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BILINEAR IDENTITIES FOR AN EXTENDED B-TYPE KADOMTSEV–PETVIASHVILI HIERARCHY

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We construct bilinear identities for wave functions of an extended B-type Kadomtsev–Petviashvili (BKP) hierarchy containing two types of (2+1)-dimensional Sawada–Kotera equations with a self-consistent source. Introducing an auxiliary variable corresponding to the extended flow for the BKP hierarchy, we find the τ-function and bilinear identities for this extended BKP hierarchy. The bilinear identities generate all the Hirota bilinear equations for the zero-curvature forms of this extended BKP hierarchy. As examples, we obtain the Hirota bilinear equations for the two types of (2+1)-dimensional Sawada–Kotera equations in explicit form.

Keywords: BKP hierarchy, self-consistent source, bilinear identity, tau function, Hirota bilinear form

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1. Introduction

The well-known Kadomtsev–Petviashvili (KP) hierarchy [1], [2] is an infinite-dimensional system of nonlinear partial differential equations (PDEs), which contains various types of soliton equations. The B-type KP (BKP) hierarchy [3], [1] is a subhierarchy of the KP system. The BKP hierarchy has many integrable structures (see, e.g., [1], [4], [5] and the references therein), such as the Lax formalism, the τ-function, and Hirota bilinear equations.

The Sato theory is fundamentally important in studying integrable systems. It reveals the infinite-dimensional Grassmannian structure of the space of τ-functions, where the τ-functions are solutions of the Hirota bilinear form of the KP hierarchy. The bilinear identity for wave and adjoint wave functions plays an important role in proving the existence of the τ-function and is also a generating function for the Hirota bilinear equations for the KP hierarchy [5]–[7].

Reductions and generalizations are important topics in the study of integrable systems. There are several possible generalizations, for example, constructing new flows to *extend* the original systems. There are various ways to introduce new flows and obtain new compatible integrable systems. In [8], the KP hierarchy was extended by appropriately combining additional flows. In [9], the KP hierarchy was extended by using introduced fractional-order pseudodifferential operators. In [10], [11], a Moyal-deformed hierarchy was extended by including additional evolution equations with respect to the deformation parameters.

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In [12], a logarithm of the difference Lax operator was defined, and the extended (2+1)-dimensional Toda hierarchy was obtained. The Hirota bilinear formalism and the relations of the extended (2+1)-dimensional and one-dimensional Toda hierarchies were later studied [13], [14]. In [15], the τ-functions and bilinear identities for the extended bigraded Toda hierarchy were studied. A discrete analogue of the symmetry-constrained KP hierarchy was presented in [16].

Soliton equations with self-consistent sources have many physical applications (see [17]–[23] and the references therein). In particular, the speed of the solitons can be changed by the sources (see, e.g., [19], [23]). It was recently shown that the construction of the soliton equation with self-consistent sources is related to the so-called squared eigenfunction symmetry and the binary Darboux transformation of the original soliton hierarchy [24], [25]. An extended BKP hierarchy was constructed in [26] based on the idea in [27] to use symmetries of the generating functions (or the squared eigenfunction symmetries), where two kinds of new multicomponent BKP hierarchy were constructed and their n-reduction and k-constraint were discussed. This kind of extended BKP hierarchy can be regarded as a generalization of the BKP hierarchy by introducing the squared eigenfunction symmetries.

The extended BKP hierarchy obtained in [26] includes two types of (2+1)-dimensional Sawada–Kotera equations with a self-consistent source (2d-SKwS-I and 2d-SKwS-II), where the 2d-SKwS-I equation coincides with that obtained in [28] by a source generating method. In [27], we proposed a method for constructing an extended KP hierarchy, and the bilinear identities for this extended KP hierarchy were then constructed in [25]. The Hirota bilinear equations were derived for all the zero-curvature forms in the extended KP hierarchy by introducing an auxiliary flow. It seems that the form of the Hirota bilinear equations in [25] is simpler than the previously obtained form [29].

Here, we construct bilinear identities for the extended BKP hierarchy in [26]. The bilinear identities generate all the Hirota bilinear forms for the zero-curvature equations of this extended BKP hierarchy. As examples, we derive the Hirota bilinear forms for the 2d-SKwS-I and 2d-SKwS-II equations. To the best of our knowledge, the Hirota bilinear equations for the 2d-SKwS-II equation have not appeared previously in the literature.

This paper is organized as follows. In Sec. 2, we construct the bilinear identities for the BKP hierarchy with a squared eigenfunction symmetry (or a "ghost symmetry"). In Sec. 3, regarding the squared eigenfunction symmetry as an auxiliary flow, we construct the bilinear identities for the extended BKP hierarchy and prove that these bilinear identities fully characterize the extended BKP hierarchy. In Sec. 4, we introduce the τ-function for the extended BKP hierarchy and find the generating functions for the Hirota bilinear form for the extended BKP hierarchy. In Sec. 5, we transform the Hirota bilinear form back to the nonlinear PDEs in the cases of the 2d-SKwS-I and 2d-SKwS-II equations, which confirms the correctness of our construction. Finally, in Sec. 6, we present conclusions and remarks.

2. Bilinear identities for the BKP hierarchy with its "ghost symmetry"

We recall some notation for pseudodifferential operators. For a pseudodifferential operator $P = \sum_{i=-\infty}^{n} a_i\partial^i$ with $\partial := \partial_x$, its nonnegative part, negative part, and adjoint operator are respectively denoted by

$$P_+ = \sum_{i=0}^{n} a_i\partial^i, \qquad P_- = \sum_{i<0}^{n} a_i\partial^i, \qquad P^* = \sum_{i=-\infty}^{n} (-\partial)^i a_i,$$

and we set $\mathrm{res}_\partial(P) = a_{-1}$. For a Laurent series $f(\lambda) = \sum_i c_i\lambda^i$, we also set $\mathrm{res}_\lambda f(\lambda) = c_{-1}$.

We consider a pseudodifferential operator $L := \partial + \sum_{i=1}^{\infty} u_i\partial^{-i}$ and set $B_n := (L^n)_+$. The BKP hierarchy is then defined by [1]

$$L_{t_n} = [B_n, L], \quad n = 1, 3, 5, \ldots, \tag{2.1a}$$

with the constraint

$$L^* = -\partial L \partial^{-1}. \tag{2.1b}$$

Constraint condition (2.1b) is equivalent to $B_n \cdot 1 = 0$, $n = 1, 3, 5, \dots$ [3], [4]. Following the idea in [30] and [25], we can introduce a new ∂_z flow as

$$\partial_z L = [r\,\partial^{-1} q_x - q\,\partial^{-1} r_x, L], \tag{2.2a}$$

where q and r satisfy

$$q_{t_n} = B_n(q), \qquad r_{t_n} = B_n(r), \quad n = 1, 3, 5, \dots, \tag{2.2b}$$

and are called eigenfunctions [7], [31]. Moreover, the adjoint eigenfunctions q^* and r^* satisfy $q^*_{t_n} = -B^*_n(q^*)$ and $r^*_{t_n} = -B^*_n(r^*)$. It is easy to see that we can take q_x and r_x as adjoint eigenfunctions. The compatibility of ∂_z flow (2.2a) and ∂_{t_n} flows (2.1b) is then ensured. The ∂_z flow describes a symmetry for the BKP hierarchy, which is called the *squared eigenfunction symmetry* [30] or *ghost flow* [32], [33].

We introduce a dressing operator

$$W = 1 + \sum_{i=1}^{\infty} w_i(z,t)\partial^{-i}, \quad t = (t_1, t_3, \dots), \quad t_1 \equiv x,$$

such that $L = W\partial W^{-1}$, where W satisfies t_n-evolution equations

$$\partial_{t_n} W = -(W\,\partial^n W^{-1})_- W = -L^n_- W, \quad n = 1, 3, 5, \dots.$$

By constraint condition (2.1b), we then have $W^*\partial W = \partial$. The dressing operator W also satisfies the equation

$$\partial_z W = (r\,\partial^{-1} q_x - q\,\partial^{-1} r_x) W. \tag{2.3}$$

We define the wave and adjoint wave functions as

$$\begin{aligned} &w(z,t,\lambda) = W e^{\xi(t,\lambda)}, \qquad w^*(z,t,\lambda) = (W^*)^{-1} e^{-\xi(t,\lambda)}, \\ &\xi(t,\lambda) = \sum_{n=0}^{\infty} t_{2n+1}\lambda^{2n+1}. \end{aligned} \tag{2.4}$$

These wave functions satisfy

$$\begin{aligned} &Lw(z,t,\lambda) = \lambda w(z,t,\lambda), \qquad &&\partial_{t_n} w(z,t,\lambda) = B_n w(z,t,\lambda), \\ &L^* w^*(z,t,\lambda) = \lambda w^*(z,t,\lambda), \qquad &&\partial_{t_n} w^*(z,t,\lambda) = -B^*_n w(z,t,\lambda). \end{aligned} \tag{2.5}$$

We recall an important lemma (see [2] for the details and proof), which we use in proving our propositions.

Lemma 1. *Let P and Q be pseudodifferential operators. Then*

$$\operatorname{res}_\partial P \cdot Q^* = \operatorname{res}_\lambda P(e^{\xi(t,\lambda)}) \cdot Q(e^{-\xi(t,\lambda)}). \tag{2.6}$$

Theorem 1. *The BKP hierarchy* (2.1) *with squared eigenfunction symmetry* (2.2) *is equivalent to the residue identities*

$$\operatorname{res}_\lambda \lambda^{-1} w(z,t,\lambda) w(z,t',-\lambda) = 1, \tag{2.7a}$$

$$\operatorname{res}_\lambda \lambda^{-1} w_z(z,t,\lambda) w(z,t',-\lambda) = q(z,t) r(z,t') - r(z,t) q(z,t'), \tag{2.7b}$$

$$\operatorname{res}_\lambda \lambda^{-1} w(z,t,\lambda) \cdot \partial_{x'}^{-1}(q(z,t') w_{x'}(z,t',-\lambda)) = q(z,t), \tag{2.7c}$$

$$\operatorname{res}_\lambda \lambda^{-1} w(z,t,\lambda) \cdot \partial_{x'}^{-1}(r(z,t') w_{x'}(z,t',-\lambda)) = r(z,t), \tag{2.7d}$$

where the inverse of ∂ is understood as a pseudodifferential operator acting on an exponential, for example, $\partial^{-1}(r w_x) = (\partial^{-1} r\, \partial W)(e^{\xi(t,\lambda)})$.

Proof. Bilinear identity (2.7a) of the BKP hierarchy was already proved in [3], [1], and z can be regarded as a fixed parameter in this case. From relations (2.2b) and (2.5), we know that the mixed partial derivatives $\partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k}$ $(k = 1, 3, 5, \dots)$ of $w(z,t,\lambda)$, $q(z,t)$, and $r(z,t)$ can also be written in terms of a differential operator $P_{m_1 \dots m_k}$:

$$\partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} w(z,t,\lambda) = P_{m_1 \dots m_k}(w(z,t,\lambda)),$$

$$\partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} q(z,t) = P_{m_1 \dots m_k}(q(z,t)),$$

$$\partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} r(z,t) = P_{m_1 \dots m_k}(r(z,t)).$$

We note that $P_{m_1 \dots m_k}$ does not contain a free term. To prove (2.7b), it remains to use the equalities

$$\begin{aligned}
&\operatorname{res}_\lambda \lambda^{-1} w_z(z,t,\lambda)\, \partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} w(z,t,-\lambda) = \\
&\qquad = \operatorname{res}_\lambda (r\, \partial^{-1} q_x - q\, \partial^{-1} r_x) W e^{\xi(t,\lambda)} P_{m_1 \dots m_k} W(-\partial^{-1}) e^{-\xi(t,\lambda)} = \\
&\qquad = \operatorname{res}_\partial (r\, \partial^{-1} q_x - q\, \partial^{-1} r_x) W \partial^{-1} W^* P^*_{m_1 \dots m_k} = \operatorname{res}_\partial (r\, \partial^{-1} q_x - q\, \partial^{-1} r_x) \partial^{-1} P^*_{m_1 \dots m_k} = \\
&\qquad = q P_{m_1 \dots m_k}\, \partial^{-1} r_x - r P_{m_1 \dots m_k}\, \partial^{-1} q_x = q P_{m_1 \dots m_k}(r) - r P_{m_1 \dots m_k}(q).
\end{aligned}$$

This proves (2.7b).

We note that for any $P_{m_1 \dots m_k}$ defined above, we have

$$\begin{aligned}
&\operatorname{res}_\lambda \partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} \lambda^{-1} w(z,t,\lambda) \cdot \partial^{-1}(q(z,t) w_x(z,t,-\lambda)) = \\
&\qquad = \operatorname{res}_\lambda P_{m_1 \dots m_k}(\lambda^{-1} w(z,t,\lambda)) \cdot \partial^{-1}(q(z,t) w_x(z,t,-\lambda)) = \\
&\qquad = \operatorname{res}_\lambda P_{m_1 \dots m_k} W \partial^{-1} e^{\xi(t,\lambda)} \cdot \partial^{-1} q(z,t) \partial W e^{-\xi(t,\lambda)} = \\
&\qquad = \operatorname{res}_\partial P_{m_1 \dots m_k} W \partial^{-1} \cdot (\partial^{-1} q(z,t) \partial W)^* = \operatorname{res}_\partial P_{m_1 \dots m_k} W \partial^{-1} W^* (-\partial) q(z,t) (-\partial)^{-1} = \\
&\qquad = \operatorname{res}_\partial P_{m_1 \dots m_k} q(z,t) \partial^{-1} = P_{m_1 \dots m_k}(q(z,t)) = \partial_{t_1}^{m_1} \partial_{t_3}^{m_3} \cdots \partial_{t_k}^{m_k} q(z,t).
\end{aligned}$$

Therefore, we have

$$q(z,t') = \operatorname{res}_\lambda \lambda^{-1} w(z,t',\lambda) \cdot \partial^{-1}(q(z,t) w_x(z,t,-\lambda))$$

and similarly

$$r(z,t') = \operatorname{res}_\lambda \lambda^{-1} w(z,t',\lambda) \cdot \partial^{-1}(r(z,t) w_x(z,t,-\lambda)).$$

Then (2.7c) and (2.7d) are proved by interchanging t and t'.

The second part of the proof of this theorem is written as the following proposition.

Proposition 1. *If the functions $q(z,t)$ and $r(z,t)$ and the wave function*

$$w(z,t,\lambda) = We^{\xi(t,\lambda)}, \quad \text{where } W = 1 + \sum_{i\geq 1} w_i(z,t)\lambda^{-i},$$

satisfy residue identities (2.7)*, then the pseudodifferential operator $L = W\partial W^{-1}$ and the functions $q(z,t)$ and $r(z,t)$ are a solution of BKP hierarchy* (2.1) *with squared eigenfunction symmetry* (2.2).

Proof. Equation (2.1) can be obtained from (2.7a) (see [3], [1]). The first and second relations in (2.2b) can be proved by taking the respective derivatives ∂_{t_n} of (2.7c) and (2.7d).

Condition (2.2a) can be derived from (2.7b) as follows. It is easy to show that $(W_z W^{-1})_+ = 0$. We note that the adjoint wave function $w^*(z,t,\lambda)$ can be written as

$$w^*(z,t,\lambda) := (W^*)^{-1} e^{-\xi(t,\lambda)} = \partial W\,\partial^{-1} e^{-\xi(t,\lambda)} = -\lambda^{-1} w_x(z,t,-\lambda).$$

From (2.7b), we then obtain

$$\operatorname{res}_\lambda w_z(z,t,\lambda) w^*(z,t',\lambda) = r(z,t) q_{x'}(z,t') - q(z,t) r_{x'} r(z,t').$$

Furthermore, we have

$$\begin{aligned}\operatorname{res}_\partial W_z W^{-1}\partial^m &= \operatorname{res}_\lambda W_z e^{\xi(t,\lambda)} (-\partial)^m W^{*-1} e^{-\xi(t,\lambda)} = \\ &= \operatorname{res}_\lambda w_z(z,t,\lambda)\,(-\partial)^m w^*(z,t,\lambda) = r\,(-\partial)^m q_x - q\,(-\partial)^m r_x,\end{aligned}$$

which means that

$$W_z W^{-1} = \sum_{m=0}^{\infty} (r\,(-\partial)^m(q_x) - q\,(-\partial)^m(r_x))\partial^{-m-1} = r\,\partial^{-1} q_x - q\,\partial^{-1} r_x.$$

Hence, (2.3) is satisfied, which implies (2.2a).

3. Bilinear identities for an extended BKP hierarchy

In [26], an extended BKP hierarchy was introduced using the squared eigenfunction symmetry as in (2.2a). Two types of BKP hierarchy with self-consistent sources were found, and their Lax representations were also obtained.

We recall that the extended BKP hierarchy here (for a fixed odd $k \neq 1$) has the form

$$\partial_{\bar{t}_k} L = [B_k + r\,\partial^{-1} q_x - q\,\partial^{-1} r_x, L], \tag{3.1a}$$

$$L_{t_n} = [B_n, L], \qquad L^* = -\partial L \partial^{-1}, \tag{3.1b}$$

$$q_{t_n} = B_n(q), \qquad r_{t_n} = B_n(r), \tag{3.1c}$$

where $n = 1,3,5,\ldots$, $n \neq k$, and q and r are eigenfunctions. This hierarchy is constructed by replacing an arbitrary fixed kth flow ∂_{t_k} with $\partial_{\bar{t}_k}$, where the $\partial_{\bar{t}_k}$ flow is a linear combination of the ∂_{t_k} and ∂_z flows.

Remark 1. For notational simplicity in this and the following sections, we continue to use $w(z,t,\lambda)$, $w^*(z,t,\lambda)$, $q(z,t)$, $r(z,t)$, L, W, etc., but they are now considered functions in extended BKP hierarchy (3.1). For example, from now on, $t=(t_1,t_3,\dots,t_{k-2},\bar{t}_k,t_{k+2},\dots)$, where the fixed k is odd.

The Hirota bilinear equations for constrained BKP hierarchy (3.1) were constructed in [5], [34]. A natural problem is to find the bilinear identities for extended BKP hierarchy (3.1) because the bilinear identities provide a systematic way to generate all the Hirota bilinear equations in extended BKP hierarchy (3.1). In this section, we explain in detail how to derive the bilinear identities for extended BKP hierarchy (3.1).

The dressing operator W is given by

$$W=1+\sum_{i=1}^{\infty}w_i(z,t)\partial^{-i},\qquad t=(t_1,t_3,\dots,t_{k-2},\bar{t}_k,t_{k+2},\dots),\quad t_1\equiv x,$$

and satisfies

$$\partial_{\bar{t}_k}W=-L_-^kW+(r\,\partial^{-1}q_x-q\,\partial^{-1}r_x)W.$$

The wave function and its adjoint are defined as in (2.4) except that now $\xi(t,\lambda)=\bar{t}_k\lambda^k+\sum_{n\neq k}t_n\lambda^n$ (here k and n are odd). The wave and adjoint wave functions satisfy

$$\begin{aligned}Lw(z,t,\lambda)&=\lambda w(z,t,\lambda), & \partial_{t_n}w(z,t,\lambda)&=B_nw(z,t,\lambda),\\ L^*w^*(z,t,\lambda)&=\lambda w^*(z,t,\lambda), & \partial_{t_n}w^*(z,t,\lambda)&=-B_n^*w(z,t,\lambda)\end{aligned}\tag{3.2}$$

(here $n\neq k$).

We then have the bilinear identities for the extended BKP hierarchy.

Proposition 2. *The bilinear identities for extended BKP hierarchy* (3.1) *are given by the sets of residue identities with an auxiliary variable* z

$$\operatorname{res}_\lambda\lambda^{-1}w(z-\bar{t}_k,t,\lambda)w(z-\bar{t}'_k,t',-\lambda)=1,\tag{3.3a}$$

$$\begin{aligned}&\operatorname{res}_\lambda\lambda^{-1}w_z(z-\bar{t}_k,t,\lambda)w(z-\bar{t}'_k,t',-\lambda)=\\&\qquad=q(z-\bar{t}_k,t)r(z-\bar{t}'_k,t')-r(z-\bar{t}_k,t)q(z-\bar{t}'_k,t'),\end{aligned}\tag{3.3b}$$

$$\operatorname{res}_\lambda\lambda^{-1}w(z-\bar{t}_k,t,\lambda)\cdot\partial_{x'}^{-1}(q(z-\bar{t}'_k,t')w_{x'}(z-\bar{t}'_k,t',-\lambda))=q(z-\bar{t}_k,t),\tag{3.3c}$$

$$\operatorname{res}_\lambda\lambda^{-1}w(z-\bar{t}_k,t,\lambda)\cdot\partial_{x'}^{-1}(r(z-\bar{t}'_k,t')w_{x'}(z-\bar{t}'_k,t',-\lambda))=r(z-\bar{t}_k,t).\tag{3.3d}$$

Proof. We note that

$$\frac{d}{d\bar{t}_k}w(z-\bar{t}_k,t,\lambda)=(\partial_{\bar{t}_k}-\partial_z)w(z,t,\lambda)\big|_{z=z-\bar{t}_k}=B_k(w(z-\bar{t}_k,t,\lambda)),\tag{3.4}$$

and (3.3a) can hence be proved as in the original BKP case [3], [1].

The proofs of (3.3b)–(3.3d) are similar to the proofs of (2.7b)–(2.7d) necessarily taking (3.4) into account.

Proposition 3. *Let*

$$w(z,t,\lambda)=We^{\xi(t,\lambda)},\quad\text{where } W=1+\sum_{i\geq1}w_i(z,t)\lambda^{-i},$$

and $q(z,t)$ *and* $r(z,t)$ *satisfy bilinear identities* (3.3)*. Then the pseudodifferential operator* $L=W\partial W^{-1}$ *and the functions* $q(z,t)$ *and* $r(z,t)$ *satisfy extended BKP hierarchy* (3.1).

Proof. It is already known [3] that (3.3a) implies the constraint $W^*\partial W = \partial$ or, equivalently, $L^* = -\partial L\partial^{-1}$ (see (3.1b)). We can define the adjoint wave function of $w(z,t,\lambda)$ as $w^*(z,t,\lambda) = (W^{-1})^* e^{-\xi(t,\lambda)}$. From $W^*\partial W = \partial$, we then obtain $w^*(z,t,\lambda) = -\lambda^{-1} w_x(z,t,-\lambda)$. Similarly to the proof of Proposition 1, we obtain $W_z = (r\,\partial^{-1} q_x - q\partial^{-1} r_x)W$ from (3.3b).

It can be shown that

$$\frac{d}{d\bar{t}_k} w(z-\bar{t}_k,t,\lambda) = \left(\frac{d}{d\bar{t}_k}W + L^k W\right) e^{\xi(t,\lambda)},$$

$$\operatorname{res}_\lambda w(z-\bar{t}_k,t,\lambda)\lambda^{-1} w_{x'}(z-\bar{t}'_k,t',-\lambda) = -\operatorname{res}_\lambda w(z-\bar{t}_k,t,\lambda) w^*(z-\bar{t}'_k,t',\lambda) = 0.$$

From the coefficients of the Taylor expansion of this equation, we then find that for any positive integer m,

$$\begin{aligned}
0 &= \operatorname{res}_\lambda \frac{d}{d\bar{t}_k} w(z-\bar{t}_k,t,\lambda)\cdot \partial^m w^*(z-\bar{t}_k,t,\lambda) = \\
&= \operatorname{res}_\lambda \left(\frac{d}{d\bar{t}_k}W + L^k W\right) e^{\xi(t,\lambda)} \cdot \partial^m (W^{-1})^* e^{-\xi(t,\lambda)} = \\
&= \operatorname{res}_\partial \left(\frac{d}{d\bar{t}_k}W + L^k W\right) \cdot \left(\partial^m (W^{-1})^*\right)^* = \\
&= \operatorname{res}_\partial \left(\frac{d}{d\bar{t}_k}W + L^k_- W\right) W^{-1}(-\partial)^m,
\end{aligned}$$

which means that $dW/d\bar{t}_k = -L^k_- W$. Therefore, $\partial_{\bar{t}_k} W = -L^k_- W + (r\,\partial^{-1} q_x - q\,\partial^{-1} r_x)W$ holds, and we can hence obtain (3.1a).

For (3.1c), the proofs can be done by directly differentiating (3.3c) and (3.3d) because we already proved (3.1b) and hence (3.2).

4. The $\boldsymbol{\tau}$-function for the extended BKP hierarchy

The existence of a τ-function for the ordinary BKP hierarchy was proved in [1]. According to [6] and [25], similar assumptions can be made for extended BKP hierarchy (3.1), i.e.,

$$\begin{aligned}
&w(z-\bar{t}_k,t,\lambda) = \frac{\tau(z-\bar{t}_k + 2/k\lambda^k, t-2[\lambda])}{\tau(z-\bar{t}_k,t)}\, e^{\xi(t,\lambda)}, \\
&q(z,t) = \frac{\sigma(z,t)}{\tau(z,t)}, \qquad r(z,t) = \frac{\rho(z,t)}{\tau(z,t)},
\end{aligned} \tag{4.1}$$

where $[\lambda] = (1/\lambda, 1/3\lambda^3, 1/5\lambda^5, \dots)$.

As in [5], we can note that $w^*(z-\bar{t}_k,t,\lambda) = -\lambda^{-1} w_x(z-\bar{t}_k,t,-\lambda)$. We have

$$\begin{aligned}
&\partial^{-1}(r(z-\bar{t}_k,t) w^*(z-\bar{t}_k,t,\lambda)) = \\
&\qquad = -\frac{1}{2\lambda\tau^2(z-\bar{t}_k,t)} \times \\
&\qquad\quad \times \left\{\rho(z-\bar{t}_k,t)\tau\left(z-\bar{t}_k - \frac{2}{k\lambda^k}, t+2[\lambda]\right) + \rho(z-\bar{t}_k,t+2[\lambda])\tau(z-\bar{t}_k,t)\right\} e^{-\xi(t,\lambda)},
\end{aligned}$$

$$\partial^{-1}(q(z-\bar t_k,t)w^*(z-\bar t_k,t,\lambda)) =$$

$$= -\frac{1}{2\lambda\tau^2(z-\bar t_k,t)}\times$$

$$\times\left\{\sigma(z-\bar t_k,t)\tau\left(z-\bar t_k-\frac{2}{k\lambda^k},t+2[\lambda]\right)+\sigma(z-\bar t_k,t+2[\lambda])\tau(z-\bar t_k,t)\right\}e^{-\xi(t,\lambda)}.$$

Substituting these expressions and (4.1) in (3.3), we obtain

$$\begin{aligned}
&\mathrm{res}_\lambda\,\lambda^{-1}\bar\tau(t-y-2[\lambda])\bar\tau(t+y+2[\lambda])\,e^{\xi(-2y,\lambda)} = \bar\tau(t-y)\bar\tau(t+y),\\
&\mathrm{res}_\lambda\,\lambda^{-1}\bar\tau_z(t-y-2[\lambda])\bar\tau(t+y+2[\lambda])\,e^{\xi(-2y,\lambda)} -\\
&\qquad - \mathrm{res}_\lambda\,\lambda^{-1}\bar\tau(t-y-2[\lambda])(\partial_z\log\bar\tau(t-y))\bar\tau(t+y+2[\lambda])\,e^{\xi(-2y,\lambda)} =\\
&\qquad\qquad = \bar\sigma(t-y)\bar\rho(t+y)-\bar\rho(t-y)\bar\sigma(t+y),\\
&\mathrm{res}_\lambda\,\lambda^{-1}\bar\tau(t-y-2[\lambda])\bar\sigma(t+y+2[\lambda])\,e^{\xi(-2y,\lambda)} = 2\bar\sigma(t-y)\bar\tau(t+y)-\bar\sigma(t+y)\bar\tau(t-y),\\
&\mathrm{res}_\lambda\,\lambda^{-1}\bar\tau(t-y-2[\lambda])\bar\rho(t+y+2[\lambda])\,e^{\xi(-2y,\lambda)} = 2\bar\rho(t-y)\bar\tau(t+y)-\bar\rho(t+y)\bar\tau(t-y).
\end{aligned}\tag{4.2}$$

Here, we set $\bar f(t) := f(z-\bar t_k,t)$, and therefore

$$\bar f(t-2[\lambda]) = f\left(z-\bar t_k+\frac{2}{k\lambda^k},t-2[\lambda]\right),\qquad \bar f(t+2[\lambda]) = f\left(z-\bar t_k-\frac{2}{k\lambda^k},t+2[\lambda]\right).$$

Letting $t\to t-y$ and $t'\to t+y$, where $y=(y_1,y_3,y_5,\dots)$, and introducing the Hirota operator D_n [35],

$$D_nf(t)\circ g(t) = (\partial_{t_n}-\partial_{t_n'})f(t)g(t')\big|_{t_n'=t_n},\qquad e^{aD_n}f(t)\circ g(t) = f(t_n+a)g(t_n-a),$$

we can write (4.2) as

$$\begin{aligned}
&\sum_{j=0} p_j(-2y)p_j(2\widetilde D)\,e^{\sum_n y_nD_n}\,\bar\tau(t)\circ\bar\tau(t) = e^{\sum_n y_nD_n}\,\bar\tau(t)\circ\bar\tau(t),\\
&\sum_{j=0} p_j(-2y)p_j(2\widetilde D)\,e^{\sum_n y_nD_n}\,\bar\tau(t)\circ\bar\tau_z(t) -\\
&\qquad - \big(\partial_z\log\bar\tau(t-y)\big)\sum_{j=0} p_j(-2y)p_j(2\widetilde D)e^{\sum_n y_nD_n}\,\bar\tau(t)\circ\bar\tau(t) =\\
&\qquad\qquad = e^{\sum_n y_nD_n}\big(\bar\rho(t)\circ\bar\sigma(t)-\bar\sigma(t)\circ\bar\rho(t)\big),\\
&\sum_{j=0} p_j(-2y)p_j(2\widetilde D)\,e^{\sum_n y_nD_n}\,\bar\sigma(t)\circ\bar\tau(t) = e^{\sum_n y_nD_n}\big(2\bar\tau(t)\circ\bar\sigma(t)-\bar\sigma(t)\circ\bar\tau(t)\big),\\
&\sum_{j=0} p_j(-2y)p_j(2\widetilde D)\,e^{\sum_n y_nD_n}\,\bar\rho(t)\circ\bar\tau(t) = e^{\sum_n y_nD_n}\big(2\bar\tau(t)\circ\bar\rho(t)-\bar\rho(t)\circ\bar\tau(t)\big),
\end{aligned}\tag{4.3}$$

where $\widetilde D = (D_1, D_3/3, D_5/5,\dots)$ and $p_j(t)$ are the Schur polynomials defined as

$$p_j(t) = \sum_{\|\alpha\|=j}\frac{t^\alpha}{\alpha!}$$

with

$$\alpha=(\alpha_1,\alpha_3,\dots),\qquad \|\alpha\|=\sum_{i=1}^{\infty} i\alpha_i,\qquad \alpha!=\alpha_1!\cdot\alpha_3!\dots,\qquad t^{\alpha}=t_1^{\alpha_1}t_3^{\alpha_3}\dots,$$

whose generating function in the general case is given by

$$\exp\Bigl\{\sum_{i=1}^{\infty} t_i\lambda^i\Bigr\}=\sum_{j=0}^{\infty} p_j(t)\lambda^j.$$

Comparing coefficients of powers of y, we rewrite (4.3) in the form

$$\frac{D^{\gamma}}{\gamma!}\bar\tau(t)\circ\bar\tau(t)=\sum_{\alpha+\beta=\gamma}\frac{(-2)^{|\alpha|}}{\alpha!\,\beta!}\,p_{\|\alpha\|}(2\widetilde{D})D^{\beta}\bar\tau(t)\circ\bar\tau(t), \tag{4.4a}$$

$$\begin{aligned}\frac{D^{\gamma}}{\gamma!}\bigl(\bar\rho(t)\circ\bar\sigma(t)-\bar\sigma(t)\circ\bar\rho(t)\bigr)={}&\\ =\sum_{\alpha+\beta=\gamma}&\frac{(-2)^{|\alpha|}}{\alpha!\,\beta!}\,p_{\|\alpha\|}(2\widetilde{D})D^{\beta}\bar\tau(t)\circ\bar\tau_z(t)-\\ -\sum_{\alpha+\beta+\delta=\gamma}&\frac{(-2)^{|\alpha|}}{\alpha!\,\beta!\,\delta!}\bigl(\partial_y^{\delta}\partial_z\log\bar\tau(t-y)\big|_{y=0}\bigr)p_{\|\alpha\|}(2\widetilde{D})D^{\beta}\bar\tau(t)\circ\bar\tau(t),\end{aligned} \tag{4.4b}$$

$$\frac{D^{\gamma}}{\gamma!}\bigl(2\bar\tau(t)\circ\bar\sigma(t)-\bar\sigma(t)\circ\bar\tau(t)\bigr)=\sum_{\alpha+\beta=\gamma}\frac{(-2)^{|\alpha|}}{\alpha!\,\beta!}\,p_{\|\alpha\|}(2\widetilde{D})D^{\beta}\bar\sigma(t)\circ\bar\tau(t), \tag{4.4c}$$

$$\frac{D^{\gamma}}{\gamma!}\bigl(2\bar\tau(t)\circ\bar\rho(t)-\bar\rho(t)\circ\bar\tau(t)\bigr)=\sum_{\alpha+\beta=\gamma}\frac{(-2)^{|\alpha|}}{\alpha!\,\beta!}\,p_{\|\alpha\|}(2\widetilde{D})D^{\beta}\bar\rho(t)\circ\bar\tau(t). \tag{4.4d}$$

Remark 2. In the case $\gamma=(1,0,0,\dots)$, the term with $y=(y_1,0,0,\dots)$ in (4.4b) can be written as

$$2D_x\tau_z\circ\tau+D_x\sigma\circ\rho=2D_z\tau_x\circ\tau+D_x\sigma\circ\rho=D_xD_z\tau\circ\tau+D_x\sigma\circ\rho=0.$$

Example 1. We consider the 2d-SKwS-I equation [28], [26], i.e., extended BKP hierarchy (3.1) with $n=3$ and $k=5$. We note that using the definition of $\bar\tau$, we can write

$$D_{\bar t_5}\bar\tau\circ\bar\tau=(D_{\bar t_5}-D_z)\tau\circ\tau|_{z=z-\bar t_k}.$$

As a result, we obtain the Hirota bilinear equations for the 2d-SKwS-I equation (hierarchy (3.1) with $n=3$ and $k=5$)

$$D_xD_z\tau\circ\tau+D_x\sigma\circ\rho=0, \tag{4.5a}$$

$$[D_x^6-5D_x^3D_{t_3}+9D_x(D_{\bar t_5}-D_z)-5D_{t_3}^2]\,\tau\circ\tau=0, \tag{4.5b}$$

$$(D_x^3-D_{t_3})\,\sigma\circ\tau=0, \tag{4.5c}$$

$$(D_x^3-D_{t_3})\,\rho\circ\tau=0, \tag{4.5d}$$

respectively using Eq. (4.4b) for y_1, (4.4a) for y_1y_5, (4.4c) for y_3, and (4.4d) for y_3.

Example 2. We consider the 2d-SKwS-II equation [26], i.e., extended BKP hierarchy (3.1) with $n = 5$ and $k = 3$. We obtain the Hirota bilinear equations

$$D_x D_z \tau \circ \tau + D_x \sigma \circ \rho = 0, \tag{4.6a}$$

$$[D_x^6 - 5D_x^3(D_{\bar{t}_3} - D_z) + 9D_x D_{t_5} - 5(D_{\bar{t}_3} - D_z)^2]\, \tau \circ \tau = 0, \tag{4.6b}$$

$$[D_x^3 - (D_{\bar{t}_3} - D_z)]\, \sigma \circ \tau = 0, \tag{4.6c}$$

$$[D_x^3 - (D_{\bar{t}_3} - D_z)]\, \rho \circ \tau = 0, \tag{4.6d}$$

$$[D_x^5 + 5D_x^2(D_{\bar{t}_3} - D_z) - 6D_{t_5}]\, \sigma \circ \tau = 0, \tag{4.6e}$$

$$[D_x^5 + 5D_x^2(D_{\bar{t}_3} - D_z) - 6D_{t_5}]\, \rho \circ \tau = 0, \tag{4.6f}$$

respectively using Eq. (4.4b) for y_1, (4.4a) for $y_1 y_5$, (4.4c) for y_3, (4.4d) for y_3, (4.4c) for y_5, and (4.4d) for y_5.

5. Back to nonlinear equations from Hirota bilinear equations

Following the strategy given in [35], we can convert the bilinear equations back into nonlinear PDEs. We consider the easily proved identities

$$\begin{aligned} &\exp\Bigl(\sum_i \delta_i D_i\Bigr)\rho \circ \tau = \exp\Bigl(2\cosh\Bigl(\sum_i \delta_i \partial_i\Bigr)\log\tau\Bigr)\exp\Bigl(\sum_i \delta_i \partial_i\Bigr)\Bigl(\frac{\rho}{\tau}\Bigr), \\ &\cosh\Bigl(\sum_i \delta_i D_i\Bigr)\tau \circ \tau = \exp\Bigl(2\cosh\Bigl(\sum_i \delta_i \partial_i\Bigr)\log\tau\Bigr). \end{aligned} \tag{5.1}$$

Using the transformations $u := 2\partial_x^2 \log\tau$, $r := \rho/\tau$, and $q := \sigma/\tau$ and expanding (5.1) with respect to δ, we can obtain a relation between the Hirota bilinear equations and the usual nonlinear form. We present two examples below.

Example 3. Hirota bilinear equations (4.5) can be converted back into nonlinear PDEs:

$$\begin{aligned} &\partial^{-1}\partial_z u + q_x r - q r_x = 0, \\ &u^{(5)} + 15u_x u^{(2)} + 15uu^{(3)} + 45u^2 u_x - 5u_{t_3}^{(2)} - 15u_x(\partial^{-1}u_{t_3}) - \\ &\qquad\qquad - 15uu_{t_3} + 9u_{\bar{t}_5} - 9u_z - 5\partial^{-1}u_{t_3 t_3} = 0, \\ &q_{t_3} = q_{xxx} + 3uq_x, \qquad r_{t_3} = r_{xxx} + 3ur_x. \end{aligned}$$

Eliminating the variable z from these equations, we obtain the 2d-SKwS-I equation [28], [26]:

$$\begin{aligned} &u_{\bar{t}_5} + \frac{1}{9}u^{(5)} + \frac{5}{3}u_x u^{(2)} + \frac{5}{3}uu^{(3)} + 5u^2 u_x - \frac{5}{9}u_{t_3}^{(2)} - \frac{5}{3}u_x(\partial^{-1}u_{t_3}) - \\ &\qquad\qquad - \frac{5}{3}uu_{t_3} - \frac{5}{9}\partial^{-1}u_{t_3 t_3} + q_{xx}r - qr_{xx} = 0, \\ &q_{t_3} = q_{xxx} + 3uq_x, \qquad r_{t_3} = r_{xxx} + 3ur_x. \end{aligned}$$

The Hirota bilinear equations for the 2d-SKwS-I equation are given by (4.5), which after z is eliminated from (4.5a) and (4.5b) coincide with the form obtained in [28].

Example 4. Hirota bilinear equations (4.6) can be converted back into nonlinear PDEs:

$$\begin{aligned}
&\partial^{-1}\partial_z u + q_x r - q r_x = 0,\\
&u^{(5)} + 15u_x u^{(2)} + 15uu^{(3)} + 45u^2 u_x - 5u^{(2)}_{\bar{t}_3} - 15u_x(\partial^{-1}u_{\bar{t}_3}) - 15uu_{\bar{t}_3} +\\
&\qquad\qquad + 9u_{t_5} + 5u^{(2)}_z + 15u_x\partial^{-1}u_z + 15uu_z - 5\partial^{-1}(\partial_{\bar{t}_3} - \partial_z)^2 u = 0,\\
&q_{\bar{t}_3} - q_z = q_{xxx} + 3uq_x, \qquad r_{\bar{t}_3} - r_z = r_{xxx} + 3ur_x,\\
&q^{(5)} + 10uq^{(3)} + 5(u_{xx} + 3u^2)q_x + 5(q_{\bar{t}_3} - q_z)^{(2)} + 5u(q_{\bar{t}_3} - q_z) +\\
&\qquad\qquad\qquad\qquad + 10q_x\partial^{-1}u_{\bar{t}_3} - 10q_x\partial^{-1}u_z - 6q_{t_5} = 0,\\
&r^{(5)} + 10ur^{(3)} + 5(u_{xx} + 3u^2)r_x + 5(r_{\bar{t}_3} - r_z)^{(2)} + 5u(r_{\bar{t}_3} - r_z) +\\
&\qquad\qquad\qquad\qquad + 10r_x\partial^{-1}u_{\bar{t}_3} - 10r_x\partial^{-1}u_z - 6r_{t_5} = 0.
\end{aligned}$$

Eliminating z and the function $q_{\bar{t}_3}$ and $r_{\bar{t}_3}$, we obtain 2d-SKwS-II equation [26]:

$$\begin{aligned}
&u_{t_5} + \frac{1}{9}u^{(5)} + \frac{5}{3}u_x u^{(2)} + \frac{5}{3}uu^{(3)} + 5u^2 u_x - \frac{5}{9}u^{(2)}_{\bar{t}_3} -\\
&\qquad - \frac{5}{3}u_x(\partial^{-1}u_{\bar{t}_3}) - \frac{5}{3}uu_{\bar{t}_3} - \frac{5}{9}\partial^{-1}u_{\bar{t}_3\bar{t}_3} =\\
&\qquad\qquad = \frac{1}{9}[10q^{(4)}r + 5q^{(3)}r_x - 5q_x r^{(3)} - 10qr^{(4)} + 5(q_x r - qr_x)_{\bar{t}_3} +\\
&\qquad\qquad\quad + 30u(q^{(2)}r - qr^{(2)}) + 30u_x(q_x r - qr_x)],\\
&q_{t_5} = q^{(5)} + 5uq^{(3)} + 5u'q^{(2)} + \left[\frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3}\partial^{-1}u_{\bar{t}_3} + \frac{5}{3}(q_x r - qr_x)\right]q_x,\\
&r_{t_5} = r^{(5)} + 5ur^{(3)} + 5u'r^{(2)} + \left[\frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3}\partial^{-1}u_{\bar{t}_3} + \frac{5}{3}(q_x r - qr_x)\right]r_x.
\end{aligned} \tag{5.2}$$

The Hirota bilinear equations for the 2d-SKwS-II equation are given by (4.6). To the best of our knowledge, Hirota bilinear form (4.6) for (5.2) has not appeared in the literature previously.

6. Conclusion and discussion

Introducing an auxiliary ∂_z flow, we have constructed bilinear identities (3.3) for extended BKP hierarchy (3.1), which was introduced in [26]. Bilinear identities (3.3) are used to generate all the Hirota bilinear equations for extended BKP hierarchy (3.1). As examples, we obtained the bilinear forms for the 2d-SKwS-I and 2d-SKwS-II equations. The correctness of the construction of these bilinear forms was confirmed by converting the bilinear equations back into the nonlinear PDEs. The Hirota bilinear equations for the 2d-SKwS-II equation were given in an explicit form, which has not been described in the literature.

A very interesting problem is to consider quasiperiodic solutions for the extended BKP hierarchy, because we have already obtained its bilinear identities. Another interesting problem is to consider the bilinear identities for some other extended hierarchies such as the two-dimensional Toda chain and the discrete KP hierarchy as well as others. We plan to study these problems in the future.

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