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Scalar curvature, Kodaira dimension and $\widehat{A}$-genus

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Abstract

Let (X, g) be a compact Riemannian manifold with quasi-positive Riemannian scalar curvature. If there exists a complex structure J compatible with g, then the Kodaira dimension of (X, J) is equal to $-\infty$ and the canonical bundle K_X is not pseudo-effective. We also introduce the complex Yamabe number $\lambda_c(X)$ for compact complex manifold X, and show that if $\lambda_c(X)$ is greater than 0, then $\kappa(X)$ is equal to $-\infty$; moreover, if X is also spin, then the Hirzebruch A-hat genus $\widehat{A}(X)$ is zero.

Contents

1 Introduction

This is a continuation of our previous paper [38], and we investigate the geometry of Riemannian scalar curvature on compact complex manifolds.

The existences of various positive scalar curvatures are obstructed. For instance, it is well-known that, if a compact Hermitian manifold has positive *Chern scalar curvature*, then the Kodaira dimension is $-\infty$. On the other hand, a classical result of Lichnerowicz (e.g. [17, Theorem 8.12]) says that if a compact Riemannian spin manifold has positive *Riemannian scalar curvature*, then the $\widehat{A}$-genus is zero. We state the first main result of this paper.

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Theorem 1.1 *Let (X, g) be a compact Riemannian manifold with quasi-positive Riemannian scalar curvature. If there exists a complex structure J compatible with g, then the canonical bundle K_X is not pseudo-effective and the Kodaira dimension $\kappa(X, J)$ is $-\infty$.*

Here quasi-positive means non-negative everywhere and strictly positive at some point. As it is well-known, in general the positivity of the Riemannian scalar curvature of (X, J, g) can not imply that of the Chern scalar curvature. As a borderline case, we obtain the second main result of this paper.

Theorem 1.2 *Let (X, g) be a compact Riemannian manifold with zero Riemannian scalar curvature. Suppose there exists a complex structure J compatible with g. Then the Kodaira dimension $\kappa(X, J)$ is either $-\infty$ or 0. Moreover, $\kappa(X, J)$ equals 0 if and only if (X, J, ω_g) is a Kähler Calabi–Yau manifold with $\mathrm{Ric}(\omega_g) = 0$.*

The proofs of Theorems 1.1 and 1.2 rely on several observations in our previous paper [38] and a new scalar curvature relation in Theorem 3.8.

Note that, on Kähler Calabi–Yau surfaces (e.g. $K3$ surfaces, bi-elliptic surfaces), there is no Riemannian metrics with quasi-positive scalar curvature (e.g. [18, Theorem A]). However, by Stolz's solution to the Gromov–Lawson conjecture ([27, Theorem A]), on a simply connected Kähler Calabi–Yau manifold with holonomy group $SU(2m + 1)$, there do exist Riemannian metrics with quasi-positive scalar curvature. On the contrary, as an application of Theorems 1.1 and 1.2, we show that those Riemannian metrics with quasi-positive scalar curvature are not compatible with the Calabi–Yau complex structures, and more generally we obtain the following result.

Corollary 1.3 *On a compact complex Calabi–Yau manifold X with torsion canonical bundle K_X, there is no Hermitian metric with quasi-positive Riemannian scalar curvature. Moreover, if X is also non-Kähler, then there is no Hermitian metric with non-negative Riemannian scalar curvature.*

It is well-known that all compact Kähler Calabi–Yau manifolds have torsion canonical bundle. On the other hand, many non-Kähler Calabi–Yau manifolds also have torsion canonical bundle. For instance, the connected sum $\#_k(\mathbb{S}^3 \times \mathbb{S}^3)$ with $k \geq 2$ ([22]).

On a compact complex manifold X of complex dimension $n \geq 2$, we introduce the complex Yamabe number $\lambda_c(X)$:

$$\lambda_c(X) = \sup_{g \text{ is Hermitian}} \quad \inf_{\widetilde{g} \text{ is conformal to } g} \frac{\int_X s_{\widetilde{g}} dV_{\widetilde{g}}}{\left(\int_X dV_{\widetilde{g}}\right)^{1-\frac{1}{n}}}, \tag{1.1}$$

where $s_{\widetilde{g}}$ is the Riemannian scalar curvature of $\widetilde{g}$. Note that in (1.1), if the supremum is taken over all Riemannian metrics, then it is the classical Yamabe number $\lambda(X)$ in conformal geometry. Hence $\lambda(X) \geq \lambda_c(X)$.

Theorem 1.4 *Let X be a compact complex manifold. If $\lambda_c(X) > 0$, then $\kappa(X) = -\infty$. Moreover, if X is also spin, then $\widehat{A}(X) = 0$.*

According to the results of Gromov–Lawson [13] and Stolz [27], on a simply connected Kähler Calabi–Yau manifold X with $\dim_{\mathbb{C}} X \geq 3$, one has $\lambda(X) > 0$ and $\widehat{A}(X) = 0$. However, we have $\lambda_c(X) \leq 0$ by Theorem 1.4.

As motivated by Theorems 1.1, 1.2, 1.4, various conjectures described in [38, Section 4] and classical works by Schoen–Yau [30–32], Gromov–Lawson [13], Stolz [27] and LeBrun [18] (see also Zhang [41]), we propose the following conjecture.

Conjecture 1.5 Let X be a compact Kähler manifold with $\kappa(X) = -\infty$. If X has a spin structure, then $\widehat{A}(X) = 0$.

Note that Conjecture 1.5 holds when $\dim_{\mathbb{C}} X = 2$ ([18,39]) or $2m+1$.

Finally, let's describe some straightforward applications of Theorem 1.1.

Proposition 1.6 *Let X be a compact Kähler threefold. If there exists a Hermitian metric with quasi-positive Riemannian scalar curvature, then X is uniruled, i.e. X is covered by rational curves.*

According to the uniruledness conjecture (e.g. [4, Conjecture]), Proposition 1.6 should be true on higher dimensional compact Kähler manifolds.

It is a long-standing open problem to determine whether the six-sphere $\mathbb{S}^6$ admits a complex structure or not. Now assuming $X := \mathbb{S}^6$ has a complex structure J. As pointed out in [16, p. 122], it is not at all clear whether $\kappa(X, J) = -\infty$, and proving this would seem to be as complicated as to show that there are no divisors on X at all. It is obvious that $c_1(X) = 0 \in H^2(X, \mathbb{Z})$ and it is also proved in [35] that $c_1^{\mathrm{BC}}(X, J) \neq 0$. In particular, K_X is not holomorphically torsion. For more related discussions, we refer to [1]. Let $\mathscr{S}$ be the space of Riemannian metrics with non-negative scalar curvature.

Theorem 1.7 *If there exists a complex structure J which is compatible with some Riemannian metric $g \in \mathscr{S}$, then K_X is not pseudo-effective and*

$$\kappa(X, J) = -\infty.$$

It is known that there is no complex structure compatible with metrics in a small neighborhood of the round metric on $\mathbb{S}^6$ (e.g. [6,20,24,33]).

2 Preliminaries

2.1 Ricci curvature on complex manifolds

Let (X, ω_g) be a compact Hermitian manifold. Locally, we write $\omega_g = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. The (first Chern-)Ricci form $\mathrm{Ric}(\omega_g)$ of (X, ω_g) has components

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{\ell}})}{\partial z^i \partial \bar{z}^j}$$

which also represents the first Chern class $c_1(X)$ of the complex manifold X (up to a constant). The Chern scalar curvature s_{C} of (X, ω_g) is given by

$$s_{\mathrm{C}} = \mathrm{tr}_{\omega_g} \mathrm{Ric}(\omega_g) = g^{i\bar{j}} R_{i\bar{j}}. \tag{2.1}$$

The total Chern scalar curvature of ω_g is

$$\int_X s_{\mathrm{C}} \cdot \omega_g^n = n \int \mathrm{Ric}(\omega_g) \wedge \omega_g^{n-1}, \tag{2.2}$$

where n is the complex dimension of X.

2.2 The Bott–Chern classes

The Bott–Chern cohomology on a compact complex manifold X is given by

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{\mathrm{Ker} d \cap \Omega^{p,q}(X)}{\mathrm{Im}\partial\overline{\partial} \cap \Omega^{p,q}(X)}.$$

Let $\mathrm{Pic}(X)$ be the set of holomorphic line bundles over X. As similar as the first Chern class map $c_1 : \mathrm{Pic}(X) \to H^{1,1}_{\overline{\partial}}(X)$, there is a *first Bott–Chern class* map

$$c_1^{\mathrm{BC}} : \mathrm{Pic}(X) \to H^{1,1}_{\mathrm{BC}}(X). \tag{2.3}$$

Given any holomorphic line bundle $L \to X$ and any Hermitian metric h on L, its curvature form Θ_h is locally given by $-\sqrt{-1}\partial\overline{\partial}\log h$. We define $c_1^{\mathrm{BC}}(L)$ to be the class of Θ_h in $H^{1,1}_{\mathrm{BC}}(X)$. For a complex manifold X, $c_1^{\mathrm{BC}}(X)$ is defined to be $c_1^{\mathrm{BC}}(K_X^{-1})$ where K_X^{-1} is the anti-canonical line bundle.

2.3 Special manifolds

Let X be a compact complex manifold.

(1) A Hermitian metric ω_g is called a Gauduchon metric if $\partial\overline{\partial}\omega_g^{n-1} = 0$. It is proved by Gauduchon ([12]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to scaling).
(2) A Hermitian metric ω_g is called a Kähler metric if $d\omega_g = 0$.
(3) X is called a Calabi–Yau manifold if $c_1(X) = 0 \in H^2(X, \mathbb{Z})$.

2.4 Kodaira dimension of compact complex manifolds

The Kodaira dimension $\kappa(L)$ of a line bundle L over a compact complex manifold X is defined to be

$$\kappa(L) := \limsup_{m\to+\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^{\otimes m})}{\log m}$$

and the *Kodaira dimension* $\kappa(X)$ of X is defined as $\kappa(X) := \kappa(K_X)$ where the logarithm of zero is defined to be $-\infty$. In particular, if

$$\dim_{\mathbb{C}} H^0(X, K_X^{\otimes m}) = 0$$

for every $m \geq 1$, then $\kappa(X) = -\infty$.

2.5 Spin manifold and $\widehat{A}$-genus

Let X be a compact oriented Riemannian manifold. It is called a spin manifold, if it admits a spin structure, i.e. its second Stiefel–Whitney class $w_2(X) = 0$. It is well-known that all compact Calabi–Yau manifolds are spin.

In the following, we shall briefly describe the definition of the $\widehat{A}$-genus of a compact oriented Riemannian manifold for readers' convenience, and for more necessary background materials, we refer to [17,23–25,37,38] and the references therein. Let $\widehat{A}_i(p_1, \ldots, p_i)$ be the multiplicative sequence of polynomials in the Pontryagin classes p_i of X belonging to the power series

$$\frac{\frac{1}{2}\sqrt{z}}{\sinh\left(\frac{1}{2}\sqrt{z}\right)} = 1 - \frac{1}{24}z + \frac{7}{2^7\cdot 3^2\cdot 5}z^2 + \cdots.$$

The first few terms are

$$\widehat{A}_1(p_1) = -\frac{1}{24}p_1, \quad \widehat{A}_2(p_1,p_2) = \frac{1}{2^7\cdot 3^2\cdot 5}\left(-4p_2 + 7p_1^2\right).$$

The $\widehat{A}$-genus, $\widehat{A}(X)$ is by definition the real number $(\sum_i \widehat{A}_i(p_1,\ldots,p_i))[X]$, where $[X]$ means evaluation of the cohomology class on the fundamental cycle of X. Since $p_i \in H^{4i}(X,\mathbb{Z})$, $\widehat{A}(X)$ is zero unless $\dim_{\mathbb{R}} X \equiv 0 (\mathrm{mod}\ 4)$. Moreover, if X is a spin manifold, $\widehat{A}(X)$ is an integer. The following result is well-known (for more historical explanations, we refer to [36, p. 420] and [17, Theorem 8.12] and the reference therein) and we shall use it frequently in the sequel:

Lemma 2.1 *On a compact spin manifold X, if it admits a Riemannian metric with quasi-positive scalar curvature, then $\widehat{A}(X)=0$.*

3 The Riemannian scalar curvature and Kodaira dimension

Let (X,ω) be a compact Hermitian manifold. We first give several computational results.

Lemma 3.1 *For any smooth real valued function $f \in C^\infty(X,\mathbb{R})$, we have*

$$\overline{\partial}^*(f\omega) = f\overline{\partial}^*\omega + \sqrt{-1}\partial f. \tag{3.1}$$

Proof For any smooth $(1,0)$-form $\eta \in \Gamma(X, T^{*1,0}X)$, we have the global inner product

$$\begin{aligned}\left(\overline{\partial}^*(f\omega),\eta\right) &= \left(f\omega,\overline{\partial}\eta\right) = \left(\omega, f\overline{\partial}\eta\right)\\ &= \left(\omega,\overline{\partial}(f\eta)\right) - \left(\omega,\overline{\partial}f\wedge\eta\right)\\ &= \left(f\overline{\partial}^*\omega,\eta\right) - \left(\omega,\overline{\partial}f\wedge\eta\right)\\ &= \left(f\overline{\partial}^*\omega,\eta\right) + \sqrt{-1}\left(\partial f,\eta\right)\end{aligned}$$

where the last identity follows from the fact that f is real valued. □

Lemma 3.2 *For any $(1,0)$ form η and real valued function $f \in C^\infty(X,\mathbb{R})$, we have*

$$\partial^*(f\eta) = f\partial^*\eta - \langle\eta,\partial f\rangle. \tag{3.2}$$

Proof For any smooth function $\varphi \in C^\infty(X)$, we have

$$\begin{aligned}\left(\partial^*(f\eta),\varphi\right) &= (f\eta,\partial\varphi) = (\eta, f\partial\varphi) = (\eta,\partial(f\varphi) - \partial f\cdot\varphi)\\ &= \left(f\partial^*\eta,\varphi\right) - (\langle\eta,\partial f\rangle,\varphi)\end{aligned}$$

and we obtain (3.2). □

Let $\omega_f = e^f\omega$ for some $f \in C^\infty(X,\mathbb{R})$. We denote by $\partial_f^*, \overline{\partial}_f^*$ and $\partial^*, \overline{\partial}^*$ the adjoint operators taking with respect to ω_f and ω respectively. The local and global inner products with respect to ω and ω_f are indicated by $\langle\bullet,\bullet\rangle$, $(\bullet,\bullet)$ and $\langle\bullet,\bullet\rangle_f$, $(\bullet,\bullet)_f$ respectively.

Lemma 3.3 *For any* $(1,0)$ *form* η *and real valued function* $f \in C^\infty(X,\mathbb{R})$*, we have*

$$\partial_f^*\eta = e^{-f}\left[\partial^*\eta - (n-1)\langle\eta, \partial f\rangle\right]. \tag{3.3}$$

Proof For any $\varphi \in C^\infty(X)$, we have

$$\left(\partial_f^*\eta, \varphi\right)_f = (\eta, \partial\varphi)_f = (e^{(n-1)f}\eta, \partial\varphi) = \left(\partial^*\left(e^{(n-1)f}\eta\right), \varphi\right)$$

where the second identity holds since η is a $(1,0)$-form. By (3.2), we obtain

$$\left(\partial_f^*\eta, \varphi\right)_f = \left(e^{(n-1)f}\partial^*\eta, \varphi\right) - \left(\langle\eta, \partial e^{(n-1)f}\rangle, \varphi\right).$$

Hence,

$$\left(\partial_f^*\eta, \varphi\right)_f = \left(e^{-f}\partial^*\eta, \varphi\right)_f - \left((n-1)e^{-f}\langle\eta, \partial f\rangle, \varphi\right)_f,$$

which verifies (3.3). □

Lemma 3.4 *We have*

$$\overline{\partial}_f^*\omega_f = \overline{\partial}^*\omega + (n-1)\sqrt{-1}\partial f. \tag{3.4}$$

Proof For any $\eta \in \Gamma(X, T^{*1,0}X)$, we have

$$\begin{aligned}\left(\overline{\partial}_f^*\omega_f, \eta\right)_f &= \left(\omega_f, \overline{\partial}\eta\right)_f = \left(e^{(n-1)f}\cdot\omega, \overline{\partial}\eta\right)\\ &= \left(\overline{\partial}^*\left(e^{(n-1)f}\cdot\omega\right), \eta\right).\end{aligned}$$

Now by (3.1), we have

$$\begin{aligned}\left(\overline{\partial}_f^*\omega_f, \eta\right)_f &= \left(e^{(n-1)f}\left[\overline{\partial}^*\omega + (n-1)\sqrt{-1}\partial f\right], \eta\right)\\ &= \left(\overline{\partial}^*\omega + (n-1)\sqrt{-1}\partial f, \eta\right)_f\end{aligned}$$

since η is a $(1,0)$ form. Therefore, we obtain (3.4). □

Lemma 3.5 *We have*

$$\begin{aligned}\sqrt{-1}\partial_f^*\overline{\partial}_f^*\omega_f &= e^{-f}\left(\sqrt{-1}\partial^*\overline{\partial}^*\omega - (n-1)\left(\Delta_d f + \mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial}f\right)\right.\\ &\quad\left. +(n-1)^2|\partial f|^2\right).\end{aligned} \tag{3.5}$$

Proof By formulas (3.2) and (3.4), we have

$$\begin{aligned}\sqrt{-1}\partial_f^*\overline{\partial}_f^*\omega_f &= \sqrt{-1}\partial_f^*\left(\overline{\partial}^*\omega + (n-1)\sqrt{-1}\partial f\right)\\ &= e^{-f}\left(\sqrt{-1}\partial^*\overline{\partial}^*\omega - \sqrt{-1}(n-1)\langle\overline{\partial}^*\omega, \partial f\rangle - (n-1)\partial^*\partial f + (n-1)^2|\partial f|^2\right).\end{aligned}$$

We also observe that

$$\sqrt{-1}\langle\overline{\partial}^*\omega, \partial f\rangle = \overline{\partial}^*\overline{\partial}f + \mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial}f. \tag{3.6}$$

Indeed, for any text function $\varphi \in C^{\infty}(X)$, we have

$$\begin{aligned}\left(\sqrt{-1}\langle\overline{\partial}^*\omega, \partial f\rangle, \varphi\right) &= \sqrt{-1}\left(\overline{\partial}^*\omega, \varphi\partial f\right)\\ &= \sqrt{-1}\left(\omega, \overline{\partial}\varphi\wedge\partial f + \varphi\overline{\partial}\partial f\right)\\ &= (\overline{\partial} f, \overline{\partial}\varphi) + (\omega, \varphi\cdot\sqrt{-1}\partial\overline{\partial} f)\\ &= (\overline{\partial}^*\overline{\partial} f, \varphi) + (\mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial} f, \varphi)\end{aligned}$$

which gives (3.6). Since $\Delta_d f = d^*df = \overline{\partial}^*\overline{\partial} f + \partial^*\partial f$, we obtain (3.5). □

The following observation is one of the key ingredients in the curvature computations.

Lemma 3.6 *Let (X, ω) be a compact Hermitian manifold. Then*

$$\langle\partial\overline{\partial}^*\omega, \omega\rangle = |\overline{\partial}^*\omega|^2 - \sqrt{-1}\partial^*\overline{\partial}^*\omega. \tag{3.7}$$

In particular, if ω is a Gauduchon metric, we have

$$\langle\partial\overline{\partial}^*\omega, \omega\rangle = |\overline{\partial}^*\omega|^2. \tag{3.8}$$

Proof For any smooth real valued function $\varphi \in C^{\infty}(X, \mathbb{R})$, we have

$$\begin{aligned}\left(\langle\partial\overline{\partial}^*\omega, \omega\rangle, \varphi\right) = \left(\partial\overline{\partial}^*\omega, \varphi\omega\right) &= \left(\overline{\partial}^*\omega, \overline{\partial}^*(\varphi\omega)\right)\\ &= \left(\overline{\partial}^*\omega, \varphi\overline{\partial}^*\omega + \sqrt{-1}\partial\varphi\right)\\ &= \left(|\overline{\partial}^*\omega|^2, \varphi\right) + \left(-\sqrt{-1}\partial^*\overline{\partial}^*\omega, \varphi\right)\end{aligned}$$

where we use formula (3.1) in the second identity. Since φ is an arbitrary smooth real function, we obtain (3.7). If ω is Gauduchon, i.e. $\partial\overline{\partial}\omega^{n-1} = 0$, we have $\partial^*\overline{\partial}^*\omega = 0$, and so (3.8) follows from (3.7). □

Corollary 3.7 *On a compact Hermitian manifold (X, ω), the Riemannian scalar curvature s and the Chern scalar curvature s_C are related by*

$$s = 2s_C - 2\sqrt{-1}\partial^*\overline{\partial}^*\omega - \frac{1}{2}|T|^2. \tag{3.9}$$

where T is the torsion tensor of the Hermitian metric ω.

Proof By Lemma 6.2 in the Appendix, we have

$$s = 2s_C + \left(\langle\partial\overline{\partial}^*\omega + \partial\partial^*\omega, \omega\rangle - 2|\overline{\partial}^*\omega|^2\right) - \frac{1}{2}|T|^2.$$

Hence, by formula (3.7) we obtain (3.9). □

Let $\omega_f = e^f\omega$ be a smooth Gauduchon metric (i.e. $\partial\overline{\partial}\omega_f^{n-1} = 0$) in the conformal class of ω for some smooth function $f \in C^{\infty}(X, \mathbb{R})$.

Theorem 3.8 *The total Chern scalar curvature of the Gauduchon metric ω_f is*

$$n\int_X \mathrm{Ric}(\omega_f)\wedge\omega_f^{n-1} = \int_X e^{(n-1)f}\cdot\left(\frac{s}{2} + \frac{|T|^2}{4}\right)\omega^n + (n-1)^2\|\partial f\|^2_{\omega_f}. \tag{3.10}$$

Proof Indeed, since ω_f satisfies $\partial\overline{\partial}\omega_f^{n-1}=0$, we have

$$
\begin{aligned}
n\int_X \mathrm{Ric}(\omega_f)\wedge\omega_f^{n-1} &= n\int_X \mathrm{Ric}(\omega)\wedge\omega_f^{n-1}\\
&= n\int_X e^{(n-1)f}\cdot\mathrm{Ric}(\omega)\wedge\omega^{n-1}=\int_X e^{(n-1)f}\cdot s_C\cdot\omega^n\\
&= \int_X e^{(n-1)f}\left(\frac{s}{2}+\frac{|T|^2}{4}\right)\omega^n+\int_X e^{(n-1)f}\cdot\sqrt{-1}\partial^*\overline{\partial}^*\omega\cdot\omega^n,
\end{aligned}
$$

where we use the scalar curvature relation (3.9) in the third identity. Since ω_f is Gauduchon, we have $\partial_f^*\overline{\partial}_f^*\omega_f=0$. By formula (3.5), we have

$$
\sqrt{-1}\partial^*\overline{\partial}^*\omega=(n-1)\left(\Delta_d f+\mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial}f\right)-(n-1)^2|\partial f|^2. \tag{3.11}
$$

It is easy to show that

$$
\int_X e^{(n-1)f}\mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial}f\cdot\omega^n=n\int_X\sqrt{-1}\partial\overline{\partial}f\wedge\omega_f^{n-1}=0
$$

and

$$
\int_X e^{(n-1)f}|\partial f|^2\omega^n=\|\partial f\|^2_{\omega_f}.
$$

Moreover,

$$
\begin{aligned}
\int_X e^{(n-1)f}\Delta_d f\omega^n &= \left(d^*df, e^{(n-1)f}\right)\\
&= (n-1)\left(df, e^{(n-1)f}df\right)\\
&= (n-1)\,(df,df)_f
\end{aligned}
$$

since df is a 1-form. Finally, we obtain

$$
\int_X e^{(n-1)f}\cdot\sqrt{-1}\partial^*\overline{\partial}^*\omega\cdot\omega^n=(n-1)^2\|df\|^2_{\omega_f}-(n-1)^2\|\partial f\|^2_{\omega_f}=(n-1)^2\|\partial f\|^2_{\omega_f}.
$$

Putting all together, we get (3.10). □

The proof of Theorem 1.1 Let ω be the Hermitian metric of (g,J). Let $\omega_f=e^f\omega$ be a smooth Gauduchon metric in the conformal class of ω. If the Riemannian scalar curvature s of ω is quasi-positive, then by formula (3.10), the total Chern scalar curvature of the Gauduchon metric ω_f is strictly positive, i.e.

$$
n\int_X \mathrm{Ric}(\omega_f)\wedge\omega_f^{n-1}>0.
$$

By [38, Theorem 1.1] and [38, Corollary 3.3], K_X is not pseudo-effective and $\kappa(X,J)=-\infty$. □

The following result follows from the proofs of Theorem 1.1 and [38, Lemma 3.2].

Corollary 3.9 *Let (X,ω) be a compact Hermitian manifold such that the background Riemannian metric has quasi-positive Riemannian scalar curvature, then there exists a (possible different) Hermitian metric $\widetilde{\omega}$ with positive Chern scalar curvature.*

The proof of Theorem 1.2 Suppose $\kappa(X, J) \geq 0$. If ω_g is not a Kähler metric, i.e. the torsion $|T|^2$ is not identically zero, then by formula (3.10), there exists a Gauduchon metric with positive total Chern scalar curvature. Hence, by [38, Corollary 3.3] we have $\kappa(X, J) = -\infty$ which is a contradiction. Therefore, ω_g is a Kähler metric and so in formula (3.10), f is a constant and $T = 0$. That means, ω_g is a Kähler metric with zero scalar curvature. Since $\kappa(X, J) \geq 0$, by [38, Corollary 1.6], X is a Calabi–Yau manifold and $\kappa(X, J) = 0$. By the Calabi–Yau theorem ([40]), there exists a Kähler Ricci-flat metric ω_{CY}, i.e. $\mathrm{Ric}(\omega_{\mathrm{CY}}) = 0$. Hence,

$$\mathrm{Ric}(\omega_g) = \mathrm{Ric}(\omega_g) - \mathrm{Ric}(\omega_{\mathrm{CY}}) = \sqrt{-1}\partial\overline{\partial}F$$

where $F = \log\left(\frac{\omega_{\mathrm{CY}}^n}{\omega_g^n}\right)$. Since ω_g has zero scalar curvature, we have

$$\Delta_{\omega_g} F = \mathrm{tr}_{\omega_g}\sqrt{-1}\partial\overline{\partial}F = 0,$$

which implies $F = \mathrm{const}$ and $\mathrm{Ric}(\omega_g) = 0$.

If (X, J, ω_g) is a Kähler Calabi–Yau manifold with $\mathrm{Ric}(\omega_g) = 0$, it is well-known that K_X is a holomorphic torsion, i.e. $K_X^{\otimes \ell} = \mathcal{O}_X$ for some positive integer ℓ. Hence, $\kappa(X, J) = 0$. □

As an application of Theorems 1.1 and 1.2, we have the following result.

Corollary 3.10 *Let (X, g) be a compact Riemannian manifold with nonnegative Riemannian scalar curvature. If there exists a complex structure J which is compatible with g, then either*

(1) $\kappa(X, J) = -\infty$; or
(2) $\kappa(X, J) = 0$ and (X, J, g) is a Kähler Calabi–Yau.

Proposition 3.11 *Suppose X is a compact complex manifold with $c_1^{\mathrm{BC}}(X) \leq 0$. Then*

(1) there exists a Hermitian metric with non-positive Riemannian scalar curvature;
(2) there is no Hermitian metric with quasi-positive Riemannian scalar curvature.

Moreover, X admits a Hermitian metric g with zero Riemannian scalar curvature if and only if (X, g) is a Kähler Calabi–Yau.

Proof Note that by definition there exists a d-closed non-positive $(1, 1)$ form η which represents $c_1^{\mathrm{BC}}(X)$. By [28, Theorem 1.3], there exists a non-Kähler Gauduchon metric ω_G such that

$$\mathrm{Ric}(\omega_G) = \eta \leq 0.$$

Hence, for any Gauduchon metric ω,

$$\int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1} = \int_X \mathrm{Ric}(\omega_G) \wedge \omega^{n-1} \leq 0. \tag{3.12}$$

(1). Since ω_G is Gauduchon, by formula (3.9), we have

$$s = 2s_{\mathrm{C}} - \frac{1}{2}|T|^2 = 2\mathrm{tr}_{\omega_G}\mathrm{Ric}(\omega_G) - \frac{1}{2}|T|^2 \leq 0.$$

(2). If there exists a Hermitian metric with quasi-positive Riemannian scalar curvature, then it induces a Gauduchon metric with positive total Chern scalar curvature which is a contradiction.

Suppose X admits a Hermitian metric g with zero Riemannian scalar curvature, then by formulas (3.10) and (3.12), we have $T = 0$ and $f = 0$, i.e. (X, ω_g) is a Kähler manifold with zero scalar curvature. Since $c_1^{\mathrm{BC}}(X) = c_1(X) \leq 0$, we have $\mathrm{Ric}(\omega_g) = 0$. □

The proof of Corollary 1.3 If K_X is a torsion, i.e. $K_X^{\otimes m} = \mathcal{O}_X$ for some $m \geq 1$, then $\kappa(X) = 0$. The first part of Corollary 1.3 follows from Theorem 1.1, and the second part follows from Theorem 1.2. □

The proof of Proposition 1.6 By Theorem 1.1, K_X is not pseudo-effective and $\kappa(X) = -\infty$. Hence by [7, Corollary 1.2] or [15, Corollary 1.4], we conclude X is uniruled. □

The proof of Theorem 1.7 Since $H^2(X, \mathbb{R}) = 0$, the Hermitian metric (g, J) is not Kähler. Then Theorem 1.7 follows from Theorem 1.1 and Theorem 1.2. □

4 The Yamabe number, $\widehat{A}$-genus and Kodaira dimension

Let (X, g_0) be a compact Riemannian manifold with real dimension $2n$. The *Yamabe invariant* $\lambda(X, g_0)$ of the conformal class $[g_0]$ is defined as

$$\lambda(X, g_0) = \inf_{g=e^f g_0, f\in C^\infty(X,\mathbb{R})} \frac{\int_X s_g dV_g}{\left(\int_X dV_g\right)^{1-\frac{1}{n}}} \tag{4.1}$$

where s_g is the Riemannian scalar curvature of g. Moreover, one can define the Yamabe number

$$\lambda(X) = \sup_{\text{all Riemannian metric } g} \lambda(X, g). \tag{4.2}$$

As analogous to (4.2), on a compact complex manifold X, one can define the complex version

$$\lambda_c(X) = \sup_{\text{all Hermitian metric } g} \lambda(X, g). \tag{4.3}$$

Theorem 4.1 *Let X be a compact complex manifold. If $\lambda_c(X) > 0$, then $\kappa(X) = -\infty$. Moreover, if X is also spin, then $\widehat{A}(X) = 0$.*

Proof Suppose $\lambda_c(X) > 0$, then there exists a Hermitian metric g_0 such that

$$\lambda(X, g_0) = \inf_{g\in[g_0]} \frac{\int_X s_g dV_g}{\left(\int_X dV_g\right)^{1-\frac{1}{n}}} > 0.$$

Let $\omega_f = e^f \omega_{g_0}$ be a Gauduchon metric in the conformal class of ω_{g_0}. Hence, ω_f has positive total Riemannian scalar curvature

$$\int_X s_f \cdot \omega_f^n > 0.$$

Moreover, by formula (3.9), the total Chern scalar curvature of ω_f is

$$\int_X (s_C)_f \cdot \omega_f^n = \int_X \frac{s_f}{2} \cdot \omega_f^n + \frac{1}{4}\int_X |T_f|_f^2 \cdot \omega_f^n > 0, \tag{4.4}$$

where we use the fact that ω_f is Gauduchon, i.e. $\partial_f^* \overline{\partial}_f^* \omega_f = 0$. Therefore, the Gauduchon metric ω_f has positive total Chern scalar curvature, and by [38, Corollary 3.2], $\kappa(X) = -\infty$. We also have $\lambda(X) \geq \lambda_c(X) > 0$. On the other hand, by a straightforward calculation ([29, Lemma 1.2]), one can show that $\lambda(X) > 0$ if and only if there exists a Riemannian metric with positive Riemannian scalar curvature. Hence, by Lichnerowicz's result (e.g. Lemma 2.1), if X is spin and $\lambda_c(X) > 0$, then $\widehat{A}(X) = 0$. □

Note that on a simply connected Kähler Calabi–Yau manifold X with $\dim_{\mathbb{C}} X = 2m+1$, one has $\lambda(X) > 0$ and $\widehat{A}(X) = 0$. However, $\lambda_c(X) \leq 0$.

Question 4.2 On a compact Kähler (or complex) manifold X, find sufficient and necessary conditions such that $\lambda(X)$ and $\lambda_c(X)$ have the same sign, or $\lambda(X) = \lambda_c(X)$.

A result along this line is

Corollary 4.3 *Let X be a simply connected compact complex manifold with $\dim_{\mathbb{C}} X \geq 3$. If $\lambda_c(X)$ has the same sign as $\lambda(X)$, then $\kappa(X) = -\infty$.*

Proof By Gromov–Lawson [13] and Stolz [27], if X is a simply connected complex manifold with $\dim_{\mathbb{C}} X \geq 3$, then X has a Riemannian metric with positive scalar curvature, hence $\lambda(X) > 0$ and so $\lambda_c(X) > 0$. By Theorem 4.1, we obtain $\kappa(X) = -\infty$. □

Finally, we want to present a nice result of LeBrun, which answers Conjecture 1.5 affirmatively when X is a compact spin Kähler surface (for related works, see also [14] and [20]):

Theorem 4.4 *[18,* Theorem A] *Let X be a compact Kähler surface, then*

$$\begin{cases} \lambda(X) > 0 \ \textit{if and only if} \ \ \kappa(X) = -\infty; \\ \lambda(X) = 0 \ \textit{if and only if} \ \ \kappa(X) = 0 \ \ \textit{or} \ \ 1; \\ \lambda(X) < 0 \ \textit{if and only if} \ \ \kappa(X) = 2. \end{cases} \tag{4.5}$$

According to Theorems 1.1, 1.2, 1.4 and [38, Theorem 1.1], there should be some analogous results for $\lambda_c(X)$ on compact Kähler manifolds, which will be addressed in future studies. For some related settings, we refer to [2,3,8,21] and the references therein.

5 Examples on compact non-Kähler Calabi–Yau surfaces

In this section, we discuss two special Calabi–Yau surfaces of class VII. One is the diagonal Hopf surface $\mathbb{S}^1 \times \mathbb{S}^3$ and the other one is the Inoue surface. It is well-known, they are non-Kähler Calabi–Yau surfaces with Kodaira dimension $-\infty$, $b_1(X) = 1$ and $b_2(X) = 0$. We show by the following example that the converses of Theorems 1.1 and 1.4 are not valid in general:

Proposition 5.1 *For every Inoue surface X, it has $\kappa(X) = -\infty$ and $\widehat{A}(X) = 0$. However, it can not support a Hermitian metric with non-negative Riemannian scalar curvature. In particular, $\lambda_c(X) \leq 0$.*

Proof Since X is a non-Kähler Calabi–Yau manifold with $b_2(X) = 0$, one can see $c_1^2 = 0$ and $c_2 = 0$ (e.g. [5, Proposition 19.2 in Chapter V]). Hence, by the index theorem, we have

$$\widehat{A}(X) = -\frac{1}{8}\tau(X) = -\frac{1}{24}(c_1^2 - c_2) = 0.$$

On the other hand, on each Inoue surface, there exists a smooth Gauduchon metric with non-positive Ricci curvature. Indeed, let $(w, z) \in \mathbb{H} \times \mathbb{C}$ be the holomorphic coordinates, then by the precise definition of each Inoue surface ([9,10,25,34]), it is easy to see that the metric $h^{-1} = [\mathrm{Im}(w)]^{-1}(dw \wedge dz) \otimes (d\overline{w} \wedge d\overline{z})$ (resp. $h^{-1} = [\mathrm{Im}(w)]^{-2}(dw \wedge dz) \otimes (d\overline{w} \wedge d\overline{z})$)

is a globally defined Hermitian metric on the anti-canonical bundle of S_M (resp. $S^+_{N,p,q,r;t}$) (e.g. [9, Section 6]). Hence, the Chern Ricci curvature of S_M is

$$-\sqrt{-1}\partial\overline{\partial}\log h^{-1} = \sqrt{-1}\partial\overline{\partial}\log[\mathrm{Im}(w)] = -\frac{\sqrt{-1}}{4}\frac{dw\wedge d\overline{w}}{[\mathrm{Im}(w)]^2},$$

which also represents $c_1^{\mathrm{BC}}(X)$. By Theorem [28, Theorem 1.3], there exists a Gauduchon metric ω_G with

$$\mathrm{Ric}(\omega_G) = -\frac{\sqrt{-1}}{4}\frac{dw\wedge d\overline{w}}{[\mathrm{Im}(w)]^2} \leq 0.$$

(Note also that the Riemannian scalar curvature of ω_G is strictly negative according to (3.9).) Hence, for any Gauduchon metric ω, the total Chern scalar curvature

$$2\int_X \mathrm{Ric}(\omega)\wedge\omega = 2\int_X \mathrm{Ric}(\omega_G)\wedge\omega < 0.$$

If X admits a Hermitian metric ω with non-negative Riemannian scalar curvature, then by formula (3.10), there exists a Gauduchon metric with positive total Chern scalar curvature. This is a contradiction. We can deduce similar contradictions for $S^{\pm}_{N,p,q,r;t}$. □

A straightforward computation shows that on diagonal Hopf surfaces $\mathbb{S}^1\times\mathbb{S}^3$, there exist Hermitian metrics $\omega_+, \omega_-, \omega_0$ with positive, negative and zero Riemannian scalar curvature respectively ([24, Section 6]). However, their Chern scalar curvatures are all positive.

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6 Appendix: The scalar curvature relation on compact complex manifolds

Let's recall some elementary settings (e.g. [24, Section 2]). Let (M, g, ∇) be a $2n$-dimensional Riemannian manifold with the Levi–Civita connection ∇. The tangent bundle of M is also denoted by $T_{\mathbb{R}}M$. The Riemannian curvature tensor of (M, g, ∇) is

$$R(X, Y, Z, W) = g\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\right)$$

for tangent vectors $X, Y, Z, W \in T_{\mathbb{R}}M$. Let $T_{\mathbb{C}}M = T_{\mathbb{R}}M\otimes\mathbb{C}$ be the complexification. We can extend the metric g and the Levi–Civita connection ∇ to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T_{\mathbb{C}}M$, we have

$$R(aX, bY, cZ, dW) = abcd\cdot R(X, Y, Z, W).$$

Let (M, g, J) be an almost Hermitian manifold, i.e., $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ with $J^2 = -1$, and for any $X, Y \in T_{\mathbb{R}}M$, $g(JX, JY) = g(X, Y)$. The Nijenhuis tensor $N_J : \Gamma(M, T_{\mathbb{R}}M)\times\Gamma(M, T_{\mathbb{R}}M) \to \Gamma(M, T_{\mathbb{R}}M)$ is defined as

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

The almost complex structure J is called *integrable* if $N_J \equiv 0$ and then we call (M, g, J) a Hermitian manifold. We can also extend J to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. Hence for any $X, Y \in T_{\mathbb{C}}M$, we still have $g(JX, JY) = g(X, Y)$. By Newlander–Nirenberg's theorem, there exists a real coordinate system $\{x^i, x^I\}$ such that $z^i = x^i + \sqrt{-1}x^I$ are local holomorphic coordinates on M. Let's define a Hermitian form $h : T_{\mathbb{C}}M \times T_{\mathbb{C}}M \to \mathbb{C}$ by

$$h(X, Y) := g(X, Y), \qquad X, Y \in T_{\mathbb{C}}M. \tag{6.1}$$

By J-invariant property of g,

$$h_{ij} := h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = 0, \quad \text{and} \quad h_{\bar{i}\bar{j}} := h\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0 \tag{6.2}$$

and

$$h_{i\bar{j}} := h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \frac{1}{2}\left(g_{ij} + \sqrt{-1}g_{iJ}\right). \tag{6.3}$$

It is obvious that $(h_{i\bar{j}})$ is a positive Hermitian matrix. Let ω be the fundamental two-form associated to the J-invariant metric g:

$$\omega(X, Y) = g(JX, Y). \tag{6.4}$$

In local complex coordinates,

$$\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j. \tag{6.5}$$

In the local holomorphic coordinates $\{z^1, \ldots, z^n\}$ on M, the complexified Christoffel symbols are given by

$$\begin{aligned}\Gamma^C_{AB} &= \sum_E \frac{1}{2}g^{CE}\left(\frac{\partial g_{AE}}{\partial z^B} + \frac{\partial g_{BE}}{\partial z^A} - \frac{\partial g_{AB}}{\partial z^E}\right)\\ &= \sum_E \frac{1}{2}h^{CE}\left(\frac{\partial h_{AE}}{\partial z^B} + \frac{\partial h_{BE}}{\partial z^A} - \frac{\partial h_{AB}}{\partial z^E}\right)\end{aligned} \tag{6.6}$$

where $A, B, C, E \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ and $z^A = z^i$ if $A = i$, $z^A = \bar{z}^i$ if $A = \bar{i}$. For example

$$\Gamma^k_{ij} = \frac{1}{2}h^{k\bar{\ell}}\left(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j}\right), \ \Gamma^k_{\bar{i}j} = \frac{1}{2}h^{k\bar{\ell}}\left(\frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^\ell}\right). \tag{6.7}$$

We also have $\Gamma^k_{\bar{i}\bar{j}} = \Gamma^{\bar{k}}_{ij} = 0$ by the Hermitian property $h_{pq} = h_{\bar{i}\bar{j}} = 0$. The complexified curvature components are

$$R^D_{ABC} = \sum_E R_{ABCE}h^{ED} = -\left(\frac{\partial \Gamma^D_{AC}}{\partial z^B} - \frac{\partial \Gamma^D_{BC}}{\partial z^A} + \Gamma^F_{AC}\Gamma^D_{FB} - \Gamma^F_{BC}\Gamma^D_{AF}\right). \tag{6.8}$$

By the Hermitian property again, we have

$$R^l_{i\bar{j}k} = -\left(\frac{\partial \Gamma^l_{ik}}{\partial \bar{z}^j} - \frac{\partial \Gamma^l_{\bar{j}k}}{\partial z^i} + \Gamma^s_{ik}\Gamma^l_{\bar{j}s} - \Gamma^s_{\bar{j}k}\Gamma^l_{is} - \Gamma^{\bar{s}}_{\bar{j}k}\Gamma^l_{i\bar{s}}\right). \tag{6.9}$$

It is computed in [24, Lemma 7.1] that

Lemma 6.1 *On the Hermitian manifold (M, h), the Riemannian Ricci curvature of the Riemannian manifold (M, g) satisfies*

$$Ric(X, Y) = h^{i\bar{\ell}}\left[R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \bar{z}^\ell}\right) + R\left(\frac{\partial}{\partial z^i}, Y, X, \frac{\partial}{\partial \bar{z}^\ell}\right)\right] \tag{6.10}$$

for any $X, Y \in T_{\mathbb{R}}M$. The Riemannian scalar curvature is

$$s = 2h^{i\overline{j}}h^{k\overline{\ell}}\left(2R_{i\overline{\ell}k\overline{j}} - R_{i\overline{j}k\overline{\ell}}\right). \tag{6.11}$$

The following result is established in [24, Corollary 4.2] (see also some different versions in [11]). For readers' convenience we include a straightforward proof without using "normal coordinates".

Lemma 6.2 *On a compact Hermitian manifold (M, ω), the Riemannian scalar curvature s and the Chern scalar curvature s_{C} are related by*

$$s = 2s_{\mathrm{C}} + \left(\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle - 2|\partial^*\omega|^2\right) - \frac{1}{2}|T|^2, \tag{6.12}$$

where T is the torsion tensor with

$$T_{ij}^k = h^{k\overline{\ell}}\left(\frac{\partial h_{j\overline{\ell}}}{\partial z^i} - \frac{\partial h_{i\overline{\ell}}}{\partial z^j}\right).$$

Proof For simplicity, we denote by

$$s_{\mathrm{R}} = h^{i\overline{j}}h^{k\overline{\ell}}R_{i\overline{\ell}k\overline{j}} \quad \text{and} \quad s_{\mathrm{H}} = h^{i\overline{j}}h^{k\overline{\ell}}R_{i\overline{j}k\overline{\ell}}.$$

Then, by formula (6.11), we have $s = 4s_{\mathrm{R}} - 2s_{\mathrm{H}}$. In the following, we shall show

$$s_{\mathrm{H}} = s_{\mathrm{C}} - \frac{1}{2}\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle - \frac{1}{4}|T|^2 \tag{6.13}$$

and

$$s_{\mathrm{R}} = s_{\mathrm{C}} - \frac{1}{2}|\partial^*\omega|^2 - \frac{1}{4}|T|^2. \tag{6.14}$$

It is easy to show that

$$\overline{\partial}^*\omega = 2\sqrt{-1}\overline{\Gamma_{ik}^k}dz^i \tag{6.15}$$

and so

$$-\frac{\partial\partial^*\omega + \overline{\partial\partial}^*\omega}{2} = \sqrt{-1}\left(\frac{\partial\Gamma_{jk}^k}{\partial z^i} + \frac{\partial\overline{\Gamma_{ik}^k}}{\partial\overline{z}^j}\right)dz^i \wedge d\overline{z}^j. \tag{6.16}$$

On the other hand, by formula (6.9), we have

$$R_{i\overline{j}k}^k = -\frac{\partial\Gamma_{ik}^k}{\partial\overline{z}^j} + \frac{\partial\Gamma_{jk}^k}{\partial z^i} + \Gamma_{jk}^{\overline{s}}\Gamma_{i\overline{s}}^k. \tag{6.17}$$

A straightforward calculation shows

$$h^{i\overline{j}}\Gamma_{jk}^{\overline{s}}\Gamma_{i\overline{s}}^k = -\frac{1}{4}|T|^2.$$

Moreover, we have

$$\left(-\frac{\partial\Gamma_{ik}^k}{\partial\overline{z}^j} + \frac{\partial\Gamma_{jk}^k}{\partial z^i}\right) - \left(\frac{\partial\Gamma_{jk}^k}{\partial z^i} + \frac{\partial\overline{\Gamma_{ik}^k}}{\partial\overline{z}^j}\right) = -\frac{\partial\Gamma_{ik}^k}{\partial\overline{z}^j} - \frac{\partial\overline{\Gamma_{ik}^k}}{\partial\overline{z}^j} = -\frac{\partial^2\log\det(g)}{\partial z^i\partial\overline{z}^j} \tag{6.18}$$

where the last identity follows from (6.7). Indeed, we have

$$\Gamma^k_{ik} + \overline{\Gamma^k_{\bar{i}k}} = h^{k\bar{\ell}} \frac{\partial h_{k\bar{\ell}}}{\partial z^i} = \frac{\partial \log \det(g)}{\partial z^i}.$$

Hence, we obtain

$$s_{\rm H} + \left\langle \frac{\partial\partial^*\omega + \overline{\partial\partial}^*\omega}{2}, \omega \right\rangle = s_{\rm C} - \frac{1}{4}|T|^2$$

which proves (6.13). Similarly, one can show (6.14). □

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