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# ABJ triality: from higher spin fields to strings

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**Abstract**

We demonstrate that a supersymmetric and parity violating version of Vasiliev's higher spin gauge theory in $AdS_4$ admits boundary conditions that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ or 6 supersymmetries. In particular, we argue that the Vasiliev theory with $U(M)$ Chan–Paton and $\mathcal{N} = 6$ boundary condition is holographically dual to the 2+1 dimensional $U(N)_k \times U(M)_{-k}$ ABJ theory in the limit of large $N, k$ and finite $M$. In this system all bulk higher spin fields transform in the adjoint of the $U(M)$ gauge group, whose bulk t'Hooft coupling is $M/N$. Analysis of boundary conditions in Vasiliev theory allows us to determine exact relations between the parity breaking phase of Vasiliev theory and the coefficients of two and three point functions in Chern–Simons vector models at large $N$. Our picture suggests that the supersymmetric Vasiliev theory can be obtained as a limit of type IIA string theory in $AdS_4 \times \mathbb{CP}^3$, and that the non-Abelian Vasiliev theory at strong bulk 't Hooft coupling smoothly turn into a string field theory. The fundamental string is a singlet bound state of Vasiliev's higher spin particles held together by $U(M)$ gauge interactions. This is illustrated by the thermal partition function of free ABJ theory on a two sphere at large $M$ and $N$ even in the analytically tractable free limit. In this system the traces or strings of the low temperature phase break up into their Vasiliev particulate constituents at a $U(M)$ deconfinement phase transition of order unity. At a higher temperature of order $T = \sqrt{\frac{N}{M}}$ Vasiliev's higher spin fields themselves break up into more elementary constituents at a $U(N)$

  

deconfinement temperature, in a process described in the bulk as black hole nucleation.

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(Some figures may appear in colour only in the online journal)

## Contents

## 1. Introduction and summary

It has long been speculated that the tensionless limit of string theory is a theory of higher spin gauge fields. One of the most important explicit and nontrivial construction of interacting higher spin gauge theory is Vasiliev's system in $AdS_4$ [1–3]. It was conjectured by Klebanov and Polyakov [4], and by Sezgin and Sundell [5, 6], that the parity invariant A-type and B-type Vasiliev theories are dual to 2+1 dimensional bosonic and fermionic $O(N)$ or $U(N)$ vector models in the singlet sector. Substantial evidence for these conjectures has been provided by comparison of three-point functions [7, 8], and analysis of higher spin symmetries [9–12].

It was noted in [13, 14] that, at large $N$, the free $O(N)$ and $U(N)$ theories described above each have a family of one parameter conformal deformations, corresponding to turning on a finite Chern–Simons level for the $O(N)$ or $U(N)$ gauge group. It was conjectured in [14] that the bulk duals of the resultant Chern–Simons vector models is given by a one parameter family of parity violating Vasiliev theories. In the bulk description parity is broken by a nontrivial phase in function $f$ in Vasiliev's theory that controls bulk interactions. This conjecture appeared to pass some nontrivial checks [14] but also faced some puzzling challenges [14]. In this paper we will find significant additional evidence in support of the proposal of [14] from the study of the bulk duals of supersymmetric vector Chern–Simons theories.

The duality between Vasiliev theory and 3d Chern–Simons boundary field theories does not rely on supersymmetry (SUSY), and, indeed, most studies of this duality have been carried out in the non-supersymmetric context. However it is possible to construct supersymmetric analogues of the type A and type B bosonic Vasiliev theories [2, 3, 6, 15, 16]. With appropriate boundary conditions, these supersymmetric Vasiliev theories preserve all higher spin symmetries and are conjectured to be dual to free boundary supersymmetric gauge theories. In the spirit of [14] it is natural to attempt to construct bulk duals of the one parameter set of interacting supersymmetric Chern–Simons vector theories obtained by turning on a finite level $k$ for the Chern–Simons terms (recall that Chern–Simons coupled gauge fields are free only in the limit $k \to \infty$). Interacting supersymmetric Chern–Simons theories differ from their free counterparts in three ways. First, as emphasized above, their Chern–Simons level is taken to be finite. According to the conjecture of [14] this is accounted for by turning on the appropriate phase in Vasiliev's equations. Second the Lagrangian includes potential terms of the schematic form $\phi^6$ and Yukawa terms of the schematic form $\phi^2\psi^2$, where $\phi$ and $\psi$ are fundamental and antifundamental scalars and fermions in the field theory. These terms may be regarded as double and triple trace deformations of the field theory; as is well known, the effect of such terms on the dual bulk theory may be accounted for by an appropriate modification of boundary conditions [17]. Lastly, supersymmetric field theories with $\mathcal{N} = 4$ and $\mathcal{N} = 6$ SUSY necessarily have two gauge groups with matter in the bifundamental. Such theories may be obtained by from theories with a single Chern–Simons coupled gauge group at level $k$ and fundamental matter by gauging a global symmetry with Chern–Simons level $-k$. In the dual bulk theory this gauging is implemented by a modification of the boundary conditions of the bulk vector gauge field [18].

These elements together suggest that it should be possible to find one parameter families of Vasiliev theories that preserve some SUSY upon turning on the parity violating bulk phase, if and only if one also modifies the boundary conditions of all bulk scalars, fermions and sometimes gauge fields in a coordinated way. In this paper we find that this is indeed the case. We are able to formulate one parameter families of parity violating Vasiliev theory (enhanced with Chan–Paton factors, see below) that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ or 6 SUSYs depending on boundary conditions. In every case we identify conjectured dual Chern–Simons vector models dual to our bulk constructions[3].

The identification of parity violating Vasiliev theory with prescribed boundary conditions as the dual of Chern–Simons vector models pass a number of highly nontrivial checks. By considering of boundary conditions alone, we will be able to determine the exact relation between the parity breaking phase $\theta_0$ of Vasiliev theory, and two and three point function coefficients of Chern–Simons vector models at large $N$. These imply non-perturbative relations

[3] A similar analysis of the breaking of higher spin symmetry by boundary conditions allows us to demonstrate that all deformations of type A or type B Vasiliev theories break all higher spin symmetries other than the conformal symmetry. We are also able to use this analysis to determine the functional form of the double trace part of higher spin currents that contain a scalar field.

among purely field theoretic quantities that are previously unknown (and presumably possible to prove by generalizing the computation of correlators in Chern–Simons-scalar vector model of [19] using Schwinger–Dyson[4] equations to the supersymmetric theories). The results also agree with the relation between $\theta_0$ and Chern–Simons 't Hooft coupling $\lambda = N/k$ determined in [14] by explicit perturbative computations at one-loop and two-loop order.

From a physical viewpoint, the most interesting Vasiliev theory presented in this paper is the $\mathcal{N} = 6$ theory. It was already suggested in [14] that a supersymmetric version of the parity breaking Vasiliev theory in $AdS_4$ should be dual to the vector model limit of the $\mathcal{N} = 6$ ABJ theory, that is, a $U(N)_k \times U(M)_{-k}$ Chern–Simons-matter theory in the limit of large $N, k$ but finite $M$. Since the ABJ theory is also dual to type IIA string theory in $AdS_4 \times \mathbb{CP}^3$ with flat $B$-field, it was speculated that the Vasiliev theory must therefore be a limit of this string theory. The concrete supersymmetric $\mathcal{N} = 6$ Vasiliev system presented in this paper allows us to turn the suggestion of [14] into a precise conjecture for a duality between three distinct theories that are autonomously well defined at least in particular limits.

The $\mathcal{N} = 6$ Vasiliev theory, conjectured below to be dual to $U(N) \times U(M)$ ABJ theory has many elements absent in more familiar bosonic Vasiliev systems. First theory is 'supersymmetric' in the bulk. This means that all fields of the theory are functions of fermionic variables $\psi_i$ $(i = 1, \ldots, 6)$ which obey Clifford algebra commutation relations $\{\psi_i, \psi_j\} = 2\delta_{ij}$ (all bulk fields are also functions of the physical spacetime variables $x_\mu$ $(\mu = 1, \ldots, 4)$ as well as Vasiliev's twistor variables $y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\beta}}$, as in bosonic Vasiliev theory). Next the star product used in the bulk equations is the usual Vasiliev star product times matrix multiplication in an auxiliary $M \times M$ space. The physical effect of this maneuver is to endow the bulk theory with a $U(M)$ gauge symmetry under which all bulk fields transform in the adjoint. Finally, for the reasons described above, interactions of the theory are also modified by a bulk phase, and bulk scalars, fermions and gauge fields obey nontrivial boundary conditions that depend on this phase.

The triality between $U(N) \times U(M)$ ABJ theory, type IIA string theory on $AdS_4 \times \mathbb{CP}^3$, and supersymmetric parity breaking Vasiliev theory may qualitatively be understood in the following manner. The propagating degrees of freedom of ABJ theory consist of bifundamental fields that we denote by $A_i$ and antibifundamental fields that we will call $B_i$. A basis for the gauge singlet operators of the theory is given by the traces $\mathrm{Tr}(A_1B_1A_2B_2 \ldots A_mB_m)$. As is well known from the study of ABJ duality, these single trace operators are dual to single string states. The basic 'partons' (the $A$ and $B$ fields) out of which this trace is composed are held together in this string state by the 'glue' of $U(N)$ and $U(M)$ gauge interactions.

Let us now study the limit $M \ll N$. In this limit the glue that joins $B$ type fields to $A$ type fields (provided by the gauge group $U(M)$) is significantly weaker than the glue that joins $A$ fields to $B$ fields (this glue is supplied by $U(N)$ interactions). In this limit the trace effectively breaks up into $m$ weakly interacting particles $A_1B_1, A_2B_2 \ldots A_mB_m$. These particles, which transform in the adjoint of $U(M)$, are the dual to the $U(M)$ adjoint fields of the dual $\mathcal{N} = 6$ Vasiliev theory. Indeed the spectrum of operators of field theory operators of the form $AB$ precisely matches the spectrum of bulk fields of the dual Vasiliev system.

If our picture is correct, the fields of Vasiliev's theory must bind together to make up fundamental IIA strings as $M/N$ is increased. We now describe a qualitative way in which this might happen. The bulk Vasiliev theory has gauge coupling $g \sim 1/\sqrt{N}$, It follows that the *bulk* 't Hooft coupling is $\lambda_{\mathrm{bulk}} = g^2M \sim M/N$. In the limit $M/N \ll 1$, the bulk Vasiliev theory is effectively weakly coupled. As $M/N$ increases, a class of multi-particle states of higher spin fields acquire large binding energies due to interactions, and are mapped to the

[4] See [14] for these equations in the Chern–Simons fermion model.

single closed string states in type IIA string theory. Roughly speaking, the fundamental string of string theory is simply the flux tube string of the non-Abelian bulk Vasiliev theory.

Note that although we claim a family of supersymmetric Vasiliev theory with Chan–Paton factors and certain prescribed boundary conditions is equivalent to string theory on $AdS_4$, we are *not* suggesting that Vasiliev's equations are the same as the corresponding limit of closed string field equations. Not all single closed string states are mapped to single higher spin particles; in fact the only closed strings that are mapped to Vasiliev's particles are those dual to the operators of the form $\mathrm{Tr}\,AB$. Closed string field theory is the weakly interacting theory of the 'glueball' bound states of the Vasiliev fields; it is not a weakly interacting description of Vasiliev's fields themselves.

We have asserted above that the glue between $B$ and $A$ partons is significantly weaker than the glue between $A$ and $B$ partons in the limit $M \ll N$. This claim may be made quantitatively precise in a calculation in the *free* ABJ theory with $M/N$ taken to be an arbitrary parameter. The computation in question is the partition function of free ABJ field theories on a sphere in the t'Hooft large $N$ and $M$ limit. We use the fact that the path integral that computes this partition function, even in the limit $k \to \infty$, is not completely free [20]. This $k = \infty$ path integral includes the effects of strong interactions between matter and the Polyakov line of $U(N)$ and $U(M)$ gauge fields. This computation of the partition function is a straightforward application of the techniques described in [20], but yields an interesting result (see section 7, and see [21, 14] for related earlier work in the context of models with fundamental matter). We find that the theory undergoes *two* phase transitions as a function of temperature. At low temperature the theory is in a confined phase. This phase may be thought of as a gas of traces of the form $\mathrm{Tr}(A_1B_1A_2B_2 \ldots A_mB_m)$, or, roughly, closed strings. Upon raising the temperature the field theory undergoes a first order phase transition at a temperature of order unity. Above the phase transition temperature, group $U(M)$ deconfines while the group $U(N)$ continues to completely confine[5] (we make this statement precise below.) The intermediate temperature phase has an effective description in terms of the partition function of a $U(M)$ gauge theory whose effective matter degrees of freedom are simply the set of adjoint 'mesons' of the form $A_iB_j$. These adjoint degrees of freedom are deconfined. In other words the traces of the low temperature phase (dual to fundamental strings of ABJ theory) split up into a free gas of smaller—but not yet indivisible units, i.e. the fields of Vasiliev's theory. Upon further raising the temperature, the theory undergoes yet another phase transition, this time of third order. This transition occurs at a temperature of order $\sqrt{\frac{N}{M}}$ and is associated with the complete 'deconfinement' of the gauge group $U(N)$. At temperatures much higher than the second phase transition temperature, the system may be thought of as a plasma of the bifundamental and anti-bifundamental letters $A_i$ and $B_j$. In other words the basic units, $\mathrm{Tr}(A_iB_j)$, of the intermediate temperature phase, split up into their basic building blocks in the high temperature phase. This extreme high temperature phase is presumably dual to a black hole in the bulk theory[6]. In the special case $M = N$ the intermediate phase never exists; the system directly transits from the string to the black hole phase. The fact that the $U(M)$ deconfinement temperature is much smaller than the $U(N)$ deconfinement temperature demonstrates that the glue between $B$ and $A$ type partons is much weaker than between $A$ and $B$ type partons. Our computations also strongly suggests that the string dual to ABJ theory has a new finite temperature phase—one composed of a gas of Vasiliev's particles—even at finite values of $\lambda$.

[5] Throughout this paper we assume without loss of generality that $M \leqslant N$.

[6] In the very high temperature limit, this phase has recently been studied in closely related supersymmetric Chern–Simons theories even away from the free limit [22] (generalizing earlier computations in nonsupersymmetric theories in [14].

Let us note a curious aspect of the conjectured duality between Vasiliev's theory and ABJ theory. The gauge groups $U(N)$ and $U(M)$ appear on an even footing in the ABJ field theory. In the bulk Vasiliev description, however, the two gauge groups play a very different role. The gauge group $U(M)$ is manifest as a gauge symmetry in the bulk. However $U(N)$ symmetry is not manifest in the bulk (just as the $U(N)$ symmetry is not manifest in the bulk dual of $\mathcal{N} = 4$ Yang–Mills); the dynamics of this gauge group that leads to the emergence of the background spacetime for Vasiliev theory. The deconfinement transition for $U(M)$ is simply a deconfinement transition of the adjoint bulk degrees of freedom, while the deconfinement transition for $U(N)$ is associated with the very different process of 'black hole formation'. If our proposal for the dual description is correct, the gauged Vasiliev theory must enjoy an $N \leftrightarrow M$ symmetry, which, from the bulk viewpoint is a sort of level–rank duality. Of course even a precise statement for the claim of such a level rank duality only makes sense if Vasiliev theory is well defined 'quantum mechanically' (i.e. away from small $M/N$) at least in the large $N$ limit.

We have noted above that Vasiliev's theory should not be identified with closed string field theory. There may, however, be a sense in which it might be thought of as an open string field theory. We use the fact that there is an alternative way to engineer Chern–Simons vector models using string theory [23], that is by adding $N_f$ D6-branes wrapped on $AdS_4 \times \mathbb{RP}^3$ inside the $AdS_4 \times \mathbb{CP}^3$, which preserves $\mathcal{N} = 3$ SUSY and amounts to adding fundamental hypermultiplets of the $U(N)_k$ Chern–Simons gauge group. In the 'minimal radius' limit where we send $M$ to zero, with flat $B$-field flux $\frac{1}{2\pi\alpha}\int_{\mathbb{CP}^1} B = \frac{N}{k} + \frac{1}{2}$, the geometry is entirely supported by the $N_f$ D6-branes [24][7]. This type IIA open+closed string theory is dual to $\mathcal{N} = 3$ Chern–Simons vector model with $N_f$ hypermultiplet flavors. The duality suggests that the open+closed string field theory of the D6-branes reduces to precisely a supersymmetric Vasiliev theory in the minimal radius limit. Note that unlike the ABJ triality, here the open string fields on the D6-branes and the non-Abelian higher spin gauge fields in Vasiliev's system both carry $U(N_f)$ Chan–Paton factors, and we expect one-to-one correspondence between single open string states and single higher spin particle states.

## 2. Vasiliev's higher spin gauge theory in $AdS_4$ and its supersymmetric extension

The Vasiliev systems that we study in this paper are defined by a set of bulk equations of motion together with boundary conditions on the bulk fields. In this section we review the structure of the bulk equations. We turn to the consideration of boundary conditions in the next section.

In this section we first present a detailed review of bulk equations of the 'standard' Vasiliev theory. We then describe non-Abelian and supersymmetric extensions of these equations. Throughout this paper we work with the so-called non-minimal version of Vasiliev's equations, which describe the interactions of a field of *each* non-negative integer spin $s$ in $AdS_4$. Under the anti-de Sitter/conformal field theory (AdS/CFT) correspondence non-minimal Vasiliev equations are conjectured to be dual to gauged $U(N)$ Chern–Simons–matter boundary theories[8].

There are exactly two 'standard' non-minimal Vasiliev theories that preserve parity symmetry. These are the type A/B theories, which are conjectured to be dual to bosonic/fermionic $SU(N)$ vector models, restricted to the $SU(N)$-singlet sector. Parity

[7] We thank Daniel Jafferis for making this important suggestion and O Aharony for related discussions.

[8] The non-minimal equations admit a consistent truncation to the so-called minimal version of Vasiliev's equations; this truncation projects out the gauge fields for odd spins and are conjectured to supply the dual to $SO(N)$ Chern–Simons boundary theories.

invariant Vasiliev theories are particular examples of a larger class of generically parity violating Vasiliev theories. These theories appear to be labeled by a real even function of one real variable. In section 2.1 we present a review of these theories. It was conjectured in [14] that a class of these parity violating theories are dual to $SU(N)$ Chern–Simons vector models.

In section 2.2 we then present a straightforward non-Abelian extension of Vasiliev's system, by introducing $U(M)$ Chan–Paton factors into Vasiliev's star product. The result of this extension is to promote the bulk gauge field to a $U(M)$ gauge field; all other bulk fields transform in the adjoint of $U(M)$. The local gauge transformation parameter of Vasiliev's theory is also promoted to a local $M \times M$ matrix field that transforms in the adjoint of $U(M)$. The nature of the boundary CFT dual to the non-Abelian Vasiliev theory depends on boundary conditions. With 'standard' magnetic type boundary conditions for all gauge fields (that set prescribed values for the field strengths restricted to the boundary) the dual boundary CFT is obtained simply by coupling $M$ copies of (otherwise non interacting) matter multiplets to the same boundary Chern–Simons gauge field. The boundary theory has a 'flavor' $U(M)$ global symmetry that acts on the $M$ identical matter multiplets.

In section 2.3 we then introduce the so-called $n$-extended supersymmetric Vasiliev theory (generalizing the special cases studied earlier in [1–3, 6, 15]). The main idea is to enhance Vasiliev's fields to functions of $n$ fermionic fields $\psi_i$ ($i = 1, \ldots, n$; we assume $n$ to be even) which obey a Clifford algebra[9]. This extension promotes the usual Vasiliev fields to $2^{n/2} \times 2^{n/2}$ dimensional matrices (or operators) that act on the $2^{n/2}$ dimensional representation of the Clifford algebra. The local Vasiliev gauge transformations are also promoted to functions of $\psi_i$, and so $2^{n/2} \times 2^{n/2}$ matrices or operators[10]. Half of the resultant fields (and gauge transformations) are fermionic; the other half are bosonic.

### *2.1. The standard parity violating bosonic Vasiliev theory*

In this section we present the 'standard' non-minimal Vasiliev equations, allowing, however, for parity violation.

*2.1.1. Coordinates.* In Euclidean space the fields of Vasiliev's theory are functions of a collection of bosonic variables $(x, Y, Z) = (x^\mu, y^\alpha, \bar{y}^{\dot\alpha}, z^\alpha, \bar{z}^{\dot\alpha})$. $x^\mu$ ($\mu = 1, \ldots, 4$) are an arbitrary set of coordinates on the four dimensional spacetime manifold. $y^\alpha$ and $z^\alpha$ are spinors under $SU(2)_L$ while $\bar{y}^{\dot\alpha}$ and $\bar{z}^{\dot\alpha}$ are spinors under a separate $SU(2)_R$. As we will see below, Vasiliev's equations enjoy invariance under *local* (in spacetime) $SO(4) = SU(2)_L \times SU(2)_R$ rotations of $y^\alpha$, $z^\alpha$, $\bar{y}^{\dot\alpha}$ and $\bar{z}^{\dot\alpha}$. This local $SO(4)$ rotational invariance, which, as we will see below is closely related to the tangent space symmetry of the first order formulation of general relativity, is only a small part of the much larger gauge symmetry of Vasiliev's theory.

*2.1.2. Star product.* Vasiliev's equations are formulated in terms of a star product. This is just the usual local product in coordinate space; whereas in auxiliary space it is

[9] We emphasize that $n$ should not be confused with the number of globally conserved supercharges $4\mathcal{N}$ (equivalently $4\mathcal{N}$ is the number of supercharges in the superconformal algebra of the dual three-dimensional CFT). $n$ characterizes only the local structure of Vasiliev's equations of motion. $\mathcal{N}$ on the other hand depends on the choice of boundary condition for bulk fields of spin 0, 1/2 and 1. As we will see $\mathcal{N} \leqslant 6$ for parity violating Vasiliev theories, as expected from the dual $CFT_3$ ($n$, or course, can be arbitrarily large ).

[10] The bulk equations of motion for $n$ extended supersymmetric Vasiliev theory is identical to those for $n = 2$ theory extended by $U(2^{\frac{n}{2}-1})$ Chan–Paton factors. However, the language of $n$ extended supersymmetric Vasiliev theory is more convenient when the boundary conditions of the problem break part of this $U(2^{\frac{n}{2}-1})$ symmetry, as will be the case later in this paper.

given by

$$f(Y,Z) * g(Y,Z) = f(Y,Z)\exp[\epsilon^{\alpha\beta}(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha})(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}) + \epsilon^{\dot\alpha\dot\beta}(\overleftarrow{\partial}_{y^{\dot\alpha}} + \overleftarrow{\partial}_{z^{\dot\alpha}})(\overrightarrow{\partial}_{y^{\dot\beta}} - \overrightarrow{\partial}_{z^{\dot\beta}})]g(Y,Z)$$
$$= \int d^2u\, d^2v\, d^2\bar u\, d^2\bar v\, e^{u^\alpha v_\alpha + \bar u^{\dot\alpha}\bar v_{\dot\alpha}} f(y+u, \bar y + \bar u, z+u, \bar z + \bar u) \times g(y+v, \bar y + \bar v, z - v, \bar z - \bar v). \tag{2.1}$$

In the last line, the integral representation of the star product is defined by the contour for $(u^\alpha, v^\alpha)$ along $e^{\pi i/4}\mathbb{R}$ in the complex plane, and $(\bar u^{\dot\alpha}, \bar v^{\dot\alpha})$ along the contour $e^{-\pi i/4}\mathbb{R}$. It is obvious from the first line of (2.1) that $1 * f = f * 1 = f$; this fact may be used to set the normalization of the integration measure in the second line. The star product is associative but non commutative; in fact it may be shown to be isomorphic to the usual Moyal star product under an appropriate change of variables. In section A.1 we describe our conventions for lowering spinor indices and present some simple identities involving the star product.

Below we will make extensive use of the so-called Kleinian operators $K$ and $\overline{K}$ defined as

$$K = e^{z^\alpha y_\alpha}, \qquad \overline{K} = e^{\bar z^{\dot\alpha}\bar y_{\dot\alpha}}. \tag{2.2}$$

They have the property (see section A.1 for a proof)

$$\begin{aligned} &K * K = \overline{K} * \overline{K} = 1,\\ &K * f(y,z,\bar y,\bar z) * K = f(-y,-z,\bar y,\bar z),\\ &\overline{K} * f(y,z,\bar y,\bar z) * \overline{K} = f(y,z,-\bar y,-\bar z). \end{aligned} \tag{2.3}$$

*2.1.3. Master fields.* Vasiliev's master fields consists of an $x$-space 1-form

$$W = W_\mu\, dx^\mu,$$

a $Z$-space 1-form

$$S = S_\alpha\, dz^\alpha + S_{\dot\alpha}\, d\bar z^{\dot\alpha},$$

and a scalar $B$, all of which depend on spacetime as well as the internal twistor coordinates which we denote collectively as $(x, Y, Z) = (x^\mu, y^\alpha, \bar y^{\dot\alpha}, z^\alpha, \bar z^{\dot\alpha})$. It is sometimes convenient to write $W$ and $S$ together as a 1-form on $(x, Z)$-space

$$\mathcal{A} = W + S = W_\mu\, dx^\mu + S_\alpha\, dz^\alpha + S_{\dot\alpha}\, d\bar z^{\dot\alpha}.$$

$\mathcal{A}$ will be regarded as a gauge connection with respect to the $*$-algebra.

We also define

$$\begin{aligned} &\hat S = S - \tfrac{1}{2} z_\alpha\, dz^\alpha - \tfrac{1}{2}\bar z_{\dot\alpha}\, d\bar z^{\dot\alpha},\\ &\hat{\mathcal{A}} = W + \hat S = \mathcal{A} - \tfrac{1}{2} z_\alpha\, dz^\alpha - \tfrac{1}{2}\bar z_{\dot\alpha}\, d\bar z^{\dot\alpha} = W_\mu\, dx^\mu + \left(-\tfrac{1}{2} z_\alpha + S_\alpha\right) dz^\alpha + \left(-\tfrac{1}{2}\bar z_{\dot\alpha} + S_{\dot\alpha}\right) d\bar z^{\dot\alpha}. \end{aligned} \tag{2.4}$$

Let $d_x$ be the exterior derivative with respect to the spacetime coordinates $x^\mu$ and denote by $d_Z$ the exterior derivative with respect to the twistor variables $(z^\alpha, \bar z^{\dot\alpha})$. We will write $d = d_x + d_Z$. We will also find it useful to define the field strength

$$\begin{aligned} \mathcal{F} &= d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} \star \hat{\mathcal{A}}\\ &= (d_x W + W * W) + (d_x \hat S + \{W, \hat S\}_*) + (\hat S * \hat S). \end{aligned} \tag{2.5}$$

Note also that

$$\hat S * \hat S = d_z S + S * S + \tfrac{1}{4}(\epsilon_{\alpha\beta}\, dz^\alpha\, dz^\beta + \epsilon_{\dot\alpha\dot\beta}\, d\bar z^{\dot\alpha}\, d\bar z^{\dot\beta}). \tag{2.6}$$

*2.1.4. Gauge transformations.* Vasiliev's master fields transform under a large set of gauge symmetries. We will see later that the $AdS_4$ vacuum solution partially Higgs or breaks this gauge symmetry group down to a subgroup of large gauge transformations—either the higher spin symmetry group or the conformal group depending on boundary conditions.

Infinitesimal gauge transformations are generated by an arbitrary real function $\epsilon(x, Y, Z)$. By definition under gauge transformations

$$\begin{aligned} &\delta\hat{\mathcal{A}} = d_x\epsilon + \hat{\mathcal{A}} * \epsilon - \epsilon * \hat{\mathcal{A}}, \\ &\delta B = -\epsilon * B + B * \pi(\epsilon). \end{aligned} \tag{2.7}$$

In other words the 1-form master field transforms as a gauge connection under the star algebra while $B$ transforms as a 'twisted' adjoint field. The operation $\pi$ that appears in (2.7) is defined as follows:

$$\pi\ (y, z, \mathrm{d}z, \bar{y}, \bar{z}, \mathrm{d}\bar{z}) = (-y, -z, -\mathrm{d}z, \bar{y}, \bar{z}, \mathrm{d}\bar{z})\ .$$

Since $\epsilon$ does not involve differentials in $(z, \bar{z})$, the action of $\pi$ on $\epsilon$ is equivalent to conjugation by $K$, namely $\pi(\epsilon) = K * \epsilon * K$. ($\pi$ acting on a 1-form in $(z_\alpha, \bar{z}_{\dot{\alpha}})$ acts by conjugation by $K$ together with flipping the sign of $\mathrm{d}z$).

It follows from (2.7) that the field strength $\mathcal{F}$ ( and so each of the three brackets on the RHS of the second line of (2.5)) transform in the adjoint. The same is true of $B * K$:

$$\begin{aligned} &\delta\mathcal{F} = [\mathcal{F}, \epsilon]_*, \\ &\delta(B \star K) = -\epsilon * (B * K) + (B * K) * \epsilon. \end{aligned} \tag{2.8}$$

When expanded in components the first line of (2.7) implies that

$$\begin{aligned} &\delta W_\mu = \partial_\mu\epsilon + W_\mu * \epsilon - \epsilon * W_\mu, \\ &\delta\hat{S}_\alpha = \hat{S}_\alpha * \epsilon - \epsilon * \hat{S}_\alpha. \end{aligned} \tag{2.9}$$

In terms of unhatted variables,

$$\begin{aligned} &\delta\mathcal{A} = \mathrm{d}\epsilon + \mathcal{A} * \epsilon - \epsilon * \mathcal{A}, \\ &\delta S_\alpha = \frac{\partial\epsilon}{\partial z^\alpha} + S_\alpha * \epsilon - \epsilon * S_\alpha. \end{aligned} \tag{2.10}$$

*2.1.5. Truncation.* The following truncation is imposed on the master fields and gauge transformation parameter $\epsilon$. Define

$$R = K\overline{K}.$$

We require

$$[R, W]_* = \{R, S\}_* = [R, B]_* = [R, \epsilon]_* = 0. \tag{2.11}$$

More explicitly, this is the statement that $W_\mu$, $B$ and $\epsilon$ are even functions of $(Y, Z)$ whereas $S_\alpha, S_{\dot{\alpha}}$ are odd in $(Y, Z)$,

$$\begin{aligned} &W_\mu(x, y, \bar{y}, z, \bar{z}) = W_\mu(x, -y, -\bar{y}, -z, -\bar{z}), \\ &S_\alpha(x, y, \bar{y}, z, \bar{z}) = -S_\alpha(x, -y, -\bar{y}, -z, -\bar{z}), \\ &S_{\dot{\alpha}}(x, y, \bar{y}, z, \bar{z}) = -S_{\dot{\alpha}}(x, -y, -\bar{y}, -z, -\bar{z}), \\ &B(x, y, \bar{y}, z, \bar{z}) = B(x, -y, -\bar{y}, -z, -\bar{z}). \\ &\epsilon(x, y, \bar{y}, z, \bar{z}) = \epsilon(x, -y, -\bar{y}, -z, -\bar{z}). \end{aligned} \tag{2.12}$$

A physical reason for the imposition of this truncation is the spin statistics theorem. As the physical fields of Vasiliev's theory are all commuting, they must also transform in the vector (rather than spinor) conjugacy class of the $SO(4)$ tangent group; the projection (2.12) ensures that this is the case. One might expect from this remark that the consistency of Vasiliev's equations requires this truncation; we will see explicitly below that this is the case.

*2.1.6. Reality conditions.* It turns out that Vasiliev's master fields admit two consistent projections that may be used to reduce their number of degrees of freedom. These two projections are a generalized reality projection (somewhat analogous to the Majorana condition for spinors) and a so-called 'minimal' truncation (very loosely analogous to a chirality truncation for spinors). These two truncations are defined in terms of two natural operations defined on the master field; complex conjugation and an operation defined by the symbol $\iota$. In this subsection we first define these two operations, and then use them to define the generalized reality projection. We will also briefly mention the minimal projection, even though we will not use the later in this paper.

Vasiliev's fields master fields admit a straightforward complex conjugation operation, $\mathcal{A} \to \mathcal{A}^*$, defined by complex conjugating each of the component fields of Vasiliev theory and also the spinor variables $Y, Z$ [11]

$$(y^\alpha)^* = \bar{y}_{\dot{\alpha}}, (\bar{y}_{\dot{\alpha}})^* = y^\alpha, (z^\alpha)^* = \bar{z}_{\dot{\alpha}}, (\bar{z}_{\dot{\alpha}})^* = z^\alpha. \tag{2.13}$$

It is easily verified that

$$(M * N)^* = M^* * N^* \tag{2.14}$$

where $M$ is an arbitrary $p$ form and $N$ and arbitrary $q$ form. In other words complex conjugation commutes with the star and wedge product, without reversing the order of either of these products. Note also that the complex conjugation operation squares to the identity.

We now turn to the definition of the operation $\iota$; this operation is defined by

$$\iota : (y, \bar{y}, z, \bar{z}, \mathrm{d}z, \mathrm{d}\bar{z}) \to (\mathrm{i}y, \mathrm{i}\bar{y}, -\mathrm{i}z, -\mathrm{i}\bar{z}, -\mathrm{i}\mathrm{d}z, -\mathrm{i}\mathrm{d}\bar{z}), \tag{2.15}$$

The signs in (2.15) are chosen[12] to ensure

$$\iota(f * g) = \iota(g) * \iota(f) \tag{2.16}$$

(see (A.7) for a proof). In other words $\iota$ reverses the order of the $*$ product. Note however that $\iota$ by definition does not affect the order of wedge products of forms. As a consequence $\iota$ picks up an extra minus sign when acting on the product of two 1-forms

$$\iota(C * D) = -\iota(D) * \iota(C)$$

(see (A.8) for a proof; the same equation is true if $C$ is a $p$ form and $D$ a $q$ form provided $p$ and $q$ are both odd; if at least one of $p$ and $q$ is even we have no minus sign).

We now define the generalized reality projection that we will require Vasiliev's master fields to obey throughout this paper (this projection defines the non-minimal Vasiliev theory which we study through this paper). The projection is defined by the conditions

$$\iota(W)^* = -W, \iota(S)^* = -S, \iota(B)^* = \overline{K} * B * \overline{K} = K * B * K. \tag{2.17}$$

The equality of the two different expressions supplied for $\iota(B)^*$ in (2.17) follows upon using the fact $B$ commutes with $R = K\overline{K}$ (see (2.11)).

[11] As complex conjugation of $SO(3, 1)$ interchanges left and right moving spinors, our definition of complex conjugation (the analytic continuation of the Lorentzian notion) must also have this property.
[12] Changing the RHS of (2.15) by an overall sign makes no difference to fields that obey (2.12).

It is easily verified that (2.17) implies that

$$\iota(\mathcal{F})^* = -\mathcal{F} \tag{2.18}$$

(see (A.13) for an expansion in components) and that

$$\iota(B * K)^* = B * \overline{K}, \qquad \iota(B * \overline{K})^* = B * K. \tag{2.19}$$

Equation (2.17) may be thought of as a combination of two separate projections. The first is the 'standard' reality projection (see (A.9)). The second is the 'minimal truncation' (A.10). As discussed in 2.12, it is consistent to simultaneously impose invariance of Vasiliev's master field under both these projections. This operation defines the minimal Vasiliev theory (dual to $SO(N)$ Chern–Simons field theories). We will not study the minimal theory in this paper.

*2.1.7. Equations of motion.* Vasiliev's gauge invariant equations of motion take the form

$$\begin{aligned} &\mathcal{F} = d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} = f_*(B * K)\,\mathrm{d}z^2 + \overline{f}_*(B * \overline{K})\,\mathrm{d}\bar{z}^2, \\ &d_x B + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) = 0. \end{aligned} \tag{2.20}$$

where $f(X)$ is a holomorphic function of $X$, $\overline{f}$ its complex conjugate, and $f_*(X)$ the corresponding $*$-function of $X$. Namely, $f_*(X)$ is defined by replacing all products of $X$ in the Taylor series of $f(X)$ by the corresponding star products.

Note that both sides of the first of (2.20) are gauge adjoints, while the second line of that equation transforms in the twisted adjoint. In section A.4 we have demonstrated that the second equation of (2.20) may be derived from the first (assuming that $f(X)$ is a non-degenerate function) using the Bianchi identity

$$d_x\mathcal{F} + [A, \mathcal{F}]_* = 0. \tag{2.21}$$

In section A.4 we have also expanded Vasiliev's equations in components to clarify their physical content. As elaborated in (A.14) and (A.15), it follows from (2.20) that the field strength $\mathrm{d}W + W * W$ is flat and that the adjoint fields $B * K$, $S_\alpha$ and $S_{\dot\alpha}$ are covariantly constant. In addition, various components of these adjoint fields commute or anticommute with each other under the star product (see A.24 for a listing). The fields $\hat{S}_\alpha$ and $\hat{S}_\beta$, however, fail to commute with each other; their commutation relations are given by

$$\begin{aligned} &[\hat{S}_\alpha, \hat{S}_\beta]_* = \epsilon_{\alpha\beta} f_*(B * K) \\ &[\hat{S}_{\dot\alpha}, \hat{S}_{\dot\beta}]_* = \epsilon_{\dot\alpha\dot\beta} \bar{f}_*(B * \bar{K}). \end{aligned} \tag{2.22}$$

Using various formulae presented in the appendix (see e.g. (A.11)) it is easily verified that the Vasiliev equations, (expanded in the appendix as (A.14) and (A.15)) map to themselves under the reality projection (2.17). The same is true of the minimal truncation projection.

*2.1.8. Equivalences from field redefinitions.* Vasiliev's equations are characterized by a single complex holomorphic function $f$. In this subsection we address the following question: to what extent to different functions $f$ label different theories?

Any field redefinition that preserves the gauge and Lorentz transformation properties of all fields, but changes the form of $f$ clearly demonstrates an equivalence of the theories with the corresponding choices of $f$. The most general field redefinitions consistent with gauge and Lorentz transformations and the form of Vasiliev's equations are

$$\begin{aligned} &B \to g_*(B * K) * K \\ &\widehat{S_z} \equiv \left(-\tfrac{1}{2}z_\alpha + S_\alpha\right)\mathrm{d}z^\alpha \to \widehat{S_z} * h_*(B * K), \\ &\widehat{S_{\bar z}} \equiv \left(-\tfrac{1}{2}\bar{z}_{\dot\alpha} + \bar{S}_{\dot\alpha}\right) * \mathrm{d}\bar{z}^{\dot\alpha} \to \widehat{S_{\bar z}} * \tilde{h}_*(-B * \overline{K}). \end{aligned} \tag{2.23}$$

Several comments are in order. First note that the field redefinitions above obviously preserve form structure and gauge transformations properties. In particular these redefinitions preserve the fact that $B*K$, $S_z$ and $S_{\bar{z}}$ transform in the adjoint representation of the gauge group. Second the field redefinitions above are purely holomorphic (e.g. $g_*$ is a function only of $B*K$ but not of $B*\overline{K}$). It is not difficult to convince oneself that this is necessary in order to preserve the holomorphic form of Vasiliev's equations. Finally we have chosen to multiply the redefined functions $S_z$ and $S_{\bar{z}}$ with functions from the right rather than the left. There is no lack of generality in this, however, as

$$\begin{aligned}
&\widehat{S}_z * h_*(B*K) = h_*(-B*K) * \widehat{S}_z, \qquad \widehat{S}_z * \overline{h}_*(B*\overline{K}) = \overline{h}_*(B*\overline{K}) * \widehat{S}_z,\\
&\widehat{S}_{\bar{z}} * h_*(B*K) = h_*(B*K) * \widehat{S}_{\bar{z}}, \qquad \widehat{S}_{\bar{z}} * \overline{h}_*(B*\overline{K}) = \overline{h}_*(-B*\overline{K}) * \widehat{S}_{\bar{z}},
\end{aligned} \tag{2.24}$$

((2.24) follows immediately from (A.24) derived in the appendix). Finally, we have inserted a minus sign into the argument of the function $\tilde{h}$ for future convenience.

The reality conditions (2.17) impose constraints on the functions $g$, $h$ and $\tilde{h}$. It is not difficult to verify that $g$ is forced to be an *odd real* function $g(X)$. $g(X)$ is forced to be odd because the complex conjugation operation turns $K$ into $\overline{K}$. When $g$ is odd, however, the truncation (2.11) may be used to turn $\overline{K}$ back into $K$. For instance, with $g_*(X) = g_1X + g_3X*X*X+\cdots$, the field redefinition is

$$B \to g_1B + g_3B*K*B*K*B+\cdots. \tag{2.25}$$

The RHS is still real because $K*B*K = \overline{K}*B*\overline{K}$ (it would not be real if $g(X)$ were not odd).

In order to examine the constraints of (2.17) on the functions $h$ and $\tilde{h}$ note that

$$\begin{aligned}
\iota(S_z * h(B*K) + S_{\bar{z}} * \tilde{h}_*(-B*\overline{K}))^* &= \overline{h}(B*\overline{K}) * (-S_{\bar{z}}) + \overline{\tilde{h}}(-B*K) * (-S_z)\\
&= -(S_{\bar{z}} * \overline{h}(-B*\overline{K}) + S_z * \overline{\tilde{h}}(B*K))
\end{aligned} \tag{2.26}$$

(where in the last step we have used (2.24)). It follows that the redefined function $\widehat{S}$ obeys the reality condition (2.17) if and only if

$$\tilde{h} = \overline{h}$$

where $\overline{h}$ is the complex conjugate of the function $h$.

The effect of the field redefinition of $B$ is simply to permit a redefinition of the argument of the function $f$ in Vasiliev's equations by an arbitrary odd real function. The effect of the field redefinition of $\widehat{S}$ may be deduced as follows. The $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ component of Vasiliev's—the assertion that $W$ is a flat connection (see (A.14))—is clearly preserved by this field redefinition. The $\mathrm{d}x \wedge \mathrm{d}Z$ components of the equation asserts that $\widehat{S}_z$ and $\widehat{S}_{\bar{z}}$ are covariantly constant. As $B*K$ and $B*\overline{K}$ are also covariantly constant (see (A.15)) the redefinition (2.23) clearly preserves this equation as well. However the $\mathrm{d}Z^2$ components of the equations become

$$\begin{aligned}
&\widehat{S}_z * h_*(B*K) * \widehat{S}_z * h_*(B*K) = f_*(B*K)\,\mathrm{d}z^2,\\
&\{\widehat{S} * h_*(B*K), \widehat{S}_{\bar{z}} * \overline{h}_*(-B*\overline{K})\}_* = 0,\\
&\widehat{S}_{\bar{z}} * \overline{h}_*(-B*\overline{K}) * \widehat{S}_{\bar{z}} * \overline{h}_*(-B*\overline{K}) = \overline{f}_*(B*\overline{K})\,\mathrm{d}\bar{z}^2.
\end{aligned} \tag{2.27}$$

Using (2.24) and the fact that $B*K$ commutes with $B*\overline{K}$ (this is obvious as $K$ and $\overline{K}$ commute), these equations may be recast as

$$\begin{aligned}
&h_*(-B*K) * (\widehat{S}_z * \widehat{S}_z) * h_*(B*K) = f_*(B*K)\,\mathrm{d}z^2,\\
&h_*(-B*K) * (\{\widehat{S}, \widehat{S}_{\bar{z}}\}_*) * \overline{h}_*(-B*\overline{K}) = 0,\\
&\overline{h}_*(B*\overline{K}) * (\widehat{S}_{\bar{z}} * \widehat{S}_{\bar{z}}) * \overline{h}_*(-B*\overline{K}) = \overline{f}_*(B*\overline{K})\,\mathrm{d}\bar{z}^2.
\end{aligned} \tag{2.28}$$

(2.28) is precisely the $dZ^2$ component of the Vasiliev equation (the third equation in (A.14)) with the replacement

$$f_*(X) \to h_*(-X)^{-1} * f_*(X) * h_*(X)^{-1}, \tag{2.29}$$

or simply $f(X) \to h(X)^{-1}h(-X)^{-1}f(X)$.

So we see that the theory is really defined by $f(X)$ up to a change of variable $X \to g(X)$ for some odd real function $g(X)$ and multiplication by an invertible holomorphic even function. Provided that the function $f(X)$ admits a power series expansion about $X = 0$ and that $f(0) \neq 0$[13], in section A.6 we demonstrate that we can use these field redefinitions to put $f(X)$ in the form

$$f(X) = \tfrac{1}{4} + X \exp(\mathrm{i}\theta(X)) \tag{2.30}$$

where $\theta(X) = \theta_0 + \theta_2 X^2 + \cdots$ is an arbitrary real even function.

Ignoring the special cases for which $f(X)$ cannot be cast into the form (2.30), the function $\theta(X)$ determines the general parity-violating Vasiliev theory.

*2.1.9. The AdS solution.* While Vasiliev's system is formulated in terms of a set of background independent equations, the perturbation theory is defined by expanding around the $AdS_4$ vacuum. In order to study this solution it is useful to establish some conventions. Let $e_0^a$ and $w_0^{ab}$ $(a, b = 1, \ldots, 4)$ denote the usual vielbein and spin connection 1-forms on any space (the index $a$ transforms under the vector representation of the tangent space $SO(4)$). We define the corresponding bi-spinor objects

$$e_{\alpha\dot\beta} = \tfrac{1}{4}\, e^a \sigma^a_{\alpha\dot\beta}, \qquad w_{\alpha\beta} = \tfrac{1}{16} w^{ab}\sigma^{ab}_{\alpha\beta}, \qquad w_{\dot\alpha\dot\beta} = -\tfrac{1}{16} w^{ab}\bar\sigma^{ab}_{\dot\alpha\dot\beta}. \tag{2.31}$$

(see section A.7 for definitions of the $\sigma$ matrices that appear in this equation.) Let $e_0$ and $\omega_0$ be the vielbein and spin connection of Euclidean $AdS_4$ with unit radius. It may be shown that (see section A.8 for some details)

$$\begin{aligned} \mathcal{A} &= W_0(x|Y) \equiv e_0(x|Y) + \omega_0(x|Y) \\ &= (e_0)_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta} + (\omega_0)_{\alpha\beta} y^\alpha y^\beta + (\omega_0)_{\dot\alpha\dot\beta} \bar y^{\dot\alpha} \bar y^{\dot\beta}, \qquad B = 0 \end{aligned} \tag{2.32}$$

solves Vasiliev's equations. We refer to this solution as the $AdS_4$ vacuum (as we will see below this preserves the $SO(2, 4)$ invariance of AdS space).

In the following we will find it convenient to work with a specific choice of coordinates and a specific choice of the vielbein field. For the metric on AdS space we work in Poincaré coordinates; the metric written in Euclidean signature takes the form

$$ds^2 = \frac{d\vec x^2 + dz^2}{z^2}. \tag{2.33}$$

We also define the vielbein 1-form fields

$$e_0^i = -\frac{dx^i}{z}, \qquad e_0^4 = -\frac{dz}{z} \tag{2.34}$$

($a$ runs over the index $i = 1, \ldots, 3$ and $a = 4$). The corresponding spin connection one form fields are given by

$$w_0^{ab} = \frac{dx^{\mathrm{i}}}{4z}[\mathrm{Tr}(\sigma^{\mathrm{iz}}\sigma^{ab}) - \mathrm{Tr}(\bar\sigma^{\mathrm{iz}}\bar\sigma^{ab})] \tag{2.35}$$

[13] This condition can probably be weakened, but cannot be completely removed. For example if $f(X)$ is an odd function, it is easy to convince oneself that it cannot be cast into the form (2.30). In this paper we will be interested in the Vasiliev duals to field theories. In the free limit, the dual Vasiliev theories to the field theory in question are given by $f(X)$ of the form (2.30) with $\theta = 0$. It follows that, at least in a power series in the field theory coupling, the Vasiliev duals to the corresponding field theories are defined by an $f(X)$ that can be put in the form (2.30).

Using (2.31) we have explicitly

$$\omega_0(x|Y) = -\frac{1}{8}\frac{dx^i}{z}(y\sigma^{iz}y + \bar{y}\bar{\sigma}^{iz}\bar{y}),$$
$$e_0(x|Y) = -\frac{1}{4}\frac{dx_\mu}{z}y\sigma^\mu\bar{y}. \quad (2.36)$$

Here our convention for contracting spinor indices is $y\sigma^\mu\bar{y} = y^\alpha(\sigma^\mu)_\alpha{}^{\dot\beta}\bar{y}_{\dot\beta}$, etc (see section A.7).

*2.1.10. Linearization around AdS.* The linearization of Vasiliev's equations around the AdS solution of the previous subsection, yields Fronsdal's equations for the fields of all spins $s = 1, 2, \ldots, \infty$ together with the free minimally coupled equation for an $m^2 = -2$ scalar field. The demonstration of this fact is rather involved; we will not review it here but instead refer the reader to [3, 25] for details. In this subsection we content ourselves with reviewing a few structural features of linearized solutions that will be of use to us in the following.

In the linearization of Vasiliev's equations around AdS, it turns out that the physical degrees of freedom are contained entirely in the master fields restricted to $Z \equiv (z_\alpha, \bar{z}_{\dot\alpha}) = 0$. The spin-$s$ degrees of freedom are contained in

$$\Omega^{(s-1+m,s-1-m)} = W_\mu(x, Y, Z = 0)|_{y^{s-1+m}\bar{y}^{s-1-m}},$$
$$C^{(2s+n,n)} = B(x, Y, Z = 0)|_{y^{2s+n}\bar{y}^n}, \quad (2.37)$$
$$C^{(n,2s+n)} = B(x, Y, Z = 0)|_{y^n\bar{y}^{2s+n}},$$

for $-(s-1) \leqslant m \leqslant (s-1)$ and $n \geqslant 0$. In particular, $W(x, Y, Z = 0)|_{y^{s-1}\bar{y}^{s-1}} = \Omega_{\alpha\dot\beta|\alpha_1\cdots\alpha_{s-1}\dot\beta_1\cdots\dot\beta_{s-1}}y^{\alpha_1}\cdots y^{A_{s-1}}\bar{y}^{\dot\beta_1}\cdots\bar{y}^{\dot\beta_{s-1}}\,dx^{\alpha\dot\beta}$ contains the rank-$s$ symmetric (double-)traceless (metric-like) tensor gauge field[14], and $B|_{y^{2s}}, B|_{\bar{y}^{2s}}$ contain the self-dual and anti-self-dual parts of the higher spin generalization of the Weyl curvature tensor (and involve up to $s$ spacetime derivatives on the symmetric tensor field). While the components of $W_\mu$ and $B$ listed above are sufficient to recover all information about the spin $s$ fields, they are not the only components of the Vasiliev field that are turned on in the linearized solution. The linearized Vasiliev equations relate the components

$$\cdots \leftarrow C^{(1,2s+1)} \leftarrow C^{(0,2s)} \leftarrow \Omega^{(0,2s-2)}\cdots \leftarrow \Omega^{(s-2,s)} \leftarrow \Omega^{(s-1,s-1)} \rightarrow$$
$$\hookrightarrow \Omega^{(s,s-2)} \rightarrow \cdots\Omega^{(2s-2,0)} \rightarrow C^{(2s,0)} \rightarrow C^{(2s+1,1)} \rightarrow \cdots. \quad (2.38)$$

Starting from $\Omega^{(s-1,s-1)}$, the arrows (to the left as well as to the right) are generated by the action of derivatives. This may schematically be understood as follows. $\Omega^{(s-1,s-1)}$ has $s-1$ symmetrized $\alpha$ type and $s-1$ symmetrized $\dot\alpha$ type indices. Acting with the derivative $\partial_{\gamma\dot\beta}$, symmetrizing $\gamma$ with all the $\alpha$ type indices but contracting $\dot\beta$ with one of the $\dot\alpha$ type indices yields an object with $s$ $\alpha$ type indices but only $s-2$ $\dot\alpha$ type indices, taking us along the right arrow from $\Omega^{(s-1,s-1)}$ in (2.38). A similar operation, interchanging the role of dotted and undotted indices takes us along to the left.

The equations for the metric-like fields $\varphi_{\mu_1\cdots\mu_s}$ of the standard form $(\Box - m^2)\varphi_{\mu_1\cdots\mu_s} + \cdots =$ (nonlinearterms) can be extracted from Vasiliev's equation by solving the auxiliary fields in terms of the metric-like fields order by order.

[14] In order to formulate Fronsdal type equations with higher spin gauge symmetry of the form $\delta\varphi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} + \cdots$, the spin-$s$ gauge field is taken to be a rank-$s$ symmetric double-traceless tensor field $\varphi_{\mu_1\cdots\mu_s}$. The trace part can be gauged away, however, leaving a symmetric rank-$s$ traceless tensor.

*2.1.11. Parity.* We wish to study Vasiliev's equations in an expansion around AdS space (with asymptotically AdS boundary conditions, as we will detail in the next section). Consider the action of a parity operation. In the coordinates of (2.33) this operation acts as $x^i \to -x^i$ (for $i = 1, \ldots, 3$). In order to fix the action of parity on the spinors $y^\alpha$, $\bar{y}^{\dot{\alpha}}$ and $z^\alpha$ and $\bar{z}^{\dot{\alpha}}$ we adopt the choice of vielbein (2.34). With this choice the vielbeins are oriented along the coordinate axes and the parity operator on spinors takes the standard flat space form $\Gamma_5\Gamma_1\Gamma_2\Gamma_3 = \Gamma_4$. Using the explicit form for $\Gamma_4$ listed in (A.29), it follows that under parity

$$\begin{aligned}
&\mathbf{P}(W(\vec{x}, z, \mathrm{d}\vec{x}, \mathrm{d}z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = W(-\vec{x}, z, -\mathrm{d}\vec{x}, \mathrm{d}z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}}),\\
&\mathbf{P}(S(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = S(-\vec{x}, z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}}), \qquad (2.39)\\
&\mathbf{P}(B(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = \pm B(-\vec{x}, z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}})
\end{aligned}$$

(while the parity transformation of the 1-form fields $W$ and $S$ are fixed by the transformations of $\mathrm{d}x^\mu$ and $\mathrm{d}Z$, the scalar $B$ can be either parity odd or parity even). With the choice of conventions adapted in section A.7, $\mathrm{i}\sigma_z = -I$. Consequently parity symmetry acts on $(Y, Z)$ by exchanging $y_\alpha \leftrightarrow -\bar{y}_{\dot{\alpha}}$, $z_\alpha \leftrightarrow -\bar{z}_{\dot{\alpha}}$, and so exchanges the two terms $f_*(B * K)\,\mathrm{d}z^2$ and $\overline{f}_*(B * \overline{K})\mathrm{d}\bar{z}^2$ in the equation of motion.

When are Vasiliev's equations invariant under parity transformations? As we have seen above, $B$ may be either parity even or odd. Thus we need either $f(X) = \overline{f}(X)$ or $f(X) = \overline{f}(-X)$. Combined with (2.30), we have

$$f_A(X) = \tfrac{1}{4} + X, \text{ (A type) or } f_B(X) = \tfrac{1}{4} + \mathrm{i}X \text{ (B type)}. \qquad (2.40)$$

They define the A-type and B-type Vasiliev theories, respectively.

Without imposing parity symmetry, however, the interactions of Vasiliev's system is governed by the function $f(X)$, or the phase $\theta(X)$. If $\theta(X)$ is not 0 or $\pi/2$, parity symmetry is violated. Parity symmetry is formally restored, however if we assign nontrivial parity transformation on $\theta(X)$ (i.e. on the coupling parameters $\theta_{2n}$) as well; there are two ways of doing this, with the scalar master field $B$ being parity even or odd:

$$\begin{aligned}
&P_A : B \to B, \quad \theta(X) \to -\theta(X), \quad \text{or}\\
&P_B : B \to -B, \quad \theta(X) \to \pi - \theta(X).
\end{aligned} \qquad (2.41)$$

This will be useful in constraining the dependence of correlation functions on the coupling parameters $\theta_{2n}$.

*2.1.12. The duals of free theories.* The bulk scalar of Vasiliev's theory turns out to have an effective mass $m^2 = -2$ in units of the AdS radius. Near the boundary $z = 0$ in the coordinates of (2.33) the equation of motion the bulk scalar field $S$ to take the form

$$S \simeq az + bz^2 \qquad (2.42)$$

while the bulk vector field takes the form

$$A_\mu \simeq a_\mu + j_\mu z. \qquad (2.43)$$

In order to completely specify Vasiliev's dynamical system we need to specify boundary conditions for the bulk scalar and vector fields (the unique consistent boundary condition of fields of higher spin is that they decay near the boundary like $z^{s+1}$).) We postpone a systematic study of boundary conditions to the next section. In this subsection we specify the boundary conditions that define, respectively, the Vasiliev dual to the theory of free bosons and free fermions.

The type A bosonic Vasiliev theory with $b = 0$ (for the unique bulk scalar) and $a_\mu = 0$ (for the unique bulk vector field) is conjectured to be dual to the theory of a single fundamental

$U(N)$ boson coupled to $U(N)$ Chern–Simons theory at infinite level $k$. The primary single trace operators of this theory have quantum numbers

$$\sum_{s=0}^{\infty}(s+1,s)$$

(the first label above refers to the scaling dimension of the operator, while the second label its spin), exactly matching the linearized spectrum of type A Vasiliev theory. In section 3.2 below we demonstrate that these are the only boundary conditions for the type A theory that preserve higher spin symmetry, the necessary and sufficient condition for these equations to be dual to the theory of free scalars [11].

The spectrum of primaries of a theory of free fermions subject to a $U(N)$ singlet condition is given by

$$(2,0)+\sum_{s=1}^{\infty}(s+1,s).$$

This is exactly the spectrum of the type B Vasiliev theory with boundary conditions $a = a_\mu = 0$. It is not difficult to convince oneself that these are the unique boundary conditions for the type B theory that preserve conformal invariance; in section 3.2 below that they also preserve the full the higher spin symmetry algebra, demonstrating that this Vasiliev system is dual to a theory of free fermions.

## 2.2. *Non-Abelian generalization*

Vasiliev's system in $AdS_4$ admits an obvious generalization to non-Abelian higher spin fields, through the introduction of Chan–Paton factors, much like in open string field theory. We simply replace the master fields $W, S, B$ by $M \times M$ matrix valued fields, and replace the $*$-algebra in the gauge transformations and equations of motion by its tensor product with the algebra of $M \times M$ complex matrices. In making this generalization we modify neither the truncation (2.11) nor the reality condition (2.17) (except that the complex conjugation in (2.17) is now defined with Hermitian conjugation on the $M \times M$ matrices). We will refer to this system as Vasiliev's theory with $U(M)$ Chan–Paton factors.

One consequence of this replacement is that the $U(1)$ gauge field in the bulk turns into a $U(M)$ gauge field, and all other bulk fields are $M \times M$ matrices that transform in the adjoint of this gauge group.

It is natural to conjecture that the non-minimal bosonic Vasiliev theory with $U(M)$ Chan–Paton factors is then dual to $SU(N)$ vector model with $M$ flavors. Take the example of A-type theory in $AdS_4$ with $\Delta = 1$ boundary condition. The dual CFT is that of $NM$ free massless complex scalars $\phi_{ia}$, $i = 1, \ldots, N$, $a = 1, \ldots, M$, restricted to the $SU(N)$-singlet sector. The conserved higher spin currents are single trace operators in the adjoint of the $U(M)$ global flavor symmetry. The dual bulk theory has a coupling constant $g \sim 1/\sqrt{N}$. The *bulk* 't Hooft coupling is then

$$\lambda = g^2 M \sim \frac{M}{N}. \tag{2.44}$$

We thus expect the bulk theory to be weakly coupled when $M/N \ll 1$. The latter will be referred to as the 'vector model limit' of quiver type theories.

At the classical level the non-Abelian generalization of Vasiliev's theory has $M^2$ different massless spin $s$ fields, and in particular $M^2$ different massless gravitons. This might appear to suggest that the dual field theory has $M^2$ exactly conserved stress tensors, in contradiction with general field theory lore for interacting field theories. In fact this is not the case. In section A.9 we argue that $1/N$ effects lift the scaling dimension of all but one of the $M^2$ apparent stress tensors for every choice of boundary conditions except the one that is dual to a theory of $M^2$ decoupled free scalar or fermionic boundary fields.

### *2.3. Supersymmetric extension*

Following [2, 3, 6, 15, 16], to construct Vasiliev's system with extended SUSY, we introduce Grassmannian auxiliary variables $\psi_i$, $i = 1, \ldots, n$, that obey Clifford algebra $\{\psi_i, \psi_j\} = 2\delta_{ij}$, and commute with all the twistor variables $(Y, Z)$. By definition, the $\psi_i$'s do not participate in the $*$-algebra. The master fields $W, S, B$, as well as the gauge transformation parameter $\epsilon$, are now all functions of $\psi_i$'s as well as of $(x^\mu, y_\alpha, \bar{y}_{\dot{\alpha}}, z_\alpha, \bar{z}_{\dot{\alpha}})$.

The operators $\psi_i$ may be thought of as $\Gamma$ matrices that act on an auxiliary $2^{n/2}$ dimensional 'spinor' space (we assume from now on that $n$ is even). Note that an arbitrary $2^{n/2} \times 2^{n/2}$ dimensional matrix can be written as a linear sum of products of $\Gamma$ matrices[15]. Consequently at this stage the extension of Vasiliev's system to allow for all fields to be functions of $\psi_i$ is simply identical to the non-Abelian extension of the previous subsection, for the special case $M = 2^{n/2}$. The construction of this subsection differs from that of the previous one in the truncation we apply on fields. The condition (2.11) continues to take the form

$$[R, W]_* = \{R, S\}_* = [R, B]_* = [R, \epsilon]_* = 0. \tag{2.45}$$

but with $R$ now defined as

$$R \equiv K\overline{K}\Gamma \tag{2.46}$$

and where

$$\Gamma \equiv i^{\frac{n(n-1)}{2}} \psi_1 \psi_2 \cdots \psi_n \tag{2.47}$$

(note that $\Gamma^2 = 1$ and that it is still true that $R * R = 1$).

While the modified truncation (2.45) looks formally similar to (2.11), it has one very important difference. As with (2.11) it ensures that those operators that commute with $\Gamma$ (i.e. are even functions of $\psi_i$) are also even functions of the spinor variables $Y, Z$. However odd functions of $\psi_i$, which anticommute with $\Gamma$, are now forced to be odd functions of $Y, Z$. Such functions transform in spinorial representations of the internal tangent space $SO(4)$. Consequently, the new projection introduces bulk spinorial fields into Vasiliev's theory, and simultaneously ensures that such fields are always anticommuting, in agreement with the spin statistics theorem.

The reality projection we impose on fields is almost unchanged compared to (2.17). We demand

$$\iota(W)^* = -W, \quad \iota(S)^* = -S, \quad \iota(B)^* = \overline{K} * B * \overline{K}\Gamma = \Gamma K * B * K. \tag{2.48}$$

The operation $\iota$ and the complex conjugation on the master fields, $\mathcal{A} \to \mathcal{A}^*$, are defined in the section 2.1, in combination with $\iota : \psi_i \to \psi_i$ but reverses the order of the product of $\psi_i$'s, and $\psi_i$'s are real under complex conjugation. We require $\iota$ to reverse the order of $\psi_i$ in order to ensure that

$$\iota(\Gamma)^* = \Gamma^{-1} = \Gamma.$$

(The reversal in the order of $\psi_i$ compensates for the sign picked up by the factor of $\mathrm{i}^{\frac{n(n-1)}{2}}$ under complex conjugation in (2.47)). The only other modification in (2.48) compared to (2.17) is in

[15] This fact gives a map from the space of $2^{n/2} \times 2^{n/2}$ dimensional matrices to constant forms on an $n$ dimensional space, where $\psi_i$ is regarded as a basis 1-form. Every $2^{n/2} \times 2^{n/2}$ dimensional matrix can be uniquely decomposed into the sum of a zero form $a_0 I$, a 1-form $a^i \psi_i$, a two form $a^{ij} \psi_i \psi_j \ldots$ an $n$ form $a_n \psi_1 \psi_2 \ldots \psi_n$. The number of basis forms is $(1+1)^n = 2^n$, precisely matching the number of independent matrix elements.

the factor on $\Gamma$ in the action on $B$; this additional factor is necessary in order for the two terms on the RHS of $\iota(B)^*$ to be the same, after using the truncation equation (2.45), given that $R$ in this section has an additional factor of $\Gamma$ as compared to the bosonic theory.

Vasiliev's equations take the form

$$\mathcal{F} = d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} = f_*(B * K)\,\mathrm{d}z^2 + \overline{f}_*(B * \overline{K}\Gamma)\,\mathrm{d}\bar{z}^2,$$
$$d_x B + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) = 0. \tag{2.49}$$

Compared to the bosonic theory, the only change in the first Vasiliev equation is the factor of $\Gamma$ in the argument of $\overline{f}$; this factor is needed in order to preserve the reality of Vasiliev equations under the operation (2.48), as it follows from (2.48) that

$$\iota(B * K)^* = \overline{K} * \overline{K} * B * \overline{K}\Gamma = B * \overline{K}\Gamma.$$

The second Vasiliev equation is unchanged in form from the bosonic theory; however the operator $\pi$ is now taken to mean conjugation by $\Gamma\overline{K}$ together with $\mathrm{d}\bar{z} \to -\mathrm{d}\bar{z}$, or equivalently, by the truncation condition (2.45) on the fields, conjugation by $K$ together with $\mathrm{d}z \to -\mathrm{d}z$. Note in particular that

$$\begin{aligned}\pi(S) &= K * S_{\bar{z}} * K + \Gamma\overline{K} * S_z * \Gamma\overline{K}\\ &= S_{\dot{\alpha}}(x| - y, \bar{y}, -z, \bar{z}, \psi)\,\mathrm{d}\bar{z}^{\dot{\alpha}} + S_\alpha(x|y, -\bar{y}, z, -\bar{z}, -\psi)\,\mathrm{d}z^\alpha\\ &= S(x|y, -\bar{y}, z, -\bar{z}, -\psi, \mathrm{d}z, -\mathrm{d}\bar{z}).\end{aligned} \tag{2.50}$$

As in the case of the bosonic theory, $f(X)$ can generically be cast into the form $f(X) = \frac{1}{4} + X\exp(\mathrm{i}\theta(X))$ by a field redefinition.

The expansion into components of the first of (2.49) is given by (A.14), with the last line of that equation replaced by

$$\hat{S} * \hat{S} = f(B * K)\,\mathrm{d}z^2 + \bar{f}(B * \overline{K}\Gamma)\bar{z}^2. \tag{2.51}$$

The expansion in components of the second line of (2.49) is given by (A.15) with no modifications.

As in the case of the bosonic theory, the second equation in (2.49) follows from the first using the Bianchi identity for the field strength. The details of the derivation differ in only minor ways from the bosonic derivation presented in section A.4.[16]

Parity acts as

$$\begin{aligned}&\mathbf{P}(W(\vec{x}, z, \mathrm{d}\vec{x}, \mathrm{d}z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = W(-\vec{x}, z, -\mathrm{d}\vec{x}, \mathrm{d}z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}}),\\ &\mathbf{P}(S(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = S(-\vec{x}, z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}}),\\ &\mathbf{P}(B(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = B(-\vec{x}, z|\mathrm{i}(\sigma_z\bar{y})_\alpha, \mathrm{i}(\sigma_z\bar{z})_\alpha, \mathrm{i}(\sigma_z y)_{\dot{\alpha}}, \mathrm{i}(\sigma_z z)_{\dot{\alpha}})\Gamma.\end{aligned} \tag{2.52}$$

The factor of $\Gamma$ in the last of (2.52) is needed in order that the theory with $f(X) = \frac{1}{4} + X$ is parity invariant.

### 2.4. The free dual of the parity preserving SUSY theory

In this subsection we consider the dual description of the parity preserving Vasiliev theory with appropriate boundary conditions. The equations we study have $f(X) = \frac{1}{4} + X$. Let us examine the bulk scalar fields which are given by the bottom component of the $B$ master field, namely $\Phi(x, \psi) = B(x|Y = Z = 0, \psi)$, which obeys the truncation condition $\Gamma\Phi\Gamma = \Phi$, i.e. $\Phi$ is even in the $\psi_i$. There are $2^{n-1}$ real scalars, half of which are parity even, the other half

[16] Equation (A.17) holds unchanged, (A.18) holds with $\overline{K} \to \overline{K}\Gamma$ these two equations are equivalent by (2.45). Equation (A.20) holds unchanged. Equation (A.22) applies with $\overline{K} \to \overline{K}\Gamma$. Equation (A.23) holds unchanged.

parity odd. We impose boundary conditions to ensure that $\Delta = 1$ for the parity even scalars and $\Delta = 2$ for the parity odd scalars (see the next sections for details). In other words the fall off near the boundary is given by (2.42), with $b = 0$ for parity even scalars, $a = 0$ for all parity odd scalars. The boundary fall off for all gauge fields is given by (2.43) with $a_\mu = 0$.

The bulk theory has also $m = 0$ spin-1/2 bulk fermions, whose boundary conditions we now specify. Recall (see e.g. [26]) that the AdS/CFT dictionary for such fermions identifies the 'source' with the coefficient of the $z^{3/2}$ fall off of the parity even part of the bulk fermionic field (the same information is also present in the $z^{5/2}$ fall off of the parity odd part of the fermion field), while the 'operator vev' is identified with the coefficient of the $z^{3/2}$ of the parity odd part of the bulk fermion field (the same information is also present in the $z^{5/2}$ fall off of the parity even part of the fermion field). We impose the standard boundary conditions that set all sources to zero, i.e. we demand that the leading $\mathcal{O}(z^{3/2})$ fall off of the fermionic field is entirely parity odd. We believe these boundary conditions preserve the fermionic higher spin symmetry (see section 5.4.1 for a partial verification) and so yield the theory dual to a free field theory.

The field content of this dual field theory is as follows; we have $2^{\frac{n}{2}-1}$ complex scalars in the fundamental representation and the same number of fundamental fermions (so that the singlets constructed out of bilinears of scalars or fermions match with the bulk scalars). We organize the fields in the boundary theory in the form

$$\phi_{iA}, \quad \psi_{i\dot{B}\alpha},$$

where $i$ is the $SU(N)$ index, $A, \dot{B}$ are chiral and anti-chiral spinor indices of an $SO(n)$ global symmetry, and $\alpha$ denotes the spacetime spinor index of $\psi_{i\dot{B}}$. The $2^{n-2} + 2^{n-2}$ $SU(N)$ singlet scalar operators, of dimension $\Delta = 1$ and $\Delta = 2$, are

$$\bar{\phi}^{iA}\phi_{iB}, \quad \bar{\psi}^{i\dot{A}}\psi_{i\dot{B}}. \tag{2.53}$$

They are dual to the bulk fields (projected to the parity even and parity odd components, respectively)

$$\Phi_+ = \Phi\frac{1+\Gamma}{2}, \qquad \Phi_- = -i\Phi\frac{1-\Gamma}{2}. \tag{2.54}$$

The free CFT has $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ bosonic flavor symmetry that act on the scalars and fermions separately, as well as $2^{n-2}$ complex fermionic symmetry currents

$$(J_{\alpha\mu})^{\dot{B}}{}_A = \bar{\psi}^{i\dot{B}}_\alpha \overleftrightarrow{\partial}_\mu \phi_{iA} + \cdots. \tag{2.55}$$

The Vasiliev bulk dual of the $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ global symmetry is given by Vasiliev gauge transformations with $\epsilon$ independent of $x, Y$ or $Z$, but an arbitrary real even function of $\psi_i$ (i.e. an arbitrary even Hermitian operator built out of $\psi_i$). Operators of this nature may be subdivided into parity even and parity odd Hermitian operators which mutually commute. The $2^{n-2}$ parity even operators of this nature generate one factor of $U(2^{\frac{n}{2}-1})$ while the complementary parity even operators generate the second factor. The two central $U(1)$ elements are generated by $I + \Gamma$ and $I - \Gamma$ respectively; these operators clearly commute with all even functions of $\psi_i$, and so commute with all other generators, establishing their central nature[17]. It is easily

[17] As an example let us consider the case $n = 4$ that is of particular interest to us below. The parity even $U(2) = U(1) \times SU(2)$ is generated by

$$(1 + \Gamma), (1 + \Gamma)\psi_4\psi_i$$

while the parity odd $U(2) = U(1) \times SU(2)$ is generated by

$$(1 - \Gamma), (1 - \Gamma)\psi_4\psi_i$$

(where $i = 1, \ldots, 3$).

verified that parity even Vasiliev scalars transform are neutral under the parity odd $U(2^{\frac{n}{2}-1})$ but transform in the adjoint of the parity even $U(2^{\frac{n}{2}-1})$ (the reverse statement is also true). On the other hand the parity even/odd spin-1/2 fields of Vasiliev theory transform in the (fundamental, antifundamental) and (fundamental, antifundamental), all in agreement with field theory expectations.

With the boundary conditions described in this section, the bulk theory may be equivalently written as the $n' = 2$ (i.e. minimally) extended supersymmetric Vasiliev theory with $U(2^{\frac{n}{2}-1})$ Chan–Paton factors and boundary conditions that preserved this symmetry. Our main interest in the bulk dual of the free theory, however, is as the starting point for the construction of the bulk dual of interacting theories. This will necessitate the introduction of parity violating phases into the theory and simultaneously modifying boundary conditions. The boundary conditions we will introduce break the $U(2^{\frac{n}{2}-1})$ global symmetry down to a smaller subgroup. In every case of interest the subgroup in question will turn out to be a subgroup of $U(2^{\frac{n}{2}-1})$ that is also a subgroup of the $SO(n)$ [18] that rotates the $\psi_i$ (here the $\psi_i$ are the fermionic fields that enter Vasiliev's construction, not the fermions of the dual boundary theory). As the preserved symmetry algebras have a natural action on $\psi_i$, the language of extended SUSY will prove considerably more useful for us in subsequent sections than the language of the non-Abelian extension of the $n = 2$ theory, which we will never adopt in the rest of this paper.

## 3. Higher spin symmetry breaking by AdS$_4$ boundary conditions

In this technical section, we will demonstrate that higher spin bulk symmetries are broken by nontrivial values of the phase function $\theta$ and by generic boundary conditions.

In this section we study mainly the bosonic Vasiliev theory. We demonstrate that higher spin symmetry is broken by generic boundary conditions and generic values of the Vasiliev phase (see [27] for an overlapping recent discussion). Higher spin symmetry is preserved *only* for the type A and type B Vasiliev theories with boundary conditions described in section 2.1.12. We will see this explicitly by showing that, in every other case, the *nonlinear* (higher) spin-$s$ gauge transformation on the bulk scalar field, at the presence of a spin-$s'$ boundary source, violates the boundary condition for the scalar field itself for every other choice of phase or boundary condition. We also use this bulk analysis together with a Ward identity to compute the coefficient $c_{ss'0}$ in the schematic equation

$$\partial^\mu J^{(s)}_\mu = c_{ss'0} J^{s'} O + \cdots$$

where the RHS includes the contributions of descendants of $J^{s'}$ and descendants of $O$. The violation of the scalar boundary condition is directly related to a double trace term in the anomalous 'conservation' law of the boundary spin-$s$ current, via a Ward identity.

This section does not directly relate to the study of the bulk duals of supersymmetric Chern Simons theories. Apart from the basic formalism for the study of symmetries in Vasiliev theory (see section 3.1 below) the only result of this subsection that we will use later in the paper are the identifications (3.28) and (3.31) presented below. The reader who is willing to take these results on faith, and who is uninterested in the bulk mechanism of higher spin symmetry breaking, could skip directly from section (3.1) to the next section.

[18] As we will see in the following, we will find it possible to choose boundary conditions to preserve up to $\mathcal{N} = 6$ SUSYs together with a flavor symmetry group which is a subgroup of $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$.

*3.1. Symmetries that preserve the AdS solution*

The asymptotic symmetry group of Vasiliev theory in $AdS_4$ is generated by gauge parameters $\epsilon(x|Y, Z, \psi_i)$ that leave the $AdS_4$ vacuum solution (2.32) invariant. $S = 0$ in the solution (2.32) is preserved if and only if the gauge transformation parameter is independent of $Z$, i.e. it takes the form $\epsilon(x|Y, \psi_i)$. As $B$ transforms homogeneously under gauge transformations, $B = 0$ (in (2.32)) is preserved under arbitrary gauge transformations. The nontrivial conditions on $\epsilon(x|Y, \psi_i)$ arise from requiring that $W = W_0$ is preserved. For this to be the case $\epsilon(x|Y, \psi_i)$ is required to obey the equation

$$D_0\epsilon(x|Y, \psi_i) \equiv d_x\epsilon(x|Y, \psi_i) + [W_0, \epsilon(x|Y, \psi_i)]_* = 0. \tag{3.1}$$

As the gauge field $W_0$ in the $AdS_4$ vacuum obeys the equation $d_xW_0 + W_0 * W_0 = 0$, $W_0$ is a flat connection and so may be written in the 'pure gauge' form.

$$W_0 = L^{-1} * \mathrm{d}L, \tag{3.2}$$

where $L^{-1}$ is the $*$-inverse of $L(x|Y)$. We may formally move to the gauge in which $W_0 = 0$;[19] $W = 0$ is preserved if and only if $\epsilon$ is independent of $x$. Transforming back to the original gauge we conclude that the most general solution to (3.1) is given by $\epsilon(x|Y)$ of the form

$$\epsilon(x|Y, \psi_i) = L^{-1}(x|Y) * \epsilon_0(Y, \psi_i) * L(x|Y). \tag{3.3}$$

where $\epsilon_0(Y)$ is independent of $x$ and is restricted, by the truncation condition, to be an even function of $y$, $\psi_i$.[20]

The gauge function $L(x|Y)$ is not uniquely defined; it may be obtained by integrating the flat connection $W_0$ along a path from a base point $x_0$ to $x$. We would then have $L(x_0|Y) = 1$ and $\epsilon_0(Y) = \epsilon(x_0|Y)$. See [28, 7] for explicit formulae for $L(x|Y)$ in Poincaré coordinates. We have used the explicit form of $L(x|Y)$ to obtain an explicit form for $\epsilon(x|Y)$. We now describe our final result, which may easily independently be verified to obey (3.1)

Let us define $y_\pm \equiv y \pm \sigma^z\bar{y}$. The $*$-contraction between $y_\pm$ and $y_\pm$ is zero, and is nonzero only between $y_\pm$ and $y_\mp$. Namely, we have

$$\begin{aligned}(y_\pm)_\alpha * (y_\pm)_\beta &= (y_\pm)_\alpha(y_\pm)_\beta,\\ (y_\pm)_\alpha * (y_\mp)_\beta &= (y_\pm)_\alpha(y_\mp)_\beta + 2\epsilon_{\alpha\beta}.\end{aligned} \tag{3.4}$$

In Poincaré coordinates, $W_0$ may be written in terms of $y_\pm$ as

$$W_0 = -\frac{\mathrm{d}x^{\mathrm{i}}}{8z}y_+\sigma^{\mathrm{i}z}y_+ + \frac{\mathrm{d}z}{8z}y_+y_-. \tag{3.5}$$

A generating function for solutions to (3.1) is given by

$$\begin{aligned}\epsilon(x|Y) &= \exp[z^{-\frac{1}{2}}\Lambda_+(\vec{x})y_+ + z^{\frac{1}{2}}\Lambda_-y_-]\\ &= \exp\left[\Lambda(x)y + \overline{\Lambda}(x)\bar{y}\right],\end{aligned} \tag{3.6}$$

where $\Lambda_+(\vec{x})$, $\Lambda(\vec{x})$, and $\overline{\Lambda}(\vec{x})$ are given in terms of constant spinors $\Lambda_0$ and $\Lambda_-$ by

$$\begin{aligned}\Lambda_+(\vec{x}) &= \Lambda_0 + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-,\\ \Lambda(x) &= z^{-\frac{1}{2}}\Lambda_+(\vec{x}) + z^{\frac{1}{2}}\Lambda_-,\\ \overline{\Lambda}(x) &= -z^{-\frac{1}{2}}\sigma^z\Lambda_+(\vec{x}) + z^{\frac{1}{2}}\sigma^z\Lambda_-.\end{aligned} \tag{3.7}$$

[19] Note that the formal gauge transformation by $L$ is not a true gauge symmetry of the theory, as it violates the AdS boundary condition. We regard it as merely a solution generating technique.

[20] This is obvious in the gauge in which $W$ vanishes. In the gauge (3.3) it follows from the truncation condition $[\epsilon, R]_* = 0$, and that the fact that $[L(x|Y), R]_* = 0$, we see that $\epsilon_0(Y)$.

$\epsilon(x|Y)$ as defined in (3.6) may directly be verified to obey the linear equation (3.1). Equation (3.6) is a generating function for solutions to that equation in the usual: upon expanding $\epsilon(x|Y)$ in a power series in the arbitrary constant spinors $\Lambda_0$ and $\Lambda_-$ the coefficients of different powers in this Taylor expansion independently obey (3.1) (this follows immediately from the linearity of (3.1)).

Notice that the various Taylor coefficients in (3.6) contains precisely all generating parameters for the universal enveloping algebra of $so(3,2)$ (in the bosonic case) or its appropriate supersymmetric extension (in the SUSY case).

Let us first describe the bosonic case. Recall that, on the boundary, the conserved currents of the higher spin algebra may be obtained by dotting a spin $s$ conserved current with $s-1$ conformal killing vectors. Let us define the 'spin $s$ charges' as the charges obtained out of the spin $s$ conserved current by this dotting process. The spin-$s$ global symmetry generating parameter, $\epsilon^{(s)}(x|Y)$, is then obtained from the terms in (3.6) of homogeneous degree $2s-2$ in $(y,\bar{y})$ (or equivalently in $\Lambda_0$ and $\Lambda_-$).

As a special case consider the 'spin 2' charges, i.e. the charges whose conserved currents correspond to the stress tensor dotted with a single conformal killing vector, i.e. the conformal generators. These generators are quadratic in $(y,\bar{y})$. These generators may be organized under the action of the boundary $SU(2)$ (i.e. the diagonal action of $SU(2)_L$ and $SU(2)_R$) as 3+3+3+1, corresponding to 3D angular momentum generators, momenta, boosts and dilations, in perfect correspondence with generators of the three dimensional conformal group $so(3,2)$.[21] Indeed the set of quadratic Hamiltonians in $Y$, with product defined by the star algebra, provides an oscillator construction of $so(3,2)$.

Let us now turn to the supersymmetric theory. The generators of the full $n$ extended superconformal algebra are given by terms that are quadratic in $(y,\psi_i)$. Terms quadratic in $y$ are conformal generators. Terms quadratic in $\psi_i$ but independent of $y$ are $SO(n)$ $R$ symmetry generators. Terms linear in both $y$ and $\psi_i$ (we denote these by $\epsilon^{(\frac{3}{2})}(x|Y)$) are SUSY and superconformal generators. More precisely the terms involving $\Lambda_0$ are Poincaré SUSY parameters, where the terms involving $\Lambda_-$ are special SUSY generators (in radial quantization with respect to the origin $\vec{x}=0$).

In the following we will make use of the following easily verified algebraic property of the generating function $\epsilon(x|Y)$ (3.6) under $*$ product,

$$\begin{aligned}\epsilon(x|Y)*f(y,\bar{y})&=\epsilon(x|Y)f(y+\Lambda,\bar{y}+\overline{\Lambda}),\\ f(y,\bar{y})*\epsilon(x|Y)&=\epsilon(x|Y)f(y-\Lambda,\bar{y}-\overline{\Lambda}).\end{aligned} \tag{3.8}$$

### *3.2. Breaking of higher spin symmetries by boundary conditions*

Any given Vasiliev theory is defined by its equations of motion together with boundary conditions for all fields. Given any particular boundary conditions one may ask the following question: which of the large gauge transformation described in the previous subsection preserve these boundary conditions? In other words which if any of the gauge transformations have the property that they return a normalizable state (i.e. a solution of Vasiliev's theory that obeys the prescribed boundary conditions) when acting on an arbitrary normalizable state? Such gauge transformations are genuine global symmetries of the system.

In this paper we will study the exact action of the large gauge transformations of the previous section on an arbitrary *linearized* solution of Vasiliev's equations. The most general such solution may be obtained by superposition of the linearized responses to arbitrary

[21] It may be checked that the Poincaré generators are obtained by simply setting $\Lambda_-$ to zero.

boundary sources. Because of the linearity of the problem, it is adequate to study these sources one at a time. Consequently we focus on the linearized solution created by a spin $s$ source at $x = 0$ on the boundary of $AdS_4$. Such a source creates a response of the $B$ field everywhere in $AdS_4$, and in particular in the neighborhood of the boundary at the point $x$. We study the higher spin gauge transformations $\epsilon^{(s')}(x|Y)$ (for arbitrary $s'$) on the $B$ master field at this point. The response to this gauge variation contains fields of various spins $s''$. As we will see below the response for $s'' > 1$ always respects the standard boundary conditions for spin $s''$ fields. However the same is not true of the response of the fields of low spins, namely $s'' = 0, \frac{1}{2}$, or 1. As we have seen in the previous section, for these fields it is possible to choose different boundary conditions, some of which turn out to be violated by the symmetry variation $\delta B$.

In the rest of this section we restrict our attention to the bosonic Vasiliev theory. The variation $\delta B$ under an asymptotic symmetry generated by $\epsilon(x|Y)$ in (3.6) is given by (2.7).

Let $B^{(s)}(x|Y)$ be the spin-$s$ component of the linearized $B(x|Y)$ sourced by a current $J^{(s)}$ on the boundary, i.e. the boundary to bulk propagator for the spin-$s$ component of the $B$ master field with the source inserted at $\vec{x} = 0$. $B^{(s)}(x|Y)$ only contains terms of order $y^{2s+n}\bar{y}^n$ and $y^n\bar{y}^{2s+n}$, $n \geqslant 0$; as we have explained above, the coefficients of these terms are spacetime derivatives of the basic spin $s$ field. We will work in Poincaré coordinates (2.33), with the spin-$s$ source located at $\vec{x} = 0$. Without loss of generality, it suffices to consider the polarization tensor for $J^{(s)}$, a three-dimensional symmetric traceless rank-$s$ tensor, of the form $\varepsilon_{\alpha_1\cdots\alpha_{2s}} = \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2s}}$, for an arbitrary polarization spinor $\lambda$. The corresponding boundary-to-bulk propagator is computed in [7]. Here we generalize it slightly to the parity violating theory, by including the interaction phase $e^{i\theta_0}$, as

$$B^{(s)}(x|Y) = \frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}}\,e^{-y\Sigma\bar{y}}[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z y)^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^{2s}], \tag{3.9}$$

where $\Sigma$ and $\mathbf{x}$ are defined as[22]

$$\Sigma \equiv \sigma^z - \frac{2z}{\vec{x}^2+z^2}\mathbf{x}, \qquad \mathbf{x} \equiv x^\mu\sigma_\mu = \vec{x}\cdot\vec{\sigma} + z\sigma^z. \tag{3.10}$$

Note that this formula is valid for spin $s > 1$, for the standard 'magnetic' boundary condition in the $s = 1$ case and for $\Delta = 1$ boundary condition in the $s = 0$ case. The variation of $B$ under the asymptotic symmetry generated by $\epsilon(x|Y)$ is given by

$$\begin{aligned}\delta B &= -\epsilon * B^{(s)} + B^{(s)} * \pi(\epsilon)\\ &= -\epsilon(x|y,\bar{y})B(x|y+\Lambda,\bar{y}+\overline{\Lambda}) + \epsilon(x|y,-\bar{y})B(x|y-\Lambda,\bar{y}+\overline{\Lambda}),\end{aligned}$$

where we made use of the properties (3.8). Using the explicit expression of the boundary-to-bulk propagator, this is

$$\begin{aligned}\delta B = &-\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}}\{e^{\Lambda y+\overline{\Lambda}\bar{y}}\,e^{-(y+\Lambda)\Sigma(\bar{y}+\overline{\Lambda})}[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y+\Lambda))^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}+\overline{\Lambda}))^{2s}]\\ &- e^{\Lambda y-\overline{\Lambda}\bar{y}}\,e^{-(y-\Lambda)\Sigma(\bar{y}+\overline{\Lambda})}[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y-\Lambda))^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}+\overline{\Lambda}))^{2s}]\}\\ = &-\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}}\,e^{-y\Sigma\bar{y}+z^{-\frac{1}{2}}\Lambda_+(1-\sigma_z\Sigma)y+z^{\frac{1}{2}}\Lambda_-(1+\sigma_z\Sigma)y}\\ &\times\{e^{(z^{-\frac{1}{2}}\Lambda_++z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+-z^{\frac{1}{2}}\Lambda_-)+z^{-\frac{1}{2}}\Lambda_+(\sigma^z-\Sigma)\bar{y}-z^{\frac{1}{2}}\Lambda_-(\sigma^z+\Sigma)\bar{y}}\end{aligned}$$

[22] In the special case $s = 0$ the terms in the square bracket reduce simply to $2\cos\theta_0$. This observation is presumably related to the fact, discussed by Maldacena and Zibhoedov [12], that the scalar and spin $s$ currents in the higher spin multiplets have different natural normalizations. In the following we will, indeed, identify the factor of $\cos\theta_0$ with the ratio of these normalizations.

$$\times\ [\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(y+z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-))^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}-\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)))^{2s}]$$
$$-\ \mathrm{e}^{-(z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-) - z^{-\frac{1}{2}}\Lambda_+(\sigma^z-\Sigma)\bar{y} + z^{\frac{1}{2}}\Lambda_-(\sigma^z+\Sigma)\bar{y}}$$
$$\times\ [\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(y-z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-))^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}-\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)))^{2s}]\}. \tag{3.11}$$

Note that although the source is a spin-$s$ current, there are nonzero variation of fields of various spins in $\delta B$. The self-dual part of the higher spin Weyl tensor, in particular, is obtained by restricting $B(x|Y)$ to $\bar{y}=0$. The variation of the self-dual part of the Weyl tensors of various spins are given by

$$\delta B|_{\bar{y}=0} = -\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}}\,\mathrm{e}^{z^{-\frac{1}{2}}\Lambda_+(1-\sigma_z\Sigma)y + z^{\frac{1}{2}}\Lambda_-(1+\sigma_z\Sigma)y}$$
$$\times\ \{\mathrm{e}^{(z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)}[\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(y+z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-))^{2s}$$
$$+\ \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-))^{2s}]$$
$$-\ \mathrm{e}^{-(z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)}[\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(y-z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-))^{2s}$$
$$+\ \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-))^{2s}]\}. \tag{3.12}$$

Now let us examine the behavior of $\delta B$ near the boundary of AdS$_4$. In the $z\to 0$ limit, the leading order terms in $z$ are given by

$$\delta B|_{\bar{y}=0} \longrightarrow -\frac{z}{|x|^{4s+2}}\,\mathrm{e}^{2z^{\frac{1}{2}}(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-)y}$$
$$\times\ \{\mathrm{e}^{\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-}[\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(z^{\frac{1}{2}}y+\Lambda_+))^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}]$$
$$-\ \mathrm{e}^{-\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ + 2\Lambda_+\Lambda_-}[\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(z^{\frac{1}{2}}y-\Lambda_+))^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}]\}. \tag{3.13}$$

The variation of the spin-$s''$ Weyl tensor, $\delta B^{(s'')}$, is extracted from terms of order $y^{2s''}$ in the above formula, which falls off like $z^{s''+1}$ as $z\to 0$. This is consistent with the boundary condition for fields of spin $s''>1$, independently of the phase $\theta_0$. As promised above, the spin $s''>1$ component of the response to an arbitrary gauge variation *automatically* obeys the prescribed boundary conditions for such field and so appears to yield no restrictions on allowed boundary conditions for the theory.

*3.2.1. Anomalous higher spin symmetry variation of the scalar.* The main difference between the scalar field and fields of arbitrary spin is that the prescribed boundary conditions for scalars involve both the leading as well as the subleading fall off of the scalar field. So while the leading fall off of the scalar field will never be faster than $z^1$ (in agreement with the general analysis above upon setting $s''=0$), this is not sufficient to ensure that the scalar field variation obeys its boundary conditions.

Let us examine the variation of the scalar field due to a higher spin gauge transformation, at the presence of a spin-$s$ source at $\vec{x}=0$ on the boundary. The spin $s''=0$ component of the symmetry variation $\delta B$ is given by (3.12) with $(y,\bar{y})$ set to zero,

$$\delta B^{(0)} = -2\frac{z}{(\vec{x}^2+z^2)^{2s+1}}\sinh[(z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)]$$
$$\times\ [\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z(\Lambda_+ + z\Lambda_-))^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}(\Lambda_+ - z\Lambda_-))^{2s}]$$
$$= \frac{4}{(\vec{x}^2+z^2)^{2s+1}}\sinh\left[2\frac{\vec{x}^2-z^2}{\vec{x}^2+z^2}(\Lambda_+\Lambda_-) + 2\frac{\Lambda_+\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+ - z^2\Lambda_-\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-}{\vec{x}^2+z^2}\right]$$
$$\times\ [\cos\theta_0(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s}z + \mathrm{i}\sin\theta_0\cdot 2s(\lambda(\Lambda_+ + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-))(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s-1}z^2$$
$$+\ \mathcal{O}(z^3)]. \tag{3.14}$$

When expanded in a power series in $\Lambda$, the RHS of (3.14) has the schematic form

$$\mathcal{O}(\Lambda^{2s+2}) \times (\text{Taylor expansion in } \Lambda^4).$$

Recall that the spin-$s'$ symmetry variation (see the previous subsection for a definition) is extracted from terms of order $2s' - 2$ in $\Lambda_\pm$. It follows that we find a scalar response to spin $s'$ gauge transformations only for $s' = s+2, s+4, \ldots$. When this is the case (i.e. when $s' - s$ is positive and even)

$$\begin{aligned}\delta_{(s')}B^{(0)} = {} & \frac{4}{(\vec{x}^2)^{2s+1}} \frac{2^{s'-s-1}}{(s'-s-1)!}\left(\Lambda_+\Lambda_- + \frac{1}{\vec{x}^2}\Lambda_+\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+\right)^{s'-s-1} \\ & \times\ [\cos\theta_0(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s}z + \mathrm{i}\sin\theta_0\cdot 2s(\lambda(\Lambda_+ + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-)) \\ & \times\ (\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s-1}z^2 + \mathcal{O}(z^3)]. \end{aligned} \tag{3.15}$$

Recall that $\Lambda_+ = \Lambda_0 + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-$, and $\Lambda_0$, $\Lambda_-$ are arbitrary constant spinors. For generic parity violating phase $\theta_0$, and $s' > s > 0$ with even $s' - s$, terms of order $z$ and $z^2$ are both nonzero, and so both $\Delta = 1$ and $\Delta = 2$ boundary conditions would be violated, leading to the breaking of spin-$s'$ symmetry.

Note that the condition $s' > s > 0$ and that $s' - s$ is even means that the broken symmetry has spin $s' > 2$. In particular the $s' = 2$ conformal symmetries are never broken[23].

The exceptional cases are when either $\cos\theta_0 = 0$ or $\sin\theta_0 = 0$. These are precisely the interaction phase of the parity invariant theories. In the A-type theory, $\theta_0 = 0$, we see that $\delta B^{(0,0)} \sim z + \mathcal{O}(z^3)$, and so $\Delta = 1$ boundary condition is preserved while $\Delta = 2$ boundary condition would be violated. This is as expected: the A-type theory with $\Delta = 1$ boundary condition is dual to the free $U(N)$ or $O(N)$ theory which has exact higher spin symmetry, whereas the A-type theory with $\Delta = 2$ boundary condition is dual to the critical theory, where the higher spin symmetry is broken at order $1/N$. For the B-type theory, $\theta_0 = \pi/2$, we see that $\delta B^{(0,0)} \sim z^2 + \mathcal{O}(z^3)$, and so the $\Delta = 2$ boundary condition is preserved, while $\Delta = 1$ boundary condition is violated. This is in agreement with the former case being dual to free fermions, and the latter dual to critical Gross–Neveu model where the higher spin symmetry is broken.

In summary, the *only* conditions under which *any* higher spin symmetries are preserved are the type A theory with $\Delta = 1$ or the type B theory with $\Delta = 2$. These are precisely the theories conjectured to be dual to the free boson and free fermion theory respectively, in agreement with the results of [11].

*3.2.2. Ward identity and current non-conservation relation.* To quantify the breaking of higher spin symmetry, we now derive a sort of Ward identity that relates the anomalous spin-$s$ symmetry variation of the bulk fields, as seen above, to the non-conservation relation of the three-dimensional spin-$s'$ current that generates the corresponding global symmetry of the boundary CFT.

Let us first word the argument in boundary field theory language. Let us consider the field theory quantity

$$\langle J^s(0)\cdots\rangle$$

where ... denote arbitrary current insertions away from the point $x^\mu$, and $\langle\rangle$ denotes averaging with the measure of the field theory path integral. On the path integral we now perform the change of variables corresponding to a spin $s'$ 'symmetry'. Let $J^{(s')}_\mu$ denote the corresponding

[23] Note that the extrapolation of this formula to the $s = 0$ case assumes $\Delta = 1$ boundary to bulk propagator, and the variation $\delta_{(s')}B^{(0)}$ is always consistent with the $\Delta = 1$ boundary condition.

current. When $J^{(s')}_\mu$ is conserved this change of variables leaves the path integral unchanged in the neighborhood of $x$ (it acts on the insertions, but we ignore those as they are well separated from $x$). When the current is not conserved, however, it changes the action by $\epsilon\,\partial^\mu J^{(s')}_\mu(y)$. Let us suppose that

$$\partial^\mu J^{(s')}_\mu(y) = \frac{1}{2}\sum_{s_1,s_2} J^{(s_1)}\mathcal{D}^{(s')}_{s_1s_2}J^{(s_2)} + \cdots, \tag{3.16}$$

where $\mathcal{D}^s_{s_1s_2}$ is a differential operator, It follows that, in the large $N$ limit, the change in the path integral induced by this change of variables is given by

$$\int \mathrm{d}^3y\langle J^{(s_1)}(y)\cdots\rangle\mathcal{D}^{(s')}_{s_1s}\langle J^s(0)J^{(s)}(y)\rangle$$

(where we have used the fact that the insertion of canonically normalized double trace operator contributes in the large $N$ limit only under conditions of maximal factorization). In other words the symmetry transformation amounts to an effective operator insertion of $J^{(s_1)}$. Specializing to the case $s_1 = 0$ we conclude that, in the presence of a spin $s$ source $J^{(s)}$, a spin $s'$ symmetry transformation should turn on a non normalizable mode for the scalar field given by

$$\mathcal{D}^{(s')}_{0s}\langle J^s(0)J^{(s)}(y)\rangle. \tag{3.17}$$

Before proceeding with our analysis, we pause to restate our derivation of (3.26) in bulk rather than field theory language. Denote collectively by $\Phi$ all bulk fields, and by $\varphi^{(s)}_{\mu\cdots}$ a particular bulk field of some spin $s$. Consider the spin-$s'$ symmetry generated by gauge parameter $\epsilon(x)$, under which $\varphi_{\mu\cdots} \to \varphi_{\mu\cdots} + \delta_\epsilon\varphi_{\mu\cdots}$. Let $\phi(\vec{x})$ be the renormalized boundary value of $\varphi(\vec{x}, z)$, namely $\varphi(\vec{x}, z) \to z^\Delta\phi(\vec{x})$ as $z \to 0$. Let us consider the expectation value of $\phi(\vec{x})$ at the presence of some boundary source $j^{\mu\cdots}$ (of some other spin $s$) located away from $\vec{x}$. The path integral is invariant under an infinitesimal field redefinition $\Phi \to \Phi + \delta_\epsilon\Phi$, where $\delta_\epsilon$ takes the form of the asymptotic symmetry variation in the bulk, but vanishes for $z$ less than a small cutoff near the boundary, so as to preserve the prescribed boundary condition, $\Phi(\vec{x}', z) \to z^{3-\Delta}j(\vec{x}') + \mathcal{O}(z^\Delta)$. From this we can write

$$\begin{aligned}0 &= \int D\Phi\Big|_{\Phi(\vec{x}',z)\to z^{3-\Delta}j(\vec{x}')+\mathcal{O}(z^\Delta)} \delta_\epsilon[\varphi^{(s_1)}(\vec{x}, z)\exp(-S[\Phi])]\\ &= \langle\delta_\epsilon\varphi^{(s_1)}(\vec{x}, z)\rangle_j - \langle\varphi^{(s_1)}(\vec{x}, z)\delta_\epsilon S\rangle_j.\end{aligned} \tag{3.18}$$

The spin-$s$ source $j$ is subject to the transversality condition $\partial_{i_1} j^{i_1\cdots i_s}_{(s)} = 0$. Now $\delta_\epsilon S$ should reduce to a boundary term,

$$\delta_\epsilon S = \int_{\partial\mathrm{AdS}} \mathrm{d}y\epsilon\partial^\mu J^{(s')}_\mu(y) = \frac{1}{2}\int_{\partial\mathrm{AdS}} \epsilon\sum_{s_1,s_2}\phi^{(s_1)}\mathcal{D}^{s'}_{s_1s_2}\phi^{(s_2)} + \cdots, \tag{3.19}$$

where $\mathcal{D}^s_{s_1s_2}$ is a differential operator, and $J_\mu$ is the boundary current associated with the global symmetry generating parameter $\epsilon$ which is now a constant along the cutoff surface, which is then taken to $z \to 0$. On the RHS of (3.19), we omitted possible higher order terms in the fields. From (3.18) we then obtain the relation

$$\begin{aligned}\langle\delta_\epsilon\varphi^{(s_1)}(\vec{x}, z)\rangle_j &= \left\langle\varphi^{(s_1)}(\vec{x}, z)\int_{\partial\mathrm{AdS}} \mathrm{d}\vec{x}'\epsilon\phi^{(s_1)}(\vec{x}')\mathcal{D}^s_{s_1s_2}\phi^{(s_2)}(\vec{x}')\right\rangle_j + (\text{higher order})\\ &= \epsilon\int_{\partial\mathrm{AdS}} \mathrm{d}\vec{x}'\langle\varphi^{(s_1)}(\vec{x}, z)\phi^{(s_1)}(\vec{x}')\rangle\mathcal{D}^s_{s_1s_2}\langle\phi^{(s_2)}(\vec{x}')\rangle_j + (\text{higher order}).\end{aligned} \tag{3.20}$$

Now specialize to the case $s_1 = 0$, i.e. $\varphi^{(s_1)}$ is the scalar field $\varphi$ subject to the boundary condition such that the dual operator has dimension $\Delta$. The anomalous symmetry variation shows up in terms of order $z^{3-\Delta}$ in $\delta_\epsilon \varphi(\vec{x}, z)$. After integrating out $\vec{x}'$ using the two-point function of $\varphi$ and taking the limit $z \to 0$, we obtain the relation

$$\langle \delta_\epsilon \varphi(\vec{x}, z)\rangle_j|_{z^{3-\Delta}} = \epsilon \mathcal{D}^s_{0s_2} \left\langle \phi^{(s_2)}(\vec{x})\right\rangle_j + (\text{higher order}). \tag{3.21}$$

Keep in mind that $j$ is the spin-$s_2$ transverse boundary source, and $\epsilon$ is the spin-$s$ global symmetry generating parameter. The differential operator $\mathcal{D}^{s'}_{s_1 s_2}$ appears in the spin-$s'$ current non-conservation relation of the form

$$\partial^\mu J^{(s)}_{\mu\cdots} = \frac{1}{2}\sum_{s_1,s_2} J^{(s_1)}_{\cdots}\mathcal{D}^s_{s_1 s_2} J^{(s_2)}_{\cdots} + (\text{total derivative}) + (\text{triple trace}). \tag{3.22}$$

In particular, the double trace term on the RHS that involves a scalar operator takes the form

$$J^{(0)}(\vec{x})\mathcal{D}^s_{0s_2} J^{(s_2)}(\vec{x}) + (\text{total derivative}). \tag{3.23}$$

Knowing the LHS of (3.21) from the gauge variation of Vasiliev's bulk fields, and using that fact that $\left\langle \phi^{(s_2)}(\vec{x})\right\rangle_j$ is given by the boundary two-point function of the spin-$s_2$ current, we can then derive $\mathcal{D}^s_{0s_2}$ using this Ward identity. In other words we have rederived (3.17).

Equation (3.17) applies to arbitrary sources $J^s$ and also to arbitrary spin $s'$ symmetry transformations. Let us assume that our sources is of the form specified in the previous subsection; all spinor indices on the source are dotted so with a constant spinor $\lambda$ which is chosen so that

$$\lambda\vec{\sigma}\sigma_z\lambda = \vec{\epsilon}'.$$

In other words our source is uniformly polarized in the $\epsilon$ direction. Let us also choose the spin $s'$ variation to be generated by the current $J^\mu_{a_1\ldots a_{2s'-2}}\Lambda_0^{a_1}\ldots\Lambda_0^{a_{2s'-2}}$ with

$$\Lambda_0\vec{\sigma}\sigma_z\Lambda_0 = \vec{\epsilon}$$

where $\vec{\epsilon}$ is a constant vector. In other words we have chosen to specialize attention to those symmetries generated by the spin $s'$ current contracted with $s' - 1$ translations in the direction $\epsilon$ rather than with a generic conformal killing vector. If we compare with the asymptotic symmetry variation the bulk scalar derived earlier we must set $\Lambda_-$ to zero and $\Lambda_+ = \Lambda_0$. It follows from the previous subsection that

$$\delta B^{(0)} = \frac{4}{(\vec{x}^2)^{2s_2+1}}\frac{1}{(s-s_2-1)!}\left(\frac{2}{\vec{x}^2}\Lambda_0\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_0\right)^{s-s_2-1} \times [\cos\theta_0(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_0)^{2s_2}z + \mathrm{i}\sin\theta_0\cdot 2s_2(\lambda\Lambda_0)(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_0)^{2s_2-1}z^2 + \mathcal{O}(z^3)]. \tag{3.24}$$

In the $\Delta = 1$ case, the anomalous variation comes from the order $z^2$ term in (3.24), giving

$$\mathcal{D}^s_{0s_2}\langle\phi^{(s_2)}(\vec{x})\rangle_j = \sin\theta_0 C_{ss_2}\frac{(\varepsilon\cdot x)^{s-s_2}(2x\cdot\varepsilon x\cdot\varepsilon' - x^2\varepsilon\cdot\varepsilon')^{s_2-1}\epsilon^{\mu\nu\rho}\varepsilon'_\mu\varepsilon_\nu x_\rho}{(\vec{x}^2)^{s+s_2+1}}, \tag{3.25}$$

Here $C_{ss_2}$ is a numerical constant that depends only on $s$ and $s_2$.

(3.25) gives a formula for the appropriate term in (3.16) when the operators that appear in this equation have two point functions

$$\langle O(0)O(x)\rangle = \frac{\alpha_0}{x^2}, \qquad \langle J^s(0)J^s(x)\rangle = \frac{\alpha_s x_-^{2s}}{x^{4s+2}}. \tag{3.26}$$

Note in particular that these two point functions are independent of the phase $\theta$. Let us now compare this relation to the results of Maldacena and Zhiboedov [12]. Those authors determined the non-conservation relation of currents of spin $s$, which in the lightcone direction to take the form

$$\partial_\mu J^{(s)\mu}{}_{-\ldots-} = \frac{\widetilde{\lambda}_b}{\sqrt{1+\widetilde{\lambda}_b^2}} \sum_{s'} a_{ss'} \epsilon_{-\mu\nu} J^{(0)} \partial_-^{s-s'-1} \partial^\mu J^{(s')\nu}{}_{-\ldots-} + \cdots, \tag{3.27}$$

where $\cdots$ stands for double trace terms involving two currents of nonzero spins, total derivatives, and triple trace terms. Note that the first term we exhibited on the RHS of (3.30) is not a primary by itself, but when combined with the total derivatives term in $\cdots$ becomes a double trace primary operator in the large $N$ limit. We have used the notation $\widetilde{\lambda}_b$ of [12] in the case of quasi-boson theory, but normalized the two-point function of $J^{(0)}$ to be independent of $\widetilde{\lambda}_b$.

Indeed with $(\mathcal{D}^s_{0s'} J^{(s')})_{-\ldots-} \sim \epsilon_{-\mu\nu} \partial_-^{s-s'-1} \partial^\mu J^{(s')\nu}{}_{-\ldots-}$, and the identification

$$\widetilde{\lambda}_b = \tan\theta_0, \tag{3.28}$$

the structure of the divergence of the current agrees with (3.25) obtained from the gauge transformation of bulk fields.

Similarly, in the $\Delta = 2$ case, the anomalous variation comes from the order $z$ term in (3.24). We have

$$\mathcal{D}^s_{0s_2} \left\langle \phi^{(s_2)}(\vec{x}) \right\rangle_j = \cos\theta_0 \widetilde{C}_{ss_2} \frac{(\varepsilon \cdot x)^{s-s'} (2x \cdot \varepsilon x \cdot \varepsilon' - x^2 \varepsilon \cdot \varepsilon')^{s'}}{(\vec{x}^2)^{s+s'+1}}. \tag{3.29}$$

This should be compared to the current non-conservation relation in the quasi-fermion theory, of the form

$$\partial_\mu J^{(s)\mu}{}_{-\ldots-} = \frac{\widetilde{\lambda}_f}{\sqrt{1+\widetilde{\lambda}_f^2}} \sum_{s'} \widetilde{a}_{ss'} J^{(0)} \partial_-^{s-s'-1} J^{(s')}{}_{-\ldots-} + (\text{total derivative}) + \cdots. \tag{3.30}$$

Once again, this agrees with the structure of (3.29), with $(\mathcal{D}^s_{0s'} J^{(s')})_{-\ldots-} \sim \partial_-^{s-s'-1} J^{(s')}{}_{-\ldots-}$, and the identification

$$\widetilde{\lambda}_f = \cot\theta_0. \tag{3.31}$$

Following the argument of [12], the double trace terms involving a scalar operator in the current non-conservation relation we derived from gauge transformation in Vasiliev's theory allows us to determine the violation of current conservation in the three-point function, $\langle(\partial\cdot J^{(s)}) J^{(s')} J^{(0)}\rangle$, and hence fix the normalization of the parity odd term in the $s - s' - 0$ three-point function.

Here we encounter a puzzle, however. By the Ward identity argument, we should also see an anomalous variation under global higher spin symmetry of a field $\varphi^{(s_1)}$ of spin $s_1 > 1$. This is not the case for our $\delta_\epsilon B^{(s_1)}$ as computed in (3.12). Presumably the resolution to this puzzle lies in the gauge ambiguity in extracting the correlators from the boundary expectation value of Vasiliev's master fields, which has not been properly understood thus far. This gauge ambiguity may also explain why one seems to find vanishing parity odd contribution to the three point function by naively applying the gauge function method of [8].[24]

*3.2.3. Anomalous higher spin symmetry variation of spin-1 gauge fields.* Since one can choose a family of mixed electric-magnetic boundary conditions on the spin-1 gauge field in $AdS_4$, such a boundary condition will generically be violated by the nonlinear asymptotic higher spin symmetry transformation as well.

Let us consider the self-dual part of the spin-1 field strength, whose variation is given in terms of $\delta_\epsilon B^{(2,0)}(\vec{x}, z|y)$, i.e. the terms in $\delta_\epsilon B$ of order $y^2$ and independent of $\bar{y}$. According to

[24] We thank S Giombi for discussions on this.

(3.13), the leading order terms in $z$, namely order $z^2$ terms, of $\delta_\epsilon B^{(2,0)}(\vec{x}, z)$ in the presence of a spin-$s$ boundary source at $\vec{x} = 0$ is given by

$$\begin{aligned}\delta_\epsilon B^{(2,0)}(\vec{x}, z|y) \longrightarrow & -\frac{z^2}{|x|^{4s+2}}\left[2\left(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-\right)y\right]^2 \sinh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right]\\ & \times [\mathrm{e}^{\mathrm{i}\theta_0}(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s} + \mathrm{e}^{-\mathrm{i}\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}]\\ & - \mathrm{e}^{\mathrm{i}\theta_0}\frac{4sz^2}{|x|^{4s+2}}\cdot\left[2\left(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-\right)y\right]\\ & \times \cosh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right](\lambda\mathbf{x}\sigma^z y)(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s-1}\\ & - \mathrm{e}^{\mathrm{i}\theta_0}\frac{2s(2s-1)z^2}{|x|^{4s+2}}\sinh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right](\lambda\mathbf{x}\sigma^z y)^2(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s-2}.\end{aligned} \tag{3.32}$$

The anti-self-dual components, $\delta_\epsilon B^{(0,2)}(\vec{x}, z|\bar{y})$, is related by complex conjugation. Note that by the linearized Vasiliev equations with parity violating phase $\theta_0$, $B^{(2,0)}$ and $B^{(0,2)}$ are related to the ordinary field strength $F_{\mu\nu}$ of the vector gauge field by

$$\begin{aligned} B^{(2,0)}(x|y) &= \mathrm{e}^{\mathrm{i}\theta_0}z^2F^+_{\mu\nu}(x)(\sigma^{\mu\nu})_{\alpha\beta}y^\alpha y^\beta,\\ B^{(0,2)}(x|\bar{y}) &= \mathrm{e}^{-\mathrm{i}\theta_0}z^2F^-_{\mu\nu}(x)(\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}.\end{aligned} \tag{3.33}$$

The factor $z^2$ here comes from the $z$-dependence of the vielbein in $\mathrm{e}^\mu_{\alpha\dot\gamma}\,\mathrm{e}^\nu_{\beta\dot\delta}\epsilon^{\dot\gamma\dot\delta}$. The two point functions of the operators dual to the gauge field in the equation above are given by

$$\langle J^\mu(0)J^\nu(x)\rangle = \frac{1}{\pi^2 g^2}\frac{\delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}}{x^4}, \tag{3.34}$$

where $g$ is the bulk gauge coupling constant. The mixed boundary condition

$$F_{ij} = \mathrm{i}\zeta\epsilon_{ijk}F_{zi} \qquad \text{at} \quad z = 0$$

is equivalent to[25]

$$\mathrm{e}^{-\mathrm{i}\rho}F^+_{zi}|_{z=0} = \mathrm{e}^{\mathrm{i}\rho}F^-_{zi}|_{z=0}, \qquad \text{where} \quad \mathrm{e}^{2\mathrm{i}\rho} \equiv \frac{1+\mathrm{i}\zeta}{1-\mathrm{i}\zeta}. \tag{3.35}$$

We see that precisely when $\theta_0 = 0$ or $\pi/2$, the standard magnetic boundary condition, i.e. $\rho = 0$ ($k = \infty$), is consistent with higher spin gauge symmetry. For generic $\theta_0$, however, there is *no* choice of $\rho$ for the boundary condition to be consistent with the higher spin symmetry variation on $\delta_\epsilon B^{(2,0)}$ and $\delta_\epsilon B^{(0,2)}$. Therefore, we see again that the parity violating phase breaks all higher spin symmetries. From this one can also derive the double trace term involving a spin-1 current in the divergence of the spin-$s$ current of the boundary theory, using the method of the previous subsection.

## 4. Partial breaking of supersymmetry by boundary conditions

In this very important section we now turn to supersymmetric Vasiliev theory. We investigate the action of asymptotic SUSY transformations on bulk fields of spin 0, 1/2, and 1. As in the case of higher spin symmetries, we find that no SUSY transformation preserves generic boundary conditions. In other words generic boundary conditions on fields violate all SUSYs. However we identify special classes of boundary conditions that that preserve $\mathcal{N} = 1, 2, 3, 4$

[25] In order to see this let us, for instance, take the special case $i = 1$. The relation becomes $\mathrm{e}^{\mathrm{i}\rho}(F_{z1} - F_{23}) = \mathrm{e}^{-\mathrm{i}\rho}(F_{z1} + F_{23})$, so that $F_{23} = \frac{\mathrm{e}^{2\mathrm{i}\rho}-1}{\mathrm{e}^{2\mathrm{i}\rho}+1}F_{z1}$.

and 6 SUSYs[26] in the next section. We go on present conjectures for CFT duals for these theories.

We emphasize that the boundary conditions presented in this section preserve SUSY when acting on *linearized* solutions of Vasiliev's theory. The study of arbitrary linearized solutions is insufficient to completely determine the boundary conditions that preserve SUSY as we now explain.

Consider a linearized solution of a bulk scalar dual to an operator of dimension unity. The solution to such a scalar field decays at small $z$ like $\mathcal{O}(z)$, and the boundary condition on this scalar asserts the vanishing of the $\mathcal{O}(z^2)$ term. However terms quadratic in $\mathcal{O}(z)$ are of $\mathcal{O}(z^2)$ at leading order, and so could potentially violate the boundary condition. It follows that the linearized boundary conditions studied presented in this section are not exact, but will be corrected at nonlinear order. Indeed we know one source of such corrections; the boundary condition deformations dual to the triple trace deformations of the dual boundary Chern–Simons theory. We ignore all such nonlinear deformations in this section (see the next section for some remarks).

### *4.1. Structure of boundary conditions*

Consider the $n$-extended supersymmetric Vasiliev theory with parity violating phase $\theta_0$. We already know that all higher spin symmetries are broken by any choice of boundary condition on fields of low spins, as expected for any interacting CFT. We also expect that any parity non-invariant CFT to have at most $\mathcal{N} = 6$ SUSY, and the question is whether the breaking of SUSYs to $\mathcal{N} \leqslant 6$ in the $n$-extended Vasiliev theory can be seen from the violating of boundary conditions by SUSY variations. The answer will turn out to be yes. In fact, we will be able to identify boundary conditions that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ and 6 SUSYs, in precise agreement with the various $\mathcal{N}$-extended supersymmetric Chern–Simons vector models that differ from one another by double and triple trace deformations.

To begin we shall describe a set of boundary condition assignments on all bulk fields of spin 0, 1/2, and 1, that will turn out to preserve various number of SUSYs and global flavor symmetries. The SUSY transformation of the bulk fields of spin 0, 1/2, and 1 are derived explicitly in terms of the master field $B(x|Y)$ in section B. For convenience we will speak of the $n$-extended parity violating supersymmetric Vasiliev theory with no extra Chan–Paton factors, though our discussion can be straightforwardly generalized to include $U(M)$ Chan–Paton factors. The bulk theory together with the prescribed boundary conditions are then conjectured to be holographically dual to supersymmetric Chern–Simons vector models with various number of SUSYs and superpotentials.

*4.1.1. Scalars.* Vasiliev's theory contains $2^{n-2}$ parity even scalar fields and an equal number of parity odd scalar fields. We expect the most general allowed boundary condition for these fields to take the form (5.6) (with $d_{abc}$ set to zero, as we restrict attention to linear analysis in this section). If we view the collection of scalar fields as a linear vector space of dimension $2^{n-1}$ then (5.6) asserts that the $z$ component of scalars lies in a particular half dimensional subspace of this vector space, while the $z^2$ component of the scalars lies in a complementary half dimensional subspace (obtained from the first space by switching the role of parity even and parity odd scalars). Now the Vasiliev master field $B$ packs all $2^{n-1}$ scalars into a single even function of $\psi_i$. In order to specify the boundary conditions on scalars, we must specify

[26] Theories with $\mathcal{N} = 5$ SUSY involve $SO$ and $Sp$ gauge groups on the boundary. Such theories presumably have bulk duals in terms of the 'minimal' Vasiliev theory, which we, however, never study in this paper. We thank O Aharony and S Yokoyama for related discussions.

the $2^{n-2}$ dimensional subspace (of the $2^{n-1}$ dimensional space of even functions of $\psi^i$) that multiply $z$ in the small $z$ expansion of these fields. We must also choose out a half dimensional subspace of functions that multiply $z^2$ (as motivated above, this subspace will always turn out to be complementary to the first).

How do we specify the subspaces of interest? The technique we adopt is the following. We choose any convenient reference subspace $S$ that has the property that $S + \Gamma S$ is the full space. Let $\gamma$ be an arbitrary Hermitian operator (built out of the $\psi_i$ fields) that acts on the subspace $S$—i.e. $\Gamma$ is the exponential of a linear combination of projectors for the basis states of $S$. An arbitrary real half dimensional subspace in the space of functions is given by $\mathrm{e}^{\mathrm{i}\gamma} S + \Gamma \mathrm{e}^{-\mathrm{i}\gamma} S$. The complementary subspace (obtained by flipping parity even and parity odd functions) is given by $\mathrm{e}^{\mathrm{i}\gamma} S - \Gamma \mathrm{e}^{-\mathrm{i}\gamma} S$. In other words the most general boundary conditions for the scalar part of $B$ takes the form

$$B^{(0)}(\vec{x}, z) = (\mathrm{e}^{\mathrm{i}\gamma} + \Gamma \mathrm{e}^{-\mathrm{i}\gamma})\tilde{f}_1(\psi) z + (\mathrm{e}^{\mathrm{i}\gamma} - \Gamma \mathrm{e}^{-\mathrm{i}\gamma})\tilde{f}_2(\psi) z^2 + \mathcal{O}(z^3) \tag{4.1}$$

where $f_1(\psi)$ and $f_2(\psi)$ represent any function—not necessarily the same—that lie within the reference real half dimensional subspace on the space of functions of $\psi$, and $\gamma$ is an operator, to be specified, that acts on this subspace. It is not difficult to verify that (section C.1) is consistent with the reality of $B$. Section C.1 may also be rewritten as

$$\begin{aligned} B^{(0)}(\vec{x}, z) = {} & z((1+\Gamma)\cos\gamma \tilde{f}_1 + (1-\Gamma)\mathrm{i}\sin\gamma \tilde{f}_1) \\ & + z^2((1-\Gamma)\cos\gamma \tilde{f}_2 + (1+\Gamma)\mathrm{i}\sin\gamma \tilde{f}_2) + \mathcal{O}(z^3), \end{aligned} \tag{4.2}$$

a form that makes the connection with (5.6) more explicit.

In the special case $\gamma = 0$, $\tilde{f}_1$ and $\tilde{f}_2$ can be arbitrary (i.e. the reference half dimensional space can be chosen arbitrarily) and (section C.1) simply asserts that parity odd scalars have dimension 1 while parity even scalars have dimension 2.

*4.1.2. Spin-1/2 fermions.* Boundary conditions for spin-1/2 fermions are specified more simply than for their scalar counterparts. The most general boundary condition relates the parity even part of any given fermion (the 'source') to the parity odd piece of all other fermions ('the vev'). The most general real boundary condition of this form is that the spin-1/2 part of $B$ takes the form

$$B^{(\frac{1}{2})}(\vec{x}, z|Y)|_{\mathcal{O}(y,\bar{y})} = z^{\frac{3}{2}}[\mathrm{e}^{\mathrm{i}\alpha}(\chi y) - \Gamma\, \mathrm{e}^{-\mathrm{i}\alpha}(\bar{\chi}\bar{y})] + \mathcal{O}(z^{\frac{5}{2}}), \quad \chi = \sigma^z \bar{\chi} \tag{4.3}$$

where $\chi$ is an arbitrary spinor and $\alpha$ is an arbitrary Hermitian operator (i.e. function of $\psi_i$). Reality of $B^{(\frac{1}{2})}$ imposes $(\chi^\alpha)^* = -\mathrm{i}\bar{\chi}_{\dot{\alpha}}$.

In the limit $\alpha = 0$ these boundary conditions simply assert that the $z^{3/2}$ fall off of the fermion is entirely parity odd. Recall that according to the standard AdS/CFT rules, the parity even component of the fermion field may be identified with the expectation value of the boundary operator, while the parity odd part is an operator deformation. When $\alpha$ (which in general is a linear operator that acts on $\chi$, $\bar{\chi}$, which are functions of $\psi$) is nonzero, the boundary conditions assert a linear relation between parity even and parity odd pieces, of the sort dual to a fermion–fermion double trace operator.

*4.1.3. Gauge fields.* The electric–magnetic mixed boundary condition on the spin-1 field is

$$B^{(1)}(\vec{x}, z|Y)|_{\mathcal{O}(y^2,\bar{y}^2)} = z^2[\mathrm{e}^{\mathrm{i}\beta}(yFy) + \Gamma\, \mathrm{e}^{-\mathrm{i}\beta}(\bar{y}\overline{F}\bar{y})] + \mathcal{O}(z^3), \quad F = -\sigma^z \overline{F} \sigma^z. \tag{4.4}$$

Here $\beta$ is equal to $\theta_0$ for the magnetic boundary condition, corresponding to ungauged flavor group in the boundary CFT (recall that $\mathrm{e}^{\mathrm{i}\theta} F$ is identified with the bulk Maxwell field strength;

see above). Once again $\beta$ is, in general, an operator that acts on $F, \overline{F}$. Reality of $B^{(1)}$ gives $\left(F^{\alpha}_{\beta}\right)^* = \bar{F}^{\dot{\beta}}_{\dot{\alpha}}$

We will see that the $\mathcal{N} = 4$ and $\mathcal{N} = 6$ boundary conditions requires taking $\beta$ to be a nontrivial linear operator that acts on $F, \overline{F}$, which amounts to gauging a flavor group with a finite Chern–Simons level.

Now to characterize the boundary condition, we simply need to give the linear operators $\alpha, \gamma, \beta$ which act on $\tilde{f}_{1,2}(\psi)$, $\chi(\psi)$, $F(\psi)$, and a set of linear conditions on $\tilde{f}_{1,2}(\psi)$.

We now proceed to enumerate boundary conditions that preserve different degrees of SUSY. In each case we also conjecture a field theory dual for the resultant Vasiliev theory. For future use we present the Lagrangians of the corresponding field theories in appendix D.

### 4.2. The $\mathcal{N} = 2$ theory with two $\square$ chiral multiplets

Let us start with $n = 4$ extended supersymmetric Vasiliev theory. The master fields depend on the auxiliary Grassmannian variables $\psi_1, \psi_2, \psi_3, \psi_4$. With $\theta(X) = 0$, $\alpha = 0$ and $\gamma = 0$ in the fermion and scalar boundary conditions, respectively, the dual CFT is the free theory of 2 chiral multiplets (in $\mathcal{N} = 2$ language) in the fundamental representation of $SU(N)$, with a total number of 16 SUSYs. Now we will turn on nonzero $\theta_0$, and describe a set of boundary conditions that preserve $\mathcal{N} = 2$ SUSY (4 supercharges) and $SU(2)$ flavor symmetry. The boundary condition for the spin-1 field is the standard magnetic one. The boundary condition for spin-1/2 and spin-0 fields are given by (B.10), (B.11), (B.18), with

$$\alpha = \gamma = \theta_0, \quad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0 \text{ or } P_{1,\psi_2\psi_3,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2}, \tag{4.5}$$

where $P_{\psi_i,\ldots}$ stands for the projection onto the subspace spanned by the monomials $\psi_i, \ldots$; $\tilde{f}_{1,2}$ are subject to the constraint that they commute with $\psi_1$, or equivalently, $\tilde{f}_{1,2}$ are spanned by $1, \psi_2\psi_3, \psi_2\psi_4, \psi_3\psi_4$. The two SUSY parameters are given by $\Lambda_+ = \Lambda_0$, $\Lambda_- = 0$, with

$$\Lambda_0 = \eta\psi_1 \text{ and } \eta\psi_1\Gamma, \tag{4.6}$$

where $\Gamma = \psi_1\psi_2\psi_3\psi_4$. $\eta$ is a constant Grassmannian spinor parameter that anti-commutes with all the $\psi_i$.

Clearly, with $\alpha = \theta_0$, (B.9) obeys the fermion boundary condition (B.10), (B.11), and (B.16) obeys the magnetic boundary condition on the spin-1 fields (B.1), (B.2). (B.17) with $\alpha = \gamma$ obeys (B.18) with $\tilde{f}_{1,2}$ of the form $\{\psi_1, \lambda\}$, or $\{\psi_1\Gamma, \lambda\}$, both of which commute with $\psi_1$. Finally, in the RHS of (B.21), all commutators of $\tilde{f}_{1,2}$ vanish, leaving the terms with anti-commutators only, which satisfy (C.21), (B.11) with $\gamma = \alpha$. Clearly, an $SU(2) \simeq SO(3)$ flavor symmetry rotating $\psi_2, \psi_3, \psi_4$ is preserved by this $\mathcal{N} = 2$ boundary condition.

It is natural to propose that the $n = 4$ extended parity violating Vasiliev theory with this boundary condition is dual to $\mathcal{N} = 2$ Chern–Simons vector model with two fundamental chiral multiplets. There is no gauge invariant superpotential in this case, while there is an $SU(2)$ flavor symmetry[27] rotating the two chiral multiplets, which is identified with the $SO(3)$ symmetry of rotations in $\psi_1$, $\psi_2$ and $\psi_3$ preserved by the boundary conditions listed above.

Let us elaborate on, for instance, the scalar boundary conditions. There are a total of eight scalars in the problem (the number of even functions of $\psi_i$). A basis for parity even scalars is given by $(1 + \Gamma)$ and $(1 + \Gamma)\psi_1\psi_i$ where $i = 1, \ldots, 3$. A basis for parity odd scalars is given by $(1 - \Gamma)$ and $(1 - \Gamma)\psi_1\psi_i$. In each case the scalars transform in the $1 + 3$ of $SU(2)$. Recall that the fundamental fields of the field theory (scalars as well as fermions) transform in the

[27] Note that the field theory is left invariant under a larger set of $U(2)$ transformations, which rotates the chiral multiplets into each other. However the diagonal $U(1)$ in $U(2)$ acts in the same way on all fundamental fields, and so is part of the $U(N)$ gauge symmetry. There is nonetheless a bulk gauge field—with $\psi$ content I—formally corresponding to this $U(1)$ factor.

1/2 of the flavor symmetry $SU(2)$; it follows that bilinears in these fields also transform in the $1 + 3$ of $SU(2)$, establishing a natural map between bulk fields and field theory operators.

The boundary conditions (4.5) assert that the coefficient of the $\mathcal{O}(z^2)$ term of the parity even scalars/vectors is equal to $\tan\theta_0$ times the coefficient of the $\mathcal{O}(z^2)$ of the corresponding parity odd scalars/vectors. Similarly the coefficient of the $\mathcal{O}(z)$ term of the parity odd scalars/vectors is equal to $\tan\theta_0$ times the coefficient of the $\mathcal{O}(z)$ of the corresponding parity even scalars/vectors. This is exactly the kind of boundary condition generated by a double trace deformation that couples the dual dimension 1 and dimension 2 operators, with equal couplings in the scalar and vector (of $SU(2)$) channels. We will elaborate on this in much more detail in the next section.

### *4.3. A family of $\mathcal{N} = 1$ theories with two $\square$ chiral multiplets*

If we keep only the SUSY generator given by

$$\Lambda_0 = \eta\psi_1, \tag{4.7}$$

then a one-parameter family of boundary conditions that preserve $\mathcal{N} = 1$ SUSY is given by

$$\alpha = \theta_0 P_1^S + \gamma P_1^A, \quad \beta = \theta_0, \quad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0, \tag{4.8}$$

where $P_1^S$ and $P_1^A$ are the projection operators that projects an odd function of the $\psi_i$ onto the subspaces spanned by

$$\psi_1\Gamma, \psi_2, \psi_3, \psi_4 (\text{all anti-commute with } \psi_1) \tag{4.9}$$

and

$$\psi_1, \psi_2\Gamma, \psi_3\Gamma, \psi_4\Gamma (\text{all commute with } \psi_1) \tag{4.10}$$

respectively. $\gamma$ is now an arbitrary phase (independent of $\psi_i$).

This family of boundary conditions is dual to $\mathcal{N} = 1$ deformations of the $\mathcal{N} = 2$ theory with two chiral flavors, by turning on an $\mathcal{N} = 1$ (non-holomorphic) superpotential that preserves the $SU(2)$ flavor symmetry (corresponding to the bulk symmetry that rotates $\psi_2, \psi_3, \psi_4$).

The same theory can also be rewritten as the $n = 2$ extended supersymmetric Vasiliev theory with $M = 2$ matrix extension. The spin-1, fermion, and scalar boundary conditions are given by

$$\alpha = \theta_0 P_{\psi_2} + \gamma P_{\psi_1}, \quad \beta = \theta_0, \quad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0. \tag{4.11}$$

It is natural to wonder about the relationship between the parameter $\gamma$ above and the field theory parameter $\omega$ (see (D.7)). General considerations leave this relationship undetermined; however we will present a conjecture for this relationship in the next section.

### *4.4. The $\mathcal{N} = 2$ theory with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet*

Now let us describe a boundary condition that preserve the two SUSYs generated by

$$\Lambda_- = 0, \quad \Lambda_0 = \eta\psi_1 \text{ and } \eta\psi_2. \tag{4.12}$$

It is given by

$$\beta = \theta_0, \quad \alpha = \theta_0(1 - P_{\psi_3\Gamma,\psi_4\Gamma}), \quad \gamma = \theta_0 P_{1,\psi_3\psi_4}, \tag{4.13}$$

where $P_{\psi_i,\ldots}$ stands for the projection onto the subspace spanned by the monomials $\psi_i, \ldots$, as before; $\tilde{f}_{1,2}$ are now subject to the constraint that they commute with *either* $\psi_1$ *or* $\psi_2$, i.e. $\tilde{f}_{1,2}$ are spanned by $1, \psi_3\psi_4, \psi_1\psi_3, \psi_1\psi_4, \psi_2\psi_3, \psi_2\psi_4$. Note that when acting on the latter four monomials, $\gamma$ vanishes, and $\tilde{f}_1$ and $\tilde{f}_2$ may be replaced by $\frac{1+\Gamma}{2}\tilde{f}_1$ and $\frac{1-\Gamma}{2}\tilde{f}_2$. Therefore, only half of the components of $\tilde{f}_{1,2}$ are independent, as required. One can straightforwardly verify that this set of boundary conditions preserves the two SUSYs (4.12). Clearly, the $U(1)$ flavor symmetry that rotates $\psi_3$, $\psi_4$ is still preserved, but there is no $SU(2)$ flavor symmetry. We also have the $U(1)$ $R$ symmetry corresponding to rotations of $\psi_1$, $\psi_2$.

The $n = 4$ Vasiliev theory with this boundary is then naturally proposed to be dual to $\mathcal{N} = 2$ Chern–Simons vector model with a fundamental and an anti-fundamental chiral flavor, with $U(1) \times U(1)$ flavor symmetry[28] (corresponding to the components of the bulk vector gauge field proportional to 1 and $\psi_3\psi_4$) besides the $U(1)$ $R$-symmetry, which means that the $\mathcal{N} = 2$ superpotential vanishes, since a nonzero superpotential would break the $U(1) \times U(1)$ flavor symmetry to a single $U(1)$.

### 4.5. A family of $\mathcal{N} = 2$ theories with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet

The boundary condition in the above section is a special point inside a one-parameter family of boundary conditions which preserved the same set of SUSYs. It is given by

$$\begin{aligned}&\beta = \theta_0, \alpha = \theta_0(1 - P_{\psi_3\Gamma,\psi_4\Gamma}) + \tilde{\alpha}(P_{\psi_3\Gamma} - P_{\psi_4\Gamma}),\\&\gamma = \theta_0 P_{1,\psi_3\psi_4} + \tilde{\alpha}P_{\psi_2\psi_4,\psi_1\psi_4},\\&P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2}.\end{aligned} \tag{4.14}$$

This one-parameter family of deformations is naturally identified with the superpotential deformation of the $\mathcal{N} = 2$ Chern–Simons vector model with a fundamental and an anti-fundamental chiral flavor. This superpotential is marginal at infinite $N$; at finite $N$ there are two inequivalent conformally invariant fixed points [29]. The $\tilde{\alpha} = 0$ point is the boundary condition on the above section, describing the $\mathcal{N} = 2$ theory with no superpotential, whereas $\tilde{\alpha} = \pm\theta_0$ give the $\mathcal{N} = 3$ point, as will be discussed in the next subsection.

### 4.6. The $\mathcal{N} = 3$ theory

The $\mathcal{N} = 3$ boundary condition that preserve SUSY generated by the parameters

$$\Lambda_- = 0, \quad \Lambda_0 = \eta\psi_1, \eta\psi_2, \text{ and } \eta\psi_3, \tag{4.15}$$

is given by

$$\beta = \theta_0, \alpha = \theta_0(1 - P_{\psi_1\psi_2\psi_3}) - \theta_0 P_{\psi_1\psi_2\psi_3}, \gamma = \theta_0, P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2}. \tag{4.16}$$

This boundary condition is dual to the $\mathcal{N} = 3$ Chern–Simons vector model with a single fundamental hypermultiplet, which may be obtained from the $\mathcal{N} = 2$ theory with a fundamental and an anti-fundamental chiral multiplet by a turning on a superpotential. The $SO(3)$ symmetry of rotations in $\psi_1$, $\psi_2$ and $\psi_3$ maps to the $SO(3)$ $R$-symmetry of the model. Notice that unlike the case studied in section 4.2, $\alpha \neq \gamma$ reflecting the fact that the $SO(3)$ $R$ symmetry, unlike a flavor symmetry, acts differently on bosons and fermions.

### 4.7. The $\mathcal{N} = 4$ theory

The $\mathcal{N} = 4$ boundary condition that preserve SUSY generated by the parameters

$$\Lambda_- = 0, \quad \Lambda_0 = \eta\psi_i, \quad i = 1, 2, 3, 4, \tag{4.17}$$

[28] One of these two $U(1)$ factors is actually part of the gauge group and so acts trivially on all gauge invariant operators.

is given by

$$\beta = \theta_0(1 - P_\Gamma), \quad \alpha = \theta_0 P_{\psi_i}, \quad \gamma = \theta_0 P_1. \tag{4.18}$$

$\tilde{f}_{1,2}$ are subject to the constraint

$$P_\Gamma \tilde{f}_{1,2} = 0. \tag{4.19}$$

Note also that the components of $\tilde{f}_{1,2}$ proportional to $\psi_i\psi_j$ are subject to the projection $\frac{1\pm\Gamma}{2}$ also, as follows automatically from (section C.1), (4.2). The boundary conditions above are invariant under the $SO(4)$ R symmetry of rotations in $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$.

This boundary condition is dual to the $\mathcal{N} = 4$ Chern–Simons quiver theory with gauge group $U(N)_k \times U(1)_{-k}$ and a single bi-fundamental hypermultiplet. The latter can be obtained from the $\mathcal{N} = 3$ $U(N)_k$ Chern–Simons vector model with one hypermultiplet flavor by gauging the $U(1)$ flavor current multiplet with another $\mathcal{N} = 3$ Chern–Simons gauge field at level $-k$ [30].

### *4.8. An one parameter family of $\mathcal{N} = 3$ theories*

There is an one parameter family of boundary conditions that preserves the same SUSY as in section 4.6,

$$\begin{aligned} &\beta = \theta_0(1 - P_\Gamma) + \tilde{\beta} P_\Gamma, \quad \alpha = \theta_0 P_{\psi_i} + \tilde{\beta}(P_{\psi_1\Gamma,\psi_2\Gamma,\psi_3\Gamma} - P_{\psi_4\Gamma}), \\ &\gamma = \theta_0 P_1 + \tilde{\beta} P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}, \\ &P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4} \tilde{f}_{1,2} = \tilde{f}_{1,2}. \end{aligned} \tag{4.20}$$

The boundary condition in section 4.6 is at $\tilde{\beta} = \theta_0$. At $\tilde{\beta} = 0$, the (4.20) coincides with (4.18), and the $\mathcal{N} = 3$ SUSY is enhanced to $\mathcal{N} = 4$.

### *4.9. The $\mathcal{N} = 6$ theory*

To construct the bulk dual of the $\mathcal{N} = 6$ ABJ vector model [31, 32], we need to double the number of matter fields in the boundary field theory, and correspondingly quadruple the number of bulk fields. This is achieved with the $n = 6$ extended supersymmetric Vasiliev theory, which in the parity even case (dual to free CFT) can have up to 64 SUSYs. We are interested in the parity violating theory, with nonzero interaction phase $\theta_0$, with a set of boundary conditions that preserve $\mathcal{N} = 6$ SUSYs[29], generated by the parameters

$$\Lambda_0 = \eta\psi_i, \quad i = 1, 2, \ldots, 6. \tag{4.21}$$

Similarly to the $\mathcal{N} = 4$ theory with one hypermultiplet, here we need to take the boundary condition on the bulk spin-1 field to be

$$\beta = \theta_0(1 - P_\Gamma) - \theta_0 P_\Gamma. \tag{4.22}$$

The spin-1/2 and spin-0 boundary conditions are given by

$$\alpha = \theta_0(1 - P_{\psi_i\Gamma}) - \theta_0 P_{\psi_i\Gamma}, \quad \gamma = \theta_0 P_{1,\psi_i\psi_j}, \tag{4.23}$$

where $P_{\psi_i\Gamma}$ for instance stands for the projection onto the subspace spanned by *all* the $\psi_i\Gamma$, $i = 1, 2, \ldots, 6$. $\tilde{f}_{1,2}$ are subject to the constraint

$$P_{\Gamma,\psi_i\psi_j\Gamma} \tilde{f}_{1,2} = 0, \tag{4.24}$$

[29] One can show that there is no boundary condition for the $n > 6$ extended supersymmetric Vasiliev theory that preserves $\mathcal{N} = n$ SUSYs. We expect that there is no $\mathcal{N} > 6$ boundary condition for the parity violating Vasiliev theory, though we have not proven this in general.

which projects out half of the components of $\tilde{f}_{1,2}$. Note that these boundary conditions enjoy invariance under the $SO(6)$ R symmetry rotations of the $\psi_i$ coordinates.

By comparing the difference between $\beta$ and $\theta_0$ with the Chern–Simons level of what would be the flavor group of the $\mathcal{N} = 3$ Chern–Simons vector model with two hypermultiplets, we will be able to identify $\theta_0$ in terms of $k$ below.

### *4.10. Another one parameter family of $\mathcal{N} = 3$ theories*

There is another one parameter family of boundary conditions that preserves the same SUSY as in section 4.6,

$$\begin{aligned}\beta &= \theta_0(1 - P_\Gamma) + \tilde{\beta}P_\Gamma,\\ \alpha &= \theta_0(P_{\psi_i,\psi_a} + P_{\psi_i\psi_j\psi_a,\psi_i\psi_a\psi_b,\psi_4\psi_5\psi_6} - P_{\psi_a\Gamma}) + \tilde{\beta}(P_{\psi_i\Gamma} - P_{\psi_1\psi_2\psi_3}),\\ \gamma &= \theta_0 P_{1,\psi_i\psi_a,\psi_a,\psi_b} - \tilde{\beta}P_{\psi_i\psi_j},\\ P_{1,\psi_i\psi_j,\psi_i\psi_a,\psi_a\psi_b}\tilde{f}_{1,2} &= \tilde{f}_{1,2},\end{aligned} \tag{4.25}$$

where $i, j = 1, 2, 3$ and $a, b = 4, 5, 6$. At $\tilde{\beta} = -\theta_0$, the (4.25) coincides with the boundary condition in 4.9, and the $\mathcal{N} = 3$ SUSY is enhanced to $\mathcal{N} = 6$.

## 5. Deconstructing the supersymmetric boundary conditions

### *5.1. The goal of this section*

As we have explained early in this paper, the Vasiliev dual to free boundary superconformal Chern–Simons theories is well known. In the previous section we have also conjectured phase and boundary condition deformations of this Vasiliev theory that describe the bulk duals of several fixed lines of superconformal Chern–Simons theories with known Lagrangians. These interacting superconformal Chern–Simons theories differ from their free counterparts in three important respects.

- The level $k$ of the $U(N)$ Chern–Simons theory is taken to infinity holding $\frac{N}{k} = \lambda$ fixed. The free theory is recovered on taking $\lambda \to 0$.
- The Lagrangian of the theory includes marginal triple trace interactions of the schematic form $(\phi^2)^3$ and double trace deformations of the form $(\phi^2)(\psi^2)$ and $(\phi\psi)^2$ (the brackets indicate the structure of color index contractions).
- In some examples including the $\mathcal{N} = 6$ ABJ theory we will also gauge a subgroup of the global symmetry group of the theory with the aid of a new Chern–Simons gauge field.

In this section we carefully compare the supersymmetric boundary conditions, determined in the previous section, with the Lagrangian of the conjectured field theory duals of these systems. This analysis allows us to understand the separate contributions of each of the three factors listed above to the boundary conditions of the previous section. It also yields some information about the relationship between the bulk deformation parameters and field theoretic quantities.

The analysis presented in this section was partly motivated by the following quantitative goal. In the previous section we have presented two one parameter sets of $\mathcal{N} = 3$ Vasiliev boundary conditions (4.20) and (4.25) at any given fixed value of the Vasiliev phase $\theta_0$. The first of these fixed lines interpolates to an $\mathcal{N} = 4$ theory while the second which interpolates to a $\mathcal{N} = 6$ theory. For each line of boundary conditions we have also conjectured a one parameter set of dual boundary field theories. In order to complete the statement of the duality between these systems we need to propose an identification of the parameter that labels boundary conditions with the parameter that labels the dual field theories. The analysis of this section was undertaken partly in order to establish this map. We have been only partly successful in this respect. While we propose a tentative identification of parameters below, there is an unresolved puzzle in the analysis that leads to this identification; as a consequence we are not confident of this identification. We leave the resolution of this puzzle to future work.

We begin this lengthy section with a review of well known effects of items (2) and (3) listed above on the bulk dual systems. With these preliminaries out of the way we then turn to the main topic of this section, namely the deconstruction of the supersymmetric boundary conditions determined in the previous subsection.

### *5.2. Marginal multitrace deformations from gravity*

As we have reviewed in the previous section, the supersymmetric Vasiliev theory contains fields of every half integer spin, including scalars with $m^2 = -2$, spin-1/2 fields with $m = 0$, and massless vectors. It is well known that the only consistent boundary conditions for the fields with spin $s > 1$ is that they decay near $z = 0$ like $z^{s+1}$.[30] On the other hand consistency permits more interesting boundary conditions for fields of spin-0, spin-1/2 and spin-1. In this section we will review the subset of these boundary conditions that preserve conformal invariance, together with their dual boundary interpretations. The discussion in this subsection is an application of well known material (see for example the references [17, 33–36, 18]—we most closely follow the approach of the paper [34]).

*5.2.1. Scalars.* The Vasiliev theories we study contain a set of scalar fields propagating in $AdS_4$, all of which have $m^2 = -2$ in AdS units. In the free theory the boundary conditions for some of these scalars, $S_a$, are chosen so that the corresponding operator has dimension 1 (these are the so-called alternate boundary conditions) while the boundary conditions for the remaining scalars, $F_\alpha$, are chosen so that its dual operator has dimension 2 (these are the so-called regular boundary conditions). See section C.1 for a detailed discussion of these boundary conditions and their dual bulk interpretation.

Let us suppose that the Lagrangian for these scalars at quadratic order takes the form[31]

$$\sum_a \frac{1}{g_a^2} \int \sqrt{g}(\partial_\mu \bar{S}_a \partial^\mu S_a - 2\bar{S}_a S_a) + \sum_\alpha \frac{1}{g_\alpha^2} \int \sqrt{g}(\partial_\mu \bar{F}_\alpha \partial^\mu F_\alpha - 2\bar{F}_\alpha F_\alpha). \tag{5.1}$$

The redefinition

$$S_a = g_a s_a, F_\alpha = g_\alpha f_\alpha$$

sets all couplings to unity as in the discussion in section C.1.

As explained in detail in section C.1.3 the action and boundary conditions of bulk scalars do not completely characterize the boundary dynamics of the system. For instance in a theory with a single regular quantized scalar and one alternately quantized scalar there exist a one parameter set of inequivalent boundary actions, each of which lead to identical boundary conditions for (appropriately redefined) bulk fields. However there is a distinguished 'simplest' set of boundary counterterms corresponding to any particular boundary condition (this is the undeformed or $\theta_0 = 0$ system described in section C.1.3). This simple counterterm has

[30] In other words the coefficient of the leading fall off is required to vanish.

[31] Vasiliev's theory is currently formulated in terms of equations of motion rather than an action. As a consequence, the values of the coupling constants $g_a$ and $g_\alpha$, for the scalars that naturally appear in Vasiliev's equations, are undetermined by a linear analysis. The study of interactions would permit the determination of the relative values of coupling constants, but we do not perform such a study in this paper.

the following distinguishing property; it yields vanishing two point functions between any operator of dimension one and any other operator of dimension two. Every other choice of counterterms yields correlators between these operators that vanish at separated points but are have non-vanishing contact term contributions.

In this section we assume that the counterterm action corresponding to the scalar boundary conditions above takes the simple ($\theta_0 = 0$) form referred to above. We will then deduce the effect of a double and triple trace deformation on the boundary conditions of bulk fields.

The two point functions of the operators dual to $s_a$ and $f_\alpha$[32] are given by[33] [37][34]

$$\begin{aligned} &\frac{1}{2\pi^2}\frac{1}{x^2}(\text{operators dual to } s_a), \\ &\frac{1}{2\pi^2}\frac{2}{x^4}(\text{operators dual to } f_\alpha). \end{aligned} \tag{5.2}$$

Later in this paper we will be interested in determining the Vasiliev dual to large $N$ theories deformed by double and triple trace scalar operators. The field theory deformations we study are marginal in the large $N$ limit and take the form

$$\int \mathrm{d}^3x \left( \frac{\pi^2}{2k^2} c_{abc}\sigma^a\sigma^b\sigma^c + \frac{2\pi}{k} d_{a\alpha}\sigma^a\phi^\alpha \right) \tag{5.3}$$

where $\sigma^a$ is proportional to the operator dual to $s_a$ and $\phi^\alpha$ is proportional to the operator dual to $f_\alpha$ (the factors in (5.3) have been inserted for future convenience). We will assume that it is known from field theoretic analysis that

$$\begin{aligned} \langle\sigma^a(x)\sigma^b(0)\rangle &= \delta^{ab}\frac{2Nh_+^a}{(4\pi)^2x^2}, \\ \langle\phi^\alpha(x)\phi^\beta(0)\rangle &= \delta^{\alpha\beta}\frac{4Nh_-^\alpha}{(4\pi)^2x^4}, \end{aligned} \tag{5.4}$$

(the factors on the RHS have been inserted for later convenience; $h_+^a$ and $h_-^\alpha$ are numbers). It follows from a comparison of (5.4) and (5.2) that the operator dual to $s^a$ is $\frac{2}{\sqrt{Nh_+^a}}\sigma^a$ while the operator dual to $f^\alpha$ is $\frac{2}{\sqrt{Nh_-^\alpha}}\phi^\alpha$

Let us suppose that at small $z$,[35]

$$s_a = s_a^{(1)}z + s_a^{(2)}z^2 + \mathcal{O}(z^3), \qquad f_\alpha = f_\alpha^{(1)}z + f_\alpha^{(2)}z^2 + \mathcal{O}(z^3). \tag{5.5}$$

It follows from the analysis of section C.1 that the marginal deformation (5.3) induces the boundary conditions

$$\begin{aligned} s_a^{(2)} &= \frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} f_\alpha^{(2)} + 3\frac{\pi^2 N^{\frac{3}{2}}\sqrt{h_+^a h_+^b h_+^c}}{16k^2} c_{abc} s_b^{(1)} s_c^{(1)}, \\ f_\alpha^{(1)} &= -\frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} s_a^{(1)}. \end{aligned} \tag{5.6}$$

[32] I.e. the two point functions for the operators for which coefficient of the $z^2$ fall off of the field $s_a$ is a source, and the operator for which the coefficient of the $z$ fall off of the field $f_\alpha$ is the source

[33] The general formula for the nontrivial prefactor is $\frac{\Gamma(\Delta+1)(2\Delta-d)}{\pi^{\frac{d}{2}}\Gamma(\Delta-d/2)\Delta}$.

[34] The Fourier transforms

$$G(k) = \int \mathrm{d}^3x \mathrm{e}^{\mathrm{i}k.x} G(x)$$

(appropriately regulated) evaluate to $\frac{1}{|k|}$ for the dimension one operator (alternate quantization), and to $-|k|$ for the dimension two operator (regular quantization). Note that these quantities are the negative inverses of each other, in agreement with the general analysis of section C.1.

[35] This expansion is in conformity with (C.9) because $\zeta = \frac{1}{2}$ for the $m^2 = -2$ scalars of Vasiliev's theory.

If we denote the boundary expansion of the original bulk fields by

$$S_a = S_a^{(1)} z + S_a^{(2)} z^2 + \mathcal{O}(z^3), \qquad F_\alpha = F_\alpha^{(1)} z + F_\alpha^{(2)} z^2 + \mathcal{O}(z^3), \tag{5.7}$$

then

$$\begin{aligned} \frac{S_a^{(2)}}{g_a} &= \frac{\pi N \sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} \frac{F_\alpha^{(2)}}{g_\alpha} + 3 \frac{\pi^2 N^{\frac{3}{2}} \sqrt{h_+^a h_+^b h_+^c}}{16k^2} c_{abc} \frac{S_b^{(1)}}{g_b} \frac{S_c^{(1)}}{g_c}, \\ \frac{F_\alpha^{(1)}}{g_\alpha} &= -\frac{\pi N \sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} \frac{S_a^{(1)}}{g_a}. \end{aligned} \tag{5.8}$$

In summary the boundary conditions (5.8) are the bulk dual of the field theory deformation (5.3).

In the rest of this subsection we ignore triple trace deformations and focus our attention entirely on the double trace deformations. As explained in section C.1, in this case the modified boundary condition in (5.7) can be undone by a rotation in the space of scalar fields. This is most easily seen in the special case that we have a single $S$ type scalar and a single $F$ type scalar so that both the $a$ and $\alpha$ indices run over a single value and can be ignored. Let us define the rotated fields

$$\frac{S'}{g_a} = \cos\theta \frac{S}{g_a} + \sin\theta \frac{F}{g_\alpha}, \qquad \frac{F'}{g_\alpha} = \cos\theta \frac{F}{g_\alpha} - \sin\theta \frac{S}{g_a} \tag{5.9}$$

with

$$\tan\theta = \frac{\pi N \sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha}. \tag{5.10}$$

Notice that the field redefinition (5.9) leaves the bulk action invariant. Moreover, it follows from (5.8) that

$$(S')^{(2)} = (F')^{(1)} = 0.$$

In other words the rotated fields $S'$ and $F'$ obey the same bulk equations and same boundary conditions in the presence of the double trace deformation as the unrotated fields $S$ and $F$ obey in their absence.

At first sight this observation leads to the following paradox. A double trace deformation by the parameter $d$ may be thought of as the result of compounding two double trace deformations of magnitude $d_1$ and $d_2$ respectively, such that $d_1+d_2 = d$. As the system after the deformation by $d_1$ is apparently self-similar to the system in its absence, it would appear to follow that the rotation that results from the deformation with $d_1 + d_2$ is simply the sum of the rotations corresponding to $d_1$ and $d_2$ respectively; in other words that the rotation angle $\theta$ is linear in $d$. This conclusion is in manifest contradiction with (5.10).

The resolution of this contradiction lies in the fact that the systems with and without the double trace deformations are not, in fact, isomorphic. The reason for this is that the boundary counterterm action does not take the simple $\theta = 0$ form in terms of rotated fields in the system with the double trace deformation (see section C.1). In the theory with double trace deformations there is, in particular, a nonzero contact term in the two point functions of the two operators with distinct scaling dimensions; this contact term is absent in the original system.

*5.2.2. Spin half fermions.* The Vasiliev theories we study include a collection of real fermions $\psi_1^a$ and $\psi_2^a$ propagating in AdS$_4$ space. It is sometimes useful to work with the complex fermions $\psi^a = \frac{\psi_1^a + i\psi_2^a}{\sqrt{2}}$ and $\bar{\psi}^a = \frac{\psi_a^1 - i\bar{\psi}_2^a}{\sqrt{2}}$. Let us suppose that the bulk action takes the form

$$\sum_a \frac{1}{g_a^2} \int \bar{\psi}^a D_\mu \Gamma^\mu \psi_a. \tag{5.11}$$

Using the rules described for instance in [26], the two point function for the operator dual to $\psi^a$ is easily computed and we find the answer

$$\frac{1}{g_a^2}\frac{\vec{x}\cdot\vec{\sigma}}{\pi^2 x^4}. \tag{5.12}$$

The same result also applies to the two point functions of the operators dual to $\psi_1^a$ and $\psi_2^a$ independently.

In analogy with the bosonic case described in the previous subsection, the formula (5.12) presumably applies only with the simplest choice of boundary counterterms [38–41]—the analogue of $\theta_0 = 0$ in section C.1.3—consistent with the boundary conditions described in [26]. Though we will not perform the required careful analysis in this paper, it seems likely that the fermionic analogue of section C.1 would find a one parameter set of inequivalent boundary actions that lead to the same boundary conditions. From the bulk viewpoint this ambiguity is likely related to the freedom associated with rotating a bulk spinor $\psi_1$ into $\Gamma_5\psi_2$ ($\Gamma_5$ is the bulk chirality matrix). We ignore this potential complication in the rest of this subsection, and focus on the simple canonical case described in [26].

Let the field theory operator proportional to $\psi^a$ be denoted by $\Psi^a$. Let us assume that we know from field theory that

$$\langle\Psi^a(x)\bar{\Psi}^b(0)\rangle = \delta^{ab}\frac{h_\psi 2N(\vec{x}\cdot\vec{\sigma})}{(4\pi)^2x^4}. \tag{5.13}$$

We will now describe the boundary conditions dual to a field theory double trace deformation. Let the fermionic fields have the small $z$ expansion

$$\begin{aligned}\psi_1^a &= z^{3/2}\left(\zeta_{1+}^a + \zeta_{1-}^a\right) + \mathcal{O}(z^{\frac{5}{2}}),\\ \psi_2^a &= z^{3/2}\left(\zeta_{2+}^a + \zeta_{2-}^a\right) + \mathcal{O}(z^{\frac{5}{2}}).\end{aligned} \tag{5.14}$$

Above the subscripts + and − denote the eigenvalue of the corresponding fermions under parity.

Using the procedure of the previous subsection, the bulk dual of the field theory double trace deformation

$$\frac{\pi}{4k}[s_{ab}(\bar{\Psi}^a+\Psi^a)(\bar{\Psi}^b+\Psi^b) - t_{ab}(\bar{\Psi}^a-\Psi^a)(\bar{\Psi}^b-\Psi^b) + u_{ab}(\bar{\Psi}^a+\Psi^a)\,\mathrm{i}(\bar{\Psi}^b-\Psi^b)]$$

is given by the modified boundary conditions

$$\begin{aligned}\frac{\zeta_{1+}^a}{g_a} &= \frac{N\pi\sqrt{h_\psi^a h_\psi^b}}{8k}\left(s_{ab}\frac{\zeta_{1-}^b}{g_b} + \frac{1}{2}u_{ab}\frac{\zeta_{2-}^b}{g_b}\right),\\ \frac{\zeta_{2+}^a}{g_a} &= \frac{N\pi\sqrt{h_\psi^a h_\psi^b}}{8k}\left(t_{ab}\frac{\zeta_{2-}^b}{g_b} + \frac{1}{2}u_{ba}\frac{\zeta_{1-}^b}{g_b}\right).\end{aligned} \tag{5.15}$$

### *5.3. Gauging a global symmetry*

As originally introduced by Witten [18], gauging a global symmetry with Chern–Simons term in the boundary CFT is equivalent to changing the boundary condition of the bulk gauge field corresponding to the boundary current of the global symmetry. We will review this relation in this subsection and in appendix B.

Let us start by considering a boundary CFT with $U(1)$ global symmetry. The current associated to this global symmetry is dual to a $U(1)$ gauge field $A_\mu$ in the bulk. In the $A_z = 0$ radial gauge, the action for the gauge field $A_\mu$ is

$$\frac{1}{4g^2}\int \frac{d^3\vec{x}dz}{z^4}F_{\mu\nu}F^{\mu\nu} = \int d^3\vec{x}dz\left(\frac{1}{2g^2}\partial_z A_i \partial_z A_i + \frac{1}{4g^2}F_{ij}F_{ij}\right). \tag{5.16}$$

Onshell the bulk action evaluates to

$$\int d^3\vec{x}\left(\frac{1}{2g^2}A_i\partial_z A_i\right) \tag{5.17}$$

where the integral is taken over a surface of constant $z$ for small $z$. The equations of motion w.r.t. the boundary gauge field impose the *electric* boundary condition

$$\frac{1}{g^2}\partial_z A_i|_{z=0} = 0. \tag{5.18}$$

Near $z = 0$, the most general solution to the gauge field equations of motion is

$$A_i = A_i^1(x) + zA_i^2(x).$$

The boundary condition (5.18) forces $A_i^2$ to vanish but allows $A_i = A_i^1$, the value of the gauge field on the cut off surface, to fluctuate freely at the boundary $z = 0$. The theory so obtained is the conceptual equivalent of the 'alternate' quantized scalar theory described in section C.1.

If we add a boundary $U(1)$ Chern–Simons term to the bulk action[36] (in Euclidean signature)

$$\frac{ik}{4\pi}\int d^3\vec{x}\epsilon_{ijk}A_i\partial_j A_k, \tag{5.19}$$

and allow arbitrary variation $\delta A_i$ at $z = 0$, the equation of motion of the boundary field $A_i$ generates the modified boundary condition

$$\frac{1}{g^2}\partial_z A_i + \frac{ik}{2\pi}\epsilon_{ijk}\partial_j A_k|_{z=0} = 0, \tag{5.20}$$

which is the electric-magnetic mixed boundary condition. By the AdS/CFT dictionary, this is also equivalent to adding the term (5.19) into the boundary theory, where $A_i$ is now interpreted as the three dimensional gauge field coupled to the $U(1)$ current.

This procedure can be straightforwardly generalized to $U(M)$. Adding the $U(M)$ Chern–Simons action on the boundary

$$\frac{ik}{4\pi}\int d^3\vec{x}\epsilon_{ijk}\,\mathrm{tr}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right). \tag{5.21}$$

modifies the electric boundary condition to

$$\frac{1}{g^2}\partial_z A_i + \frac{ik}{2\pi}\epsilon_{ijk}(\partial_j A_k + A_jA_k)|_{z=0} = 0. \tag{5.22}$$

Note that this mixed boundary condition is still gauge invariant.

Of course $\partial_z A_i$ is determined in terms of $A_i$ by the equations of motion. As the equations of motion are linear, the relation between these quantities is linear—but nonlocal—and takes the form

$$\partial_z A_i(q) = G_{ij}(q)A_j(q).$$

The function $G_{ij}(q)$ has a simple physical interpretation; it is the two point function of the current operator (with natural normalization) in the theory at $k = \infty$ (at this value of $k$ the

[36] This is the same as adding a term in the bulk action proportional to $\int F \wedge F$ as this term is the total derivative of the Chern–Simons term.

boundary condition (5.22) is simply the standard Dirichlet boundary condition). A simple computation yields

$$\langle J_i(p)J_j(-q)\rangle = \frac{1}{2g^2}G_{ij}(q)\delta^3(p-q) = -\frac{|p|}{2g^2}\left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)(2\pi)^3\delta^3(p-q). \tag{5.23}$$

Note that here we have normalized the current coupled to the Chern–Simons gauge field according to the convention for non-Abelian gauge group generators, $\mathrm{Tr}(t^at^b) = \frac{1}{2}\delta^{ab}$ for generators $t^a$, $t^b$ in the fundamental representation. This is also the normalization convention we use to define the Chern–Simons level $k$ (which differs by a factor of 2 from the natural convention for $U(1)$ gauge group).

Recall that (5.23) yields the two point functions of the 'ungauged' theory—i.e. the theory with $k = \infty$. Our analysis of the dual boundary theory to this ungauged system, we find it convenient to work with currents normalized so that

$$\langle J_i(p)J_j(-q)\rangle = -\frac{\widetilde{N}|p|}{32}\left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)(2\pi)^3\delta^3(p-q). \tag{5.24}$$

Our convention is such that in the free theory $\widetilde{N}$ counts the total number of complex scalars plus fermions (i.e. the two point function for the charge current for a free complex scalar is equal to that of the free complex fermion and is given by (5.24) upon setting $\widetilde{N} = 1$, see appendix F). In order that (5.23) and (5.24) match we must identify

$$g^2 = \frac{16}{\widetilde{N}},$$

so that the effective boundary conditions on gauge fields become

$$\frac{\pi\widetilde{N}}{8k}\partial_z A_i + \mathrm{i}\epsilon_{ijk}\partial_j A_k|_{z=0} = 0. \tag{5.25}$$

In summary, gauging of the global symmetry is affected by the boundary conditions (5.25). Note that the boundary conditions (5.25) constrain only the boundary field strength $F_{ij}$. Holonomies around noncontractable cycles are unconstrained and must be integrated over. In the finite temperature theory the integral over the Polyakov line of $U(M)$ enforces the $U(M)$, as we study in detail in section 7.

### *5.4. Deconstruction of boundary conditions: general remarks*

*5.4.1. The bulk dual of the finite Chern Simons coupling.* With essential preliminaries taken care of we now turn to the main topic of this subsection, namely the deconstruction of the supersymmetric boundary conditions of the previous section.

The Vasiliev dual of free SUSY theories was described in section 2.4. What is the Vasiliev dual to the free field theory deformed *only* by turning on a finite Chern–Simons–t'Hooft coupling $\lambda = \frac{N}{k}$? The deformation we study is unaccompanied by any potential and Yukawa terms—in particular those needed to preserve SUSY—and so is not supersymmetric. Consequently the comparisons between SUSY Lagrangians and boundary conditions, presented later in this section, does not directly address the question raised here. As we will see, however, the answer to this question is partly constrained by symmetries, and receives indirect inputs from our analysis of SUSY theories below.

We first recall that it was conjectured in [14] that the bulk dual to turning on $\lambda$ involves a modification of the *bulk* Vasiliev equations by turning on an appropriate parity violating phase, $\theta(X)$, as a function of $\lambda$. The results of the previous section clearly substantiate this conjecture[37]. It is possible, however, that in addition to turning on the phase, a nonzero Chern–Simons coupling also results in modified boundary conditions on bulk scalars and fermions. We now proceed to investigate this possibility.

A consideration of symmetries greatly constrains possible modifications of boundary conditions. Recall that the Vasiliev dual to free SUSY theories possesses a $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ global symmetry. In the dual boundary theory the $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ symmetry rotates the fundamental bosons and fermions respectively, and is preserved by turning on a nonzero Chern–Simons coupling. A constant phase in Vasiliev's equations also preserves this symmetry. It follows that all accompanying boundary condition deformations must also preserve this symmetry.

Parity even and odd bulk scalars respectively transform in the (adjoint + singlet, singlet) and (singlet, adjoint+singlet) representations of the $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ symmetry. The only conformally invariant modifications of boundary condition that preserve this symmetry are those dual to the double trace coupling of the parity odd and parity even singlet scalars, and that dual to the triple trace deformation of three parity even singlet scalars.

The conjectures of the previous section strongly constrain the double trace type deformation of boundary conditions induced by the Chern–Simons coupling[38]. Let us, for instance, compare Lagrangian and boundary conditions of the fixed line of $\mathcal{N} = 1$ theories described in the previous subsection. The double trace scalar potential in these theories is listed in (5.45) below and vanishes at $\omega = -1$. On the other hand the rotation $\gamma$ in the scalar boundary conditions for the dual Vasiliev system is listed in (4.11), and vanishes for the dual of $\omega = -1$. In other words the Vasiliev dual to the Chern–Simons theory with no scalar potential obeys boundary conditions such that all 'parity even' scalars continue to have $\Delta = 1$ boundary conditions, while all 'parity odd' scalars continue to have $\Delta = 2$ boundary conditions. While the argument presented above holds only for $n = 2$, the result continues to apply at $n = 4$ and $n = 6$ as well, as we will see in more detail in the detailed comparisons later[39].

We turn now to the fermions. Bulk fermions transform in the (fundamental, antifundamental) and (antifundamental, fundamental) of the free symmetry algebra. There is, of course, a natural double trace type singlet boundary condition deformation with this field content (this deformation has the same effect on boundary conditions as a double trace field theory term $(\phi_a\bar{\psi}^b)(\psi_b\bar{\phi}^a)$ where $a$ and $b$ are global symmetry indices and brackets denote the structure of gauge contractions). Perhaps surprisingly, we will now argue that merely turning on the Chern–Simons term *does* induce such a boundary condition deformation. More precisely,

[37] As those results are valid only for the linearized theory, they unfortunately cannot distinguish between a constant phase and a more complicated phase function; we return to this issue later.

[38] Our analysis of boundary conditions in the previous section was insensitive to triple trace type boundary conditions, and so does not constrain the triple trace type modification.

[39] For the case $n = 4$ consider, for instance, the $\mathcal{N} = 2$ theory with two fundamental chiral multiplets. The free theory has a $U(2) \times U(2)$ symmetry. The interacting theory preserves the diagonal $SU(2)$ subgroup of this symmetry (corresponding to rotations of the two chiral multiplets). The parity odd and even single trace operators in this theory each transform in the $1+3$ representations of this symmetry. The allowed double trace deformations of this interacting theory couple the parity even 3 with the parity odd 3 and the parity even scalar with the parity odd scalar. It so happens that these two terms appear with the same coefficient in both the field theory potential (D.6) and the corresponding Vasiliev boundary conditions (the fact that these terms appear with the same coefficient in (4.5) is simply the fact that the singlet monomial $I$, appears on the same footing as the triplet monomials $\psi_2\psi_3$, $\psi_3\psi_4$, $\psi_4\psi_2$ in the scalar boundary conditions). These facts together demonstrate that the Chern–Simons term (which could have acted only on the singlet double trace term and so would have 'split the degeneracy' between singlets and triplets) has no double trace type effect on scalar boundary conditions.

it turns out that the bulk theory with trivial boundary conditions on fermions corresponds to a quantum field theory with fermion double trace potential equal to

$$-\frac{6\pi}{k}\bar{\Psi}\Psi$$

for every single trace Fermionic operator.

We present a heuristic argument for this conclusion in appendix E by comparing the Lagrangian and boundary conditions of the line of $\mathcal{N} = 1$ theories with a single chiral multiplet. However the most convincing argument for this conclusion is that it leads to consistent results between the Lagrangian and boundary conditions in *every* case we study in detail later in this section.

In order to compensate for the shift described above, will find it useful, in our analysis below, to compare Fermionic boundary conditions with a shifted field theory Lagrangian: one in which we add by hand the double trace term $\frac{6\pi}{k}\bar{\Psi}\Psi$ for every single trace fermionic field. Bulk fermionic fields have trivial boundary conditions only when the double trace deformations of the corresponding fermionic operators vanish in the shifted field theory Lagrangian.

*5.4.2. Special points in moduli space for scalars.* If we wish to specify the bulk dual for a 3D CFT, it is insufficient to specify the bulk action and the boundary conditions for bulk scalars (see section C.1). In order to specify the correlators of the dual theory we must, in addition, specify the precise nature of the boundary dynamics that gives rise the resultant boundary conditions. Inequivalent boundary dynamics that lead to the same boundary conditions result in distinct correlation functions; in particular to different counterterms in correlators.

Of the set of all boundary actions that lead to a particular boundary condition, one is particularly simple ($\theta_0 = 0$ in section C.1.3); this choice of boundary counterterms ensures that correlators between dimension one and dimension two operators vanish identically (including contact terms). Let us suppose that the dual of a particular quantum field theory is governed by this simple boundary dynamics. Then the dual of this theory deformed by a scalar double trace deformation cannot, in general, also be governed by the same simple boundary dynamics (see section C.1.3).

In the moduli space of field theories obtained from one another by double trace deformations, it follows that there is a special point at which boundary scalar dynamics is governed by the simple $\theta_0 = 0$ rule. It certainly seems natural to conjecture that this special theory is governed by a Lagrangian with no double trace terms, i.e. the pure Chern–Simons theory described in the previous subsection. As we will explain below, this assumption unfortunately appears to clash with an at least equally natural assumption about the AdS/CFT implementation of the boundary Chern–Simons gauging of a global symmetry, as we review below.

*5.4.3. Identification of bulk and boundary Chern–Simons terms.* As we have explained in section 5.3, it is very natural to simply identify the boundary field theoretic Chern–Simons term with a Chern–Simons term for the boundary value of bulk gauge fields. If we make this assumption then it follows that the boundary conditions for bulk vector uniquely specify its boundary dynamics and the comparison of gauge field structures between the bulk and the boundary establish a map between moduli spaces of field theories and the Vasiliev dual. As we have mentioned in the previous subsection, however, the results obtained in this manner clash with those obtained from the 'natural' identification of the specially simple field theory as far as scalar double trace operators are concerned. As we explain, one way out of this conundrum is to abandon the 'natural' assumption of the previous subsection. However we do not propose a definitive resolution to this clash in this paper, leaving this for future work.

In the rest of this section we present a detailed comparison between double trace deformations of the field theory Lagrangian and boundary conditions of the dual Vasiliev theory, for the various theories we study, starting with those theories that allow a nontrivial matching of gauge field terms.

### *5.5. $\mathcal{N} = 3$ fixed line with one hypermultiplet*

In this section we present a detailed comparison of the Lagrangian (section D.7) of a fixed line of one hypermultiplet $\mathcal{N} = 3$ theories with boundary conditions (4.20) of its conjectured Vasiliev dual.

*5.5.1. Boundary conditions for the vector.* As described in the section 5.3, the Chern–Simons gauging of the boundary global current results in modifying the boundary conditions for the dual gauge field in the bulk. The modified boundary condition are given by (5.25) which can also be written as

$$\epsilon_{ijk}F_{jk} = \frac{\mathrm{i}\pi\tilde{N}}{4k}F_{zi}. \tag{5.26}$$

The form of boundary conditions for gauge field used in section 4

$$B^{(1)}(\vec{x}, z|Y)|_{\mathcal{O}(y^2,\bar{y}^2)} = z^2[\mathrm{e}^{\mathrm{i}\beta}(yFy) + \Gamma\,\mathrm{e}^{-\mathrm{i}\beta}(\bar{y}\bar{F}\bar{y})] + \mathcal{O}(z^3) \tag{5.27}$$

are equivalent to

$$\epsilon_{ijk}F_{jk} = 2\mathrm{i}\tan(\beta - \theta_0)F_{zi}. \tag{5.28}$$

Comparing (5.26) and (5.28) we get

$$\tan(\beta - \theta_0) = \frac{\pi\tilde{N}}{8k}. \tag{5.29}$$

From (4.20) we have

$$\beta = \theta_0 + (\tilde{\beta} - \theta_0)P_\Gamma,$$

where $\tilde{\beta}$ is the free parameter that parameterizes the fixed line of boundary conditions (4.20). In particular case of vectors proportional to $P_\Gamma\beta = \tilde{\beta}$. Comparing (5.26), (5.28) and (5.29) it follows that

$$\tan(\tilde{\beta} - \theta_0) = \frac{k_1}{k_2}\tan\theta_0, \tag{5.30}$$

where

$$\tan\theta_0 = \frac{\pi\tilde{N}}{8k_1} = \frac{\pi N h_A}{2k_1}. \tag{5.31}$$

Here $h_A$ is the ratio of the two point function of current at the ungauged $\mathcal{N} = 3$ point ($k_2 = \infty$) to the two point function in the free theory. (5.30) establishes a clear map between the parameter $\tilde{\beta}$ that labels boundary conditions in (4.20) and the parameter $\frac{k_1}{k_2}$ that labels the fixed line of dual field theories.

*5.5.2. Scalar double trace deformation.* In this subsection we compare the scalar double trace operators in the field theory Lagrangian (section D.7) with the boundary conditions for scalar fields (4.20) in the Vasiliev dual.

The scalar double trace deformation in the Lagrangian (section D.7) is given by

$$V_s = \frac{2\pi}{k_1}\Phi_+^a\Phi_-^b\eta_{ab} + \frac{2\pi}{k_2}\left(\Phi_+^0\Phi_-^0 + \Phi_+^a\Phi_-^b\eta_{ab}\right), \\ = -\frac{2\pi}{k_1}\Phi_+^0\Phi_-^0 + \frac{2\pi}{k_1}\left(1+\frac{k_1}{k_2}\right)\Phi_+^i\Phi_-^i. \tag{5.32}$$

This potential interpolates between that of the $\mathcal{N}=3$ ungauged theory ($k_2=\infty$) and $\mathcal{N}=4$ theory ($k_2=-k_1$). The two point function of $\Phi_\pm^a$ are twice of those given in (F.4) and thus matches with (5.4). The boundary conditions for scalar fields are described by the rotation angle

$$\gamma = \theta_0 P_1 + \tilde\beta P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}. \tag{5.33}$$

The double trace term $\frac{2\pi}{k_1}(1+\frac{k_1}{k_2})\Phi_+^i\Phi_-^i$ couples two $SO(3)$ vectors. The rotation angle that multiples $P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}$ in (5.8) is determined by the coefficient of this term. The precise relationship between these may be obtained as follows. Let us suppose that the formula (5.8) applies starting from some as yet unknown point, $\tilde\beta=\tilde\beta_0$, in the moduli space of theories. In other words we hypothesize that $\theta_0=0$ (in the language of section C.1.3) for the point in moduli space with $\tilde\beta=\tilde\beta_0$. Let us also suppose that $k_2=(k_2)_0$ corresponding field theory. It follows then from (5.8), (5.33) and (5.32) that (see below for the numerical values of the proportionality constants)

$$\tan(\tilde\beta-\tilde\beta_0) \propto \frac{1}{k_2} - \frac{1}{(k_2)_0}.$$

*Case:* $\tilde\beta_0=0$. Purely from the viewpoint of the scalars it is natural to conjecture that $\tilde\beta_0=0$ and $(k_2)_0=-k_1$. This conjecture is motivated by the following observations. The contact term in the two point function between $\Phi_+^i$ and $\Phi_-^i$ vanishes in the field theory dual to bulk boundary conditions governed by the parameter $\tilde\beta_0$. At leading order in boundary perturbation theory (i.e. at order $1/k$) a naive computation yields a contact term proportional to the double trace coupling of $\Phi_+^i$ and $\Phi_-^i$. Thus appears to imply that the special field theory have a vanishing double trace term; this occurs at the $\mathcal{N}=4$ point and so $\tilde\beta_0=0$. If we make this assumption it then follows that that

$$\tan\tilde\beta = \tan\theta_0\left(1+\frac{k_1}{k_2}\right), \qquad \text{with} \quad \tan\theta_0 = \frac{N\pi}{2k_1}\sqrt{h_+h_-}, \tag{5.34}$$

where $h_+$ and $h_-$ is the ratio of two point function for $\Phi_+$ and $\Phi_-$ respectively in the interacting ($\mathcal{N}=4$ point) to free theory. Unfortunately (5.34) conflicts with (5.30), so both relations cannot be simultaneously correct.

*Case:* $\tilde\beta_0=\theta_0$. The conflict with (5.36) vanishes if we instead assume that

$$\tilde\beta_0=\theta_0. \tag{5.35}$$

This is dual to the 'ungauged' $\mathcal{N}=3$ theory and so it follows that and $(k_2)_0=\infty$. Under this assumption it follows that

$$\tan(\tilde\beta-\theta_0) = \tan\theta_0\left(\frac{k_1}{k_2}\right), \qquad \text{with} \quad \tan\theta_0 = \frac{N\pi}{2k_1}\sqrt{h_+h_-}, \tag{5.36}$$

where $h_+$ and $h_-$ is the ratio of two point function for $\Phi_+$ and $\Phi_-$ respectively in the interacting ('ungauged' $\mathcal{N}=3$ point) to free theory. Note that (5.36) perfectly matches (5.31) if $h_A=\sqrt{h_+h_-}$. It is plausible that SUSY enforces this relationship on field theory operators, but we will not attempt to independently verify this relationship in this paper.

Perhaps the simplest resolution of the clash between (5.34) and (5.30) is obtained by setting $\tilde{\beta}_0 = \theta_0$. Before accepting this suggestion we must understand why the contact term in the scalar–scalar two point function vanishes at the $\mathcal{N} = 3$ rather than at the $\mathcal{N} = 4$ point (where the double trace term in the Lagrangian vanishes). As discussions relating to contact terms are famously full of pitfalls; we postpone the detailed study of this question to later work.

*Coefficient of the scalar double trace deformation.* The double trace term in (5.32) that couples two $SO(3)$ scalars is $\frac{2\pi}{k_1}\Phi^0_+\Phi^0_-$. Note that the coefficient of this term is independent of $k_2$, which matches with the fact that the coefficient of $P_1$ in (5.33) is independent of $\tilde{\beta}$.

If we assume that $\tilde{\beta}_0 = \theta_0$ for this term as well we once again find the second of (5.34), where $h_+$ and $h_-$ have the same meaning as in (5.34), except that the two point function in question is that of the scalar operator $\phi^0$. We conclude that $\phi^a$ and $\phi^0$ have equal values of $h_+h_-$. If, instead, $\tilde{\beta} = 0$ then a very similar equation holds; the only difference is that $h_+h_-$ would then compute ratios of the interacting and free two point functions at the $\mathcal{N} = 4$ point.

*5.5.3. Fermionic double trace deformation.* The fermionic double trace deformation for this fixed line is given by

$$V_3 = \frac{2\pi}{k_1}\left(\frac{1}{2}\bar{\Psi}^a\Psi^b\delta^{ab} - 2\bar{\Psi}^0\Psi^0 - \bar{\Psi}^0\bar{\Psi}^0 - \Psi^0\Psi^0\right) + \frac{2\pi}{k_2}\left(\bar{\Psi}^a\Psi^b\eta^{ab} + \frac{1}{2}\bar{\Psi}^a\bar{\Psi}^b\eta_{ab} + \frac{1}{2}\Psi^a\Psi^b\eta^{ab}\right). \tag{5.37}$$

Adding $\delta V_f = \frac{3\pi}{k}\bar{\psi}^a\psi^a$ in order to account the effect of finite Chern–Simons level as described earlier, we obtain the shifted potential

$$V_3 + \delta V_f = -\frac{\pi}{k_1}(\Psi^a - \bar{\Psi}^a)(\Psi^b - \bar{\Psi}^b)\delta^{ab} + \frac{\pi}{k_1}\left(1 + \frac{k_1}{k_2}\right)(\bar{\Psi}^a + \Psi^a)\eta_{ab}(\bar{\Psi}^b + \Psi^b). \tag{5.38}$$

The two point function of $\langle\bar{\Psi}^a\Psi^b\rangle$ is twice of the that given in (F.4) because $\Psi^a$ are constructed out of field doublets and thus matches with (5.13).

The rest of the analysis closely mimics the study of scalar double trace deformations presented in the previous subsection. We associate (in the boundary conditions) the projector $P^a_\psi$ with the real Lagrangian deformation $[\mathrm{i}(\psi^a - \bar{\psi}^a)]^2$ and $P_{\Gamma\psi^a}$ with the other real Lagrangian deformation $(\psi^a + \bar{\psi}^a)^2$. As for the scalar double trace deformations, (5.15) yields results consistent with (5.30) if and only if we assume that (5.15) applies for deformations about the special point $\tilde{\beta} = \theta_0$. Given this assumption (4.20) and (5.15) matches with the identification (5.36) with $\sqrt{h_+h_-} = h_\psi$ and $h_\psi$ interpreted as the ratio of $\langle\bar{\Psi}^a\Psi^b\rangle$ at $\mathcal{N} = 3$ point to the free theory[40].

## 5.6. $\mathcal{N} = 3$ fixed line with 2 hypermultiplets

In this section we compare the Lagrangian for the fixed line of two hypermultiplet theories presented in (section D.9) with the boundary conditions (4.25) of the conjectured Vasiliev duals.

[40] If, on the other hand, (5.15) had applied for deformations around $\tilde{\beta} = 0$ we would instead have found agreement with (5.34) with $\sqrt{h_+h_-} = h_\psi$, where $h_\psi$ would have been interpreted as the ratio of $\langle\bar{\Psi}^a\Psi^b\rangle$ at $\mathcal{N} = 4$. Of course these results contradict (5.30).

The field theories under study interpolate between the ungauged $\mathcal{N} = 3$ theory ($k_2 = \infty$) and the $\mathcal{N} = 6$ theory (at $k_2 = -k_1$).

*5.6.1. Vector field boundary conditions.* The comparison here is very similar to that performed in the previous subsection, and our presentation will be brief. Making the natural assumptions spelt out in the previous section, the gauge field boundary conditions listed in (4.25) assert that

$$\beta = \theta_0 + (\tilde{\beta} - \theta_0)P_\Gamma.$$

Using (5.29) we find

$$\tan(\tilde{\beta} - \theta_0) = \frac{k_1}{k_2} \tan 2\theta_0. \tag{5.39}$$

with the identification

$$\tan(2\theta_0) = \frac{\pi \tilde{N}}{8k_1} = \frac{\pi N h_A}{k_1}$$

where $h_A$ is interpreted as the ratio of the two point function of the flavor current in the ungauged $\mathcal{N} = 3$ theory to the free theory.

*5.6.2. Scalar double trace deformation.* The scalar double trace deformation for this case, in the notation defined in appendix (section D.9), is given by

$$\begin{aligned} V_s &= \frac{\pi}{k_1} \Phi_+^{Ii} \Phi_-^{Jj} \eta^{IJ} \eta_{ij} - \frac{2\pi}{k_2} \Phi_+^{I0} \Phi_-^{J0} \eta^{IJ} \\ &= \frac{\pi}{k_1} \left( \Phi_+^{Ii} \Phi_-^{Jj} \eta^{IJ} \eta_{ij} + 2\Phi_+^{I0} \Phi_-^{J0} \eta^{IJ} \right) - \frac{2\pi}{k_1} \left( 1 + \frac{k_1}{k_2} \right) \Phi_+^{I0} \Phi_-^{J0} \eta^{IJ}. \end{aligned} \tag{5.40}$$

Due the fact that $\Phi_+^{Ii}$ and $\bar{\Phi}_-^{Ii}$ are made of two field doublets, there free two point function are four times of those given in (F.4) and thus twice of those given in (5.4). The boundary conditions of the dual scalars listed in (section D.9) is governed by

$$\gamma = \theta_0 P_{1,\psi_i\psi_a,\psi_a\psi_b} - \tilde{\beta} P_{\psi_i\psi_j}, \quad P_{1,\psi_i\psi_j,\psi_i\psi_a,\psi_a\psi_b} \tilde{f}_{1,2} = \tilde{f}_{1,2}. \tag{5.41}$$

As in the previous section the coefficient of the double trace deformations (5.40) and the boundary conditions of scalars in (5.41) are both respectively independent of $k_2$ and $\tilde{\beta}$ in every symmetry channel but one (i.e. (vector, scalar) under $SU(2) \times SU(2)$). Comparing coefficients in this special channel we find that (5.41) and (section D.9) agree with (5.29) if and only if we assume that (5.8) applies for deformations of $\tilde{\beta}$ away from the special point $\tilde{\beta}_0 = \theta$ at which point $k_2 = \infty$.

$$\tan(\tilde{\beta} - \theta_0) = \tan 2\theta_0 \left( \frac{k_1}{k_2} \right) \qquad \text{with} \quad \tan 2\theta_0 = \frac{\pi N}{k_1} \sqrt{h_+ h_-}, \tag{5.42}$$

with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N} = 3$ ungauged point to free theory.

On the other hand upon assuming $\tilde{\beta}_0 = 0$ we find

$$\tan(\tilde{\beta} + \theta_0) = \tan 2\theta_0 \left( 1 + \frac{k_1}{k_2} \right) \qquad \text{with} \quad \tan 2\theta_0 = \frac{\pi N}{k_1} \sqrt{h_+ h_-}, \tag{5.43}$$

with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N} = 6$ point to free theory. This is in contradiction with (5.39).

We now turn to the comparison of the double trace terms and boundary conditions in all other channels (i.e. (scalar, scalar), (vector, vector) and (scalar, vector) under $SO(3) \times SO(3)$. In each case if we assume that (5.8) applies starting from the special point $\tilde{\beta}_0 = \theta_0$, we find the second of (5.42) with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N} = 3$ ungauged point to free theory for the appropriate scalar. This suggests that the product $h_+h_-$ is the same for scalars in all four symmetry channels; this product is also equal to $h_A^2$. It is possible that this equality is consequence of $\mathcal{N} = 3$ SUSY of the field theory; we leave the verification of this suggestion to future work.

*5.6.3. Fermionic double trace deformation.* The fermionic double trace deformation for this case, in the notation defined in appendix (section D.9), after compensating by a for the Chern–Simons shift[41], is given by

$$
\begin{aligned}
V_f + \delta V_f = {} & \frac{\pi}{k_1}(\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij} + \bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij} + (\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij} + \Psi^{0i}\Psi^{0j}\eta_{ij})) \\
& + \frac{\pi}{k_2}(\bar{\Psi}^{I0} + \Psi^{I0})(\bar{\Psi}^{J0} + \Psi^{J0})\eta_{IJ}. \\
= {} & \frac{\pi}{k_1}(\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij} + \bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij} + (\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij} + \Psi^{0i}\Psi^{0j}\eta_{ij}) \\
& - (\bar{\Psi}^{I0} + \Psi^{I0})(\bar{\Psi}^{J0} + \Psi^{J0})\eta_{IJ}) + \frac{\pi}{k_1}\left(1 + \frac{k_1}{k_2}\right)(\bar{\Psi}^{I0} + \Psi^{I0})(\bar{\Psi}^{J0} + \Psi^{J0})\eta_{IJ}.
\end{aligned}
\tag{5.44}
$$

The two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ is twice of that given by (5.13).

The bulk boundary conditions are generated by

$$\alpha = \theta_0(P_{\psi_i,\psi_a} + P_{\psi_i\psi_j\psi_a,\psi_i\psi_a\psi_b,\psi_4\psi_5\psi_6} - P_{\psi_a\Gamma}) + \tilde{\beta}(P_{\psi_i\Gamma} - P_{\psi_1\psi_2\psi_3}).$$

Consistency requires us to assume that (5.15) applies for deviations away from $\tilde{\beta} = 0$ (i.e. from the ungauged $\mathcal{N} = 3$ theory). Applying (5.15) we recover (5.42) provided $h_\psi = \sqrt{h_+h_-}$ where $h_\psi$ is the ratio the two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ at the ungauged $\mathcal{N} = 3$ point to free theory[42].

## 5.7. Fixed line of $\mathcal{N} = 1$ theories

We now turn to the comparison of the Lagrangian (D.7) of the large $N$ fixed line of $\mathcal{N} = 1$ field theories with the boundary conditions (4.8) (a beta function is generated at finite $N$, the zeros of this beta function are the two ends of the line we study below). We restrict attention to the case $M = 1$. The field content of the theory is a single complex scalar $\phi$ together with a single complex fermion $\psi$.

*5.7.1. Scalar double trace terms.* The (scalar)(scalar) double trace potential in (D.7) is given by

$$\frac{2\pi(1+\omega)}{k}\bar{\phi}\phi\bar{\psi}\psi. \tag{5.45}$$

$\omega = -1$ is the $\mathcal{N} = 1$ theory with no superpotential while $\omega = 1$ is the $\mathcal{N} = 2$ theory. The two point functions of the constituent single trace operators, $\bar{\phi}\phi$ and $\bar{\psi}\psi$, are given, in the free theory, by (F.4) (note that this corresponds to $h_+ = h_- = \frac{1}{2}$ in (5.4)).

[41] The compensating factor in this case is $\delta V_f = \frac{3\pi}{2k_1}\bar{\Psi}^{Ii}\Psi^{Ii}$.

[42] If, instead, (5.15) had applied starting from $\tilde{\beta} = 0$ we would have found consistency with (5.43) provided $h_\psi = \sqrt{h_+h_-}$ where $h_\psi$ interpreted as the ratio the two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ at $\mathcal{N} = 6$ point to free theory. This result contradicts the gauge field matching and so cannot apparently cannot be correct.

The $n = 2$ Vasiliev dual to this system is conjectured to have boundary conditions listed in (4.11). Specifically the boundary conditions require $B$ to take the form

$$B(x,z) = zf_1(x)((1+\Gamma)\cos\gamma + \mathrm{i}(1-\Gamma)\sin\gamma) + \mathrm{i}f_2(x)z^2((1-\Gamma)\cos\gamma + \mathrm{i}(1+\Gamma)\sin\gamma) \tag{5.46}$$

where $f_1$ and $f_2$ are real constants and $\gamma$ ranges from zero (for the $\mathcal{N} = 1$ theory with no superpotential) to $\gamma = \theta_0$ (for the $\mathcal{N} = 2$ theory). Notice that the shift change in phase between these two points is $\theta_0$, while the change in the coefficient of the corresponding double trace term in the Lagrangian (5.45) is $\frac{4\pi}{k}$.

In order to establish a map between the Lagrangian parameter $\omega$ and the boundary condition parameter $\gamma$ we need to know the location of the special point, $\gamma_0$, in $\gamma$ parameter space from which (5.8) applies (this is the point with $\theta_0 = 0$ in the language of section C.1.3). Unlike the previous subsections, in this case we have no information from the gauge field boundary conditions, so the best we can do is to make a guess. We consider two cases.

*Case* $\gamma_0 = \theta_0$. The results of the previous subsection suggest that $\gamma_0 = \theta_0$ so that the special point in the moduli space of Vasiliev theories is the $\mathcal{N} = 2$ theory. If this is the case then

$$\tan(\theta_0 - \gamma) = \tan\theta_0 \frac{1-\omega}{2}$$

where

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+h_-}}{2} \tag{5.47}$$

and $h_+$ gives the ratio of the interacting and free two point functions of $\bar\phi\phi$ for the $\mathcal{N} = 2$ theory.

*Case* $\gamma_0 = 0$. Purely from the point of view of the scalar part of the Lagrangian, the most natural assumption is $\gamma_0 = 0$ in which case

$$\tan\gamma = \tan\theta_0 \frac{1+\omega}{2}$$

where

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+h_-}}{2} \tag{5.48}$$

and $h_+$ gives the ratio of the interacting and free two point functions of $\bar\phi\phi$ for the $\mathcal{N} = 1$ theory with no superpotential.

*5.7.2. Fermion double trace terms.* The (fermion)(fermion) double trace potential term after accounting for the shift described in

$$\begin{aligned} V_f + \delta V_f &= V_f + \frac{6\pi}{k}\bar\psi\phi\bar\phi\psi \\ &= \frac{\pi(\omega+1)}{k}(\bar\psi\phi + \bar\phi\psi)^2 - \frac{2\pi}{k}(\bar\psi\phi - \bar\phi\psi)^2. \end{aligned} \tag{5.49}$$

Here $\omega = -1$ corresponds to the undeformed $\mathcal{N} = 1$ theory and $\omega = 1$ corresponds to the $\mathcal{N} = 2$ theory. The two point function of the operator $\bar\psi\phi$ and $\bar\phi\psi$ are given in (F.4). Note that this corresponds to $h_\psi = \frac{1}{2}$ in (5.4). The boundary condition for fermions are given by (B.10) with

$$\alpha = \theta_0 P_{\psi_2} + \gamma P_{\psi_1}.$$

As explained in the previous section, the coefficient of the $P_{\psi_2}$ in the boundary conditions is associated with the coefficient of double trace deformation $(\mathrm{i}(\bar{\psi}\phi - \bar{\phi}\psi))^2$ while the coefficient of $P_{\psi_1}$ is associated with the double trace deformation $(\bar{\psi}\phi + \bar{\phi}\psi)^2$. Note that this matches with the fact that coefficient of the former are constant along the line while those of the later change along the fixed line.

Using the analysis of section (5.2.2) we can get a more quantitative match. As in the previous section it is natural to assume—and we conjecture—that if (5.15) applies starting from the $\mathcal{N} = 2$ point, at which the first term in (5.49) has coefficient $\frac{2\pi}{k}$. With this assumption

$$\tan(\theta_0 - \gamma) = \tan\theta_0 \frac{1-\omega}{2}, \qquad \text{with} \quad \tan\theta_0 = \frac{\pi\lambda h_\psi}{2}, \tag{5.50}$$

where $h_\psi$ is the ratio of interacting to free two point function $\langle\bar{\psi}\phi\bar{\phi}\psi\rangle$ in $\mathcal{N} = 2$ theory.

If, on the other hand (5.15) were to apply starting from the pure $\mathcal{N} = 1$ point we would find

$$\tan\gamma = \tan\theta_0 \frac{1+\omega}{2}, \qquad \text{with} \quad \tan\theta_0 = \frac{\pi\lambda h_\psi}{2} \tag{5.51}$$

where $h_\psi$ is the ratio of interacting and free two point function $\langle\bar{\psi}\phi\bar{\phi}\psi\rangle$ in $\mathcal{N} = 1$ theory with no superpotential. The results of the previous two subsections appear to disfavor this possibility over the one presented in the previous paragraph.

### 5.8. $\mathcal{N} = 2$ theory with two chiral multiplets

In the final subsection of this section we turn to the comparison of the Lagrangian (section D.1) (with $M = 2$) of the $\mathcal{N} = 2$ theory with two fundamental chiral multiplets with the boundary conditions (4.5). The theory we study admits no marginal superpotential deformations, and so appears as a fixed point rather than a fixed line at any given value of $k_1$.

*5.8.1. Scalar double trace deformation.* The scalar double trace deformation in (section D.1) is given by

$$V_s = \frac{2\pi}{k}\Phi_+^a\Phi_-^a, \tag{5.52}$$

where $\Phi_+^a = \bar{\phi}^i\phi_j(\sigma^a)_i^j$, $\Phi_-^a = \bar{\psi}^i\psi_j(\sigma^a)_i^j$ and $a$ runs over 0,1,2,3. In appendix F we have computed the two point functions of the operators $\Phi_+^a$ and $\Phi_-^a$ in free field theory; the result is given by (F.4) with an extra factor of two to account for the fact that the operators $\Phi_\pm^a$ are constructed out of field doublets. In other words the two point functions of $\Phi_\pm^a$ exactly agree with those presented in (5.4) with $h_+$ and $h_-$ interpreted as the ratio of the two point functions of $\Phi_\pm$ in the interacting theory and the free theory[43]. With this interpretation (5.8) predicts the boundary conditions of the bulk scalars with $d_{a\alpha} = 1$ (both for the singlet of $SU(2)$ as well as the triplet). Comparing these equations with the actual boundary conditions

$$\gamma = \theta_0, P_{1,\psi_2\psi_3,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2},$$

we conclude that $g_a = g_\alpha$ both for singlet scalars as well as for $SU(2)$ triplet scalars.

In order to make a quantitative comparison between the Lagrangian and boundary conditions we need to make an assumption about which point in the moduli space of double trace deformations (5.8) applies from. Given the results of the previous subsections it is natural to guess that (5.8) applies for double trace deformations away from the $\mathcal{N} = 2$

[43] Here it is ambiguous what is the interacting theory i.e. what is the value of $k$ in theory without the double trace deformations, from where (5.8) applies.

theory. Assuming that the theory with no double trace deformation has trivial scalar boundary conditions, we conclude that

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+ h_-}}{2} \tag{5.53}$$

where $h_\pm$ are the ratios of two point functions of the scalar operators in the $\mathcal{N} = 2$ and free theories. This equation must hold separately for singlet as well as $SU(2)$ vector sector. It seems very likely that $h_+ = h_- = h_s$ for all scalars in which case

$$\tan\theta_0 = \frac{\pi\lambda h_s}{2}. \tag{5.54}$$

*5.8.2. Fermion double trace deformation.* The fermion double trace deformation in this case is given by

$$V_f = \frac{\pi}{k}\bar{\Psi}^a\Psi^a, \tag{5.55}$$

where $\Psi^a = \bar{\phi}^i\psi_j(\sigma^a)_i^j$, $\bar{\Psi}^a = \bar{\psi}^i\phi_j(\sigma^a)_i^j$ and $a$ runs over 0,1,2,3. In order to compare this double trace potential with boundary conditions, however, we must remove the effect of the Chern–Simons term. In other words we should expect the fermion boundary conditions to match with an effective fermion double trace potential given by

$$\delta S = \frac{4\pi}{k}\bar{\Psi}^a\Psi^a.$$

(it is easily verified that a shift by $-\frac{3\pi}{k}$ in the coefficient of $\bar{\Psi}^a\Psi^a$ is equivalent to a shift of $-\frac{6\pi}{k}$ in the coefficient of each fermion). The two point functions of these fields is given by (see appendix F)

$$\langle\Psi^a(x)\bar{\Psi}^b(0)\rangle = \frac{N\delta^{ab}h_\psi}{8\pi^2}\frac{\vec{x}\cdot\vec{\sigma}}{x^4},$$

where $h_\psi$ is the ratio of the two point function in the interacting and free theories.

This matches onto the analysis leading up to (5.15) if we set $s = t = 4$ and $u = 0$. Here we assume that (5.15) applies for deformations about the $\mathcal{N} = 2$ point. In this application of (5.15) all factors of $g_a$ relate to fields that are related by $SO(4)$ invariance, and so must be equal. Consequently factors of $g_a$ cancel from that equation. Comparing (5.15) with $s = t = 4$ and $u = 0$ with the actual fermion boundary conditions, in this case

$$\alpha = \theta_0,$$

we recover the equation

$$\tan\theta_0 = \frac{\pi\lambda h_\psi}{2}. \tag{5.56}$$

We see that (5.56) is consistent with (5.53) provided $h_\psi = \sqrt{h_+h_-}$, with $h_\psi$ interpreted as the ratio of the two point function in the $\mathcal{N} = 2$ and free theories. It seems very likely to us that in fact $h_\psi = h_+ = h_- = h_s$.

## 6. The ABJ triality

Having established the supersymmetric Vasiliev theories with various boundary conditions dual to Chern–Simons vector models, we will now use the relation between deformations of the boundary conditions and double trace deformations in the boundary CFT to extract some nontrivial mapping of parameters. In the case of $\mathcal{N} = 6$ theory, the triality between ABJ vector model, Vasiliev theory, and type IIA string theory suggests a bulk–bulk duality between Vasiliev theory and type IIA string field theory. We will see that the parity breaking phase $\theta_0$ of Vasiliev theory can be identified with the flux of flat Kalb–Ramond $B$-field in the string theory.

### *6.1. From $\mathcal{N} = 3$ to $\mathcal{N} = 4$ Chern–Simons vector models*

Let us consider the $\mathcal{N} = 3$ $U(N)_k$ Chern–Simons vector model with one hypermultiplets. Upon gauging the diagonal $U(1)$ flavor symmetry with another Chern–Simons gauge field at level $-k$, one obtains the $\mathcal{N} = 4$ $U(N)_k \times U(1)_{-k}$ theory. In section 5.5.1, by comparing the boundary conditions, we have found the relation

$$\tan\theta_0 = \frac{\pi\widetilde{N}}{8k} = \frac{\pi\lambda h_A}{2}. \tag{6.1}$$

By comparing the structure of three-point functions with the general results of [12], we see that $\tan\theta_0$ is identified with $\widetilde{\lambda}$ of [12]. Therefore, by consideration of SUSY breaking by AdS boundary conditions, we determine the relation between the parity breaking phase $\theta_0$ of Vasiliev theory and the Chern–Simons level of the dual $\mathcal{N} = 3$ or $\mathcal{N} = 4$ vector model to be

$$\widetilde{\lambda} = \frac{\pi\widetilde{N}}{8k}. \tag{6.2}$$

Recall that $\widetilde{N}$ is defined as the coefficient of the two-point function of the $U(1)$ flavor current $J_i$ in the $\mathcal{N} = 3$ Chern–Simons vector model, normalized so that $\widetilde{N}$ is 4 for each *free* hypermultiplet. In notation similar to that of the previous section $\widetilde{N} = 4Nh_A$ where $h_A$ is the ratio of the two point function of the flavor currents in the interacting and free theory. Consequently (6.2) may be rewritten as

$$\widetilde{\lambda} = \frac{\pi\lambda h_A}{2}. \tag{6.3}$$

After gauging this current with $U(1)$ Chern–Simons gauge field $\widetilde{A}_\mu$ at level $-k$, passing to the $\mathcal{N} = 4$ theory, the new $U(1)$ current which may be written as $J_{\text{new}} = -k * \mathrm{d}\widetilde{A}$ has a different two-point function than $J_i$, as can be seen from section 3.1. The two-point function of $J_{\text{new}}$ also contains a parity odd contact term, as was pointed out in [18].

We would also like to determine the relation between $\theta_0$ and $\lambda = N/k$, which cannot be fixed directly by the consideration of SUSY breaking by boundary conditions. The two-loop result of [14] on the parity odd contribution to the three-point functions also applies to correlators of singlet currents made out of fermion bilinears in supersymmetric Chern–Simons vector models, since the double trace and triple terms do not contribute to the parity odd terms in the three-point function at this order. From this we learn that $\theta_0 = \frac{\pi}{2}\lambda + \mathcal{O}(\lambda^3)$. Parity symmetry would be restored if we also send $\theta(X) \to -\theta(X)$ under parity, and in particular $\theta_0 \to -\theta_0$. Further, in the supersymmetric Vasiliev theory, $\theta_0$ should be regarded as a periodically valued parameter, with periodicity $\pi/2$. This is because the shift $\theta_0 \to \theta_0 + \frac{\pi}{2}$ can be removed by the field redefinition $\mathcal{A} \to \psi_1\mathcal{A}\psi_1$, $B \to -\mathrm{i}\psi_1 B\psi_1$, where $\psi_1$ is any one of the Grassmannian auxiliary variables. Note that the factor of i in the transformation of the master field $B$ is required to preserve the reality condition. Essentially, $\theta_0 \to \theta_0 + \frac{\pi}{2}$ amounts to exchanging bosonic and fermionic fields in the bulk.

Giveon–Kutasov duality [42] states that the $\mathcal{N} = 2$ $U(N)_k$ Chern–Simons theory with $N_f$ fundamental and $N_f$ anti-fundamental chiral multiplets is equivalent to the IR fixed point of the $\mathcal{N} = 2$ $U(N_f + k - N)_k$ theory with the same number of fundamental and anti-fundamental chiral multiplets, together with $N_f^2$ mesons in the adjoint of the $U(N_f)$ flavor group, and a cubic superpotential coupling the mesons to the fundamental and anti-fundamental superfields. Specializing to the case $N_f = 1$ (or small compared to $N, k$), this duality relates the 'electric' theory: $\mathcal{N} = 2\ U(N)_k$ Chern–Simons vector model with $N_f$ pairs of $\square, \overline{\square}$ chiral multiplets at large $N$, to the 'magnetic' theory obtained by replacing $\lambda \to 1 - \lambda$ and rescaling the value of $N$, together with turning on a set of double trace deformations and flowing to the critical point. In the holographic dual of this vector model, the double trace deformation in the definition of the magnetic theory simply amounts to changing the boundary condition on a set of bulk scalars and fermions. This indicates that the bulk theory with parity breaking phase $\theta_0(\lambda)$ should be equivalent to the theory with phase $\theta_0(1 - \lambda)$, suggesting that the identification

$$\theta_0 = \frac{\pi}{2}\lambda \tag{6.4}$$

is in fact exact in the duality between Vasiliev theory and $\mathcal{N} = 2$ Chern–Simons vector models of the Giveon–Kutasov type. By turning on a further superpotential deformation, this identification can be extended to the $\mathcal{N} = 3$ theory as well. Together with (6.3), (6.4) then implies that relation $\tan(\frac{\pi}{2}\lambda) = \frac{\pi\widetilde{N}}{8k} = \frac{\pi\lambda h_A}{2}$ in the $\mathcal{N} = 3$ Chern–Simons vector model in the planar limit. Note that in the $k \to \infty$ limit where the theory becomes free, this relation becomes the simply $\widetilde{N} = 4N$, which follows from our normalization convention of the spin-1 flavor current.

A similar comparison between double trace deformations of scalar operators and the change of scalar boundary condition in the bulk Vasiliev theory lead to the same identification between $\theta_0$ and $\widetilde{N}$, $k$. Note that in the supersymmetric Chern–Simons vector model, $\widetilde{N}$ by our definition is the two-point function coefficient of a flavor current, which is related to the two-point function coefficient of gauge invariant scalar operators by SUSY. However, our $\widetilde{N}$ is a priori normalized *differently* from that of Maldacena and Zhiboedov [12], where $\widetilde{N}$ was defined as the coefficient of two-point function of higher spin currents, normalized by the corresponding higher spin charges[44].

A high nontrivial check would be to prove the relations (6.3) and (6.4) directly in the field theory using the Schwinger–Dyson equations considered in [14]. In the case of Chern–Simons-scalar vector model, this computation is performed in [19]. It is found in [19] that the relation $\theta_0 = \pi\lambda/2$ holds, whereas the scalar two-point function is precisely proportional to $k \tan\theta_0$ up to a numerical factor that depends on the number of matter fields[45], remarkably coinciding with our finding in the supersymmetric theory by consideration of boundary conditions and holography. We leave it to future work to establish these relations in the supersymmetric theory using purely large $N$ field theoretic technique.

### *6.2. ABJ theory and a triality*

Now let us consider the $\mathcal{N} = 3\ U(N)_k$ Chern–Simons vector model with two hypermultiplets. Upon gauging the diagonal $U(1)$ flavor symmetry with another Chern–Simons gauge field at level $-k$, one obtains the $\mathcal{N} = 6\ U(N)_k \times U(1)_{-k}$ ABJ theory. By comparing the boundary conditions, in section 5.6.1, we have found the formula

$$\tan(2\theta_0) = \frac{\pi\widetilde{N}}{8k} = \pi\lambda h_A, \tag{6.5}$$

where $\widetilde{N}$ is the coefficient of the two-point function of the $U(1)$ flavor current in the $\mathcal{N} = 6$ theory, and $h_A$, as usual, is the ratio of the flavor current two point function in the interacting

[44] We thank Ofer Aharony for discussions on this point.

[45] The authors of [19] adopted the natural field theory normalization for the scalar operator, which would agree with our normalization for the flavor current, and differ from the normalization of [12] by a factor $\cos^2\theta_0$.

and free theory. Note that the factor of 2 in the argument of $\tan(2\theta_0)$ is precisely consistent with the fact that in the $k \to \infty$ limit, the $U(1)$ flavor current which is made out of twice as the $\mathcal{N} = 2$ theory of one hypermultiplet considered in the previous subsection, so that $\widetilde{N}$ is enhanced by a factor of 2 (namely, $\widetilde{N} = 8N$ in the free limit).

Now we can complete our dictionary of 'ABJ triality'. We propose that the $U(N)_k \times U(M)_{-k}$ ABJ theory, in the limit of large $N, k$ and fixed $M$, is dual to the $n = 6$ extended supersymmetric Vasiliev theory with $U(M)$ Chan–Paton factors, parity breaking phase $\theta_0$ that is identified with $\frac{\pi}{2}\lambda$, and the $\mathcal{N} = 6$ boundary condition described in section 4.2.6. The *bulk* 't Hooft coupling can be identified as $\lambda_{\rm bulk} \sim M/N$. In the strong coupling regime where $\lambda_{\rm bulk} \sim \mathcal{O}(1)$, we expect a set of bound states of higher spin particles to turn into single closed string states in type IIA string theory in $AdS_4 \times \mathbb{CP}^3$ with flat Kalb–Ramond $B_{\rm NS}$-field flux

$$\frac{1}{2\pi\alpha'}\int_{\mathbb{CP}^1} B_{\rm NS} = \frac{N-M}{k} + \frac{1}{2}. \tag{6.6}$$

In the limit $N \gg M$, we have the identification

$$\theta_0 = \frac{\pi}{2}\lambda = \frac{1}{4\alpha'}\int_{\mathbb{CP}^1} B_{\rm NS} - \frac{\pi}{4}. \tag{6.7}$$

Note that this is consistent with $B_{\rm NS} \to -B_{\rm NS}$ under parity transformation. This suggests that the RHS of Vasiliev's equation of motion involving the $B$-master field should be related to worldsheet instanton corrections in string theory (in the suitable small radius/tensionless limit).

### 6.3. Vasiliev theory and open-closed string field theory

A direct way to engineer $\mathcal{N} = 3$ Chern–Simons vector model in string theory was proposed in [23]. Starting with the $U(N)_k \times U(M)_{-k}$ ABJ theory, one adds $N_f$ fundamental $\mathcal{N} = 3$ hypermultiplets of the $U(N)$. In the bulk type IIA string theory dual, this amounts to adding $N_f$ D6-branes wrapping $AdS_4 \times \mathbb{RP}^3$, which preserve $\mathcal{N} = 3$ SUSY. The vector model is then obtained by taking $M = 0$. The string theory dual would be the 'minimal radius' $AdS_4 \times \mathbb{CP}^3$, supported by the $N_f$ D6-branes and flat Kalb–Ramond $B$-field with

$$\frac{1}{2\pi\alpha'}\int_{\mathbb{CP}^1} B_{\rm NS} = \frac{N}{k} + \frac{1}{2}. \tag{6.8}$$

In this case, our proposed dual $n = 4$ Vasiliev theory in $AdS_4$ with $\mathcal{N} = 3$ boundary condition carries $U(N_f)$ Chan–Paton factors, as does the open string field theory on the D6-branes. This lead to the natural conjecture that the open–closed string field theory of the D6-branes in the 'minimal' $AdS_4 \times \mathbb{CP}^3$ with flat $B$-field is the *same* as the $n = 4$ Vasiliev theory, at the level of classical equations of motion. It would be fascinating to demonstrate this directly from type IIA string field theory in $AdS_4 \times \mathbb{CP}^3$, say using the pure spinor formalism [43–45].

## 7. The partition function of free ABJ theory on $S^2$ as a matrix integral

The ABJ theory is a supersymmetric Chern–Simons theory based on the gauge group $U(N) \times U(M)$, at level $k$ (for $U(N)$) and $-k$ (for $U(M)$) respectively. In addition to the gauge fields, this theory possesses four chiral multiplets $A_1, A_2, B_1, B_2$ (in $d = 3$ $\mathcal{N} = 2$ language). While $A_1$ and $A_2$ transform in the fundamental times antifundamental of $U(N) \times U(M)$, $B_1$ and $B_2$ transform in the antifundamental times fundamental of the same gauge group. The chiral fields all have canonical kinetic terms, and interact with each other via a superpotential proportional to $\epsilon^{ij}\epsilon^{mn}\,{\rm Tr}\, A_i B_m A_j B_n$. While ABJ Lagrangian classically enjoys invariance under the $\mathcal{N} = 6$ superconfomal algebra (an algebra with 24 fermionic generators) for all values of parameters, it has been argued that, quantum mechanically, the theory exists as a superconformal theory only for $k \geqslant |N - M|$.

In this section we will study the partition function of the *free* ABJ theory on $S^3$. In other words we study the free theory and compute

$$Z = \mathrm{Tr}(x^E). \tag{7.1}$$

In more conventional notation $x = \mathrm{e}^{-\beta}$ and $Z$ is the usual thermal partition function at $T = \frac{1}{\beta}$. Here we study the limit $k \to \infty$. In this limit the ABJ theory is free and its partition function is given by the simple formula [46, 20, 47]

$$Z = \int DUDV \exp\left[\sum_{n=1}^{\infty} \frac{(F_B(x^n) + (-1)^{n+1} F_F(x^n))}{n} (\mathrm{Tr}\, U^n \,\mathrm{Tr}\, V^{-n} + \mathrm{Tr}\, V^n \,\mathrm{Tr}\, U^{-n})\right]. \tag{7.2}$$

Here $U$ is an $N \times N$ unitary matrix, $V$ is an $M \times M$ unitary matrix. $F_B(x)$ and $F_F(x)$ are the bifundamental letter partition functions (equal to the antibifundamental letter partition function) over bosonic and fermionic fields respectively. The letter partition function receives contribution from all the basic bifundamental (antibifundamental)fields and there derivatives after removing contribution from equation of motion and are given by

$$\begin{aligned} F_B(x) &= \mathrm{Tr}_{\mathrm{bosons}}(x^{\Delta}), \text{ (where } \Delta \text{ is the dilatation operator)} \\ &= \frac{4x^{\frac{1}{2}}(1-x^2)}{(1-x)^3}, \\ F_F(x) &= \mathrm{Tr}_{\mathrm{fermions}}(x^{\Delta}) = \frac{8x(1-x)}{(1-x)^3}, \\ F(x) &= \mathrm{Tr}_{\mathrm{all}}(x^{\Delta}) = F_B(x) + F_F(x) = \frac{4\sqrt{x}}{(1-\sqrt{x})^2}. \end{aligned} \tag{7.3}$$

In the rest of this section we will study the matrix integral (7.2) as a function of $x$ in the large $M$ and $N$ limit. More precisely, we will focus on the limit $N \to \infty$ and $M \to \infty$ with

$$A = \frac{N}{M}$$

held fixed. Note that we will always assume $A > 1$.

### 7.1. Exact solution of a truncated toy model

The summation over $n$ in (7.15) makes the matrix model in that equation quite complicated to study for $F(x) > 1$. While this matrix model is in principle 'exactly solvable' using the work of [48], the implicit solution thus obtained can be turned into explicit formulae only in special limits (see below for more discussion). Instead of plunging into a discussion of this exact solution, in the rest of this section, we will analyze the model in various limits and approximations; these exercises will clearly reveal the qualitative nature of the solution to the matrix model (7.15).

In this section we analyze a toy model whose solution will qualitatively describe the full phase structure of (7.15). In quantitative terms we will explain below that solution of toy model presented in this subsection agrees with the exact solution of the matrix model when $x = x_c$, and can be used as the starting point for developing a perturbative expansion of this solution in a power series in $x - x_c$. In other words the toy model presented in this subsection qualitatively captures the phase structure of the full matrix model; it also gives a quantitatively correct description of the first phase transition.

The toy model we study is the matrix model obtained from (7.2) by truncating to the $n = 1$ part of its action,

$$Z_t = \int DUDV \exp[F(x)(\mathrm{Tr}\, U\ \mathrm{Tr}\, V^{-1} + \mathrm{Tr}\, V\ \mathrm{Tr}\, U^{-1})]. \tag{7.4}$$

The general saddle point solution to (7.4) obtained extremely easily. Let us assume that (7.4) has a saddle point solution on which $\mathrm{Tr}\, U = N\rho_1$ and $\mathrm{Tr}\, V = M\chi_1$. The eigenvalue distribution for $U$ is then the saddle point solution to the auxiliary matrix model

$$\int DU \exp\left[N\frac{F(x)}{A}\chi_1(\mathrm{Tr}\, U + \mathrm{Tr}\, U^{-1})\right]. \tag{7.5}$$

In a similar fashion the eigenvalue distribution of $V$ is given as the solution to the auxiliary matrix integral

$$\int DV \exp[MAF(x)\rho_1(\mathrm{Tr}\, V + \mathrm{Tr}\, V^{-1})]. \tag{7.6}$$

The matrix integrals (7.5) and (7.6) are of the famous Gross–Witten–Wadia form [49, 50]. Here we briefly review the solution Gross–Witten–Wadia model. The relevant matrix integral is that of an $N \times N$ unitary matrix $W$, defined as

$$\mathcal{I} = \int DW \exp\left[\frac{N}{\lambda}\mathrm{Tr}(W + W^{-1})\right] \tag{7.7}$$

where $\lambda$ is coupling constant. In the large $N$ limit this model undergoes phase transition at $\lambda = 2$. For $\lambda > 2$ the eigenvalue density distribution is

$$\rho(\theta) = \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\theta\right).$$

We call this phase as the 'wavy' phase as the eigenvalue distribution sinusoidal and non-vanishing over the entire $\theta$ circle $-\pi < \theta \leqslant \pi$. For $\lambda < 2$ the eigenvalue distribution is given by

$$\rho(\theta) = \frac{2}{\pi\lambda}\cos\left(\frac{\theta}{2}\right)\left(\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}$$

where

$$\sin^2\frac{\theta_c}{2} = \frac{\lambda}{2}, \quad \text{and} - \theta_c < \theta < \theta_c.$$

We call this phase the 'clumped' phase as the eigenvalue distribution is non-vanishing only in subset of $\theta$ circle. Figure 1 shows the eigenvalue distribution for the two phases.

It follows from the results just presented that the eigenvalue distributions in the models (7.5) and (7.6) are given by

$$\rho_U(\theta) = \frac{1}{2\pi}\left(1 + \frac{2\chi_1 F(x)}{A}\cos(\theta)\right), \text{i.e. } \mathrm{Tr}\, U = N\frac{\chi_1 F(x)}{A} \text{ when } \frac{2\chi_1 F(x)}{A} < 1,$$

$$\rho_U(\theta) = \frac{2\chi_1 F(x)}{\pi A}\cos\frac{\theta}{2}\sqrt{\frac{A}{2\chi_1 F(x)} - \sin^2\frac{\theta}{2}},$$

$$\text{i.e } \mathrm{Tr}\, U = N\left(1 - \frac{A}{4\chi_1 F(x)}\right) \text{ when } \frac{2\chi_1 F(x)}{A} > 1; \tag{7.8}$$

$$\rho_V(\theta) = \frac{1}{2\pi}\left(1 + 2\rho_1 AF(x)\cos(\theta)\right), \text{i.e. } \mathrm{Tr}\, V = M\rho_1 F(x)A \text{ when } 2\rho_1 AF(x) < 1,$$

$$\rho_V(\theta) = \frac{2\rho_1 AF(x)}{\pi}\cos\frac{\theta}{2}\sqrt{\frac{1}{2\rho_1 AF(x)} - \sin^2\frac{\theta}{2}},$$

$$\text{i.e. } \mathrm{Tr}\, V = M\left(1 - \frac{1}{4A\rho_1 F(x)}\right) \text{ when } 2\rho_1 AF(x) > 1. \tag{7.9}$$

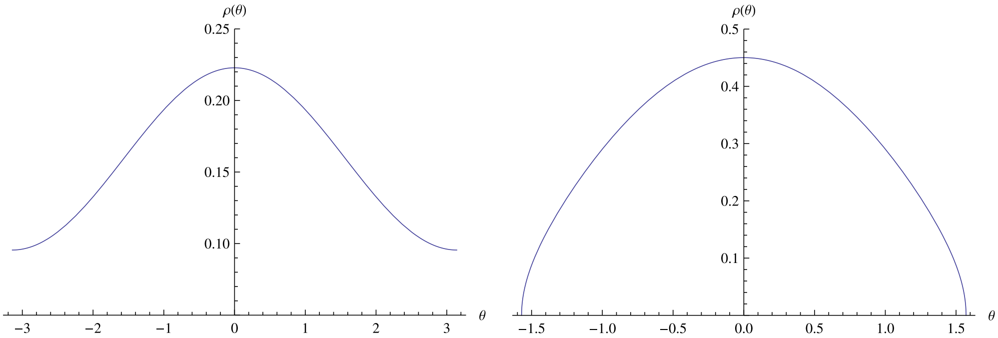


**Figure 1.** Eigenvalue distribution for *wavy* ($\lambda = 5$) and *clumped* ($\lambda = 1$) phases of Gross–Witten–Wadia model.

To complete the solution to the model we must impose the self consistency-conditions $\mathrm{Tr}\, U = N\rho_1$ on (7.8) and $\mathrm{Tr}\, V = M\chi_1$ on (7.9). Without loss of generality let us assume that $N \geqslant M$ so that $A \geqslant 1$. As we have explained above, flat (constant) eigenvalue distributions for both $U$ and $V$ are always solutions; this solution is stable for $F(x) < 1$ and unstable for $F(x) > 1$. It is also easy to check that if the $U$ eigenvalue distribution is flat then the same must be true for the $V$ eigenvalue distribution, and vice versa. In addition we have four possibilities; each of the $U$ and $V$ matrix models may be in either the wavy or the clumped phases. We consider each in turn.

*7.1.1. U wavy, V wavy.* In this case the self-consistency equations are

$$\rho_1 = \frac{\chi_1 F(x)}{A}, \qquad \chi_1 = \rho_1 F(x) A.$$

If $\rho_1$ and $\chi_1$ are nonzero, we have a solution only when $F(x) = 1$; on this solution $\rho_1 = \frac{\chi_1}{A}$ where $\chi_1 \leqslant \frac{1}{2}$ (for self-consistency with the assumption that $V$ is wavy) but is otherwise arbitrary.

*7.1.2. U wavy, V clumped.* In this case the self-consistency equations are

$$\rho_1 = \frac{\chi_1 F(x)}{A}, \qquad \chi_1 = 1 - \frac{1}{4A\rho_1 F(x)}.$$

These equations admit real solutions only when $F(x) > 1$. The solution is given by

$$\begin{aligned} \chi_1 &= \frac{1}{2F(x)}\left(F(x) + \sqrt{F^2(x) - 1}\right), \\ \rho_1 &= \frac{1}{2A}\left(F(x) + \sqrt{F^2(x) - 1}\right). \end{aligned} \tag{7.10}$$

Clearly this solution exists only when $F(x) > 1$. The assumption that $U$ is wavy is true only when

$$F(x) + \sqrt{F^2(x) - 1} < A. \tag{7.11}$$

In other words, the solution (7.10) is self-consistent when

$$1 < F(x) < \frac{1}{2}\left(A + \frac{1}{A}\right).$$

**Table 1.** Nature of eigenvalue distribution for $U$ and $V$ matrices in different temperature regimes.

| | $U$ | $V$ |
|---|---|---|
| $F(x) < 1$ | Flat ($\rho_1 = 0$) | Flat ($\chi_1$ equals; 0) |
| $1 < F(x) < \frac{1}{2}\left(A + \frac{1}{A}\right)$ | Wavy | Clumped |
| $F(x) > \frac{1}{2}\left(A + \frac{1}{A}\right)$ | Clumped | Clumped |

*7.1.3. U clumped, V wavy.* It may be verified that there are no solutions of this nature when $A > 1$.

*7.1.4. U clumped, V clumped.* In this case the self-consistency equations are

$$\rho_1 = 1 - \frac{A}{4\chi_1 F(x)}, \qquad v = 1 - \frac{1}{4A\rho_1 F(x)}.$$

The solution to these equations is given by

$$\begin{aligned} \rho_1 &= \frac{\sqrt{(A^2 - 4AF(x) - 1)^2 - 16AF(x)} - A^2 + 4AF(x) + 1}{8AF(x)}, \\ \chi_1 &= \frac{\sqrt{(A^2 - 4AF(x) - 1)^2 - 16AF(x)} + A^2 + 4AF(x) - 1}{8AF(x)}. \end{aligned} \tag{7.12}$$

This solution is consistent with the assumption that $U$ is wavy provided that

$$\rho_1 \geqslant \tfrac{1}{2}.$$

In other words this solution exists only when

$$F(x) > \frac{1}{2}\left(A + \frac{1}{A}\right).$$

*7.1.5. Summary of solution.* In summary, any given temperature (except for the special case $F(x) = 1$) the toy model has a unique stable saddle point. This saddle point is listed in table 1. The model starts out in the flat–flat phase, transits to wavy–clumped via a first order phase transition at $F(x) = 1$ and then transits to clumped–clumped via a third order phase transition at $F(x) = \frac{1}{2}\left(A + \frac{1}{A}\right)$.

*7.1.6. Solution obtained by first integrating out U.* In the first two phases listed in table 1 above, the eigenvalue distribution for the matrix $U$ is ungapped. In these phases the free energy is stationary with respect to a variation of the Fourier modes, $\rho_n$, of the eigenvalue distribution of $U$. These two phases may, therefore, simply be studied in an effective matrix model for the matrix $V$, obtained by integrating $\rho_n$ out classically, using their equations of motion. The part of the action in (7.4) that depends on $\rho_n$ is given by

$$N^2\left[\frac{F(x)}{A}\left(\rho_1\chi_{-1} + \rho_{-1}\chi_1\right) - \sum_{n=1}^{\infty}\frac{\rho_n\rho_{-n}}{n}\right].$$

On-shell we find $\rho_1 = \frac{F(x)}{A}\chi_1$, $\rho_{-1} = \frac{F(x)}{A}\chi_{-1}$, and $\rho_n = 0$ ($|n| \geqslant 2$). Integrating out $\rho_n$ we obtain the following effective matrix model for $V$ (note this is accurate only at leading order in $N$)

$$Z_t = \int DV \exp[F(x)^2 \operatorname{Tr} V \operatorname{Tr} V^{-1}]. \tag{7.13}$$

This model was solved in section 6.4 of [20] ($m_1^2 - 1$ in that paper is our $F(x)^2$ and $b$ of that paper should be set to zero). The solution takes the following form. For $F(x)^2 < 1$ the $V$ eigenvalue distribution is flat, in agreement with table 1. At $F(x)^2 = 1$ the model undergoes a phase transition. The $V$ eigenvalue distribution is clumped in the high temperature phase. Using equations (6.11) and (6.18) of [20], it is easily verified that the $V$ eigenvalue distribution agrees with that given in (7.9) and (7.10). In particular the value of $\chi_1$ on this solution is given by (7.10). From the fact that $\rho_1 = \frac{F(x)}{A}\chi_1$, it follows also that $\rho_1$ on the solution takes the value presented in (7.10). Consequently the assumption of this section, namely that the $U$ eigenvalue distribution is wavy rather than clumped, is self-consistent only when the inequality (7.11) is true. When this inequality is violated, our system undergoes a further phase transition (in agreement with table 1). However this phase transition and the resultant high temperature phase are not accurately captured by the effective model (7.13).

### *7.2. Effective description of the low and intermediate temperature phase of the full model*

We will argue self-consistently below that the qualitative features of the phase diagram of the toy model are also true of the full matrix model (7.2). The full model also undergoes two transitions; the first from uniform–uniform to wavy–clumped and the second from wavy–clumped to clumped–clumped. Exactly as in the previous section, the low and intermediate temperature phases of the full model may be analyzed by integrating out the Fourier modes, $\rho_n$, of the holonomy $U$ using their equations of motion. Performing this integration (see the previous section for procedure) we obtain the effective matrix model

$$Z = \int DV \exp\left[\sum_{n=1}^{\infty} \frac{(F_B(x^n) + (-1)^{n+1}F_F(x^n))^2}{n} \operatorname{Tr} V^n \operatorname{Tr} V^{-n}\right]. \tag{7.14}$$

Equation (7.14) has a simple physical interpretation. Note that (7.14) is the free partition function of a gauge theory based on the gauge group $U(M)$. Our effective theory has only adjoint matter, with effective bosonic letter partition function $F_B(x)^2 + F_F(x)^2$ and effective fermionic letter partition function $2F_B(x)F_F(x)$. But this is exactly the partition function for mesonic fields of the sort $AB$. In other words (7.14) describes a phase in which the gauge group $U(N)$ is completely confined, so that the effective letters $A$ and $B$ only appear in the combination $AB$; composite letters in the adjoint of $U(M)$. The entire effect of the integration over the $U(N)$ holonomy is to effect this complete confinement. The effective description of this phase is in terms of a single gauge group, $U(M)$, and adjoint letters $AB$.

This matrix model (7.14) has been studied in detail, in a perturbation expansion in $x - x_c$, in section 5.5 of [20]. The qualitative behavior is similar to the toy model studied in the previous section.

#### *7.2.1. $F(x) < 1$.* In the case that $F(x) < 1$ the saddle point is given by

$$\operatorname{Tr} V^n = 0$$

for all $n \neq 0$. This also implies that

$$\operatorname{Tr} U^n = 0$$

for all $n \neq 0$. The free energy in this phase vanishes at leading order in $N$. At first subleading order, the partition function (7.2) is obtained by computing the one loop determinant about this saddle point and is given by [20]

$$Z = \prod_{n=1}^{\infty} \frac{1}{1 - (F_B(x^n) + (-1)^{n+1}F_F(x^n))^2}. \tag{7.15}$$

Note that the result (7.15) diverges when $F_B(x^n) + (-1)^{n+1}F_F(x^n) = 1$ for any $n$. Now $F(x)$ is a monotonically increasing function of $x$ with $F(0) = 0$ and $F(1) = \infty$. As $x^n < x$ for $x \in (0, 1)$ it follows that $F(x) < 1$ implies that $F(x^n) < 1$ for all positive $n$. In other words, as $x$ is increased from zero (i.e. the temperature of the system is increased from zero) the partition function (7.15) first diverges when $F(x) = 1$, i.e. at $x = x_c = 17 - 12\sqrt{2} = 0.0294\cdots$. As explained in [20], this divergence has a simple physical interpretation. The effective potential for the mode $\chi_1$ is proportional to $(1-F(x)^2)|\chi_1|^2$. This potential develops a zero at $F(x) = 1$ and is tachyonic for $F(x) > 1$.[46]

*7.2.2. $F(x) > 1$.* The $V$ eigenvalue distribution gets increasingly clumped as $x$ is increased. Recall that the $U$ eigenvalue distribution is determined by the equations

$$\rho_n = \frac{F_B(x^n) + (-1)^{n+1}F_F(x^n)}{A}\chi_n,$$

which gives the eigenvalue distribution for $U$ as

$$\rho_u(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{F_B(x^n) + (-1)^{n+1}F_F(x^n)}{A}\chi_n\cos\theta\right).$$

As $x$ is increased this eigenvalue distribution eventually goes negative at $\theta = \pi$ for $x > x_{c_2}$ where $x_{c_2}$ is a second critical temperature that we will not in generality be able analytically compute in this paper. For $x > x_{c_2}$ the effective action (7.2) is no longer accurate. As in the toy model of the previous section, of course, the physics of this second phase transition is the clumping of the $U$ eigenvalue distribution.

*7.2.3. The second phase transition at large $A$.* When $A$ is very large, the second phase transition occurs at a high temperature of order $\sqrt{A}$ (as we will see later). This results in a key simplification; when the $U$ eigenvalue distribution undergoes the phase transition, the $V$ eigenvalue distribution is well approximated by a $\delta$ function. In other words $\frac{1}{M}\mathrm{Tr}\,V^n = 1$ for all $n$. Consequently, at leading order in the $\frac{1}{A}$ expansion, the second phase transition is well described by the matrix model

$$Z = \int DU \exp\left[\frac{N}{A}\sum_{n=1}^{\infty}\frac{F_B(x^n) + (-1)^{n+1}F_F(x^n)}{n}\left(\mathrm{Tr}\,U^n + \mathrm{Tr}\,U^{-n}\right)\right]. \tag{7.16}$$

When $x$ is of order unity, the saddle point to this matrix model is wavy with

$$\frac{\mathrm{Tr}\,U^n}{N} = \frac{\mathrm{Tr}\,U^{-n}}{N} = \frac{F_B(x^n) + (-1)^{n+1}F_F(x^n)}{A},$$

so that

$$\rho_u(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{F_B(x^n) + (-1)^{n+1}F_F(x^n)}{A}\cos n\theta\right]. \tag{7.17}$$

[46] We can see all this directly in the full matrix model (7.2) involving both the $U$ and the $V$ variables. The potential for the modes $\mathrm{Tr}\,U$ and $\mathrm{Tr}\,V$ is given by

$$\mathrm{Tr}\,U\,\mathrm{Tr}\,U^{-1} + \mathrm{Tr}\,V\,\mathrm{Tr}\,V^{-1} - F(x)\,\mathrm{Tr}\,U\,\mathrm{Tr}\,V^{-1} - F(x)\,\mathrm{Tr}\,V\,\mathrm{Tr}\,U^{-1}.$$

Let $\mathrm{Tr}\,U = N\rho_1$ and $\mathrm{Tr}\,V = M\chi_1$ then this potential can be written as

$$V(\rho_1, \chi_1) = M^2[|A\rho_1 - F(x)\chi_1|^2 + (1 - F(x)^2)|\chi_1|^2].$$

$\rho_1 = \chi_1 = 0$ is a stable minimum of this potential for $F(x) < 1$. At $F(x) = 1$ the potential develops a flat direction that evolves into an unstable direction for $F(x) > 1$. It follows that the trivial solution studied in this subsection is unstable for $F(x) > 1$ providing an explanation for the divergence on $F(x) \to 1$. At $F(x) = 1$ the system undergoes a phase transition to another phase. As we will see later this phase transition is of first order.

The leading contribution to free energy computed using this saddle point distribution

$$
\begin{aligned}
Z &= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(\mathrm{Tr}\,U^n\,\mathrm{Tr}\,U^{-n} - \frac{NF_B(x^n)+(-1)^{n+1}F_F(x^n)}{A}\left(\mathrm{Tr}\,U^n+\mathrm{Tr}\,U^{-n}\right)\right)\right]\\
&= \exp\left[-N^2\sum_{n=1}^{\infty}\frac{1}{n}\left(\rho_n^2 - 2\rho_n\frac{F_B(x^n)+(-1)^{n+1}F_F(x^n)}{A}\right)\right]\\
&= \exp\left[-N^2\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\rho_n - \frac{F_B(x^n)+(-1)^{n+1}F_F(x^n)}{A}\right)^2 - \left(\frac{F_B(x)^n+(-1)^{n+1}F_F(x^n)}{A}\right)^2\right]\right].
\end{aligned}
\tag{7.18}
$$

The first term in the sum vanishes on the saddle point density distribution and we get

$$
Z = \exp\left[M^2\sum_{n=1}^{\infty}\frac{1}{n}\left(F_B(x)^n + (-1)^{n+1}F_F(x^n)\right)^2\right].
\tag{7.19}
$$

Equation (7.19) has a simple interpretation. Products of 'letters' of the form $AB$ are singlets under the gauge group $U(N)$, but transform in the adjoint of $U(M)$. The partition function over all bosonic operators is simply given by $M^2\left(F_B^2(x)+F_F^2(x)\right)$, while the partition function over all fermionic letters of the same form is given by $2M^2(F_B(x)F_F(x))$. (7.19) is precisely the Bose/Fermi exponentiation (multi-particling) of these single meson partition functions. In other words (7.19) is the partition function over a gas of non-interacting mesons of the form $AB$. In the limit described in this subsection it follows that the intermediate temperature phase may be thought of as a phase in which the gauge group $U(N)$ is completely confined while the gauge group $U(M)$ is completely deconfined.

At high temperatures $T \gg 1$ the eigenvalue distribution (7.17) attains its minimum at $\theta = \pi$. This minimum value decreases below zero when

$$
\sum_{n=1}^{\infty}\left((-1)^{n+1}F_B(x^n) + F_F(x^n)\right) > \frac{A}{2}.
\tag{7.20}
$$

As $A$ is assumed large in this section, this condition can only be met in the limit that $x \to 1$, i.e. in the large $T$ limit. At leading order in the large temperature limit $x - 1 = -\frac{1}{T}$ (recall $x = \mathrm{e}^{-1/T}$) and

$$
F_B(x^n) \approx \frac{8}{T^2n^2}, \qquad F_F(x) \approx \frac{8}{T^2n^2}, \qquad F(x^n) \approx \frac{16T^2}{n^2},
$$

and the eigenvalue distribution (7.17) reduces to

$$
\rho(\theta) = \frac{1}{2\pi}\left(1 + \sum_{n=1}^{\infty}\frac{32T^2}{A(2n+1)^2}\cos n\theta\right).
$$

In this approximation the condition (7.20) for the eigenvalue distribution to go negative is given by

$$
T^2 > T_c^2 = \frac{A}{4\pi^2}.
\tag{7.21}
$$

As it makes no sense for an eigenvalue distribution to be negative, it follows that the $U$ matrix undergoes the clumping transition at $T = T_c$. Note that $T_c$ is of order $\sqrt{A}$, and so is large, as promised at the beginning of this section.

The condition (7.20) gives an expression for the phase transition temperature that may be power series expanded in $\frac{1}{\sqrt{A}}$ ((7.21) is the leading term in that expansion). However (7.20) was itself derived under the approximation that the $V$ eigenvalue distribution is a delta function.

In reality (see below) the $V$ eigenvalue distribution has a width of order $\frac{1}{T^2}$ which is $\sim \frac{1}{A}$ near the phase transition temperature. It is possible to systematically account for the broadening of the $V$ eigenvalue distribution (and thereby develop a systematic procedure for computing the phase transition temperature to arbitrary order in $\frac{1}{A}$). We demonstrate how this works in appendix G by computing the first correction to (7.20) resulting from the finite width of the eigenvalue distribution of the matrix $V$.

*7.2.4. Effect of interactions on first phase transition.* The most general form of effective action in large $N$ perturbation theory is

$$Z = \int DUDV \exp\left[\sum_{n=1}^{\infty}\left(A_n^{(0)}(x)(\mathrm{Tr}\,U^n\,\mathrm{Tr}\,V^{-n} + \mathrm{Tr}\,V^n\,\mathrm{Tr}\,U^{-n})\right)\right.$$
$$+ \sum_{m,n} A_{m,n}^{(1)}(\mathrm{Tr}\,V^m\,\mathrm{Tr}\,V^n\,\mathrm{Tr}\,U^{-m-n} + \mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^n\,\mathrm{Tr}\,V^{-m-n})$$
$$+ \sum_{m,n,p} A_{m,n,p}^{(2)}(\mathrm{Tr}\,V^m\,\mathrm{Tr}\,V^n\,\mathrm{Tr}\,V^p\,\mathrm{Tr}\,U^{-m-n-p} + \mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^n\,\mathrm{Tr}\,V^p\,\mathrm{Tr}\,V^{-m-n-p}$$
$$\left. + \mathrm{Tr}\,U^m\,\mathrm{Tr}\,U^n\,\mathrm{Tr}\,U^p\,\mathrm{Tr}\,V^{-m-n-p}) + \cdots\right]. \tag{7.22}$$

Moving to the Fourier basis and integrating out the $U$ modes we get an effective adjoint matrix model the $V$ matrix

$$Z_{\mathrm{eff}} = \int DV \exp\left[\sum_{n=1}^{n} B_n^{(0)}(x)\,\mathrm{Tr}\,V^n\,\mathrm{Tr}\,V^{-n} + \sum_{m,n} B_{m,n}^{(1)}\,\mathrm{Tr}\,V^m\,\mathrm{Tr}\,V^n\,\mathrm{Tr}\,V^{-m-n}\right.$$
$$\left. + \sum_{m,n,p} B_{m,n,p}^{(2)}\,\mathrm{Tr}\,V^m\,\mathrm{Tr}\,V^n\,\mathrm{Tr}\,V^p\,\mathrm{Tr}\,V^{-m-n-p} + \cdots\right] \tag{7.23}$$

where the $B$ coefficients can be determined in terms of the $A$ coefficients appearing in (7.22). As explained in the [20] the only interaction terms relevant for the phase transition in this adjoint matrix model are

$$\mathrm{Tr}\,V^2(\mathrm{Tr}\,V^{-1})^2, \quad \mathrm{Tr}\,V^{-2}(\mathrm{Tr}\,V)^2, \quad (\mathrm{Tr}\,V\,\mathrm{Tr}\,V^{-1})^2.$$

Now we will determine the coefficient of these term in the effective adjoint model in terms of the coefficients appearing in the original matrix model. The relevant part of the original action in Fourier modes is

$$\frac{S}{N^2} = -\left(\rho_1\rho_{-1} + \frac{1}{2}\rho_2\rho_{-2}\right) - \frac{1}{A^2}\left(\chi_1\chi_{-1} + \frac{1}{2}\chi_2\chi_{-2}\right) + \frac{m_1(x)}{A}(\rho_1\chi_{-1} + \rho_{-1}\chi_1)$$
$$+ \frac{m_2(x)}{A}(\rho_2\chi_{-2} + \rho_{-2}\chi_2) + \frac{a}{A}(\rho_1\rho_{-1}\chi_1\chi_{-1})$$
$$+ \frac{b}{A}(\rho_1^2\chi_{-1}^2 + \rho_{-1}^2\chi_1^2) + \frac{c}{A}(\rho_2\chi_{-1}^2 + \rho_{-2}\chi_1^2). \tag{7.24}$$

Here $\rho_n$ and $\chi_n$ are the Fourier mode of eigenvalue distribution for $U$ and $V$ matrices respectively and $A = \frac{N}{M}$. Also $c \sim \lambda$ while $a, b \sim \lambda^2$ where $\lambda$ is the 't Hooft coupling. The coefficients $m_1(x)$ and $m_2(x)$ reduces to $F(x)$ and $F_B(x^2) + (-1)F_F(x^2)$ respectively in the free theory. The equation of motion for $\rho_1$ and $\rho_2$ are

$$\rho_1 = -\frac{m_1(x)}{A}\left[\frac{\chi_1\left(-1 + \frac{a}{A}\chi_1\chi_{-1}\right) - \frac{2b}{A}\chi_1^2\chi_{-1}}{\left(-1 + \frac{a}{A}\chi_1\chi_{-1}\right)^2 - \left(\frac{2b}{A}\chi_1\chi_{-1}\right)^2}\right],$$
$$\rho_2 = \frac{m_2(x)}{A}\chi_2 - \frac{2c}{A}\chi_1^2.$$

Linearizing in $a$ and $b$ and substituting back the action we get the effective adjoint model (keeping only the terms relevant to phase transition)

$$\frac{S_{\text{eff}}}{N^2} = \left[ -\left(\frac{1-m_1(x)^2}{A^2}\right)\chi_1\chi_{-1} - \left(\frac{1-m_2(x)^2}{2A^2}\right)\chi_2\chi_{-2} + \frac{2c}{A^2}m_2(x)(\chi_2\chi_{-1}^2 + \chi_{-2}\chi_1^2) + \frac{(a+2b)m_1(x)^2 + 4Ac^2}{A^3}\chi_1^2\chi_{-1}^2 \right]. \tag{7.26}$$

Now we can again integrate out $\chi_2$ to get

$$\frac{S_{\text{eff}}}{N^2} = -\left(\frac{1-m_1(x)^2}{A^2}\right)\chi_1\chi_{-1} + \left(\frac{(a+2b)m_1(x)^2}{A^3} + \frac{4c^2(1+m_2(x)^2)}{A^2(1-m_2(x)^2)}\right)\chi_1^2\chi_{-1}^2. \tag{7.27}$$

The phase structure described by this effective action was described in [20]. The first order transition of the free theory splits into two phase transitions ; the first of second order (when $m_1(x) = 1$) and the next of third order at a higher temperature—if the coefficient of quartic term is negative. However the transition remains a single transition of first order if it is positive; this transition occurs at $m_1(x) < 1$ (see appendix H). In appendix H we demonstrate all this in a somewhat more quantitative fashion by studying interaction in a truncated toy model.

## 8. Conclusion

In this paper, we proposed the higher spin gauge theories in $AdS_4$ described by supersymmetric extensions of Vasiliev's system and appropriate boundary conditions that are dual to a large class of supersymmetric Chern–Simons vector models. The parity violating phase $\theta_0$ in Vasiliev theory plays the key role in identifying the boundary conditions that preserve or break certain SUSYs. In particular, our findings are consistent with the following conjecture: starting with the duality between parity invariant Vasiliev theory and the dual free supersymmetric $U(N)$ vector model at large $N$, turning on Chern–Simons coupling for the $U(N)$ corresponds to turning on the parity violating phase $\theta_0$ in the bulk, and at the same time induces a change of fermion boundary condition as described in section 5.4.1. We conjectured that the relation $\theta_0 = \frac{\pi}{2}\lambda$, where $\lambda = N/k$ is the 't Hooft coupling of the boundary Chern–Simons theory, suggested by two-loop perturbative calculation in the field theory and Giveon–Kutasov duality and ABJ self-duality, is exact.

Turning on various scalar potential and scalar–fermion coupling in the Chern–Simons vector model amounts to double trace and triple trace deformations, which are dual to deformation of boundary conditions on spin 0 and spin 1/2 fields in the bulk theory. Gauging a flavor symmetry of the boundary theory with Chern–Simons amounts to changing the boundary condition on the bulk spin-1 gauge field from the magnetic boundary condition to a electric–magnetic mixed boundary condition. Consideration of SUSY breaking by boundary conditions allowed us to identify precise relations between $\theta_0$, the Chern–Simons level $k$, and two-point function coefficient $\widetilde{N}$ in $\mathcal{N} = 3$ Chern–Simons vector models.

While substantial evidence for the dualities proposed in this paper is provided by the analysis of linear boundary conditions, we have not analyzed in detail the nonlinear corrections to the boundary conditions, which are responsible for the triple trace terms needed to preserve SUSY. Furthermore, we have not nailed down the bulk theory completely, due to the possible non-constant terms in the function $\theta(X) = \theta_0 + \theta_2 X^2 + \theta_4 X^4 + \cdots$ that controls bulk interactions and breaks parity. It seems that $\theta_2$, $\theta_4$ etc cannot be removed merely by field redefinition, and presumably contribute to five and higher point functions at bulk tree level, and yet their presence would not affect the preservation of SUSY. This non-uniqueness at higher order in the bulk equation of motion is puzzling, as we know of no counterpart of it in the dual boundary CFT. Perhaps clues to resolving this puzzle can be found by explicit computation of say the contribution of $\theta_2$ to the five-point function. It is possible that a thorough analysis of the near boundary behavior of solutions to Vasiliev's equations (via a Graham–Fefferman type analysis) could be useful in this regard.

We have also encountered another puzzle that applies to Vasiliev duals of all Chern–Simons field theories, not necessarily supersymmetric. Our analysis of the bulk Vasiliev description of the breaking of higher spin symmetry correctly reproduced those double trace terms in the divergence of higher spin currents that involve a scalar field on the RHS. However we were unable to reproduce the terms bilinear in two higher spin currents. The reason for this failure was very general; when acting on a state the higher spin symmetry generators never appear to violate the boundary conditions for any field except the scalar. It would be reassuring to resolve this discrepancy.

The triality between ABJ theory, $n = 6$ Vasiliev theory with $U(M)$ Chan–Paton factors, and type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ suggests a concrete way of embedding Vasiliev theory into string theory. In particular, the $U(M)$ Vasiliev theory is controlled by its *bulk* 't Hooft coupling $\lambda_{\text{bulk}} = g^2 M \sim M/N$. We see clear indication from the dual field theory that at strong $\lambda_{\text{bulk}}$, the non-Abelian higher spin particles form color neutral bound states, that are single closed string excitations. Vice versa, in the small radius limit and with near critical amount of flat Kalb–Ramond $B$-field on $\mathbb{CP}^3$, the type IIA strings should break into multi-particle states of higher spin fields. This picture is further supported by the study of thermal partition function of ABJ theory in the free limit. The dual field theory mechanism for the disintegration of the string is very general, and so should apply more generally to the dual string theory description of any field theory with bifundamental matter, when the rank of one of the gauge groups is taken to be much smaller than the other[47].

In this paper we have computed the thermal phase diagram of ABJ theory in the free limit. This phase diagram has three distinct phases; a low temperature string like phase, an intermediate temperature thermal Vasiliev like phase and a high temperature black hole like phase. It would be very interesting to extend these computations to the interacting theory. Order by order in $M/N$ such computations may be technically feasible nonperturbatively in $\lambda = \frac{N}{K}$ following the methods employed in [14] and [22].

It has been argued that the vacuum of the ABJ model spontaneously breaks SUSY for $k < N - M$ [32]. Requiring the existence of a supersymmetric vacuum, the maximum value of t'Hooft coupling in a theory with $M \neq N$ is $\frac{N}{k_{\min}} = \frac{1}{1-\frac{M}{N}}$. As the radius of the dual AdS space in string units is proportional to a positive power of the t'Hooft coupling, it follows that ABJ theories have a weakly curved string description only in the limit $\frac{M}{N} \to 1$. It is interesting that, in the free computations performed above, the new intermediate phase (a free gas of Vasiliev particles) continued to exist all the way up to $\frac{M}{N} = 1$. If this continues to be the case in the strongly interacting theory, it may be possible to access this new phase at strong coupling via a string worldsheet computation. We find this a fascinating possibility.

More generally, the recasting of ABJ theory as a Vasiliev theory suggests that it would be interesting, purely within field theory, to study ABJ theory in a power expansion in $M/N$ but nonperturbatively in $\lambda$. At $\frac{M}{N} = 0$ this would require a generalization of the results of [11] and [12] to the supersymmetric theory. It may then be possible to systematically correct this solution in a power series in $M/N$. This would be fascinating to explore.

Perhaps the most surprising recipe in this web of dualities is that the full classical equation of motion of the bulk higher spin gauge theory can be written down explicitly and exactly,

[47] We thank K Narayan for discussions on this point.

thanks to Vasiliev's construction. One of the outstanding questions is how to derive Vasiliev's system directly from type IIA string field theory in $AdS_4 \times \mathbb{CP}^3$, or to learn about the structure of the string field equations (in AdS) from Vasiliev's equations. As already mentioned, a promising approach is to consider the open-closed string field theory on D6-branes wrapped on $AdS_4 \times \mathbb{RP}^3$, which should directly reduce to $n = 4$ Vasiliev theory in the minimal radius limit. It would also be interesting to investigate whether—and in what guise—the huge bulk gauge symmetry of Vasiliev's description survives in the bulk string sigma model description of the same system. We leave these questions to future investigation.

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## Appendix A. Details and explanations related to section 2

### *A.1. Star product conventions and identities*

It follows from the definition of the star product that

$$\begin{aligned} y^\alpha * y^\beta &= y^\alpha y^\beta + \epsilon^{\alpha\beta}; \qquad [y^\alpha, y^\beta]_* = 2\epsilon^{\alpha\beta} \\ z^\alpha * z^\beta &= z^\alpha z^\beta - \epsilon^{\alpha\beta}; \qquad [z^\alpha, z^\beta]_* = -2\epsilon^{\alpha\beta} \\ y^\alpha * z^\beta &= y^\alpha z^\beta - \epsilon^{\alpha\beta}; \qquad z^\alpha * y^\beta = z^\alpha y^\beta + \epsilon^{\alpha\beta}; \; [y^\alpha, z^\beta]_* = 0. \end{aligned} \tag{A.1}$$

Identical equations (with obvious modifications) apply to the bar variables. Spinor indices are lowered using the $\epsilon$ tensor as follows

$$z_\alpha = z^\beta \epsilon_{\beta\alpha}, \quad \epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta_\alpha^\beta. \tag{A.2}$$

Note that for an arbitrary function $f$ we have

$$\begin{aligned} z^\alpha * f &= z^\alpha f + \epsilon^{\alpha\beta}(\partial_{y^\beta} f - \partial_{z^\beta} f) \\ f * z^\alpha &= z^\alpha f + \epsilon^{\alpha\beta}(\partial_{y^\beta} f + \partial_{z^\beta} f) \end{aligned}. \tag{A.3}$$

Using (A.3) we the following (anti)commutator

$$\begin{aligned} [z^\alpha, f]_* &= -2\epsilon_{\alpha\beta}\partial_{z^\beta} f \\ \{z^\alpha, f\}_* &= 2z^\alpha f + 2\epsilon^{\alpha\beta}\partial_{y^\beta} f \end{aligned}. \tag{A.4}$$

It follows from (A.1) that

$$[z_\alpha, f]_* = -2\frac{\partial f}{\partial z^\alpha}, \quad [y^\alpha, f]_* = 2\epsilon^{\alpha\beta}\frac{\partial f}{\partial y^\beta}, \quad [y_\alpha, f]_* = 2\frac{\partial f}{\partial y^\alpha}. \tag{A.5}$$

Similar expressions (with obvious modifications) are true for (anti)commutators with $\bar{y}$ and $\bar{z}$. Substituting $f = K$ into (A.3) and using $\partial_{y^\alpha} K = -z_\alpha K$, one obtains

$$\{z^\alpha, K\}_* = 0, \quad \text{i.e. } K * z^\alpha * K = -z^\alpha. \tag{A.6}$$

In a similar manner we find

$$\{y^\alpha, K\}_* = 0, \quad \text{i.e. } K * y^\alpha * K = -y^\alpha.$$

On the other hand $K$ clearly commutes with $\bar{y}_{\dot\alpha}$ and $\bar{z}_{\dot\alpha}$. The second line of (2.3) follows immediately from these observations.

The first line of (2.3) is also easily verified.

### *A.2. Formulas relating to ι operation*

We present a proof of (2.16)

$$\begin{aligned}
\iota(f * g) &= (f(Y,Z)\exp[\epsilon^{\alpha\beta}(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha})(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}) \\
&\quad + \epsilon^{\dot\alpha\dot\beta}(\overleftarrow{\partial}_{\bar{y}^{\dot\alpha}} + \overleftarrow{\partial}_{\bar{z}^{\dot\alpha}})(\overrightarrow{\partial}_{\bar{y}^{\dot\beta}} - \overrightarrow{\partial}_{\bar{z}^{\dot\beta}})]g(Y,Z))_{(Y,Z)\to(\tilde{Y},\tilde{Z})} \\
&= f(\tilde{Y},\tilde{Z})\exp[-\epsilon^{\alpha\beta}(\overleftarrow{\partial}_{y^\alpha} - \overleftarrow{\partial}_{z^\alpha})(\overrightarrow{\partial}_{y^\beta} + \overrightarrow{\partial}_{z^\beta}) \\
&\quad - \epsilon^{\dot\alpha\dot\beta}(\overleftarrow{\partial}_{y^{\dot\alpha}} - \overleftarrow{\partial}_{z^{\dot\alpha}})(\overrightarrow{\partial}_{y^{\dot\beta}} + \overrightarrow{\partial}_{z^{\dot\beta}})]g(\tilde{Y},\tilde{Z}) \\
&= \iota(g) * \iota(f)
\end{aligned} \tag{A.7}$$

where $(Y, Z) = (y, \bar{y}, z, \bar{z})$ and $(\tilde{Y}, \tilde{Z}) = (\mathrm{i}y, \mathrm{i}\bar{y}, -\mathrm{i}z, -\mathrm{i}\bar{z}, -\mathrm{i}\,\mathrm{d}z, -\mathrm{i}\,\mathrm{d}\bar{z})$.

We now demonstrate that

$$\iota(C * D) = -\iota(D) * \iota(C)$$

if $C$ and $D$ are each 1-forms.

$$\begin{aligned}
\iota(C * D) &= \iota(C_M * D_N\,\mathrm{d}X^M\,\mathrm{d}X^N)) = \iota(D_N) * \iota(C_M)\iota(\mathrm{d}X^M)\iota(\mathrm{d}X^N) \\
&= -\iota(D_N) * \iota(C_M)\iota(\mathrm{d}X^N)\iota(\mathrm{d}X^M) = -\iota(D) * \iota(C).
\end{aligned} \tag{A.8}$$

### *A.3. Different projections on Vasiliev's master field*

One natural projection one might impose on the Vasiliev master field is to restrict to real fields where reality is defined by

$$\mathcal{A} = \mathcal{A}^*. \tag{A.9}$$

This projection preserves the reality of the field strength (i.e. $\mathcal{F}$ is real if $\mathcal{A}$ is). As we will see below, however, the projection (A.9) does not have a natural extension to the supersymmetric Vasiliev theory, and is not the one we will adopt in this paper.

The second 'natural' projection on Vasiliev's master fields is given by

$$\iota(W) = -W, \quad \iota(S) = -S, \quad \iota(B) = K * B * K. \tag{A.10}$$

Note that the various components of $\mathcal{F}$ transform homogeneously under this projection

$$\begin{aligned}
&\iota(d_x W + W * W) = -(d_x W + W * W), \\
&\iota(d_x\hat{S} + \{W, \hat{S}\}_*) = -(d_x\hat{S} + \{W, \hat{S}\}_*), \\
&\iota(\hat{S} * \hat{S}) = -(\hat{S} * \hat{S}),
\end{aligned} \tag{A.11}$$

(the signs in (A.10) were chosen to ensure that all the quantities in (A.11) transform homogeneously). Note also that

$$\iota(B*K) = B*K, \qquad \iota(B*\bar{K}) = B*\bar{K} \tag{A.12}$$

(we have used $K*K=1$).

As we have explained in the main text, in this paper we impose the projection (2.17) on all fields. (2.17) may be thought of as the product of the projections (A.9) and (A.10). As we have mentioned in the main text $\mathcal{F}$ transforms homogeneously under this truncation (see (2.18)); in components

$$\begin{aligned}&\iota(d_xW + W*W)^* = -(d_xW + W*W),\\&\iota(d_x\hat{S} + \{W,\hat{S}\}_*)^* = -(d_x\hat{S} + \{W,\hat{S}\}_*),\\&\iota(\hat{S}*\hat{S})^* = -(\hat{S}*\hat{S}).\end{aligned} \tag{A.13}$$

### A.4. More about Vasiliev's equations

Expanded in components the first equation in (2.20) reads

$$\begin{aligned}&d_xW + W*W = 0,\\&d_x\hat{S} + \{W, \quad \hat{S}\}_* = 0,\\&\hat{S}*\hat{S} = f_*(B*K)\,\mathrm{d}z^2 + \bar{f}_*(B*\bar{K})\,\mathrm{d}\bar{z}^2.\end{aligned} \tag{A.14}$$

The second equation reads

$$\begin{aligned}&d_xB + W*B - B*\pi(W) = 0,\\&\hat{S}*B - B*\pi(\hat{S}) = 0.\end{aligned} \tag{A.15}$$

We will now demonstrate that the second equation in (2.20) follows from the first (i.e. that (A.15) follows from (A.14)). Using (2.21) and the first of (2.20) we conclude that

$$d_x(f_*(B*K)\,\mathrm{d}z^2 + \bar{f}_*(B*\bar{K})\,\mathrm{d}\bar{z}^2) + \hat{A}*(f_*(B*K)\,\mathrm{d}z^2 + \bar{f}_*(B*\bar{K})\,\mathrm{d}\bar{z}^2) = 0. \tag{A.16}$$

The components of (A.16) proportional to $\mathrm{d}x\,\mathrm{d}z^2$ yield,

$$d_xB*K + [W, B*K]_* = 0 \tag{A.17}$$

Multiplying this equation by $K$ from the right and using $K*W*K = \pi(W)$ we find the first of (A.15).

The components of (A.16) proportional to $\mathrm{d}x\mathrm{d}\bar{z}^2$ yield

$$d_xB*\bar{K} + [W, B*\bar{K}]_* = 0 \tag{A.18}$$

Multiplying this equation by $\bar{K}$ from the right and using $\bar{K}*W*\bar{K} = \bar{K}*W*\bar{K} = \pi(W) =$ (the second step uses the truncation condition (2.11) on $W$) we once again find the first of (A.15).

The term in (A.16) proportional to $\mathrm{d}z^2\,\mathrm{d}\bar{z}$ and $\mathrm{d}z\,\mathrm{d}\bar{z}^2$ may be processed as follows. Let

$$\hat{S} = \hat{S}_z + \hat{S}_{\bar{z}} \tag{A.19}$$

where $\hat{S}_z$ is proportional to $\mathrm{d}z$ and $\hat{S}_{\bar{z}}$ is proportional to $\mathrm{d}\bar{z}$. The part of (A.16) proportional to $\mathrm{d}z^2\,\mathrm{d}\bar{z}$ yields

$$[S_{\bar{z}}, B*K]_* = 0. \tag{A.20}$$

Multiplying this equation with $K$ from the right and using $K * \hat{S}_{\bar{z}} * K = \pi(\hat{S}_{\bar{z}})$ we find

$$\hat{S}_{\bar{z}} * B - B * \pi(\hat{S}_{\bar{z}}) = 0. \tag{A.21}$$

Finally, the part of (A.16) proportional to $dz\, d\bar{z}^2$ yields

$$[S_z, B * \bar{K}]_* = 0. \tag{A.22}$$

Multiplying this equation with $\bar{K}$ from the right and using

$$\bar{K} * \hat{S}_z * \bar{K} = \bar{\pi}(\hat{S}_z) = \pi(\hat{S}_z)$$

(where we have used (2.12)) we find

$$\hat{S}_z * B - B * \pi(\hat{S}_z) = 0. \tag{A.23}$$

Adding together (A.21) and (A.23) we find the second of (A.15).

The fact that $z$ and $\bar{z}$ each have only two components, mean that there are no terms in (A.16) proportional to $dz^3$ or $d\bar{z}^3$, so we have fully analyzed the content of (A.16).

### *A.5. Onshell (anti) commutation of components of Vasiliev's master field*

In this subsection we list some useful commutation and anticommutation relations between the adjoint fields $S_z$, $S_{\bar{z}}$, $B * K$ and $B * \bar{K}$. The relations we list can be derived almost immediately from Vasiliev's equations; we list them for ready reference

$$\begin{aligned}
&[B * K, B * \bar{K}]_* = 0\\
&\{S_z, S_{\bar{z}}\}_* = 0\\
&[S_{\bar{z}}, B * K]_* = 0\\
&[S_z, B * \bar{K}]_* = 0\\
&\{S_{\bar{z}}, B * K\}_* = 0\\
&\{S_z, B * \bar{K}\}_* = 0.
\end{aligned} \tag{A.24}$$

The derivation of these equations is straightforward. The first equation follows upon expanding the commutator and noting that $K * B * \bar{K} = \bar{K} * B * K$ (this follows from (2.11) together with the obvious fact that $K$ and $\bar{K}$ commute). The second equation in (A.24) follows upon inserting the decomposition (A.19) into the third equation in (A.14). The third and fourth equations in (A.24) are simply (A.20) and (A.22) rewritten.

The fifth equation in (A.24) may be derived from the third equation as follows

$$\begin{aligned}
&S_{\bar{z}} * B * K = B * K * S_{\bar{z}}\\
&\Rightarrow S_{\bar{z}} * B = B * K * S_{\bar{z}} * K\\
&\Rightarrow S_{\bar{z}} * B = -B * \bar{K} * S_{\bar{z}} * \bar{K}\\
&\Rightarrow S_{\bar{z}} * B * \bar{K} = -B * \bar{K} * S_{\bar{z}}.
\end{aligned} \tag{A.25}$$

In the third line of (A.25) we have used the truncation condition (2.11).

The sixth equation in (A.24) is derived in a manner very similar to the fifth equation.

### *A.6. Canonical form of f(X) in Vasiliev's equations*

In this subsection we demonstrate that we can use the change of variables $X \to g(X)$ for some odd real function $g(X)$ together with multiplication by an invertible holomorphic even function to put any function $f(X)$ in the form (2.30), at least provided that the function $f(X)$ admits a power series expansion about $X = 0$ and that $f(0) \neq 0$.

An arbitrary function $f(X)$ may be decomposed into its even and odd parts

$$f(X) = f_e(X) + f_o(X).$$

If $f_e(X)$ in invertible then the freedom of multiplication with an even complex function may be used to put $f(X)$ in the form

$$f(X) = 1 + \tilde{f}_o(X)$$

where $\tilde{f}_o(X) = \frac{f_o(X)}{f_e(X)}$. Clearly $\tilde{f}_o(X)$ is an odd function that admits a power series expansion. At least in the sense of a formal power series expansion of all functions, it is easy to convince oneself that any such function may be written in the form $g(X)e^{i\theta(X)}$ where $g(X)$ is an a real odd function and $\theta(X)$ is a real even function. We may now use the freedom of variable redefinitions to work with the variable $g(X)$ instead of $X$. This redefinition preserves the even nature of $\theta(X)$ and casts $f(X)$ in the form (2.30).

### *A.7. Conventions for SO(4) spinors*

Euclidean $SO(4)$ $\Gamma$ matrices may be chosen as

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix} \tag{A.26}$$

where $a = 1, \ldots, 4$ and

$$\sigma_a = (\sigma_i, iI), \qquad \bar{\sigma}_a = -\sigma_2\sigma_a^T\sigma_2 = (\sigma_i, -iI) \tag{A.27}$$

(where $i = 1, \ldots, 3$ and $\sigma^i$ are the usual Pauli matrices). In the text below we will often refer to the fourth component of $\sigma^\mu$ as $\sigma^z$; in other words

$$\sigma^z = iI$$

(we adopt this cumbersome notation to provide easy passage to different conventions). The chirality matrix $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ is given by

$$\Gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{A.28}$$

$\Gamma$ matrices act on the spinors

$$\begin{pmatrix} \chi_\alpha \\ \bar{\zeta}^{\dot{\beta}} \end{pmatrix}$$

whereas the row spinors that multiply $\Gamma$ from the left have the index structure

$$(\chi^\alpha \quad \bar{\zeta}_{\dot{\beta}}).$$

As a consequence we assign the index structure $(\sigma_a)_{\alpha\dot{\beta}}$ and $\bar{\sigma}^{\dot{\alpha}\beta}$. It is easy to check that

$$[\Gamma_a, \Gamma_b] = 2\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix} \tag{A.29}$$

where

$$\sigma_{ab} = \frac{1}{2}(\sigma_a\bar{\sigma}_b - \sigma_b\bar{\sigma}_a), \qquad \bar{\sigma}_{ab} = \frac{1}{2}(\bar{\sigma}_a\sigma_b - \bar{\sigma}_b\sigma_a)$$
$$\Rightarrow \sigma_{ij} = i\epsilon_{ijk}\sigma^k, \bar{\sigma}_{ij} = i\epsilon_{ijk}\bar{\sigma}^k, \sigma_{i4} = -i\sigma_i, \bar{\sigma}_{i4} = i\sigma_i. \tag{A.30}$$

Clearly the index structure above is $(\sigma_{ab})_\alpha{}^\beta$ and $(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$. Spinor indices are raised and lowered according to the conventions

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = \psi_\beta\epsilon^{\beta\alpha}, \quad \epsilon^{12} = \epsilon_{12} = 1.$$

The product of a chiral spinor $y^\alpha$ and an antichiral spinor $\bar{y}^{\dot{\beta}}$ is a vector. By convention we define the associated vector as

$$V_\mu = y^\alpha(\sigma_\mu)_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}}. \tag{A.31}$$

The product of a chiral spinor $y$ with itself is a self-dual antisymmetric 2 tensor which we take to be

$$V_{ab} = y^\alpha(\sigma_{ab})_\alpha^\beta y_\beta. \tag{A.32}$$

Similarly the product of an antichiral spinor with itself is an anti-self-dual two tensor which we take to be

$$V_{ab} = \bar{y}_{\dot{\alpha}}(\bar{\sigma}_{ab})^{\dot{\alpha}}_{\dot{\beta}}\bar{y}^{\dot{\beta}}. \tag{A.33}$$

### *A.8. AdS$_4$ solution*

In this appendix we will show that

$$W_0 = (e_0)_{\alpha\dot{\beta}}y^\alpha\bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta}y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} \tag{A.34}$$

with the AdS$_4$ values for the vielbein and spin connection, satisfies the Vasiliev equation

$$d_x W_0 + W_0 * W_0 = 0. \tag{A.35}$$

Substituting (A.34) in (A.35) and collecting terms quadratic in $y$ and $\bar{y}$ we get

$$\begin{aligned}
y^\alpha\bar{y}^{\dot{\alpha}} : \quad & d_x e_{\alpha\dot{\beta}} + 4\omega_\alpha{}^\beta \wedge e_{\beta\dot{\beta}} - 4e_{\alpha\dot{\gamma}} \wedge \omega^{\dot{\gamma}}{}_{\dot{\beta}} = 0\\
y^\alpha y^\beta : \quad & d_x\omega_\alpha{}^\beta - 4\omega_\alpha{}^\gamma \wedge w_\gamma{}^\beta - e_{\alpha\dot{\alpha}} \wedge e_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} = 0\\
y^{\dot{\alpha}}y^{\dot{\beta}} : \quad & d_x\omega^{\dot{\alpha}}{}_{\dot{\beta}} + 4\omega^{\dot{\alpha}}{}_{\dot{\gamma}} \wedge \omega^{\dot{\gamma}}{}_{\dot{\beta}} - e_{\alpha\dot{\alpha}} \wedge e_{\beta\dot{\beta}}\epsilon^{\alpha\beta} = 0.
\end{aligned} \tag{A.36}$$

Let us consider the Vasiliev gauge transformations generated by

$$\epsilon(x|Y) = C_{1ab}(y\sigma_{ab}y) + C_{2ab}(\bar{y}\bar{\sigma}_{ab}\bar{y}).$$

Under these the vielbein and spin connection changes by

$$\begin{aligned}
\delta e_{\alpha\dot{\alpha}} &= -4C_{1ab}(\sigma_{ab})_\alpha{}^\beta e_{\beta\dot{\alpha}} - 4C_{2ab}e_{\alpha\dot{\beta}}(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}}\\
\delta\omega_\alpha^\beta &= d_x C_{1ab}(\sigma_{ab})_\alpha{}^\beta - 8C_{1ab}\omega_\alpha^\gamma(\sigma_{ab})_\gamma{}^\beta\\
\delta\omega^{\dot{\alpha}}_{\dot{\beta}} &= d_x C_{2ab}(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} + 8C_{2ab}\omega^{\dot{\alpha}}{}_{\dot{\gamma}}(\bar{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\beta}}.
\end{aligned} \tag{A.37}$$

Notice that these are just the rotation of the vielbeins in the tangent space. The two homogeneous terms in $\delta e$ are just the rotation by under $SU(2)_L$ and $SU(2)_R$ of $SO(4)$ that acts on the tangent space. As expected under such rotation the spin connection transforms inhomogeneously. Substituting (A.37) in (A.36) it is easily verified that (A.36) transforms homogeneously.

In fact the first equation in (A.36) is just the torsion free condition while the second and third equation expresses the self-dual and anti-self-dual part of curvature two form in term of vielbeins. Substituting the AdS$_4$ values of vielbeins and spin connection (2.36) one can easily check that (A.36) are satisfied.

Converting (A.36) from bispinor notation to $SO(4)$ vector notation using the following conversion

$$e_{\alpha\dot{\beta}} = 2e_a(\sigma_a)_{\alpha\dot{\beta}}, \quad \omega_\alpha{}^\beta = \tfrac{1}{16}\omega_{ab}(\sigma_{ab})_\alpha{}^\beta, \quad \omega^{\dot{\alpha}}{}_{\dot{\beta}} = -\tfrac{1}{16}\omega_{ab}(\sigma_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{A.38}$$

we get

$$\begin{aligned} T_a &\equiv d_x e_a + \omega_{ab} \wedge e_b = 0 \\ R_{ab} &\equiv d_x\omega_{ab} + \omega_{ac} \wedge \omega_{cb} + 64 e_a \wedge e_b = 0. \end{aligned} \tag{A.39}$$

### *A.9. Exploration of various boundary conditions for scalars in the non-Abelian theory*

The same theory in $AdS_4$ with $\Delta = 2$ boundary condition on the $U(M)$-*singlet* bulk scalar is dual to the critical point of the $SU(N)$ vector model with $M$ flavors and the double trace deformation by $(\bar{\phi}^{ia}\phi_{ia})^2$. Alternatively, this critical point may be defined by introducing a Lagrangian multiplier $\alpha$ and adding the term

$$\alpha\bar{\phi}^{ia}\phi_{ia} \tag{A.40}$$

to the Lagrangian of the vector model[48]. As in the case of the $M = 1$ critical vector model, higher spin symmetry is broken by $1/N$ effects. Note that the $SU(M)$ part of the spin-2 current is also broken by $1/N$ effects, i.e. there are no interacting colored massless gravitons, as expected. To see this explicitly from the boundary CFT, let us consider the spin-2 current

$$(J^{(2)}_{\mu\nu})^a{}_b = \frac{1}{2}\bar{\phi}^{ia} \overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu \phi_{ib} - 2\partial_{(\mu}\bar{\phi}^{ia}\partial_{\nu)}\phi_{ib} + \delta_{\mu\nu}\partial^\rho\bar{\phi}^{ia}\partial_\rho\phi_{ib}. \tag{A.41}$$

Using the classical equation of motion

$$\Box\phi_i = \alpha\phi_i, \tag{A.42}$$

we have

$$\partial^\mu\left(J^{(2)}_{\mu\nu}\right)^a{}_b = (\partial_\nu\alpha)\bar{\phi}^{ia}\phi_{ib} - \alpha\partial_\nu(\bar{\phi}^{ia}\phi_{ib}). \tag{A.43}$$

While the $SU(M)$-singlet part of $J_{\mu\nu}$, being the stress-energy tensor, is conserved ($\bar{\phi}^{ia}\phi_{ia}$ is set to zero by $\alpha$-equation of motion), the $SU(M)$ non-singlet part of $J_{\mu\nu}$ is not conserved, and acquires an anomalous dimension of order $1/N$ at the leading nontrivial order in the $1/N$ expansion. In the bulk, the colored gravitons become massive, and their longitudinal components are supplied by the bound state of the singlet scalar and a colored spin-1 field.

One could also consider the theory in $AdS_4$ with $\Delta = 2$ boundary condition on *all* bulk scalars, that is, on both the singlet and adjoint of the $SU(M)$ bulk gauge group. The dual CFT is the critical point defined by turning on the double trace deformation $\bar{\phi}^{ia}\phi_{ib}\bar{\phi}^{jb}\phi_{ja}$ and flow to the IR, or by introducing the Lagrangian multiplier $\Lambda_a{}^b$, and the term

$$\Lambda_a{}^b\bar{\phi}^{ia}\phi_{ib} \tag{A.44}$$

in the CFT Lagrangian. Now the classical equations of motion

$$\Box\phi_{ia} = \Lambda_a{}^b\phi_{ib}, \bar{\phi}^{ia}\phi_{ib} = 0, \tag{A.45}$$

imply the divergence of the colored spin-2 currents

$$\partial^\mu(J^{(2)}_{\mu\nu})^a{}_b = \Lambda_b{}^c\bar{\phi}^{ia} \overleftrightarrow{\partial}_\nu \phi_{ic} - \Lambda_c{}^a\bar{\phi}^{ic} \overleftrightarrow{\partial}_\nu \phi_{ib} = \Lambda_b{}^c(J^{(1)}_\nu)^a{}_c - \Lambda_c{}^a(J^{(1)}_\nu)^c{}_b. \tag{A.46}$$

Once again, the $SU(M)$ non-singlet spin-2 current is no longer conserved. In this case, the colored gravitons in the bulk are massive because their longitudinal component are supplied by the two-particle state of colored scalar and spin-1 fields.

[48] The critical point can be conveniently defined using dimensional regularization.

## Appendix B. Supersymmetry transformations on bulk fields of spin 0, 1/2, and 1

We begin by rewriting the magnetic boundary condition on the spin-1 bulk fields in the supersymmetric Vasiliev theory. With the magnetic boundary condition, the $2^{n-1}$ vector gauge fields are dual to ungauged $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ '$R$-symmetry' currents of boundary CFT that rotate the bosonic and fermionic flavors separately. Supersymmetrizing Chern–Simons coupling will generally break this flavor symmetry to a subgroup. We will see this as the violation of magnetic boundary condition by the SUSY variation of the bulk spin-1 fields. If we do not gauge the flavor symmetries of the Chern–Simons vector model, then all bulk vector fields should be assigned the magnetic boundary condition. We will see later that in this case only up to $\mathcal{N} = 3$ SUSY can be preserved, whereas by relaxing the magnetic boundary condition on some of the bulk vector fields, it will be possible to preserve $\mathcal{N} = 4$ or 6 SUSY.

In terms of Vasiliev's master field $B$ which contains the field strength, the general electric-magnetic boundary condition may be expressed as

$$B|_{\mathcal{O}(y^2,\bar{y}^2)} \to z^2[\mathrm{e}^{\mathrm{i}\beta}(yFy) + \mathrm{e}^{-\mathrm{i}\beta}(\bar{y}\overline{F}\bar{y})\Gamma], \quad z \to 0, \tag{B.1}$$

where $F \equiv F_{\mu\nu}\sigma^{\mu\nu}$ and its complex conjugate $\overline{F}$ are functions of $\psi_i$, and are constrained by the linear relation

$$F = -\sigma^z\overline{F}\sigma^z. \tag{B.2}$$

With this choice of boundary condition, the boundary to bulk propagator for the spin-1 components of the $B$ master field is given by the standard one,

$$\begin{aligned} B^{(1)} &= \frac{z^2}{(\vec{x}^2+z^2)^3}\,\mathrm{e}^{-y\Sigma\bar{y}}[\mathrm{e}^{\mathrm{i}\beta}(\lambda\mathbf{x}\sigma^z y)^2 + \mathrm{e}^{-\mathrm{i}\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma] \\ &\equiv \widetilde{B}^{(1)}[\mathrm{e}^{\mathrm{i}\beta}(\lambda\mathbf{x}\sigma^z y)^2 + \mathrm{e}^{-\mathrm{i}\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma]. \end{aligned} \tag{B.3}$$

It indeed obeys (B.2), with $F$ and $\overline{F}$ given by

$$\begin{aligned} F_\alpha{}^\beta &= -(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)_\alpha(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)^\beta, \\ \overline{F}_{\dot\alpha}{}^{\dot\beta} &= -(\lambda\sigma^z\vec{x}\cdot\vec{\sigma}\sigma^z)_{\dot\alpha}(\lambda\sigma^z\vec{x}\cdot\vec{\sigma}\sigma^z)^{\dot\beta} = -(\lambda\vec{x}\cdot\vec{\sigma})_{\dot\alpha}(\lambda\vec{x}\cdot\vec{\sigma})^{\dot\beta}, \end{aligned} \tag{B.4}$$

and

$$(\sigma^z\overline{F}\sigma^z)_\alpha{}^\beta = -(\lambda\vec{x}\cdot\vec{\sigma})_{\dot\alpha}(\lambda\vec{x}\cdot\vec{\sigma})^{\dot\beta}(\sigma^z)_\alpha{}^{\dot\alpha}(\sigma^z)^\beta{}_{\dot\beta} = (\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)_\alpha(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)^\beta = -F_\alpha{}^\beta. \tag{B.5}$$

In the next four subsections, we give the explicit formulae for the SUSY variation $\delta_\epsilon$ (i.e. spin 3/2 gauge transformation of Vasiliev's system) of bulk fields of spin 0, 1/2, 1, sourced by boundary currents of spin 0, 1/2, 1.

### *B.1. $\delta_\epsilon$: spin 1 → spin 1/2*

Let us start with the $B$ master field sourced by a spin-1 boundary current at $\vec{x} = 0$, i.e. the spin-1 boundary to bulk propagator $B^{(1)}(x|Y)$, and consider its variation under SUSY, which is generated by $\epsilon(x|Y)$ of degree one in $Y = (y, \bar{y})$:

$$\begin{aligned} \delta_\epsilon B^{(1)}(x|Y) = &-\epsilon * \mathrm{e}^{\mathrm{i}\beta}(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} + \mathrm{e}^{\mathrm{i}\beta}(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} * \pi(\epsilon) \\ &-\epsilon * \mathrm{e}^{-\mathrm{i}\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} + \mathrm{e}^{-\mathrm{i}\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} * \pi(\epsilon). \end{aligned} \tag{B.6}$$

Carrying out the $*$ products explicitly, we find

$$\begin{aligned} &-\epsilon * (\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} + (\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} * \pi(\epsilon) \\ &\quad = -(\Lambda y + \overline{\Lambda}\bar{y}) * (\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} + (\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} * (-\Lambda y + \overline{\Lambda}\bar{y}) \\ &\quad = -\{y_\alpha, (\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}\}_*\{\Lambda^\alpha, \lambda^\beta\lambda^\gamma\} - [y_\alpha, (\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}]_*[\Lambda^\alpha, \lambda^\beta\lambda^\gamma] \end{aligned}$$

$$
\begin{aligned}
&- [\bar{y}_{\dot\alpha}, (\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)}]_* \{\overline{\Lambda}^{\dot\alpha}, \lambda^\beta\lambda^\gamma\} - \{\bar{y}_{\dot\alpha}, (\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)}\}_* [\overline{\Lambda}^{\dot\alpha}, \lambda^\beta\lambda^\gamma] \\
&= -2\{\Lambda y, \lambda^\beta\lambda^\gamma\}(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)} - 2[\Lambda\partial_y, \lambda^\beta\lambda^\gamma](\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)} \\
&\quad - 2\{\overline{\Lambda}\partial_{\bar y}, \lambda^\beta\lambda^\gamma\}(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)} - 2[\overline{\Lambda}\bar{y}, \lambda^\beta\lambda^\gamma](\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma \widetilde{B}^{(1)} \\
&= 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda\mathbf{x}\sigma^z y)^2\}\widetilde{B}^{(1)} + 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda\mathbf{x}\sigma^z y)^2]\widetilde{B}^{(1)} \\
&\quad - 4[(\mathbf{x}\sigma^z\Lambda)_\beta, \lambda^\beta(\lambda\mathbf{x}\sigma^z y)]\widetilde{B}^{(1)},
\end{aligned}
\tag{B.7}
$$

and

$$
\begin{aligned}
&-\epsilon * (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} + (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} * \pi(\epsilon) \\
&= -2\{\Lambda y, \lambda^\beta\lambda^\gamma\Gamma\}(\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)} - 2[\Lambda\partial_y, \lambda^\beta\lambda^\gamma\Gamma](\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)} \\
&\quad - 2\{\overline{\Lambda}\partial_{\bar y}, \lambda^\beta\lambda^\gamma\Gamma\}(\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)} - 2[\overline{\Lambda}\bar{y}, \lambda^\beta\lambda^\gamma\Gamma](\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)} \\
&= 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\}\widetilde{B}^{(1)} + 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma]\widetilde{B}^{(1)} \\
&\quad - 4\{(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta, \lambda^\beta(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\}\widetilde{B}^{(1)}.
\end{aligned}
\tag{B.8}
$$

Note that the commutators and anti-commutators in above formula are due to the $\psi_i$-dependence only, and do not involve $*$ product. $\delta_\epsilon B^{(1)}$ contains SUSY variation of fields of spin 1/2 and 3/2. We will focus on the variation spin 1/2 fields, since they can be subject to a family of different boundary conditions, corresponding to turning on fermionic double trace deformations (i.e. (fermion singlet)$^2$) in the boundary CFT. So we restrict to terms linear in $(y, \bar{y})$,

$$
\begin{aligned}
\delta B^{(1)}\big|_{\mathcal{O}(y,\bar y)} &= -4[(\mathbf{x}\sigma^z\Lambda)_\beta, \lambda^\beta(\lambda\mathbf{x}\sigma^z y)]\widetilde{B}^{(1)} - 4\{(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta, \lambda^\beta(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\}\widetilde{B}^{(1)} \\
&\to -4\mathrm{e}^{\mathrm{i}\beta}\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^3}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)_\beta, \lambda^\beta(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z y)] \\
&\quad +4\mathrm{e}^{-\mathrm{i}\beta}\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^3}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)_\beta, \lambda^\beta(\lambda\vec{x}\cdot\vec{\sigma}\bar{y})]\Gamma
\end{aligned}
\tag{B.9}
$$

where in the second line we kept the leading terms, of order $z^{\frac{3}{2}}$, in the $z \to 0$ limit.

### B.2. $\delta_\epsilon$: spin 1/2 $\to$ spin 1

The general conformally invariant boundary condition on spin 1/2 fermions, in terms of Vasiliev's $B$ master field, takes the form

$$
B|_{\mathcal{O}(y,\bar y)} \to z^{\frac{3}{2}}[\mathrm{e}^{\mathrm{i}\alpha}(\chi y) - \Gamma\,\mathrm{e}^{-\mathrm{i}\alpha}(\bar\chi\bar y)],
\tag{B.10}
$$

Here $\chi$ and its complex conjugate $\bar\chi$ are chiral and anti-chiral spinors that are odd functions of the Grassmannian variables $\psi_i$. They are further constrained by the linear relation

$$
\chi = \sigma^z\bar\chi.
\tag{B.11}
$$

$\alpha$ is generally a linear operator that acts on the vector space spanned by odd monomials in the $\psi_i$, i.e. it assigns phase angles to fermions in the bulk $R$-symmetry multiplet. A choice of the spin-1/2 fermion boundary condition is equivalent to a choice of the 'phase angle' operator $\alpha$.

The fermion boundary to bulk propagator that satisfies the above boundary condition is:

$$
\begin{aligned}
B^{(\frac{1}{2})} &= \frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\,\mathrm{e}^{-y\Sigma\bar y}[\mathrm{e}^{\mathrm{i}\alpha}(\lambda\mathbf{x}\sigma^z y) - \Gamma\,\mathrm{e}^{-\mathrm{i}\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar y)] \\
&\equiv [\mathrm{e}^{\mathrm{i}\alpha}(\lambda\mathbf{x}\sigma^z y) - \Gamma\,\mathrm{e}^{-\mathrm{i}\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar y)]\widetilde{B}^{(\frac{1}{2})}.
\end{aligned}
\tag{B.12}
$$

Here the linear operator $\alpha$ is understood to act on $\lambda$ only, the latter being an odd function of the $\psi_i$.

Next, we make super transformation on the fermion boundary to bulk propagator. The SUSY transformation reads

$$\delta B^{(\frac{1}{2})} = -e^{i\alpha}\epsilon * (\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} + e^{i\alpha}(\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon)$$
$$- e^{-i\alpha}\epsilon * (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} + e^{-i\alpha}(\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon), \quad \text{(B.13)}$$

where $\epsilon = \Lambda y + \bar{\Lambda}\bar{y}$, $\Lambda$ is an odd SUSY parameter $\eta$ multiplied by an odd function of the $\psi_i$'s. $\eta$ in particular anti-commutes with all the $\psi_i$, and therefore anti-commutes with $\lambda$ which involves an odd number of the $\psi_i$.

Carrying out the $*$ algebra, we have

$$-\epsilon * (\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} + (\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon) = 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda \mathbf{x}\sigma^z y)\}\widetilde{B}^{(\frac{1}{2})}$$
$$+ 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda \mathbf{x}\sigma^z y)]\widetilde{B}^{(\frac{1}{2})} - 2[(\mathbf{x}\sigma^z\Lambda)_\beta, \lambda^\beta]\widetilde{B}^{(\frac{1}{2})}, \quad \text{(B.14)}$$

and

$$-\epsilon * (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} + (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon) = 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\}\widetilde{B}^{(\frac{1}{2})}$$
$$+ 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma]\widetilde{B}^{(\frac{1}{2})} - 2\{(\sigma^z \mathbf{x}\sigma^z\overline{\Lambda})_\beta, \lambda^\beta\Gamma\}\widetilde{B}^{(\frac{1}{2})}. \quad \text{(B.15)}$$

The SUSY variation of the spin-1 field strengths are extracted from $\mathcal{O}(y^2, \bar{y}^2)$ terms in $\delta B^{(\frac{1}{2})}$, namely

$$\delta_\epsilon B^{(\frac{1}{2})}(x|Y)|_{\mathcal{O}(y^2,\bar{y}^2)} = 2\{\overline{\Lambda}\Sigma y - \Lambda y, e^{i\alpha}(\lambda \mathbf{x}\sigma^z y)\}\widetilde{B}^{(\frac{1}{2})} - 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, \Gamma e^{-i\alpha}(\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})]\widetilde{B}^{(\frac{1}{2})}$$
$$\to -4\frac{z^2}{(\vec{x}^2+z^2)^3}\{\Lambda_0 \vec{x}\cdot\vec{\sigma}\sigma^z y, e^{i\alpha}(\lambda\sigma^z \vec{x}\cdot\vec{\sigma} y)\}$$
$$- 4\frac{z^2}{(\vec{x}^2+z^2)^3}[\Lambda_0 \vec{x}\cdot\vec{\sigma}\bar{y}, \Gamma e^{-i\alpha}(\lambda \vec{x}\cdot\vec{\sigma}\bar{y})]. \quad \text{(B.16)}$$

In the second line, we have taken the small $z$ limit and kept the leading terms, of order $z^2$.

### *B.3. $\delta_\epsilon$: spin 1/2 → spin 0*

The SUSY variation of the scalar field due to a spin-1/2 fermionic boundary source is extracted from $\delta_\epsilon B^{(\frac{1}{2})}$ of the previous subsection, restricted to $y = \bar{y} = 0$:

$$\delta_\epsilon B^{(\frac{1}{2})}|_{y,\bar{y}=0}(\vec{x}, z) = -2[(\mathbf{x}\sigma^z\Lambda)_\beta, e^{i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} - 2\Gamma[(\sigma^z \mathbf{x}\sigma^z\overline{\Lambda})_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})}$$
$$+ 2z^{-\frac{1}{2}}\Gamma[(\sigma^z \mathbf{x}\Lambda_+)_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} - 2z^{\frac{1}{2}}\Gamma[(\sigma^z \mathbf{x}\Lambda_-)_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})}$$
$$= 2(e^{i\alpha} + \Gamma e^{-i\alpha})\frac{z}{(\vec{x}^2+z^2)^2}[(\sigma^z \vec{x}\cdot\vec{\sigma}\Lambda_+)_\beta, \lambda^\beta]$$
$$- 2(e^{i\alpha} - \Gamma e^{-i\alpha})\frac{z^2}{(\vec{x}^2+z^2)^2}[(\Lambda_+)_\beta, \lambda^\beta]$$
$$- 2(e^{i\alpha} - \Gamma e^{-i\alpha})\frac{z^2}{(\vec{x}^2+z^2)^2}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-)_\beta, \lambda^\beta] + \mathcal{O}(z^3). \quad \text{(B.17)}$$

In the last two lines, $\alpha$ as a linear operator is understood to act on $\lambda$ only (and not on $\Lambda_\pm$).

### *B.4. $\delta_\epsilon$: spin 0 → spin 1/2*

The general conformally invariant linear boundary condition on the bulk scalars $B^{(0)}(\vec{x}, z) = B(\vec{x}, z|y = \bar{y} = 0)$ may be expressed as

$$B^{(0)}(\vec{x}, z) = (e^{i\gamma} + \Gamma e^{-i\gamma})\tilde{f}_1 z + (e^{i\gamma} - \Gamma e^{-i\gamma})\tilde{f}_2 z^2 + \mathcal{O}(z^3) \quad \text{(B.18)}$$

in the limit $z \to 0$. Here $\tilde{f}_1, \tilde{f}_2$ are real and even function in $\psi_i$, and are subject to a set of linear relations that eliminate half of their degrees of freedom. The phase $\gamma$ is generally a linear operator acting on the space spanned by even monomials in the $\psi_i$'s (analogously to $\alpha$ in the fermion boundary condition). We will determine our choice of $\gamma$ and the linear constraints on $\tilde{f}_{1,2}$ later.

The boundary-to-bulk propagator for the scalar components of the $B$ master field, subject to the above boundary condition, is now written as

$$B^{(0)} = f_1(\psi)\widetilde{B}^{(0)}_{\Delta=1} + f_2(\psi)\widetilde{B}^{(0)}_{\Delta=2}, \tag{B.19}$$

where

$$f_1(\psi) = (\mathrm{e}^{\mathrm{i}\gamma} + \Gamma\mathrm{e}^{-\mathrm{i}\gamma})\tilde{f}_1(\psi), \qquad f_2(\psi) = (\mathrm{e}^{\mathrm{i}\gamma} - \Gamma\mathrm{e}^{-\mathrm{i}\gamma})\tilde{f}_2(\psi). \tag{B.20}$$

A straightforward calculation gives the SUSY variation of the spin-1/2 fermion due to a scalar boundary source at $\vec{x} = 0$,

$$\begin{aligned}
\delta_\epsilon \widetilde{B}^{(0)}(\vec{x}, z)|_{\mathcal{O}(y,\bar{y})} \to & -4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\{\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, f_1\} - 4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}[\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, f_1] \\
& + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}[\Lambda_+\sigma^z\bar{y}, f_2] + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\{\Lambda_+ y, f_2\} \\
= & -4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}(\mathrm{e}^{\mathrm{i}\gamma}\{\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, \tilde{f}_1\} - \Gamma\,\mathrm{e}^{-\mathrm{i}\gamma}[\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, \tilde{f}_1] \\
& + \mathrm{e}^{\mathrm{i}\gamma}[\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, \tilde{f}_1] - \Gamma\,\mathrm{e}^{-\mathrm{i}\gamma}\{\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, \tilde{f}_1\}) \\
& + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}(\mathrm{e}^{\mathrm{i}\gamma}[\Lambda_+\sigma^z\bar{y}, \tilde{f}_2] + \Gamma\,\mathrm{e}^{-\mathrm{i}\gamma}\{\Lambda_+\sigma^z\bar{y}, \tilde{f}_2\} \\
& + \mathrm{e}^{\mathrm{i}\gamma}\{\Lambda_+ y, \tilde{f}_2\} + \Gamma\,\mathrm{e}^{-\mathrm{i}\gamma}[\Lambda_+ y, \tilde{f}_2]).
\end{aligned} \tag{B.21}$$

We have taken the small $z$ limit, and kept terms of order $z^{\frac{3}{2}}$. Again, in the last two lines $\gamma$ as a linear operator should be understood as acting on $\tilde{f}_{1,2}(\psi)$ only and not on $\Lambda$.

## Appendix C. The bulk dual of double trace deformations and Chern–Simons gauging

### *C.1. Alternate and regular boundary conditions for scalars in $AdS_{d+1}$*

In this section we review the AdS/CFT implementation alternate and regular boundary conditions for scalars, in the presence of multitrace deformations. The material reviewed here is well known (see e.g. [17, 33–36, 18]—we most closely follow the approach of the paper [34]); our brief review focuses on aspects we will have occasion to use in the main text of our paper.

*C.1.1. Multi-trace deformations in large N field theories.* In this brief subsection we will address the following question: how is the generating function of correlators of a large $N$ field theory modified by the addition of a multi-trace deformation to the action of the theory?

Consider any large $N$ field theory whose single trace operators are denoted by $O_i$. Let $W(J)$ denote the generating function of correlators[49]

$$\langle \mathrm{e}^{J_i O_i}\rangle = \mathrm{e}^{-W[J_i]}. \tag{C.2}$$

[49] More precisely this equation should have read

$$\langle \mathrm{e}^{\int \mathrm{d}^d x J_i(x) O^i(x)}\rangle = \mathrm{e}^{-W[J_i(x)]}. \tag{C.1}$$

However for ease of readability, in all the formal discussions of this section we will use compact notation in which we suppress the position dependence of operators and fields, and do not explicitly indicate integration.

Note that $W[J_i]$ is of order $N^2$ in a matrix type large $N$ theory, while it is of order $N$ in a vector type large $N$ theory. For formal purposed below we will find it useful to Legendre transform $W$ to define an effective action for the operators $O_i$

$$I[O^i] = W[J_i] + O^i J_i. \tag{C.3}$$

$I[O^i]$ is a function only of $O^i$ (and not of $J_i$) in the following sense. The RHS of (C.3) is viewed as an action for the dynamical variable $J_i$. The equation of motion for $J_i$ follows from varying this action and is

$$\frac{\partial W}{\partial J_i} = -O^i. \tag{C.4}$$

The RHS of (C.3) is evaluated with the onshell value of $J_i$.

$I[O^i]$ plays the role of the effective action for the trace operators $O^i$. In the large $N$ limit the dynamics of the operators $O^i$ is generated by the classical dynamics of the action $I(O^i)$.

Of course $W[J^i]$ may equally be thought of as the Legendre transform of $I[O^i]$

$$W[J_i] = I[O^i] - O^i J_i, \tag{C.5}$$

where $O^i$ is the function of $J^i$ obtained by solving the equation of motion

$$\frac{\partial I}{\partial O^i} = J_i. \tag{C.6}$$

Now let us suppose that the action $S$ of the original large $N$ field theory is deformed by the addition of a multitrace term $S \to S + P(O^i)$ where $P(O^i)$ is an arbitrary function of $O^i$. The effective action for this deformed theory is simply given by $\tilde{I}(O^i)$

$$\tilde{I}(O^i) = I(O^i) + P(O^i). \tag{C.7}$$

The generating function of correlators of the deformed theory is once again given by the Legendre transform (C.5) with $I[O^i]$ replaced by $\tilde{I}[O^i]$.

*C.1.2. Bulk dual to multi trace deformations in regular and alternate quantization.* Consider a real scalar field propagating in $\mathrm{AdS}_{d+1}$ according to the action

$$S = \frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{g}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2). \tag{C.8}$$

It is well known that these scalars admit two distinct conformally invariant boundary conditions–sometimes referred to as alternate and standard quantization–in the mass range $-\left(\frac{\mathrm{d}^2}{4} - 1\right) > m^2 > -\frac{\mathrm{d}^2}{4}$. In this subsection we will review the very well known rules for the computation of correlation functions for scalars with alternate and standard boundary conditions.

The action (C.8) is ambiguous as it generically receives divergent contributions from the boundary, as we now explain. We use coordinates so that the metric of AdS space is given by (2.33). Near $z = 0$ the general solution to the equation motion from (C.8) takes the form

$$\phi = \frac{\phi_1 z^{\frac{\mathrm{d}}{2}-\zeta}}{2\zeta} + \phi_2 z^{\frac{\mathrm{d}}{2}+\zeta}, \tag{C.9}$$

where $\zeta$ is the positive root of the equation $\zeta^2 = m^2 + \frac{\mathrm{d}^2}{4}$. Let us cut of the action (C.8) at a small value, $z_c$ of the coordinate $z$. Onshell (C.8) evaluates to

$$S = -\frac{1}{2}\int \mathrm{d}^d x \frac{1}{z_c^{d-1}}\,\phi\,\partial_z\phi, \tag{C.10}$$

where the integral is evaluated over the boundary surface $z = z^c$. It is easily verified that the action $S$ has a divergence proportional to $z_c^{2\zeta}$ when evaluated on the generic solution (C.9). To cure this divergence we supplement (C.8) with a diffeomorphically invariant boundary action for the $d$ dimensional boundary field $\phi(z_c, x)$

$$\delta S = \frac{1}{2}\int \mathrm{d}^d x\sqrt{g}\left(\frac{\mathrm{d}}{2}-\zeta\right)\phi^2 \tag{C.11}$$

where, once again, the integral is taken over the boundary surface $z = z_c$ and $g$ is the induced metric on this boundary. It is easily verified that

$$S + \delta S = -\frac{1}{2}\int \mathrm{d}^d x\phi_1(x)\phi_2(x). \tag{C.12}$$

Regularity in the interior of AdS relates $\phi_2$ to $\phi_1$. The relationship is clearly linear and so takes the form

$$\phi_2(x) = \int \mathrm{d}^d x G(x-y)\phi_1(y). \tag{C.13}$$

In the rest of this subsection we use abbreviated notation so that (C.14) is written as $S = -\frac{1}{2}\phi_1\phi_2$ and (C.13) is written as $\phi_2 = G\phi_1$. It follows that the onshell action is given by

$$S = -\tfrac{1}{2}\phi_1 G\phi_1. \tag{C.14}$$

In the case of alternate quantization the boundary action (C.14), thought of as a functional of the dynamical field $\phi_1 = \lim_{z_c\to 0}\frac{\phi}{z_c^{\frac{d}{2}-\zeta}}$, is identified with the single trace effective action $I[O]$ defined in (C.3). The generator of correlators of this theory is obtained by coupling $\phi_1 = \frac{\phi}{z_c^{\frac{d}{2}-\zeta}}$ to a source $J$:

$$S = -\tfrac{1}{2}\phi_1 G\phi_1 - J\phi_1. \tag{C.15}$$

The resulting equation of motion for $\phi_1$ yields

$$G\phi_1 = -J. \tag{C.16}$$

Integrating out $\phi_1$ we find the action

$$S = JG^{-1}J.$$

It follows that the two point function of the dual operator is $-G^{-1}$. It also follows from (C.16) that

$$\phi_2 = -J.$$

in particular $\phi_2$ vanishes wherever $J$ vanishes. Consequently, alternate quantization is associated with the boundary condition $\phi_2 = 0$.

The multi trace deformation $P(O)$ of the dual theory is implemented, in alternate quantization, by adding the term $P(\phi_1)$to the boundary effective action (C.14), in perfect imitation of (C.7). Correlation functions of the deformed theory are obtained by the Legendre transform of this augmented boundary action. The resultant equation of motion is $G\phi_1 + J - P'(\phi_1) = 0$ yields the bulk boundary conditions

$$\phi_2 + J - P'(\phi_1) = 0.$$

In the case of regular quantization we supplement the action (C.14) with an additional degree of freedom $\tilde{\phi}_2$ so that the full boundary action takes the form

$$S = -\tfrac{1}{2}\phi_1 G\phi_1 + \tilde{\phi}_2\phi_1. \tag{C.17}$$

The dynamical field $\phi_1$ is then integrated out using its equation of motion

$$G\phi_1 = \tilde{\phi}_2. \tag{C.18}$$

On shell, therefore $\tilde{\phi}_2 = \phi_2$. The resultant action

$$S = \tfrac{1}{2}\tilde{\phi}_2 G^{-1}\tilde{\phi}_2 \tag{C.19}$$

as a function of $\tilde{\phi}_2$ is identified with $I(O)$ in (C.3). The generator of correlators of the theory is obtained by coupling $\tilde{\phi}_2$ to a source $J$

$$S = \tfrac{1}{2}\tilde{\phi}_2 G^{-1}\tilde{\phi}_2 - J\tilde{\phi}_2,$$

and then integrating this field out according to its equations of motion. This allows us, in particular, to identify the two point function of the dual theory with $G$. Note also that the resultant equation of motion, $G^{-1}\tilde{\phi}_2 = J$ implies

$$\phi_1 = J,$$

so that $\phi_1$ vanishes wherever $J$ vanishes. In other words standard quantization is associated with the boundary condition $\phi_1 = 0$. The multitrace deformation $P(O)$ of the dual theory is implemented, in standard quantization, by adding $P(\tilde{\phi}_2)$ to the action (C.19). The resultant boundary condition is

$$\phi_1 - J + P'(\phi_2) = 0.$$

*C.1.3. Marginal multitrace deformation with two scalar field in opposite quantization.* Consider two scalar fields in $AdS_4$, $\phi$ and $\chi$, with $\phi$ quantized with alternate quantization and $\chi$ with regular quantization. In the compact notation defined in earlier subsection, the generating function of correlation function of the dual field theory deformed by double trace operator $\tan\theta_0 O_1 O_2$ is

$$S = -\tfrac{1}{2}G\phi_1^2 - \tfrac{1}{2}G\chi_1^2 + \chi_1\tilde{\chi}_2 - J_1\phi_1 - J_2\tilde{\chi}_2 + \tan\theta_0\tilde{\chi}_2\phi_1. \tag{C.20}$$

The action is linear in $\tilde{\chi}_2$; the equation of motion for this field immediately yields

$$J_2 = \frac{1}{\cos\theta_0}(\sin\theta_0\phi_1 + \cos\theta_0\chi_1). \tag{C.21}$$

Using (C.21) to eliminate $\phi_1$ in favor of $\chi_1$, $S$ simplifies to a function of $\phi_1$. The resultant equation of motion yields

$$J_1 = -\frac{1}{\cos\theta_0}G(\cos\theta_0\phi_1 - \sin\theta_0\chi_1). \tag{C.22}$$

Using $G\phi_1 = \phi_2$ and $G\chi_1 = \chi_2$, (C.22) may be rewritten as

$$J_1 = -\frac{1}{\cos\theta_0}(\cos\theta_0\phi_2 - \sin\theta_0\chi_2). \tag{C.23}$$

Upon setting $J_1 = J_2 = 0$, (C.21) and (C.23) express the boundary conditions of the trace deformed model. These boundary conditions may, most succinctly be expressed as follows. Let us define new 'rotated' bulk fields

$$\phi' = \cos\theta_0\phi - \sin\theta_0\chi, \qquad \chi' = \sin\theta_0\phi + \cos\theta_0\chi.$$

Note that the rotated fields have same bulk action as the original fields. The boundary conditions (C.21) and (C.23) reduce to

$$\phi_2' = 0, \chi_1' = 0.$$

In summary dual to the double trace deformed field theory has the same action as well as boundary conditions for $\phi'$ and $\chi'$ as the dual to the undeformed theory had for $\phi$ and $\chi$. Despite this fact, the double trace deformed theory is *not* field redefinition equivalent to the original theory. This can be seen in many ways. Most simply, the full action (C.20) does not have a simple rotational invariance, and does not take a simple form when re-expressed in terms of $\phi'$ and $\chi'$. This lack of equivalence also shows itself up in the generator of two point functions of the operators dual to $\phi'$ and $\chi'$. This generating function is obtained by plugging (C.21) and (C.22) into (C.20); we find

$$-S = -\cos^2\theta_0 \frac{J_1^2}{2G} + \cos^2\theta_0 \frac{J_2^2 G}{2} + \sin\theta_0\cos\theta_0 J_1 J_2. \tag{C.24}$$

The fact that $\theta_0$ does not disappear from (C.24) demonstrates the lack of equivalence of the trace deformed model from the trace undeformed model ($\theta_0 = 0$). Note in particular that the double trace deformed theory has a contact cross two point function

$$\langle O_\phi(x) O_\chi(y)\rangle = \sin\theta_0\cos\theta_0\delta(x-y),$$

which is absent in the trace undeformed theory. On the other hand the direct correlators $\langle O_\phi(x) O_\phi(y)\rangle$ and $\langle O_\chi(x) O_\chi(y)\rangle$ have the same spacetime structure in the deformed and undeformed theories, but have different normalizations.

### *C.2. Gauging a U(1) symmetry*

Let us begin with a three dimensional CFT with a $U(1)$ global symmetry, generated by the current $J_i$, where $i$ is the three-dimensional vector index. This theory will be referred to as $\mathrm{CFT}_\infty$, as opposed to the theory obtained by gauging the $U(1)$ with Chern–Simons gauge field at level $k$, which we refer to as $\mathrm{CFT}_k$. Suppose $\mathrm{CFT}_\infty$ is dual to a weakly coupled gravity theory in $\mathrm{AdS}_4$. The global $U(1)$ current $J_i$ of the boundary CFT is dual to a gauge field $A_\mu$ in the bulk. The two-derivative part of the bulk action for the gauge field is

$$\frac{1}{4}\int \frac{\mathrm{d}^3\vec{x}\,\mathrm{d}z}{z^4} F_{\mu\nu}F^{\mu\nu} = \int \mathrm{d}^3\vec{x}\,\mathrm{d}z \left(\frac{1}{2}F_{zi}F_{zi} + \frac{1}{4}F_{ij}F_{ij}\right). \tag{C.25}$$

Working in the radial gauge $A_z = 0$, we have

$$F_{zi} = \partial_z A_i, \qquad F_{ij} = \partial_i A_j - \partial_j A_i. \tag{C.26}$$

Consider the linearized, i.e. free, equation of motion

$$\left(\partial_z^2 + \partial_j^2\right)A_i - \partial_i\partial_j A_j = 0, \tag{C.27}$$

together with the constraint

$$\partial_z\partial_i A_i = 0. \tag{C.28}$$

Near the boundary, a solution to the equation of motion has two possible asymptotic behaviors, $A_i \sim z + \mathcal{O}(z^2)$, or $A_i \sim 1 + \mathcal{O}(z^2)$. Equivalently, they can be expressed in gauge invariant form as the magnetic boundary condition

$$F_{ij}|_{z=0} = 0, \tag{C.29}$$

and the electric boundary condition

$$F_{zi}|_{z=0} = 0, \tag{C.30}$$

respectively. With the magnetic boundary condition, $A_\mu$ is dual to a $U(1)$ global current in the boundary CFT, i.e. $\mathrm{CFT}_\infty$. The family of $\mathrm{CFT}_k$, on the other hand, is dual to the same bulk theory with the mixed boundary condition (still conformally invariant)

$$\left.\left(\frac{1}{2}\epsilon_{ijk}F_{jk} + \frac{\mathrm{i}\alpha}{k}F_{zi}\right)\right|_{z=0} = 0. \tag{C.31}$$

Here $\alpha$ is a constant. It will be determined in terms of the two-point function of the current $J_i$.

Let us now solve the bulk Green function with the mixed boundary condition. The bulk linearized equation of motion with a point source at $z = z_0$, after a Fourier transformation in the boundary coordinates $\vec{x}$, is

$$\left(\partial_z^2 - p^2\right)A_i + p_i p_j A_j = \delta(z - z_0)\xi_i. \tag{C.32}$$

Due to the constraint (C.28), the source $\xi_i$ is restricted by $p_i\xi_i = 0$. The boundary condition is

$$\left(\epsilon_{ijk}p_j A_k + \frac{\alpha}{k}\partial_z A_i\right)\bigg|_{z=0} = 0. \tag{C.33}$$

Without loss of generality, let us consider the case $\vec{p} = (0, 0, p)$, and assume $p = p_3 > 0$. The Green equation is now written as

$$\begin{aligned} &\partial_z^2 A_3 = 0,\\ &\left(\partial_z^2 - p^2\right)A_i = \delta(z - z_0)\xi_i, \quad i = 1, 2, \end{aligned} \tag{C.34}$$

and the boundary condition as

$$\partial_z A_3|_{z=0} = 0, \qquad \left(p\epsilon_{ij}A_j - \frac{\alpha}{k}\partial_z A_i\right)\bigg|_{z=0} = 0, \quad i = 1, 2. \tag{C.35}$$

The $z$-independent part of $A_3$ can be gauged away. We may then take the solution

$$\begin{aligned} &A_3 = 0,\\ &A_i = \theta(z - z_0)[g_i(p) + h_i(p)]\,\mathrm{e}^{-p(z-z_0)} + \theta(z_0 - z)[g_i(p)\,\mathrm{e}^{-p(z-z_0)} + h_i(p)\,\mathrm{e}^{p(z-z_0)}], \end{aligned} \tag{C.36}$$

where $g_i(p)$ and $h_i(p)$ obey

$$\begin{aligned} &-p(g_i + h_i) - (-pg_i + ph_i) = \xi_i.\\ &\epsilon_{ij}(g_j\,\mathrm{e}^{pz_0} + h_j\,\mathrm{e}^{-pz_0}) + \frac{\alpha}{k}(g_i\,\mathrm{e}^{pz_0} - h_i\,\mathrm{e}^{-pz_0}) = 0. \end{aligned} \tag{C.37}$$

The solutions are

$$g_i = \frac{\mathrm{e}^{-2pz_0}}{2\left(1 + \frac{\alpha^2}{k^2}\right)p}\left[\left(1 - \frac{\alpha^2}{k^2}\right)\xi_i + 2\frac{\alpha}{k}\epsilon_{ij}\xi_j\right], \qquad h_i = -\frac{\xi_i}{2p}. \tag{C.38}$$

The nontrivial components of Green's function are thus given by

$$\begin{aligned} &G_{33} = 0,\\ &G_{ij} = \frac{1}{2p}\left[\mathrm{e}^{-p(z+z_0)}\frac{\left(1 - \frac{\alpha^2}{k^2}\right)\delta_{ij} + 2\frac{\alpha}{k}\epsilon_{ij}}{1 + \frac{\alpha^2}{k^2}}\right] - \frac{\delta_{ij}}{2p}[\theta(z - z_0)\,\mathrm{e}^{-p(z-z_0)} + \theta(z_0 - z)\,\mathrm{e}^{p(z-z_0)}]. \end{aligned} \tag{C.39}$$

In particular, we find the change of the bulk Green function due to the changing of the boundary condition,

$$G_{ij}^{(k)} - G_{ij}^{(\infty)} \equiv \Delta_{ij}(p, z, z_0) = \frac{\alpha}{kp}\frac{\epsilon_{ij} - \frac{\alpha}{k}\delta_{ij}}{1 + \frac{\alpha^2}{k^2}}\,\mathrm{e}^{-p(z+z_0)}. \tag{C.40}$$

The boundary to bulk propagator for $k = \infty$ can be obtained by taking $z_0 \to 0$ limit on $z_0^{-1}G^{(\infty)}$, giving

$$\begin{aligned} &K_{33} = 0,\\ &K_{ij} = -\mathrm{e}^{-pz}\delta_{ij}. \end{aligned} \tag{C.41}$$

We observe that $\Delta_{ij}$ factorizes into the product of two boundary to bulk propagators, $K(p,z)$ and $K(p,z_0)$, multiplied by

$$M_{ij}(p) = \frac{\alpha}{kp}\frac{\epsilon_{ij} - \frac{\alpha}{k}\delta_{ij}}{1+\frac{\alpha^2}{k^2}}. \tag{C.42}$$

This is reminiscent of the change of scalar propagator due to boundary conditions [51, 10]. So far we worked in the special case $p = p_3$. Restoring rotational invariance, (C.42) is replaced by

$$\begin{aligned} M_{ij}(p) &= \frac{\alpha}{k|p|}\frac{\epsilon_{ijk}\frac{p^k}{|p|} - \frac{\alpha}{k}\left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)}{1+\frac{\alpha^2}{k^2}} \\ &= \frac{\alpha/k}{1+\alpha^2/k^2}\epsilon_{ijk}\frac{p^k}{p^2} - \frac{\alpha^2/k^2}{1+\alpha^2/k^2}\frac{\delta_{ij}-\frac{p_ip_j}{p^2}}{|p|}. \end{aligned} \tag{C.43}$$

In the boundary CFT, the change of boundary condition amounts to coupling the $U(1)$ current $J^i$ to a boundary gauge field $A_i$ at Chern–Simons level $k$. $M_{ij}(p)$ is proportional to the two-point function of $A_i$ in the Lorentz gauge $\partial_j A^j = 0$. Namely,

$$\langle A_i(p)A_j(-q)\rangle = \frac{32}{\widetilde{N}}M_{ij}(p)(2\pi)^3\delta^3(p-q), \tag{C.44}$$

where $\widetilde{N}$ is the overall normalization factor in the two-point function of the current $J_i$,

$$\langle J_i(p)J_j(-q)\rangle = -\frac{\widetilde{N}|p|}{32}\left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)(2\pi)^3\delta^3(p-q). \tag{C.45}$$

Our convention is such that in the free theory $\widetilde{N}$ counts the total number of complex scalars and fermions. Note that here we are normalizing the current coupled to the Chern–Simons gauge field according to the convention for non-Abelian gauge group generators, $\mathrm{Tr}(t^at^b) = \frac{1}{2}\delta^{ab}$ for generators $t^a, t^b$ in the fundamental representation. This is also the normalization convention we use to define the Chern–Simons level $k$ (which differs by a factor of 2 from the natural convention for $U(1)$ gauge group).

To see this, note that the inverse of the matrix $M_{ij}$ in (C.42), restricted to directions transverse to $\vec{p} = p_3\hat{e}_3$, is

$$\left(M_\perp^{-1}\right)_{ij} = \frac{kp}{\alpha}\epsilon_{ij} + \delta_{ij}p. \tag{C.46}$$

After restoring rotational invariance, this is

$$\left(M_\perp^{-1}\right)_{ij} = \frac{k}{\alpha}\epsilon_{ijk}p^k + \left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)|p| \tag{C.47}$$

which for $\alpha = \frac{\pi}{8}\widetilde{N}$ precisely matches $32\widetilde{N}^{-1}$ times the kinetic term of the Chern–Simons gauge field plus the contribution to the self-energy of $A_i$ from $\langle J_i(p)J_j(-p)\rangle_{\mathrm{CFT}_\infty}$.

## Appendix D. Supersymmetric Chern–Simons vector models at large $N$

In this appendix, we review the Lagrangian of Chern–Simons vector models with various numbers of SUSYs and/or superpotentials. The scalar potentials and scalar–fermion coupling resulting from the coupling to auxiliary fields in the Chern–Simons gauge multiplet and superpotentials can be expressed in terms of bosonic or fermionic singlets under the $U(N)$ Chern–Simons gauge group as double trace or triple trace terms. These can be matched with the change of boundary conditions in the holographically dual Vasiliev theories in $AdS_4$, described in section 4.

*D.1. $\mathcal{N} = 2$ theory with $M$ $\square$ chiral multiplets*

The action of the $\mathcal{N} = 2$ pure Chern–Simons theory in Lorentzian signature is

$$S_{\rm CS}^{\mathcal{N}=2} = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A^3 - \bar{\chi}\chi + 2D\sigma\right), \tag{D.1}$$

where $\chi, \bar{\chi}$ and $D, \sigma$ are fermionic and bosonic auxiliary fields. The $M$ chiral multiplets in the fundamental representation couple to the gauge multiplet through the action

$$S_m = \int \sum_{i=1}^{M}[D_\mu\bar{\phi}^i D^\mu\phi_i + \bar{\psi}^i(\not{D} + \sigma)\psi_i + \bar{\phi}^{\rm i}(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F]. \tag{D.2}$$

We will focus on the matter coupling

$$\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi + 2D\sigma) + \int \sum_{i=1}^{M}[\bar{\psi}^i\sigma\psi_i + \bar{\phi}^{\rm i}(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F]. \tag{D.3}$$

Integrating out the auxiliary fields, we obtain the scalar potential and scalar–fermion coupling,

$$V = \frac{4\pi^2}{k^2}\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k\bar{\phi}^k\phi_i + \frac{4\pi}{k}\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j + \frac{2\pi}{k}\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i. \tag{D.4}$$

For the purpose of comparing with vector models of other numbers of SUSYs, it is useful to consider the $M = 2$ case. Let us define bosonic and fermionic gauge invariant bilinears in the matter fields,

$$\Phi_+^a = \bar{\phi}^i\phi_j(\sigma^a)^j{}_i, \qquad \Phi_-^a = \bar{\psi}^i\psi_j(\sigma^a)^j{}_i, \qquad \Psi^i{}_j = \bar{\phi}^i\psi_j, \tag{D.5}$$

where $\sigma^a = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)$. The non-supersymmetric theory with two flavors and without matter self-interaction $V$ would have had $SU(2)_b \times SU(2)_f$ flavor symmetry acting on the bosons and fermions separately. With respect to this symmetry, $\Phi_+^a$, $\Phi_-^a$ and $\Psi^i{}_j$ are in the representation $(\mathbf{1}\oplus\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1}\oplus\mathbf{3})$ and $(\mathbf{2}, \mathbf{2})$ respectively. Expressed in terms of the bosonic and fermionic singlets, $V$ can be written as

$$V = \frac{\pi^2}{2k^2}\Phi_+^a\Phi_+^b\Phi_+^c\,\mathrm{Tr}(\sigma^a\sigma^b\sigma^c) + \frac{2\pi}{k}\Phi_+^a\Phi_-^a + \frac{2\pi}{k}\bar{\Psi}^i{}_j\Psi^j{}_i. \tag{D.6}$$

Note that the (fermion singlet)$^2$ terms is invariant under $SU(2)_b \times SU(2)_f$, whereas the (bosonic singlet)$^2$ term and the scalar potential explicitly break $SU(2)_b \times SU(2)_f$ to the diagonal flavor $SU(2)$.

Indeed, the boundary conditions of the conjectured holographic dual described in section 4.3.1 are such that the fermionic boundary condition (characterized by $\gamma$) is invariant under the $SO(4) \sim SU(2)_b \times SU(2)_f$ that rotates the four Grassmannian variables of supersymmetric Vasiliev theory, while the scalar boundary condition only preserve an $SU(2) \sim SO(3)$ subgroup.

*D.2. $\mathcal{N} = 1$ theory with $M$ $\square$ chiral multiplets*

The $\mathcal{N} = 2$ theory in the previous section admits a one-parameter family of exactly marginal deformations that preserves $\mathcal{N} = 1$ SUSY. The matter coupling of this $\mathcal{N} = 1$ theory is given by

$$\begin{aligned} V = {}& \frac{4\pi^2\omega^2}{k^2}\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k\bar{\phi}^k\phi_i + \frac{2\pi(1+\omega)}{k}\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j + \frac{2\pi\omega}{k}\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i \\ & + \frac{\pi(\omega-1)}{k}(\bar{\psi}^i\phi_j\bar{\psi}^j\phi_i + \bar{\phi}^i\psi_j\bar{\phi}^j\psi_i), \end{aligned} \tag{D.7}$$

where $\omega$ is a real deformation parameter. The $\mathcal{N} = 2$ theory is given by $\omega = 1$.

*D.3. The $\mathcal{N} = 2$ theory with $M\,\square$ chiral multiplets and $M\,\overline{\square}$ chiral multiplets*

Now we turn to the $\mathcal{N} = 2$ Chern–Simons vector model with an equal number $M$ of fundamental and anti-fundamental chiral multiplets. This model differs from the $\mathcal{N} = 2$ theory with $2M$ fundamental chiral multiplets through the scalar–fermion coupling and scalar potential only. The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi + 2D\sigma) + \sum_{i=1}^{M}[\bar{\psi}^i\sigma\psi_i + \bar{\phi}^{\mathrm{i}}(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F]$$
$$+ \sum_{i=1}^{M}[-\tilde{\psi}^i\sigma\bar{\tilde{\psi}}_i + \tilde{\phi}^{\mathrm{i}}(\sigma^2 + D)\bar{\tilde{\phi}}_i - \tilde{\psi}^i\chi\bar{\tilde{\phi}}_i - \tilde{\phi}^i\bar{\chi}\bar{\tilde{\psi}}_i - \tilde{F}\bar{\tilde{F}}]. \tag{D.8}$$

Integrating out the auxiliary fields, we obtain

$$V_d = \frac{4\pi^2}{k^2}(\bar{\phi}^k\phi_i\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k - \bar{\phi}^k\bar{\tilde{\phi}}_i\tilde{\phi}^i\bar{\tilde{\phi}}_j\tilde{\phi}^j\phi_k - \bar{\phi}^k\bar{\tilde{\phi}}_i\tilde{\phi}^i\phi_j\bar{\phi}^j\phi_k + \tilde{\phi}^k\bar{\tilde{\phi}}_i\tilde{\phi}^i\bar{\tilde{\phi}}_j\tilde{\phi}^j\bar{\tilde{\phi}}_k)$$
$$+ \frac{4\pi}{k}(\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j - \bar{\phi}^j\bar{\tilde{\psi}}_i\tilde{\phi}^i\psi_j - \tilde{\psi}^j\phi_i\bar{\psi}^i\bar{\tilde{\phi}}_j + \tilde{\psi}^i\bar{\tilde{\psi}}_j\tilde{\phi}^j\bar{\tilde{\phi}}_i)$$
$$+ \frac{2\pi}{k}(\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i - \bar{\psi}^i\bar{\tilde{\phi}}_j\tilde{\phi}^j\psi_i - \tilde{\psi}^i\phi_j\bar{\phi}^j\bar{\tilde{\psi}}_i + \tilde{\psi}^i\bar{\tilde{\phi}}_j\tilde{\phi}^j\bar{\tilde{\psi}}_i). \tag{D.9}$$

*D.4. The $\mathcal{N} = 3$ theory with $M$ hypermultiplets*

The $\mathcal{N} = 3$ Chern–Simons vector model with $M$ hypermultiplets can be obtained from the $\mathcal{N} = 2$ theory described in the previous subsection by adding the superpotential [52, 29]

$$W = -\frac{k}{8\pi}\,\mathrm{tr}\,\varphi^2 + \tilde{\Phi}^i\varphi\Phi_i \tag{D.10}$$

where $\varphi$ is an auxiliary $\mathcal{N} = 2$ chiral superfield. Integrating out $\varphi$, we obtain a quartic superpotential

$$W = \frac{2\pi}{k}(\tilde{\Phi}^i\Phi_j)(\tilde{\Phi}^j\Phi_i). \tag{D.11}$$

After integrating over the superspace, we obtain

$$\int \mathrm{d}^2\theta W + \mathrm{c.c.} = \frac{2\pi}{k}[2\tilde{\phi}^i\phi_j(\tilde{\phi}^jF_i + \tilde{F}^j\phi_i + \tilde{\psi}^j\psi_i) + (\tilde{\psi}^i\phi_j + \tilde{\phi}^i\psi_j)(\tilde{\psi}^j\phi_i + \tilde{\phi}^j\psi_i) + \mathrm{c.c.}]. \tag{D.12}$$

Integrating out the auxiliary fields $F, \tilde{F}$, the $W$-term potential is

$$V_w = \frac{2\pi}{k}[2(\tilde{\phi}^i\phi_j)(\tilde{\psi}^j\psi_i) + (\tilde{\psi}^i\phi_j + \tilde{\phi}^i\psi_j)(\tilde{\psi}^j\phi_i + \tilde{\phi}^j\psi_i) + \mathrm{c.c.}]$$
$$+ \frac{16\pi^2}{k^2}(\bar{\phi}^j\bar{\tilde{\phi}}_i)(\tilde{\phi}^i\phi_k)(\tilde{\phi}^k\bar{\tilde{\phi}}_j) + \frac{16\pi^2}{k^2}(\bar{\phi}^j\bar{\tilde{\phi}}_i)(\bar{\phi}^i\phi_k)(\tilde{\phi}^k\phi_j). \tag{D.13}$$

The total potential is given by the $D$-term plus $W$-term potentials:

$$V = V_d + V_w. \tag{D.14}$$

To make the $SO(3)$ $R$-symmetry manifest, we rewrite the potential in terms of the $SO(3)$ doublets:

$$(\phi_i^A) = \begin{pmatrix}\phi_i\\ \bar{\tilde{\phi}}_i\end{pmatrix}, \qquad (\psi_{A,i}) = \begin{pmatrix}\psi_i\\ \bar{\tilde{\psi}}_i\end{pmatrix}. \tag{D.15}$$

The $D$-term and $W$-term potentials are

$$V_d = \frac{4\pi^2}{k^2}[(\bar{\phi}_1\phi^1)(\bar{\phi}_1\phi^1)(\bar{\phi}_1\phi^1) - (\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^2)(\bar{\phi}_2\phi^1)$$
$$- (\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^1)(\bar{\phi}_1\phi^1) + (\bar{\phi}_2\phi^2)(\bar{\phi}_2\phi^2)(\bar{\phi}_2\phi^2)]$$
$$+ \frac{4\pi}{k}[(\bar{\phi}_1\phi^1)(\bar{\psi}^1\psi_1) - (\bar{\phi}_1\psi_2)(\bar{\phi}_2\psi_1) - (\bar{\psi}^2\phi^1)(\bar{\psi}^1\phi^2) + (\bar{\psi}^2\psi_2)(\bar{\phi}_2\phi^2)]$$
$$+ \frac{2\pi}{k}[(\bar{\psi}^1\phi^1)(\bar{\phi}_1\psi_1) - (\bar{\psi}^1\phi^2)(\bar{\phi}_2\psi_1) - (\bar{\psi}^2\phi^1)(\bar{\phi}_1\psi_2) + (\bar{\psi}^2\phi^2)(\bar{\phi}_2\psi_2)], \tag{D.16}$$

and

$$V_w = \frac{2\pi}{k}[2(\bar{\phi}_2\phi^1)(\bar{\psi}^2\psi_1) + (\bar{\psi}^2\phi^1 + \bar{\phi}_2\psi_1)(\bar{\psi}^2\phi^1 + \bar{\phi}_2\psi_1) + \text{c.c.}]$$
$$+ \frac{16\pi^2}{k^2}(\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^1)(\bar{\phi}_2\phi^2) + \frac{16\pi^2}{k^2}(\bar{\phi}_1\phi^2)(\bar{\phi}_1\phi^1)(\bar{\phi}_2\phi^1). \tag{D.17}$$

We have also suppressed the flavor indices. The total potential can be written in a $SO(3)$ $R$-symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{D.18}$$

where $V_1$ contains the double trace operator of the form (bosonic singlet)$^2$,

$$V_1 = \frac{4\pi}{k}(\bar{\phi}_A\phi^B)(\bar{\psi}^A\psi_B), \tag{D.19}$$

$V_2$ is the scalar potential in the form of a triple trace term,

$$V_2 = \frac{16\pi^2}{3k^2}(\bar{\phi}_A\phi^B)(\bar{\phi}_B\phi^C)(\bar{\phi}_C\phi^A) - \frac{4\pi^2}{3k^2}(\bar{\phi}_B\phi^C)(\bar{\phi}_A\phi^B)(\bar{\phi}_C\phi^A), \tag{D.20}$$

$V_3$ is the double trace term of the form (fermionic singlet)$^2$,

$$V_3 = -\frac{2\pi}{k}(\bar{\psi}^A\phi_B)(\bar{\phi}^B\psi_A) + \frac{4\pi}{k}(\bar{\psi}^A\phi_A)(\bar{\phi}^B\psi_B)$$
$$+ \frac{2\pi}{k}(\bar{\psi}^A\phi_A)(\bar{\psi}^B\phi_B) + \frac{2\pi}{k}(\bar{\phi}^A\psi_A)(\bar{\phi}^B\psi_B), \tag{D.21}$$

where $\phi_A$, $\psi^A$ are defined as

$$\phi_A = \phi^B\epsilon_{BA}, \psi^A = \epsilon^{AB}\psi_B, \tag{D.22}$$

and $\epsilon^{AB}$, $\epsilon_{AB}$ are antisymmetric tensors with $\epsilon_{12} = \epsilon^{12} = 1$.

For reference in main text we will record the double trace part of the potential in $SO(3)$ vector notation. Let us define

$$\begin{aligned}
\Phi^a_+ &= \bar{\phi}_A\phi^B(\sigma^a)^A_B \quad \Leftrightarrow \quad \bar{\phi}_A\phi^B = \tfrac{1}{2}\Phi^a_+(\bar{\sigma}^a)^B_A\\
\Phi^a_- &= \bar{\psi}^A\psi_B(\sigma^a)^B_A \quad \Leftrightarrow \quad \bar{\psi}^A\psi_B = \tfrac{1}{2}\Phi^a_-(\bar{\sigma}^a)^A_B\\
\Psi^a &= \bar{\phi}_A\psi_B(\epsilon\sigma^a)^{AB} \quad \Leftrightarrow \quad \bar{\phi}_A\psi_B = -\tfrac{1}{2}\Psi^a(\sigma^a\epsilon)_{AB}\\
\bar{\Psi}^a &= -\bar{\psi}^A\phi^B(\sigma^a\epsilon)_{AB} \quad \Leftrightarrow \quad \bar{\psi}^A\phi^B = \tfrac{1}{2}\bar{\Psi}^a(\epsilon\bar{\sigma}^a)^{AB}
\end{aligned} \tag{D.23}$$

where

$$(\sigma^a)^B_A = (\sigma^i, \mathrm{i}I)^B_A, \qquad (\bar{\sigma}^a)^B_A = (\epsilon(\sigma^a)^T\epsilon)^B_A = (\sigma^a, -\mathrm{i}I)^B_A, \epsilon^{12} = \epsilon_{12} = 1.$$

Here $\sigma^i$ are Pauli sigma matrices. The $a, b$ indices runs over 1, 2, 3, 0. $A, B$ runs over 1, 2. $\Psi^a$ and $\bar{\Psi}^a$ transform under the as vectors of $SO(4)$ which under $SO(3)$ transform as singlet ($a = 0$) and a vector ($a = 1, 2, 3$) while $\phi^A$, $\psi_A$ transform as doublets of $SU(2)$:

$$\begin{aligned} V_1 &= \frac{2\pi}{k}\Phi_+^a\Phi_-^b\eta_{ab}, \\ V_3 &= \frac{2\pi}{k}\left(\frac{1}{2}\bar{\Psi}^a\Psi^b\delta^{ab} - 2\bar{\Psi}^0\Psi^0 - \bar{\Psi}^0\bar{\Psi}^0 - \Psi^0\Psi^0\right). \end{aligned} \tag{D.24}$$

*D.5. A family of $\mathcal{N} = 2$ theories with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet*

If we deformed the superpotential in the above subsection as

$$W = \frac{2\pi\omega}{k}(\tilde{\Phi}^i\Phi_j)(\tilde{\Phi}^j\Phi_i), \tag{D.25}$$

the $\mathcal{N} = 3$ SUSY is broken to $\mathcal{N} = 2$. In this case, the potential is

$$V = V_1 + V_2 + V_3, \tag{D.26}$$

where $V_1$ contains the double trace operator of the form (bosonic singlet)$^2$,

$$V_1 = \frac{4\pi}{k}[(\bar{\phi}_1\phi^1)(\bar{\psi}^1\psi_1) + (\bar{\phi}_2\phi^2)(\bar{\psi}^2\psi_2) + \omega(\bar{\phi}_2\phi^1)(\bar{\psi}^2\psi_1) + \omega(\bar{\phi}_1\phi^2)(\bar{\psi}^1\psi_2)], \tag{D.27}$$

$V_2$ is the scalar potential in the form of a triple trace term,

$$\begin{aligned} V_2 = \frac{4\pi^2}{k^2}&[(\bar{\phi}_1\phi^1)(\bar{\phi}_1\phi^1)(\bar{\phi}_1\phi^1) - (\bar{\phi}_2\phi^1)(\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^2) \\ &- (\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^1)(\bar{\phi}_1\phi^1) + (\bar{\phi}_2\phi^2)(\bar{\phi}_2\phi^2)(\bar{\phi}_2\phi^2)] \\ &+ \frac{16\pi^2\omega}{k^2}(\bar{\phi}_1\phi^2)(\bar{\phi}_2\phi^1)(\bar{\phi}_2\phi^2) + \frac{16\pi^2\omega}{k^2}(\bar{\phi}_1\phi^2)(\bar{\phi}_1\phi^1)(\bar{\phi}_2\phi^1). \end{aligned} \tag{D.28}$$

$V_3$ is the double trace term of the form (fermionic singlet)$^2$,

$$\begin{aligned} V_3 = \frac{2\pi}{k}&[(\bar{\psi}^1\phi^1)(\bar{\phi}_1\psi_1) - (\bar{\psi}^1\phi^2)(\bar{\phi}_2\psi_1) - (\bar{\psi}^2\phi^1)(\bar{\phi}_1\psi_2) + (\bar{\psi}^2\phi^2)(\bar{\phi}_2\psi_2)] \\ &+ \frac{4\pi}{k}[-(\bar{\phi}_1\psi_2)(\bar{\phi}_2\psi_1) - (\bar{\psi}^2\phi^1)(\bar{\psi}^1\phi^2)] \\ &+ \frac{2\pi\omega}{k}[(\bar{\psi}^2\phi^1)(\bar{\psi}^2\phi^1) + 2(\bar{\phi}_2\psi_1)(\bar{\psi}^2\phi^1) + (\bar{\phi}_2\psi_1)(\bar{\phi}_2\psi_1) \\ &+ (\bar{\phi}_1\psi_2)(\bar{\phi}_1\psi_2) + 2(\bar{\psi}^1\phi^2)(\bar{\phi}_1\psi_2) + (\bar{\psi}^1\phi^2)(\bar{\psi}^1\phi^2)]. \end{aligned} \tag{D.29}$$

*D.6. The $\mathcal{N} = 4$ theory with one hypermultiplet*

As shown by [30], $\mathcal{N} = 3$ $U(N)_k$ Chern–Simons vector model with $M$ hypermultiplets can be deformed to an $\mathcal{N} = 4$ quiver type Chern–Simons matter theory by gauging (a subgroup of) the flavor group with another $\mathcal{N} = 3$ Chern–Simons gauge multiplet, at the opposite level $-k$. Here we will focus on the case where the entire $U(M)$ is gauged, so that the resulting $\mathcal{N} = 4$ theory has $U(N)_k \times U(M)_{-k}$ Chern–Simons gauge group and a single bifundamental hypermultiplet. This $\mathcal{N} = 4$ theory will still be referred to as a vector model, as we will be thinking of the 't Hooft limit of taking $N, k$ large and $M$ kept finite. As we have seen, turning on the finite Chern–Simons level for the flavor group $U(M)$ amounts to simply changing the boundary condition on the $U(M)$ vector gauge fields in the bulk Vasiliev theory.

The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\begin{aligned}\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi+2D\sigma)-\frac{k}{4\pi}\mathrm{Tr}(-\bar{\hat{\chi}}\hat{\chi}+2\hat{D}\hat{\sigma})&\\ +[\bar{\psi}\sigma\psi+\bar{\phi}(\sigma^2-D)\phi+\bar{\psi}\bar{\chi}\phi+\bar{\phi}\chi\psi-\hat{\sigma}\bar{\psi}\psi&\\ +(\hat{\sigma}^2+\hat{D})\bar{\phi}\phi-\bar{\psi}\phi\bar{\hat{\chi}}-\hat{\chi}\bar{\phi}\psi-2\hat{\sigma}\bar{\phi}\sigma\phi-\bar{F}F]&\\ +[-\tilde{\psi}\sigma\bar{\tilde{\psi}}+\tilde{\phi}(\sigma^2+D)\bar{\tilde{\phi}}-\tilde{\psi}\chi\bar{\tilde{\phi}}-\tilde{\phi}\bar{\chi}\bar{\tilde{\psi}}+\hat{\sigma}\tilde{\psi}\bar{\tilde{\psi}}&\\ +(\hat{\sigma}^2-\hat{D})\tilde{\phi}\bar{\tilde{\phi}}+\bar{\hat{\chi}}\tilde{\phi}\bar{\tilde{\psi}}+\tilde{\psi}\bar{\tilde{\phi}}\hat{\chi}-2\hat{\sigma}\tilde{\phi}\sigma\bar{\tilde{\phi}}-\tilde{F}\bar{\tilde{F}}],&\end{aligned}\tag{D.30}$$

where we suppressed the both $SU(N)$ and $SU(M)$ indices. Integrating out the auxiliary fields, we obtain the potential:

$$\begin{aligned}V=\frac{2\pi}{k}\bar{\phi}_A\phi^A\bar{\psi}^B\psi_B+\frac{4\pi^2}{3k^2}(\bar{\phi}_A\phi^B\bar{\phi}_B\phi^C\bar{\phi}_C\phi^A+\bar{\phi}_A\phi^A\bar{\phi}_B\phi^B\bar{\phi}_C\phi^C-2\bar{\phi}_B\phi^C\bar{\phi}_A\phi^B\bar{\phi}_C\phi^A)&\\ +\frac{2\pi}{k}(-\bar{\psi}^A\phi^B\bar{\phi}_B\psi_A+\bar{\phi}^A\psi^B\bar{\phi}_A\psi_B+\bar{\psi}^A\phi^B\bar{\psi}_A\phi_B).&\end{aligned}\tag{D.31}$$

The complex scalar $\phi^A$ and the fermion $\psi_A$ transform as $(2,1)$ and $(1,2)$ under the $SO(4)=SU(2)\times SU(2)$ $R$-symmetry. The potential (D.31) is manifestly invariant under the R-symmetry.

For reference to main text we now record the double trace part of this potential in $SO(4)$ vector notation. Using the definitions (D.23), the (scalar singlet)$^2$ part($V_1$) and (fermion singlet)$^2$ part($V_3$) are given by

$$\begin{aligned}V_1&=-\frac{2\pi}{k}\Phi^0_+\Phi^0_-,\\ V_2&=-\frac{\pi}{k}(\bar{\Psi}^a\Psi^a+\bar{\Psi}^a\bar{\Psi}^a+\Psi^a\Psi^a).\end{aligned}\tag{D.32}$$

### *D.7. $\mathcal{N}=3$ $U(N)_{k_1}\times U(M)_{k_2}$ theories with one hypermultiplet*

The $\mathcal{N}=4$ theory in the previous section sits in a discrete one parameter family of $\mathcal{N}=3$ $U(N)_{k_1}\times U(M)_{k_2}$ theories with one hypermultiplet. The potential can be written in an $SO(3)$ $R$-symmetry manifest way:

$$V=V_1+V_2+V_3,\tag{D.33}$$

where $V_1$ contains the double trace operator of the form (bosonic singlet)$^2$,

$$V_1=\frac{4\pi}{k_1}\bar{\phi}_A\phi^B\bar{\psi}^A\psi_B+\frac{2\pi}{k_2}[\bar{\phi}_A\phi^A\bar{\psi}_B\psi^B+2\bar{\phi}_A\phi^B\bar{\psi}^A\psi_B],\tag{D.34}$$

$V_2$ is the scalar potential in the form of triple trace term. $V_3$ is the double trace term of the form (fermionic singlet)$^2$,

$$\begin{aligned}V_3=\frac{2\pi}{k_1}[-\bar{\psi}^A\phi_B\bar{\phi}^B\psi_A+2\bar{\psi}^A\phi_A\bar{\phi}^B\psi_B+\bar{\psi}^A\phi_A\bar{\psi}^B\phi_B+\bar{\phi}^A\psi_A\bar{\phi}^B\psi_B]&\\ +\frac{2\pi}{k_2}[2\bar{\psi}^A\phi^B\bar{\phi}_A\psi_B+\bar{\psi}^A\phi_B\bar{\psi}^B\phi_A+\bar{\phi}_A\psi^B\bar{\phi}_B\psi^A].&\end{aligned}\tag{D.35}$$

In the notation defined in (D.23) $V_1$ and $V_3$ becomes

$$V_1 = \frac{2\pi}{k_1}\Phi^a_+\Phi^b_-\eta_{ab} + \frac{2\pi}{k_2}\left(\Phi^0_+\Phi^0_- + \Phi^a_+\Phi^b_-\eta_{ab}\right),$$

$$V_3 = \frac{2\pi}{k_1}\left(\frac{1}{2}\bar\Psi^a\Psi^b\delta^{ab} - 2\bar\Psi^0\Psi^0 - \bar\Psi^0\bar\Psi^0 - \Psi^0\Psi^0\right) + \frac{2\pi}{k_2}\left(\bar\Psi^a\Psi^b\eta^{ab} + \frac{1}{2}\bar\Psi^a\bar\Psi^b\eta_{ab} + \frac{1}{2}\Psi^a\Psi^b\eta^{ab}\right). \tag{D.36}$$

### D.8. The $\mathcal{N} = 6$ theory

The above $\mathcal{N} = 4$ theory can be generalized to a quiver $\mathcal{N} = 3$ theory with $\tilde n$ hypermultiplets by starting with the $\mathcal{N} = 3$ $U(N)_k$ Chern–Simons vector model with $\tilde nM$ hypermultiplets and only gauging the $U(M)$ subgroup, of the $U(\tilde nM)$ flavor group, at level $-k$ with another $\mathcal{N} = 3$ Chern–Simons gauge multiplet. The resulting theory has $SU(\tilde n)$ flavor symmetry. For generic value of $\tilde n$, the theory has $\mathcal{N} = 3$ sypersymmetry, but for $\tilde n = 1, 2$, the theory exhibits $\mathcal{N} = 4, 6$ sypersymmetry, respectively. Let us focus on the $\tilde n = 2$ case. The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\begin{aligned}
&\frac{k}{4\pi}\mathrm{Tr}(-\bar\chi\chi + 2D\sigma) - \frac{k}{4\pi}\mathrm{Tr}(-\bar{\hat\chi}\hat\chi + 2\hat D\hat\sigma)\\
&\quad + [\bar\psi_a\sigma\psi^a + \bar\phi_a(\sigma^2 - D)\phi^a + \bar\psi_a\bar\chi\phi^a + \bar\phi_a\chi\psi^a - \hat\sigma\bar\psi_a\psi^a\\
&\quad + (\hat\sigma^2 + \hat D)\bar\phi_a\phi^a - \bar\psi_a\phi^a\bar{\hat\chi} - \hat\chi\bar\phi_a\psi^a - 2\hat\sigma\bar\phi_a\sigma\phi^a - \bar F_aF^a]\\
&\quad + [-\tilde\psi_{\dot a}\sigma\bar{\tilde\psi}^{\dot a} + \tilde\phi_{\dot a}(\sigma^2 + D)\bar{\tilde\phi}^{\dot a} - \tilde\psi_{\dot a}\chi\bar{\tilde\phi}^{\dot a} - \tilde\phi_{\dot a}\bar\chi\bar{\tilde\psi}^{\dot a} + \hat\sigma\tilde\psi_{\dot a}\bar{\tilde\psi}^{\dot a}\\
&\quad + (\hat\sigma^2 - \hat D)\tilde\phi_{\dot a}\bar{\tilde\phi}^{\dot a} + \bar{\hat\chi}\tilde\phi_{\dot a}\bar{\tilde\psi}^{\dot a} + \tilde\psi_{\dot a}\bar{\tilde\phi}^{\dot a}\hat\chi - 2\hat\sigma\tilde\phi_{\dot a}\sigma\bar{\tilde\phi}^{\dot a} - \tilde F_{\dot a}\bar{\tilde F}^{\dot a}],
\end{aligned} \tag{D.37}$$

where $a, \dot a = 1, 2$ are the $SU(2)\times SU(2)$ indices. There is also an superpotential

$$W = -\frac{2\pi}{k}\,\mathrm{Tr}(\tilde\Phi^{\dot a}\Phi^b\tilde\Phi_{\dot a}\Phi_b). \tag{D.38}$$

After integrating over the superspace, we obtain

$$\begin{aligned}
&\int \mathrm{d}^2\theta W + \mathrm{c.c.}\\
&= -\frac{2\pi}{k}[2\tilde\phi^{\dot a}\phi^b(\tilde\phi_{\dot a}F_b + \tilde F_{\dot a}\phi_b + \tilde\psi_{\dot a}\psi_b) + (\tilde\psi^{\dot a}\phi^b + \tilde\phi^{\dot a}\psi^b)(\tilde\psi_{\dot a}\phi_b + \tilde\phi_{\dot a}\psi_b) + \mathrm{c.c.}].
\end{aligned} \tag{D.39}$$

After integrating out all the auxiliary fields, the resulting potential can be written in a $SO(6)$ R-symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{D.40}$$

where $V_1$ contains the double trace operator of the form (bosonic singlet)$^2$,

$$\begin{aligned}
V_1 &= -\frac{2\pi}{k}(\bar\phi_{1a}\phi^{1a}\bar\psi^{2\dot b}\psi_{2\dot b} + \bar\phi_{1a}\phi^{1a}\bar\psi^{1b}\psi_{1b} + \bar\phi_{2\dot a}\phi^{2\dot a}\bar\psi^{2\dot b}\psi_{2\dot b} + \bar\phi_{2\dot a}\phi^{2\dot a}\bar\psi^{1b}\psi_{1b})\\
&\quad + \frac{4\pi}{k}(\bar\phi_{2\dot a}\phi^{1b}\bar\psi^{2\dot a}\psi_{1b} + \bar\phi_{1b}\phi^{2\dot a}\bar\psi^{1b}\psi_{2\dot a} + \bar\phi_{1a}\phi^{1b}\bar\psi^{1a}\psi_{1b} + \bar\phi_{2\dot a}\phi^{2\dot b}\bar\psi^{2\dot a}\psi_{2\dot b})\\
&= -\frac{2\pi}{k}\bar\phi_A\phi^A\bar\psi^B\psi_B + \frac{4\pi}{k}\bar\phi_A\phi^B\bar\psi^A\psi_B
\end{aligned} \tag{D.41}$$

where we have rewrite the potential in terms of the $SO(3)$ doublets (D.15), and $A, B = (11, 12, 21, 22)$ are the $SO(6)$ spinor indices. $V_2$ is the scalar potential in the form of triple trace term. $V_3$ is the double trace term of the form (fermionic singlet)$^2$,

$$V_3 = \frac{2\pi}{k}(\bar{\psi}^A\phi^B\bar{\phi}_B\psi_A - 2\bar{\psi}^A\phi^B\bar{\phi}_A\psi_B) + \frac{2\pi}{k}(\epsilon_{ABCD}\bar{\psi}^A\phi^B\bar{\psi}^C\phi^D + \epsilon^{ABCD}\bar{\phi}_A\psi_B\bar{\phi}_C\psi_D) \tag{D.42}$$

where $\epsilon_{11,12,21,22} = \epsilon^{11,12,21,22} = 1$.

### *D.9. $\mathcal{N} = 3$ $U(N)_{k_1} \times U(M)_{k_2}$ theories with two hypermultiplets*

The $\mathcal{N} = 6$ theory in the previous section sits in a discrete one parameter family of $\mathcal{N} = 3$ $U(N)_{k_1} \times U(M)_{k_2}$ theories with two hypermultiplets. The superpotential of these theories are

$$W = \frac{2\pi}{k_1}\mathrm{Tr}(\tilde{\Phi}^a\Phi_b\tilde{\Phi}^b\Phi_a) + \frac{2\pi}{k_2}\mathrm{Tr}(\tilde{\Phi}^a\Phi_a\tilde{\Phi}^b\Phi_b), \tag{D.43}$$

where $a, b = 1, 2$ are the $SU(2)$ flavor indices. The potential can be written in an $SO(3)$ $R$-symmetry and $SU(2)$ flavor symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{D.44}$$

where $V_1$ contains the double trace operator of the form (bosonic singlet)$^2$,

$$V_1 = \frac{4\pi}{k_1}\bar{\phi}_{Aa}\phi^{Bb}\bar{\psi}^A_b\psi^a_B + \frac{2\pi}{k_2}(\bar{\phi}_{Aa}\phi^{Aa}\bar{\psi}_{Bb}\psi^{Bb} + 2\bar{\phi}_{Aa}\phi^{Ba}\bar{\psi}^A_b\psi^b_B) \tag{D.45}$$

where we have rewrite the potential in terms of the $SO(3)$ doublets (D.15), and $A, B = 1, 2$ are the $SO(3)_R$ spinor indices. $V_2$ is the scalar potential in the form of triple trace term. $V_3$ is the double trace term of the form (fermionic singlet)$^2$,

$$\begin{aligned} V_3 = {} & \frac{2\pi}{k_1}(\bar{\psi}^{Aa}\phi^{Bb}\bar{\phi}_{Bb}\psi_{Aa} - 2\bar{\psi}^{Aa}\phi^{Bb}\bar{\phi}_{Ab}\psi_{Ba}) \\ & + \frac{2\pi}{k_1}\epsilon_{AB}\epsilon_{CD}\bar{\psi}^A_a\phi^{Bb}\bar{\psi}^C_b\phi^{Da} + \frac{2\pi}{k_1}\epsilon^{AB}\epsilon^{CD}\bar{\phi}_{Aa}\psi^b_B\bar{\phi}_{Cb}\psi^a_D \\ & + \frac{4\pi}{k_2}\bar{\psi}^A_a\phi^{Ba}\bar{\phi}_{Ab}\psi^b_B + \frac{2\pi}{k_2}\epsilon_{AD}\epsilon_{CB}\bar{\psi}^A_a\phi^{Ba}\bar{\psi}^C_b\phi^{Db} + \frac{2\pi}{k_2}\epsilon^{AD}\epsilon^{CB}\bar{\phi}^a_A\psi_{aB}\bar{\phi}^b_C\psi_{Db}. \end{aligned} \tag{D.46}$$

Now we record the double trace parts of the potential in vector notation of $SO(3)_R \times SU(2)_{\mathrm{flavor}}$ symmetry. Let us define

$$\begin{aligned} \Phi^{Ii}_+ &= \bar{\phi}_{Aa}\phi^{Bb}(\sigma^I)^A{}_B(\sigma^i)^a{}_b &\Leftrightarrow\quad \bar{\phi}_{Aa}\phi^{Bb} &= \tfrac{1}{4}\Phi^{Ii}_+(\sigma^I)^B{}_A(\sigma^i)^b{}_a \\ \Phi^{Ii}_- &= \bar{\psi}^A_a\psi^b_B(\sigma^I)^B{}_A(\sigma^i)^a{}_b &\Leftrightarrow\quad \bar{\psi}^A_a\psi^b_B &= \tfrac{1}{4}\Phi^{Ii}_-(\sigma^I)^A{}_B(\sigma^i)^b{}_a \\ \Psi^{Ii} &= \bar{\phi}_{Aa}\psi^b_B(\sigma^I\epsilon)^{AB}(\sigma^i)^a{}_b &\Leftrightarrow\quad \bar{\phi}_{Aa}\psi^b_B &= -\tfrac{1}{4}\Psi^{Ii}(\epsilon\sigma^I)_{AB}(\bar{\sigma}^i)^b{}_a \\ \bar{\Psi}^{Ii} &= -\bar{\psi}^A_a\phi^{Bb}(\epsilon\bar{\sigma}^I)_{AB}(\bar{\sigma}^i)^a{}_b &\Leftrightarrow\quad \bar{\psi}^A_a\phi^{Bb} &= -\tfrac{1}{4}\bar{\Psi}^{Ii}(\bar{\sigma}^I\epsilon)_{AB}(\sigma^i)^b{}_a. \end{aligned} \tag{D.47}$$

Here both set of indices $I, J$ as well $i, j$ run over 1, 2, 3, 0. $I, J$ are the vector indices of $SO(3)_R$ while $i, j$ are vector indices of $SU(2)_{\mathrm{flavor}}$. The 0 component corresponds to the singlet while 1, 2, 3 represents the vector part. In this notation the double trace potential part of the becomes

$$\begin{aligned} V_1 &= \frac{\pi}{k_1}\Phi^{Ii}_+\Phi^{Jj}_-\eta^{IJ}\eta_{ij} - \frac{2\pi}{k_2}\Phi^{I0}_+\Phi^{J0}_-\eta^{IJ}, \\ V_3 &= \frac{2\pi}{k_1}\left(-\frac{1}{4}\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij} + \frac{1}{2}\bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij} + \frac{1}{2}\left(\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij} + \Psi^{0i}\Psi^{0j}\eta_{ij}\right)\right) \\ &\quad + \frac{2\pi}{k_2}\left(\bar{\Psi}^{I0}\Psi^{J0}\eta^{IJ} + \frac{1}{2}\bar{\Psi}^{I0}\bar{\Psi}^{J0}\eta^{IJ} + \frac{1}{2}\Psi^{I0}\Psi^{J0}\eta^{IJ}\right). \end{aligned} \tag{D.48}$$

The double potentials for $\mathcal{N} = 6$ theory is obtained from (D.48) on setting $k_2 = -k_1 = -k$.

## Appendix E. Argument for a Fermionic double trace shift

In this appendix compare the boundary conditions and Lagrangian for the fixed line of $\mathcal{N} = 1$ theories to argue for the effective shift of fermionic boundary conditions induced by the Chern–Simons term.

Let us use the notation $\bar{\phi}\psi = \Psi$ and $\bar{\psi}\phi = \bar{\Psi}$ for field theory single trace operators. We know that a double trace deformation proportional to $(\Psi + \bar{\Psi})^2$ is dual to fermion boundary condition (4.8) with $\alpha \propto P_{\psi_1}$. On the other hand the double trace deformation $(\mathrm{i}\Psi - \mathrm{i}\bar{\Psi})^2$ is dual to the fermion boundary condition with $\alpha \propto P_{\psi_2}$. Now in the zero potential theory ($w = -1$) the relevant terms in (D.7) are

$$-\frac{2\pi}{k}(\Psi\Psi + \bar{\Psi}\bar{\Psi} + \Psi\bar{\Psi}),$$

while $\alpha = \theta_0 P_{\psi_2}$. At the $\mathcal{N} = 2$ point, on the other hand, the fermion double trace term is

$$+\frac{2\pi}{k}\Psi\bar{\Psi}$$

while $\alpha = \theta_0(P_{\psi_1} + P_{\psi_2})$. Subtracting these two data points we conclude that the double trace deformation by

$$\frac{2\pi}{k}(\Psi + \bar{\Psi})^2$$

is dual to a boundary condition deformation with $\alpha = \theta_0 P_{\psi_1}$. By symmetry it must also be that the double trace deformation by

$$-\frac{2\pi}{k}(\Psi - \bar{\Psi})^2$$

is dual to a boundary condition deformation with $\alpha = \theta_0 P_{\psi_2}$. Adding these together, it follows that a double trace deformation by

$$\frac{8\pi}{k}\bar{\Psi}\Psi$$

is dual to the boundary condition deformation with $\alpha = \theta_0(P_{\psi_1} + P_{\psi_2})$. But the $\mathcal{N} = 2$ theory with this boundary condition has a double trace potential equal only to

$$\frac{2\pi}{k}\bar{\Psi}\Psi.$$

For consistency, it must be that the Chern–Simons interaction itself induces a change in fermion boundary conditions equal to that one would have obtained from a double trace deformation

$$-\frac{6\pi}{k}\bar{\Psi}\Psi. \tag{E.1}$$

## Appendix F. Two-point functions in free field theory

Consider the action for free $SU(N)$ theory of a boson and a fermion in the fundamental representation, in flat three dimensional Euclidean space

$$S = \int \mathrm{d}^3x(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\psi}\sigma^\mu\partial_\mu\psi) \tag{F.1}$$

where the $SU(N)$ in indices are suppressed and will continue to be in what follows. The Green functions for the scalar and fermions are given by

$$\begin{aligned} G_s(x) &= \langle\bar{\phi}(x)\phi(0)\rangle = \frac{1}{4\pi|x|} \\ G_f(x) &= \langle\bar{\psi}(x)\psi(0)\rangle = \frac{x.\sigma}{4\pi|x|^3} \end{aligned} \tag{F.2}$$

Let us define the 'single trace' operators

$$\Phi_+ = \bar{\phi}\phi,\ \Phi_- = \bar{\psi}\psi,\ \Psi = \bar{\phi}\psi,\ \bar{\Psi} = \bar{\psi}\phi,\ J_B^\mu = \mathrm{i}\bar{\phi}\partial^\mu\phi - \partial^\mu\bar{\phi}\phi,\ J_F^\mu = \mathrm{i}\bar{\psi}\sigma^\mu\psi. \tag{F.3}$$

In the free theory

$$\begin{aligned}
\langle\Phi_+(x)\Phi_+(0)\rangle &= \frac{N}{(4\pi)^2x^2},\\
\langle\Phi_-(x)\Phi_-(0)\rangle &= \frac{2N}{(4\pi)^2x^4},\\
\langle\Psi(x)\bar{\Psi}(0)\rangle &= \frac{N(x.\sigma)}{(4\pi)^2x^4}\\
J_B^\mu(x)J_B(0)^\nu &= \frac{N}{8\pi^2}\frac{\delta^{\mu\nu}-\frac{2x^\mu x^\nu}{x^2}}{x^4}\\
J_F^\mu(x)J_F(0)^\nu &= \frac{N}{8\pi^2}\frac{\delta^{\mu\nu}-\frac{2x^\mu x^\nu}{x^2}}{x^4}.
\end{aligned} \tag{F.4}$$

## Appendix G. Corrections at large $A$

The expression for $T_c$ presented in (7.20) receives corrections in a power series expansion in $\frac{1}{A}$. In this appendix we compute the first correction to the expression for the second phase transition temperature presented in (7.20) at small $\frac{1}{A}$.

Equation (7.20) receives corrections once we take into account the fact that the $V$ eigenvalue distribution is not quite a delta function in the neighborhood of the phase transition. To compute the leading correction to eigenvalue distribution of $V$-matrices we substitute

$$\frac{1}{N}\,\mathrm{Tr}\,U^n = \frac{1}{N}\,\mathrm{Tr}\,U^{-n} = \frac{F(x^n)}{A}$$

for odd $n$, and

$$\frac{1}{N}\,\mathrm{Tr}\,U^n = \frac{1}{N}\,\mathrm{Tr}\,U^{-n} = \frac{F_B(x^n)-F_F(x^n)}{A}$$

for even $n$ (see (7.17)). It follows that the effective matrix integral for V-matrices is given by

$$Z = \int DV\exp\left[\frac{N}{A}\sum_{n=1}^{n}\frac{(F_B(x^n)+(-1)^nF_F(x^n))^2}{n}(\mathrm{Tr}\,V^n+\mathrm{Tr}\,V^{-n})\right]. \tag{G.1}$$

The saddle point equation for this model is

$$2\sum_{n=1}^{\infty}(F_B(x^n)+(-1)^nF_F(x^n))^2\sin(n\alpha) = \mathcal{P}v\int \mathrm{d}\beta\cot\left(\frac{\alpha-\beta}{2}\right). \tag{G.2}$$

To leading order in $\frac{1}{A}$ the $V$-eigenvalues are clumped into a delta function around zero. To first subleading order we expect that the eigenvalues will spread but only in a small region around zero and vanishes outside. Since all the eigenvalues are small the above saddle point equation reduces to Hermitian Wigner model

$$\left(\sum_{n=1}^{\infty}n\,(F_B(x^n)+(-1)^nF_F(x^n))^2\right)\alpha = \mathcal{P}v\int \mathrm{d}\beta\frac{\rho(\beta)}{\alpha-\beta}. \tag{G.3}$$

The solution to the above Wigner model is

$$\rho_v(\alpha) = \frac{2}{a^2}\sqrt{a^2-\alpha^2}, \qquad a^2 = \frac{2}{\sum_{n=1}^{\infty}n(F_B(x^n)+(-1)^nF_F(x^n))^2}. \tag{G.4}$$

Using this one can compute

$$\frac{1}{M}\,\mathrm{Tr}\,V^n = \frac{1}{M}\,\mathrm{Tr}\,V^{-n} = \frac{2}{an}J_1(an), \tag{G.5}$$

where $J_1(x)$ is Bessel function. Substituting these into (7.2) and using saddle point approximation one gets the corrected eigenvalue distribution for $U$-matrices to be

$$\rho_u(\theta) = \frac{1}{2\pi}\left(1 + \sum_{n=1}^{\infty} \frac{4\,(F_B(x^n) + (-1)^n F_F(x^n))\,J_1(an)}{an}\cos n\theta\right). \tag{G.6}$$

At leading order at high temperatures we substitute $x \to 1 - \frac{1}{T}$. The leading correction to the $U$ eigenvalue distribution (from the finite width of the $V$ eigenvalue distribution) is given by

$$\delta\rho_u(\theta) \to \frac{1}{2\pi A}\left[64T^2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\left(\frac{J_1(an)}{an} - \frac{1}{2}\right)\cos n\theta\right], \qquad a \to \frac{1}{\sqrt{112\zeta(3)}T^2}. \tag{G.7}$$

where $\zeta$ is the Riemann zeta function with $\zeta(3) = 1.202$.

The shift in the eigenvalue distribution evaluated at $\pi$ is given, to leading order in large $T$, by

$$-\frac{1}{2\pi A}32T^2 a \int_0^{\infty} \frac{\mathrm{d}x}{x^2}\left(\frac{J_1(x)}{x} - \frac{1}{2}\right).$$

This results is a shift of the phase transition temperature (about the result (7.20) at leading order at large $A$)

$$\delta T_c^2 = -\frac{8}{\sqrt{112\zeta(3)}} \int_0^{\infty} \frac{\mathrm{d}x}{x^2}\left(\frac{J_1(x)}{x} - \frac{1}{2}\right).$$

Thus the finite width of the $V$ eigenvalue distribution gives rise to a fractional correction of order $\frac{1}{A}$ second phase transition temperature.

## Appendix H. Truncated toy matrix model including interaction effects

In this appendix we study the toy model

$$\begin{aligned} Z = \int DUDV \exp[&-F(x)(\mathrm{Tr}\,U\,\mathrm{Tr}\,V^{-1} + \mathrm{Tr}\,V\,\mathrm{Tr}\,U^{-1}) - a\,\mathrm{Tr}\,U\,\mathrm{Tr}\,U^{-1}\,\mathrm{Tr}\,V\,\mathrm{Tr}\,V^{-1} \\ &- b((\mathrm{Tr}\,U)^2(\mathrm{Tr}\,V^{-1})^2 + (\mathrm{Tr}\,U^{-1})^2(\mathrm{Tr}\,V)^2)] \end{aligned} \tag{H.1}$$

in a neighborhood of $F(x) = 1$, with $a$ and $b$ taken to be small.

The saddle point equations for the for $U$ and $V$ eigenvalues are

$$\begin{aligned} &\mathcal{P}v\int \mathrm{d}\lambda_1\rho_u(\lambda_1)\cot\left(\frac{\lambda-\lambda_1}{2}\right) + \frac{2\chi_1}{A}(F(x) + (a+2b)\rho_1\chi_1)\sin\lambda = 0, \\ &\mathcal{P}v\int \mathrm{d}\alpha_1\rho_v(\alpha_1)\cot\left(\frac{\alpha-\alpha_1}{2}\right) + \frac{2\rho_1}{A}(F(x) + (a+2b)\rho_1\chi_1)\sin\alpha = 0. \end{aligned} \tag{H.2}$$

These equations are of the Gross–Witten–Wadia form with the Gross–Witten–Wadia coupling dependent on the $\rho_1$ and $\chi_1$ themselves. We now search for self-consistent solutions to these equations.

### *H.1. U flat, V flat*

we see that the $\rho_n = \chi_n = 0$ for all $n$, is always a solutions.

*H.1.1. U flat, V wavy or vice-versa.* Substituting either $\rho_1$ or $\chi_1$ to zero we see that the other one is necessarily zero. Thus flat–wavy or wavy–flat is not a solution.

*H.2. U wavy, V wavy:*

In this case we will have

$$\begin{aligned} \left(\lambda_{GW}^{(u)}\right)^{-1} &= \rho_1 = \frac{\chi_1}{A}(F(x) + (a+2b)\rho_{-1}\chi_1), \\ \left(\lambda_{GW}^{(v)}\right)^{-1} &= \chi_1 = A\rho_1(F(x) + (a+2b)\rho_1\chi_{-1}). \end{aligned} \tag{H.3}$$

These equations may be used solve for $\chi_1$ and $\rho_{-1}$; without loss of generality we may choose $\chi_1$ and $\rho_1$ each to be real so that $\chi_1 = \chi_{-1}$ and $\rho_1 = \rho_{-1}$. We find

$$\frac{\chi_1}{A} = \rho_1 = \sqrt{\frac{1-F(x)}{A(a+2b)}}. \tag{H.4}$$

When $a+2b$ is positive this solution only makes sense for $F(x) \leqslant 1$. On the other hand when $a+2b$ is negative, the solution only makes sense for $F(x) \geqslant 1$.

Consistency of the solution (positivity of eigenvalue density distribution) further requires

$$\chi_1 \leqslant \tfrac{1}{2}, \quad \rho_1 \leqslant \tfrac{1}{2}. \tag{H.5}$$

As $\chi_1 \geqslant \rho_1$ the first of these two conditions is stronger. When $a+2b$ is positive this condition amounts to the requirement that

$$F(x) \geqslant 1 - \frac{a+2b}{4A}.$$

When $a+2b$ is negative this condition amounts to the requirement that

$$F(x) \leqslant 1 - \frac{a+2b}{4A}.$$

In summary, when $a+2b$ is positive the wavy–wavy solution exists for

$$1 - \frac{a+2b}{4A} \leqslant F(x) \leqslant 1.$$

At the lower end of this range the $V$ eigenvalue distribution is on the border of being gapped, while at the upper end of this range the $U$ and $V$ eigenvalue distributions are both flat.

When $a+2b$ is negative, on the other hand, the wavy–wavy solution exists for

$$1 \leqslant F(x) \leqslant 1 - \frac{a+2b}{4A}.$$

At the lower end of this range the $V$ eigenvalue distribution becomes flat, while at the upper end of this range the $V$ eigenvalue distribution is at the edge of being gapped.

*H.3. U wavy, V clumped*

In this case we have

$$\begin{aligned} \rho_1 &= \left(\lambda_{\mathrm{GW}}^{(u)}\right)^{-1} = \frac{\chi_1}{A}[F(x) + (a+2b)\rho_1\chi_1], \\ \chi_1 &= 1 - \frac{\lambda_{\mathrm{GW}}^{(v)}}{4} = 1 - \frac{1}{4A\rho_1[F(x)+(a+2b)\rho_1\chi_1]}. \end{aligned} \tag{H.6}$$

We may solve for $\rho_1$ in terms of $\chi_1$ and then obtain an equation for $\chi_1$ as follows

$$\begin{aligned} \rho_1 &= \frac{-\chi_1 F(x)}{-A + (a+2b)\chi_1^2}, \\ \chi_1 &= 1 + \frac{(-A+(a+2b)\chi_1^2)^2}{4A^2\chi_1 F(x)^2}. \end{aligned} \tag{H.7}$$

Again consistency of solution requires

$$\rho_1 \leqslant \tfrac{1}{2} \text{ and } \chi_1 \geqslant \tfrac{1}{2}. \tag{H.8}$$

The second condition is satisfied if and only if

$$F(x) \geqslant 1 - \frac{a+2b}{4A}. \tag{H.9}$$

When this inequality is saturated, the $V$ eigenvalue distribution is on the border between wavy and clumped. The first condition is saturated at a higher temperature when the $U$ eigenvalue distribution first begins to clump (i.e. near the second phase transition of the free model). As the quartic interaction are not particularly important for this transition, we do not study this transition in detail.

It is possible to verify that solutions of the $U$ clumped $V$ wavy form do not exist. As mentioned above, clumped–clumped solutions do exist, but the quartic interaction terms do not play an important role in determining their properties, and we do not consider them further here.

### *H.4. Summary*

The quartic interaction terms of this subsection qualitatively modify the nature of the first phase transition of the free theory.

When $a + 2b$ is positive the flat-flat configuration is the only solution to the saddle point equations when $F(x) < 1 - \frac{a+2b}{4A}$. At this temperature two new solutions are nucleated. The first is a wavy–wavy solution is a local maximum and so is unstable throughout the range of its existence. The second is a wavy–clumped solution and is locally a solution. At $F(x) =$ the free energy of the wavy–clumped solution decrease below that of the flat–flat solution and the system undergoes a first order phase transition. At the higher temperature $F(x) = 1$ the wavy–wavy solution merges with the flat–flat solution and ceases to exist thereafter. At higher temperatures the flat–flat solution is unstable and the wavy–clumped solution is the unique stable saddle point. At still higher temperatures this saddle point undergoes a third order phase transition to the clumped–clumped saddle.

When $a + 2b$ is negative the flat–flat configuration is the only solution to the saddle point equations when $F(x) < 1$. At this temperature the flat–flat saddle goes unstable, but a wavy–wavy solution is nucleated, and is stable at higher temperatures. The system undergoes a second order phase transition (from the flat–flat saddle to the wavy–wavy saddle) at $F(x) = 1$. At $F(x) = 1 - \frac{a+2b}{4A}$ (recall this is a higher temperature than $F(x) = 1$ because $a + 2b$ is negative) the wavy–wavy saddle turns into a wavy–clumped saddle through a third order phase transition. At still higher temperatures the wavy–clumped saddle point undergoes a third order phase transition to the clumped–clumped saddle.

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