *There is an natural isomorphism* $K_{\mathbb{P}} \cong \pi^*(K_X \otimes \det E) \otimes \mathcal{O}(-r)$

Proof. The proof of first two propositions are the same as toric complete intersections [12]. □

From now on, we assume Y_f is smooth with codimension r.

Definition 5.2. *The variable cohomology* $H^d_{var}(Y_f)$ *is defined to be cokernel of*

$$H^d(X) \to H^d(Y_f).$$

Using the same argument for toric complete intersections in [12], we have the following propositions

Proposition 5.3. *There is an long exact sequence of mixed Hodge structures*

$$
\begin{aligned}
(5.1)\qquad 0 \to H^{n-r-1}(X) \to & H^{n+r-1}(X) \\
& \to H^{n+r-1}(X - Y_f) \to H^d_{var}(Y_f) \otimes \mathbb{C}(r) \to 0
\end{aligned}
$$

Here $\mathbb{C}(r)$ is the r-th Tate twist.

Proposition 5.4. *The map $\pi\colon \mathbb{P} - \widetilde{Y}_f \to X - Y_f$ is fibration with fibers isomorphic to $\mathbb{C}^{r-1}$ and induces an isomorphism of mixed Hodge structures $\pi^*\colon H^{n+r-1}(X - Y_f) \to H^{n+r-1}(\mathbb{P} - \widetilde{Y}_f)$.*

We now consider the Calabi-Yau case, equivalently $\det E \cong K_X^{-1}$, for simplicity. Then we have the vanishings in the Hodge filtration

$$F^{n+r-1-k} H^{n+r-1}(\mathbb{P} - \widetilde{Y}_f) = 0 \quad \text{for } k < r-1$$

and isomorphisms

$$
\begin{aligned}
H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \to & H^0(\mathbb{P}, K_{\mathbb{P}} \otimes \mathcal{O}(r)) \\
& \to F^n H^{n+r-1}(\mathbb{P} - \widetilde{Y}_f) \to F^{n-r} H^{n-r}(Y_f) \cong \mathbb{C}.
\end{aligned}
$$

Proposition 5.5. *The constant function* 1 *is sent to holomorphic top form ω_f on Y_f via this sequence of isomorphisms. Then ω_f is the same as ω in Definition 2.4.*

Consider the principle bundle adjunction formula for base space $\mathbb{P}$. Let (P, H, ν, χ) is a Calabi-Yau bundle over $\mathbb{P}$. The image of 1 in $H^0(\mathbb{P}, K_{\mathbb{P}} \otimes \mathcal{O}(r))$ has the form $\frac{\Omega}{f^r}$ on principle bundle P. If we write f as universal section, then similar calculation can recover the differential operators in Theorem 3.2.

6. Solution rank

Now we discuss the solution rank for the system. There are two versions of solution rank formula for hypersurfaces. One is in terms of Lie algebra homology, see [1]. One is in terms of perverse sheaves on X, see [4]. Here we have similar description for zero loci of vector bundle sections.

6.1. Lie algebra homology description

We fix some notation. Let

$$R = \mathbb{C}[V]/I(\mathbb{P}(E^\vee))$$

be the coordinate ring of $\mathbb{P}$. Let $Z\colon \hat{\mathfrak{g}} \to \mathrm{End}(V)$ be the extended representation by e acting as identity. We extend the character $\beta\colon \hat{\mathfrak{g}} \to \mathbb{C}$ by assigning $\beta(e) = r$.

Definition 6.1. *We define $\mathcal{D}_{V^\vee}$-module structure on $R[a]e^f \cong R[a_1, \ldots, a_N]$ as follows. The functions a_i act as left multiplications on $R[a_1, \ldots, a_N]$. The action of ∂_{a_i} on $R[a_1, \ldots, a_N]$ is $\partial_{a_i} + a_i^\vee$.*

It is straightforward to check that this $\mathcal{D}_{V^\vee}$-module $R[a]e^f$ is isomorphic to

$$\mathcal{D}_{V^\vee}/\mathcal{D}_{V^\vee} J(\mathbb{P}(E^\vee)).$$

So we have the following $\mathcal{D}_{V^\vee}$-module isomorphism.

Theorem 6.2. *There is a canonical isomorphism of $\mathcal{D}_{V^\vee}$-modules*

$$\tau \cong R[a]e^f / Z^\vee(\hat{\mathfrak{g}}) R[a] e^f.$$

This leads to the Lie algebra homology description of (classical) solution sheaf

Corollary 6.3. *If the action of G on $\mathbb{P}(E^\vee)$ has finitely many orbits, then the stalk of the solution sheaf at $b \in V^\vee$ is*

$$sol(\tau) \cong Hom_{\mathcal{D}}(Re^{f(b)}/Z^\vee(\hat{\mathfrak{g}})Re^{f(b)}, \mathcal{O}_b) \cong H_0(\hat{\mathfrak{g}}, Re^{f(b)}).$$

6.2. Perverse sheaves description

We follow the notation in [4].

1) Let $\mathbb{L}^\vee$ be the total space of $\mathcal{O}(1)$ and $\mathring{\mathbb{L}}^\vee$ the complement of the zero section.
2) Let $ev\colon V^\vee \times \mathbb{P} \to \mathbb{L}^\vee, \quad (a, x) \mapsto a(x)$ be the evaluation map.
3) Assume $\mathbb{L}^\perp = \ker(ev)$ and $U = V^\vee \times \mathbb{P} - \mathbb{L}^\perp$. Let $\pi^\vee\colon U \to V^\vee$. Notice that U is the complement of the zero locus of the universal section.

4) Let $\mathcal{D}_{\mathbb{P},\beta} = (\mathcal{D}_{\mathbb{P}} \otimes k_\beta) \otimes_{U\mathfrak{g}} k$, where k_β is the 1-dimensional $\mathfrak{g}$-module with character β and k is the trivial $\mathfrak{g}$-character.

5) Let $\mathcal{N} = \mathcal{O}_{V^\vee} \boxtimes \mathcal{D}_{\mathbb{P},\beta}$.

We have the following description of τ. See Theorem 2.1 in [4].

Theorem 6.4. *There is a canonical isomorphism*

$$\tau \cong H^0\pi_+^\vee(\mathcal{N}|_U).$$

Proof. The proof follows the same arguments of Theorem 2.1 in [4]. The only difference is that the universal section $f = a_i \otimes a_i^\vee$ defines a trivialization of $\mathcal{O}_{V^\vee} \boxtimes \mathcal{O}(1)$. So f^{-r} instead of f^{-1} defines a nonzero section of $\mathcal{O}_{V^\vee} \boxtimes K_{\mathbb{P}}$. Hence we have an isomorphism

$$\mathcal{O}_U f^{-r} \cong \omega_{U/V^\vee}.$$

Let $\mathcal{R} := \mathcal{O}_{V^\vee}/\mathcal{O}_{V^\vee} J(\mathbb{P})$ be a $\mathcal{D}_{V^\vee} \times \hat{\mathfrak{g}}$-module. Then from Theorem 6.2, we have

$$\tau \cong (\mathcal{R} \otimes k_\beta) \otimes_{\hat{\mathfrak{g}}} k.$$

The $\mathcal{D}_{V^\vee} \times \hat{\mathfrak{g}}$-morphism in the technical Lemma 2.6 in [4] is now changed to

$$\phi\colon \mathcal{R} \otimes k_\beta \to \mathcal{O}_U f^{-r}$$

by setting

$$\phi(a \otimes b) = \frac{(-1)^l (l+r)!}{f^{l+r}} a \otimes b$$

Here we identify $\mathcal{R}$ with $\mathcal{O}_{V^\vee} \otimes \mathcal{S}$ and $\mathcal{S}$ is the graded coordinate ring of $(\mathbb{P}, \mathcal{O}(1))$. The element $b \in \mathcal{S}$ has degree l. Since $\beta(e) = r$, we have an isomorphism induced by ϕ

$$\tau \cong (\mathcal{R} \otimes k_\beta) \otimes_{\hat{\mathfrak{g}}} k \cong (\mathcal{O}_U f^{-r}) \otimes_{\mathfrak{g}} k.$$

□

A direct corollary is the following

Corollary 6.5. *If $\beta(\mathfrak{g}) = 0$, there is a canonical surjective map*

$$\tau \to H^0 \pi_+^\vee \mathcal{O}_U.$$

In terms of period integral, we have an injective map

$$H_{n+r-1}(\mathbb{P} - \tilde{Y}_b) \to Hom(\tau, \mathcal{O}_{V^\vee, b})$$

given by

$$\gamma \mapsto \int_\gamma \frac{\Omega}{f^r}.$$

We have similar solution rank formula. We assume G-action on $\mathbb{P}$ has finitely many orbits. Let $\mathcal{F} = Sol(\mathcal{D}_{\mathbb{P},\beta}) = RHom_{\mathcal{D}^{an}}(\mathcal{D}_{\mathbb{P},\beta}^{an}, \mathcal{O}_{\mathbb{P}}^{an})$ be a perverse sheaf on $\mathbb{P}$. Under Riemann-Hilbert correspondence, we have the perverse sheaf version of Theorem 6.4.

Corollary 6.6. *Denote by N the dimension of V. The non-derived solution sheaf of τ is isomorphic to*

$$Hom_{\mathcal{D}_{V^\vee}}(\tau, \mathcal{O}_{V^\vee}) \cong H^{-N}({}^pR^0\pi_!^\vee(\mathbb{C}[N] \boxtimes \mathcal{F}))$$

where ${}^pR^0\pi_!^\vee$ is the 0-th perverse cohomology of $\pi_!^\vee$ and H^{-N} is the $(-N)$-th sheaf cohomology.

Hence we have the following description of solution rank

Corollary 6.7. *Let $b \in V^\vee$. Then the solution rank of τ at b is*

$$\dim H_c^0(U_b, \mathcal{F}|_{U_b}).$$

Now we apply the solution rank formulas to different cases.

6.3. Irreducible homogeneous vector bundles

In this subsection, we assume X is homogeneous G-variety and the lifted G-action on $\mathbb{P}$ is also transitive. In other words, we have $X = G/R$ and $\mathbb{P} = G/R'$ with $R/R' \cong \mathbb{P}^{r-1}$. Then we have the following corollary

Corollary 6.8. *If $\beta(\mathfrak{g}) = 0$, then the solution rank of τ at point $b \in V^\vee$ is given by* $\dim H_{n+r-1}(X - Y_b)$.

Proof. The solution sheaf in this case is $\mathcal{F} \cong \mathbb{C}[n+r-1]$. So the solution rank is

$$\dim H_c^0(U_b, \mathbb{C}[n+r-1]) = \dim H_{n+r-1}(U_b) = \dim H_{n+r-1}(X - Y_b). \quad \square$$

6.4. Complete intersections

We first fix some assumptions.

1) Let E split as direct sum of homogeneous G-line bundles $L_1, \ldots, L_r$.
2) Let $\tilde{G} = G \times (\mathbb{C}^*)^{r-1}$ acting on $\mathbb{P}$ as example 4.1.
3) Let G-action of X have finitely many orbits. Then $\tilde{G}$-action on $\mathbb{P}$ has finitely many orbits.
4) We further assume $\beta(\mathfrak{g}) = 0$. This implies $\beta(\tilde{\mathfrak{g}}) = 0$.

In this case, there are always some invariant divisors on $\mathbb{P}$ given as follows. Let $[t_1, \ldots, t_r]$ be the local homogeneous coordinates on $\mathbb{P}$ in $\mathbb{P}^{r-1}$-direction. Each t_i comes from the coordinate on $L_i^\vee$. Then $t_i = 0$ defines globally a divisor on $\mathbb{P}$. We denote it by D_i. The complement of $\cup_i D_i$ is denoted by $\mathring{\mathbb{P}}$. Let $j\colon \mathring{\mathbb{P}} \to \mathbb{P}$ be the open embedding. We treat the following two special cases with X being homogeneous or toric.

Let $X = G/R$ be a homogeneous G-variety. Then we have the isomorphism

Proposition 6.9. $\mathcal{D}_{\mathbb{P},\beta} \cong j_! j^! \mathcal{D}_{\mathbb{P},\beta}$.

Proof. The proposition follows from the proof of Corollary 3.3 in [5]. The key observation is that Lemma 3.2 and Corollary 3.3 in [5] rely on two conditions. One is that the toric divisors D have normal crossing singularities in the ambient space X. The other condition is that the log tangent bundle $T_X(-\log D)$ is globally generated. (This is called a log homogeneous variety, See section 7 or [2] for the definition.) The total space $\mathbb{P}$ satisfies these conditions with divisor $D = \sum_i D_i$ at infinity. So this proposition holds. $\square$

So we have the following description of solution rank the same as Theorem 3.5 in [5].

Theorem 6.10. *The solution rank at point $b \in V^\vee$ is given by*

$$H_{n+r-1}(\mathbb{P} - \widetilde{Y}_b, (\mathbb{P} - \widetilde{Y}_b) \cap (\cup_i D_i)).$$

This theorem is not satisfying because the final cohomology is not directly related to X. Let $Y_{1,b}, \ldots, Y_{r,b}$ be the zero locus of the L_i component of section s_b. From the geometric realization of some solutions as period integrals of rational forms along the cycles in the complement of $Y_{1,b} \cup \cdots \cup Y_{r,b}$, we have the following conjecture:

Conjecture 6.11. *There is a natural isomorphism of solution sheaves as period integrals*

$$Hom_{\mathcal{D}_{V^\vee}}(\tau_V, \mathcal{O})|_b \cong H_n(X - (Y_{1,b} \cup \cdots \cup Y_{r,b})).$$

Remark 6.12. *This conjecture has been proved recently by Lee, Lian and Zhang [7].*

7. Chain integrals in log homogeneous varieties

Let X be a complex variety with normal crossing divisor D. The log D tangent bundle $T_X(-\log D)$ is a subsheaf of T_X defined as follows. If $z_1, \ldots, z_n$ is the local coordinate of X and D is the union of hyperplanes defined by $z_1 = 0, \ldots, z_r = 0$, then the generating sections of $T_X(-\log D)$ are $z_1\partial_1, \ldots, z_r\partial_r$, $\partial_{r+1}, \ldots, \partial_n$. Then we say X is log homogeneous if $T_X(-\log D)$ is globally generated. Let $\mathfrak{g}$ be $H^0(X, T_X(-\log D)$ and G be the corresponding simply connected complex Lie group. Then X is a G-variety. Let L be a G-equivariant line bundle defined on X. We assume L is very ample and $L + K_X$ is base point free. Then the period integrals of sections in $W^\vee = H^0(X, L + K_X)$on hypersurface families cut out by $V^\vee = H^0(X, L)$ satisfy the tautological systems τ_{VW} with character $\beta = 0$. See [9] and [4]. The tautological system τ in this case is holonomic because G-action on X is stratified by D with finitely many orbits. See [2] for discussion of log homogeneous varieties.

We consider the solution rank of τ_{VW} in this case. Following the same proof of Theorem 3.5 in [5], we have the following description of solution rank of τ_{VW}.

Theorem 7.1. *Let $b \neq 0$ be a section in $H^0(X, L)$ and $c \neq 0$ a section in $H^0(X, L + K_X)$. Then there is a natural isomorphism from the stalk of solution sheave of τ_{VW} at point (b, c) to the following homology group*

$$Hom_{\mathcal{D}_{V^\vee \times W^\vee}}(\tau_{VW}, \mathcal{O})|_{(b,c)} \cong H_n(X - Y_b, (X - Y_b) \cap (\cup D))$$

given by period integrals.

Acknowledgements

B. Lian and S.-T. Yau are partially supported by the Simons Collaboration on Homological Mirror Symmetry 2015.

References

[1] S. Bloch, A. Huang, B. H. Lian, V. Srinivas, and S.-T. Yau, *On the holonomic rank problem*, Journal of Differential Geometry **97** (2014), no. 1, 11–35.

[2] M. Brion, *Log homogeneous varieties*, `arXiv:math/0609669`, (2006).

[3] I. M. Gel'fand, A. V. Zelevinskii, and M. M. Kapranov, *Hypergeometric functions and toral manifolds*, Functional Analysis and Its Applications **23** (1989), no. 2, 94–106.

[4] A. Huang, B. H. Lian, and X. Zhu, *Period integrals and the Riemann–Hilbert correspondence*, Journal of Differential Geometry **104** (2016), no. 2, 325–369.

[5] A. Huang, B. H. Lian, X. Zhu, and S.-T. Yau, *Chain integral solutions to tautological systems*, Math. Res. Lett **23** (2016), no. 6, 1721–1736.

[6] G. Lancaster and J. Towber, *Representation-functors and flag-algebras for the classical groups I*, Journal of Algebra **59** (1979), no. 1, 16–38.

[7] T.-J. Lee, B. Lian, and D. Zhang, *A remark on solutions of tautological systems for complete intersections*, in preperation (2019).

[8] B. H. Lian, R. Song, and S.-T. Yau, *Periodic integrals and tautological systems*, Journal of the European Mathematical Society **15** (2013), no. 4, 1457–1483.

[9] B. H. Lian and S.-T. Yau, *Period integrals of CY and general type complete intersections*, Inventiones Mathematicae (2013), 1–55.

[10] B. H. Lian and M. Zhu, *On the hyperplane conjecture for periods of Calabi-Yau hypersurfaces in* $\mathbb{P}^n$, `arXiv:1610.07125`, (2016).

[11] W. Lichtenstein, *A system of quadrics describing the orbit of the highest weight vector*, Proceedings of the American Mathematical Society **84** (1982), no. 4, 605–608.

[12] A. R. Mavlyutov, *Cohomology of complete intersections in toric varieties*, Pacific Journal of Mathematics **191** (1999), no. 1, 133–144.

[13] H. Sumihiro, *A theorem on splitting of algebraic vector bundles and its applications*, Hiroshima Mathematical Journal **12** (1982), no. 2, 435–452.

Department of Mathematics, Brandeis University
Waltham, MA 02454, USA
E-mail address: anhuang@brandeis.edu

Department of Mathematics, Brandeis University
Waltham, MA 02454, USA
E-mail address: lian@brandeis.edu

Department of Mathematics, Harvard University
Cambridge, MA 02138, USA
E-mail address: yau@math.harvard.edu

Department of Mathematics, University of Pennsylvania
Philadelphia, PA 19104, USA
E-mail address: yucl18@upenn.edu

Received May 11, 2019
Accepted December 23, 2019