A Non-Perturbative Definition of the Standard Models

Juven Wang[1,2] and Xiao-Gang Wen[3]

[1] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*
[2] *Center of Mathematical Sciences and Applications, Harvard University, MA 02138, USA*
[3] *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

The Standard Models contain chiral fermions coupled to gauge theories. It has been a long-standing problem to give such gauged chiral fermion theories a quantum non-perturbative definition. By classification of quantum anomalies (including perturbative local anomalies and non-perturbative global anomalies) and symmetric interacting invertible topological orders via a mathematical cobordism theorem for differentiable and triangulable manifolds, and by the *existence* of symmetric gapped boundary (designed for the mirror sector) on the trivial symmetric invertible topological orders, we propose that Spin(10) chiral fermion theories with Weyl fermions in 16-dimensional spinor representations can be defined on a 3+1D lattice without fermion doubling, and subsequently dynamically gauged to be a Spin(10) chiral gauge theory. As a result, the Standard Models from the 16n-chiral fermion SO(10) Grand Unification can be defined non-perturbatively via a 3+1D local lattice model of bosons or qubits. Furthermore, we propose that Standard Models from the 15n-chiral fermion SU(5) Grand Unification can be regularized by a 3+1D local lattice model of fermions.

CONTENTS

I. INTRODUCTION AND DEFINITIONS

The *Standard Models* [1–3], gauge theories with the Lie algebra $u(1) \times su(2) \times su(3)$ in 3+1D, coupled to fermions and bosons, are believed to describe elementary particles.[1] In the standard *Standard Model*, there are 15 of 2-component complex Weyl fermions per family. The SU(5) Grand Unification [5] has 15 complex Weyl fermions per family. There are also non-standard Standard Models, such as the one from the SO(10) Grand Unification [6] which has 16 complex Weyl fermions per family. But for a long time, the Standard Models were only defined via a perturbative expansion, which is known not to converge. So the Standard Models were not yet known

[1] Elementary particles include fermions from quarks and leptons, and bosons from gauge mediators and Higgs particle. Gravitons are not yet discovered experimentally. In addition, in our work, we do not consider any dynamical gravity; we only consider anomalies of gauge fields or gravitational non-dynamical background fields. The local Lie algebra of standard Standard Models is $u(1)\times su(2)\times su(3)$, but the global structure Lie group can be $\frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_q}$ where $q = 1, 2, 3, 6$, see a recent overview [4] on this issue. In fact, as we will show later that for SO(10) and SU(5) Grand Unifications, it is more natural to study the case $q = 6$. Also, we denote the d-dimensional space and 1-dimensional time as $d + 1$D, and denote the d-dimensional space as dD.

to be well-defined quantum theories. This is related to the long-standing gauged chiral fermion problem: How to define a chiral fermion theory, with the parity violation [7], coupled to the gauge field, non-perturbatively and in the same dimension, as a well-defined quantum theory with a finite-dimensional Hilbert space for a finite-size system (for details, see Appendix A 1), but without suffered from fermion doublings [8]. In this work, we use the term *gauged chiral fermion theory* to mean chiral fermion theory coupled to a non-dynamical background gauge field. In fact, the gauge theories focused in this article are mostly non-dynamical, unless mentioned otherwise.

There were many previous pioneer attempts, such as a lattice gauge approach [9], Ginsparg-Wilson fermion approach [10], Domain-wall fermion approach [11, 12], and Overlap-fermion approach [13, 14]. In the Ginsparg-Wilson fermion approach, the to-be-gauged symmetry is not strictly an onsite symmetry but only a quasi-local symmetry (see Def. II and [15–18], the quasi-local symmetry is still a non-onsite symmetry), thus it is much challenging to gauge. (The abelian chiral gauge theory is achieved by Ref. [14], however, the non-abelian case is still an open question.) In the Domain-wall fermion approach, we have an extra dimension, where the dynamical gauge fields can propagate. The Overlap-fermion approach is a reformulation of the Domain-wall fermion approach. The above approaches normally start with a spacetime Euclidean lattice path integral and implement the Ginsparg-Wilson fermion.

In contrast, in our work, we do *not* formulate a spacetime lattice path integral *nor* Ginsparg-Wilson fermion. Instead, we consider a discretized spatial lattice Hamiltonian with a continuous time, with additional criteria (see Def. I): (1) with a tensor product Hilbert space, (2) with all interaction terms bounded by a finite range of lattice spacings (called short-range interactions), (3) we only discuss onsite symmetries (see Def. II). Below we refer to our setup as a *local lattice model*.[2]

In this work, we aim to show nontrivial evidence that the gauged chiral fermion problem in both the 16n-fermion and the 15n-fermion Standard Models can be solved via a generalized lattice gauge approach under *local lattice model* (Def. I). In the standard lattice gauge approach, the fermions do not interact directly. The generalized lattice gauge approach simply adds an extra direct fermion interaction or an indirect fermion interaction via some Higgs fields. A generalized lattice gauge approach, called the mirror fermion approach, was proposed in 1986 [19, 20]. In such an approach, one starts with a lattice model containing chiral fermions (named the chiral sector or the normal sector) and a chiral conjugated mirror sector (the mirror sector), with a to-be-gauged symmetry acting as an onsite symmetry. Then, *one includes a proper fermion interaction* [21, 22] *in such a local lattice model, attempting to gap out the mirror sector completely, without breaking the onsite symmetry and without affecting the low energy properties of the normal sector.* This is the key step, which will be referred to as *gapping out the mirror sector without breaking the (to-be-gauged) symmetry*. Last, one can gauge the onsite symmetry to obtain a gauged chiral fermion theory, regularized by a local lattice model.[3]

Ref. 19 proposed a way to gap out the mirror sector without breaking the symmetry, by introducing composite fermion fields formed by mirror fermion fields, and by adding symmetric mass terms between composite fermion fields and the mirror fermion fields to make all those fermion fields formally massive. However, such a proposal cannot work in general. Even we can make all the fermion fields formally massive, it does not imply we can fully gap out the mirror sector. This is because, even for some models with a *perturbative local anomaly*,[4] one can find composite fermion fields formed by mirror fermion fields and to make all those fermion fields formally massive (see the Appendix of Ref. 23 arXiv version). Also, the extensive studies on the previous mirror fermion proposal [24–27] had difficulties to demonstrate that interactions can fully gap out the mirror sector without breaking the symmetry and without modifying the low energy dynamics of the normal sector. It was pointed out in Ref. 28 that "attempts to decouple lattice fermion doubles by the method of Swift and Smit cannot succeed." Consequently, many people gave up the mirror fermion approach.

Recently, Ref. 23 *conjectured* a new gapping condition for the mirror sector:

Proposition i: *Consider a mirror sector in $d+1D$ with a symmetry group G. The mirror sector can be gapped out without breaking the symmetry G if (1) there exist (possibly G-symmetry breaking) mass terms that make all the fermions massive; and (2) $\pi_n(G/G_{grnd}) = 0$ for $n \leq d+2$, where G_{grnd} is the unbroken symmetry group.*

The above Ref. 23's claim is obtained based on the following assumption (not rigorously proven so far):

[2] For a concrete lattice model, we mostly focus on a spatial lattice Hamiltonian. However, our arguments and Propositions 1, 2 and 3 are more general than a Hamiltonian picture, they are also applicable to quantum field theory and spacetime path integral approaches.

[3] Colloquially, we refer a lattice regularization as the same as a lattice realization. When we say a field theory can be regularized (on the boundary of a lattice in one higher dimension), we also mean a field theory can be realized.

[4] We overview the concepts of anomalies, including *perturbative local anomaly* and *non-perturbative global anomaly* in Appendix A. In the main text, however, we purposefully reduce the information on anomalies but focus on the mathematically much well-defined concepts called the *cobordism theory*.

Proposition ii: *A $d+1$D G-symmetric non-linear σ-model with topologically trivial target space $M = G/G_{grnd}$ (i.e. $\pi_n(M) = 0$ for $n \leq d+2$) allows a gapped G-symmetric ground state.*

Applying the above Propositions, Ref. 23 *claimed* that 3+1D Spin(10) chiral fermion theory with Weyl fermions in a 16-dimensional spinor representation can be defined via an *interacting* local lattice model with a Spin(10) onsite symmetry which can be gauged.[5] The 16-fermion Standard Model (*i.e.* SO(10) Grand Unification or SO(10) Grand Unified Theory ≡ SO(10) GUT) can then be obtained from a 3+1D Spin(10) chiral gauge theory, coupled to Spin(10) chiral Weyl fermions in the 16-dimensional representation of Spin(10).

Purpose of our present work: The homotopy group argument in Ref. 23 only *proposed* a sufficient condition. There are mirror sectors (thus also normal sectors) that do not satisfy the condition, but that mirror sectors can still be gapped out without breaking the symmetry and without altering low energy physics in the normal sector. In this work, we are going to prescribe a more general condition, to capture the cases missed by Ref. 23:

Proposition 1. *Consider a continuum field theory in $d+1$D with an internal symmetry group G_f.[6] If the following two conditions hold: (i) If the field theory can be regularized as the low energy effective theory of a boundary of a gapped local lattice model in one higher dimension $d+2$D with a bulk onsite symmetry G_f, and (ii) if the gapped ground state of the bulk lattice model represents a trivial cobordism invariant in $d+2$D; then the $d+1$D field theory can be regularized as the low energy effective theory of a local lattice model in the same dimension $d+1$D with an* **onsite** *internal symmetry G_f.*

Our above statement used the following assumption:

Proposition 2. *A gapped local lattice model with an onsite internal symmetry G_f in $d+2$D must exist a G_f-symmetric gapped $d+1$D boundary (that does not break the G_f symmetry), if its gapped bulk ground state in $d+2$D represents a trivial cobordism invariant in $d+2$D.*

The existence of a symmetric gapped boundary is based on the belief that the bulk with a trivial cobordism invariant can be smoothly deformed into a symmetric product state without closing the gap. The symmetric product state always have a symmetric gapped boundary. See Appendix A and D for further details.

To obtain Proposition 1, we have to apply Proposition 2. We first regularize the field theory as a boundary (also referred as the normal sector) of the gapped lattice model in one higher dimension, then assume the lattice model has a finite thickness, and make the boundary on the other side (also referred as the mirror sector) to be the symmetric gapped boundary ensured by Proposition 2.

Using the above statements, we will show that a 3+1D Spin(10) chiral fermion theory with Weyl fermions in a 16-dimensional spinor representation can be defined via an *interacting* local lattice model with a Spin(10) onsite internal symmetry which can be gauged. In addition, we will show that a 3+1D SU(5) chiral fermion theory with Weyl fermions in 5-dimensional and 10-dimensional representations can be defined via an *interacting* local lattice model with an SU(5) onsite symmetry which can be gauged.

Last, we remark that to fully characterize the global symmetry in a fermion system, we need to specify the full internal global symmetry group G_f and how the fermion number parity $\mathbb{Z}_2^f$ is embedded in G_f. So we can denote the fermion symmetry as $G_f \supset \mathbb{Z}_2^f$. In our case, the full internal symmetry is actually $G_f = \text{Spin}(10)$, while SO(10) is the quotient group $\text{Spin}(10)/\mathbb{Z}_2^f = \text{SO}(10)$. So in this work, we use the name: a Spin(10) chiral fermion model (rather than an SO(10) chiral fermion model which was sometimes used by others).

II. COBORDISM THEORY AND SYMMETRIC GAPPED BOUNDARY

Let us first explain the cobordism theory used in Proposition 1 and 2. Based on a theorem of Freed-Hopkin [29] and an extended generalization [30–32] (including higher symmetries [33–40]) there is a 1-to-1 correspondence between "the deformation class of invertible topological quantum field theories (iTQFTs[7]) [41, 42] with symmetry (including higher symmetries)" and "a cobordism group."[8] More precisely, there is a 1-to-1 correspondence (isomorphism "≅") between the following two well-

[5] In this work, a *local lattice model* is a lattice model of bosons and/or fermions with short-range interactions and a tensor-product structured Hilbert space, see (Def. I).

[6] An internal symmetry may or may not be an onsite symmetry. But an onsite symmetry must be an internal symmetry.

[7] It is called an invertible TQFT because its partition function $\mathbf{Z}(M^D)$ on any closed manifold M^D must have its absolute value $|\mathbf{Z}(M^D)| = 1$, namely $\mathbf{Z}(M^D) = e^{i\theta}$ can only be a complex phase. On a closed spatial manifold M^{D-1}, it always has a single ground state $\mathbf{Z}(M^{D-1} \times S^1) = 1$ with no topological ground state degeneracy. Thus, $\mathbf{Z}(M) = e^{i\theta}$ has an inverted phase $\mathbf{Z}^\dagger(M) = e^{-i\theta}$ that can be defined as its complex conjugated iTQFT. The combined iTQFT $\mathbf{Z}(M) \cdot \mathbf{Z}^\dagger(M) = 1$ is the trivial iTQFT (*i.e.* the trivial gapped vacuum).

[8] By all symmetric iTQFTs, their classifications and characterizations depend on the category of manifolds that can detect them. The categories of manifolds can be: TOP (topological manifolds), PL (piecewise linear manifolds), or DIFF (differentiable thus equivalently smooth manifolds), etc. These categories are different, and they are related by the inclusions:

$$\text{TOP} \supseteq \text{PL} \supseteq \text{DIFF}. \tag{1}$$

defined "mathematical objects" (these "objects" turn out to form the abelian group structures):

$$\left\{\begin{array}{c}\text{Deformation classes of the reflection positive}\\ D\text{-dimensional extended invertible}\\ \text{topological field theories (iTQFT) with}\\ \text{symmetry group } G=\frac{G_{\text{spacetime}}\times G_f}{N_{\text{shared}}}\end{array}\right\}$$
$$\cong [MT(G),\Sigma^{D+1}I\mathbb{Z}]_{\text{tors}}. \tag{2}$$

The $MT(G)$ is the Madsen-Tillmann spectrum [43] of the G group, the Σ is the suspension, the $I\mathbb{Z}$ is the Anderson dual spectrum, and the $\Sigma^{D+1}I\mathbb{Z}$ is the $D+1$-th suspension of the spectrum. The tors means taking only the finite group sector (*i.e.* the torsion group). The right-hand side is the torsion subgroup of the homotopy classes of maps from a Thom-Madsen-Tillmann spectrum [43, 44] to a shift of the Anderson dual to the sphere spectrum. The spacetime symmetry $G_{\text{spacetime}}$ and the internal symmetry G_f, mod out the shared common normal subgroup N_{shared}, is combined to a G structure:

$$G=\frac{G_{\text{spacetime}}\times G_f}{N_{\text{shared}}}. \tag{3}$$

This also means the pertinent iTQFTs of (2) are defined on manifolds with G structure.

In condensed matter physics, this *roughly* means that

Proposition 3. *There is a 1-to-1 correspondence [29] between "the invertible gapped states with an internal symmetry G_f (including higher symmetries [30, 33])" that can be regularized on a lattice with G_f realized as an onsite symmetry [45] in its own dimensions and "the group elements as the corresponding generators in a cobordism group for the internal symmetry G_f," at least in lower dimensions.*[9]

There is a logic gap here to establish Proposition 3, since by (2), we only know there is a 1-to-1 correspondence between "the iTQFTs with symmetry" and "the cobordism invariants from a cobordism group." We do not yet know if there is a 1-to-1 correspondence between "the lattice invertible topological order with symmetry" and "the iTQFTs with symmetry." In particular, we do not mathematically and rigorously prove how to construct a lattice Hamiltonian realization for each iTQFT with symmetry classified by a cobordism group. (We remark that some of the "lattice invertible topological orders with symmetry on a lattice" are also called the Symmetry Protected Topological/Trivial states (SPTs) [15, 16, 46, 47], if they can deform to a trivial tensor product state under *local unitary transformations* after explicitly breaking the symmetry.) Regardless of a logic gap in the rigorous mathematical sense, the broad literature suggests strong physical evidence that

(a): The classification of iTQFT [29, 30, 48–50] so far matches with the classification of lattice invertible topological orders and lattice SPTs [47, 51–53]. Many such iTQFTs can thus be constructed on the lattice Hamiltonian.

(b): Moreover, in (2), we only focus on iTQFTs definable on differentiable and triangulable manifolds, thus those iTQFTs may be regularized by the same lattice from the simplicial complex of triangulable manifolds.

In summary, based on the support of (a) and (b), below we propose and assume that a refined and rigorous version of Proposition 3 is true.

In contrast, triangulable manifolds are smooth manifolds at least for dimensions up to $D=4$ (*i.e.* the "if and only if" statement is true below $D\leq 4$). The concept of piecewise linear (PL) and smooth DIFF structures are equivalent in dimensions $D\leq 6$. Thus all symmetric iTQFT classified by the cobordant properties of smooth manifolds have a triangulation (thus a lattice regularization) on a simplicial complex (thus a UV [ultraviolet] competition on a lattice). This implies a correspondence between "the symmetric iTQFTs (on smooth manifolds)" and "the symmetric invertible topological orders (on triangulable manifolds)" for $D\leq 4$. This leads to our application of this mathematical fact on the lattice regularization of symmetric iTQFTs and symmetric invertible topological orders for various Standard Models of particle physics. In this work, we only focus on the smooth differentiable (DIFF) manifolds and their associated all possible iTQFTs. The tools we use in either case would be a certain version of cobordism theory suitable for a specific category of manifolds.

[9] We clarify that, before gauging, the G_f symmetry discussed in our setup must be an onsite internal symmetry of the lattice model (see Appendix A). Certainly, this does not exclude the possibility that the lattice model may have a larger symmetry. We stress that the G_f in the cobordism calculations is the onsite internal symmetry for

$$\text{the } G_f\text{-}\textit{symmetric deformation class} \text{ of the Hamiltonians.} \tag{4}$$

Thus, we consider many G_f-symmetric Hamiltonians under the G_f-symmetric preserving deformations. The onsite G_f always needs to be preserved in order to be gauged later. For example, we choose an internal symmetry group $G_f=\text{Spin}(10)$ for SO(10) GUTs. However, we point out that the full symmetry group G used in cobordism calculations also include the emergent spacetime symmetry at low energy infrared (IR) as $G_{\text{spacetime}}=\text{Spin}(D)$ (for a D-dimensional Euclidean spacetime). Thus, for the Spin(10) fermion model, in Eq. (3), we have $N=\mathbb{Z}_2^f$, so $G=(\frac{\text{Spin}(D)\times\text{Spin}(10)}{\mathbb{Z}_2^f})$.
Given a lattice model, there can be a larger symmetry $G_{\text{onsite}}\supset G_f$. Such as some lattice models in Sec. III and in Appendix B, we have

$$G_{\text{onsite}}\supset \text{U}(16N_f)\supset \text{U}(16)\supset G_f=\text{Spin}(10),$$

for some flavor number N_f. There are also other space group symmetries on a lattice, say $G_{\text{lattice,space}}$, while $G_{\text{lattice,space}}$ is typically smaller than the emergent $G_{\text{spacetime}}$, so usually

$$G_{\text{lattice,space}}\subset G_{\text{spacetime}}.$$

Overall, all these "symmetries" are *not* crucial to our discussion, except the only key symmetries are the *onsite internal* G_f symmetry, and the overall G.

Proposition 2 can be obtained from Proposition 3. There can be two kinds of gapped fermion systems on a lattice, those with topological excitations (which may be fractionalized) and those without topological excitations (*i.e.* all the excitations correspond to the original fermions or bosons). By definition, the gapped states with topological excitations, are the lattice *non-invertible* topological orders. The gapped states without topological excitations are the "lattice *invertible* topological orders with symmetry." According to Proposition 3, if a "lattice invertible topological order with symmetry" has a trivial cobordism invariant, then it must be a "trivial lattice invertible topological order with symmetry."[10] In other words, there exists a symmetry preserving local unitary transformation that deforms the "trivial lattice invertible topological order with symmetry" into a "trivial tensor product state with symmetry" [54], where its gapped symmetric boundary can always be constructed. (We provide more steps along with these logical arguments in Appendix A 2.) Crucially, this is precisely why the cobordism approach allows us to obtain the gapping condition for the mirror sector.

Proposition 1 can be obtained from Proposition 2, if we can show that the normal sector or the mirror sector can be regularized as some boundary states of a gapped local lattice model. We will provide such a local lattice model construction for the Spin(10) chiral fermion theory in Sec. III and in Sec. B as an example.

Although we propose Proposition 1, 2, and 3, we do not require the complete versions of all these Propositions to establish our claim of a local lattice model with *a chiral fermion low-energy spectrum*. We only require the weaker Proposition 1, let us clarify:

- Proposition 1 's "the existence of a fully gapped boundary" is a *static* statement. On the other hand, "the gapless sector can be fully gapped out without breaking the symmetry" is a *dynamic* statement, more changeling than Proposition 1. But the two statements are related; their detailed relations are given in Appendix A 2 and in D, based on physical intuitions of phase boundaries and quantum phase transitions. In fact, we only require the weaker *static* statement in Proposition 1's "the existence of a fully gapped boundary" in order to establish the gapped mirror sector.

- To use Proposition 3, we only require a local lattice construction for the cobordism class whose boundary gives rise to the normal sector or mirror sector. To establish Spin(10), Spin(18), and SU(5) chiral fermion theories, we only require a local lattice construction of the trivial cobordism class (the identity element 0 in the cobordism group). They happen to be a trivial bulk gapped insulator that we certainly can construct their local lattice model with a gapless normal sector on the boundary (Sec. III).

In the following sections, we also provide the physics interpretations of the classifications of all 4+1D iTQFTs whose boundaries are associated with the 3+1D Spin(10) and Spin(18) chiral fermion theories (for SO(10) and SO(18) GUTs) in Sec. III,[11] and the 3+1D SU(5) chiral fermion theories (for SU(5) GUTs) in Sec. IV. We relegate the mathematical calculation details on algebraic topology in Appendix E. (See also Ref. 55.)

III. SPIN(N) CHIRAL FERMION THEORY, AND SO(10) AND SO(18) GRAND UNIFICATION

We now construct a local lattice Hamiltonian model. A 3+1D two-component Weyl fermion described by a Hamiltonian (Model 1 defined in Appendix A 1)

$$H = \psi^\dagger \,\mathrm{i}\, \sigma^j \partial_j \psi, \quad \sigma^{1,2,3} \text{ are Pauli matrices,} \tag{5}$$

can be regularized on the boundary of a fermion hopping model on a 4D spatial cubic lattice with a Hamiltonian operator [23]

$$\hat{H}_{\text{hop}} = \sum_{\boldsymbol{ij}} (t^{ab}_{\boldsymbol{ij}} \hat{c}^\dagger_{a,\boldsymbol{i}} \hat{c}_{b,\boldsymbol{j}} + h.c.), \tag{6}$$

which has 4 fermion orbitals $(a, b = 1, \cdots, 4)$ per site ($\boldsymbol{i}, \boldsymbol{j}$ for sites). The $h.c.$ contains the hermitian conjugate term. The 4×4 hopping matrices $t_{\boldsymbol{ij}}$ are given by

$$\begin{aligned} &H_{4\text{D}}(k_1,k_2,k_3,k_4) \qquad (7)\\ &= 2[\Gamma^1 \sin(k_1) + \Gamma^2 \sin(k_2) + \Gamma^3 \sin(k_3) + \Gamma^4 \sin(k_4)] \\ &\quad + 2\Gamma^5[\cos(k_1) + \cos(k_2) + \cos(k_3) + \cos(k_4) - 3] \end{aligned}$$

in the momentum $\boldsymbol{k}$-space, where $\Gamma^1 = \sigma^1 \otimes \sigma^3$, $\Gamma^2 = \sigma^2 \otimes \sigma^3$, $\Gamma^3 = \sigma^3 \otimes \sigma^1$, $\Gamma^4 = \sigma^0 \otimes \sigma^2$, and $\Gamma^5 = \sigma^0 \otimes \sigma^3$, which obey $\{\Gamma^i, \Gamma^j\} = 2\delta_{ij}$. If the 4D lattice is formed by two layers of 3D cubic lattices, the one-body Hamiltonian in the (k_1, k_2, k_3)-space is given by the following 8-by-8 matrix

$$\begin{aligned} H_{3\text{D}}(k_1,k_2,k_3) &= \begin{pmatrix} M_1 & M_2 \\ M_2^\dagger & M_1 \end{pmatrix}, \quad \text{where} \\ M_1 &= 2[\Gamma^1 \sin(k_1) + \Gamma^2 \sin(k_2) + \Gamma^3 \sin(k_3)] \\ &\quad + 2\Gamma^5[\cos(k_1) + \cos(k_2) + \cos(k_3) - 3], \\ M_2 &= -\mathrm{i}\,\Gamma^4 + \Gamma^5. \end{aligned} \tag{8}$$

[10] A "trivial lattice invertible topological order with symmetry" means the "trivial gapped vacuum with symmetry" in quantum field theory, or the "symmetric gapped direct product state" in condensed matter.

[11] To be precise, in order to embed the standard Standard Model-like spacetime-and-internal symmetry group with this $G = \frac{G_{\text{spacetime}} \times G_f}{N_{\text{shared}}}$ structure (3) to the SO(10) Grand Unification's $\frac{\text{Spin}(D)\times\text{Spin}(10)}{\mathbb{Z}_2^f}$, it is natural to consider an alternative standard Standard Model spacetime-and-internal group $\text{Spin}(D) \times \frac{\text{U}(1)\times\text{SU}(2)\times\text{SU}(3)}{\mathbb{Z}_6}$, while their gauge Lie algebra is still $u(1) \times su(2) \times su(3)$. Here Spin(10) and $\frac{\text{U}(1)\times\text{SU}(2)\times\text{SU}(3)}{\mathbb{Z}_6}$ are their gauge groups respectively. See more discussions in footnote 18.

One can directly check that the above 3D fermion hopping model gives rise to a two-component massless complex Weyl fermion on each of the two 3D surfaces of the 4D lattice. The Weyl fermion on one boundary is a left-hand Weyl fermion and on the other boundary is a right-hand Weyl fermion. We have a similar result when the 4D lattice is formed by many layers of 3D cubic lattices.

The 16 copies of the local lattice model (7) give rise to the 3+1D Weyl fermions in the 16-dimensional spinor representation of the Spin(10) on the lattice boundary's low energy spectrum. The ground state of the 4+1D hopping model is

- a "lattice invertible topological state (invertible topological order whose low energy is an iTQFT) with a $\text{Spin}(10) \supset \mathbb{Z}_2^f$ symmetry," since it has no non-trivial topological excitations.
- a lattice non-trivial 4+1D Spin(10) *non-interacting free* fermionic SPT state [56–58], which belongs to the 16-th class (or the 16n-th) in the $\mathbb{Z}$ classification.

But such a state may correspond to

- a trivial state for Spin(10) SPT state in the lattice *interacting* fermionic SPT systems [49, 52, 59–61] and to a trivial cobordism class (in 1, 2 and 3) [29, 30], which belongs to the 0-th class in the classification.

If so, the 4+1D hopping model can have a symmetric gapped boundary, and the 3+1D Spin(10) chiral Weyl fermions (5) for the mirror sector can be gapped by interactions without breaking the symmetry by Proposition 1 and 2.

To show that the 4+1D hopping model gives rise to a trivial Spin(10) SPT state in *interacting* fermion systems, we use a recent conjectured complete classification of interacting fermionic invertible topological orders [29, 30, 49, 50, 61–63] with onsite symmetry, via a twisted version of the spin cobordism theory of Freed-Hopkin [29]. This classification includes all known interacting fermionic SPT states and all known interacting fermionic invertible topological orders on a lattice [52, 59, 60, 64].

We first note that, for fermions with the full symmetry $G_f \supset \mathbb{Z}_2^f$ in the D spacetime dimensions, they transform as $G = \frac{\text{Spin}(D)\times G_f}{\mathbb{Z}_2^f}$ under the combined spacetime symmetry $G_{\text{spacetime}} = \text{Spin}(D)$ rotation and the internal G_f transformation, where a double-counted fermion parity symmetry $\mathbb{Z}_2^f$ is mod out. This shared normal subgroup $\mathbb{Z}_2^f$ is due to the fact that rotating a fermion by 2π in the spacetime (namely, the spin-statistics) gives rise to the same fermion parity minus sign for the fermion operator $\psi \to -\psi$.

To classify the iTQFT whose boundary can have a 3+1D Spin(10) chiral fermion theory, we focus on the following cobordism group

$$\Omega^{D=5}_{\frac{(\text{Spin}(D=5)\times\text{Spin}(10))}{\mathbb{Z}_2^f}} \equiv \text{TP}_{D=5}\Big(\frac{(\text{Spin}(D=5)\times\text{Spin}(10))}{\mathbb{Z}_2^f}\Big). \tag{9}$$

More generally, we find that 4+1D fermionic invertible topological orders with $G_f = \text{Spin}(N) \supset \mathbb{Z}_2^f$ onsite global symmetry for $N \geq 7$ are classified by the 5-th cobordism group [30]:

$$\Omega^{D=5}_{\frac{(\text{Spin}(D=5)\times\text{Spin}(N))}{\mathbb{Z}_2^f}} = \mathbb{Z}_2, \qquad N \geq 7. \tag{10}$$

Beware that we *define* the cobordism group, classifying symmetric fermionic invertible topological orders, as

$$\begin{aligned} \Omega^D_G &\equiv \Omega^D_{(\frac{G_{\text{spacetime}}\times G_f}{N_{\text{shared}}})} \\ &\equiv \text{TP}_D(G) \equiv [MT(G), \Sigma^{D+1} I\mathbb{Z}], \end{aligned} \tag{11}$$

which stands for the homotopy classes of maps from Thom-Madsen-Tillmann spectrum [43, 44] $MT(G)$ to the $D+1$-th suspension of the Anderson dual spectrum $\Sigma^{D+1} I\mathbb{Z}$. Our notations follow Refs. [29, 30, 50] and [65]: TP abbreviates "Topological Phases" classifying the symmetric invertible topological orders (or invertible topological quantum field theories), N_{shared} is the shared normal subgroup of $G_{\text{spacetime}}$ and G_f.

The cobordism group of topological phases (TP) defined in [29] as $\text{TP}_D(G)$ classifies the deformation classes of reflection positive invertible d-dimensional extended topological field theories with symmetry group G_D. The *cobordism group* $\text{TP}_D(G) \equiv \Omega^D_G$ and the *bordism group* Ω^G_D are related by a short exact sequence

$$0 \to \text{Ext}^1(\Omega^G_D, \mathbb{Z}) \to \text{TP}_D(G) \equiv \Omega^D_G \to \text{Hom}(\Omega^G_{D+1}, \mathbb{Z}) \to 0, \tag{12}$$

with Ext denotes the extension functor, see Appendix E.

In contrast, we *do not* define the cobordism group as the usual definition of Pontryagin dual of the torsion subgroup ($\equiv$ tors) of the bordism group Ω^G_D as the homomorphism (Hom) map to U(1):

$$\text{Hom}(\Omega^{G,\text{tors}}_D, \text{U}(1)), \tag{13}$$

although the torsion (*i.e.* finite group) sectors of (11) and (13) are equivalent. Mathematical details for the above result are presented in Refs. 30, 50, and 55.[12] We classify the deformation classes of invertible topological quantum field theories (further precisely, the reflection positive invertible extended topological field theories) via Ω^D_G, by classifying the cobordant differentiable and triangulable manifolds with a stable G-structure, via associating them to the homotopy groups of Thom-Madsen-Tillmann spectra [43, 44], thanks to a theorem in Ref. 29.

To be precise, here the spin cobordism theory is believed to completely classify all the fermionic iTQFTs.

[12] In contrast, Ref. 66 computes a different bordism group $\Omega_{D=5}^{(\text{Spin}(D=5)\times\text{Spin}(10))} = 0$ which detects no anomaly. Instead, we study the bordism group $\Omega_{D=5}^{(\text{Spin}(D=5)\times\text{Spin}(10))/\mathbb{Z}_2^f} = \mathbb{Z}_2$, whose manifold generator can detect the recently discovered new SU(2) anomaly [67].

By applying this spin cobordism theory, we classify 4+1D Spin(N) SPT states and 4+1D Spin(N) symmetric invertible fermionic topological orders. In fact, in this context, the 3+1D Spin(N) fermion theories already include all possible 3+1D Spin(N) chiral fermion theories that we need. To this end, we will especially focus on the 3+1D Spin(10) chiral fermion theories with Weyl fermions in a 16-dimensional spinor representation.

The above $\mathbb{Z}_2$ classification in (10) implies that there is only one non-trivial 4+1D invertible fermionic topological order with a Spin(N) onsite symmetry. We find that such a topological phase is characterized by a 5-dimensional topological invariant [30] written in terms of a bulk partition function on a 5-manifold M^5,

$$\mathbf{Z} = \mathrm{e}^{\mathrm{i}\,\pi \int_{M^5} \mathrm{w}_2(TM)\cup \mathrm{w}_3(TM)}, \tag{14}$$

where $\mathrm{w}_n(TM)$ is the n^{th}-Stiefel-Whitney class for the tangent bundle of $4+1$D spacetime manifold M^5, and the $\cup$ is the cup product (which we may omit writing $\cup$) [68]. We note that on M^5, we have a $\frac{\mathrm{Spin}(D=5)\times \mathrm{Spin}(N)}{\mathbb{Z}_2^f}$ connection — a mixed gravitational and gauge connection, rather than a pure gravitational Spin($D=5$) connection, such that $\mathrm{w}_2(TM) = \mathrm{w}_2(V_{\mathrm{SO}(N)})$ and $\mathrm{w}_3(TM) = \mathrm{w}_3(V_{\mathrm{SO}(N)})$, where $\mathrm{w}_n(V_{\mathrm{SO}(N)})$ is the n^{th}-Stiefel-Whitney class for an SO(N) gauge bundle.[13] Thus, M^5 may not be a spin manifold (note that a spin manifold requires $\mathrm{w}_2(TM)=0$), generally called a non-spin manifold.

We can detect the 4+1D cobordism invariant $\mathrm{e}^{\mathrm{i}\,\pi \int_{M^5} \mathrm{w}_2(TM)\mathrm{w}_3(TM)}$ for the 4+1D invertible fermionic topological order by study its boundary state. In particular, if the 4+1D state has a boundary described by 3+1D Spin(N) chiral Weyl fermion theory, then we can detect the 4+1D cobordism invariant via the Spin(N) representation of the chiral Weyl fermions on the boundary. Here we use a fact that the 4+1D cobordism invariant can be detected by restricting to a SU(2) = Spin(3) subgroup of Spin(N) [67]: Let n_j be the number of isospin-j representations of SU(2) = Spin(3) $\subseteq$ Spin(N) for 3+1D boundary chiral Weyl fermions, then the 4+1D cobordism invariant $\mathrm{e}^{\mathrm{i}\,\pi \int_{M^5} \mathrm{w}_2(TM)\mathrm{w}_3(TM)}$ is absent if

$$\sum_{r=0}^{\infty} n_{2r+\frac{1}{2}} \in \mathbb{Z}_{\text{even}}, \quad \sum_{r=0}^{\infty} n_{4r+\frac{3}{2}} \in \mathbb{Z}_{\text{even}}. \tag{15}$$

[13] In the context of anomalies (see Appendix A for details), the boundary of this 4+1D Spin(N)-SPT state may have a mixed anomaly of SO(N)-gauge bundle and spacetime geometry/gravity, and we can use the this 3+1D anomaly on the boundary to detect the bulk invertible topological order. Namely, we find that *there is only one possible candidate of the 3+1D anomaly for interacting fermion systems with a* Spin(N) *symmetry* ($N \geq 7$), which is a non-perturbative global mixed gauge-gravity (*i.e.* gauge-diffeomorphism) anomaly characterized by (14).

To see how the representation of Spin(N) reduces to the representations of SU(2) = Spin(3), let us describe the representation of Spin(N) (the spinor representation of Spin(N)), assuming N = even. We first introduce γ-matrices γ_a, $a = 1, \cdots, N$:

$$\begin{aligned} \gamma_{2k-1} &= \underbrace{\sigma^0 \otimes \cdots \otimes \sigma^0}_{\frac{N}{2}-k\ \sigma^0\text{'s}} \otimes \sigma^1 \otimes \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k-1\ \sigma^3\text{'s}}, \\ \gamma_{2k} &= \underbrace{\sigma^0 \otimes \cdots \otimes \sigma^0}_{\frac{N}{2}-k\ \sigma^0\text{'s}} \otimes \sigma^2 \otimes \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{k-1\ \sigma^3\text{'s}}, \end{aligned} \tag{16}$$

$k = 1, \cdots, \frac{N}{2}$, which satisfy $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ and $\gamma_a^\dagger = \gamma_a$. Here σ^0 is the 2-by-2 identity matrix and σ^l with $l = 1,2,3$ are the Pauli matrices. The $\frac{N(N-1)}{2}$ hermitian matrices $\gamma_{ab} = \frac{\mathrm{i}}{2}[\gamma_a, \gamma_b] = \mathrm{i}\gamma_a\gamma_b,\ a < b$, generate a $2^{N/2}$-dimensional representation of Spin(N). The above $2^{N/2}$-dimensional representation is reducible. To obtain an irreducible representation, we introduce

$$\gamma_{\text{FIVE}} = (-\mathrm{i})^{N/2}\gamma_1 \cdots \gamma_N = \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{\frac{N}{2}\ \sigma^3\text{'s}}. \tag{17}$$

We have $(\gamma_{\text{FIVE}})^2 = 1$, its trace $\mathrm{Tr}(\gamma_{\text{FIVE}}) = 0$, and $\{\gamma_{\text{FIVE}}, \gamma_a\} = [\gamma_{\text{FIVE}}, \gamma_{ab}] = 0$. This allows us to obtain two $2^{N/2-1}$-dimensional irreducible representations: one for $\gamma_{\text{FIVE}} = 1$ and the other for $\gamma_{\text{FIVE}} = -1$.

Now, let us consider an SU(2) = Spin(3) subgroup of Spin(N), generated by $\gamma_{12} = I \otimes \sigma^0 \otimes \sigma^3$, $\gamma_{23} = I \otimes \sigma^1 \otimes \sigma^1$, and $\gamma_{31} = I \otimes \sigma^1 \otimes \sigma^2$. We see that the $2^{N/2-1}$-dimensional representation of Spin(N) becomes $2^{N/2-2}$ isospin-1/2 representations of SU(2).

Summarizing the above results, we see that the $2^{N/2-1}$ copies of 4+1D hopping model (7) formed by many (but finite) layers of 3D cubic lattices has a 3+1D boundary chiral Weyl fermion in the $2^{N/2-1}$-dimensional representation of Spin(N) on one boundary (the normal sector), and a conjugate 3+1D chiral Weyl fermion on the other boundary (the mirror sector). For an even $N \geq 8$, the 3+1D boundary chiral Weyl fermions only reduces to an even number of isospin-1/2 representations, and, according to (15), the 4+1D cobordism invariant $\mathrm{e}^{\mathrm{i}\,\pi \int_{M^5} \mathrm{w}_2(TM)\mathrm{w}_3(TM)}$ is absent. Thus the corresponding 4+1D bulk state is a trivial fermionic iTQFT (*i.e.* the identity element as the trivial cobordism class in Proposition 1 and 2) with a Spin(N) symmetry.

In this case, the mirror sector can be chosen as a Spin(N)-symmetric gapped boundary (Proposition 1), or the mirror sector can be gapped out without breaking the Spin(N) symmetry by introducing a proper symmetric fermion interaction on the boundary (Proposition 2). Since the 4+1D hopping model has many layers and the 3+1D boundary massless chiral Weyl fermion has no symmetric *relevant* deformation operators (in the renormalization group sense, e.g., there is no Spin(N) symmetric mass term), the symmetric interaction in the mirror sector on one boundary will not affect the low energy dynamics of the 3+1D massless chiral Weyl fermion in the normal sector on the other boundary.[14]

Now we apply a well-known lattice method

Finite-width/layer lattice dimensional reduction:

An $n+1D$ lattice model with finite layers along one extra direction (a finite width w) can be dimensionally reduced to an nD lattice model via absorbing the degrees of freedom along w to the orbital in nD. (18)

Thus by (18), the 4+1D hopping model with finite layers can be viewed as a 3+1D lattice model with *finite orbitals per site*, the 3+1D Spin(N) chiral Weyl fermion theory in the $2^{N/2-1}$-dimensional representation can be regularized by a lattice model in the same dimension without breaking the Spin(N) symmetry for even $N \geq 8$.

In particular, for $N = 10$, the Spin(10) chiral fermion theory with Weyl fermions in a 16-dimensional spinor representation (similarly, for Spin(18) chiral fermion theory in a 256-dimensional spinor representation) can be regularized by a local lattice fermion model in the same dimension. After regularizing the Spin(10) chiral fermion theory as a lattice fermion model in the same dimension (3+1D) with an onsite Spin(10) symmetry, we can gauge the onsite Spin(10) symmetry to obtain a gauged Spin(10) chiral fermion theory,[15] again regularized by a lattice model in the same dimension.

We remark that, in fact, for $N = 3$, the 4+1D fermionic invertible topological orders with a Spin(3)=SU(2) internal global symmetry are classified by the cobordism group of $(\mathrm{Spin}(D=5) \times \mathrm{Spin}(N=3))/\mathbb{Z}_2^f$ [30, 67]:

$$\Omega^{D=5}_{\frac{(\mathrm{Spin}(D=5)\times \mathrm{Spin}(3))}{\mathbb{Z}_2^f}} = \Omega^{D=5}_{\frac{(\mathrm{Spin}(D=5)\times \mathrm{SU}(2))}{\mathbb{Z}_2^f}} = (\mathbb{Z}_2)^2. \quad (19)$$

The corresponding cobordism invariant is given by

$$\mathbf{Z} = \mathrm{e}^{\mathrm{i}\,\alpha\pi \int_{M^5} \mathrm{Arf}\ \ \widetilde{\mathrm{w}}_3(TM)}\, \mathrm{e}^{\mathrm{i}\,\beta\pi \int_{M^5} \mathrm{w}_2(TM)\mathrm{w}_3(TM)}. \quad (20)$$

Here Arf is the Arf invariant [69], which characterizes the 1+1D fermionic chain whose open ends host Majorana zero modes [70]. This 1+1D fermionic chain is also known as Kitaev chain [70] whose low energy physics is governed by a 1+1D invertible fermionic topological order. The $\widetilde{\mathrm{w}}_3(TM)$ is a *twisted* version of the 3rd-Stiefel-Whitney class $\mathrm{w}_3(TM)$. The above cobordism invariant can be detected by the SU(2) representations of 3+1D boundary chiral Weyl fermions, and α, β in (20) are given by [67]

$$\alpha = \sum_{r=0}^{\infty} n_{2r+\frac{1}{2}} \bmod 2, \qquad \beta = \sum_{r=0}^{\infty} n_{4r+\frac{3}{2}} \bmod 2. \quad (21)$$

In this work, we only suggest that there exists a symmetric short-range non-perturbative interaction that can fully gap out the mirror sector without breaking the Spin(10) symmetry.[16] Our approach only proves the symmetric gapped boundary exists (via Proposition 1 and 2), but does not provide a prescription to design such an interaction. The approach in Ref. 23 and 71 proposes a design: The interaction in the mirror sector is given by the smooth orientation fluctuations of Higgs field (thus beyond the Higgs mechanism [18, 72]), where a constant orientation will gap out all the mirror fermions. But the validity of the design requires confirmation by numerical simulations. A first step is taken in Ref. 71 for a 1+1D system. In such a design, crucially the mass of the mirror fermions induced by the Higgs field must be comparable with the fermion bandwidth. Some other gapping-mirror-fermion approaches have also been proposed recently [17, 72–77]. Many previous calculations [26, 27] checking the mirror fermion approach choose an induced energy gap (*i.e.* an effective mass) to

---

[14] In the previous paragraph, we had determined that the 4+1D hopping model (7) without higher-order interactions, whose boundary hosts a 3+1D chiral Weyl fermion in the $2^{N/2-1}$-dimensional representation of Spin(N), has a trivial cobordism class in the bulk. Readers may wonder whether the bulk's cobordism class would change under the interactions that we required? The answer is no. To recall, our setup follows:
(i) Bulk does not include non-perturbative interactions. Bulk only allows small perturbative interactions if any. So the bulk gap does not close, and can never be closed.
(ii) Only on the boundary, we can add "arbitrary Spin(N) preserving interactions" (both small perturbative or large non-perturbative interactions).
In summary, since the bulk gap does not close, the bulk phase remains the same trivial cobordism class, which stays valid before and after adding boundary interactions.

[15] To gauge the onsite symmetry, one way is by inserting gauge variables on the 1-dimensional links between local sites. This is known as the *hard-gauge*, such that the outcome gauge theory does not have a tensor product Hilbert space (Def. I) thus it is *not* a local lattice model that we aim for. However, we can further maintain a tensor product Hilbert space (Def. I) by designing the *soft-gauge*. We relegate the details of the *soft-gauge* via a local lattice model in Appendix B. See discussions on *hard-gauge* and *soft-gauge* in Ref. 18.

[16] There exists such a symmetric gapping interaction preserving Spin(10). Moreover, for 16 chiral Weyl fermions at IR, there can be a U(16) global symmetry, with U(16) ⊃ Spin(10). It is possible that additional constraints happen on what interactions we can engineer in the quotient space $\frac{\mathrm{U}(16)}{\mathrm{Spin}(10)}$ (also a homogeneous space) without breaking Spin(10). We provide some further guidelines for designing interactions:

- To gap the mirror sector by adding non-perturbative lattice-scale interactions on the boundary, we may need to look for a larger symmetry G_+ than Spin(10), such that
$$\mathrm{Spin}(10) \subseteq G_+ \subseteq \mathrm{U}(16).$$
This G_+ can be useful for constructing exactly solvable and integrable models.
- However, we can further weakly break G_+ down to Spin(10), by small perturbative interactions that are not merely *irrelevant* perturbations in the renormalization group (RG) and field theory sense but also with *small* lattice-scale couplings. Thus we can break the redundant symmetry outside Spin(10) without changing the quantum dynamics.

be much bigger than the bandwidth (*i.e.* at the infinite coupling limit). The infinite coupling limit in the mirror sector generates a dead layer, a neighbor layer next to the mirror sector would become the new mirror sector with fermion doublings [8], which would fail to produce a chiral fermion/gauge theory at low energies.

IV. SU(5) CHIRAL FERMION THEORY AND SU(5) GRAND UNIFICATION

Above we have discussed the lattice regularization of a Spin(10) gauged chiral fermion theory. To consider a lattice regularization of a SU(5) gauged chiral fermion theory (with $G_f = \mathbb{Z}_2^f \times \mathrm{SU}(5)$, but only SU(5) will be gauged), we classify the 4+1D invertible fermionic topological order with $G_f = \mathbb{Z}_2^f \times \mathrm{SU}(5)$ symmetry by a cobordism group defined in Eq. (11) (note that $\mathrm{Spin}(D=5) \supset \mathbb{Z}_2^f$) [30, 55]:

$$\Omega^{D=5}_{\mathrm{Spin}(D=5)\times \mathrm{SU}(5)} \equiv \mathrm{TP}_D(\mathrm{Spin}(D=5)\times \mathrm{SU}(5)) = \mathbb{Z}, \tag{22}$$

where the topological invariant is given by the SU(5) Chern-Simons 5-form, associated with *perturbative local anomalies* captured by perturbative Feynman diagram calculations in 3+1D.

Again, such a cobordism invariant and the associated invertible topological order can be detected by the boundary chiral fermions: if the 3+1D boundary SU(5) chiral fermion theory is free from any of the $\mathbb{Z}$ class of SU(5) perturbative local anomaly, then the corresponding cobordism invariant and the 4+1D bulk invertible topological order are trivial. Thus, by Proposition 1 and 2, *any* SU(5) *gauged chiral fermion theory that can be regularized at the boundary of a 4+1D gapped local lattice model, can be regularized by a 3+1D local lattice model via the method (18), provided that the* SU(5) *gauge theory is free of the* SU(5) *perturbative anomalies* (see also Prop. I in Appendix A 1). In particular, the SU(5) grand unified theory [5] can be regularized by a lattice. This implies that its induced 15-fermion Standard Model can be regularized by a lattice fermion model.

V. IMPLICATIONS AND CONCLUSIONS

In fact, an $n+1$D G-symmetric iTQFT given by a cobordism class in Proposition 1 and 2 corresponds to an nD 't Hooft anomaly of G-symmetry (see footnote 4 and the details of anomalies in Appendix A). So a trivial cobordism class in $n+1$D for G-symmetry means all-'t Hooft-anomaly-free in nD for the full G-symmetry. Namely, by far we only show that anomaly-free gauged chiral fermion theories can be defined on a lattice with non-dynamical background gauge fields (Model 1 and Model 2 in Appendix A 1), regularized with onsite symmetries in its own dimensions (via (18)). However, we can obtain a dynamical chiral gauge theory (Model 3 in Appendix A 1) by dynamically gauging the *onsite* symmetry: introducing dynamical gauge link variables between local sites (e.g. dynamically sum over gauge inequivalent configurations in the partition function) — this is a *hard-gauge* model but not a local lattice model, see footnote 15. We can further apply the *soft-gauge* method [18] to obtain a local lattice model, see Appendix B. We emphasize if all gauge invariant operators are bosonic, the above dynamical lattice gauge theory coupled to fermions is actually a local lattice bosonic model in disguise, as one can see from the slave-particle/parton approach [36, 78–81].

We remark that the dynamical Spin(10) chiral gauge theory coupled to Weyl fermions in the 16-dimensional spinor representation is a local bosonic theory, since all gauge invariant operators are bosonic.[17] The lattice regularization that realizes the dynamical Spin(10) chiral gauge theory is also a local bosonic model (see Appendix B). In other words, the Spin(10) dynamical chiral gauge theory with Weyl fermions in a 16-dimensional representation, and the induced 16-fermion Standard Model, can be regularized as the low energy effective theory of a local lattice model of qubits (since any local bosonic lattice model can be viewed as a lattice model of qubits). Based on the stability of cobordism group of Eq. (10) for $N \geq 7$, our result directly applies to a $\mathrm{Spin}(N=18)$ chiral gauge theory [72, 82], which is also a local bosonic model. Thus our study implies that all elementary particles (except the graviton) can be viewed as originated from qubits [83–85]. It is a concrete realization of "it from qubit [86]," representing an Ultra Unification of all gauge interactions and matter fermions in term of quantum information (*i.e.* qubits).

The statement that all elementary particles arise from bosonic qubits has a falsifiable experimental prediction: all fermions and their fermionic bound states must carry non-trivial gauge charge [36, 87]. As a result, the "Standard Model" from a lattice qubit model cannot just have a $\frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_q}$ gauge group, since such a Standard Model indeed has fermionic bound states that carry *no* gauge charge. Thus, the "standard model" from a lattice qubit model must have a larger gauge group, e.g. adding a new $[\mathbb{Z}_2]$ gauge sector,[18] where we gain a new cosmic string (whose spacetime trajectory is a 2-dimensional

[17] In Appendix B, we provide the explicit slave-particle/parton construction for a 4+1D *local bosonic* lattice model, whose boundary can give rise to the dynamical Spin(10) chiral gauge theory coupled to Weyl fermions (Model 3) in the 16-dimensional representation.

[18] See Footnote 11, we can show that

$$\frac{\mathrm{Spin}(D)\times\mathrm{Spin}(10)}{\mathbb{Z}_{q'}} \supset \mathrm{Spin}(D)\times\mathrm{SU}(5) \supset \mathrm{Spin}(D)\times\frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_6},$$

where $q'=1$ or 2, while

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \supset \frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_6}$$

worldsheet) — the flux line of the new $[\mathbb{Z}_2]$ gauge field [88].

In contrast, the dynamical SU(5) chiral gauge theory coupled to Weyl fermions in the 5- and 10-dimensional representations is a fermionic theory definable on *spin manifolds*, since some gauge invariant operators are fermionic. The lattice regularization that realizes the dynamical SU(5) chiral gauge theory is also a local fermionic model (which is not a local lattice model of qubits). The Standard Model from local fermionic lattice models can have $\frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_q}$ as its gauge group, see Footnote 18. It does not require extra gauge groups.

In our work, we have shown that a Spin(10) (or SU(5)) chiral gauge theory with 16 (or 15) Weyl fermions can be regularized by lattice, since the mirror sector can be fully gapped by Spin(10) (or SU(5)) symmetric interactions without spontaneously breaking the symmetry. However, it is possible that the mirror sector can be fully gapped by interactions with a larger symmetry G_+ without spontaneously breaking the symmetry G_+. In this case, after gauging Spin(10) (or SU(5)), the chiral gauge theory may have an exact global symmetry G_Q (on a lattice scale or a UV cutoff scale such as an effective Planck scale) sitting as a quotient group satisfying the short exact sequences:

$$\begin{aligned} &1 \to \mathrm{Spin}(10) \to G_+ \to G_Q \to 1, \\ \text{or} \quad &1 \to \mathrm{SU}(5) \to G_+ \to G_Q \to 1. \end{aligned} \tag{23}$$

The reason G_Q is still a global symmetry after gauging Spin(10) (or SU(5)) is that because G_+ can be chosen as an *onsite* symmetry on the UV cutoff scale and G_+ is anomaly-free on the $d+1$D. Once a normal subgroup is gauged, the G_Q is still anomaly-free and unbroken thus can still be made *onsite* in $d+1$D.

Lastly, we comment on the dynamics of these dynamical chiral gauge theories (Model 3, as highly long-range entangled states). At the low energy of these chiral gauge theories, there could be emergent symmetries (e.g. higher-form symmetries [33] or higher symmetries in general [30]) having new 't Hooft anomalies. However, emergent new anomalies only mean the emergent symmetries cannot be strictly regularized locally on-site, on-link, on-n-simplex, etc., which, we emphasize, is a rather distinct issue deviated from regularizing chiral fermion theories which we solved earlier. After regularizing chiral fermion theories on a lattice, and after dynamically gauging, the emergent new anomalies *only* constrain the dynamics of gauge theories (e.g. gapless near a quantum critical fixed point, or emergent symmetry spontaneously broken, etc.). We aim to address the dynamics of gauge theories in future work.

and

$$\mathrm{Spin}(10) \supset \mathrm{SU}(5) \supset \frac{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)}{\mathbb{Z}_6}.$$

For $q'=2$, when we gauge the [Spin(10)], we also require to gauge the $[\mathbb{Z}_2^f \times \mathrm{SU}(5)]$ in the embedded smaller group $\mathrm{Spin}(D)\times\mathrm{SU}(5) \subset \frac{\mathrm{Spin}(D)\times\mathrm{Spin}(10)}{\mathbb{Z}_{q'=2}}$. The dynamically gauging $[\mathbb{Z}_2^f]$ symmetry produces the new $[\mathbb{Z}_2]$ gauge sector.

VI. ACKNOWLEDGEMENT

We thank E. Witten for many helpful discussions and comments, and for a collaboration on Ref. 67. JW thanks Z. Wan for related collaborations on Ref. 55.[19] JW is supported by Corning Glass Works Foundation Fellowship and NSF Grant PHY-1606531. XGW is partially supported by NSF grant DMS-1664412. This work is also supported by NSF Grant PHY-1306313, PHY-0937443, DMS-1308244, DMS-0804454, DMS-1159412 and Center for Mathematical Sciences and Applications at Harvard University.

Appendix A: Definitions of Terminology and Discussions based on Anomalies

In the main text, we have described our results without directly mentioning the quantum anomaly (footnote 4). However, in literature, many people discuss the gauge chiral fermion problem in terms of anomaly. In this section, we will discuss our approach using the concept of the anomaly. We will carefully define several different anomalies. We will also carefully define several concepts of chiral fermion field theory and the concepts of lattice theory as a well-defined quantum theory.

1. Detailed definitions of some relevant concepts

We should clarify several related concepts of Spin(10) chiral fermion field theories and models as follows:

Model 1: *Without gauging or before gauging* Spin(10) symmetry, the theory is a "Spin(10) chiral fermion theory" with the full internal global symmetry $G_f = \mathrm{Spin}(10) \supset \mathbb{Z}_2^f$. In this case, we call the anomaly associated with the global symmetry G_f as the 't Hooft anomaly of G_f. We classify the *'t Hooft anomaly* [91] of G_f in Eq. (10) and Eq. (11).

Model 2: We may twist the Spin(10) symmetry via a non-dynamical background Spin(10) gauge field, known as the *symmetry twist*. We name such a theory

[19] Ref. 66 (which appears around the same period of time of the present work) and two other later works, Ref. 55 and 89, check more systematically the co/bordism groups relevant for other Standard Models and Grand Unifications. Furthermore, Ref. 90 examines the anomaly and cobordism constraints from Ref. 55 of these models, and explores the potential new physics beyond the Standard Model.

as a "Spin(10) gauged chiral fermion theory." The anomaly of G classified in (10) becomes the *background gauge anomaly*, which is the same as the 't Hooft anomaly in nature.

Model 3: *After dynamical gauging* Spin(10) symmetry via dynamical weakly fluctuating Spin(10) gauge field, the theory becomes a "Spin(10) chiral gauge theory." In this case, it is a standard terminology to call the anomaly, descending from Model 1's 't Hooft anomaly and after gauging Spin(10), as the *dynamical gauge anomaly*. We will thus also study the dynamical gauge anomaly of Spin(10), thanks to Eq. (10). If any model possesses any dynamical gauge anomaly, then this theory is inconsistent thus ill-defined.

Ref. 23 adopted a new viewpoint (or a new definition) of anomalies proposed in Ref. 92 (see Def. 1), for *interacting quantum* theories, which in turn leads to a classification of anomalies. Before we proceed, we should clarify some conventions of terminology as the definitions:

Def. I: **Well-defined quantum theories** are *quantum theories* defined with a finite-dimensional Hilbert space and a finite-dimensional Hamiltonian matrix for a finite-size system in the real space. In this work, we only focus on this class of quantum theories.

Local lattice models are interacting or non-interacting lattice models whose many-body Hilbert space $\mathcal{V}$ has the following tensor product decomposition

$$\mathcal{V} = \bigotimes_i \mathcal{V}_i \tag{A1}$$

where $\mathcal{V}_i$ is a finite-dimensional Hilbert space for each lattice site.

By *interacting* models, we mean the Hamiltonian contains certain higher-order terms beyond the quadratic terms of fundamental lattice operators (such as quartic fermionic or spin operator terms beyond the quadratic terms).

By *non-interacting* models (or the so-called *free* or *quadratic* models), we mean the Hamiltonian contains at most the quadratic terms (thus easily diagonalizable and solvable) of fundamental lattice operators.

Local interactions: By local interactions, we mean all interaction terms in the Hamiltonian (or a Lagrangian in the path integral) must be bounded by a *finite range* of lattice spacings. We call these types as local, finite-range or short-range interactions. We do *not* allow infinite-range interactions, *nor* the interactions with strength exponentially-decay to zero only at infinite. For any interaction term of our lattice model, it must be bounded by a finite spatial range, say if the operators act on any site i to j, then the locality means that "$|i - j| \leq$ a finite distance."

We emphasize that conventional *lattice gauge theories* with dynamical gauge fields are usually *not* local lattice models: Since there is a non-local gauge constraint, thereby the tensor product decomposition (A1) is violated. In this work, we do *not* use models of conventional *lattice gauge theories*, but limit ourselves to only *local lattice models*.

Def. II: The **onsite symmetry**, for such local lattice models, is defined as a global internal symmetry, whose symmetry transformation operator has the following tensor product decomposition

$$U = \bigotimes_i U_i, \tag{A2}$$

where U_i is a unitary operator acting on $\mathcal{V}_i$.

Def. III: **Well-defined quantum field theory** (living on the boundary of lattice model): When we mention a "well-defined" quantum field theory (QFT), we always mean a *limited class* of QFTs which can be regularized (*i.e.* regularized) as the low energy effective boundary theory of a gapped local lattice model (see Def. I) in one higher dimension (so-called the bulk). The global symmetry, if any, is regularized as an onsite symmetry (see Def. II) for the full bulk-boundary system. Such a QFT has at most the b-anomaly to be defined later in Def. 1. A "well-defined" QFT cannot have the r-anomaly to be defined later in Def. 2.

Def. IV: **All lattice obstruction-free** (required to be regularizable in the same dimension): The above defined QFTs include $d+1$D QFTs that can be regularized by a lattice model in the *same* dimension $d+1$D (with the symmetry, if any, regularized as an onsite symmetry or a local on-n-simplex symmetry[20]), because the gapped bulk in one higher dimension (of Def. III) can be a decoupled gapped tensor product state. This leads to a concept of *all lattice obstruction-free*: By saying a $d+1$D QFT is *all lattice obstruction-free*, we always mean a $d{+}1$D well-defined quantum field theory in Def. III, which can be regularized as the low energy effective boundary theory of a $d+2$D gapped tensor product state (*i.e.* a gapped trivial vacuum) on a one-higher-dimensional lattice. Note that a tensor product state (*i.e.* a trivial state, with *neither*

[20] Here we only focus on the well-defined G-symmetric QFTs with ordinary G-global symmetries (the 0-form symmetry in the sense of generalized global symmetries [33]). If there is a generalized higher global symmetry [33], then we need to modify the "lattice onsite symmetry realization" to the "lattice local on-n-simplex symmetry realization."

short-range nor long-range entanglement of Def. V) in a local lattice model is defined as

$$|\Psi\rangle = \bigotimes_i |\psi_i\rangle, \quad |\psi_i\rangle \in \mathcal{V}_i, \tag{A3}$$

which can be gapped and decoupled from its boundary theory. In contrast, the generic state is more general and is not necessarily a tensor product, such as

$$|\Psi\rangle = \sum_{\{c_{\{i\}}\}} c_{\{i\}} \Big(\bigotimes_{i} |\psi_i\rangle \Big), \quad |\psi_i\rangle \in \mathcal{V}_i, \tag{A4}$$

with generic complex normalizable coefficients $c_{\{i\}}$.

Def. V: **Gapped system and entanglement:** By a gapped tensor product state, we mean that the tensor product state is a unique ground state (with an energy E_0) of some lattice bulk Hamiltonian system whose energy spectrum has a finite energy gap $\Delta_E = E_{\text{excited}} - E_0 > 0$ separated from all excited states E_{excited}. Below the energy gap Δ_E, the system behaves as a gapped trivial vacuum (or a gapped trivial insulator in condensed matter) with no entanglement.

On the other hand, general gapped systems ($\Delta_E = E_{\text{excited}} - E_0 > 0$) can generically posses short-range or long-range entanglements.

Short-range entangle states, short-range entanglements (SRE) and SPT states are defined as those gapped quantum *ground states* which can be deformed via local unitary transformations (LUT) to a trivial tensor product state once we remove, *part of* or *all of*, the internal global symmetries [16]. Namely, along the deformations to a trivial tensor product state, the LUT may break some internal global symmetry of the state. Gapped SRE states are also named to be *SPT states*.

Long-range entangle states, long-range entanglements (LRE) and topological orders are defined as those gapped quantum *ground states* which *cannot* be deformed via local unitary transformations (LUT) to a trivial tensor product state, even if we remove all internal global symmetries. Gapped LRE states are also named to be *topological orders*.

By this definition Def. V, we can also rephrase Def. IV as a well-defined quantum field theory (in Def. III) is all lattice obstruction-free (Def. IV) if it can be regularized as the low energy effective boundary theory of a gapped bulk lattice system whose bulk has no LRE (*i.e.* no topological order) and no SRE (*i.e.* no SPT state), thus the bulk has no entanglement structure at all as a gapped trivial tensor product state. Readers should be cautious that although this gapped bulk alone has *no entanglement*, the boundary theory (such as an all anomaly-free QFT) can be *highly-entangled* and can have *gapless states*.

Def. VI: **All anomaly-free** (*i.e.* here free of all invertible bosonic and fermionic b-anomalies):

The recent development suggests that all anomaly-free conditions of $d+1$D G-symmetric QFT can be understood as the QFT can live on the boundary of a trivial cobordism class of a trivial invertible topological quantum field theories (iTQFT) from a corresponding cobordism group [29] or its higher-symmetry and higher-classifying space generalization [30–32]:

$$\Omega_G^{d+2}. \tag{A5}$$

Namely, the trivial iTQFT is the trivial element 0 in the Ω_G^{d+2}. Let us explain this development below.

't Hooft anomaly is a property that the global symmetry of the theory cannot be made onsite on a lattice, thus there is an obstruction to gauge the non-onsite symmetry, which is called the anomalous symmetry (Model 1) [18, 92]. Dynamical gauge anomaly is a property that its theory is ill-defined (discussed in Model 3). How to classify the property of non-onsite global symmetries or seemly ill-defined theories?

The previous anomaly inflow picture relates the anomalous *non-interacting field theories* or *non-interacting lattice models* (Def. I) to the boundary of one higher dimensional bulk [93, 94]. Ref. 95 systematically described anomalies in field theories in terms of topological invariants in one higher dimension (such as the index of a Dirac operator), which turn out to be cobordism invariants [96]. However, to construct an *interacting* lattice regularization of a field theory, we need to classify anomalies in *interacting* field theories and *interacting* lattice models. Ref. 92 attempts to classify anomalies in interacting lattice models, via topological orders and symmetry-protected topological states (SPTs) of interacting lattice models in one higher dimension.

Let us introduce a few different concepts of anomalies as the definitions of terminology:

Def. 1: **b-anomaly** (≡ boundary defined anomaly): There are anomalous theories that can be regularized as the low energy effective boundary theory of a gapped local interacting *lattice* model in one higher dimension, where the global symmetry, if any, is regularized as an *onsite* symmetry for the whole bulk-boundary coupled system. However, the effective symmetry, if any, on the effective boundary theory alone is *non-onsite*. There is an obstruction to gauge the non-onsite symmetry [18, 92], because the standard gauging only works for an onsite symmetry: Because there is no canonical way to input the gauge variables on the links between "non-local sites" where the non-onsite symmetry acts. The obstruction of gauging is the same phenomenon happened in 't Hooft anomalies. We will call this kind of anomalies as the *b-anomalies*, which include the 't Hooft anomalies (associated with some internal global symmetry), gravitational anomalies (associated with no internal global symmetry), and their mixed anomalies.

Def. 2: **r-anomalies** ($\equiv$ radical anomaly):
There are also anomalous theories that cannot be regularized as the low energy effective boundary theory of any gapped local lattice model in one higher dimension. We will call this kind of anomalies as the *r-anomalies*, which include the dynamical gauge anomalies. A theory with an r-anomaly is simply an ill-defined quantum theory.

However, for a dynamical gauge theory with an r-anomaly, very often, we un-gauge the theory to turn the dynamical gauge field (on the link or on n-simplex) into a global symmetry transformation (onsite or on $(n-1)$-simplex). The resulting ungauged quantum theory may have a b-anomaly ('t Hooft anomaly) instead of an r-anomaly. (See examples below.)

Def. 3: **invertible v.s. non-invertible anomalies**:
There are invertible anomalies that can be canceled by other anomalies. (The anomalies discussed in the field theory literature are mostly invertible anomalies.) Invertible anomalies form an abelian group, such as an infinite integer group $\mathbb{Z}$ (*i.e.* a perturbative local anomaly, captured by a Feynman diagram loop calculation) or a finite group $\mathbb{Z}_n$ of some positive integer n (*i.e.* a non-perturbative global anomaly), or the product groups of $\mathbb{Z}$ and $\mathbb{Z}_n$. The invertible anomaly labeled by an abelian group element g can be canceled by an inverted anomaly labeled by an inverted abelian group element g^{-1}. There are also non-invertible anomalies[41, 92, 97–100] that cannot be canceled by any other anomalies.

Def. 4: **bosonic v.s. fermionic anomalies**:
There are bosonic anomalies where the local operators in the corresponding anomalous theories are all bosonic [101, 102]. There are fermionic anomalies where some local operators in the corresponding anomalous theories are fermionic. For example, a Spin(10) chiral Weyl fermion theory has an internal symmetry $\text{Spin}(10) \supset \mathbb{Z}_2^f$ containing the fermion parity, thus we will need to classify possible *fermionic anomalies* of the interacting fermionic theory (later in Eq. (10)) in order to classify all of its anomalies.[21]

For more examples,

- A 1+1D chiral complex Weyl fermion theory with a Hamiltonian, $H = \mathrm{i}\psi^\dagger \partial_x \psi$ and a 1-component complex Weyl spinor ψ, has

  a fermionic invertible b-anomaly.

  It is invertible because the anomaly has a $\mathbb{Z}$ class as a group classification.

- A 3+1D Weyl fermion doublet coupled to a probed (thus non-dynamical) SU(2) background gauge field has the Witten SU(2) anomaly [103] as a type of 't Hooft anomaly of the SU(2) global symmetry, which is

  a fermionic invertible b-anomaly.

  It is fermionic because the $\text{SU}(2) \supset \mathbb{Z}_2^f$ has the fermion parity at its $\mathbb{Z}_2$ center. It is invertible because the anomaly has a $\mathbb{Z}_2$ class as a group classification.

- A 3+1D Weyl fermion doublet coupled to a dynamical SU(2) gauge field has the Witten SU(2) anomaly [103], which is

  a bosonic invertible r-anomaly.

  It is bosonic since all the local operators are gauge invariant and bosonic. Namely, the SU(2) ($\supset \mathbb{Z}_2^f$) is dynamically gauged, thus the fermion parity $\mathbb{Z}_2^f$ is also gauged and the full theory is bosonic. It is an r-anomaly, since the Weyl fermion coupled to this SU(2) gauge theory cannot be regularized as a boundary of any gapped local bosonic lattice model [67]. However, if we un-gauge the SU(2) of this ill-defined gauge theory, then its bosonic invertible r-anomaly (Def. 2) becomes the previous fermionic invertible b-anomaly.

- A $\mathbb{Z}_2$ gauge theory in 2+1D or above with only $\mathbb{Z}_2$ charge excitations has

  a bosonic non-invertible b-anomaly,

  regularized as a boundary theory of a one-higher-dimensional $\mathbb{Z}_2$ gauge theory which is a topological quantum field theory (TQFT).

---

[21] However, once the $[\text{Spin}(10)] \supset \mathbb{Z}_2^f$ is gauged thus the fermion parity $\mathbb{Z}_2^f$ is gauged in the Spin(10) chiral Weyl fermion theory, it becomes a Spin(10) chiral gauge theory, where all local gauge-invariant operators are bosons.

The classification in Ref. 92 is a classification of all b-anomalies in terms of the topological orders [104] or symmetry-protected topological (SPT) states [15, 16, 46] in local *lattice models* in one higher dimension. A b-anomaly is invertible if it is characterized by an SPT state or an invertible topological order [41, 42, 105, 106] in one higher dimension. In this work, we will only focus on the invertible b-anomalies and their classifications.

From now on, by *anomalous field theory*, we will specifically mean a well-defined quantum field theory (Def. III) with at most some invertible b-anomalies (defined in Def. 1). In this work, we only study well-defined *quantum field theories* (Def. III) as the effective low energy theory of the boundary of local lattice models (Def. I). So, we exclude theories with the r-anomaly (defined in Def. 2) since they are not well-defined quantum theories (by the norm of both Def. I and Def. 1, and the standard lore).

According to the above classification, an anomaly-free (Def. VI) well-defined quantum field theory (Def. III)[22] is nothing but a boundary theory of a gapped trivial state (a tensor product state) on a one-higher-dimensional *lattice*, which means *all lattice obstruction-free* that can be also regularizable in the same dimension (Def. IV).

The generalization of the anomaly inflow to a lattice model with interactions is crucial to obtain this result, since some of the key concepts, like the tensor product state and the onsite symmetry, require a lattice (providing the locality of sites) to define.

With the above terminology definitions, we claim a proposition (Prop.):

Prop. I: Any well-defined quantum field theory (Def. III) if that is

all anomaly free (Def. VI)

with a list of conditions in the footnote 22, then it is

all lattice obstruction-free (Def. IV)

required to be regularizable in the same dimension: Namely, any well-defined QFT that is *all* anomaly-free can be regularized by a local *interacting* lattice model in the same dimension, where the global symmetry is regularized as an onsite symmetry (or generalized local on-n-simplex symmetries) [92].[23]

This result can be used to solve the gauged chiral fermion problem via the mirror fermion approach [23]: Given a $d+1$D gauged chiral fermion theory with a gauge group $G_f \supset \mathbb{Z}_2^f$, we first un-gauge, and obtain a $d+1$D chiral fermion theory with an internal global symmetry group $G_f \supset \mathbb{Z}_2^f$. Then, we find a gapped $d+2$D lattice model with a symmetry $G_f \supset \mathbb{Z}_2^f$ whose boundary regularizes the un-gauged $d+1$D chiral fermion theory (the Model 1). The symmetry G_f is regularized as an onsite symmetry of the $d+2$D lattice model. Next, we determine if the ground state of the bulk gapped $d+2$D lattice model has a trivial topological order and a trivial SPT state or not. If the $d+2$D ground state indeed has no topological order and no SPT state (which is a trivial tensor product state by Def. IV), then the $d+1$D un-gauged chiral fermion theory can be regularized as the low energy effective theory of a $d+1$D local lattice model. Also, the $d+1$D gauged chiral fermion theory can be regularized as the low energy effective theory of a $d+1$D local lattice model after gauging the onsite symmetry G_f.

To show the above claim, we can choose the $d+1$D lattice model to be a slab of the $d+2$D lattice model with *a finite number of layers* in the extra dimension. In such a model, the normal sector (or the chiral fermion sector) lives on one surface of the slab and the mirror fermion sector lives on the other surface of the slab. If the normal sector is free of all anomalies, it implies that the $d+2$D bulk is actually a trivial gapped phase. If so, the mirror sector *can be chosen to be* a symmetric gapped boundary and can be fully gapped out without breaking the onsite symmetry [15, 16, 18]. A detailed explanation is given in Section A 2. Since the $d+2$D slab has only finite layers, the $d+2$D slab is actually a $d+1$D lattice model with *finite orbitals per site*. Last, we gauge

[22] Thus here the *all anomaly-free* condition for a well-defined quantum field theory (Def. III) specifically satisfies:

- free of b-anomalies in Def. 1,
- free of all invertible anomalies in Def. 3,
- free of bosonic and fermionic anomalies in Def. 4.

Above we propose that:

$$\text{If “all anomaly free”}_{\text{(Def. VI)}} \to \text{then “all lattice obstruction-free.”}_{\text{(Def. IV)}} \quad \text{(A6)}$$

However, some well-defined QFTs (Def. III) can be regularized on the boundary of one higher-dimensional lattice model, e.g., even if they have b-anomalies in Def. 1.

Thus, there is a subtlety about the converse statement. Only when we restrict the "all lattice obstruction-free" requiring QFT to be regularizable in the same dimension and all symmetries regularized strictly locally (Def. IV), then the converse statement is also true:

$$\text{If “all lattice obstruction-free”}_{\text{(Def. IV)}} \to \text{then “all anomaly free.”}_{\text{(Def. VI)}} \quad \text{(A7)}$$

In this work, when we classify invertible 't Hooft anomalies of global symmetries G, we use the cobordism group Ω_G^{d+2} in (A5) whose category of manifolds are only smooth and differentiable manifolds. Therefore, we can apply a known mathematical fact that all those smooth and differentiable manifolds are triangulable manifolds, via the Morse theory. Thus the anomalies captured in Ω_G^{d+2} of *smooth and differentiable* manifolds can be triangulated on a lattice of *triangulable* manifolds. See more comments on Sec. II.

[23] See Footnote 20 for the comment on the local symmetry realizations on the lattice.

the onsite symmetry to obtain a gauged chiral fermion theory.

Thus, the above understanding suggests:

Prop. II: Any $d+1$D gauged chiral fermion theory (Model 2), that can be regularized as the low energy effective boundary theory of a $d+2$D gapped local lattice model in one higher dimension (Def. III), can be regularized as the low energy effective theory of a local lattice model in the same $d+1$D dimension (Def. IV), as long as the theory is free of all anomalies (given by Def. VI and Footnote 22).

We remark that for a certain anomalous $d+1$D chiral fermion theory with an internal symmetry group G_f, their corresponding $d+2$D topological/SPT orders may have a gapped boundary that does not break the G_f symmetry, but has a non-trivial G_f-symmetric anomalous boundary topological order [18, 107] — the low energy theory of topological order may be a $d+1$D G_f-symmetric topological quantum field theories (TQFT) canceling the same 't Hooft anomaly of $d+1$D chiral fermion theory.

For such an anomalous $d+1$D chiral fermion theory, we can have a lattice model in the same $d+1$D dimension that exactly regularizes all the low energy particles of the anomalous chiral fermion theories. However, the full low energy effective theory of the lattice model will contain an extra gauge field for a finite gauge group G_{extra}, prescribing the non-trivial anomalous $d+1$D topological order and TQFT. Thus, if we only concern about low energy particles, even some anomalous gauged chiral fermion theories can be regularized by lattice models in the same dimension [23]. But the lattice models will also produce an extra $d+1$D G_{extra}-gauge theory with *no* additional low energy particles, but may give rise to additional extended objects such as string and brane excitations from the TQFT.

It is well-known that a Spin(10) chiral fermion theory (Model 1) is free of all *perturbative* 't Hooft anomalies; similarly, it is also well-known that a Spin(10) chiral gauge theory (Model 3) is free of all *perturbative* dynamical gauge anomalies [108, 109]. But it is not known before if the Spin(10) chiral fermion theory (Model 1) is free of all other *non-perturbative global anomalies* (of 't Hooft anomalies) or not. Thus, it is also not known in the past literature if the Spin(10) chiral gauge theory (Model 3) is free of all other *non-perturbative global anomalies* (as dynamical gauge anomalies) or not.

Ref. 23 provides an argument that the Spin(10) chiral fermion theory is free of all anomalies, by proposing a sufficient condition: *A gauged chiral fermion theory in a $d+1$-dimensional spacetime with a gauge group G_f is free of all anomalies if* (0) *it can be regularized as a low energy effective boundary theory of a gapped local lattice model in one higher dimension* (Def. III), (1) *there exists a non-zero Higgs field that makes all the fermions massive, and* (2) $\pi_n(G_f/G_{grnd}) = 0$ *for* $0 \le n \le d+2$, *where* G_{grnd} *is the unbroken gauge symmetry group for the non-zero Higgs field.* The chiral fermions satisfying the above conditions can be gapped out by direct interactions or boson-induced interactions without breaking the G_f symmetry, even when the fermion mass term is forbidden by the symmetry. This new mechanism to give fermions an effective energy gap (or an effective mass) is referred to as "mass without mass term [72]" (an induced energy gap by interactions without a quadratic mass term, beyond the ordinary Higgs mechanism [18]). But the above statement is based on an assumption that a smooth orientation fluctuation of Higgs field can give rise to a symmetric disordered phase. Some other related approaches have also been proposed [17, 72–74, 77].

In this work, we do *not* require the proposed conditions of Ref. 23 above, nor need the assumption of new fluctuating Higgs fields in Ref. 23. Instead, we will independently and rigorously show that the above Spin(10) chiral fermion theory (Model 1) is indeed *free of all 't Hooft anomalies* by a cobordism group approach (Eq. (10)), and thus it can be defined on a 3+1D lattice; which can become a Spin(10) gauged chiral fermion theory (Model 2) by coupling to a Spin(10) background gauge field, or become a Spin(10) chiral gauge theory (Model 3) by dynamically gauging Spin(10).

2. Gapped boundary of a state with a trivial invertible topological order with symmetry

In the following, we show that:
(1) There *exists* a 3+1D gapped boundary for the above lattice model without breaking the Spin(10) symmetry at the low energy.
(2) There *exist* non-perturbative interactions to gap the mirror world chiral fermions without breaking the Spin(10) symmetry.

The focus of this section is on showing the *existence* (in the mathematical sense), instead of proving the *constructions* (which may not be *unique* for the *uniqueness* in the mathematical sense). In section III, we provide the 16 copies of the lattice model (7) give rise to the 3+1D Weyl fermions in the 16-dimensional spinor representation of the Spin(10) on the lattice boundary, see Fig 1.

In order to show Prop. I which consequently includes also Prop. II, we break down this proposition into several relatedly helpful sub-propositions. For any well-defined $d+1$D QFT (defined in Def. III) that is *all* anomaly-free (defined in Def. VI) with an internal symmetry G_f, which can live on the boundary of $d+2$D bulk regularized lattice model, we aim to show that (which we focus on the spatial dimension $d=3$):

Prop. i: There exists a *symmetric gapped boundary* for the corresponding $d+2$D bulk regularized lattice model. This $d+1$D symmetric gapped boundary does not break any internal symmetry G_f of the whole bulk-boundary system, and does not contribute any ground state degeneracy (neither symmetry-breaking degeneracy, nor topological degeneracy [110, 111]).

Prop. ii: There exist *non-perturbative symmetric interactions* to fully gap this well-defined all anomaly-free $d+1$D QFT, via deforming the QFT by adding any all anomaly-free gapless or gapped sectors, while still preserving the full G_f internal symmetry, without any symmetry-breaking and without contributing any degeneracy (neither symmetry-breaking degeneracy nor topological degeneracy).

We will see that showing Prop. i is sufficient enough to show that Prop. I is also true. In other words, we only need Prop. i but do not need to prove Prop. ii, in order to prove Prop. I.

• To show Prop. i, we first note that by a symmetric gapped boundary, we also mean that the ground state energy E_0 (of this whole bulk-boundary system) to its higher energy excited states (at energy $E_1, \ldots,$ etc.), are separated by a finite energy gap $\Delta_E = E_1 - E_0 > 0$. Of course, by defining the energy gap $\Delta_E > 0$ here, we should first set-up a toy-model system with only such a $d+1$D symmetric gapped boundary and a fully gapped $d+2$D bulk. (We either have only this gapped boundary and without other boundaries, or other boundaries are also fully gapped.)

If the gapped d+2D bulk has also a *symmetric gapless* boundary (say on a d+1D boundary A) other than the *symmetric gapped* boundary of Prop. i (say on another d+1D boundary B). Thus the gapless boundary A contributes to the low energy spectrum at the infrared (IR) of a tiny energy sub-gap

$$\delta_{E,\mathrm{A}} \simeq \exp(-L/\xi) \tag{A8}$$

which scales exponentially over the linear system size L over the correlation length ξ; the gapped boundary B contributes to the energy spectrum only at the higher energy at a deeper ultraviolet (UV) of a finite energy gap

$$\Delta_{E,\mathrm{B}} \simeq \Delta_E > 0, \tag{A9}$$

mentioned earlier. Then the whole bulk-boundary system would become *gapless* instead of being gapped.

The important issue is that when the $d+2$D gapped bulk has no entanglements (*i.e.* no LRE nor SRE by Def. V), then "the d+1D symmetric gapless boundary A" and "the d+1D symmetric gapped boundary B" actually cannot affect each other, thus are isolated from each other. See more in Appendix C.

To proceed showing Prop. i, if the bulk regularized lattice is in the gapped trivial phase (*i.e.* has a gapped trivial tensor product ground state), we can make a boundary by first deforming the bulk ground state (by symmetry-preserving LUT in Def. V) into a tensor product state. Such a deformation does *not* close the energy gap since the bulk state is already in the symmetric gapped trivial phase. The trivial tensor product state *always* can have a gapped boundary respect to a trivial vacuum[24] — by saying so, we mean that we set the energy scale of the trivial vacuum (normally to below some energy scale such as a finite energy $\Delta_E > 0$, or below an infinite energy gap $\Delta_E \to \infty$) to be the same as the energy scale of the gapped boundary (say on B) $\Delta_E \simeq \Delta_{E,\mathrm{B}} > 0$ in Eq. (A9). We note that the above deformation respects the onsite symmetry (if any), and the resulting tensor product state also respects the onsite symmetry. The gapped boundary does not break the onsite symmetry, thus has no *symmetry-breaking degeneracy*. Since the symmetric gapped boundary has no entanglements, thus has no *topological degeneracy* (because topological degeneracy [110, 111] are due to LRE defined in Def. V).

The above completes our proof of Prop. i.

In Appendix D, we can also show Prop. i by a second viewpoint: a derivation from the *classification of quantum phases of matter and their phase transitions*.

This second viewpoint from the *classification of quantum phases of matter* shows that there is *no need for an energy-gap closing phase transition*. By maintaining a finite energy gap Δ_E between two phases, there must exist a *symmetric gapped boundary* between two phases, thus we have given an alternative proof of Prop. i.

The slight conceptual difference between the first viewpoint and the second viewpoint is that, the first is about the *one-spatial-dimensional-lower phase boundary* in $d+1$D between two $d+2$D phases, while the second is about no need for the *phase transition* in $d+2$D between two $d+2$D phases in a quantum phase diagram (at zero temperature $T=0$) by tuning a certain coupling g.

• To show Prop. I, we consider a $d+2$D bulk regularized lattice model that regularizes the $d+1$D QFT as its boundary theory by Def. III. We choose the bulk lattice model to be a slab of finite thickness, such that one boundary of the slab regularizes the QFT (Boundary A), and the other boundary is a symmetric gapped boundary (Boundary B) in Prop. i. We apply the Prop. i proven earlier. Here a slab of finite thickness is always achievable for this system (especially for the gauged chiral fermion problem of Model 1 and Model 2), because of the isolation between two $d+1$D boundaries A and B due to Appendix C's Remark (i) on the isolation of the energy scale and Remark (ii) on the isolation of the mutual entanglement, see Appendix C on the energy scale and mutual entanglement between gapless and gapped boundaries.

---

[24] The trivial tensor product state *always* can have a gapped boundary respect to a trivial vacuum, because the trivial tensor product state is itself the same phase indistinguishable as the trivial vacuum. Thus its gapped boundary simply is the trivial gapped domain wall between the same phase [112, 113].

Thus the low energy physics of the $d+2$D slab is described by this $d+1$D QFT. A lattice model of this $d+2$D slab of a finite thickness can be constructed explicitly as a lattice model in one lower dimension ($d+1$D), by rewriting the "quantum Hilbert space associated with different lattice sites along the finite width thickness w (*i.e.* an extra small dimension along w)" to "quantum Hilbert space associated with finite orbitals per site" in $d+1$D.

This completes our proof of Prop. I.[25]

3. A deformation class of all anomaly-free well-defined QFTs

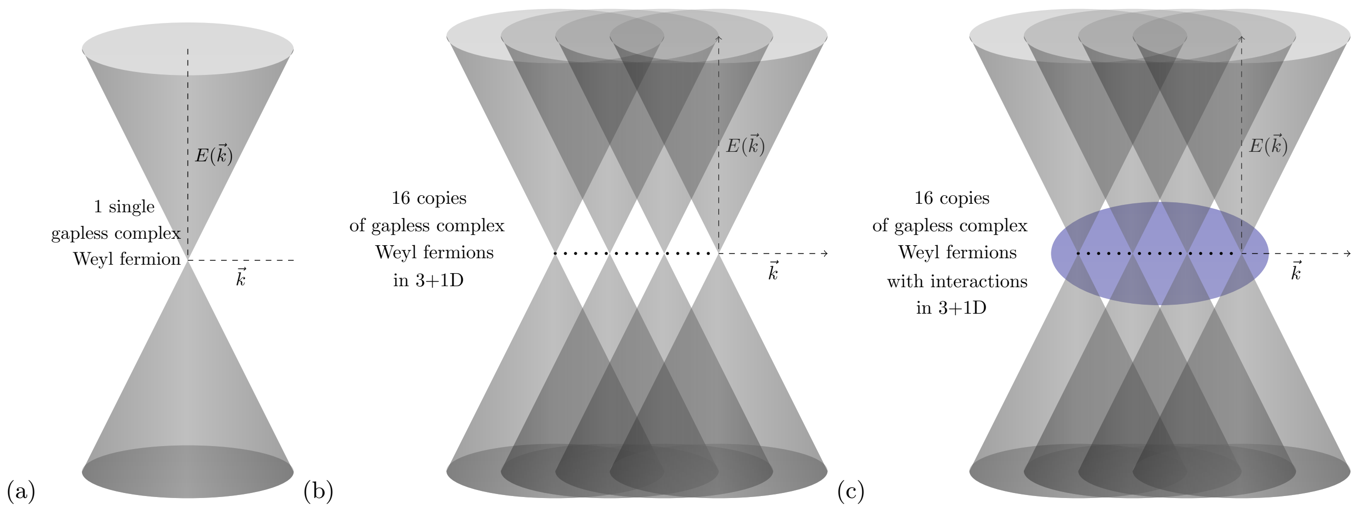

FIG. 1.
(a) A lattice construction of a single Weyl fermion is given in Section III, the subfigure shows the gapless energy spectrum $E(\vec{k})$ of Brillouin zone in the schematic 3-dimensional momentum $\vec{k}=(k_x,k_y,k_z)$-space with a linear dispersion $|E(\vec{k})| \propto c|\vec{k}|$ for some effective speed of light c.
(b) The 16 copies of the same lattice model (7) give rise to the 3+1D Weyl fermions at the low energy in the 16-dimensional spinor representation of the Spin(10) on the lattice boundary shown in Section III. The 16 gapless Weyl points (schematically the 16 dots $\bullet$) may be separated but can be tuned to the same point on the $\vec{k}$-space Brillouin zone. We show this Spin(10) chiral Weyl fermion theory is free from all 't Hooft anomalies via a cobordism theory in Sec. II.
(c) There are two ways to obtain the symmetric gapped boundary for the bulk of the 16 copies of the lattice model: First, via Prop. i, there exists a *symmetric gapped boundary* for the corresponding $d+2$D bulk regularized lattice model (without the need to access from gapping out the gapless theories from interactions. Second, via Prop. ii, there exist *non-perturbative symmetric interactions* to fully gap this well-defined all anomaly-free Spin(10) chiral fermion theory with 16 Weyl fermions in 16-dimensional spinor representation of the Spin(10). (Schematic interactions are drawn in the shaded blue region.) In this work, we only prove Prop. i, but we suggest some supportive evidence for Prop. ii but without proving Prop. ii. However, applying only Prop. i (but without requiring Prop. ii) is sufficient enough for us to construct the Spin(10) chiral fermion theory on the lattice via Prop. I.

• For Prop. ii, we again consider a bulk lattice model that regularizes the QFT as its boundary theory by Def. III. Since the same bulk model can also have a symmetric gapped boundary according to Prop. i, thus we ask: How to modify the symmetric interactions in the QFT to make it into a fully symmetric gapped theory describing the symmetric gapped boundary?

One key ingredient is that there are more degrees of freedom given by the full Hilbert space to help us reach the goal. We may be able to access the symmetric gapped phase not only within this specific well-defined *all anomaly-free* QFT, but also higher energy spectrum by engineering all possible degrees of freedom and their symmetry-preserving local interactions.

We do not have a direct proof of Prop. ii, but we have several supportive evidence to argue Prop. ii should be true:

(α) "*The deformation classes of QFTs*" advocated by Seiberg [114]: Given a continuum QFT with some energy scale Λ, with a given *global symmetry* and derivable *'t Hooft anomaly* of global symemtry. We are allowed to add arbitrary degrees of freedom and new fields preserving the symmetries (and selection rules) and with

[25] As we said earlier, we do not need Prop. ii to show Prop. I.

no additional anomaly (without modifying the original 't Hooft anomaly) at some energies. The new degrees of freedom *do not* directly affect the dynamics at lower energies. Next, we can deform the parameters of this larger theory with the new degrees of freedom, by making the new degrees of freedom interacting with the original QFT, which *do* affect the dynamics. This is a much larger space of theories, which can land into different new phases with different dynamics. Seiberg names all these possible deformations of QFT as a *deformation class of the QFT.*
Seiberg [114] conjectured that given two QFTs, say partition functions $\mathbf{Z}_1$ and $\mathbf{Z}_2$, in the same spacetime dimension with:
(i) the same global symmetry (and selection rules),
(ii) the same 't Hooft anomalies,
we can always add new degrees of freedom at short distances so that we can interpolate between two QFTs: The two QFTs, $\mathbf{Z}_1$ and $\mathbf{Z}_2$, are in the same *deformation class of QFT*. In other words, this also means that the deformation class of QFT can be determined and defined by the symmetries and the 't Hooft anomalies of QFT. Seiberg's conjecture is in fact shown to be true for many examples.

What we claim on Prop. ii is indeed a special case of Seiberg's proposal [114]: We consider the *deformation class of the anomaly-free well-defined QFT*, containing the trivial gapped phase (e.g. a symmetric gapped Spin(10) boundary) and a gapless phase (e.g. a symmetric gapless Spin(10) chiral fermion theory), both have (i) the same global symmetry (and selection rules) and (ii) no 't Hooft anomalies (we will show via a cobordism theory in Sec. II). If Seiberg's proposal [114] is true, our proposal must also be true.

(β) We can start from the $d+1$D symmetric *gapless all-anomaly-free* theory, and adding new $d+1$D symmetric *gapped all-anomaly-free* sectors (this is analogous to Seiberg's proposal [114]).

Moreover, we can also add additional *gapless all-anomaly-free* sectors in the *various possible representations* (Rep) of symmetry.[26] The symmetry organizes the (both gapped and gapless) energy eigenstates in the energy spectrum into *various possible representations* of the symmetry group, whose selection rules constrain the interactions and dynamics between states in the energy spectrum.

Based on (α) and (β), we propose that Prop. ii is also true.

Appendix B: Construct a local bosonic lattice model realizing a 3+1D Spin(10) gauged chiral fermion theory

Below we will use the slave-particle/parton approach [78–80] to explicitly construct a 4+1D *local bosonic* lattice model, whose boundary can give rise to the dynamical Spin(10) chiral gauge theory coupled to Weyl fermions (Model 3) in the 16-dimensional representation. We start with a fermionic model on a 4D cubic lattice. On each site we have $16N_f$ complex fermions $\hat{\psi}_{\alpha,m}$, with $\alpha = 1,\cdots,16$, and $m = 1,\cdots,N_f$.[27] So on each side, there are 2^{16N_f} states. Now we project into the even fermion subspace on each site, and turn the fermionic model into a bosonic model with 2^{16N_f-1} states per site. The Hamiltonian for such a bosonic model is given by

$$\hat{H}_1 = \sum_{\langle ij\rangle}\sum_{\alpha\beta}(\hat{\chi}_{ij}^{\alpha\beta})^\dagger\hat{\chi}_{ij}^{\alpha\beta} + \sum_i(-)^i\hat{n}_i, \tag{B1}$$
$$\hat{\chi}_{ij}^{\alpha\beta} = \sum_m \hat{\psi}^\dagger_{\alpha,m,i}\hat{\psi}_{\beta,m,j}, \quad \hat{n}_i = \sum_{m,\alpha}\hat{\psi}^\dagger_{\alpha,m,i}\hat{\psi}_{\alpha,m,i}.$$

The above model has $[\mathrm{U}(16)]^{N_{\text{site}}}$ local symmetry. In the large N_f limit, $\hat{\chi}_{ij}^{\alpha\beta}$ is weakly fluctuating and can be replaced by $\chi(\mathrm{e}^{\,\mathrm{i}\,A_{ij}})_{\alpha\beta} = \langle\hat{\chi}_{ij}^{\alpha\beta}\rangle$ expectation value and A_{ij} is a 16×16 Hermitian matrix to describe the U(16) gauge fluctuation. This leads to the following emergent U(16) gauge theory (at a mean-field level)

$$\hat{H}_1^{\text{mean}} = \sum_{\langle ij\rangle}\sum_{\alpha\beta,m}\Big[\hat{\psi}^\dagger_{\alpha,m,i}\chi^*(\mathrm{e}^{-\,\mathrm{i}\,A_{ij}})_{\alpha\beta}\hat{\psi}_{\beta,m,j} + h.c.\Big] + \sum_i(-)^i\hat{n}_i. \tag{B2}$$

The ground state is given by $A_{ij} = 0$. The emergent fermions are in a fully gapped product state and the bosonic model $\hat{H}_1$ gives rise to a U(16) gauge theory at low energies.

Next, we reduce the U(16) gauge theory to the Spin(10) gauge theory by adding a term

$$\hat{H}_2 = \sum_{i,m}\Gamma^{\alpha\beta\gamma\lambda}\hat{\psi}_{\alpha,m,i}\hat{\psi}_{\beta,m,i}\hat{\psi}_{\gamma,m,i}\hat{\psi}_{\lambda,m,i} + h.c. \tag{B3}$$

to break the $[\mathrm{U}(16)]^{N_{\text{site}}}$ local symmetry to a $[\mathrm{Spin}(10)]^{N_{\text{site}}}$ local symmetry where the fermions $\psi_{\alpha,m,i}$ form the 16-dimensional spinor representation. Here $\Gamma^{\alpha\beta\gamma\lambda}$ is the antisymmetric tensor which is invariant under the Spin(10) transformations. The 4+1D bosonic

[26] For example, the trivial Rep of pairs of left and right moving 3+1D Weyl fermion ψ_L and ψ_R in the trivial Rep of Spin(10). Then adding their mass term, e.g. $m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$ (only for these additional trivial Rep gapless sectors) do not break the Spin(10) symmetry.

[27] We introduce a new flavor parameter N_f, so that we gain a benefit to do a large N_f analysis for $N_f \gg 1$ or further $N_f \to \infty$.

model $\hat{H}_1 + \hat{H}_2$ will give rise to an emergent Spin(10) gauge theory with the fermions in a fully gapped product state. Those fermions are also gapped on the boundary.

To have gapless Weyl fermions on the boundary, we add the third term

$$\hat{H}_3 = \sum_{\boldsymbol{ij}} (t^{ab}_{\boldsymbol{ij}} \hat{c}^\dagger_{\alpha,a,\boldsymbol{i}} \hat{c}_{\beta,b,\boldsymbol{j}} \hat{\chi}^{\alpha\beta}_{ij} + h.c.). \tag{B4}$$

Now on each site, we have fermions $\hat{\psi}_{\alpha,m}$ and $\hat{c}_{\alpha,a}$, but we still project into the subspace with even fermion per site. The $\hat{H}_1 + \hat{H}_2 + \hat{H}_3$ acts within this subspace. So the model is still a lattice bosonic model. When $t_{\boldsymbol{ij}}$ is given by Eq. (7), the model

$$\hat{H}_1 + \hat{H}_2 + \hat{H}_3 \tag{B5}$$

will give rise to emergent massless Weyl fermions on the boundary coupled to the Spin(10) gauge field.

Consider a 4+1D slab of the local bosonic lattice model described by $\hat{H}_1 + \hat{H}_2 + \hat{H}_3$. In the main text, based on the complete classification of 't Hooft anomaly of the group $G = (\mathrm{Spin}(5) \times \mathrm{Spin}(10))/\mathbb{Z}_2^f$ (in Eq. (10) and Eq. (14)), we have shown that there exists a symmetric gapped boundary by Proposition 1 and 2 (or there exists a symmetric boundary gapping interaction, called $\hat{H}_{\text{int.bdry.non-pert}}$), that allows us to gap out the boundary Weyl fermions (without inducing a boundary 3+1D topological order [18]) on one of the surfaces of the slab.

In this case, by including such interactions $\hat{H}_{\text{int.bdry.non-pert}}$ into (B5), we propose a new Hamiltonian

$$\hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_{\text{int.bdry.non-pert}}. \tag{B6}$$

Then by applying Eq. (18)'s finite-width/layer lattice dimensional reduction, the 4+1D slab, with a *finite* width in the extra dimension, indeed becomes a 3+1D *local bosonic* lattice model that regularizes a 3+1D dynamical Spin(10) gauge theory coupled to Weyl fermions in the 16-dimensional spinor representation.

Appendix C: Energy scale and mutual entanglement between gapless and gapped boundaries

The important issue is that when the $d + 2$D gapped bulk has no entanglements (*i.e.* no LRE nor SRE by Def. V), then "the d+1D symmetric gapless boundary A" and "the d+1D symmetric gapped boundary B" actually cannot affect each other, thus are isolated from each other, in the sense that:

(i): **Energy scale**: Boundary A and Boundary B are *decoupled* below the energy scale $\simeq \Delta_{E,\mathrm{B}}$. But when the energy is above the scale $\gtrsim \Delta_{E,\mathrm{B}}$, the energy spectra of A, B, and the bulk may affect and mix with each other together.

(ii): **Mutual entanglement**: Although the d+1D symmetric gapless boundary A is highly-entangled (due to the low-lying massless chiral fermions as the energy gapless spectrum), and the symmetric gapped boundary B is trivially gapped with no entanglements as a tensor product state on Boundary B. Thanks to the trivial gapped bulk, Boundary A and Boundary B on two sides have no entanglements in between. More precisely, if we choose a $d+1$D bipartite cut inside the $d+2$D gapped bulk, we get a zero bipartite Von Neumann entanglement entropy

$$\begin{aligned} S_{\mathrm{EE}} &= S(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] \\ &= -\operatorname{Tr}[\rho_B \log \rho_B] = S(\rho_B) = 0 \end{aligned} \tag{C1}$$

for the mutual entanglement between two sides. (This understanding is consistent with the entanglement structure discussed in Def. V.)

For the *lattice regularization of the gauged chiral fermion problem*, we should emphasize that our statements in (i) and (ii) apply to Model 1 (a chiral fermion theory) and Model 2 (a gauged chiral fermion theory). However, we do not intend to apply our statements in (i) and (ii) to Model 3 (a chiral gauge theory) — once we dynamically gauge the internal global symmetry for the bulk-boundary coupled system, then "the bulk, Boundary A and Boundary B" form altogether highly-entangle quantum states (as a dynamical gauge theory). The S_{EE} of the dynamically gauged system (Model 3), based on the previous bipartite cut in Eq. (C1), is generically nonzero.

Appendix D: Show the existence of symmetric gapped boundary via quantum phase transitions

A derivation can also be obtained from the *classification of quantum phases of matter and their phase transitions*. To give a proof of Prop. i, all we need to show is that there exists a LUT deformation path (Def. V) between two bulk gapped quantum phases:
(1) Bulk phase: The $d+2$D bulk regularized lattice model which has a symmetric gapped trivial tensor product ground state, with a finite energy gap Δ_E.
(2) Trivial gapped vacuum phase (mentioned above).
Such that this LUT deformation path satisfies the criteria:
(*1*) does not close the energy gaps between Bulk Phase and Trivial Vacuum Phase (*i.e.* no gap closing, thus no gapless modes and no zero-mode degeneracy [=ground state degeneracy]).
(*2*) does not break the internal global symmetry given by the Bulk Phase. (*i.e.* $G_f = \mathrm{Spin}(10)$ for the Spin(10) gauged chiral fermion problem.)
This LUT deformation path can be regarded as a path labeled by g in the quantum phase diagram (at zero temperature $T = 0$) by tuning a parameter (*i.e.* a coupling

constant) g of the lattice Hamiltonian $\hat{H}(g)$ such that the ground state $|\Psi_{\text{g.s.}}(g)\rangle$ is unitarily evolving under this LUT along the deformation path. Then we can prove the claim of (*1*) and (*2*), either by "*proof by a contradiction*," or by directly "*constructing such a LUT deformation path*."

"*A proof by a contradiction*": Suppose, given by engineering arbitrary symmetry-preserving (e.g. G_f) local interactions for the lattice Hamiltonian, such a path in the phase diagram is still impossible between two phases (the bulk phase and the trivial gapped vacuum phase). Then there must be a phase transition between two phases, and the two phases should be different quantum phases — in fact, they should be *different* SPT phases within the G_f symmetry. But as we emphasize that both two phases have symmetric gapped trivial tensor product ground states, they must be in the *same trivial* SPT phase, thus the same trivial gapped vacuum, at least below the energy gap Δ_E of the bulk phase. This leads to a contradiction, thus we end the proof successfully.

"*Constructing such a LUT deformation path*": This path construction is basically what we had in the earlier proof. Since both phases are symmetric gapped trivial phases (both a trivial SPT phase and a trivial gapped vacuum respect to the G_f symmetry), the LUT deformation path is simply the deformation to make both symmetric gapped trivial phases become *exactly the same symmetric gapped trivial tensor product states* in a certain "*canonical basis*" respect the G_f symmetry. (Normally it is known as the *symmetric disordered* phase, where the canonical basis is chosen to be the *dual* variable of the symmetry breaking basis.)

Appendix E: Cobordism theory and classification of all possible invertible anomalies related to SU(5), SO(10), and SO(18) Grand Unifications

Here we provide the cobordism group calculations classifying all potential invertible 't Hooft anomalies of SU(5), Spin(10) and Spin(18) chiral fermion theories. Our calculations are crucial for showing all gauge anomaly free conditions for SU(5), SO(10) and SO(18) Grand Unifications. Notice that other related work [66] computes $\Omega_D^{\text{Spin}\times\text{SU}(5)}$ and $\Omega_D^{\text{Spin}\times\text{Spin}(10)}$ based on a different method, Atiyah-Hirzebruch spectral sequence (AHSS), while our work focus on $\Omega_D^{\text{Spin}\times\text{SU}(5)}$ and $\Omega_D^{\text{Spin}\times\text{Spin}(10)/\mathbb{Z}_2^f}$, also based on a more powerful Adams spectral sequence. See also Ref. 55.

1. Adams spectral sequence

The Adams spectral sequence shows:

$$\text{Ext}^{s,t}_{\mathcal{A}_p}(\text{H}^*(Y,\mathbb{Z}_p),\mathbb{Z}_p) \;\Rightarrow\; \pi_{t-s}(Y)^\wedge_p, \tag{E1}$$

where Ext denotes the extension functor, $\mathcal{A}_p$ is the mod p Steenrod algebra, and Y is any spectrum. The $\text{H}^*(Y,\mathbb{Z}_p)$ is an $\mathcal{A}_p$-module whose internal degree t is given by the $*$. The $\pi_{t-s}(Y)^\wedge_p$ is the p-completion of the $(t-s)$-th homotopy group of the spectrum Y. We note that, for any finitely generated abelian group $\mathcal{G}$, then $\mathcal{G}^\wedge_p = \lim_{n\to\infty}\mathcal{G}/p^n\mathcal{G}$ is the p-completion of $\mathcal{G}$; if $\mathcal{G}$ contains an infinite group $\mathbb{Z}$, then the $\mathcal{G}^\wedge_p$ is the ring of p-adic integers. Here the $\mathcal{G}$ is meant to be substituted by a homotopy group $\pi_{t-s}(Y)^\wedge_p$ in (E2). Here are some explanations and inputs:

1. Here the double-arrow "⇒" means "convergent to." The E_2 page contains groups $\text{Ext}^{s,t}$ with double indices (s,t), we reindex the bidegree by $(t-s,s)$. There are differentials d_2 in E_2 page which are arrows from $(t-s,s)$ to $(t-s-1,s+2)$. That is, $\text{Ext}^{s,t}\to\text{Ext}^{s+2,t+1}$. Take $\text{Ker}d_2/\text{Im}d_2$ at each $(t-s,s)$, then we get the E_3 page. Repeat this procedure, we get E_4 page, E_5 page and so on. Finally E_r page equals E_{r+1} page (there are no differentials) for $r\geq N$, we call this E_N page as the E_∞ page, we can read the result π_D at $D=t-s$. See further details discussed in Ref. 30's Sec. 2.3.

2. In Adams spectral sequence, we consider $\text{Ext}^{s,t}_R(L,\mathbb{Z}_p)$. Here we have the ring or the algebra $R=\mathcal{A}_p$ or $\mathcal{A}_2(1)$ for $p=2$, and the L is a R-module. The $\mathcal{A}_2(1)$ is the subalgebra of $\mathcal{A}_2$ generated by the Steenrod square Sq^1 and Sq^2. The index s refers to the degree of resolution, and the index t is the internal degree of the R-module L. Ext groups are defined by firstly taking a projective R-resolution $P_\bullet$ of L, then secondly computing the (co)homology group of the (co)chain complex $\text{Hom}(P_\bullet,\mathbb{Z}_p)$. A $P_\bullet$ is a resolution, which is an exact sequence of modules. Here a projective R-resolution $P_\bullet$ is an exact sequence of R-modules $\cdots\to P_s\to P_{s-1}\to\cdots\to P_0\to L$ where P_s is projective for $s\geq 0$.

2. Thom-Madsen-Tillmann spectrum and Pontryagin-Thom isomorphism

For $Y=MTG$, where MTG is the Thom-Madsen-Tillmann spectrum MTG of a group G, the Adams spectral sequence shows:

$$\text{Ext}^{s,t}_{\mathcal{A}_p}(\text{H}^*(MTG,\mathbb{Z}_p),\mathbb{Z}_p) \;\Rightarrow\; \pi_{t-s}(MTG)^\wedge_p = (\Omega^G_{D=t-s})^\wedge_p. \tag{E2}$$

The last equality is by the generalized Pontryagin-Thom isomorphism, we have an equality between the D-th bordism group of G given by Ω_D^G and the D-th homotopy group of MTG given by $\pi_D(MTG)$, namely

$$\Omega_D^G = \pi_D(MTG). \tag{E3}$$

We also compute the cobordism group of topological phases (TP) defined in [29] as

$$\mathrm{TP}_D(G). \tag{E4}$$

The $\mathrm{TP}_D(G)$ classifies deformation classes of reflection positive invertible d-dimensional extended topological field theories with symmetry group G_D. The $\mathrm{TP}_D(G)$ and the bordism group Ω_D^G are related by a short exact sequence

$$0 \to \mathrm{Ext}^1(\Omega_D^G, \mathbb{Z}) \to \mathrm{TP}_D(G) \to \mathrm{Hom}(\Omega_{D+1}^G, \mathbb{Z}) \to 0. \tag{E5}$$

We can compute the E_2 page of $\mathcal{A}_2(1)$-module based on Lemma 11 of [30]. More precisely, in order to compute $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(L_2, \mathbb{Z}_2)$, we find a short exact sequence of $\mathcal{A}_2(1)$-modules

$$0 \to L_1 \to L_2 \to L_3 \to 0, \tag{E6}$$

then we apply Lemma 11 of [30] to compute $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(L_2, \mathbb{Z}_2)$ by the given data of $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(L_1, \mathbb{Z}_2)$ and $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(L_3, \mathbb{Z}_2)$. Our strategy is choosing L_1 to be the direct sum of suspensions of $\mathbb{Z}_2$ on which Sq^1 and Sq^2 act trivially, then we take L_3 to be the quotient of L_2 by L_1. We can use this procedure again and again until $\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(L_3, \mathbb{Z}_2)$ is determined.

If $G = \mathrm{Spin} \times G'$, then its classifying space $\mathrm{B}G = \mathrm{B}(\mathrm{Spin} \times G') = \mathrm{BSpin} \times \mathrm{B}G'$. By definition, the Madsen-Tillmann spectrum $MTG = \mathrm{Thom}(\mathrm{B}G, -V)$ where V is the induced virtual bundle of dimension 0 by the map $\mathrm{B}G \to \mathrm{BO}$. By the properties of Thom space (see the discussions in Ref. 30's Sec. 1.3), we have

$$MT(\mathrm{Spin} \times G') = M\mathrm{Spin} \wedge (\mathrm{B}G')_+. \tag{E7}$$

The $\wedge$ is the smash product.

Below we will use the (E2) and (E3) to compute the D-th bordism group of G given by Ω_D^G. Then we will use the (E5) and the techniques around (E6) to compute the D-th cobordism group of topological phases of G given by $\mathrm{TP}_D(G)$.

3. Cobordism groups and topological phases for Spin × SU(5): SU(5) Grand Unification

We consider $G = \mathrm{Spin} \times \mathrm{SU}(5)$ for the Georgi-Glashow SU(5) Grand Unification [5], the Thom-Madsen-Tillmann spectrum MTG of the group G is

$$MTG = M\mathrm{Spin} \wedge (\mathrm{BSU}(5))_+. \tag{E8}$$

The T in MTG means the G-structures are on tangent bundles instead of normal bundles. For Spin, the Thom-Madsen-Tillmann spectrum $MT\mathrm{Spin} = M\mathrm{Spin}$ is equivalent to the Thom spectrum which splits $M\mathrm{Spin} = ko \vee \Sigma^8 ko \vee \cdots$. The ko is the (-1)-connected cover of the real K-theory spectrum. The $\wedge$ is the smash product and the $\vee$ is the wedge sum. The $(\mathrm{BSU}(5))_+$ is the disjoint union of the classifying space BSU(5) and a point.[28]

For the dimension $D = t - s < 8$, since there is no odd torsion,[29] by $MTG = M\mathrm{Spin} \wedge X$, then the D-th homotopy group $\pi_D(MTG) = \pi_D(ko \wedge X)$ for $D < 8$. So for the dimension $D = t - s < 8$, we have

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(X, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow (\Omega_{D=t-s}^G)_2^\wedge. \tag{E9}$$

Hence for $MTG = M\mathrm{Spin} \wedge (\mathrm{BSU}(5))_+$, for the dimension $D = t - s < 8$, by (E9), we have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BSU}(5), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin} \times \mathrm{SU}(5)}. \tag{E10}$$

---

[28] For a topological space X, it is a standard convention to denote that X_+ as the disjoint union of X and a point. Note that the reduced cohomology of X_+ is exactly the ordinary cohomology of X.

[29] By computation using the mod p Adams spectral sequence for an odd prime p, we find there is no odd torsion.

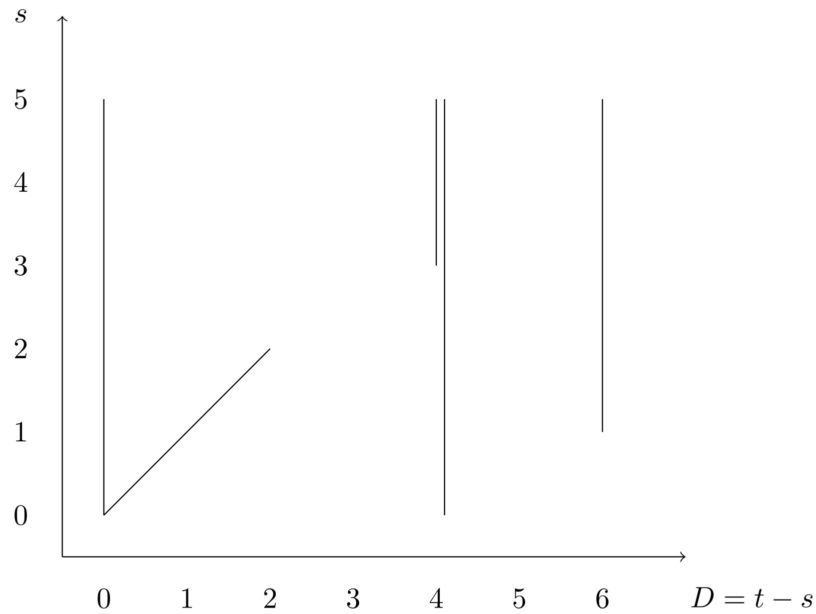

FIG. 2. Adams chart for $\Omega_D^{\mathrm{Spin}\times\mathrm{SU}(5)}$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BSU}(5),\mathbb{Z}_2)$ below degree 6 is shown in Ref. 55's Sec. 6's Figure 29, and the E_2 page is shown in Figure 2. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Appendix A of Ref. 55.

In Adams chart, the horizontal axis labels the integer degree $D = t - s$ and the vertical axis labels the integer degree s. The differential $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is an arrow starting at the bidegree $(t-s, s)$ with direction $(-1, r)$. $E_{r+1}^{s,t} := \frac{\mathrm{Ker} d_r^{s,t}}{\mathrm{Im} d_r^{s-r,t-r+1}}$ for $r \geq 2$. There exists N such that $E_{N+k} = E_N$ stabilized for $k > 0$, we denote the stabilized page $E_\infty := E_N$.

To read the result from the Adams chart in Figure 2, we look at the stabilized E_∞ page, one dot indicates a finite group $\mathbb{Z}_p$, a vertical finite line segment connecting n dots indicates a finite group $\mathbb{Z}_{p^n}$. But when $n = \infty$, the infinite line connecting infinite dots indicates an infinite group, an integer $\mathbb{Z}$. Here p is given by the mod p Steenrod algebra $\mathcal{A}_p$ in (E2). Here in Figure 2, $p = 2$, we can read from the Adams chart $\Omega_0^{\mathrm{Spin}\times\mathrm{SU}(5)} = \mathbb{Z}$ (an infinite line), $\Omega_1^{\mathrm{Spin}\times\mathrm{SU}(5)} = \mathbb{Z}_2$ (a dot), $\Omega_2^{\mathrm{Spin}\times\mathrm{SU}(5)} = \mathbb{Z}_2$ (a dot), $\Omega_3^{\mathrm{Spin}\times\mathrm{SU}(5)} = 0$ (nothing), $\Omega_4^{\mathrm{Spin}\times\mathrm{SU}(5)} = \mathbb{Z}^2$ (two infinite lines), $\Omega_5^{\mathrm{Spin}\times\mathrm{SU}(5)} = 0$ (nothing), and $\Omega_6^{\mathrm{Spin}\times\mathrm{SU}(5)} = 0$ (an infinite line).

a. *Classification of all invertible anomalies of* Spin × SU(5) *fermion theories*

By (E2) and (E3), we obtain the bordism group $\Omega_D^{\mathrm{Spin}\times\mathrm{SU}(5)}$ shown in Table I, focusing on $D = 4, 5, 6$.

Bordism group		
D	$\Omega_D^{\mathrm{Spin}\times\mathrm{SU}(5)}$	generators
4	$\mathbb{Z}^2$	$\frac{\sigma}{16}, c_2$
5	0	
6	$\mathbb{Z}$	$\frac{c_3}{2}$

TABLE I. Bordism group $\Omega_D^{\mathrm{Spin}\times\mathrm{SU}(5)}$. σ is the signature of manifold. The c_j is the j-th Chern class of the associated vector bundle of SU(n). Note that $c_3 = \mathrm{Sq}^2 c_2 = (w_2 + w_1^2)c_2 = 0 \mod 2$ on Spin 6-manifolds. Actually $\Omega_D^{\mathrm{Spin}\times\mathrm{SU}(n)} = \Omega_D^{\mathrm{Spin}\times\mathrm{SU}(n+1)}$ for $n \geq 3$ and $0 \leq D \leq 6$. See also Ref. 55.

By (E5) and (E6), we obtain the cobordism group $\mathrm{TP}_D(\mathrm{Spin}\times\mathrm{SU}(5))$ shown in Table II, focusing on $D = 4, 5$.

Cobordism group		
d	$\mathrm{TP}_D(\mathrm{Spin}\times\mathrm{SU}(5))$	generators
4	0	
5	$\mathbb{Z}$	$\frac{1}{2}\mathrm{CS}_5^{\mathrm{SU}(5)}$

TABLE II. Topological phase classification (≡ TP) as a cobordism group $\mathrm{TP}_D(\mathrm{Spin}\times\mathrm{SU}(5))$, following Table I. See also Ref. 55.

4. Cobordism groups and topological phases for $\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}$ and $\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^f}$: SO(10) and SO(18) Grand Unification

We consider $G=\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}$ for the Fritzsch-Minkowski SO(10) Grand Unification [6], the Thom-Madsen-Tillmann spectrum MTG of the group G is

$$MTG = M\mathrm{Spin}\wedge\Sigma^{-10}M\mathrm{SO}(10). \tag{E11}$$

The T in MTG means the G-structures are on tangent bundles instead of normal bundles. In this case, we have $w_2(TM)=w_2(V_{\mathrm{SO}(10)})$.

For the dimension $D=t-s<8$, since there is no odd torsion (see the footnote 29), by $MTG=M\mathrm{Spin}\wedge X$, then $\pi_D(MTG)=\pi_D(ko\wedge X)$ for $D<8$; so for the dimension $D=t-s<8$, from (E2), we have

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^*(X,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow(\Omega^G_{D=t-s})^\wedge_2. \tag{E12}$$

Hence, we have the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^{*+10}(M\mathrm{SO}(10),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{D=t-s}^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}}. \tag{E13}$$

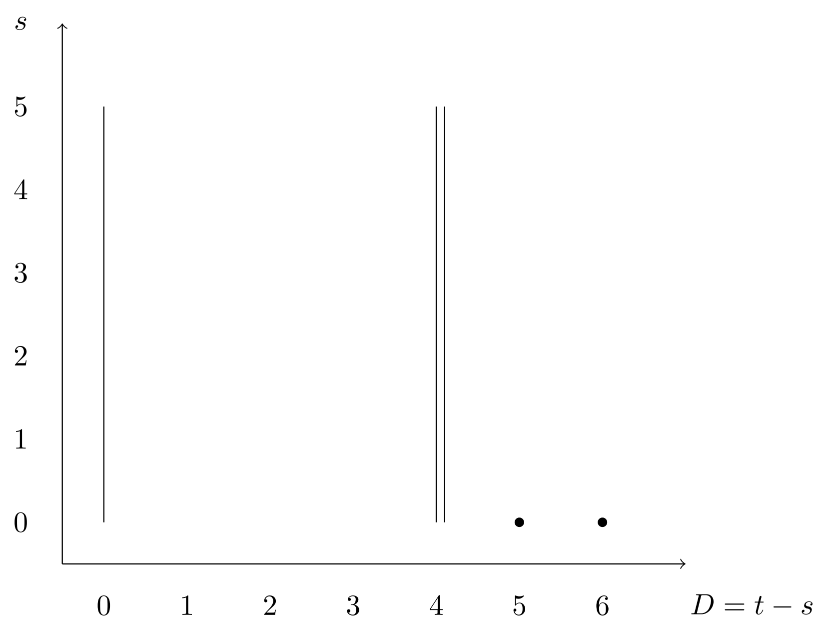

FIG. 3. Adams chart for $\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}}$, also for $\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^F}}$.

Actually we find [55]

$$\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^F}}=\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^F}}=\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}}=\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n+1)}{\mathbb{Z}_2^F}},$$

for $n\geq 7$ and $0\leq D\leq 6$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+10}(M\mathrm{SO}(10), \mathbb{Z}_2)$ below degree 6 is shown in Ref. 55's Sec. 6's Figure 27, and the E_2 page is shown in Figure 3. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the E_2 page shown in Appendix A of Ref. 55.

To read the result from the Adams chart Figure 3, we look at the stabilized E_∞ page, one dot indicates a finite group $\mathbb{Z}_p$, a vertical finite line segment connecting n dots indicates a finite group $\mathbb{Z}_{p^n}$. But when $n = \infty$, the infinite line connecting infinite dots indicates an infinite group, an integer $\mathbb{Z}$. Here p is given by the mod p Steenrod algebra $\mathcal{A}_p$ in (E2). Here in Figure 3, $p = 2$, we can read from the Adams chart $\Omega_0^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = \mathbb{Z}$ (an infinite line), $\Omega_1^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = 0$ (nothing), $\Omega_2^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = 0$ (nothing), $\Omega_3^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = 0$ (nothing), $\Omega_4^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = \mathbb{Z}^2$ (two infinite lines), $\Omega_5^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = \mathbb{Z}_2$ (a dot), and $\Omega_6^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}} = \mathbb{Z}_2$ (a dot).

a. *Classification of all invertible anomalies of* $\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}$ *and* $\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^f}$ *fermion theories*

By (E2) and (E3), we obtain the bordism group $\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}}$ shown in Table III, focusing on $D = 5$.

Bordism group		
D	$\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f}}$	generators
5	$\mathbb{Z}_2$	$\mathrm{w}_2(TM)\mathrm{w}_3(TM) = \mathrm{w}_2(V_{\mathrm{SO}(10)})\mathrm{w}_3(V_{\mathrm{SO}(10)})$

TABLE III. Bordism group. The same result holds for $\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^F}}$ and $\Omega_D^{\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F}}$ with $n \geq 7$ and $0 \leq D \leq 6$. See Ref. 55.

By (E5) and (E6), we obtain the cobordism group $\mathrm{TP}_D(\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f})$ shown in Table IV, focusing on $D = 5$.

Cobordism group		
D	$\mathrm{TP}_D(\frac{\mathrm{Spin}\times\mathrm{Spin}(10)}{\mathbb{Z}_2^f})$	generators
5	$\mathbb{Z}_2$	$\mathrm{w}_2(TM)\mathrm{w}_3(TM) = \mathrm{w}_2(V_{\mathrm{SO}(10)})\mathrm{w}_3(V_{\mathrm{SO}(10)})$

TABLE IV. Topological phase classification ($\equiv$ TP) as a cobordism group, following Table III. Same result for $\mathrm{TP}_D(\frac{\mathrm{Spin}\times\mathrm{Spin}(18)}{\mathbb{Z}_2^F})$ and $\mathrm{TP}_D(\frac{\mathrm{Spin}\times\mathrm{Spin}(n)}{\mathbb{Z}_2^F})$ with $n \geq 7$ and $0 \leq D \leq 5$. See also Ref. 55.

[1] S. L. Glashow, Nuclear Physics **22**, 579 (1961).
[2] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).
[3] A. Salam and J. Ward, Physics Letters **13**, 168 (1964).
[4] D. Tong, JHEP **07**, 104 (2017), arXiv:1705.01853 [hep-th].
[5] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438 (1974).
[6] H. Fritzsch and P. Minkowski, Ann. Phys. **93**, 193 (1975).
[7] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956).
[8] H. B. Nielsen and M. Ninomiya, Phys. Lett. **105B**, 219 (1981).
[9] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[10] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).
[11] D. B. Kaplan, Phys. Lett. B **288**, 342 (1992), hep-lat/9206013.
[12] Y. Shamir, Nucl. Phys. B **406**, 90 (1993).
[13] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993).
[14] M. Lüscher, Nucl. Phys. B **549**, 295 (1999), hep-lat/9811032.
[15] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011), arXiv:1106.4752.
[16] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013), arXiv:1106.4772.
[17] J. Wang and X.-G. Wen, (2013), arXiv:1307.7480 [hep-lat].
[18] J. Wang, X.-G. Wen, and E. Witten, Phys. Rev. **X8**, 031048 (2018), arXiv:1705.06728 [cond-mat.str-el].
[19] E. Eichten and J. Preskill, Nucl. Phys. B **268**, 179 (1986).
[20] I. Montvay, Nucl. Phys. B Proc. Suppl. **29**, 159 (1992), hep-lat/9205023.
[21] P. D. V. Swift, Phys. Lett. B **145**, 256 (1984).
[22] J. Smit, Acta Phys. Pol. **B17**, 531 (1986).
[23] X.-G. Wen, Chinese Phys. Lett. **30**, 111101 (2013), arXiv:1305.1045.
[24] M. F. Golterman, D. N. Petcher, and E. Rivas, Nucl. Phys. B **395**, 596 (1993), hep-lat/9206010.
[25] L. Lin, Phys. Lett. B **331**, 449 (1994), hep-lat/9403014.
[26] M. F. L. Golterman and Y. Shamir, Phys. Rev. D **51**, 3026 (1995), hep-lat/9409013.
[27] C. Chen, J. Giedt, and E. Poppitz, JHEP **04**, 131 (2013), arXiv:1211.6947 [hep-lat].
[28] T. Banks and A. Dabholkar, Phys. Rev. D **46**, 4016 (1992), hep-lat/9204017.
[29] D. S. Freed and M. J. Hopkins, (2016), arXiv:1604.06527.
[30] Z. Wan and J. Wang, Ann. Math. Sci. Appl. **4**, 107 (2019), arXiv:1812.11967 [hep-th].
[31] Z. Wan, J. Wang, and Y. Zheng, (2019), arXiv:1912.13504 [hep-th].
[32] Z. Wan and J. Wang, (2019), arXiv:1912.13514 [hep-th].
[33] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energ. Phys. **2015**, 172 (2015), arXiv:1412.5148.
[34] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[35] X.-G. Wen, Phys. Rev. Lett. **90**, 016803 (2003), quant-ph/0205004.
[36] M. Levin and X.-G. Wen, Phys. Rev. B **67**, 245316 (2003), cond-mat/0302460.
[37] B. Yoshida, Ann. Phys. **326**, 15 (2011), arXiv:1007.4601.
[38] H. Bombín, Commun. Math. Phys. **327**, 387 (2014), arXiv:1107.2707.
[39] Z. Nussinov and G. Ortiz, Proceedings of the National Academy of Sciences **106**, 16944 (2009), arXiv:cond-mat/0605316.
[40] Z. Nussinov and G. Ortiz, Ann. Phys. **324**, 977 (2009), arXiv:cond-mat/0702377.
[41] L. Kong and X.-G. Wen, (2014), arXiv:1405.5858.
[42] D. S. Freed, (2014), arXiv:1406.7278.
[43] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, Acta Math. **202**, 195 (2009), arXiv:math/0605249.
[44] R. Thom, Commentarii Mathematici Helvetici **28**, 17 (1954).
[45] X.-G. Wen, Phys. Rev. B **99** (2019), 10.1103/physrevb.99.205139, arXiv:1812.02517.
[46] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009), arXiv:0903.1069.
[47] T. Senthil, Ann. Rev. Condensed Matter Phys. **6**, 299 (2015), arXiv:1405.4015 [cond-mat.str-el].
[48] A. Kapustin, (2014), arXiv:1403.1467 [cond-mat.str-el].
[49] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, J. High Energ. Phys. **2015**, 1 (2015), arXiv:1406.7329.
[50] M. Guo, P. Putrov, and J. Wang, Ann. Phys. **394**, 244 (2018), arXiv:1711.11587.
[51] C. Wang and T. Senthil, Phys. Rev. **B89**, 195124 (2014), [Erratum: Phys. Rev.B91,no.23,239902(2015)], arXiv:1401.1142 [cond-mat.str-el].
[52] Q.-R. Wang and Z.-C. Gu, Phys. Rev. X **8**, 011055 (2018), arXiv:1703.10937.
[53] Q.-R. Wang and Z.-C. Gu, (2018), arXiv:1811.00536 [cond-mat.str-el].
[54] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010), arXiv:1004.3835.
[55] Z. Wan and J. Wang, (2019), arXiv:1910.14668 [hep-th].
[56] A. Schnyder, S. Ryu, A. Furusaki, and A. Ludwig, Phys. Rev. B **78**, 195125 (2008), arXiv:0803.2786 [cond-mat.mes-hall].
[57] A. Kitaev, in *Advances in Theoretical Physics: Landau Memorial Conference, Chernogolovka, Russia, 2008*, Vol. AIP Conf. Proc. No. 1134, edited by V. Lebedev and M. Feigelman (AIP, Melville, NY, 2009) p. 22, arXiv:0901.2686.
[58] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, AIP Conference Proceedings (2009), 10.1063/1.3149481, arXiv:0905.2029 [cond-mat.mes-hall].

[59] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **90**, 115141 (2014), arXiv:1201.2648.
[60] D. Gaiotto and A. Kapustin, Int. J. Mod. Phys. A **31**, 1645044 (2016), arXiv:1505.05856.
[61] A. Kapustin and R. Thorngren, J. High Energ. Phys. **2017**, 80 (2017), arXiv:1701.08264.
[62] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo, H. Lin, P. Gao, and S.-T. Yau, Progress of Theoretical and Experimental Physics **2018**, 053A01 (2018), arXiv:1801.05416.
[63] M. Guo, K. Ohmori, P. Putrov, Z. Wan, and J. Wang, Commun. Math. Phys. **376**, 1073 (2020), arXiv:1812.11959 [hep-th].
[64] T. Lan, C. Zhu, and X.-G. Wen, (2018), arXiv:1809.01112.
[65] Z. Wan, J. Wang, and Y. Zheng, Annals Phys. **414**, 168074 (2020), arXiv:1812.11968 [hep-th].
[66] I. Garca-Etxebarria and M. Montero, JHEP **08**, 003 (2019), arXiv:1808.00009 [hep-th].
[67] J. Wang, X.-G. Wen, and E. Witten, J. Math. Phys. **60**, 052301 (2019), arXiv:1810.00844 [hep-th].
[68] J. Milnor and J. Stasheff, *Characteristic Classes, by Milnor and Stasheff*, Annals of Mathematics Studies, No. 76 (Princeton University Press, 1974).
[69] C. Arf, J. Reine Angew. Math., **183**, 148 (1941).
[70] A. Y. Kitaev, Phys.-Usp. **44**, 131 (2001), cond-mat/0010440.
[71] M. DeMarco and X.-G. Wen, (2017), arXiv:1706.04648.
[72] Y. BenTov and A. Zee, Phys. Rev. D **93**, 065036 (2016), arXiv:1505.04312.
[73] Y.-Z. You, Y. BenTov, and C. Xu, (2014), arXiv:1402.4151.
[74] Y.-Z. You and C. Xu, Phys. Rev. B **91**, 125147 (2015), arXiv:1412.4784.
[75] Y. Kikukawa, PTEP **2019**, 073B02 (2019), arXiv:1710.11101 [hep-lat].
[76] Y. Kikukawa, PTEP **2019**, 113B03 (2019), arXiv:1710.11618 [hep-lat].
[77] J. Wang and X.-G. Wen, Phys. Rev. **D99**, 111501 (2018), arXiv:1807.05998 [hep-lat].
[78] G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).
[79] I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988).
[80] E. Dagotto, E. Fradkin, and A. Moreo, Phys. Rev. B **38**, 2926 (1988).
[81] R. C. Ball, Phys. Rev. Lett. **95**, 176407 (2005), cond-mat/0409485.
[82] F. Wilczek and A. Zee, Phys. Rev. **D25**, 553 (1982).
[83] X.-G. Wen, Phys. Rev. Lett. **88**, 011602 (2001), hep-th/01090120.
[84] X.-G. Wen, Phys. Rev. D **68**, 065003 (2003), hep-th/0302201.
[85] M. Levin and X.-G. Wen, Phys. Rev. B **73**, 035122 (2006), hep-th/0507118.
[86] J. A. Wheeler, "Information, physics, quantum: The search for links," in *Feynman and Computation* (CRC Press, 1989) Chap. Information, Physics, Quantum: The Search for Links, pp. 309–336.
[87] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005), cond-mat/0404617.
[88] X.-G. Wen, ISRN Condensed Matter Physics **2013**, 1 (2013), arXiv:1210.1281.
[89] J. Davighi, B. Gripaios, and N. Lohitsiri, (2019), arXiv:1910.11277 [hep-th].
[90] J. Wang, (2020), arXiv:2006.16996 [hep-th].
[91] G. 't Hooft, NATO Adv. Study Inst. Ser. B Phys. **59**, 135 (1980).
[92] X.-G. Wen, Phys. Rev. D **88**, 045013 (2013), arXiv:1303.1803.
[93] C. Callan and J. Harvey, Nucl. Phys. B **250**, 427 (1985).
[94] D. S. Freed, J. Differential Geom. **80**, 45 (2008), hep-th/0607134.
[95] E. Witten, Commun.Math. Phys. **100**, 197 (1985).
[96] X. Dai and D. S. Freed, J. Math. Phys. **35**, 5155 (1994), hep-th/9405012.
[97] L. Kong, X.-G. Wen, and H. Zheng, (2015), arXiv:1502.01690.
[98] D. Fiorenza and A. Valentino, Commun. Math. Phys. **338**, 1043 (2015), arXiv:1409.5723.
[99] S. Monnier, Commun. Math. Phys. **338**, 1327 (2015), arXiv:1410.7442.
[100] W. Ji and X.-G. Wen, Phys. Rev. Research **1**, 033054 (2019), arXiv:1905.13279.
[101] J. Wang, L. H. Santos, and X.-G. Wen, Phys. Rev. **B91**, 195134 (2015), arXiv:1403.5256 [cond-mat.str-el].
[102] A. Kapustin and R. Thorngren, (2014), arXiv:1404.3230 [hep-th].
[103] E. Witten, Phys. Lett. B **117**, 324 (1982).
[104] X. G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).
[105] A. Kapustin, (2014), arXiv:1404.6659.
[106] J. Wang, Z.-C. Gu, and X.-G. Wen, Phys. Rev. Lett. **114**, 031601 (2015), arXiv:1405.7689.
[107] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013), arXiv:1209.3058.
[108] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[109] J. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).
[110] J. Wang and X.-G. Wen, Phys. Rev. **B91**, 125124 (2015), arXiv:1212.4863 [cond-mat.str-el].
[111] A. Kapustin, Phys. Rev. **B89**, 125307 (2014), arXiv:1306.4254 [cond-mat.str-el].
[112] A. Kitaev and L. Kong, Communications in Mathematical Physics **313**, 351 (2012).
[113] T. Lan, J. C. Wang, and X.-G. Wen, Phys. Rev. Lett. **114**, 076402 (2015), arXiv:1408.6514 [cond-mat.str-el].
[114] N. Seiberg, (Talk at Strings 2019 "Thoughts About Quantum Field Theory") (2019).