Journal of Nonlinear Mathematical Physics, Vol. 20, No. 2 (June 2013), 214-228

Bilinear Identities and Hirota's Bilinear Forms for an Extended Kadomtsev-Petviashvili Hierarchy

Runliang Lin[1], Xiaojun Liu[2*], and Yunbo Zeng[1]

[1] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China*

[2] *Department of Applied Mathematics, China Agricultural University, Beijing 100083, P.R. China*

rlin@math.tsinghua.edu.cn, xjliu@cau.edu.cn, yzeng@math.tsinghua.edu.cn

Received 1 November 2012
Accepted 5 March 2013

In this paper, we construct the bilinear identities for the wave functions of an extended Kadomtsev-Petviashvili (KP) hierarchy, which is the KP hierarchy with particular extended flows. By introducing an auxiliary parameter, whose flow corresponds to the so-called squared eigenfunction symmetry of KP hierarchy, we find the tau-function for this extended KP hierarchy. It is shown that the bilinear identities will generate all the Hirota's bilinear equations for the zero-curvature forms of the extended KP hierarchy, which includes two types of KP equation with self-consistent sources (KPSCS). The Hirota's bilinear equations obtained in this paper for the KPSCS are in different forms by comparing with the existing results.

Keywords: KP hierarchy; bilinear identity; Hirota's bilinear form.

2010 Mathematics Subject Classification: 37K10

1. Introduction

Sato theory has fundamental importance in the study of integrable systems (see [6] and references therein). It reveals the infinite dimensional Grassmannian structure of space of τ-functions, where the τ-functions are solutions for the Hirota's bilinear form of Kadomtsev-Petviashvili (KP) hierarchy. The key point to this important discovery is a bilinear residue identity for wave functions called *bilinear identity*. Bilinear identity plays an important role in the proof of existence for τ-function. It also serves as the generating function of the Hirota's bilinear equations for KP hierarchy [4, 29, 39]. In this paper, we will construct the bilinear identity for an extended KP hierarchy introduced in [26].

The study of integrable generalization is an important subject in the study of integrable systems. Several approaches for the generalizations have been developed, e.g., constructing new flows to *extend* original systems. There are many ways to introduce new flows to make a new compatible integrable system. In [42], the KP hierarchy is extended by properly combining additional flows. In [17], extension of KP hierarchy is formulated by introducing fractional order pseudo-differential operators. In [8, 9], Dimakis and Muller-Hoissen extended the Moyal-deformed hierarchies by including additional evolution equations with respect to the deformation parameters. In [3],

*Corresponding author.

Carlet, Dubrovin and Zhang defined a logarithm of the difference Lax operator and got the extended $(2+1)$D Toda lattice hierarchy. Later, the Hirota's bilinear formalism and the relations of extended $(2+1)$D Toda lattice hierarchy and extended 1D Toda lattice hierarchy have been studied [35, 40]. In [20], Li, *et al.*, studied the τ-functions and bilinear identities for the extended bi-graded Toda lattice hierarchy.

Recently, we proposed a different and general approach to extend $(2+1)$-dimensional integrable hierarchies by using the symmetries generating functions (or the squared eigenfunction symmetries) [26]. This kind of extended KP hierarchy can be thought as the generalizations of the KP hierarchy by squared eigenfunction symmetries (or *ghost flows*) [1, 36–38]. The reason for constructing such kind of extended KP hierarchy is that it includes two types of KP equation with self-consistent sources (KPSCS-I and KPSCS-II), which has important applications in hydrodynamics, plasma physics and solid state physics (see, e.g., [10, 11, 15, 21, 22, 31–34, 44, 45]). The extended KP hierarchy also includes the well-known k-constrained KP hierarchy [5, 12, 18, 19] and some $(1+1)$-dimensional soliton equations with sources as reductions. It has been shown that many $(2+1)$-dimensional integrable systems can be extended in this way [16, 23, 25, 30, 43]. We also proposed a generalized dressing approach to construct *Wronskian* type solutions (including soliton solutions) for the extended hierarchies [16, 24, 27, 28, 43].

KPSCS-I and KPSCS-II can also be written in Hirota bilinear forms. In [14, 41], Hu, Wang, *et al.*, proposed a *source generation procedure* to construct the Hirota bilinear form for KPSCS (I and II) on the basis of the bilinear form for the original KP equation.

However, it turns out that it is not an easy task to find Hirota bilinear form for KPSCS (I and II). More generally, it remains unsolved to give the Hirota bilinear form for each zero-curvature equation in the extended KP hierarchy. The most natural way to solve this problem is to consider the bilinear identities of the extended KP hierarchy. Because of the importance of the bilinear identities in Sato theory, the existence of bilinear identities could be an important open question for the extended KP hierarchy.

In this paper, we will construct the bilinear identities for the extended KP hierarchy. Moreover, it will serve as a generating function of all the Hirota bilinear forms for the zero-curvature equations in the extended KP hierarchy. In particular, it generates Hirota bilinear forms for KPSCS (both type I and II). The squared eigenfunction symmetry plays a crucial role in our construction. It seems that the Hirota's bilinear equations obtained in this paper for KPSCS are simpler comparing with the results by Hu and Wang [15].

This paper is organized as follows. In Section 2, the bilinear identities for KP hierarchy with a squared eigenfunction symmetry are constructed. In Section 3, by making the squared eigenfunction symmetry as an auxiliary flow, we construct the bilinear identities for the extended KP hierarchy. In addition, we prove that any wave function satisfying bilinear identities will be a wave function for the extended KP hierarchy. In Section 4, we find the τ-function for the extended KP hierarchy. We also find the generation functions for Hirota bilinear form for the extended KP hierarchy. Subsequently, several examples including the Hirota bilinear form for the KPSCS (I and II) and a higher order system in the hierarchy are given. In Section 5, we show how to go back from Hirota bilinear forms to nonlinear partial differential equations (PDEs) for KPSCS, which verifies the correctness of our construction. In the last section, we will give conclusion and remarks, and discuss some problems for further exploration.

2. Bilinear Identities for KP hierarchy with Squared Eigenfunction Symmetry

Consider the system given in [38], defined by assuming that $L = \partial + \sum_{i=1}^{\infty} u_i \partial^{-i}$ satisfies both the ordinary KP flows

$$\partial_{t_n} L = [L_+^n, L], \tag{2.1a}$$

and a new ∂_z-flow given by

$$\partial_z L = [q\partial^{-1} r, L]. \tag{2.1b}$$

Here the Lax operator L is a pseudo-differential operator (see [7] for a detailed introduction). The "$+$" sign in subscript part of L_+^n indicates the projection to the non-negative part of L^n with respect to the power of ∂. There are a series of time variables t_n, and we can set $x = t_1$ according to (2.1a).

A sufficient condition for the commutativity of ∂_z- and ∂_{t_n}-flow is obtained by imposing q and r to be an eigenfunction and an adjoint eigenfunction of KP hierarchy, respectively. Namely, we should assume

$$\partial_{t_n} q = L_+^n(q), \tag{2.1c}$$

$$\partial_{t_n} r = -(L_+^n)^*(r), \tag{2.1d}$$

for all $n \in \mathbb{N}$. The ∂_z-flow describes a symmetry for KP hierarchy, which is called the *squared eigenfunction symmetry* [38] or a *ghost flow* [1,2].

It is well-known that dressing operator $W = \sum_{i\geqslant 0} w_i \partial^{-i}$ ($w_0 = 1$) plays a very important role in KP theory. Here w_i ($i > 0$) are functions of all t_n and z. Suppose that the t_n-evolution equations for W are given by

$$\partial_{t_n} W = -\left(W\partial^n W^{-1}\right)_- W, \quad (n \in \mathbb{N}). \tag{2.2}$$

Then Lax operator L is related to the dressing operator W by $L = W\partial W^{-1}$. We define the wave and adjoint wave function as

$$w(z,\mathbf{t},\lambda) = We^{\xi(\mathbf{t},\lambda)}, \tag{2.3a}$$

$$w^*(z,\mathbf{t},\lambda) = (W^*)^{-1} e^{-\xi(\mathbf{t},\lambda)} \tag{2.3b}$$

where $\xi(\mathbf{t},\lambda) = \sum_{i\geqslant 1} t_i \lambda^i$ and $\mathbf{t} = (t_1, t_2, \ldots)$. At this moment, we temporally ignore the parameter z. Then the original KP theory states that a residue identity, called *bilinear identity* (2.4), holds for (adjoint) wave functions:

$$\mathrm{Res}_\lambda\, w(z,\mathbf{t},\lambda)\, w^*(z,\mathbf{t}',\lambda) = 0, \tag{2.4}$$

where $\mathbf{t}' = (t_1', t_2', \ldots)$ and the residue of λ can be simply considered as the coefficient of λ^{-1} in the Laurent expansion. Since we have ignored the parameter z in w and w^*, the proof of (2.4) is the same as the original one given in [6]. Notice that the parameter z takes the same value for w and w^* in (2.4). The key step in the proof of (2.4) is an important lemma. We recall it here [7]:

Lemma 2.1.

$$\mathrm{Res}_\partial\, P \cdot Q^* = \mathrm{Res}_\lambda\, P(e^{\xi(\mathbf{t},\lambda)}) \cdot Q(e^{-\xi(\mathbf{t},\lambda)}), \tag{2.5}$$

where P and Q are pseudo-differential operators. The residue with respect to ∂ on the left hand side of (2.5) *is defined as the coefficient of ∂^{-1} for a pseudo-differential operator.*

Actually, it is not difficult to prove that the bilinear identity (2.4) is equivalent to the KP hierarchy (2.1a) (see [7] for the proof). Namely, if we start with (2.1a) or (2.2), we can prove that wave functions defined by (2.3) will satisfy the bilinear identity (2.4). Conversely, if we have wave and adjoint wave functions satisfying bilinear identity (2.4), then we can immediately know that the dressing operator W corresponding to the wave function w will satisfy (2.2), which makes the operator L defined by dressing form $W\partial W^{-1}$ as a Lax operator for KP hierarchy.

Then, if we consider KP hierarchy with squared eigenfunction symmetry, we can see that the dressing operator should satisfy another equation:

$$\partial_z W = q\partial^{-1} rW. \tag{2.6}$$

In this case, we have:

Theorem 2.1. *The KP hierarchy with a squared eigenfunction symmetry* (2.1) *is equivalent to the following residue identities*

$$\mathrm{Res}_\lambda\ w(z,\mathbf{t},\lambda)\ \cdot\ w^*(z,\mathbf{t}',\lambda) = 0, \tag{2.7a}$$
$$\mathrm{Res}_\lambda\ w_z(z,\mathbf{t},\lambda)\ \cdot\ w^*(z,\mathbf{t}',\lambda) = q(z,\mathbf{t})r(z,\mathbf{t}'), \tag{2.7b}$$
$$\mathrm{Res}_\lambda\ w(z,\mathbf{t},\lambda)\ \cdot\ \partial^{-1}\left(q(z,\mathbf{t}')w^*(z,\mathbf{t}',\lambda)\right) = -q(z,\mathbf{t}), \tag{2.7c}$$
$$\mathrm{Res}_\lambda\ \partial^{-1}\left(r(z,\mathbf{t})w(z,\mathbf{t},\lambda)\right)\ \cdot\ w^*(z,\mathbf{t}',\lambda) = r(z,\mathbf{t}'), \tag{2.7d}$$

where the inverse of ∂ is understood as pseudo-differential operator acting on an exponential function, e.g., $\partial^{-1}(rw) = (\partial^{-1}rW)(e^{\xi(\mathbf{t},\lambda)})$.

Proof. Frist, we will prove the residue identities (2.7) from (2.1), (2.2) and (2.6). Equation (2.7a) is easy to prove, since L satisfies the evolution equations of original KP hierarchy.

To prove (2.7b), we only need to show

$$\mathrm{Res}_\lambda\ w_z(z,\mathbf{t},\lambda)\cdot(\partial_{t_1}^{m_1}\cdots\partial_{t_k}^{m_k}w^*(z,\mathbf{t},\lambda)) = q(z,\mathbf{t})\,\partial_{t_1}^{m_1}\cdots\partial_{t_k}^{m_k}r(z,\mathbf{t})$$

for arbitrary $k \geqslant 1$ and $m_i \geqslant 0$. Here we have a key observation: the action of mixed partial derivatives $\partial_{t_1}^{m_1}\cdots\partial_{t_k}^{m_k}$ on w^* can be written as $P_{m_1\cdots m_k}w^*(z,\mathbf{t},\lambda)$, and

$$\partial_{t_1}^{m_1}\cdots\partial_{t_k}^{m_k}r(z,\mathbf{t}) = P_{m_1\cdots m_k}\left(r(z,\mathbf{t})\right),$$

where $P_{m_1\cdots m_k}$ is a differential operator in ∂. Then by noticing $w_z(z,\mathbf{t},\lambda) = q\partial^{-1}rw(z,\mathbf{t},\lambda)$ and Lemma 2.1, we have

$$\begin{aligned}
&\mathrm{Res}_\lambda\ w_z(z,\mathbf{t},\lambda)\,\partial_{t_1}^{m_1}\cdots\partial_{t_k}^{m_k}w^*(z,\mathbf{t},\lambda)\\
=&\mathrm{Res}_\lambda\ q(z,\mathbf{t})\partial^{-1}r(z,\mathbf{t})w(z,\mathbf{t},\lambda)\cdot P_{m_1\cdots m_k}w^*(z,\mathbf{t},\lambda)\\
=&\mathrm{Res}_\partial\ q(z,\mathbf{t})\partial^{-1}r(z,\mathbf{t})P^*_{m_1\cdots m_k}\\
=&q(z,\mathbf{t})P_{m_1\cdots m_k}(r(z,\mathbf{t})).
\end{aligned}$$

Analogously, we have another residue identity:

$$\mathrm{Res}_\lambda\ w(z,\mathbf{t},\lambda)\,w_z^*(z,\mathbf{t}',\lambda) = q(z,\mathbf{t})r(z,\mathbf{t}'). \tag{2.8}$$

However this equation is equivalent to (2.7b), so this equation is not included in (2.7).

The residue identities (2.7c) and (2.7d) can be easily derived from (2.7b) (and (2.8)) if one substitutes $w_z = q\partial^{-1}rw$ and $w_z^* = -r\partial^{-1}qw^*$ into (2.7b) (or (2.8)) and eliminate q or r.

The proof for the reverse part in this theorem is written as the following proposition. $\square$

Proposition 2.1. *If there are functions $q(z,\mathbf{t})$, $r(z,\mathbf{t})$,*

$$w(z,\mathbf{t},\lambda) = \left(1+\sum_{i\geqslant 1} w_i\lambda^{-i}\right)e^{\xi(\mathbf{t},\lambda)} \quad \text{and} \quad w^*(z,\mathbf{t},\lambda) = \left(1+\sum_{i\geqslant 1} w_i^*\lambda^{-i}\right)e^{-\xi(\mathbf{t},\lambda)}$$

satisfying the identities (2.7)*, then there exists a pseudo-differential operator W and Lax operator $L := W\partial W^{-1}$ such that L, q and r satisfy* (2.1).

Proof. Supposing $W = 1+\sum_{i\geqslant 1} w_i\partial^{-i}$, $\widetilde{W} = 1+\sum_{i\geqslant 1} w_i^*\partial^{-i}$, it is easy to see $w(z,\mathbf{t},\lambda) = W(e^{\xi(\mathbf{t},\lambda)})$ and $w^*(z,\mathbf{t},\lambda) = \widetilde{W}(e^{-\xi(\mathbf{t},\lambda)})$. Then from (2.7a) and Lemma 2.1, we know the ∂-residue $\mathrm{Res}_\partial\, W\widetilde{W}^*\partial^m$ vanishes for $m\geqslant 1$, which implies the negative part $(W\widetilde{W}^*)_-$ is 0. Furthermore, the non-negative part of $W\widetilde{W}^*$ is 1. So we have proved that $W\widetilde{W}^* = 1$, which means $\widetilde{W} = (W^*)^{-1}$.

Now, let's define $L := W\partial W^{-1}$. We will prove $W_{t_n} = -L_-^n W$, which implies (2.1a). From the definition form of W, it is easy to see that $(W_{t_n} + L_-^n W)_+ = 0$. Again from (2.7a) we have

$$\begin{aligned}
&\mathrm{Res}_\partial(W_{t_n}W^{-1}+L_-^n)_-\partial^m = \mathrm{Res}_\partial(W_{t_n}W^{-1}+(W\partial^nW^{-1})_-)_-\partial^m\\
&= \mathrm{Res}_\lambda W_{t_n}\mathrm{e}^{\xi}\cdot(-\partial)^m(W^*)^{-1}\mathrm{e}^{-\xi} + \mathrm{Res}_\partial(W\partial^nW^{-1}-(W\partial^nW^{-1})_+)\partial^m\\
&= \mathrm{Res}_\lambda W_{t_n}\mathrm{e}^{\xi}\cdot(-\partial)^m(W^*)^{-1}\mathrm{e}^{-\xi} + \mathrm{Res}_\lambda W\partial^n\mathrm{e}^{\xi}\cdot(-\partial)^m W^{*-1}\mathrm{e}^{-\xi}\\
&= \mathrm{Res}_\lambda w_{t_n}(z,\mathbf{t},\lambda)(-\partial)^m w^*(z,\mathbf{t},\lambda) = 0, \quad (\text{for } m>0)
\end{aligned}$$

so we obtain $W_{t_n} = -L_-^n W$.

Next, we need to prove $W_z = q\partial^{-1}rW$, which implies (2.1b). It is easy to see that $(W_zW^{-1})_+ = 0$, and by using (2.7b) we have

$$\begin{aligned}
&\mathrm{Res}_\partial W_zW^{-1}\partial^m = \mathrm{Res}_\lambda W_z\mathrm{e}^{\xi}\cdot(-\partial)^m W^{*-1}\mathrm{e}^{-\xi}\\
&= \mathrm{Res}_\lambda w_z(z,\mathbf{t},\lambda)\cdot(-\partial)^m w^*(z,\mathbf{t},\lambda) = q(-\partial)^m r \quad (\text{for } m>0),
\end{aligned}$$

which means $W_zW^{-1} = \sum_{m=0}^{\infty} q(-\partial)^m(r)\partial^{-m-1} = q\partial^{-1}r$, namely, $W_z = q\partial^{-1}rW$.

As the final step, we will prove (2.1c) and (2.1d). We have already proved $W_{t_n} = -L_-^n W$, which implies $w_{t_n} = L_+^n(w)$. Then according to (2.7c), we find

$$\begin{aligned}
q_{t_n}(z,\mathbf{t}) &= -\mathrm{Res}_\lambda\; w_{t_n}(z,\mathbf{t},\lambda)\;\cdot\partial^{-1}\left(q(z,\mathbf{t}')w^*(z,\mathbf{t}',\lambda)\right)\\
&= L_+^n\left(-\mathrm{Res}_\lambda w(z,\mathbf{t},\lambda)\;\cdot\partial^{-1}\left(q(z,\mathbf{t}')w^*(z,\mathbf{t}',\lambda)\right)\right)\\
&= L_+^n(q(z,\mathbf{t})).
\end{aligned}$$

In a similar way, we can find (2.1d) by using (2.7d). $\square$

3. Bilinear Identity for the Extended KP Hierarchy

In [26], we defined a new extended KP hierarchy. The key idea is to combine a specific k-th order flow of KP hierarchy with the squared eigenfunction symmetry flow like (2.1b). In this way, we found two types of KP hierarchy with self-consistent sources and their Lax representations.

In [14, 15, 41], the Hirota's bilinear equations for some $(2+1)$D integrable equations with sources are constructed by *source generation procedure*. For the case of KP equation, they found that these Hirota's bilinear equations correspond to the first and second type of KP equation with self-consistent sources.

It is well-known that the bilinear identities are generating equations for Hirota bilinear equations of the KP hierarchy. The natural questions are how to find the bilinear identities for the extended KP hierarchy and how to derive Hirota's bilinear equations from them. So in this section, we will give a detailed construction on how to derive the bilinear identities for the extended KP hierarchy (3.1).

We remind that the extended KP hierarchy (for a fixed k)

$$\partial_{t_n} L = [L^n_+, L] \quad (n \neq k) \tag{3.1a}$$

$$\partial_{\bar{t}_k} L = [L^k_+ + q\partial^{-1} r, L] \tag{3.1b}$$

$$\partial_{t_n} q = L^n_+(q) \tag{3.1c}$$

$$\partial_{t_n} r = -(L^n_+)^*(r) \tag{3.1d}$$

is constructed by replacing an arbitrary fixed k-th flow ∂_{t_k} by $\partial_{\bar{t}_k}$ where the $\bar{t}_k$-flow is a combination of ∂_{t_k}- and ∂_z-flow given by (3.1b). Functions q and r are eigenfunction and adjoint eigenfunction satisfying (3.1c) and (3.1d), respectively. Note that q and r depend on $\bar{t}_k$ rather than t_k in the extendeded KP hierarchy (3.1).

Remark 3.1. In order to simplify the notions, we will still use the symbols $w(z,\mathbf{t},\lambda)$, $w^*(z,\mathbf{t},\lambda)$, $q(z,\mathbf{t})$, $r(z,\mathbf{t})$, L, W, etc., in this and the following sections, but they correspond to the case of the extended KP hierarchy (3.1), for example, from now on, $\mathbf{t} = (t_1, t_2, \ldots, t_{k-1}, \bar{t}_k, t_{k+1}, \ldots)$.

To find the bilinear identity for the extended KP hierarchy (3.1), the key idea is supposing that L depends on the auxiliary variable z, whose evolution is given by (2.1b).

Then we assume that the dressing operator for L with auxiliary variable z is given by $W = 1 + \sum_{i\geqslant 1} w_i(z,\mathbf{t})\partial^{-i}$. And the evolution of W with respect to z and $\bar{t}_k$ are given by (2.2) and

$$\partial_{\bar{t}_k} W = -L^k_- W + q\partial^{-1} r W. \tag{3.2}$$

The definition forms for *wave function* and *adjoint wave function* are the same as (2.3) except for the functions depending on $\bar{t}_k$ rather than t_k, e.g., $\xi(\mathbf{t},\lambda) = \bar{t}_k\lambda^k + \sum_{i\neq k} t_i\lambda^i$.

Then we can prove the following theorem.

Theorem 3.1. *The bilinear identity for the extended KP hierarchy* (3.1) *is given by the following sets of residue identities with auxiliary variable z:*

$$\mathrm{Res}_\lambda\, w(z-\bar{t}_k,\mathbf{t},\lambda)\, \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = 0, \tag{3.3a}$$

$$\mathrm{Res}_\lambda\, w_z(z-\bar{t}_k,\mathbf{t},\lambda)\, \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = q(z-\bar{t}_k,\mathbf{t}) r(z-\bar{t}'_k,\mathbf{t}'), \tag{3.3b}$$

$$\mathrm{Res}_\lambda\, w(z-\bar{t}_k,\mathbf{t},\lambda)\, \cdot \partial^{-1}\left(q(z-\bar{t}'_k,\mathbf{t}') w^*(z-\bar{t}'_k,\mathbf{t}',\lambda)\right) = -q(z-\bar{t}_k,\mathbf{t}), \tag{3.3c}$$

$$\mathrm{Res}_\lambda\, \partial^{-1}\left(r(z-\bar{t}_k,\mathbf{t}) w(z-\bar{t}_k,\mathbf{t},\lambda)\right)\, \cdot w^*(z-\bar{t}'_k,\mathbf{t}',\lambda) = r(z-\bar{t}'_k,\mathbf{t}'), \tag{3.3d}$$

where the inverse of ∂ is understood as pseudo-differential operator acting on an exponential function, e.g., $\partial^{-1}(rw) = (\partial^{-1} rW)(e^\xi)$.

Proof. We notice that $\frac{d}{d\bar{t}_k} w(z-\bar{t}_k,\mathbf{t},\lambda) = w_{\bar{t}_k}(z-\bar{t}_k,\mathbf{t},\lambda) - w_z(z-\bar{t}_k,\mathbf{t},\lambda) = L^k_+(w)$. The λ-residue of $[\frac{d}{d\bar{t}_k} w(z-\bar{t}_k,\mathbf{t},\lambda)] w^*(z-\bar{t}_k,\mathbf{t},\lambda)$ simply vanishes according to Lemma 2.1. The same is true for the λ-residues of arbitrary mixed derivatives $[\frac{d^{\sum m_i}}{dt_1^{m_1}dt_2^{m_2}\cdots}w(z-\bar{t}_k,\mathbf{t},\lambda)]w^*(z-\bar{t}_k,\mathbf{t},\lambda)$. Then identity (3.3a) holds.

The proofs of (3.3b)-(3.3d) are almost the same as the proofs of (2.7b)-(2.7d). One should notice again that $\frac{d}{d\bar{t}_k}w(z-\bar{t}_k,\mathbf{t},\lambda)=L_+^k(w)$. □

Theorem 3.2. *Suppose that there are (adjoint) wave functions*

$$w(z,\mathbf{t},\lambda)=\left(\sum_{i\geqslant 0}w_i(z,\mathbf{t})\lambda^{-i}\right)e^{\xi(\mathbf{t},\lambda)},\quad w^*(z,\mathbf{t},\lambda)=\left(\sum_{i\geqslant 0}\tilde{w}_i(z,\mathbf{t})\lambda^{-i}\right)e^{-\xi(\mathbf{t},\lambda)},$$

(with $w_0=\tilde{w}_0=1$), $q(z,\mathbf{t})$ and $r(z,\mathbf{t})$ satisfying the bilinear relations (3.3), then the pseudo-differential operator $L=W\partial W^{-1}$, $W=\sum_{i\geqslant 0}w_i(z,\mathbf{t})\partial^{-i}$, and functions q and r give a solution to the extended KP hierarchy (3.1).

Proof. Let us define $\tilde{W}:=\sum_{i\geqslant 0}\tilde{w}_i(z,\mathbf{t})\partial^{-i}$. With the same argument as in Theorem 2.1, we will find that $\tilde{W}=(W^{-1})^*$. Then from (3.3b) and Lemma 2.1, we find that for any $m>0$,

$$\mathrm{Res}_\partial W_zW^{-1}(-\partial)^m=q\partial^m(r),$$

which means $(W_zW^{-1})_-=q\partial^{-1}r$. Notice that the non-negative part of W_zW^{-1} vanishes, so we have $W_z=q\partial^{-1}rW$.

Next, from the coefficient of Taylor expansion of (3.3a), we find

$$\mathrm{Res}_\lambda\frac{d}{d\bar{t}_k}w(z-\bar{t}_k,\mathbf{t},\lambda)\cdot\partial^m w^*(z-\bar{t}_k,\mathbf{t},\lambda)=0.$$

By realizing that $\frac{d}{d\bar{t}_k}w(z-\bar{t}_k,\mathbf{t},\lambda)$ is $(\frac{d}{d\bar{t}_k}W(z-\bar{t}_k,\mathbf{t})+L^kW)\exp(\xi(\mathbf{t},\lambda))$ and using Lemma 2.1, we find

$$\mathrm{Res}_\partial(\frac{d}{d\bar{t}_k}W(z-\bar{t}_k,\mathbf{t})+L_-^kW)W^{-1}(-\partial)^m=0,\quad(\text{for } m>0)$$

which means $\frac{d}{d\bar{t}_k}W(z-\bar{t}_k,\mathbf{t})=-L_-^kW$. Hence $\partial_{\bar{t}_k}W(z,\mathbf{t})=(-L_-^k+q\partial^{-1}r)W$.

For equations (3.1c) and (3.1d), the proof can be done by differentiating (3.3c) and (3.3d) directly. Since $\partial_{t_n}w=L_+^n(w)$, $\partial_{t_n}w^*=-(L_+^n)^*(w^*)$, the rest of proof is obvious. □

4. Tau-Function for the Extended KP Hierarchy

The existence of τ-function for original bilinear identity of KP hierarchy is proved in [6]. In our case, the wave function $w(z-\bar{t}_k,\mathbf{t},\lambda)$ and $w^*(z-\bar{t}_k,\mathbf{t},\lambda)$ satisfy exactly the same bilinear identity (3.3a) as the original KP case if one considers z as an additional parameter. So it is reasonable to assume the existence of τ-function and make the following ansatz:

$$w(z-\bar{t}_k,\mathbf{t},\lambda)=\frac{\tau(z-\bar{t}_k+\frac{1}{k\lambda^k},\mathbf{t}-[\lambda])}{\tau(z-\bar{t}_k,\mathbf{t})}\cdot\exp\xi(\mathbf{t},\lambda),\tag{4.1a}$$

$$w^*(z-\bar{t}_k,\mathbf{t},\lambda)=\frac{\tau(z-\bar{t}_k-\frac{1}{k\lambda^k},\mathbf{t}+[\lambda])}{\tau(z-\bar{t}_k,\mathbf{t})}\cdot\exp(-\xi(\mathbf{t},\lambda)),\tag{4.1b}$$

where $[\lambda] = \left(\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \cdots\right)$. According to [4], we should make further assumptions:

$$q(z,\mathbf{t}) = \frac{\sigma(z,\mathbf{t})}{\tau(z,\mathbf{t})}, \qquad r(z,\mathbf{t}) = \frac{\rho(z,\mathbf{t})}{\tau(z,\mathbf{t})}. \tag{4.1c}$$

Then, similar with [4], we have the following results.

$$\partial^{-1}(r(z-\bar{t}_k,\mathbf{t})w(z-\bar{t}_k,\mathbf{t},\lambda)) = \frac{\rho(z-\bar{t}_k+\frac{1}{k\lambda^k},\mathbf{t}-[\lambda])}{\lambda\tau(z-\bar{t}_k,\mathbf{t})}e^{\xi(\mathbf{t},\lambda)}, \tag{4.1d}$$

$$\partial^{-1}(q(z-\bar{t}_k,\mathbf{t})w^*(z-\bar{t}_k,\mathbf{t},\lambda)) = \frac{-\sigma(z-\bar{t}_k-\frac{1}{k\lambda^k},\mathbf{t}+[\lambda])}{\lambda\tau(z-\bar{t}_k,\mathbf{t})}e^{-\xi(\mathbf{t},\lambda)}. \tag{4.1e}$$

To find Hirota's bilinear equations for extended KP hierarchy (3.1), we substitute (4.1) into (3.3), and get

$$\mathrm{Res}_\lambda\ \bar{\tau}\big(z,\mathbf{t}-[\lambda]\big)\ \bar{\tau}\big(z,\mathbf{t}'+[\lambda]\big)\ e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = 0, \tag{4.2a}$$

$$\begin{aligned}&\mathrm{Res}_\lambda\ \bar{\tau}_z\big(z,\mathbf{t}-[\lambda]\big)\ \bar{\tau}\big(z,\mathbf{t}'+[\lambda]\big)\ e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)}\\ &\quad -\mathrm{Res}_\lambda\ \bar{\tau}\big(z,\mathbf{t}-[\lambda]\big)\ (\partial_z\log\bar{\tau}(z,\mathbf{t}))\ \bar{\tau}\big(z,\mathbf{t}'+[\lambda]\big)\ e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)}\\ &\qquad = \bar{\sigma}(z,\mathbf{t})\ \bar{\rho}(z,\mathbf{t}'),\end{aligned} \tag{4.2b}$$

$$\mathrm{Res}_\lambda\ \lambda^{-1}\bar{\tau}\big(z,\mathbf{t}-[\lambda]\big)\ \bar{\sigma}\big(z,\mathbf{t}'+[\lambda]\big)\ e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = \bar{\sigma}(z,\mathbf{t})\ \bar{\tau}(z,\mathbf{t}'), \tag{4.2c}$$

$$\mathrm{Res}_\lambda\ \lambda^{-1}\bar{\rho}\big(z,\mathbf{t}-[\lambda]\big)\ \bar{\tau}\big(z,\mathbf{t}'+[\lambda]\big)\ e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)} = \bar{\rho}(z,\mathbf{t}')\ \bar{\tau}(z,\mathbf{t}). \tag{4.2d}$$

Here the bar $\bar{}$ over a function $f(z,\mathbf{t})$ is defined as $\bar{f}(z,\mathbf{t}) \equiv f(z-\bar{t}_k,\mathbf{t})$, e.g, $\bar{\tau}\big(z,\mathbf{t}-[\lambda]\big) \equiv \tau\big(z-(\bar{t}_k-\frac{1}{k\lambda^k}),\mathbf{t}-[\lambda]\big)$, $\bar{\tau}\big(z,\mathbf{t}'+[\lambda]\big) \equiv \tau\big(z-(\bar{t}_k+\frac{1}{k\lambda^k}),\mathbf{t}'+[\lambda]\big)$. After setting $\mathbf{t}$ as $\mathbf{t}+\mathbf{y}$ and $\mathbf{t}'$ as $\mathbf{t}-\mathbf{y}$ in (4.2) with $\mathbf{y}=(y_1,y_2,\ldots)$, We can write (4.2) as the following systems with Hirota bilinear derivatives $\tilde{D}$ and D_i's:

$$\sum_{i\geqslant 0} p_i(2\mathbf{y})p_{i+1}(-\tilde{D})\exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\tau}(z,\mathbf{t})\cdot\bar{\tau}(z,\mathbf{t}) = 0, \tag{4.3a}$$

$$\begin{aligned}&\sum_{i\geqslant 0} p_i(2\mathbf{y})p_{i+1}(-\tilde{D})\exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\tau}_z(z,\mathbf{t})\cdot\bar{\tau}(z,\mathbf{t})\\ &\quad -\sum_{i\geqslant 0} p_i(2\mathbf{y})\left(\partial_z\log\bar{\tau}(z,\mathbf{t}+\mathbf{y})\right)p_{i+1}(-\tilde{D})\bar{\tau}(z,\mathbf{t}+\mathbf{y})\cdot\bar{\tau}(z,\mathbf{t}-\mathbf{y})\\ &\qquad = \exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\sigma}(z,\mathbf{t})\cdot\bar{\rho}(z,\mathbf{t})\end{aligned} \tag{4.3b}$$

$$\sum_{i\geqslant 0} p_i(2\mathbf{y})p_i(-\tilde{D})\exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\tau}(z,\mathbf{t})\cdot\bar{\sigma}(z,\mathbf{t}) = \exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\sigma}(z,\mathbf{t})\cdot\bar{\tau}(z,\mathbf{t}), \tag{4.3c}$$

$$\sum_{i\geqslant 0} p_i(2\mathbf{y})p_i(-\tilde{D})\exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\rho}(z,\mathbf{t})\cdot\bar{\tau}(z,\mathbf{t}) = \exp\left(\sum_{j\geqslant 1} y_jD_j\right)\bar{\tau}(z,\mathbf{t})\cdot\bar{\rho}(z,\mathbf{t}), \tag{4.3d}$$

where $\tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \cdots)$, D_i is the well-known Hirota bilinear derivative $D_i f \cdot g = f_{t_i} g - f g_{t_i}$, and $p_i(\mathbf{t})$ is the i-th *Schur polynomial*, whose generating function is given by

$$\exp \sum_{i=1}^{\infty} y_i \lambda^i = \sum_{i=0}^{\infty} p_i(\mathbf{y}) \lambda^i.$$

Remark 4.1. The zero-th order term in (4.3b) (with $y_j = 0\ \forall\ j$) can be written in the following forms with Hirota's operator

$$\sigma\rho + D_x \tau_z \cdot \tau = \sigma\rho + D_z \tau_x \cdot \tau = \sigma\rho + \frac{1}{2} D_x D_z \tau \cdot \tau = 0.$$

Note that (4.3) is the generating equations of Hirota's bilinear equations for the extended KP hierarchy (3.1), where the dependence of auxiliary parameter z can be eliminated if one wants to convert bilinear equations to PDEs. In the following sections we will give several examples to show the Hirota's bilinear forms for the nonlinear evolution equations in extended KP hierarchy and how it can be transformed back to nonlinear PDEs (which will correspond to the well-known KPSCS-I and II, etc.).

Example 4.1 (First type of KP equation with a source (KPSCS-I) [15, 26, 31, 33], i.e., the extended KP hierarchy (3.1) for $n = 2$ and $k = 3$). The Hirota equations for the KPSCS-I can be obtained as

$$\begin{aligned}
&D_x \tau_z \cdot \tau + \sigma\rho = 0, && \text{by (4.3b) with } y_j = 0, && (4.4a)\\
&(D_x^4 + 3D_{t_2}^2 - 4D_x(D_{\bar{t}_3} - D_z))\tau \cdot \tau = 0, && \text{by (4.3a) in } y_3, && (4.4b)\\
&(D_{t_2} + D_x^2)\tau \cdot \sigma = 0, && \text{by (4.3c) in } y_2, && (4.4c)\\
&(D_{t_2} + D_x^2)\rho \cdot \tau = 0, && \text{by (4.3d) in } y_2. && (4.4d)
\end{aligned}$$

where D_z is Hirota's derivative, i.e., $D_z f(z) \cdot g(z) = f_z g - f g_z$. Note that from the definition of $\bar{\tau}$, we know that $D_{\bar{t}_3} \bar{\tau} \cdot \bar{\tau} = (D_{\bar{t}_3} - D_z)\tau \cdot \tau$, which interprets the appearance of this term in the second equation.

Example 4.2 (Second type of KP with a source (KPSCS-II) [15, 26, 31, 33], i.e., the extended KP hierarchy (3.1) for $n = 3$ and $k = 2$). The Hirota equations for the KPSCS-II can be obtained as

$$\begin{aligned}
&D_x \tau_z \cdot \tau + \sigma\rho = 0, && \text{by (4.3b) with } y_j = 0, && (4.5a)\\
&(D_x^4 + 3(D_{\bar{t}_2} - D_z)^2 - 4D_x D_{t_3})\tau \cdot \tau = 0, && \text{by (4.3a) in } y_3, && (4.5b)\\
&((D_{\bar{t}_2} - D_z) + D_x^2)\tau \cdot \sigma = 0, && \text{by (4.3c) in } y_2, && (4.5c)\\
&((D_{\bar{t}_2} - D_z) + D_x^2)\rho \cdot \tau = 0. && \text{by (4.3d) in } y_2, && (4.5d)\\
&(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\tau \cdot \sigma = 0, && \text{by (4.3c) in } y_3, && (4.5e)\\
&(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\rho \cdot \tau = 0, && \text{by (4.3d) in } y_3. && (4.5f)
\end{aligned}$$

It seems that the Hirota bilinear equations (4.5) obtained here for KPSCS-II are simpler than the results by Hu and Wang [15].

Example 4.3 (The extended KP hierarchy (3.1) for $n = 4$ and $k = 2$ [26]). The Hirota bilinear equations for this system can be obtained as

$$D_x\tau_z \cdot \tau + \sigma\rho = 0, \qquad \text{by (4.3b) with } y_j = 0, \quad (4.6a)$$
$$\left(D_x^4 + 3(D_{\bar{t}_2} - D_z)^2 - 4D_xD_{t_3}\right)\tau \cdot \tau = 0, \qquad \text{by (4.3a) in } y_3, \quad (4.6b)$$
$$\left(3D_xD_{t_4} - 2(D_{\bar{t}_2} - D_z)D_{t_3} - D_x^3(D_{\bar{t}_2} - D_z)\right)\tau \cdot \tau = 0, \qquad \text{by (4.3a) in } y_4, \quad (4.6c)$$
$$\left((D_{\bar{t}_2} - D_z) + D_x^2\right)\tau \cdot \sigma = 0, \qquad \text{by (4.3c) in } y_2, \quad (4.6d)$$
$$\left((D_{\bar{t}_2} - D_z) + D_x^2\right)\rho \cdot \tau = 0, \qquad \text{by (4.3d) in } y_2, \quad (4.6e)$$
$$\left(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z)\right)\tau \cdot \sigma = 0, \qquad \text{by (4.3c) in } y_3, \quad (4.6f)$$
$$\left(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z)\right)\rho \cdot \tau = 0, \qquad \text{by (4.3d) in } y_3, \quad (4.6g)$$
$$\left(18D_{t_4} + D_x^4 - 6D_x^2(D_{\bar{t}_2} - D_z) + 3(D_{\bar{t}_2} - D_z)^2 + 8D_xD_{t_3}\right)\tau \cdot \sigma = 0, \qquad \text{by (4.3c) in } y_4, \quad (4.6h)$$
$$\left(18D_{t_4} + D_x^4 - 6D_x^2(D_{\bar{t}_2} - D_z) + 3(D_{\bar{t}_2} - D_z)^2 + 8D_xD_{t_3}\right)\rho \cdot \tau = 0, \qquad \text{by (4.3d) in } y_4. \quad (4.6i)$$

5. Back to Nonlinear Equations from Hirota's Bilinear Equations

To see the nonlinear PDEs corresponding to the Hirota's bilinear equations ((4.4), (4.5) and (4.6)) in previous examples, we convert the bilinear equations back to nonlinear PDEs in this section. It is not as simple as generating special bilinar equations. We use the method given by [13].

Consider the following identities

$$\exp\left(\sum_i \delta_i D_i\right)\rho \cdot \tau = e^{2\cosh(\sum_i \delta_i\partial_i)\log\tau} \cdot e^{\sum_i \delta_i\partial_i}(\rho/\tau), \tag{5.1a}$$

$$\cosh\left(\sum_i \delta_i D_i\right)\tau \cdot \tau = e^{2\cosh(\sum_i \delta_i\partial_i)\log\tau}. \tag{5.1b}$$

Remark 5.1. Note that in this section, $D_i f \cdot g \stackrel{\text{def}}{=} f_{t_i} g - f g_{t_i}$.

The identities (5.1) are easy to prove. One should notice the definitions of operator D_i, for example, (5.1a) is proved by checking from left hand side:

$$\begin{aligned}
\text{l.h.s.} &= \rho(t_1 + \delta_1, t_2 + \delta_2, \cdots)\tau(t_1 - \delta_1, t_2 - \delta_2, \cdots), \\
\text{r.h.s.} &= \exp\left[(e^{\sum_i \delta_i\partial_i} + e^{-\sum_i \delta_i\partial_i})\log\tau\right] \cdot \frac{\rho(t_1 + \delta_1, t_2 + \delta_2, \cdots)}{\tau(t_1 + \delta_1, t_2 + \delta_2, \cdots)} \\
&= \exp\left[\log\tau(t_1 + \delta_1, t_2 + \delta_2, \cdots) + \log\tau(t_1 - \delta_1, t_2 - \delta_2, \cdots)\right] \\
&\quad \cdot \frac{\rho(t_1 + \delta_1, t_2 + \delta_2, \cdots)}{\tau(t_1 + \delta_1, t_2 + \delta_2, \cdots)} \\
&= \rho(t_1 + \delta_1, t_2 + \delta_2, \cdots)\tau(t_1 - \delta_1, t_2 - \delta_2, \cdots).
\end{aligned}$$

Using the transformations $u \stackrel{\text{def}}{=} \partial_x^2 \log\tau$ ($x \equiv t_1$), $r \stackrel{\text{def}}{=} \rho/\tau$, $q \stackrel{\text{def}}{=} \sigma/\tau$, we can rewrite (5.1) into the following form.

$$\frac{1}{\tau^2}\sum_{n=0}^{\infty}\frac{(\sum_i \delta_i D_i)^n}{n!}\rho\cdot\tau = \exp\left[2\sum_{n=1}^{\infty}\frac{\sum_i(\delta_i\partial_i)^{2n}}{(2n)!}\partial^{-2}u\right]\cdot e^{\sum_i \delta_i\partial_i}r, \tag{5.2a}$$

$$\frac{1}{\tau^2}\sum_{n=0}^{\infty}\frac{(\sum_i \delta_i D_i)^{2n}}{(2n)!}\tau\cdot\tau = \exp\left[2\sum_{n=1}^{\infty}\frac{(\sum_i \delta_i\partial_i)^{2n}}{(2n)!}\partial^{-2}u\right]. \tag{5.2b}$$

Thus, relations between the Hirota bilinear derivatives and the usual partial derivatives are established. We list some of these relations in Table 1.

Table 1. The relations between Hirota derivatives on tau-function and the usual derivatives.

$\frac{D_1^2\tau\cdot\tau}{\tau^2}$ $2u$	$\frac{D_1D_2\tau\cdot\tau}{\tau^2}$ $2\partial^{-1}u_2$	$\frac{D_1D_3\tau\cdot\tau}{\tau^2}$ $2\partial^{-1}u_3$	$\frac{D_1D_4\tau\cdot\tau}{\tau^2}$ $2\partial^{-1}u_4$
$\frac{D_2D_3\tau\cdot\tau}{\tau^2}$ $2\partial^{-2}u_{2,3}$	$\frac{D_1^3D_2\tau\cdot\tau}{\tau^2}$ $2u_{1,2}+12u(\partial^{-1}u_2)$	$\frac{D_1^4\tau\cdot\tau}{\tau^2}$ $2u_{1,1}+12u^2$	$\cdots$ $\cdots$
$\frac{D_1\rho\cdot\tau}{\tau^2}$ r_1	$\frac{D_2\rho\cdot\tau}{\tau^2}$ r_2	$\frac{D_3\rho\cdot\tau}{\tau^2}$ r_3	$\frac{D_4\rho\cdot\tau}{\tau^2}$ r_4
$\frac{D_1^2\rho\cdot\tau}{\tau^2}$ $r_{1,1}+2ur$	$\frac{D_1D_2\rho\cdot\tau}{\tau^2}$ $r_{1,2}+2r\partial^{-1}u_2$	$\frac{D_1D_3\rho\cdot\tau}{\tau^2}$ $r_{1,3}+2r\partial^{-1}u_3$	$\frac{D_2^2\rho\cdot\tau}{\tau^2}$ $r_{2,2}+2r\partial^{-2}u_{2,2}$
$\frac{D_1^3\rho\cdot\tau}{\tau^2}$ $r_{1,1,1}+6ur_1$	$\frac{D_1^2D_2\rho\cdot\tau}{\tau^2}$ $r_{1,1,2}+2ur_2+4r_1\partial^{-1}u_2$	$\frac{D_1^4\rho\cdot\tau}{\tau^2}$ $r_{1,1,1,1}+12ur_{1,1}$ $+2(6u^2+u_{1,1})r$	$\cdots$ $\cdots$

Note: The subscripts $i, j, \ldots$ of $u_{i,j,\ldots}$ and $r_{i,j,\ldots}$ denote the derivatives with respect to the variables $t_i, t_j, \ldots$.

Example 5.1. The Hirota's bilinear equations (4.4) in Example 4.1 can be translated back to nonlinear PDEs as ($y \equiv t_2$)

$$\begin{aligned}
&\partial_z\partial^{-1}u + qr = 0,\\
&(u_{xxx} + 12uu_x - 4u_{\bar{t}_3} + 4u_z)_x + 3u_{yy} = 0,\\
&q_y = q_{xx} + 2uq,\\
&r_y = -r_{xx} - 2ur.
\end{aligned}$$

Co-published by Atlantis Press and Taylor & Francis

If we eliminate auxiliary variable z from the above equations, we get the first type of KP equation with a source [15, 26, 31, 33]

$$(4u_{\bar{t}_3} - u_{xxx} - 12uu_x)_x - 3u_{yy} + 4(qr)_{xx} = 0, \tag{5.3a}$$
$$q_y = q_{xx} + 2uq, \tag{5.3b}$$
$$r_y = -r_{xx} - 2ur, \tag{5.3c}$$

whose soliton solutions can be obtained by dressing method [27] and Hirota method [15].

Example 5.2. The Hirota's bilinear equations (4.5) in Example 4.2 can be translated back to nonlinear PDEs as ($y \equiv \bar{t}_2$, $t \equiv t_3$)

$$\begin{aligned}
&\partial_z\partial^{-1}u + qr = 0,\\
&(u_{xxx} + 12uu_x - 4u_{t_3})_x + 3(\partial_y - \partial_z)^2 u = 0,\\
&q_y - q_z = q_{xx} + 2uq,\\
&r_y - r_z = -r_{xx} - 2ur,\\
&-4q_t + q_{xxx} + 6uq_x + 3q_{xy} - 3q_{xz} + 6q\partial_y\partial^{-1}u - 6q\partial_z\partial^{-1}u = 0,\\
&4r_t - r_{xxx} - 6ur_x + 3r_{xy} - 3r_{xz} + 6r\partial_y\partial^{-1}u - 6r\partial_z\partial^{-1}u = 0.
\end{aligned}$$

By eliminating auxiliary variable z, we reach the second type of KP equation with a source [15, 26, 31, 33]

$$(4u_t - u_{xxx} - 12uu_x)_x - 3u_{yy} = 3\left[(qr)_y + (q_{xx}r - qr_{xx})\right]_x, \tag{5.4a}$$
$$q_t = q_{xxx} + 3uq_x + \frac{3}{2}u_x q + \frac{3}{2}q\partial^{-1}u_y + \frac{3}{2}q^2 r, \tag{5.4b}$$
$$r_t = r_{xxx} + 3ur_x + \frac{3}{2}u_x r - \frac{3}{2}r\partial^{-1}u_y - \frac{3}{2}r^2 q, \tag{5.4c}$$

whose soliton solutions can be obtained by dressing method [27] and Hirota method [15].

Example 5.3. The Hirota's bilinear equations (4.6) in Example 4.3 can be translated to nonlinear PDEs as ($y \equiv \bar{t}_2$)

$$\begin{aligned}
&\partial_z\partial^{-1}u + qr = 0,\\
&2u_{xx} + 12u^2 + 6\partial^{-2}[(\partial_y - \partial_z)^2 u] - 8\partial^{-1}u_{t_3} = 0,\\
&6\partial^{-1}u_{t_4} - 4\partial^{-2}(u_{t_3,y} - u_{t_3,z}) - 12u\partial^{-1}(u_y - u_z) - 2(u_{xy} - u_{xz}) = 0,\\
&q_y = q_z + q_{xx} + 2uq,\\
&r_y = r_z - r_{xx} - 2ur,\\
&4q_{t_3} = q_{xxx} + 6uq_x + 3\left(q_{xy} - q_{xz} + 2q\partial^{-1}(u_y - u_z)\right),\\
&4r_{t_3} = r_{xxx} + 6ur_x - 3\left(r_{xy} - r_{xz} + 2r\partial^{-1}(u_y - u_z)\right),\\
&18q_{t_4} = q_{xxxx} + 12uq_{xx} + 24q_x\partial^{-1}(u_y - u_z) + \big(12u^2 + 2u_{xx} + 16\partial^{-1}u_{t_3}\\
&\qquad + 6\partial^{-2}(\partial_y - \partial_z)^2 u\big)q + 12u(q_y - q_z) + 6(q_{xxy} - q_{xxz}) + 3(\partial_y - \partial_z)^2 q + 8q_{xt_3},\\
&18r_{t_4} = -r_{xxxx} - 12ur_{xx} + 24r_x\partial^{-1}(u_y - u_z) - \big(12u^2 + 2u_{xx} + 16\partial^{-1}u_{t_3}\\
&\qquad + 6\partial^{-2}(\partial_y - \partial_z)^2 u\big)r + 12u(r_y - r_z) + 6(r_{xxy} - r_{xxz}) - 3(\partial_y - \partial_z)^2 r - 8r_{xt_3}.
\end{aligned}$$

Eliminating the ∂_z and ∂_{t_3} terms, we get the extended KP hierarchy (3.1) for $n = 4$ and $k = 2$ [26]

$$\begin{aligned}2u_{t_4} =&\partial^{-2}u_{yyy} + u_{xxy} + 4(u^2)_y + 4(\partial^{-1}u_y)u_x + 8(uqr)_x + \partial^{-1}(qr)_{yy}\\&+ 2(q_{xx}r + qr_{xx})_x + (rq_x - qr_x)_y \qquad (5.5a)\\q_{t_4} =&q_{xxxx} + 4uq_{xx} + (4u_x + 4qr + 2\partial^{-1}u_y)q_x + (\partial^{-2}u_{yy} + \partial^{-1}(qr)_y\\&+ 4u^2 + 2u_{xx} + u_y)q, \qquad (5.5b)\\-r_{t_4} =&r_{xxxx} + 4ur_{xx} + (4u_x - 4qr - 2\partial^{-1}u_y)r_x + (\partial^{-2}u_{yy} + \partial^{-1}(qr)_y\\&+ 4u^2 + 2u_{xx} - u_y)r, \qquad (5.5c)\end{aligned}$$

whose Wronskian type solution (including soliton solutions) can be obtained by dressing method [27]. The correctness of (5.5) can be verified from the zero-curvature equation of extended KP hierarchy (3.1) with $n = 4$ and $k = 2$:

$$B_{4,\bar{t}_2} - B_{2,t_4} + [B_4, B_2 + q\partial^{-1}r]_+ = 0, \qquad (5.6a)$$

$$\partial_{t_4}q = B_4(q), \qquad (5.6b)$$

$$\partial_{t_4}r = -B_4^*(r). \qquad (5.6c)$$

In the expressions of (5.6), if we eliminate u_2 and u_3 in ∂^0 term by using ∂^2 and ∂ terms, we can verify that (5.6) is exactly (5.5).

6. Conclusion and Discussions

In this paper, we constructed the bilinear identities (3.3) for the extended KP hierarchy (3.1) defined in [26]. The bilinear identities (3.3) are used to derive the Hirota's bilinear equations (4.3) for all the zero-curvature forms of the extended KP hierarchy (3.1). We have shown that the Hirota's bilinear forms in Example 4.1 and Example 4.2 correspond to the KP equation with a self-consistent source (KPSCS-I and II). After translating the Hirota equations back to nonlinear PDEs, the correctness of our bilinear forms (3.3) are verified. Another forms of Hirota's bilinear equations for KPSCS-I and KPSCS-II have been given in [15] by using *source generation procedure*, where the auxiliary functions P_i and Q_i are introduced. In this paper, by introducing an auxiliary flow (∂_z-flow), the bilinear identity for the whole extended KP hierarchy and a simpler Hirota's form are obtained. To show the validity of our method, we gave an extra example in the extended KP hierarchy (Example 4.3).

There are some important applications of the bilinear identities for extended KP hierarchy. As we know, the quasi-periodic solutions for KP hierarchy can be constructed by using method in algebraic geometry, where the construction of wave functions (or Baker-Akhiezer functions as in quasi-periodic cases) are intimately related with bilinear identities, Riemann surfaces and divisors on it. It is very interesting to consider the quasi-periodic solutions for the extended KP hierarchy (3.1) when bilinear identities have been obtained in this paper. Another interesting problem is to consider the bilinear identities for other extended hierarchies, such as BKP, CKP, 2D Toda and discrete KP, etc. We will investigate these problems in the future.

Acknowledgement

The authors would like to thank R. Conte and M. Mañas for valuable discussions. This work is supported by National Natural Science Foundation of China (11171175, 11201477). XL is also supported by Chinese Universities Scientific Fund (2011JS041). RL is supported in part by the joint project between Chinese NSFC and French CNRS in 2011.

References

[1] H. Aratyn, E. Nissimov and S. Pacheva, Method of squared eigenfunction potentials in integrable hierarchies of KP type, *Commun. Math. Phys.* **193** (1998) 493–525. (arXiv:solv-int/9701017)

[2] H. Aratyn, E. Nissimov and S. Pacheva, Multi-component matrix KP hierarchies as symmetry-enhanced scalar KP hierarchies and their Darboux-Bäcklund solutions, *Bäcklund and Darboux transformations*, CRM Proc. Lect. Notes. (Amer. Math. Soc. 2001) 109–120. (arXiv:solv-int/9904024)

[3] G. Carlet, B. Dubrovin and Y.J. Zhang, The extended Toda hierarchy, *Moscow Math. J.* **4** (2004) 313–332. (arXiv:nlin/0306060)

[4] Y. Cheng and Y.J. Zhang, Bilinear equations for the constrained KP hierarchy, *Inverse Probl.* **10** (1994) L11–L17.

[5] Y. Cheng, Modifying the KP, the *n*-th constrained KP hierarchies and their Hamiltonian structures, *Commun. Math. Phys.* **171** (1995) 661–682.

[6] E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation groups for soliton equations, *Nonlinear Integrable Systems – Classical Theory and Quantum Theory*, (M. Jimbo and T. Miwa, 1983) 39–119.

[7] L.A. Dickey, *Soliton equations and Hamiltonian systems*, (World Scientific, Singapore, 2003).

[8] A. Dimakis, F. Müller-Hoissen, Extension of Moyal-deformed hierarchies of soliton equations, *XI International Conference Symmetry Methods in Physics*, (C. Burdik, O. Novratil and S. Posta, 2004). (arXiv:nlin/0408023)

[9] A. Dimakis and F. Müller-Hoissen, Explorations of the extended ncKP hierarchy, *J. Phys. A: Math. Gen.* **37** (2004) 10899–10930. (arXiv:hep-th/0406112)

[10] E.V. Doktorov and V.S. Shchesnovich, Nonlinear evolutions with singular dispersion laws associated with a quadratic bundle, *Phys. Lett. A*, **207** (1995) 153-158.

[11] P.G. Grinevich and I.A. Taimanov, Spectral conservation laws for periodic nonlinear equations of the Melnikov type, *Amer. Math. Soc. Transl.* **224(2)** (2008) 125–138. (arXiv:0801.4143)

[12] G.F. Helminck and J. van de Leur, An analytic description of the vector constrained KP hierarchy, *Commun. Math. Phys.* **193** (1998) 627. (arXiv:solv-int/9706004)

[13] R. Hirota, *The direct method in soliton theory*, (Cambridge University Press, Cambridge, 2004).

[14] X.B. Hu and H.Y. Wang, Construction of dKP and BKP equations with self-consistent sources, *Inverse Probl.* **22** (2006) 1903–1920.

[15] X.B. Hu and H.Y. Wang, New type of Kadomtsev-Petviashvili equation with self-consistent sources and its bilinear Bäcklund transformation, *Inverse Probl.* **23** (2007) 1433–1444.

[16] Y.H. Huang, X.J. Liu, Y.Q. Yao and Y.B. Zeng, A new extended matrix KP hierarchy and its solutions, *Theor. Math. Phys.* **167** (2011) 590–605. (arXiv:1011.4430)

[17] M. Kamata and A. Nakamula, Riemann Liouville integrals of fractional order and extended KP hierarchy, *J. Phys. A: Math. Gen.* **35** (2002) 9657–9670. (arXiv:nlin/0207037)

[18] B.G. Konopelchenko and W. Strampp, The AKNS hierarchy as symmetry constraint of the KP hierarchy, *Inverse Probl.* **7** (1991) L17–L24.

[19] B.G. Konopelchenko and W. Strampp, New reductions of the Kadomtsev-Petviashvili and two-dimensional Toda lattice hierarchies via symmetry constraints, *J. Math. Phys.* **33** (1992) 3676-3686.

[20] C.Z. Li, J.S. He, K. Wu and Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda Hierarchy, *J. Math. Phys.* **50** (2010) 043514. (arXiv:0906.0624)

[21] R.L. Lin, Y.B. Zeng and W.X. Ma, Solving the KdV hierarchy with self-consistent sources by inverse scattering method, *Physica A* **291** (2001) 287–298.

[22] R.L. Lin, H.S. Yao and Y.B. Zeng, Restricted flows and the soliton equation with self-consistent sources, *SIGMA* **2** (2006) 096. (arXiv:nlin/0701003)

[23] R.L. Lin, X.J. Liu and Y.B. Zeng, A new extended q-deformed KP hierarchy, *J. Nonlinear Math. Phys.* **15** (2008) 333–347. (arXiv:0810.1862)

[24] R.L. Lin, H. Peng and M. Manas, The q-deformed mKP hierarchy with self-consistent sources, Wronskian solutions and solitons, *J. Phys. A: Math. Theor.* **43** (2010) 434022. (`arXiv:1005.3878`)

[25] X.J. Liu, Y.B. Zeng and R.L. Lin, An extended two-dimensional Toda lattice hierarchy and two-dimensional Toda lattice with self-consistent sources, *J. Math. Phys.* **49** (2008) 093506.

[26] X.J. Liu, Y.B. Zeng, R.L. Lin, A new extended KP hierarchy, *Phys. Lett. A* **372** (2008) 3819–3823. (`arXiv:0710.4015`)

[27] X.J. Liu, R.L. Lin, B. Jin and Y.B. Zeng, A generalized dressing approach for solving the extended KP and the extended mKP hierarchy, *J. Math. Phys.* **50** (2009) 053506. (`arXiv:0905.1402`)

[28] X.J. Liu, C. Gao, Generalized dressing method for the extended two-dimensional Toda lattice hierarchy and its reductions, *Sci. China Math.* **54** (2010) 365–380.

[29] I. Loris and R. Willox, Bilinear form and solutions of the k-constrained Kadomtsev-Petviashvili hierarchy, *Inverse Probl.* **13** (1997) 411–420.

[30] W.X. Ma, An extended Harry Dym hierarchy, *J. Phys. A: Math. Theor.* **43** (2010) 165202.

[31] V.K. Mel'nikov, On Equations for wave interactions, *Lett. Math. Phys.* **7** (1983) 129–136.

[32] V.K. Mel'nikov, Some new nonlinear evolution equations integrable by the inverse problem method, *Math. USSR SB+* **49** (1984) 461–489.

[33] V.K. Mel'nikov, A direct method for deriving a multi-soliton solution for the problem of interaction of waves on the $x-y$ plane, *Commun. Math. Phys.* **112** (1987) 639–652.

[34] V.K. Mel'nikov, Interaction of solitary waves in the system described by the Kadomtsev-Petviashvili equation with a self-consistent source, *Commun. Math. Phys.* **126** (1989) 201–215.

[35] T.E. Milanov, Hirota quadratic equations for the extended Toda hierarchy, *Duke Math. J.* **138** (2007) 161–178. (`arXiv:math/0501336`)

[36] W. Oevel, Darboux theorems and Wronskian formulas for integrable systems I. Constrained KP flows, *Physica A* **195** (1993) 533–576.

[37] W. Oevel and W. Strampp, Constrained KP hierarchy and Bi-Hamiltonian structures, *Commun. Math. Phys.* **157** (1993) 51–81.

[38] W. Oevel and W. Schief, Squared eigenfunctions of the (modified) KP hierarchy and scattering problems of Loewner type, *Rev. Math. Phys.* **6** (1994) 1301-1338.

[39] H.F. Shen and M.H. Tu, On the constrained B-type Kadomtsev-Petviashvili hierarchy: Hirota bilinear equations and Virasoro symmetry, *J. Math. Phys.* **52** (2011) 032704.

[40] K. Takasaki, Two extensions of 1D Toda hierarchy, *J. Phys. A: Math. Theor.* **43** (2010) 434032. (`arXiv:1002.4688`)

[41] H.Y. Wang, X.B. Hu and Gegenhasi: 2D Toda lattice equation with self-consistent sources: Casoratian type solutions, bilinear Bäcklund transformation and Lax pair, *J. Compu. Appl. Math.* **202** (2007) 133–143.

[42] C.S. Xiong, The generalized KP hierarchy. *Lett. Math. Phys.* **36** (1996) 223–229. (`arXiv:hep-th/9211098`)

[43] Y.Q. Yao, X.J. Liu and Y.B. Zeng, A new extended discrete KP hierarchy and a generalized dressing method, *J. Phys. A: Math. Theor.* **42** (2009) 454026. (`arXiv:0907.2783`)

[44] Y.B. Zeng, W.X. Ma and R.L. Lin, Integration of the soliton hierarchy with self-consistent sources, *J. Math. Phys.* **41** (2000) 5453–5489.

[45] D.J. Zhang and D.Y. Chen, The N-soliton solutions of the sine-Gordon equation with self-consistent sources, *Physica A* **321** (2003) 467–481.