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# RC-POSITIVITY, VANISHING THEOREMS AND RIGIDITY OF HOLOMORPHIC MAPS

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*Abstract* Let $M$ and $N$ be two compact complex manifolds. We show that if the tautological line bundle $\mathcal{O}_{T^*_M}(1)$ is not pseudo-effective and $\mathcal{O}_{T^*_N}(1)$ is nef, then there is no non-constant holomorphic map from $M$ to $N$. In particular, we prove that any holomorphic map from a compact complex manifold $M$ with RC-positive tangent bundle to a compact complex manifold $N$ with nef cotangent bundle must be a constant map. As an application, we obtain that there is no non-constant holomorphic map from a compact *Hermitian manifold* with *positive holomorphic sectional curvature* to a Hermitian manifold with non-positive holomorphic bisectional curvature.

*Keywords*: RC-positivity; vanishing theorems; rigidity

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## Contents



## 1. Introduction

The classical Schwarz–Pick lemma states that any holomorphic map from the unit disc in the complex plane into itself decreases the Poincaré metric. This was extended by

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Ahlfors [1] to maps from the disc into a hyperbolic Riemann surface and by Chern [9] and Lu [18] to higher-dimensional complex manifolds. A major advance was Yau's Schwarz lemma [39], which says that a holomorphic map from a complete Kähler manifold with *Ricci curvature* bounded below into a Hermitian manifold with holomorphic bisectional curvature bounded above by a negative constant is distance-decreasing up to a constant depending only on these bounds. In particular, there is no non-trivial holomorphic map from compact Kähler manifolds with positive Ricci curvature to Hermitian manifolds with non-positive holomorphic bisectional curvature. Later generalizations were mainly in two directions: relaxing the curvature hypothesis or the Kähler assumption. In philosophy, holomorphic maps from 'positively curved' complex manifolds to 'non-positively curved' complex manifolds should be constant. For more details, we refer to the recent paper [29] of Tosatti and the references therein. There are also some other generalizations along this line, for instance, on complex analyticity of harmonic maps (e.g. [14, 28]).

In this paper, we obtain a rigidity theorem on holomorphic maps between complex manifolds, which recovers many classical rigidity theorems along this line in differential geometry. The curvature condition of the domain manifold is only required to be *RC-positive*. This curvature notion was introduced in our previous paper [33], and it is significantly weaker than the positivity of Ricci curvature. For instance, a *complex manifold* with *positive holomorphic sectional curvature* is RC-positive. One of the key ingredients in our proofs relies on the Leray–Grothendieck spectral sequence and isomorphisms of various cohomology groups, which is quite different from classical methods in differential geometry. As it is well known, the latter is based on various maximum principles (e.g. [38]).

In [33], we introduced a terminology called 'RC-positivity'. A Hermitian holomorphic vector bundle $(\mathscr{E}, h^{\mathscr{E}})$ over a complex manifold $X$ is called *RC-positive* (resp. RC-negative) if for any $q \in X$ and any non-zero vector $v \in \mathscr{E}_q$, there exists *some* non-zero vector $u \in T_qX$ such that

$$R^{\mathscr{E}}(u, \overline{u}, v, \overline{v}) > 0 \quad (\text{resp.} \ < 0).$$

It is easy to see that for a Hermitian line bundle $(\mathscr{L}, h^{\mathscr{L}})$, it is RC-positive if and only if its Ricci curvature $-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}$ has at least one positive eigenvalue at each point of $X$. This terminology has many nice properties. For instance, quotient bundles of RC-positive bundles are also RC-positive; subbundles of RC-negative bundles are still RC-negative. On the other hand, it is obvious that a compact complex manifold with positive holomorphic sectional curvature has RC-positive tangent bundle. By using the Calabi–Yau theorem [40], we proved in [33, Corollary 3.8] that the holomorphic tangent bundles of Fano manifolds can admit RC-positive Kähler metrics. This curvature notion should be closely related to the pseudo-effectiveness of vector bundles defined by Păun and Takayama in [24] (see also [10, 23] and Theorem 2.4). Moreover, it can also be regarded as a differential geometric interpretation of the positive $\alpha$-slope investigated by Campana and Păun in [7]. The properties of RC-positive vector bundles are studied in [33] and § 2.

The geometry of vector bundles is usually characterized by their tautological line bundles. Let $\mathscr{E}$ be a holomorphic vector bundle and $\mathbb{P}(\mathscr{E}^*)$ be its projective bundle.

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The tautological line bundle is denoted by $\mathcal{O}_{\mathscr{E}}(1)$. For instance, $\mathscr{E}$ is called ample (resp. nef) if the tautological line bundle $\mathcal{O}_{\mathscr{E}}(1)$ is ample (resp. nef) over $\mathbb{P}(\mathscr{E}^*)$ [13]. There are many methods to construct Hermitian metrics on line bundles (e.g. on $\mathcal{O}_{\mathscr{E}}(1)$) with various weak positivities. However, it is still a challenging problem to construct Hermitian metrics on vector bundles with desired curvature properties. For instance, it is a long-standing open problem [11] to construct positive Hermitian metrics on ample vector bundles.

The main result of this paper is the following rigidity theorem.

**Theorem 1.1.** *Let $M$ and $N$ be two compact complex manifolds. If the tautological line bundle $\mathcal{O}_{T^*_M}(-1)$ is RC-positive and $\mathcal{O}_{T^*_N}(1)$ is nef, then any holomorphic map from $M$ to $N$ is constant.*

Theorem 1.1 has an equivalent algebraic version.

**Theorem 1.2.** *Let $M$ and $N$ be two compact complex manifolds. If the tautological line bundle $\mathcal{O}_{T^*_M}(1)$ is not pseudo-effective and $\mathcal{O}_{T^*_N}(1)$ is nef, then any holomorphic map from $M$ to $N$ is constant.*

Let us explain the curvature conditions in Theorems 1.1 and 1.2. A line bundle is called pseudo-effective if it possesses a (possibly) singular Hermitian metric whose curvature is semi-positive in the sense of current. When $M$ is a Riemann surface, $\mathcal{O}_{T^*_M}(1)$ is not pseudo-effective if and only if $T^*_M$ is not pseudo-effective, i.e. $M \cong \mathbb{P}^1$. In this case, Theorem 1.2 is classical. In a higher-dimensional case, $\mathcal{O}_{T^*_M}(1)$ is not pseudo-effective if and only if $\mathcal{O}_{T^*_M}(1)$ is RC-negative or, equivalently, the dual line bundle $\mathcal{O}_{T^*_M}(-1)$ is RC-positive (Theorem 2.4). Roughly speaking, it says that $T_M$ has a 'positive direction' at each point of $M$. As we discussed before, the RC-positivity of $\mathcal{O}_{T^*_M}(-1)$ is a very weak curvature condition. For example, it can be implied by the positivity of holomorphic sectional curvature (e.g. Proposition 2.4). Moreover, a compact complex manifold with RC-positive $\mathcal{O}_{T^*_M}(-1)$ is not necessarily Kähler. For the curvature requirement on the target manifold $N$, $\mathcal{O}_{T^*_N}(1)$ is nef if and only if the cotangent bundle $T^*_N$ is nef. For instance, all submanifolds of abelian varieties have nef cotangent bundles. The proofs of Theorems 1.1 and 1.2 rely on vanishing theorems for twisted vector bundles (Theorem 3.1) which are established by using the Le Potier isomorphism (Leray–Grothendieck spectral sequence) and characterizations of RC-positive vector bundles obtained in [32, 33], which are significantly different from classical methods in differential geometry.

We say that $M$ has *RC-positive tangent bundle* if $M$ admits a smooth Hermitian metric $\omega_g$ such that $(T_M, \omega_g)$ is RC-positive. We show in Proposition 2.6 that if $T_M$ is RC-positive, then $\mathcal{O}_{T^*_M}(-1)$ is RC-positive. As an application of Theorem 1.1, we obtain the following result.

**Theorem 1.3.** *Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If $M$ has RC-positive tangent bundle and $N$ has nef cotangent bundle, then $f$ is a constant map.*

There are many Kähler and non-Kähler complex manifolds with RC-positive tangent bundles. We just list some of them for readers' convenience.

- Fano manifolds [33, Corollary 3.8];
- manifolds with positive second Chern–Ricci curvature [33, Corollary 3.7];
- Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$ ([16, formula (6.4)]);
- complex manifolds with positive holomorphic sectional curvature.

The following differential geometric version of Theorem 1.3 is of particular interest, which recovers several classical rigidity theorems in complex differential geometry.

**Corollary 1.4.** *Let $M$ be a compact complex manifold with RC-positive tangent bundle and $N$ be a Hermitian manifold with non-positive holomorphic bisectional curvature, then any holomorphic map from $M$ to $N$ is a constant map.*

**Remark 1.5.** A compact complex manifold with RC-positive tangent bundle can contain no rational curves. For instance, Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$.

**Remark 1.6.** As we pointed out before, one of the key ingredients in the proofs is the Leray–Grothendieck spectral sequence. In Appendix A, we include a discussion on classical methods for readers' convenience.

As a special case of Corollary 1.4, we obtain the following.

**Corollary 1.7.** *Let $(M, \omega_g)$ be a compact Hermitian manifold with positive holomorphic sectional curvature and $(N, h)$ be a Hermitian manifold with non-positive holomorphic bisectional curvature. Then there is no non-constant meromorphic map from $M$ or its blowing-up to $N$.*

**Remark 1.8.** The notion of positive holomorphic sectional curvature is very natural in differential geometry, but it seems to be mysterious in literature. Recently, we proved in [33, Theorem 1.7] that a compact Kähler manifold with positive holomorphic sectional curvature must be projective and rationally connected, which confirms a well-known conjecture [41, Problem 47] of Yau. However, the geometry of *compact complex manifolds* with positive holomorphic sectional curvature is still not clear. For some related topics, we refer to [2, 5, 6, 8, 17, 19–22, 25, 30, 31] and the references therein. A project on the geometry of complete non-compact complex manifolds with RC-positive curvature is also carried out and we have obtained some results analogous to Yau's classical work [39].

## 2. Background materials

Let $(\mathscr{E}, h)$ be a Hermitian holomorphic vector bundle over a complex manifold $X$ with Chern connection $\nabla$. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on $X$ and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of $\mathscr{E}$. The curvature tensor $R^{\mathscr{E}} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(\mathscr{E}))$ has components

$$R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}. \tag{2.1}$$

(Here and henceforth, we sometimes adopt the Einstein convention for summation.) If $(X, \omega_g)$ is a Hermitian manifold, then $(T_X, g)$ has Chern curvature components

$$R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 g_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{\ell}}}{\partial \overline{z}^j}. \tag{2.2}$$

The Chern–Ricci curvature $\mathrm{Ric}(\omega_g)$ of $(X, \omega_g)$ is represented by

$$R_{i\overline{j}} = g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}}$$

and the second Chern–Ricci curvature $\mathrm{Ric}^{(2)}(\omega_g)$ has components

$$R^{(2)}_{k\overline{\ell}} = g^{i\overline{j}} R_{i\overline{j}k\overline{\ell}}.$$

**Definition 2.1.** A Hermitian holomorphic vector bundle $(\mathscr{E}, h)$ over a complex manifold $X$ is called *Griffiths positive at point* $q \in X$ if for any non-zero vector $v \in \mathscr{E}_q$ and any non-zero vector $u \in T_qX$, we have

$$R^{\mathscr{E}}(u, \overline{u}, v, \overline{v}) > 0. \tag{2.3}$$

$(\mathscr{E}, h)$ is called *Griffiths positive* if it is Griffiths positive at every point of $X$.

As analogous to Griffiths positivity, we introduced in [33] the following concept.

**Definition 2.2.** A Hermitian holomorphic vector bundle $(\mathscr{E}, h)$ over a complex manifold $X$ is called *RC-positive at point* $q \in X$ if for each non-zero vector $v \in \mathscr{E}_q$, there exists *some* non-zero vector $u \in T_qX$ such that

$$R^{\mathscr{E}}(u, \overline{u}, v, \overline{v}) > 0. \tag{2.4}$$

$(\mathscr{E}, h)$ is called *RC-positive* if it is RC-positive at every point of $X$.

**Remark 2.3.** Similarly, one can define semi-positivity, negativity, etc.

In [32, Theorem 1.4], we obtained an equivalent characterization for RC-positive line bundles, which plays a key role in this paper.

**Theorem 2.4.** *Let $X$ be a compact complex manifold and $\mathscr{L}$ be a holomorphic line bundle over $X$. Then the following statements are equivalent:*

(1) *$\mathscr{L}$ is RC-positive;*

(2) *the dual line bundle $\mathscr{L}^*$ is not pseudo-effective.*

As an application of Theorem 2.4, we have the following.

**Corollary 2.5.** *Let $X$ be a compact complex manifold. If $\mathscr{L}$ is an RC-positive line bundle over $X$, then*

$$H^0(X, \mathscr{L}^*) = 0. \tag{2.5}$$

**Proof.** Suppose $H^0(X, \mathscr{L}^*) \neq 0$, then $\mathscr{L}^*$ is $\mathbb{Q}$-effective, and so it is pseudo-effective. By Theorem 2.4, this is a contradiction. □

The points of the projective bundle $\mathbb{P}(\mathscr{E}^*)$ of $\mathscr{E} \to X$ can be identified with the hyperplanes of $\mathscr{E}$. Note that every hyperplane $\mathscr{V}$ in $\mathscr{E}_z$ corresponds bijectively to the line of linear forms in $\mathscr{E}_z^*$ which vanish on $\mathscr{V}$. Let $\pi : \mathbb{P}(\mathscr{E}^*) \to X$ be the natural projection. There is a tautological hyperplane subbundle $\mathscr{S}$ of $\pi^*\mathscr{E}$ such that

$$\mathscr{S}_{[\xi]} = \xi^{-1}(0) \subset \mathscr{E}_z$$

for all $\xi \in \mathscr{E}_z^* \setminus \{0\}$. The quotient line bundle $\pi^*\mathscr{E}/\mathscr{S}$ is denoted by $\mathcal{O}_{\mathscr{E}}(1)$ and is called the *tautological line bundle* associated with $\mathscr{E} \to X$. Hence, there is an exact sequence of vector bundles over $\mathbb{P}(\mathscr{E}^*)$

$$0 \to \mathscr{S} \to \pi^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1) \to 0. \tag{2.6}$$

A holomorphic vector bundle $\mathscr{E} \to X$ is called *ample* (resp. *semi-ample, nef*) if the line bundle $\mathcal{O}_{\mathscr{E}}(1)$ is ample (resp. semi-ample, nef) over $\mathbb{P}(\mathscr{E}^*)$.

Suppose $\dim_{\mathbb{C}} X = n$ and $r = \mathrm{rank}(\mathscr{E})$. Let $\pi$ be the projection $\mathbb{P}(\mathscr{E}^*) \to X$ and $\mathscr{L} = \mathcal{O}_{\mathscr{E}}(1)$. Let $(e_1, \ldots, e_r)$ be the local holomorphic frame on $\mathscr{E}$ and the dual frame on $\mathscr{E}^*$ is denoted by $(e^1, \ldots, e^r)$. The corresponding holomorphic coordinates on $\mathscr{E}^*$ are denoted by $(W_1, \ldots, W_r)$. If $(h_{\alpha\overline{\beta}})$ is the matrix representation of a smooth Hermitian metric $h^{\mathscr{E}}$ on $\mathscr{E}$ with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then the induced Hermitian metric $h^{\mathscr{L}}$ on $\mathscr{L}$ can be written as

$$h^{\mathscr{L}} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta}. \tag{2.7}$$

The curvature of $(\mathscr{L}, h^{\mathscr{L}})$ is

$$R^{\mathscr{L}} = \sqrt{-1}\partial\overline{\partial} \log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right), \tag{2.8}$$

where $\partial$ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(\mathscr{E}^*)$. We fix a point $p \in \mathbb{P}(\mathscr{E}^*)$, then there exist local holomorphic coordinates $(z^1, \ldots, z^n)$ centered at point $q = \pi(p)$ and local holomorphic basis $\{e_1, \ldots, e_r\}$ of $\mathscr{E}$ around $q$ such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3). \tag{2.9}$$

Without loss of generality, we assume that $p$ is the point $(0, \ldots, 0, [a_1, \ldots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \ldots, r-1$. By formulas (2.8) and (2.9),

$$R^{\mathscr{L}}(p) = \sqrt{-1} \sum R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \omega_{\mathrm{FS}}, \tag{2.10}$$

where $|a|^2 = \sum_{\alpha=1}^r |a_\alpha|^2$ and $\omega_{\mathrm{FS}} = \sqrt{-1} \sum_{A,B=1}^{r-1} \left(\frac{\delta_{AB}}{|a|^2} - \frac{a_B \overline{a}_A}{|a|^4}\right) dw^A \wedge d\overline{w}^B$ is the Fubini–Study metric on the fiber $\mathbb{P}^{r-1}$. The following result is one of the key ingredients in this paper.

**Proposition 2.6.** *Let $X$ be a compact complex manifold. If $(\mathscr{E}, h^{\mathscr{E}})$ is an RC-positive vector bundle over $X$, then $\mathcal{O}_{\mathscr{E}^*}(-1)$ is an RC-positive line bundle over $\mathbb{P}(\mathscr{E})$.*

**Proof.** By (2.10), the induced metric on $\mathcal{O}_{\mathscr{E}^*}(-1)$ over $\mathbb{P}(\mathscr{E})$ has curvature form

$$R^{\mathcal{O}_{\mathscr{E}^*}(-1)} = -\left(\sqrt{-1}\sum R^{\mathscr{E}^*}_{i\bar{j}\alpha\bar{\beta}} \frac{a_\beta \bar{a}_\alpha}{|a|^2} dz^i \wedge d\bar{z}^j + \omega_{\mathrm{FS}}\right).$$

On the other hand, $R^{\mathscr{E}^*} = -(R^{\mathscr{E}})^t$ and so

$$R^{\mathcal{O}_{\mathscr{E}^*}(-1)} = \sqrt{-1}\sum R^{\mathscr{E}}_{i\bar{j}\alpha\bar{\beta}} \frac{a_\alpha \bar{a}_\beta}{|a|^2} dz^i \wedge d\bar{z}^j - \omega_{\mathrm{FS}}.$$

Hence, $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive as long as $(\mathscr{E}, h^{\mathscr{E}})$ is RC-positive. □

**Remark 2.7.** We also have the following results:

(1) If $\mathscr{L}_1$ is an RC-positive line bundle and $\mathscr{L}_2$ is a pseudo-effective line bundle, then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is RC-positive.

(2) Let $(\mathscr{E}, h^{\mathscr{E}})$ be an RC-positive vector bundle and $(\mathscr{F}, h^{\mathscr{F}})$ be a Griffiths semi-positive vector bundle. The Hermitian vector bundle $(\mathscr{E} \otimes \mathscr{F}, h^{\mathscr{E}} \otimes h^{\mathscr{F}})$ is *not* necessarily RC-positive unless $\mathrm{rank}(\mathscr{E}) = 1$.

**Remark 2.8.** It is easy to see that if $(\mathscr{E}, h^{\mathscr{E}})$ is Griffiths positive (resp. semi-positive), then the tautological line bundle $\mathcal{O}_{\mathscr{E}}(1)$ is positive (resp. semi-positive). Whether the converse is valid is a long-standing open problem (the so-called Griffiths conjecture). In the same vein, we wonder whether the RC-positivity of $\mathcal{O}_{\mathscr{E}^*}(-1)$ can imply the RC-positivity of $\mathscr{E}$.

The following well-known lemma is called *the Le Potier isomorphism* [15]. Its proof relies on the Leray–Grothendieck spectral sequence, and we refer to [27, Theorem 5.16] and the references therein.

**Lemma 2.9.** *Let $\mathscr{E}$ be a holomorphic vector bundle over a complex manifold $X$ and $\mathscr{F}$ be a coherent sheaf on $X$. Then for all $p, q \geqslant 0$*

$$H^q(X, \Omega^p_X \otimes \mathscr{E} \otimes \mathscr{F}) \cong H^q\left(\mathbb{P}(\mathscr{E}^*), \Omega^p_{\mathbb{P}(\mathscr{E}^*)} \otimes \mathcal{O}_{\mathscr{E}}(1) \otimes \pi^*\mathscr{F}\right), \tag{2.11}$$

*where $\pi : \mathbb{P}(\mathscr{E}^*) \to X$ is the projection. In particular,*

$$H^0(X, \mathscr{E}) \cong H^0(\mathbb{P}(\mathscr{E}^*), \mathcal{O}_{\mathscr{E}}(1)). \tag{2.12}$$

By using the Le Potier isomorphism, we obtain vanishing theorems for vector bundles.

**Lemma 2.10.** *Let $\mathscr{E}$ be a holomorphic vector bundle over a compact complex manifold $X$. If $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive, then*

$$H^0(X, \mathscr{E}^*) = 0. \tag{2.13}$$

*In particular, if $\mathscr{E}$ is RC-positive, then $\mathscr{E}^*$ has no non-trivial holomorphic sections.*

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**Proof.** It follows from Corollary 2.5, Proposition 2.6 and Lemma 2.9. □

The following concept is a generalization of the RC-positivity for line bundles, which is also well known in literatures.

**Definition 2.11.** Let $\mathscr{L}$ be a holomorphic line bundle over a complex manifold $X$. $\mathscr{L}$ is called *$k$-positive* if there exists a smooth Hermitian metric $h^{\mathscr{L}}$ such that the Chern curvature $R^{\mathscr{L}} = -\sqrt{-1}\partial\bar{\partial}\log h^{\mathscr{L}}$ has at least $(\dim X - k)$ positive eigenvalues at every point on $X$.

It is easy to see that $\mathscr{L}$ is $(\dim X - 1)$-positive if and only if it is RC-positive. In [3, Theorem 14], Andreotti and Grauert proved the following fundamental vanishing theorem.

**Lemma 2.12.** *Let $\mathscr{L}$ be a $k$-positive line bundle over a compact complex manifold $X$. Then for any vector bundle $\mathscr{F}$ over $X$, there exists a positive integer $m_0 = m_0(\mathscr{F})$ such that*

$$H^q(X, \mathscr{L}^{\otimes m} \otimes \mathscr{F}) = 0 \tag{2.14}$$

*for all $q > k$ and $m \geqslant m_0$.*

## 3. Vanishing theorems for tensor product of vector bundles

The main result of this section is the following vanishing theorem.

**Theorem 3.1.** *Let $\mathscr{E}$ and $\mathscr{F}$ be two holomorphic vector bundles over a compact complex manifold $X$. If $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive over $\mathbb{P}(\mathscr{E})$ and $\mathcal{O}_{\mathscr{F}}(1)$ is nef over $\mathbb{P}(\mathscr{F}^*)$, then*

$$H^0(X, \mathscr{E}^* \otimes \mathscr{F}^*) = 0. \tag{3.1}$$

By Theorem 2.4, we have a variant of Theorem 3.1.

**Theorem 3.2.** *Let $\mathscr{E}$ and $\mathscr{F}$ be two holomorphic vector bundles over a compact complex manifold $X$. If $\mathcal{O}_{\mathscr{E}^*}(1)$ is not pseudo-effective and $\mathcal{O}_{\mathscr{F}}(1)$ is nef, then*

$$H^0(X, \mathscr{E}^* \otimes \mathscr{F}^*) = 0. \tag{3.2}$$

**Remark 3.3.** Theorem 3.1 does not hold in general if $\mathcal{O}_{\mathscr{F}}(1)$ is pseudo-effective and $\operatorname{rank}(\mathscr{F}) > 1$. It should hold if we refine this notion a bit more (e.g. [10, 24]).

By using Proposition 2.6, we obtain another application of Theorem 3.1.

**Theorem 3.4.** *Let $\mathscr{E}$ and $\mathscr{F}$ be two holomorphic vector bundles over a compact complex manifold $X$. If $\mathscr{E}$ is RC-positive and $\mathscr{F}$ is nef, then*

$$H^0(X, \mathscr{E}^* \otimes \mathscr{F}^*) = 0. \tag{3.3}$$

Before giving the proof of Theorem 3.1, we need several lemmas.

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**Lemma 3.5.** *Let $f : X \to Y$ be a holomorphic* submersion *between two complex manifolds. If $\mathscr{L}$ is an RC-positive line bundle over $Y$, then $f^*(\mathscr{L})$ is also RC-positive.*

**Proof.** Suppose $\dim X = m$ and $\dim Y = n$. Let $\{z^i\}_{i=1}^m$ and $\{w^\alpha\}_{\alpha=1}^n$ be the local holomorphic coordinates on $X$ and $Y$, respectively. Let $h$ be a smooth RC-positive metric on $\mathscr{L}$ and $R = -\sqrt{-1}\partial\overline{\partial}\log h$. It is easy to see that the curvature tensor of $(f^*(\mathscr{L}), f^*h)$ is given by

$$R_{\alpha\overline{\beta}}\frac{\partial f^\alpha}{\partial z^i}\frac{\partial\overline{f}^\beta}{\partial\overline{z}^j}dz^i\wedge d\overline{z}^j. \tag{3.4}$$

Since $(\mathscr{L}, h)$ is RC-positive, at any point $p \in Y$, there exists a non-zero local vector $v = (v^1, \ldots, v^n)$ such that $\sum R_{\alpha\overline{\beta}}v^\alpha\overline{v}^\beta > 0$. Since $f$ is a smooth submersion, the rank of the matrix $\left(\frac{\partial f^\alpha}{\partial z^i}\right)$ is equal to $n = \dim Y$. Therefore, there exists a non-zero vector $u = (u^1, \ldots u^m)$ such that $\left(\frac{\partial f^\alpha}{\partial z^i}\right)u = v$. Hence, $(f^*(\mathscr{L}), f^*h)$ is RC-positive. □

**Remark 3.6.** Lemma 3.5 also holds for $k$-positive line bundles.

**Lemma 3.7.** *Let $f : X \to Y$ be a holomorphic map between two compact complex manifolds. If $\mathscr{L}$ is a nef line bundle over $Y$, then $f^*(\mathscr{L})$ is also nef.*

**Proof.** It follows from the definition of nefness and formula (3.4). □

**Lemma 3.8.** *Let $f : X \to Y$ be a holomorphic map between two compact complex manifolds. If $\mathscr{E}$ is a holomorphic vector bundle over $Y$ such that $\mathcal{O}_{\mathscr{E}}(1)$ is nef, then $\mathcal{O}_{f^*\mathscr{E}}(1)$ is also nef.*

**Proof.** We have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{f^*\mathscr{E}}(1) & \xrightarrow{(f^\#)^*} & \mathcal{O}_{\mathscr{E}}(1) \\
\downarrow & & \downarrow \\
\mathbb{P}(f^*(\mathscr{E}^*)) & \xrightarrow{f^\#} & \mathbb{P}(\mathscr{E}^*) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}$$

Lemma 3.8 follows from the above diagram and Lemma 3.7. □

**Lemma 3.9.** *Let $\mathscr{E}$ be a holomorphic vector bundle over a compact complex manifold $X$. If $\mathcal{O}_{\mathscr{E}}(1)$ is* $(\dim X - 1)$*-positive over $\mathbb{P}(\mathscr{E}^*)$, then*

$$H^0(X, \mathscr{E}^*) = 0. \tag{3.5}$$

**Proof.** If $\mathcal{O}_{\mathscr{E}}(1)$ is $(\dim X - 1)$-positive over $\mathbb{P}(\mathscr{E}^*)$, then by Lemma 2.12, for any vector bundle $\mathscr{F}$ on $\mathbb{P}(\mathscr{E}^*)$, there exists some positive integer $m_0 = m_0(\mathscr{F})$ such that

$$H^q(\mathbb{P}(\mathscr{E}^*), \mathcal{O}_{\mathscr{E}}(m)\otimes\mathscr{F}) = 0$$

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for all $q > \dim X - 1$ and $m \geqslant m_0$. In particular, if we take $q = n = \dim X$ and $\mathscr{F} = \Omega^n_{\mathbb{P}(\mathscr{E}^*)}$, by Lemma 2.9 and the Serre duality,

$$H^n(\mathbb{P}(\mathscr{E}^*), \mathcal{O}_{\mathscr{E}}(m) \otimes \Omega^n_{\mathbb{P}(\mathscr{E}^*)}) \cong H^n(X, \mathrm{Sym}^{\otimes m}\mathscr{E} \otimes \Omega^n_X) \cong H^0(X, \mathrm{Sym}^{\otimes m}\mathscr{E}^*) = 0.$$

In particular, for large $m$, we have

$$H^0(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathscr{E}^*}(m)) = 0.$$

Hence, $H^0(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathscr{E}^*}(1)) = 0$ and $H^0(X, \mathscr{E}^*) = 0$. □

**Theorem 3.10.** *Let $X$ be a compact complex manifold. If $(\mathscr{L}, h^{\mathscr{L}})$ is an RC-positive line bundle and $\mathscr{E}$ is a holomorphic vector bundle with nef tautological line bundle $\mathcal{O}_{\mathscr{E}}(1)$, then*

$$H^0\left(X, \mathscr{E}^* \otimes \mathscr{L}^*\right) = 0.$$

**Proof.** Let $\pi : \mathbb{P}(\mathscr{E}^*) \to X$ be the natural projection. Since $\pi$ is a submersion, by Lemma 3.5, $\pi^*\mathscr{L}$ is RC-positive.

*Claim* 1. $\pi^*\mathscr{L} \otimes \mathcal{O}_{\mathscr{E}}(1)$ is a $(\dim X - 1)$-positive line bundle over $\mathbb{P}(\mathscr{E}^*)$.

Fix a smooth Hermitian metric $h^{\mathscr{E}}$ on $\mathscr{E}$ and a smooth Hermitian metric $\omega$ on $\mathbb{P}(\mathscr{E}^*)$. The induced metric on $\mathcal{O}_{\mathscr{E}}(1)$ is denoted by $h^{\mathcal{O}_{\mathscr{E}}(1)}$. Since the restriction of $h^{\mathcal{O}_{\mathscr{E}}(1)}$ on each fiber $\mathbb{P}^{r-1}$ is a Fubini–Study metric, by curvature formula (2.10), there exist a Hermitian metric $\omega_X$ on $X$ and two positive constants $c_1, c_2$ such that

$$-\sqrt{-1}\partial\overline{\partial}\log h^{\mathcal{O}_{\mathscr{E}}(1)} + c_1\pi^*(\omega_X) \geqslant c_2\omega. \tag{3.6}$$

Let $\lambda(x)$ be the largest eigenvalue function of the curvature tensor $-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}$ of $(\mathscr{L}, h^{\mathscr{L}})$ with respect to the Hermitian metric $\omega_X$ on $X$ and

$$c_3 = \min_{x\in X}\lambda(x). \tag{3.7}$$

Since $X$ is compact and $-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}$ is RC-positive, we deduce $c_3 > 0$. Moreover, at any point $q \in X$, there exists a non-zero vector $u_0 \in T_qX$ such that

$$\left(-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}\right)(u_0, \overline{u}_0) \geqslant c_3|u_0|^2_{\omega_X}. \tag{3.8}$$

Since $\pi : \mathbb{P}(\mathscr{E}^*) \to X$ is a holomorphic submersion, by Lemma 3.5, $h_1 = \pi^*(h^{\mathscr{L}})$ is an RC-positive metric on $\pi^*\mathscr{L}$. Moreover, for any point $p \in \mathbb{P}(\mathscr{E}^*)$ with $\pi(p) = q \in X$, there exists a non-zero vector $u_1 \in T_p\mathbb{P}(\mathscr{E}^*)$ such that $\pi_*(u_1) = u_0 \in T_qX$ and

$$\left(-\sqrt{-1}\partial\overline{\partial}\log h_1\right)(u_1, \overline{u}_1) = \left(-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}\right)(u_0, \overline{u}_0) \geqslant c_3|u_0|^2_{\omega_X} > 0. \tag{3.9}$$

We fix a small number $\varepsilon > 0$ such that

$$\frac{c_3}{2} - c_1\varepsilon > 0. \tag{3.10}$$

On the other hand, since $\mathcal{O}_{\mathscr{E}}(1)$ is nef, there exists a smooth Hermitian metric $h_0$ on $\mathcal{O}_{\mathscr{E}}(1)$ such that the curvature of $(\mathcal{O}_{\mathscr{E}}(1), h_0)$ satisfies

$$-\sqrt{-1}\partial\overline{\partial}\log h_0 \geqslant -\varepsilon c_2\omega \tag{3.11}$$

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over $\mathbb{P}(\mathscr{E}^*)$. Let $h = \left(h^{\mathcal{O}_{\mathscr{E}}(1)}\right)^{\varepsilon} \cdot h_0^{(1-\varepsilon)}$ be a smooth Hermitian metric on $\mathcal{O}_{\mathscr{E}}(1)$. Then

$$\left(\pi^*\mathscr{L} \otimes \mathcal{O}_{\mathscr{E}}(1), h_1 \otimes h\right)$$

is $(\dim X - 1)$-positive, i.e. the curvature tensor $R = -\sqrt{-1}\partial\bar{\partial}\log(h_1 h)$ has at least $r$-positive eigenvalues at each point of $\mathbb{P}(\mathscr{E}^*)$. Indeed,

$$R = \varepsilon\left(-\sqrt{-1}\partial\bar{\partial}\log h^{\mathcal{O}_{\mathscr{E}}(1)}\right) + (1-\varepsilon)\left(-\sqrt{-1}\partial\bar{\partial}\log h_0\right) + \left(-\sqrt{-1}\partial\bar{\partial}\log h_1\right). \tag{3.12}$$

By (3.6), (3.9), (3.10), (3.11) and (3.12), we have

$$\begin{aligned} R(u_1, u_1) &\geqslant \varepsilon(c_2|u_1|^2_{\omega} - c_1|u_1|^2_{\pi^*\omega_X}) - (1-\varepsilon)\varepsilon c_2|u_1|^2_{\omega} + c_3|u_1|^2_{\pi^*\omega_X} \\ &\geqslant \frac{c_3}{2}|u_1|^2_{\pi^*\omega_X} = \frac{c_3}{2}|u_0|^2_{\omega_X} > 0. \end{aligned}$$

Along the fiber $\mathbb{P}^{r-1}$ direction, for any $u_2 \in T_p\mathbb{P}(\mathscr{E}^*)$ with $\pi_*(u_2) = 0 \in T_qX$, we have

$$R(u_2, u_2) \geqslant \varepsilon(c_2|u_2|^2_{\omega} - c_1|u_2|^2_{\pi^*\omega_X}) - (1-\varepsilon)\varepsilon c_2|u_2|^2_{\omega} + c_3|u_2|^2_{\pi^*\omega_X} \geqslant c_2\varepsilon^2|u_2|^2_{\omega}.$$

Since the map $\pi_* : T_p\mathbb{P}(\mathscr{E}) \to T_qX$ is surjective, $\dim\ker(\pi_*) = r-1$ and $u_1 \notin \ker(\pi_*)$, we deduce that the curvature tensor $R = -\sqrt{-1}\partial\bar{\partial}\log(h_1 h)$ has at least $r$ positive eigenvalues at each point of $\mathbb{P}(\mathscr{E}^*)$.

*Claim* 2. The tautological line bundle $\mathcal{O}_{\mathscr{L}\otimes\mathscr{E}}(1)$ is $(\dim X - 1)$-positive over $\mathbb{P}(\mathscr{L}^* \otimes \mathscr{E}^*)$. Indeed, it follows from the fact that $i : \mathbb{P}(\mathscr{L}^* \otimes \mathscr{E}^*) \to \mathbb{P}(\mathscr{E}^*)$ is an isomorphism and

$$\mathcal{O}_{\mathscr{L}\otimes\mathscr{E}}(1) = i^*\left(\mathcal{O}_{\mathscr{E}}(1) \otimes \pi^*(\mathscr{L})\right). \tag{3.13}$$

By Lemma 3.9, we obtain $H^0(X, \mathscr{E}^* \otimes \mathscr{L}^*) = 0$. The proof of Theorem 3.10 is completed. □

**The proof of Theorem 3.1.** Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the projection. By Lemma 2.9,

$$H^0(X, \mathscr{E}^* \otimes \mathscr{F}^*) \cong H^0\left(X, \pi_*\left(\mathcal{O}_{\mathscr{E}^*}(1) \otimes \pi^*\mathscr{F}^*\right)\right) \cong H^0\left(Y, \mathcal{O}_{\mathscr{E}^*}(1) \otimes \pi^*\mathscr{F}^*\right), \tag{3.14}$$

where $Y = \mathbb{P}(\mathscr{E})$. Since $\mathcal{O}_{\mathscr{F}}(1)$ is nef, by Lemma 3.8, $\mathcal{O}_{\pi^*\mathscr{F}}(1)$ is nef. Let $\mathscr{L} = \mathcal{O}_{\mathscr{E}^*}(-1)$, $\mathscr{W} = \pi^*\mathscr{F}$ and $\widetilde{\pi} : \mathbb{P}(\mathscr{W}) \to Y$. Since $\mathscr{L}$ is an RC-positive line bundle and $\mathcal{O}_{\mathscr{W}}(1)$ is nef, by Lemma 2.9 and Theorem 3.10,

$$H^0(X, \mathscr{F}^* \otimes \mathscr{E}^*) \cong H^0\left(Y, \pi^*\mathscr{F}^* \otimes \mathcal{O}_{\mathscr{E}^*}(1)\right) = H^0(Y, \mathscr{W}^* \otimes \mathscr{L}^*) = 0. \tag{3.15}$$

The proof of Theorem 3.1 is completed. □

## 4. RC-positivity and rigidity of holomorphic maps

In this section, we prove the main results of this paper, i.e. Theorem 1.1 (= Theorem 4.1), Theorem 1.3 (= Theorem 4.2) and Corollary 1.7 (= Corollary 4.3).

**Theorem 4.1.** *Let $M$ and $N$ be two compact complex manifolds. If $\mathcal{O}_{T^*_M}(-1)$ is an RC-positive line bundle and $\mathcal{O}_{T^*_N}(1)$ is nef, then any holomorphic map from $M$ to $N$ is constant.*

**Proof.** Let $\mathscr{E} = T_M \otimes f^*(T_N^*)$ and $\{z_i\}$, $\{w_\alpha\}$ be the local holomorphic coordinates on $M$ and $N$, respectively. Let

$$s = \partial f = f_i^\alpha dz_i \otimes e_\alpha \in \Gamma(M, \mathscr{E}^*),$$

where $e_\alpha = f^* \frac{\partial}{\partial w_\alpha}$. Since $f$ is a holomorphic map, $s$ is a holomorphic section of $\mathscr{E}$, i.e. $s \in H^0(M, \mathscr{E}^*)$. Since $\mathcal{O}_{T_N^*}(1)$ is nef, by Lemma 3.8, we know $\mathcal{O}_{f^*(T_N^*)}(1)$ is also nef. By Theorem 3.1, $H^0(M, \mathscr{E}^*) = 0$. Hence, $f$ is a constant map. □

In particular, we have the following.

**Theorem 4.2.** *Let $M$ be a compact complex manifold with RC-positive tangent bundle $T_M$ and $N$ be a compact complex manifold with nef cotangent bundle. Then any holomorphic map from $M$ to $N$ is constant.*

**Proof.** By Proposition 2.6, if $T_M$ is RC-positive, then $\mathcal{O}_{T_M^*}(-1)$ is RC-positive. Theorem 4.2 follows from Theorem 4.1. □

Let $M, N$ be compact complex manifolds of complex dimensions $m$ and $n$, respectively. Recall that a *meromorphic map* $f : M \to N$ is given by an irreducible analytic subset (the graph of $f$) $\Gamma \subset M \times N$ together with a proper analytic subset $S \subset M$ and a *holomorphic* map $f : M \setminus S \to N$ such that $\Gamma$ restricted to $(M - S) \times N$ is exactly the graph of $f$.

**Corollary 4.3.** *Let $(M, \omega_g)$ be a compact Hermitian manifold with positive holomorphic sectional curvature and $(N, h)$ be a Hermitian manifold with non-positive holomorphic bisectional curvature. Then there is no non-constant meromorphic map from $M$ or its blowing-up to $N$.*

**Proof.** Let $f : M \to N$ be a meromorphic map. By a theorem of Griffiths [12, Theorem II] and Shiffman [26, Theorem 2], when the target manifold has non-positive holomorphic sectional curvature, then $f$ is holomorphic. It is easy to see that if $\omega_g$ has positive holomorphic sectional curvature, then $(T_M, \omega_g)$ is RC-positive. By Theorem 4.2, there is no non-constant holomorphic map from $M$ to $N$.

Let $\widetilde{M}$ be a blowing-up of $M$ along some submanifold and $\pi : \widetilde{M} \to M$ be the canonical map. If $\widetilde{f} : \widetilde{M} \to N$ is a meromorphic map, then it is holomorphic. Moreover, it induces a meromorphic map $f : M \to N$. Hence, $f$ is a constant map. By Aronszajin's principle [4], $\widetilde{f}$ is also constant. □

We have shown in [33, Corollary 3.7] that if a complex manifold has positive second Chern–Ricci curvature, then it is RC-positive.

**Corollary 4.4.** *Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If $(M, g)$ has positive second Chern–Ricci curvature* $\mathrm{Ric}^{(2)}(g)$ *and $N$ has nef cotangent bundle, then $f$ is a constant.*

Hence, the following classical result is a special case of Corollary 4.4 (e.g. [37]).

**Corollary 4.5.** *Let $(M, g)$ be a compact Hermitian manifold with positive second Chern–Ricci curvature* $\mathrm{Ric}^{(2)}(g)$ *and $(N, h)$ be a Hermitian manifold with non-positive holomorphic bisectional curvature, then $f$ is a constant.*

We need to point out that, as a straightforward consequence of [33, Theorem 1.7], one has the following.

**Theorem 4.6.** *Let $(M, \omega_g)$ be a compact Kähler manifold with positive holomorphic sectional curvature and $N$ be a complex manifold without any rational curve (e.g. $N$ has a Hermitian metric with non-positive holomorphic sectional curvature, or $N$ is hyperbolic). Then any holomorphic map from $M$ to $N$ is a constant map.*

**Proof.** It is proved in [33, Theorem 1.7] that if $(M, \omega_g)$ is a compact Kähler manifold with positive holomorphic sectional curvature, then $M$ is rationally connected, i.e. any two points of $M$ can be connected by a rational curve. Since $N$ contains no rational curve, the image of a holomorphic map $f : M \to N$ must be a point. □

We would like to propose questions on rigidity results in a general setting. For instance, we have the following.

**Question 4.7.** Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If $T_M$ is RC-positive and $N$ is Kobayashi hyperbolic, is $f$ necessarily a constant map?

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## Appendix A. Yau's Schwarz calculation and rigidity of holomorphic maps

In this section, we review classical differential geometric methods (a model version of Yau's Schwarz calculation) on the proof of rigidity of holomorphic maps. We shall see clearly that the main results in this paper (e.g. Corollary 1.4) cannot be proved by using purely differential geometric methods. The following result is essentially well known (e.g. [9, 18, 39]).

**Lemma A.1.** *Let $f : (M, g) \to (N, h)$ be a holomorphic map between two Hermitian manifolds. Then in the local holomorphic coordinates $\{z^i\}$ and $\{w^\alpha\}$ on $M$ and $N$, respectively, we have the identity*

$$\partial\overline{\partial} u = \langle \nabla df, \nabla df \rangle + \left( R^g_{i\overline{j}k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f^\alpha_p \overline{f^\beta_q} - R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \left( f^\alpha_i \overline{f^\beta_j} \right) \left( g^{p\overline{q}} f^\gamma_p \overline{f^\delta_q} \right) \right) dz^i \wedge d\overline{z}^j$$

*and*

$$\Delta_g u = |\nabla df|^2 + \left( g^{i\overline{j}} R^g_{i\overline{j}k\overline{\ell}} \right) g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f^\alpha_p \overline{f^\beta_q} - R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \left( g^{i\overline{j}} f^\alpha_i \overline{f^\beta_j} \right) \left( g^{p\overline{q}} f^\gamma_p \overline{f^\delta_q} \right),$$

*where $u = \mathrm{tr}_{\omega_g}(f^*\omega_h)$, $f^\alpha_i = \frac{\partial f^\alpha}{\partial z^i}$, where $f$ is represented by $w^\alpha = f^\alpha(z)$ locally, $\nabla$ is the induced connection on the bundle $\mathscr{E} = T^*_M \otimes f^*(T_N)$.*

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To simplify the formulations, at a given point $p \in M$ and $q = f(p) \in N$, we choose $g_{i\overline{j}} = \delta_{ij}$ and $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$. Hence, we have

$$\partial\overline{\partial}u = \langle \nabla df, \nabla df \rangle + \left( \sum_{k,\ell,\alpha} R^g_{i\overline{j}k\overline{\ell}} f^\alpha_k \overline{f^\alpha_\ell} - \sum_{\alpha,\beta,\gamma,\delta,k} R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \left( f^\alpha_i \overline{f^\beta_j} \right) \left( f^\gamma_k \overline{f^\delta_k} \right) \right) dz^i \wedge d\overline{z}^j \tag{A 1}$$

and

$$\Delta_g u = |\nabla df|^2 + \sum_{k,\ell,\alpha} R^{(2)}_{k\overline{\ell}} f^\alpha_k \overline{f^\alpha_\ell} - \sum_{\alpha,\beta,\gamma,\delta,k,i} R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \left( f^\alpha_i \overline{f^\beta_i} \right) \left( f^\gamma_k \overline{f^\delta_k} \right), \tag{A 2}$$

where $R^{(2)}_{k\overline{\ell}} = g^{i\overline{j}} R^g_{i\overline{j}k\overline{\ell}}$. If $M$ is compact, by applying the standard maximum principle to (A 2), we obtain Corollary 4.5. One may wonder whether Corollary 1.4 can be obtained by applying a similar maximum principle to equation (A 1). Suppose $u$ attains a maximum at some point $p \in X$. Then for any vector $v = (v^1, \dots v^n)$, by formula (A 1), at point $p \in X$, we have

$$0 \geqslant \sum_{i,j} \frac{\partial^2 u}{\partial z^i \partial \overline{z}^j} v^i \overline{v}^j \geqslant \sum_{i,j} \left( \sum_{k,\ell,\alpha} R^g_{i\overline{j}k\overline{\ell}} f^\alpha_k \overline{f^\alpha_\ell} \right) v^i \overline{v}^j. \tag{A 3}$$

Recall that if $(T_M, g)$ is RC-positive, then for any non-zero vector $\xi = (\xi^1, \dots, \xi^n)$, there exists some non-zero vector $\eta = (\eta^1, \dots, \eta^n)$ (*it may depend on* $\xi$!) such that $R^g_{i\overline{j}k\overline{\ell}} \eta^i \overline{\eta}^j \xi^k \overline{\xi}^\ell > 0$. Apparently, in (A 3), there are many vectors indexed by $\alpha$, and, in general, there does not exist a uniform vector $v$ such that the right-hand side of (A 3) is positive. A refined notion called 'uniform RC-positivity' would work for this analytical proof. By using similar ideas, we also investigated rigidity of harmonic maps into Riemannian manifolds in [35] (see also [34, 36]).

The relationship between the Leray–Grothendieck spectral sequence in algebraic geometry and maximum principles in differential geometry will be systematically investigated.

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