Non-Abelian String and Particle Braiding in Topological Order: Modular SL(3,$\mathbb{Z}$) Representation and 3+1D Twisted Gauge Theory

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String and particle braiding statistics are examined in a class of topological orders described by discrete gauge theories with a gauge group G and a 4-cocycle twist ω_4 of G's cohomology group $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ in 3 dimensional space and 1 dimensional time (3+1D). We establish the topological spin and the spin-statistics relation for the closed strings, and their multi-string braiding statistics. The 3+1D twisted gauge theory can be characterized by a representation of a modular transformation group SL$(3,\mathbb{Z})$. We express the SL$(3,\mathbb{Z})$ generators S^{xyz} and T^{xy} in terms of the gauge group G and the 4-cocycle ω_4. As we compactify one of the spatial directions z into a compact circle with a gauge flux b inserted, we can use the generators S^{xy} and T^{xy} of an SL$(2,\mathbb{Z})$ subgroup to study the dimensional reduction of the 3D topological order $\mathcal{C}^{\text{3D}}$ to a direct sum of degenerate states of 2D topological orders $\mathcal{C}_b^{\text{2D}}$ in different flux b sectors: $\mathcal{C}^{\text{3D}} = \oplus_b \mathcal{C}_b^{\text{2D}}$. The 2D topological orders $\mathcal{C}_b^{\text{2D}}$ are described by 2D gauge theories of the group G twisted by the 3-cocycles $\omega_{3(b)}$, dimensionally reduced from the 4-cocycle ω_4. We show that the SL$(2,\mathbb{Z})$ generators, S^{xy} and T^{xy}, fully encode a particular type of three-string braiding statistics with a pattern that is the connected sum of two Hopf links. With certain 4-cocycle twists, we discover that, by threading a third string through two-string unlink into three-string Hopf-link configuration, Abelian two-string braiding statistics is promoted to non-Abelian three-string braiding statistics.

CONTENTS

I. INTRODUCTION

In the 1986 Dirac Memorial Lectures, Feynman explained the braiding statistics of fermions by demonstrating the plate trick and the belt trick.[1] Feynman showed that the wavefunction of a quantum system obtains a mysterious (-1) sign by exchanging two fermions, which is associated with the fact that an extra 2π twist or rotation is required to go back to the original state. However, it is known that there is richer physics in deconfined topological phases of 2+1D and 3+1D spacetime.[2] (Here $d+1$D is d-dimensional space and 1-dimensional time, while dD is d-dimensional space.) In 2+1D, there are "anyons" with exotic braiding statistics for point particles.[3] In 3+1D, Feynman only had to consider bosonic or fermionic statistics for point particles, without worrying about anyonic statistics. Nonetheless, there are string-like excitations, whose braiding process in 3+1D can enrich the statistics of deconfined topological phases. In this work, we aim to systematically address the string and particle braiding statistics in deconfined gapped phases of 3+1D *topological orders*. Namely, we aim to determine what statistical phase the wavefunction of the whole system gains under the string and particle braiding process.

Since the discovery of 2+1D topological orders[4–6](see Ref.7 for an overview), we have now gained quite systematic ways to classify and characterize them, by using the induced representations of the mapping class group of the $\mathbb{T}^2$ torus (the modular group $\mathrm{SL}(2,\mathbb{Z})$ and the gauge/Berry phase structure of ground states[6,8,9]) and the topology-dependent ground state degeneracy,[6,10,11] using the unitary fusion categories,[12–19] and using simple current algebra,[20–23] a pattern of zeros,[24–29] and field theories.[30–34] Our better understanding of topologically ordered states also holds the promises of applying their rich quantum phenomena, including fractional statistics[3] and non-Abelian anyons, to topological quantum computation.[35]

However, our understanding of 3+1D topological orders is in its infancy and far from systematic. This motivates our work attempting to address:

Q1: "*How do we* (*at least partially*) *classify and characterize 3D topological orders*?"

By *classification*, we mean counting the number of distinct phases of topological orders and giving them a proper label. By *characterization*, we mean to describe their properties in terms of physical observables. Here our approach to studying dD topological orders is to simply generalize the above 2D approach, to use the ground state degeneracy (GSD) on d-torus $\mathbb{T}^d=(S^1)^d$, and the associated representations of the mapping class group (MCG) of $\mathbb{T}^d$ (recently proposed in Refs.19 and 38),

$$\mathrm{MCG}(\mathbb{T}^d)=\mathrm{SL}(d,\mathbb{Z}). \tag{1}$$

(Refer to Appendix A 4 and Reference cited therein for a brief review of the computation of 2D topological orders.) For 3D, the mapping class group $\mathrm{SL}(3,\mathbb{Z})$ is generated by the modular transformation $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$:[37]

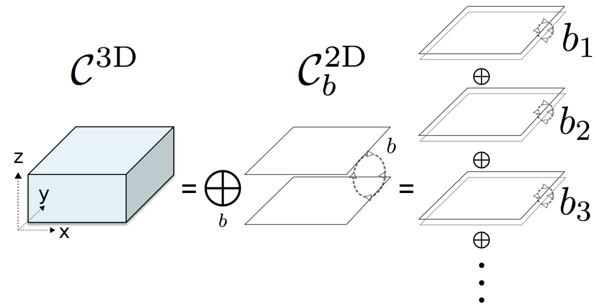


FIG. 1. The 3D topological order $\mathcal{C}^{3\mathrm{D}}$ can be regarded as the direct sum of 2D topological orders $\mathcal{C}_b^{2\mathrm{D}}$ in different sectors b, as $\mathcal{C}^{3\mathrm{D}}=\oplus_b\mathcal{C}_b^{2\mathrm{D}}$, when we compactify a spatial direction z into a circle. **This idea is general and applicable to $\mathcal{C}^{3\mathrm{D}}$ without a gauge theory description.** However, when $\mathcal{C}^{3\mathrm{D}}$ allows a gauge group G description, the b stands for a group element (or the conjugacy class for the non-Abelian group) of G. Thus b acts as a gauge flux along the dashed arrow - - -▷ in the compact direction z. Thus, $\mathcal{C}^{3\mathrm{D}}$ becomes the direct sum of different $\mathcal{C}_b^{2\mathrm{D}}$ under distinct gauge fluxes b.

$$\hat{\mathsf{S}}^{xyz}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\qquad \hat{\mathsf{T}}^{xy}=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{2}$$

What are examples of 3D topological orders? One class of them is described by a discrete gauge theory with a finite gauge group G. Another class is described by the *twisted gauge theory*,[36] a gauge theory G with a 4-cocycle twist $\omega_4\in\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of G's fourth cohomology group. But the twisted gauge theory characterization of 3D topological orders is not one-to-one: different pairs (G,ω_4) can describe the same 3D topological order. In this work, we will use $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$ of $\mathrm{SL}(3,\mathbb{Z})$ to characterize the topological *twisted discrete gauge theory* with finite gauge group G, which has topology-dependent ground state degeneracy. The twisted gauge theories describe a large class of 3D gapped quantum liquids in condensed matter. Although we will study the $\mathrm{SL}(3,\mathbb{Z})$ modular data of the ground state sectors of gapped phases, these data can capture the gapped excitations such as particles and strings. (This strategy is widely-used especially in 2D.) There are **two main issues** that we will focus on addressing. The first is the **dimensional reduction** from 3D to 2D of $\mathrm{SL}(3,\mathbb{Z})$ modular transformation and cocycles to study 3D topological order. The second is the **non-Abelian three-string braiding statistics** from a twisted discrete gauge theory of an Abelian gauge group.

($\star$1) **Dimensional Reduction from 3D to 2D: for $\mathrm{SL}(3,\mathbb{Z})$ modular S, T matrices and cocycles** - For the first issue, our general philosophy is as follows:

"Since 3D topological orders are foreign and unfamiliar to us, we will *dimensionally reduce 3D topological orders to several sectors of 2D topological orders* in the *Hilbert space of ground states* (*not in the real space*, see Fig.1). Then we will be able to *borrow the more familiar 2D topological orders to understand 3D topological orders*."

We will compute the matrices S^{xyz} and T^{xy} that generate the $\mathrm{SL}(3,\mathbb{Z})$ representation in the quasi-(particle or string)-excitations basis of 3+1D topological order. We find an explicit expression of S^{xyz} and T^{xy}, in terms of the gauge group G and the 4-cocycle ω_4, for both Abelian and non-Abelian gauge groups. (A calculation using a different novel approach, the universal wavefunction overlap for the normal untwisted gauge theory, is studied in Ref.39.) We note that $\mathrm{SL}(3,\mathbb{Z})$ contains a subgroup $\mathrm{SL}(2,\mathbb{Z})$, which is generated by $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$, where

$$\hat{\mathsf{S}}^{xy} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

In the most generic cases of topological orders (potentially *without a gauge group description*), the matrices S^{xy} and T^{xy} can still be block diagonalized as the sum of several sectors in the quasi-excitations basis, each sector carrying an index of b,

$$\mathsf{S}^{xy} = \oplus_b \mathsf{S}^{xy}_b, \qquad \mathsf{T}^{xy} = \oplus_b \mathsf{T}^{xy}_b, \tag{4}$$

The pair $(\mathsf{S}^{xy}_b, \mathsf{T}^{xy}_b)$, generating an $\mathrm{SL}(2,\mathbb{Z})$ representation, describes a 2D topological order $\mathcal{C}^{2\mathrm{D}}_b$. This leads to a dimension reduction of the 3D topological order $\mathcal{C}^{3\mathrm{D}}$:

$$\mathcal{C}^{3\mathrm{D}} = \oplus_b \mathcal{C}^{2\mathrm{D}}_b. \tag{5}$$

In the more specific case, when the topological order allows a gauge group G description which we focus on here, we find that the b stands for a gauge flux for group G (that is, b is a group element for an Abelian G, while b is a conjugacy class for a non-Abelian G).

The physical picture of the above dimensional reduction is the following (see Fig.1): If we compactify one of the 3D spatial directions (say the z direction) into a small circle, the 3D topological order $\mathcal{C}^{3\mathrm{D}}$ can be viewed as a direct sum of 2D topological orders $\mathcal{C}^{2\mathrm{D}}_b$ with (accidental) degenerate ground states at the lowest energy.

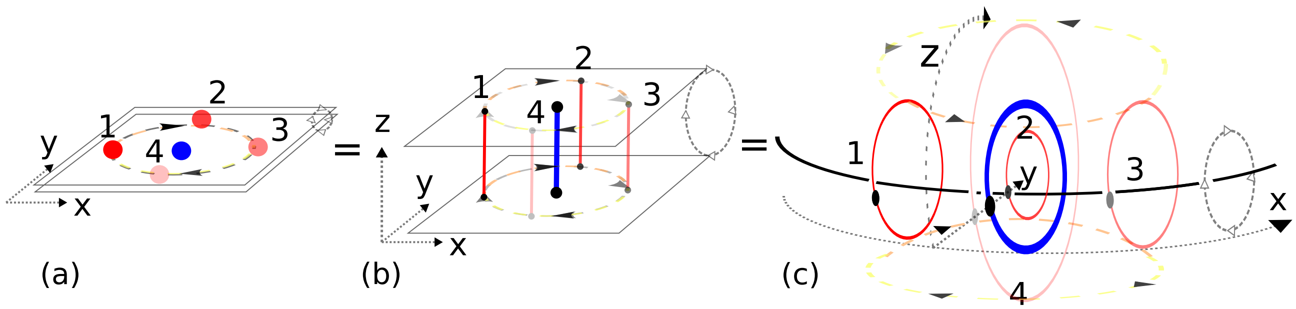


FIG. 2. Mutual braiding statistics following the path $1 \to 2 \to 3 \to 4$ along time evolution (see Sec.III C 2): (a) From a 2D viewpoint of dimensional reduced $\mathcal{C}^{2\mathrm{D}}_b$, the 2π braiding of two particles is shown. (b) The compact z direction extends two particles to two closed (red, blue) strings. (c) An equivalent 3D view, the b flux (along the arrow - - -▷) is regarded as the monodromy caused by a third (black) string. We identify the coordinates x, y and a compact z to see that the full-braiding process is *one (red) string going inside to the loop of another (blue) string, and then going back from the outside.* For Abelian topological orders, the mutual braiding process between two excitations (A and B) in Fig.2(a) yields a statistical Abelian phase $e^{\mathrm{i}\theta_{(\mathrm{A})(\mathrm{B})}} \propto \mathsf{S}^{xy}_{(\mathrm{A})(\mathrm{B})}$ proportional to the 2D's S^{xy} matrix. The dimensional-extended equivalent picture Fig.2(c) implies that the loop-braiding yields a phase $e^{\mathrm{i}\theta_{(\mathrm{A})(\mathrm{B}),b}} \propto \mathsf{S}^{xy}_{b\,(\mathrm{A})(\mathrm{B})}$ of Eq.(34) (up to a choice of canonical basis), where b is the flux of the black string. We clarify that in both (b) and (c) our strings may carry both flux and charge. If a string carries only a pure charge, then it is effectively a point particle in 3D. If a string carries a pure flux, then it is effectively a loop of a pure string in 3D. If a string carries both charge and flux (as a dyon in 2D), then it is a loop with string fluxes attached with some charged particles in 3D. Therefore our Fig.2(c)'s string-string braiding actually represents several braiding processes: the particle-particle, particle-loop and loop-loop braidings, all processes are threaded with a background (black) string.

In this work, we focus on a generic finite Abelian gauge group $G = \prod_i Z_{N_i}$ (isomorphic to products of cyclic groups) with generic cocycle twists from the group cohomology.[36] We examine the 3+1D twisted gauge theory twisted by 4-cocycle $\omega_4 \in \mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$, and reveal that it is a direct sum of 2+1D twisted gauge theories twisted by a dimensionally-reduced 3-cocycle $\omega_{3(b)} \in \mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ of G's third cohomology group, namely

$$\mathcal{C}^{3\mathrm{D}}_{G,\omega_4} = \oplus_b \mathcal{C}^{2\mathrm{D}}_{G_b,\omega_{3(b)}}. \tag{6}$$

Surprisingly, even for an Abelian group G, we find that such a *twisted Abelian gauge theory* can be dual to a twisted or untwisted *non-Abelian gauge theory*. We study this fact for 3D as an extension of the 2D examples in Ref.42. By this equivalence, we are equipped with (both untwisted and twisted) non-Abelian gauge theory to study its non-Abelian braiding statistics.

($\star$2) **Non-Abelian three-string braiding statistics** - We are familiar with the 2D braiding statistics: there is only *particle-particle braiding*, which yields bosonic, fermionic or anyonic statistics by braiding a particle around another particle.[3] We find that the 3D topological order introduces both particle-like and string-like excitations. We aim to address the question:

Q2: "*How do we characterize the braiding statistics of strings and particles in 3+1D topological orders*?"

The possible braiding statistics in 3D learned in the past literature are as follows:

(i) *Particle-particle braiding*, which can only be bosonic or fermionic due to the absence of nontrivial braid group in 3D for point particles.

(ii) *Particle-string braiding*, which is the Aharonov-Bohm effect of $\mathbb{Z}_N$ gauge theory, where a particle such as a $\mathbb{Z}_N$ charge braiding around a string (or a vortex line) as $\mathbb{Z}_N$ flux, obtaining a $e^{\mathrm{i}\frac{2\pi}{N}}$ phase of statistics.[3,43]

(iii) *String-string braiding*, where a closed string (a red loop), shown in Fig.2(c) excluding the background black string, wraps around a blue loop. The related idea known as loop-loop braiding forming the *loop braid group* has been proposed mathematically.[44] (See also some earlier studies in Ref.45 and 46.)

However, we will address some extra *new braiding statistics among three closed strings*:

(iv) *Three-string braiding*, shown in Fig.2(c), where a closed string (a red loop) wraps around another closed string (a blue loop) but the two loops are both threaded by a third loop (the black string). This braiding configuration is discovered recently by Ref.40, also a related work in Ref.41 for a twisted Abelian gauge theory.

The new ingredient of our work on braiding statistics can be summarized as follows: We consider the string and particle braiding of general twisted gauge theories with the most generic finite Abelian gauge group $G=\prod_u Z_{N_u}$, labeled by the data (G,ω_4). We provide a 3D to 2D dimensional reduction approach to realize the three-string braiding statistics of Fig.2. We first show that the $\mathrm{SL}(2,\mathbb{Z})$ representations $(\mathsf{S}_b^{xy},\mathsf{T}_b^{xy})$ fully encode this particular type of Abelian three-closed-string statistics shown in Fig.2. We further find that, for a twisted gauge theory with an Abelian $(Z_N)^4$ group, certain 4-cocycles (called as Type IV 4-cocycles) will make the twisted theory to be a non-Abelian theory. More precisely, **while the two-string braiding statistics of unlink is Abelian, the three-string braiding statistics of Hopf links, obtained from threading the two strings with the third string, will become non-Abelian.** We also demonstrate that (S_b^{xy} encodes this three-string braiding statistics.

Our article is organized as follows. In Sec.II, we address the third question: **Q3**: "*How to formulate or construct certain 3+1D topological orders on the lattice*?" We outline a lattice formulation of twisted gauge theories in terms of 3D twisted quantum double models, which generalize the Kitaev's 2D toric code and quantum double models. Our model is the lattice Hamiltonian formulation of Dijkgraaf-Witten theory,[36] and we provide the *spatial lattice* as well as the *spacetime lattice path integral* pictures. In Sec.III, we answer **Q4**: "*What are the generic expressions of SL(3,$\mathbb{Z}$) modular data*?" We compute the modular $\mathrm{SL}(3,\mathbb{Z})$ representations of S and T matrices, using both the spacetime path integral approach and the Representation Theory approach. In Secs.III C and IV, we address: **Q5**: "*What is the physical interpretation of* SL(3,$\mathbb{Z}$) *modular data in 3D*?" We use the modular $\mathrm{SL}(3,\mathbb{Z})$ data to characterize the braiding-statistics of particles and strings. In Sec.V, we discuss the link and knot patterns of string-braiding systematically, and end with a conclusion. In addition to the main text, we organize the following information in the Appendix: (i) group cohomology and cocycles; (ii) projective representation; (iii) some examples of classification of topological orders; and (iv) direct calculations of S and T using cocycle path integrals.

(Note: We adopt the name *strings* for the vision of incorporating the excitations from both the closed strings (loops) and open strings. Such excitations can have fusion or braiding process. In this work, however, we focus only on the closed string case. Our notation for a finite cyclic group is either Z_N or $\mathbb{Z}_N$, though they are equivalent mathematically. We use Z_N to denote the gauge group G, the discrete gauge Z_N flux, or the Z_N variables, but $\mathbb{Z}_N$ to denote only the classes of group cohomology or topological order classification. We denote $\gcd(N_i,N_j)\equiv N_{ij}$, $\gcd(N_i,N_j,N_k)\equiv N_{ijk}$, $\gcd(N_i,N_j,N_k,N_l)\equiv N_{ijkl}$, where gcd stands for the greatest common divisor. We also have $|G|$ as the order of the group, and $\mathbb{R}/\mathbb{Z}=\mathrm{U}(1)$. We may use subindex n for ω_n to indicate n-cocycle. In principle, we will use **types** to count the number of cocycles in cohomology groups. But we will use **classes** to count the number of distinct phases in topological orders. Normally **the types overcount the classes**. We use the hat symbol $\hat{\mathsf{S}}$ and $\hat{\mathsf{T}}$ for modular matrices acting on the real space in the x,y,z directions, so $\hat{\mathsf{S}}^{xyz}\cdot(x,y,z)=(z,x,y)$ and $\hat{\mathsf{T}}^{xy}\cdot(x,y,z)=(x+y,y,z)$; while we use the symbols S and T to denote modular matrices in the quasi-excitation basis.)

II. TWISTED GAUGE THEORY AND COCYCLES OF GROUP COHOMOLOGY

In this section, we aim to address the question:

Q3: "*How to formulate or construct certain 3+1D topological orders on the lattice*?"

We will consider 3+1D twisted discrete gauge theories. Our motivation to study the discrete gauge theory is that it is topological and exhibits Aharonov-Bohm phenomena (see Ref.3 and 43). One approach to formulating a discrete gauge theory is the lattice gauge theory.[47] A famous example in both high energy and condensed matter communities is the Z_2 discrete gauge theory in 2+1D (also called the Z_2 toric code, Z_2 spin liquids, Z_2 topological order[49]). Kitaev's toric code and quantum double model[48] provides a simple Hamiltonian,

$$H = -\sum_v A_v - \sum_p B_p, \tag{7}$$

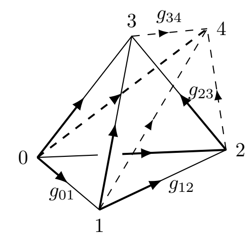
where a space lattice formalism is used, and A_v is the vertex operator acting on the vertex v, B_p is the plaquette (or face) term to ensure the zero flux condition on each plaquette. Both A_v and B_p consist of only Pauli spin operators for the Z_2 model. Such ground states of the Hamiltonian are found to be Z_2 gauge theory with $|G|^2 = 4$-fold topological degeneracy on the $\mathbb{T}^2$ torus. Its generalization to a *twisted* Z_2 *gauge theory* is the Z_2 double-semions model, captured by the framework of the Levin-Wen *string-net model*.[12,49]

A. Dijkgraaf-Witten topological gauge theory

For a more generic twisted gauge theory, there is indeed another way using the spacetime lattice formalism to construct them by the Dijkgraaf-Witten topological gauge theory.[36] One can formulate the path integral $\mathbf{Z}$ (or partition function) of a $(d+1)$D gauge theory (dD space, 1D time) of a gauge group G as,

$$\mathbf{Z} = \sum_\gamma e^{\mathrm{i}S[\gamma]} = \sum_\gamma e^{\mathrm{i}2\pi\langle \omega_{d+1}, \gamma(\mathcal{M}_{\text{tri}})\rangle (\text{mod} 2\pi)}$$
$$= \frac{|G|}{|G|^{N_v}} \frac{1}{|G|} \sum_{\{g_{ab}\}} \prod_i (\omega_{d+1}{}^{\epsilon_i}(\{g_{ab}\})) \,|_{v_{c,d}\in T_i} \tag{8}$$

where we sum over all mappings $\gamma : \mathcal{M} \to BG$, from the spacetime manifold $\mathcal{M}$ to BG, the classifying space of G. In the second equality, we triangulate $\mathcal{M}$ to $\mathcal{M}_{\text{tri}}$ with the edge $[v_a v_b]$ connecting the vertex v_a to the vertex v_b. The action $\langle \omega_{d+1}, \gamma(\mathcal{M}_{\text{tri}})\rangle$ evaluates the cocycles ω_{d+1} on the spacetime $(d+1)$-complex $\mathcal{M}_{\text{tri}}$. By the relation between the topological cohomology class of BG and the cohomology group of G: $H^{d+2}(BG, \mathbb{Z}) = \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$,[36,51] we can simply regard ω_{d+1} as the $d+1$-cocycles of the cohomology group $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ (see more details in Appendix A). The group elements g_{ab} are assigned at the edge $[v_a v_b]$. The $|G|/|G|^{N_v}$ factor is to mod out the redundant gauge equivalence configuration, with the number of vertices N_v. Another extra $|G|^{-1}$ factor mods out the group elements evolving in the time dimension. The cocycle ω_{d+1} is evaluated on all the $d+1$-simplex T_i (namely a $d+2$-cell) triangulation of the spacetime complex. In the case of our 3+1D, we have the 4-cocycle ω_4 evaluated at the 4-simplex (or 5-cell) as



$$= \omega_4{}^{\epsilon}(g_{01}, g_{12}, g_{23}, g_{34}). \tag{9}$$

Here the cocycle ω_4 satisfies cocycle condition: $\delta\omega_4 = 1$, which ensures the path integral $\mathbf{Z}$ on the 4-sphere S^4 (the surface of the 5-ball) will be trivial as 1. This is a feature of topological gauge theory. The ϵ is the $\pm$ sign of the orientation of the 4-simplex, which is determined by the sign of the volume determinant of the 4-simplex evaluated by $\epsilon = \text{sgn}(\det(\vec{01}, \vec{02}, \vec{03}, \vec{04}))$.

We utilize Eq.(8) to calculate the path integral amplitude from an initial state configuration $|\Psi_{in}\rangle$ on the spatial manifold evolving along the time direction to the final state $|\Psi_{out}\rangle$, see Fig.3. In general, the calcuation can be done for the mapping class group MCG on any spatial manifold $\mathcal{M}_{space}$ as MCG($\mathcal{M}_{space}$). Here we focus on $\mathcal{M}_{space} = \mathbb{T}^3$ and MCG($\mathbb{T}^3$) = SL(3, $\mathbb{Z}$), as the modular transformation. We first note that $|\Psi_{in}\rangle = \hat{\mathsf{O}}|\Psi_{\text{B}}\rangle$, such a generic SL(3, $\mathbb{Z}$) transformation $\hat{\mathsf{O}}$ under SL(3, $\mathbb{Z}$) representation can be absolutely generated by $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$ of Eq.(2),[37] thus $\hat{\mathsf{O}} = \hat{\mathsf{O}}(\hat{\mathsf{S}}^{xyz}, \hat{\mathsf{T}}^{xy})$ as a function of $\hat{\mathsf{S}}^{xyz}, \hat{\mathsf{T}}^{xy}$. The calculation of the modular SL(3, $\mathbb{Z}$) transformation from $|\Psi_{in}\rangle$ to $|\Psi_{out}\rangle = |\Psi_{\text{A}}\rangle$ by filling the 4-cocycles ω_4 into the spacetime-complex-triangulation renders the amplitude of the matrix element $\mathsf{O}_{(\text{A})(\text{B})}$:

$$\mathsf{O}(\mathsf{S}^{xyz}, \mathsf{T}^{xy})_{(\text{A})(\text{B})} = \langle \Psi_{\text{A}} | \hat{\mathsf{O}}(\hat{\mathsf{S}}^{xyz}, \hat{\mathsf{T}}^{xy}) | \Psi_{\text{B}} \rangle, \tag{10}$$

both space and time are discretely triangulated, so this is a spacetime lattice formalism.

B. Canonical basis and the generalized twisted quantum double model $D^\omega(G)$ to 3D triple basis

So far we have answered the question **Q3** using the *spacetime-lattice path integral*. Our next goal is to *construct its Hamiltonian on the space lattice*, and to find a good basis representing its quasi-excitations, such that we can efficiently read the information of $\mathsf{O}(\mathsf{S}^{xyz}, \mathsf{T}^{xy})$ in this *canonical basis*. We will outline the *twisted quantum double model* generalized to 3D as the exactly soluble model in the next subsection, where the canonical basis can diagonalize its Hamiltonian.

Canonical basis - For a gauge theory with the gauge group G, one may naively think that a good basis for the amplitude Eq.(10) is the group elements $|g_x, g_y, g_z\rangle$, with $g_i \in G$ as the flux labeling three directions of $\mathbb{T}^3$. However, this flux-only label $|g_x, g_y\rangle$ is known to be improper

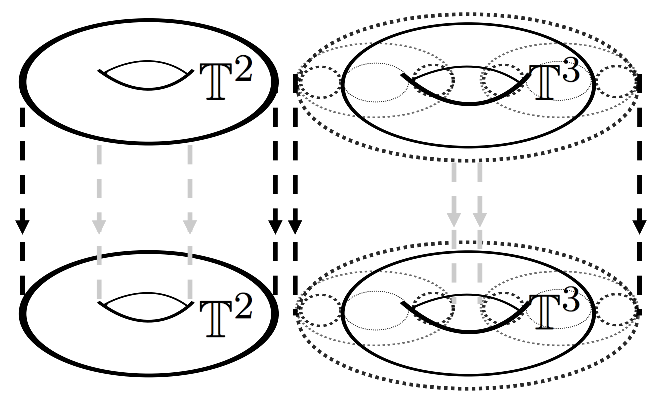
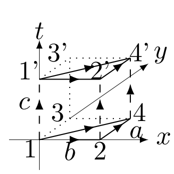


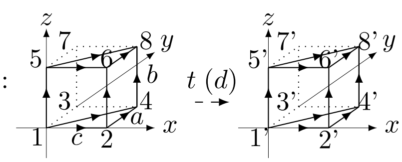
FIG. 3. The illustration for $\mathsf{O}_{(\mathrm{A})(\mathrm{B})} = \langle \Psi_{\mathrm{A}} | \hat{\mathsf{O}} | \Psi_{\mathrm{B}} \rangle$. Evolution from an initial state configuration $|\Psi_{in}\rangle$ on the spatial manifold (from the top) along the time direction (the dashed line - - -) to the final state $|\Psi_{out}\rangle$ (at the bottom). For the spatial $\mathbb{T}^d$ torus, the mapping class group $\mathrm{MCG}(\mathbb{T}^d)$ is the modular $\mathrm{SL}(d, \mathbb{Z})$ transformation. We show schematically the time evolution on the spatial $\mathbb{T}^2$, and $\mathbb{T}^3$. The $\mathbb{T}^3$ is shown as a $\mathbb{T}^2$ attached an S^1 circle at each point.

on the $\mathbb{T}^2$ torus already - the canonical basis labeling particles in 2D is $|\alpha, a\rangle$, requiring both the charge α (as the representation) and the flux a (the group element or the conjugacy class of G). We propose that the proper way to label excitations for a 3+1D twisted discrete gauge theory for any finite group G in the canonical basis requires one charge α and two fluxes, a and b:

$$|\alpha, a, b\rangle = \frac{1}{\sqrt{|G|}} \sum_{\substack{g_y \in C^a, g_z \in C^b \\ g_x \in Z_{g_y} \cap Z_{g_z}}} \mathrm{Tr}[\widetilde{\rho}_\alpha^{g_y, g_z}(g_x)] |g_x, g_y, g_z\rangle. \tag{11}$$

which is the finite group discrete Fourier transformation on $|g_x, g_y, g_z\rangle$. This is a generalization of the 2D result in Ref.42 and a very recent 3D Abelian case in Ref.41. Here α is the charge of the representation (Rep) label, which is the $\mathsf{C}_{a,b}^{(2)}$ Rep of the centralizers Z_a, Z_b of the conjugacy classes C^a, C^b. (For an Abelian G, the conjugacy class is the group element, and the centralizer is the full G.) $\mathsf{C}_{a,b}^{(2)}$ Rep means an inequivalent unitary irreducible projective representation of G. $\widetilde{\rho}_\alpha^{a,b}(c)$ labels this inequivalent unitary irreducible projective $\mathsf{C}_{a,b}^{(2)}$ Rep of G. $\mathsf{C}_{a,b}^{(2)}$ is an induced 2-cocycle, dimensionally-reduced from the 4-cocycle ω_4. We illustrate $\mathsf{C}_{a,b}^{(2)}$ in terms of geometric pictures in Eqs. (12) and (13).

$\mathsf{C}_a(b, c):$  , (12)

$\mathsf{C}_{a,b}^{(2)}(c, d):$  . (13)

The reduced 2-cocycle $\mathsf{C}_a(b, c)$ is from the 3-cocycle ω_3 in Eq.(12), which triangulates a half of $\mathbb{T}^2$ and with a time interval I. The reduced 2-cocycle $\mathsf{C}_a(b, c)$ is from 4-cocycle ω_4 in Eq.(13), which triangulates a half of $\mathbb{T}^3$ and with a time interval I. The dashed arrow $\dashrightarrow$ stands for the time t evolution.

The $\widetilde{\rho}_\alpha^{g_y, g_z}(g_x)$ values are determined by the $\mathsf{C}_{a,b}^{(2)}$ projective representation formula:

$$\widetilde{\rho}_\alpha^{a,b}(c) \widetilde{\rho}_\alpha^{a,b}(d) = \mathsf{C}_{a,b}^{(2)}(c, d) \widetilde{\rho}_\alpha^{a,b}(cd). \tag{14}$$

The trace term $\mathrm{Tr}[\widetilde{\rho}_\alpha^{g_y, g_z}(g_x)]$ is called *the character* in the math literature. One can view the charge α_x along x direction, the flux a, b along the y, z. Other details and the calculations of $\mathsf{C}_{a,b}^{(2)}$ Rep with many examples can be found in Appendix A.

We first recall that, in 2D, a reduced 2-cocycle $\mathsf{C}_a(b, c)$ comes from a slant product $i_a \omega(b, c)$ of 3-cocycles,[42] which is geometrically equivalent to filling three 3-cocycles in a triangular prism of Eq.(12). This is known to present the **projective representation** $\widetilde{\rho}_\alpha^a(b) \widetilde{\rho}_\alpha^a(c) = \mathsf{C}_a(b, c) \widetilde{\rho}_\alpha^a(bc)$, because the induced 2-cocycle belongs to the second cohomology group $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$.[42,52–54] (See its explicit triangulation and a novel use of the projective representation in Sec VI.B. of Ref.55.)

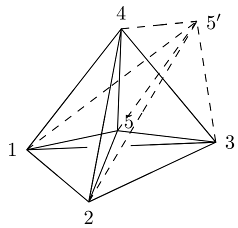
Similarly, in 3D, a reduced 2-cocycle $\mathsf{C}_a(b, c)$ comes from doing *twice* of the slant products of 4-cocycles forming the geometry of Eq.(13), and renders

$$\mathsf{C}_{a,b}^{(2)} = i_b(\mathsf{C}_a(c, d)) = i_b(i_a \omega(c, d)), \tag{15}$$

presenting the $\mathsf{C}_{a,b}^{(2)}$-projective representation in Eq.(14), where $\widetilde{\rho}_\alpha^{a,b}(c)$: $(Z_a, Z_b) \to \mathrm{GL}\,(Z_a, Z_b)$ can be written as a matrix in the general linear (GL) group. This 3D generalization for the canonical basis in Eq.(11) is not only natural, but also consistent to 2D when we turn off the flux along z direction (e.g. set $b = 0$). which reduces 3D's $|\alpha, a, b\rangle$ to $|\alpha, a\rangle$ in the 2D case.

Generalizing 2D twisted quantum double model $D^\omega(G)$ to 3D: twisted quantum triple model? – A natural way to combine the Dijkgraaf-Witten theory with Kitaev's quantum double model Hamiltonian approach will enable us to study the Hamiltonian formalism for the twisted gauge theory, which is achieved in Ref.50,54 for 2+1D, named as the *twisted quantum double model*. In 2D, the widely-used notation $D^\omega(G)$ implies the twisted quantum double model with its gauge group G and its cocycle twist ω. It is straightforward to generalize their results to 3+1D.

To construct the Hamiltonian on the 3D spatial lattice, we follow Ref.50 with the form of the twisted quantum double model Hamiltonian of Eq.(7) and put the system on the $\mathbb{T}^3$ torus. However, some modification for 3D are adopted: the vertex operator $A_v = |G|^{-1} \sum_{[vv']=g\in G} A_v^g$ acts on the vertices of the lattice by lifting the vertex point v to v' living in an extra (fourth) dimension as Eq.(16),



(16)

and one computes the 4-cocycle filling amplitude as $\mathbf{Z}$ in Eq.(8). To evaluate Eq.(16)'s A_v operator acting on the vertex 5, one effectively lifts 5 to 5′, and fill 4-cocycles ω into this geometry to compute the amplitude $\mathbf{Z}$ in Eq.(8). For this specific 3D spatial lattice surrounding vertex 5 with $1, 2, 3$, and 4 neighboring vertices, there are four 4-cocycles ω filling in the amplitude of $A_5^{[55']}$.

A plaquette operator $B_p^{(1)}$ still enforces the zero flux condition on each 2D face (a triangle p) spanned by three edges of a triangle. This will ensure zero flux on each face (along the Wilson loop of a 1-form gauge field). Moreover, zero flux conditions are required if higher form gauge flux are presented. For example, for 2-form field, one adds an additional $B_p^{(2)}$ to ensure the zero flux on a 3-simplex (a tetrahedron p). Thus, $\sum_p B_p$ in Eq.(7) becomes $\sum_p B_p^{(1)} + \sum_p B_p^{(2)} + \dots$

Analogous to Ref.50, the local operators A_v, B_p of the Hamiltonian have nice commuting properties: $[A_v^g, A_u^h] = 0$ if $v \neq u$, $[A_v^g, B_p] = [B_p, B'_p] = 0$, and also $A_v^{g=[vv']} A_{v'}^h = A_v^{gh}$. Notice that A_g defines a ground sate projection operator $\mathsf{P}_v = |G|^{-1} \sum_g A_v^g$ if we consider a $\mathbb{T}^3$ torus triangulated in a cube with only a point v (all eight points are identified). It can be shown that both A_g and P as projection operators project other states to the ground state $|\alpha, a, b\rangle$, and $\mathsf{P}|\alpha, a, b\rangle = |\alpha, a, b\rangle$ and $A_v|\alpha, a, b\rangle \propto |\alpha, a, b\rangle$. Since $[A_v^g, B_p] = 0$, one can simultaneously diagonalize the Hamiltonian Eq.(7) by this *canonical basis* $|\alpha, a, b\rangle$ as the *ground state basis*.

A similar 3D model has been studied recently in Ref.41. There the zero flux condition is imposed in both the vertex operator as well as the plaquette operator. Their Hilbert space thus is more constrained than that of Ref.50 or ours. However, in the ground state sector, we expect that the physics is the same. It is less clear to us whether the name, **twisted quantum double model** and its notation $D^\omega(G)$, are still proper usages in 3D or higher dimensions. With the quantum double basis $|\alpha, a\rangle$ in 2D generalized to a triple basis $|\alpha, a, b\rangle$ in 3D, we are tempted to call it the **twisted quantum triple model** in 3D. It awaits mathematicians and mathematical physicists to explore more details in the future.

C. Cocycle of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ and its dimensional reduction

To study the twisted gauge theory of a finite Abelian group, we now provide the explicit data on cohomology group and 4-cocycles.[56] Here $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(G, \mathrm{U}(1))$ by $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$, as the $(d+1)$th-cohomology group of G over G module $\mathrm{U}(1)$. Each class in $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ corresponds to a distinct $(d+1)$-cocycle. The different 4-cocycles label the distinct topological terms of 3+1D twisted gauge theories. (However, different topological terms may share the same data for topological orders, such as the same modular data S^{xyz} and T^{xy}. Thus different topological terms may describe the same topological order.) The 4-cocycles ω_4 are 4-cochains, but additionally satisfy the cocycle condition $\delta\omega = 1$. The 4-cochain is a mapping $\omega_4(a, b, c, d)$: $(G)^4 \to \mathrm{U}(1)$, which inputs $a, b, c, d \in G$, and outputs a $\mathrm{U}(1)$ phase. Furthermore, distinct 4-cocycles are not identified by any 4-coboundary $\delta\Omega_3$. (Namely, distinct cocycles ω_4 and ω'_4 do *not* satisfy $\omega_4/\omega'_4 = \delta\Omega_3$, for any 3-cochain Ω_3.) The 4-cochain satisfies the group multiplication rule: $(\omega_4 \cdot \omega'_4)(a, b, c, d) = \omega_4(a, b, c, d) \cdot \omega'_4(a, b, c, d)$, thus forms a group C^4, the 4-cocycle further forms its subgroup Z^4, and the 4-coboundary further forms a Z^4's subgroup B^4 (since $\delta^2 = 1$). In short, $\mathrm{B}^4 \subset \mathrm{Z}^4 \subset \mathrm{C}^4$. The fourth cohomology group is a kernel Z^4 (the group of 4-cocycle) mod out the image B^4 (the group of 4-coboundaries) relation: $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \mathrm{Z}^4/\mathrm{B}^4$. We derive the fourth cohomology group of a generic finite Abelian $G = \prod_{i=1}^k Z_{N_i}$ as

$$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \prod_{1 \le i < j < l < m \le k} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{ijl}})^2 \times \mathbb{Z}_{N_{ijlm}}. \quad (17)$$

We construct generic 4-cocycles (not identified by 4-coboundaries) for each type, summarized in Table I.

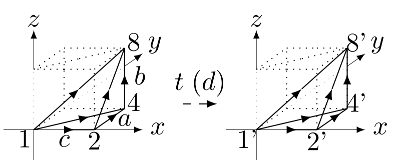
We name the Type II 1st and Type II 2nd 4-cocycles for those with topological term indices: $p_{\mathrm{II}(ij)}^{(1st)} \in \mathbb{Z}_{N_{ij}}$ and $p_{\mathrm{II}(ij)}^{(2nd)} \in \mathbb{Z}_{N_{ij}}$ of Eq.(17). There are Type III 1st and Type III 2nd 4-cocycles for topological term indices: $p_{\mathrm{III}(ijl)}^{(1st)} \in \mathbb{Z}_{N_{ijl}}$ and $p_{\mathrm{III}(ijl)}^{(2nd)} \in \mathbb{Z}_{N_{ijl}}$. There is also Type IV 4-cocycle topological term index: $p_{\mathrm{IV}(ijlm)} \in \mathbb{Z}_{N_{ijlm}}$.

Since we earlier alluded to the relation Eq.(5), $\mathcal{C}^{\mathrm{3D}} = \oplus_b \mathcal{C}_b^{\mathrm{2D}}$, between 3D topological orders (described by 4-

$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$	4-cocycle name	4-cocycle form	Induced 3-cocycle $\mathsf{C}_b(a,c,d)$
$\mathbb{Z}_{N_{ij}}$	Type II 1st $p^{(1st)}_{\mathrm{II}(ij)}$	$\omega^{(1st,ij)}_{4,\mathrm{II}}(a,b,c,d)=\exp\big(\frac{2\pi\mathrm{i}p^{(1st)}_{\mathrm{II}(ij)}}{(N_{ij}\cdot N_j)}(a_ib_j)(c_j+d_j-[c_j+d_j])\big)$	Type I, II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ij}}$	Type II 2nd $p^{(2nd)}_{\mathrm{II}(ij)}$	$\omega^{(2nd,ij)}_{4,\mathrm{II}}(a,b,c,d)=\exp\big(\frac{2\pi\mathrm{i}p^{(2nd)}_{\mathrm{II}(ij)}}{(N_{ij}\cdot N_i)}(a_jb_i)(c_i+d_i-[c_i+d_i])\big)$	Type I, II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijl}}$	Type III 1st $p^{(1st)}_{\mathrm{III}(ijl)}$	$\omega^{(1st,ijl)}_{4,\mathrm{III}}(a,b,c,d)=\exp\big(\frac{2\pi\mathrm{i}p^{(1st)}_{\mathrm{III}(ijl)}}{(N_{ij}\cdot N_l)}(a_ib_j)(c_l+d_l-[c_l+d_l])\big)$	two Type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijl}}$	Type III 2nd $p^{(2nd)}_{\mathrm{III}(ijl)}$	$\omega^{(2nd,ijl)}_{4,\mathrm{III}}(a,b,c,d)=\exp\big(\frac{2\pi\mathrm{i}p^{(2nd)}_{\mathrm{III}(ijl)}}{(N_{li}\cdot N_j)}(a_lb_i)(c_j+d_j-[c_j+d_j])\big)$	two Type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijlm}}$	Type IV $p_{\mathrm{IV}(ijlm)}$	$\omega^{(ijlm)}_{4,\mathrm{IV}}(a,b,c,d)=\exp\big(\frac{2\pi\mathrm{i}p_{\mathrm{IV}(ijlm)}}{N_{ijlm}}a_ib_jc_ld_m\big)$	Type III of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$

TABLE I. Cohomology group $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ and 4-cocycles ω_4 for a generic finite Abelian group $G=\prod_{i=1}^k Z_{N_i}$. The first column lists the types in $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of Eq.(17). The second column lists the topological term indices for 3+1D *twisted gauge theory*. (When all indices $p_{\dots}=0$, it becomes the *normal untwisted gauge theory*.) The third column lists the explicit 4-cocycle functions $\omega_4(a,b,c,d)$: $(G)^4\to \mathrm{U}(1)$. Here $a=(a_1,a_2,\dots,a_k)$, with $a\in G$ and $a_i\in Z_{N_i}$. (Same notations for b,c,d.) We define the mod N_j relation by $[c_j+d_j]\equiv c_j+d_j \pmod{N_j}$. The last column lists the induced 3-cocycles from the slant product $\mathsf{C}_b(a,c,d)\equiv i_b\omega_4(a,c,d)$ in terms of Type I, II, III 3-cocycles of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ listed in Table XII.

cocycles) as the direct sum of sectors of 2D topological orders (described by 3-cocycles), we wish to see how the dimensionally-reduced 3-cocycle from 4-cocycles can hint at the $\mathcal{C}^{\mathrm{2D}}_b$ theory of 2D. The slant product $\mathsf{C}_b(a,c,d)\equiv i_b\omega_4(a,c,d)$ are organized in the last column in Table I. The geometric interpretation of the induced 3-cocycle $\mathsf{C}_b(a,c,d)\equiv i_b\omega_4(a,c,d)$ is derived from the 4-cocycle ω_4:

$\mathsf{C}_b(a,c,d)$:  (18)

The combination of Eq.(18) (with four 4-cocycles filling) times the contribution of Eq.(12) (with three 3-cocycles filling) will produce Eq.(13) with twelve 4-cocycles filling. Luckily, the Type II and III ω_4 have a simpler form of $\mathsf{C}_b(a,c,d)=\omega_4(a,b,c,d)/\omega_4(b,a,c,d)$, while the reduced form of Type IV ω_4 is more involved.[56]

This indeed promisingly suggests the relation in Eq.(6), $\mathcal{C}^{\mathrm{3D}}_{G,\omega_4}=\oplus_b\mathcal{C}^{\mathrm{2D}}_{G,\omega_{3(b)}}$ with $G_b=G$ the original group. If we view b as the gauge flux along the z direction, and compactify z into a circle, then a single winding around z acts as a monodromy defect carrying the gauge flux b (group elements or conjugacy classes).[55,57,58] This implies a geometric picture in Fig.4.

One can tentatively write down a relation,

$$\mathcal{C}^{\mathrm{3D}}_{G,\omega_4}=\mathcal{C}^{\mathrm{2D}}_{G,1(\mathrm{untwist})}\oplus_{b\neq 0}\mathcal{C}^{\mathrm{2D}}_{G,\omega_{3(b)}}. \tag{19}$$

There is a zero flux $b=0$ sector $\mathcal{C}^{\mathrm{2D}}_{G,1(\mathrm{untwist})}$ (with $\omega_3=1$) where the 2D gauge theory with G is untwisted. There are other direct sums of $\mathcal{C}^{\mathrm{2D}}_{G,\omega_{3(b)}}$ with nonzero b flux insertion that have twisted $\omega_{3(b)}$.

However, different cocycles can represent the same topological order with the equivalent modular data, in the next section, we should examine this Eq.(19) more carefully not in terms of cocycles, but in terms of the modular data S^{xyz} and T^{xy}.

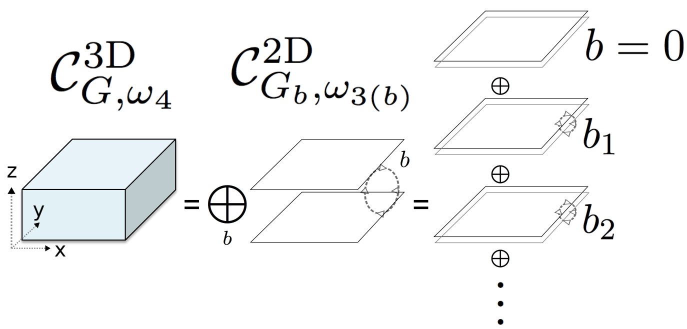


FIG. 4. Combine the reasoning in Eq.(18) and Fig.1, we obtain the physical meaning of dimensional reduction from a 3+1D twisted gauge theory as a 3D topological order to several sectors of 2D topological orders: $\mathcal{C}^{\mathrm{3D}}_{G,\omega_4}=\oplus_b\mathcal{C}^{\mathrm{2D}}_{G,\omega_{3(b)}}$. Here b stands for the gauge flux (Wilson line operator) of gauge group G. Here ω_3 are dimensionally reduced 3-cocycles from 4-cocycles ω_4. Note that there is a zero flux $b=0$ sector with $\mathcal{C}^{\mathrm{2D}}_{G,(\mathrm{untwist})}=\mathcal{C}^{\mathrm{2D}}_G$.

III. REPRESENTATION FOR S^{xyz} AND T^{xy}

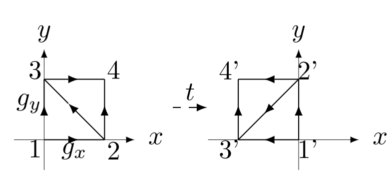
The modular transformations $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, $\hat{\mathsf{S}}^{xyz}$ of Eq.(2),(3) act on the 3D real space as

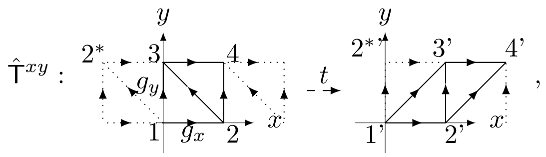
$$\hat{\mathsf{S}}^{xy}\cdot(x,y,z)=(-y,x,z), \tag{20}$$

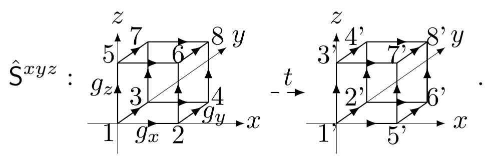
$$\hat{\mathsf{T}}^{xy}\cdot(x,y,z)=(x+y,y,z), \tag{21}$$

$$\hat{\mathsf{S}}^{xyz}\cdot(x,y,z)=(z,x,y). \tag{22}$$

More explicitly, we present triangulations of them:

$\hat{\mathsf{S}}^{xy}$:  , (23)

 (24)

 (25)

The modular transformation $\mathrm{SL}(2,\mathbb{Z})$ is generated by $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$, while the $\mathrm{SL}(3,\mathbb{Z})$ is generated by $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$. The dashed arrow $\dashrightarrow$ represents the time evolution (as in Fig.3) from $|\Psi_{in}\rangle$ to $|\Psi_{out}\rangle$ under $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, $\hat{\mathsf{S}}^{xyz}$ respectively. The $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$ transformations on a $\mathbb{T}^3$ torus's x-y plane with the z direction untouched are equivalent to its transformations on a $\mathbb{T}^2$ torus.

Q4: "*What are the generic expressions of SL(3,$\mathbb{Z}$) modular data?*"

First, in Sec III A, we apply the *cocycle approach* using the *spacetime path integral* with $\mathrm{SL}(3,\mathbb{Z})$ transformation acting along the time evolution to formulate the $\mathrm{SL}(3,\mathbb{Z})$ modular data, and then in Sec III B we use the more powerful *Representation* (Rep) *Theory* to determine the general expressions of those data in terms of (G,ω_4).

A. Path Integral and Cocycle approach

The cocycles approach uses the spacetime lattice formalism, where we triangulate the spacetime complex of a 4-manifold $\mathcal{M}=\mathbb{T}^3\times I$, (a $\mathbb{T}^3$ torus times a time interval I) of Eq.(23),(24),(25) into 4-simplices. We then apply the path integral $\mathbf{Z}$ in Eq.(8) and the amplitude form in Eq.(10) to obtain

$$\mathsf{T}^{xy}_{(\mathrm{A})(\mathrm{B})}=\langle\Psi_{\mathrm{A}}|\hat{\mathsf{T}}^{xy}|\Psi_{\mathrm{B}}\rangle, \tag{26}$$

$$\mathsf{S}^{xy}_{(\mathrm{A})(\mathrm{B})}=\langle\Psi_{\mathrm{A}}|\hat{\mathsf{S}}^{xy}|\Psi_{\mathrm{B}}\rangle, \tag{27}$$

$$\mathsf{S}^{xyz}_{(\mathrm{A})(\mathrm{B})}=\langle\Psi_{\mathrm{A}}|\hat{\mathsf{S}}^{xyz}|\Psi_{\mathrm{B}}\rangle, \tag{28}$$

$$\mathrm{GSD}=\mathrm{Tr}[\mathsf{P}]=\sum_{A}\langle\Psi_{\mathrm{A}}|\mathsf{P}|\Psi_{\mathrm{A}}\rangle. \tag{29}$$

Here $|\Psi_{\mathrm{A}}\rangle$ and $|\Psi_{\mathrm{B}}\rangle$ are ground state bases on the $\mathbb{T}^d$ torus, for example, they are $|\alpha,a\rangle$ (with α charge and a flux) in 2+1D and $|\alpha,a,b\rangle$ (with α charge and a,b fluxes) in 3+1D. We also include the data of GSD, where the P is the projection operator to ground states discussed in Sec.II B. In the case of d-D GSD on $\mathbb{T}^d$ (e.g. 3D GSD on $\mathbb{T}^3$), we simply compute the $\mathbf{Z}$ amplitude filling in $\mathbb{T}^d\times S^1=\mathbb{T}^{d+1}$. There is no short cut here except doing explicit calculations.[56]

B. Representation Theory approach

The cocycle approach in Sec.III A provides nice physical intuition about the modular transformation process. However, the calculation is tedious. There is a powerful approach simply using Representation Theory, we will present the general formula of $\hat{\mathsf{S}}^{xys}$, $\hat{\mathsf{T}}^{xy}$, $\hat{\mathsf{S}}^{xy}$ data in terms of (G,ω_4) directly. The three steps are outlined as follows:
(i) Obtain the Eq.(15)'s $\mathsf{C}^{(2)}_{a,b}$ by doing the slant product twice from 4-cocycle ω_4, or triangulating Eq.12. (ii) Derive $\widetilde{\rho}^{a,b}_{\alpha}(c)$ of $\mathsf{C}^{(2)}_{a,b}$-projective representation in Eq.(14), which $\widetilde{\rho}^{a,b}_{\alpha}(c)$ is a general linear matrix.
(iii) Write the modular data in the canonical basis $|\alpha,a,b\rangle$, $|\beta,c,d\rangle$ of Eq.(11).

After some long computations,[56] we find the most general formula S^{xyz} for a group G (both Abelian or non-Abelian) with cocycle twist ω_4:

$$\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)}=\frac{1}{|G|}\langle\alpha_x,a_y,b_z|\sum_w \mathsf{S}^{xyz}_w|\beta_{x'},c_{y'},d_{z'}\rangle=\frac{1}{|G|}\sum_{\substack{g_y\in C^a\cap Z_{g_z}\cap Z_{g_x},\\ g_z\in C^b\cap C^c,\\ g_x\in Z_{g_y}\cap Z_{g_z}\cap C^d}}\mathrm{Tr}\widetilde{\rho}^{g_y,g_z}_{\alpha_x}(g_x)^*\ \mathrm{Tr}\widetilde{\rho}^{g_z,g_x}_{\beta_y}(g_y)\delta_{g_x,h_{z'}}\delta_{g_y,h_{x'}}\delta_{g_z,h_{y'}}. \tag{30}$$

Here C^a,C^b,C^c,C^d are conjugacy classes of the group elements $a,b,c,d\in G$. In the case of a non-Abelian G, we should regard a,b as its conjugacy class C^a,C^b in $|\alpha,a,b\rangle$. Z_g means the centralizer of the conjugacy class of g. For an Abelian G, it simplifies to

$$\begin{aligned}\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)}&=\frac{1}{|G|}\mathrm{Tr}\widetilde{\rho}^{a,b}_{\alpha}(d)^*\ \mathrm{Tr}\widetilde{\rho}^{b,d}_{\beta}(a)\delta_{b,c}\equiv\frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{b,c}\\ &=\frac{1}{|G|}\mathrm{Tr}\widetilde{\rho}^{a_y,b_z}_{\alpha_x}(d_{z'})^*\mathrm{Tr}\widetilde{\rho}^{b_z,d_{z'}}_{\beta_{x'}}(a_y)\delta_{b_z,c_{y'}}\equiv\frac{1}{|G|}\mathsf{S}^{\alpha_x,\beta_y}_{d_x,a_y,b_z}\delta_{b_z,c_{y'}}.\end{aligned}$$

We write $\beta_{x'}=\beta_y$, $d_{z'}=d_x$ due to the coordinate identification under $\hat{\mathsf{S}}^{xyz}$. The assignment of the directions of gauge fluxes (group elements) are clearly expressed in the second line. We may use the first line expression for simplicity.

We also provide the most general formula of T^{xy} in the $|\alpha,a,b\rangle$ basis:

$$\mathsf{T}^{xy}=\mathsf{T}^{a_y,b_z}_{\alpha_x}=\frac{\mathrm{Tr}\widetilde{\rho}^{a_y,b_z}_{\alpha_x}(a_y)}{\dim(\alpha)}\equiv\exp(\mathrm{i}\Theta^{a_y,b_z}_{\alpha_x}). \tag{31}$$

Here dim(α) means the dimension of the representation, equivalently the rank of the matrix of $\widetilde{\rho}_{\alpha_x}^{a,b}(c)$. Since SL$(2,\mathbb{Z})$ is a subgroup of SL$(3,\mathbb{Z})$, we can express the SL$(2,\mathbb{Z})$'s S^{xy} by SL$(3,\mathbb{Z})$'s S^{xyz} and T^{xy} (an expression for both the real spatial basis and the canonical basis):

$$\mathsf{S}^{xy} = ((\mathsf{T}^{xy})^{-1}\mathsf{S}^{xyz})^3(\mathsf{S}^{xyz}\mathsf{T}^{xy})^2\mathsf{S}^{xyz}(\mathsf{T}^{xy})^{-1}. \quad (32)$$

For an Abelian G, and when $\mathsf{C}_{a,b}^{(2)}(c,d)$ is a 2-coboundary (cohomologically trivial), the dimensionality of Rep is dim(Rep) $\equiv$ dim$(\alpha) = 1$, and the S^{xy} is simplified:

$$\mathsf{S}^{xy}_{(\alpha,a,b)(\beta,c,d)} = \frac{1}{|G|}\frac{\mathrm{tr}\widetilde{\rho}_{\alpha}^{a,b}(ac^{-1})^*}{\mathrm{tr}\widetilde{\rho}_{\alpha}^{a,b}(a)}\frac{\mathrm{tr}\widetilde{\rho}_{\beta}^{c,d}(ac^{-1})}{\mathrm{tr}\widetilde{\rho}_{\beta}^{c,d}(c)}\delta_{b,d}. \quad (33)$$

We can verify the above results by first computing the cocycle path integral approach in Sec.III A, and substituting from the flux basis to the canonical basis by Eq.(11). We have made several consistent checks, by comparing our $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, $\hat{\mathsf{S}}^{xyz}$ to: (1) the known 2D case for the untwisted theory of a non-Abelian group,[42] (2) the recent 3D case for the untwisted theory of a non-Abelian group,[39] (3) the recent 3D case for the twisted theory of an Abelian group.[41] And our expression works for all cases: the (un)twisted theory of (non-)Abelian group. More detailed calculations are provided in Appendix B.

C. Physics of S and T in 3D

The S^{xy} and T^{xy} in 2D are known to have precise physical meanings. At least for Abelian topological orders, there is no ambiguity that S^{xy} in the quasiparticle basis provides the mutual statistics of two particles (winding one around the other by 2π), while T^{xy} in the quasiparticle basis provides the self statistics of two identical particles (winding one around the other by π). Moreover, the intimate spin-statistics relation shows that the statistical phase $e^{\mathrm{i}\Theta}$ gained by interchanging two identical particles is equal to the spin s by $e^{\mathrm{i}2\pi s}$. Fig.5 illustrates the *spin-statistics relation*.[59] Thus, people also call T^{xy} in 2D as the *topological spin*. Here we ask:

Q5: "*What is the physical interpretation of* SL(3,$\mathbb{Z}$) *modular data in 3D*?"

Our approach again is by dimensional reduction of Fig.1, via Eq.(4) and Eq.(5): $\mathsf{S}^{xy} = \oplus_b \mathsf{S}_b^{xy}$, $\mathsf{T}^{xy} = \oplus_b \mathsf{T}_b^{xy}$, $\mathcal{C}^{\mathrm{3D}} = \oplus_b \mathcal{C}_b^{\mathrm{2D}}$, reducing the 3D physics to the direct sum of 2D topological phases in different flux sectors, so we can retrieve the familiar physics of 2D to interpret 3D.

For our case with a gauge group description, the b (subindex of S_b^{xy}, T_b^{xy}, $\mathcal{C}_b^{\mathrm{2D}}$) labels the gauge flux (group element or conjugacy class C^b) winding around the compact z direction in Fig.1. This b flux can be viewed as the by-product of a *monodromy defect* causing a *branch cut* (*a symmetry twist*[55,57,58,67]), such that the wavefunction will gain a *phase* by winding around the compact z direction. Now we further regard the b flux as a *string threading around* in the background, so that winding around this background string (e.g. the black string threading in Fig.2(c),6(c),7(c)) gains the b flux effect if there is a nontrivial winding on the compact direction z. The arrow - - -▷ along the compact z schematically indicates such a b flux effect from the background string threading.

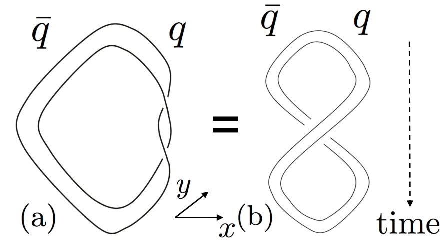


FIG. 5. Both process (a) and process (b) start from the creation of a pair of particle q and anti-particle $\bar{q}$, but the wordlines evolve along time to the bottom differently. Process (a) produces a phase $e^{\mathrm{i}2\pi s}$ due to 2π rotation of q, with spin s. Process (b) produces a phase $e^{\mathrm{i}\Theta}$ due to the exchange statistics. The homotopic equivalence by deformation implies $e^{\mathrm{i}2\pi s} = e^{\mathrm{i}\Theta}$.

1. T_b^{xy} *and topological spin of a closed string*

We apply the above idea to interpret T_b^{xy}, shown in Fig.6. From Eq.(31), we have $\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z} \equiv \exp(\mathrm{i}\Theta_{\alpha_x}^{a_y,b_z})$ with a fixed b_z label for a given b_z flux sector. For each b, T_b^{xy} acts as a familiar 2D T matrix $\mathsf{T}_{\alpha_x}^{a_y}$, which provides the topological spin of a quasiparticle (α,a) with charge α and flux a, in Fig.6(a).

From the 3D viewpoint, however, this $|\alpha,a\rangle$ particle is actually a closed string compactified along the compact z direction. Thus, in Fig.6(b), the self-2π rotation of the topological spin of a quasiparticle $|\alpha,a\rangle$ is indeed the self-2π rotation of a framed closed string. (Physically we understand that there the string can be *framed* with arrows, because the inner texture of the string excitations are allowed in a condensed matter system, due to defects or the finite size lattice geometry.) Moreover, from an equivalent 3D view in Fig.6(c), we can view the self-2π rotation of a framed closed string as the *self-*2π *flipping* of a framed closed string, which flips the string inside-out and then outside-in back to its original status. This picture works for both the $b=0$ zero flux sector and the $b\neq 0$ sector under the background string threading. **We thus propose T_b^{xy} as the topological spin of a framed closed string, threaded by a background string carrying a monodromy b flux.**

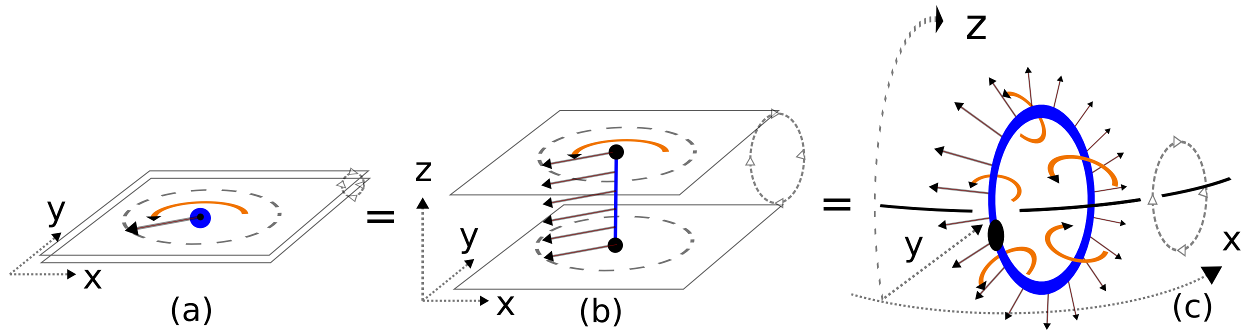


FIG. 6. Topological spin of (a) a particle by 2π-self rotation in 2D, (b) a framed closed-string by 2π-self rotation in 3D with a compact z, (c) a closed-string (blue) by 2π-self flipping, threaded by a background (black) string creating monodromy b flux (along the arrow - - -▷), under a single Hopf link 2_1^2 configuration. All above equivalent pictures describe the physics of topological spin in terms of T_b^{xy}. For Abelian topological orders, the spin of an excitation (say A) in Fig.6(a) yields an Abelian phase $e^{\mathrm{i}\Theta_{(\mathrm{A})}} = \mathsf{T}^{xy}_{(\mathrm{A})(\mathrm{A})}$ proportional to the diagonal of the 2D's T^{xy} matrix. The dimensional-extended equivalent picture Fig.6(c) implies that the loop-flipping yields a phase $e^{\mathrm{i}\Theta_{(\mathrm{A}),b}} = \mathsf{T}^{xy}_{b\;(\mathrm{A})(\mathrm{A})}$ of Eq.(31) (up to a choice of canonical basis), where b is the flux of the black string.

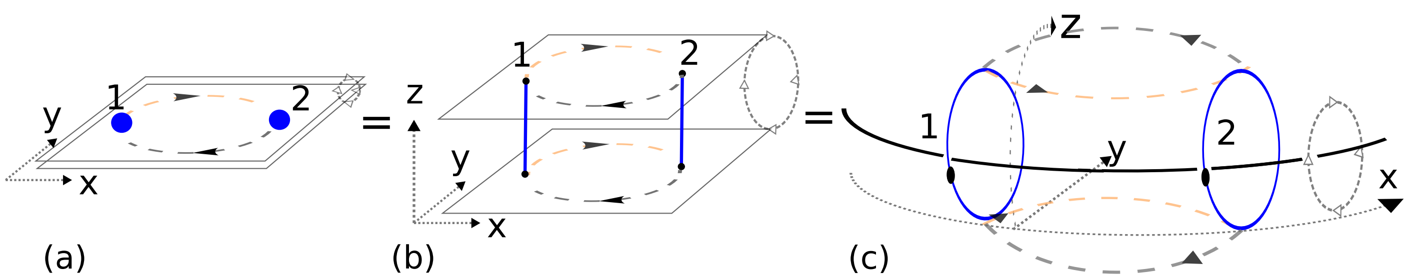


FIG. 7. Exchange statistics of (a) two identical particles at positions 1 and 2 by a π winding (half-winding), (b) two identical strings by a π winding in 3D with a compact z, (c) two identical closed-strings (blue) with a π-winding around, both threaded by a background (black) string creating monodromy b flux, under the Hopf links $2_1^2\#2_1^2$ configuration. Here figures (a)(b)(c) describe the equivalent physics in 3D with a compact z direction. The physics of exchange statistics of a closed string turns out to be related to the topological spin in Fig.6, discussed in Sec.III C 3.

2. S_b^{xy} *and three-string braiding statistics*

Similarly, we apply the same philosophy to do 3D to 2D reduction for S_b^{xy}, each effective 2D threading with a distinct gauge flux b. We can obtain S_b^{xy} from Eq.(32) with SL$(3,\mathbb{Z})$ modular data. Here we will focus on interpreting S_b^{xy} in the Abelian topological order. Writing S_b^{xy} in the canonical basis $|\alpha,a,b\rangle$, $|\beta,c,d\rangle$ of Eq.(11), we find that, true for Abelian topological order

$$\mathsf{S}_b^{xy} = \mathsf{S}^{xy}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\text{2D}\;\alpha,\beta}_{a,c\;(b)}\delta_{b,d}. \tag{34}$$

As we predict the generality in Eq.(4), the S_b^{xy} here is diagonalized with the b and d identified (as the z-direction flux created by the background string threading). For a given fixed b flux sector, the only free indices are $|\alpha,a\rangle$ and $|\beta,c\rangle$, all collected in $\mathsf{S}^{\text{2D}\;\alpha,\beta}_{a,c\;(b)}$. (Explicit data will be presented in Sec.IV B) Our interpretation is shown in Fig.2. From a 2D viewpoint, S_b^{xy} gives the full 2π braiding statistics data of two quasiparticle $|\alpha,a\rangle$ and $|\beta,c\rangle$ excitations in Fig.2(a). However, from the 3D viewpoint, the two particles are actually two closed strings compactified along the compact z direction. Thus, the full-2π braiding of two particles in Fig.2(a) becomes that of two closed-strings in Fig.2(b). More explicitly, an equivalent 3D view in Fig.2(c), we identify the coordinates x, y, z carefully to see such a full-braiding process is that *one (red) string going inside to the loop of another (blue) string, and then going back from the outside.*

The above picture works again for both the $b = 0$ zero flux sector as well as the $b \neq 0$ sector under the background string threading. When $b \neq 0$, the third (black) background string in Fig.2(c) threading through the two (red, blue) strings. The third (black) string creates the monodromy defect/branch cut on the background, and carrying b flux along z acting on two (red, blue) strings which have nontrivial winding on the third string. This three-string braiding was first emphasized in a recent paper,[40] here we make further connection to the data S_b^{xy} and understand its physics in a 3D to 2D under b flux sectors.

We have proposed and shown that S_b^{xy} can capture the physics of three-string braiding statistics with two strings threaded by a third background string causing b flux monodromy, where the three strings have the linking configuration as the connected sum of two Hopf links $2_1^2 \# 2_1^2$.

3. *Spin-Statistics relation for closed strings*

Since a spin-statistics relation for 2D particles is shown by Fig.5, we may wonder, by using our 3D to 2D reduction picture, whether a *spin-statistics relation for a closed string* holds?

To answer this question, we should compare the *topological spin* picture of $\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y, b_z} \equiv \exp(\mathrm{i}\Theta_{\alpha_x}^{a_y,b_z})$ to the *exchange statistic* picture of two closed strings in Fig.7. Fig.7 essentially takes a *half-braiding* of the S_b^{xy} process of Fig.2, and considers doing half-braiding on the same excitations in $|\alpha, a, b\rangle = |\beta, c, d\rangle$. In principle, one can generalize the framed worldline picture of particles in Fig.5 to the framed worldsheet picture of closed-strings. (ps. The framed worldline is like a worldsheet, the framed worldsheet is like a worldvolume.) This interpretation shows that the topological spin of Fig.6 and the exchange statistics of Fig.7 carry the same data, namely

$$\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z} = (\mathsf{S}^{\text{2D}\ \alpha_x,\alpha_x}_{a_y,a_y\ (b_z)})^{\frac{1}{2}} \text{ or } (\mathsf{S}^{\text{2D}\ \alpha_x,\alpha_x}_{a_y,a_y\ (b_z)})^{\frac{1}{2}*} \quad (35)$$

from the data of Eq.(31),(34). The equivalence holds, up to a (complex conjugate *) sign caused by the orientation of the rotation and the exchange.

In Sec.(IV B), we will show, for the twisted gauge theory of Abelian topological orders, such an interpretation Eq.35 is correct and agrees with our data. We term this as the **spin-statistics relation for a closed string**.

In this section, we have obtained the explicit formulas of S^{xyz}, T^{xy}, S^{xy} in Sec.III A,III B, and as well as captured the physical meanings of S_b^{xy}, T_b^{xy} in Sec.III C 3. Before concluding, we note that the full understanding of S^{xyz} seems to be intriguingly related to the 3D nature. It is *not* obvious to us that the use of 3D to 2D reduction can capture all physics of S^{xyz}. We will come back to comment this issue in the Sec.V.

IV. SL$(3,\mathbb{Z})$ MODULAR DATA AND MULTI-STRING BRAIDING

A. Ground state degeneracy and Particle, String types

We now proceed to study the topology-dependent ground state degeneracy (GSD), modular data S, T of 3+1D twisted gauge theory with finite group $G = \prod_i Z_{N_i}$. We shall comment that the GSD on $\mathbb{T}^2$ of 2D topological order counts the number of quasi-particle excitations, which from the Representation (Rep) Theory is simply counting the number of charges α and fluxes a forming the quasi-particle basis $|\alpha, a\rangle$ spanned the ground state Hilbert space. **In 2D, GSD counts the number of types of quasi-particles** (or anyons) as well as **the number of basis $|\alpha, a\rangle$.** For higher dimension, GSD on $\mathbb{T}^d$ of d-D topological order still counts **the number of canonical basis** $|\alpha, a, b, \dots\rangle$, however, may over count the number of types of **particles (with charge), strings (with flux)**, etc excitations. From a untwisted Z_N field theory perspective, the fluxed string may be described by a 2-form B field, and the charged particle may be described by a 1-form A field, with a BF action $\int BdA$. As we can see the fluxes a, b are over-counted.

We suggest that counting the number of types of particles of d-dimensions is equivalent to Fig.8 process, where

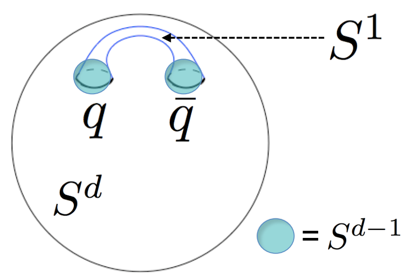


FIG. 8. Number of particle types = GSD on $S^{d-1} \times S^1$.

we dig a ball B^d with a sphere S^{d-1} around the particle q, which resides on S^d. And we connect it through a S^1 tunnel to its anti-particle $\bar{q}$. This process causes creation-annihilation from vacuum, and counts how many types of q sectors is equivalent to:

$$\text{the number of particle types} = \text{GSD on } S^{d-1} \times I. \quad (36)$$

with $I \simeq S^1$ for this example. For the spacetime integral, one evaluates Eq.(29) on $\mathcal{M} = S^{d-1} \times S^1 \times S^1$.

For counting closed string excitations, one may naively use $\mathbb{T}^2$ to enclose a string, analogously to using $\mathbb{S}^2$ to enclose a particle in 3D. Then, one may deduce the number of string types = GSD on $\mathbb{T}^2 \times S^1 \overset{?}{=} \mathbb{T}^3$, and that of spacetime integral on $\mathbb{T}^4$, as we already mentioned earlier which is *incorrect* and overcounting. We suggest,

$$\text{the number of string types} = \mathsf{S}^{xy}, \mathsf{T}^{xy}\text{'s number of blocks}, \quad (37)$$

whose blocks are labeled by b as the form of Eq.4. We will show the counting by Eq.(36), (37) in explicit examples in the next.

B. Abelian examples: 3D twisted $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ gauge theories with Type II, III 4-cocycles

We first study the most generic 3+1D finite Abelian twisted gauge theories with Type II, III 4-cocycle twists in Table I. It is general enough for us to consider $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ with non-vanished gcd N_{ij}, N_{ijl}. The Type II, III (both their 1st and 2nd kinds) twisted gauge theory have GSD$= |G|^3$ on the spatial $\mathbb{T}^3$ torus. As such the canonical basis $|\alpha, a, b\rangle$ of the ground state sector labels the charge (α along x) and two fluxes (a, b along y, z), each of the three has $|G|$ kinds. Thus, naturally from the Rep Theory viewpoint, we have GSD$= |G|^3$. However, as mentioned in Sec.IV A, the $|G|^3$ overcounts the number of strings and particles. By using Eq.(36),(37), we find there are $|G|$ types of particles and $|G|$ types of strings. The canonical basis $|\alpha, a, b\rangle$ (GSD on $\mathbb{T}^3$) counts twice the flux sectors.

In Table II, we list their S^{xyz} by computing Eq.(30), where we denote $a = (a_1, a_2, a_3, \dots)$, with $a_j \in Z_{N_j}$, and the same notation for other b, c, d fluxes:

$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	4-cocycle	$\mathsf{S}^{\alpha,\beta}_{d,a,b}$	Induced S^{xy}_b
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\exp\big(\sum_k \frac{2\pi i}{N_k}(\beta_k a_k - \alpha_k d_k)\big) \cdot \exp\big(\frac{2\pi i p^{(1st)}_{\text{II}(12)}}{(N_{12} \cdot N_2)}(a_1 d_2 + a_2 d_1) b_2 - 2 a_2 b_1 d_2\big)$	Type I, II of $\mathcal{H}^3$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\exp\big(\sum_k \frac{2\pi i}{N_k}(\beta_k a_k - \alpha_k d_k)\big) \cdot \exp\big(\frac{2\pi i p^{(2nd)}_{\text{II}(12)}}{(N_{12} \cdot N_1)}(a_1 d_2 + a_2 d_1) b_1 - 2 a_1 b_2 d_1\big)$	Type I, II of $\mathcal{H}^3$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\exp\big(\sum_k \frac{2\pi i}{N_k}(\beta_k a_k - \alpha_k d_k)\big) \cdot \exp\big(\frac{2\pi i p^{(1st)}_{\text{III}(123)}}{(N_{12} \cdot N_3)}(a_1 b_2 - a_2 b_1) d_3 + (b_2 d_1 - b_1 d_2) a_3\big)$	two Type IIs of $\mathcal{H}^3$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\exp\big(\sum_k \frac{2\pi i}{N_k}(\beta_k a_k - \alpha_k d_k)\big) \cdot \exp\big(\frac{2\pi i p^{(2nd)}_{\text{III}(123)}}{(N_{31} \cdot N_2)}(a_3 b_1 - a_1 b_3) d_2 + (b_1 d_3 - b_3 d_1) a_2\big)$	two Type IIs of $\mathcal{H}^3$

TABLE II. $\mathsf{S}^{xyz} = \mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{bc}$ modular data of 3+1D twisted gauge theories with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. In the last column, the $\mathcal{H}^3$ is the shorthand of $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$; the induced S^{xy}_b is shown in Table IV.

Here we extract the $\mathsf{S}^{\alpha,\beta}_{d,a,b}$ part of S^{xyz} ignoring the $|G|^{-1}$ factor:

$$\mathsf{S}^{xyz} = \mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{b,c}. \tag{38}$$

The S–matrix reads $g_{x_k} = d_k$, $g_{y_k} = a_k$ in Eq.(30).

In Table III, we show T^{xy}. Here for Abelian G, with $\mathsf{C}^{(2)}_{a,b}(c,d)$ is a 2-coboundary (cohomologically trivial) thus dim(Rep) $= 1$, we compute S^{xy} by Eq.(33) and that reduces to Eq.(34) $\mathsf{S}^{xy}_b = (\mathsf{S}^{xy})_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\text{2D}\ \alpha,\beta}_{a,c\ (b)}\delta_{b,d}$. In Table IV, we show S^{xy} in terms of $\mathsf{S}^{\text{2D}\ \alpha,\beta}_{a,c\ (b)}$ for simplicity.

$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	4-cocycle	$\mathsf{T}^{a,b}_{\alpha}$	Induced T^{xy}_b
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\exp\big(\sum_k \frac{2\pi i}{N_k}\alpha_k \cdot a_k\big) \cdot \exp\big(\frac{2\pi i p^{(1st)}_{\text{II}(12)}}{(N_{12} \cdot N_2)}(a_2 b_1 - a_1 b_2)(a_2)\big)$	Type I, II of $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\exp\big(\sum_k \frac{2\pi i}{N_k}\alpha_k \cdot a_k\big) \cdot \exp\big(\frac{2\pi i p^{(2nd)}_{\text{II}(12)}}{(N_{12} \cdot N_1)}(a_1 b_2 - a_2 b_1)(a_1)\big)$	Type I, II of $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\exp\big(\sum_k \frac{2\pi i}{N_k}\alpha_k \cdot a_k\big) \cdot \exp\big(\frac{2\pi i p^{(1st)}_{\text{III}(123)}}{(N_{12} \cdot N_3)}(a_2 b_1 - a_1 b_2)(a_3)\big)$	two Type IIs of $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\exp\big(\sum_k \frac{2\pi i}{N_k}\alpha_k \cdot a_k\big) \cdot \exp\big(\frac{2\pi i p^{(2nd)}_{\text{III}(123)}}{(N_{31} \cdot N_2)}(a_1 b_3 - a_3 b_1)(a_2)\big)$	two Type IIs of $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$

TABLE III. T^{xy} modular data of the 3+1D twisted gauge theories with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. We can view this in terms of the index b for blocks of $\mathsf{T}^{xy}_b = \mathsf{T}^{a_y, b_z}_{\alpha_x}$, with the flux b along the compact z direction.

Several remarks follow:

(1) For an untwisted gauge theory (topological term

ω_4	$\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)} = \text{tr}\widetilde{\rho}^{a,b}_{\alpha}(a^2c^{-1})^* \ \text{tr}\widetilde{\rho}^{c,b}_{\beta}(ac^{-2})$
II 1st	$\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(1st)}_{\text{II}(12)}}{(N_{12}\cdot N_2)}\ b_1(2a_2c_2-2a_2^2-2c_2^2)+b_2(2a_1a_2+2c_1c_2-a_1c_2-a_2c_1)\big)$
II 2nd	$\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(2nd)}_{\text{II}(12)}}{(N_{12}\cdot N_1)}\ b_2(2a_1c_1-2a_1^2-2c_1^2)+b_1(2a_1a_2+2c_1c_2-a_1c_2-a_2c_1)\big)$
III 1st	$\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(1st)}_{\text{III}(123)}}{(N_{12}\cdot N_3)}\ b_1(a_2c_3+a_3c_2-2a_2a_3-2c_2c_3)+b_2(2a_1a_3+2c_1c_3-a_1c_3-a_3c_1)\big)$
III 2nd	$\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(2nd)}_{\text{III}(123)}}{(N_{31}\cdot N_2)}\ b_3(a_1c_2+a_2c_1-2a_1a_2-2c_1c_2)+b_1(2a_3a_2+2c_3c_2-a_3c_2-a_2c_3)\big)$

TABLE IV. S^{xy} modular data of the 3+1D twisted gauge theories with $G = Z_{N_1}\times Z_{N_2}\times Z_{N_3}$. There are two more columns ($\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$, induced S^{xy}_b) not shown here, since the data simply duplicates Table II's first and fourth column. The basis chosen here is not canonical for excitations, in the sense that particle braiding around trivial vacuum still gain a non-trivial statistic phase. Finding the proper canonical basis for each b block of S^{xy}_b can be done by the method of Ref.60.

$p_{..}=0$), which is the direct product of Z_N gauge theory or Z_N toric code, its statistics has the form $\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}(\beta_k a_k - \alpha_k d_k)\big)$ and $\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\alpha_k\cdot a_k\big)$. This shall be described by the BF theory of $\int BdA$ action. With α,β as the charge of particles (1-form gauge field A), a,b as the flux of string(2-form gauge field B). This essentially describes **the braiding between a pure-particle and a pure-string**.

(2) Both S^{xy}, T^{xy} have block diagonal forms as S^{xy}_b, T^{xy}_b respect to the b flux (along z) correctly reflects what Eq.(4) preludes already.

(3) T^{xy} is in $\text{SL}(3,\mathbb{Z})$ canonical basis automatically and full-diagonal, but S^{xy} may not be in the canonical basis for each blocks of S^{xy}_b, due to its $\text{SL}(2,\mathbb{Z})$ nature. We can find the proper basis in each b block by Ref.60 method. Nevertheless, the eigenvalues of S^{xy} in Table IV are still proper and invariant regardless any basis.

(4) **Characterization of topological orders**: We can further compare the 3D S^{xy}_b data to $\text{SL}(2,\mathbb{Z})$'s data of 2D S^{xy} of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ in Table XII. (see Appendix A for data) All of the dimensional reduction of these data (S^{xy}_b in Table II, IV, and T^{xy}_b in Table III) agree with 3-cocycle (induced from 4-cocycle ω_4) in Table I's last column. Gathering all data, we conclude that Eq.(19) becomes explicitly. For example, Type II twists for $G=(Z_2)^2$ as,

$$\mathcal{C}^{\text{3D}}_{(Z_2)^2,1} = 4\mathcal{C}^{\text{2D}}_{(Z_2)^2,1} \tag{39}$$

$$\mathcal{C}^{\text{3D}}_{(Z_2)^2,\omega_{4,\text{II}}} = \mathcal{C}^{\text{2D}}_{(Z_2)^2} \oplus \mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{I}}} \oplus 2\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{II}}} \tag{40}$$

Such a Type II $\omega_{4,\text{II}}$ can produce a $b=0$ sector of (Z_2 toric code $\otimes$ Z_2 toric code) of 2D as $\mathcal{C}^{\text{2D}}_{(Z_2)^2}$, some $b\neq 0$ sector of (Z_2 double-semions $\otimes$ Z_2 toric code) as $\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{I}}}$ and another $b\neq 0$ sector $\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{II}}}$, for example. This procedure can be applied to other types of cocycle twists.

(5) **Classification of topological orders**:
We shall interpret the decomposition in Eq.(19) as the implication of classification. Let us do the *counting of number of phases* in the simplest example of Type II, $G=Z_2\times Z_2$ twisted theory. There are four types in $(p^{(1st)}_{\text{II}(12)}, p^{2nd}_{\text{II}(12)}) \in \mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^2$. However, we find there are **only two distinct topological orders** out of four. One is the trivial $(Z_2)^2$ gauge theory as Eq.(39), the other is the nontrivial type as Eq.(40). There are two ways to see this, (i) from the full S^{xyz}, T^{xy} data. (ii) viewing the sector of S^{xy}_b, T^{xy}_b under distinct fluxes b, which is from a $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ perspective. We should beware that in principle *tagging particles, strings or gauge groups is not allowed*, so one can identify many seemingly-different orders by relabeling their excitations. We will give more examples of counting 2D and 3D topological orders in Appendix A.

(6) **Spin-statistics relation of closed strings** in Eq.(35) is verified to be correct here, while we take the complex conjugate in Eq.(35). This is why we draw the orientation of Fig.6,7 oppositely. Interpreting T^{xy} as the **topological spin** also holds.

(7) **Cyclic relation for S^{xyz} in 3D**: For all the above data (Type II, Type III), there is a special cyclic relation for $\mathsf{S}^{\alpha,\beta}_{a,b,d}$ when the charge labels are equal $\alpha=\beta$ (e.g. for pure fluxes $\alpha=\beta=0$, namely for pure strings):

$$\mathsf{S}^{\alpha,\alpha}_{a,b,d}\cdot\mathsf{S}^{\alpha,\alpha}_{b,d,a}\cdot\mathsf{S}^{\alpha,\alpha}_{d,a,b} = 1. \tag{41}$$

However, such a cyclic relation does not hold (even at the zero charge) for $\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)}$, namely $\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)}\cdot\mathsf{S}^{\text{2D }\alpha,\beta}_{c,b\ (a)}\cdot\mathsf{S}^{\text{2D }\alpha,\beta}_{b,a\ (c)}\neq 1$ in general. Some other cyclic relations are studied recently in Ref.40 and 41, for which we have not yet made detailed comparisons but the perspectives may be different. In Ref.41, their cyclic relation is determined by triple linking numbers associated with the membrane operators. In Ref.40, their cyclic relation is related to the loop braiding of Fig.2, which has its relevancy instead to $\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)}$, not our cyclic relation of $\mathsf{S}^{\alpha,\beta}_{a,b,d}$ for 3D. We will comment more about the difference and the subtlety of S^{xy} and S^{xyz} in Sec.V B.

C. Non-Abelian examples: 3D twisted $(Z_n)^4$ gauge theories with Type IV 4-cocycle

We now study a more interesting example, a generic 3+1D finite Abelian twisted gauge theory with Type IV 4-cocycle twists with $p_{ijlm} \neq 0$ in Table I. For generality, our formula also incorporates Type IV twists together with the aforementioned Type II, III twists. So all 4-cocycle twists will be discussed in this subsection. Differ from the previous example of Abelian topological order with Abelian statistics in Sec.IV B, we will show Type IV 4-cocycle $\omega_{4,\text{IV}}$ will cause the *gauge theory becomes non-Abelian, having non-Abelian statistics even if the original G is Abelian.* Our inspiration rooted in a 2D example for Type III 3-cocycle twist in Table XII will cause a similar effect, discovered in Ref.42. In general, one can consider $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3} \times Z_{N_4}$ with non-vanished gcd N_{1234}; however, we will focus on $G = (Z_n)^4$ with $N_{1234} = n$, with n is prime for simplicity. From $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n^{21}$, we have n^{21} types of theories, while n^{20} are Abelian gauge theories, and $n^{20} \cdot (n-1)$ types with Type IV ω_4 show non-Abelian statistics.

Ground state degeneracy (GSD)-

We compute the GSD of gauge theories with a Type IV twist on the spatial $\mathbb{T}^3$ torus, truncated from $= |G|^3 = |n^4|^3 = n^{12}$ to:

$$\text{GSD}_{\mathbb{T}^3,\text{IV}} = \big(n^8 + n^9 - n^5\big) + \big(n^{10} - n^7 - n^6 + n^3\big) \quad (42)$$

$$\equiv \text{GSD}^{Abel}_{\mathbb{T}^3,\text{IV}} + \text{GSD}^{nAbel}_{\mathbb{T}^3,\text{IV}} \quad (43)$$

(We derive the above only for a prime n. The GSD truncation is less severe and is in between $\text{GSD}_{\mathbb{T}^3,\text{IV}}$ and $|G|^3$ for a non-prime n.) As such, the canonical basis $|\alpha, a, b\rangle$ of the ground state sector on $\mathbb{T}^3$ *no longer* has $|G|^3$ labels with the $|G|$ number charge and two pairs of $|G| \times |G|$ number of fluxes as in Sec.IV B. This truncation is due to the nature of non-Abelian physics of Type IV $\omega_{4,\text{IV}}$ twisted. We explain our notation in Eq.(43); the $(n)Abel$ indicates the contribution from (non-)Abelian excitations. From the Rep Theory viewpoint, we can recover the truncation back to $|G|^3$ by carefully reconstructing the *quantum dimension of excitations.* We obtain

$$\begin{aligned}
&|G|^3 = \big(\text{GSD}^{Abel}_{\mathbb{T}^3,\text{IV}}\big) + \big(\text{GSD}^{nAbel}_{\mathbb{T}^3,\text{IV}}\big) \cdot n^2 \quad (44)\\
&= \{n^4 + n^5 - n\} \cdot n^4 \cdot (1)^2\\
&+\{(n^4)^2 - n^5 - n^4 + n\} \cdot n^2 \cdot (n)^2\\
&= \{\text{Flux}^{Abel}_{\text{IV}}\} \cdot n^4 \cdot (\dim_1)^2 + \{\text{Flux}^{nAbel}_{\text{IV}}\} \cdot n^2 \cdot (\dim_n)^2
\end{aligned}$$

The $\dim_m$ means the dimension of Rep as dim(Rep) is m, which is also the *quantum dimension* of excitations. Here we have a dimension 1 for Abelian and n for non-Abelian. In summary, we understand the decomposition precisely in terms of each (non-)Abelian contribution as follows:

$$\begin{cases}
\text{flux sectors} = |G|^2 = |n^4|^2 = \text{Flux}^{Abel}_{\text{IV}} + \text{Flux}^{nAbel}_{\text{IV}}\\
\text{GSD}_{\mathbb{T}^3,\text{IV}} = \text{GSD}^{Abel}_{\mathbb{T}^3,\text{IV}} + \text{GSD}^{nAbel}_{\mathbb{T}^3,\text{IV}}\\
\dim(\text{Rep})^2 = 1^2, n^2\\
\text{Numbers of charge Rep} = n^4, n^2.
\end{cases} \quad (45)$$

Actually, the canonical basis $|\alpha, a, b\rangle$ (GSD on $\mathbb{T}^3$) still works, the sum of Abelian $\text{Flux}^{Abel}_{\text{IV}}$ and non-Abelian $\text{Flux}^{nAbel}_{\text{IV}}$ counts the flux number of a, b as the unaltered $|G|^2$. The charge Rep α is unchanged with a number of $|G| = n^4$ for Abelian sector with a rank-1 matrix (1-dim linear or projective) representation, however, the charge Rep α is truncated to a smaller number n^2 for non-Abelian sector also with a larger rank-n matrix (n-dim projective) representation.

Another view on $\text{GSD}_{\mathbb{T}^3,\text{IV}}$ can be inspired by a generic formula like Eq.(4)

$$\text{GSD}_{\mathcal{M}' \times S^1} = \oplus_b \text{GSD}_{b,\mathcal{M}'} = \sum_b \text{GSD}_{b,\mathcal{M}'}, \quad (46)$$

where we sum over GSD in all different b flux sectors, with b flux along S^1. Here we can take $\mathcal{M}' \times S^1 = \mathbb{T}^3$ and $\mathcal{M}' = \mathbb{T}^2$. For non-Type IV (untwisted, Type II, III) ω_4 case, we have $|G|$ sectors of b flux and each has $\text{GSD}_{b,\mathbb{T}^2} = |G|^2$. For Type IV ω_4 case $G = (Z_n)^4$ with a prime n, we have

$$\begin{aligned}
&\text{GSD}_{\mathbb{T}^3,\text{IV}}\\
&= |G|^2 + (|G| - 1) \cdot |Z_n|^2 \cdot (1 \cdot |Z_n|^3 + (|Z_n|^2 - 1) \cdot n)\\
&= n^8 + (n^4 - 1) \cdot n^2 \cdot (1 \cdot n^3 + (n^3 - 1) \cdot n). \quad (47)
\end{aligned}$$

As we expect, the first part is from the zero flux $b = 0$, which is the normal untwisted 2+1D $(Z_n)^4$ gauge theory (toric code) as $\mathcal{C}^{2D}_{(Z_n)^4}$ with $|G|^2 = n^8$ on 2-torus. The remaining $(|G| - 1)$ copies are inserted with nonzero flux ($b \neq 0$) as $\mathcal{C}^{2D}_{(Z_n)^4,\omega_3}$ with Type III 3-cocycle twists of Table XII. In some case but not all cases, $\mathcal{C}^{2D}_{(Z_n)^4,\omega_3}$ is $\mathcal{C}^{2D}_{(Z_n)_{\text{untwist}} \times (Z_n)^3_{\text{twist}},\omega_3}$. In either case, the $\text{GSD}_{b,\mathbb{T}^2}$ for $b \neq 0$ has the same decomposition always equivalent to a untwisted Z_n gauge theory with $GSD_{\mathbb{T}^2} = n^2$ direct product with

$$\text{GSD}_{\mathbb{T}^2,\omega_{3,\text{III}}} = (1 \cdot n^3 + (n^3 - 1) \cdot n) \quad (48)$$

$$\equiv \text{GSD}^{Abel}_{\mathbb{T}^2,\omega_{3,\text{III}}} + \text{GSD}^{nAbel}_{\mathbb{T}^2,\omega_{3,\text{III}}}, \quad (49)$$

which we generalize the result derived for 2+1D Type III ω_3 twisted theory with $G = (Z_2)^3$ in Ref.42 to $G = (Z_n)^3$ of a prime n. One can repeat the counting for 2+1D as Eq.(44)(45), see Appendix A.

To summarize, from the GSD counting, we already foresee there exist **non-Abelian strings in 3+1D Type IV twisted gauge theory, with a quantum dimension n.** Those non-Abelian strings (fluxes) carries dim(Rep) $= n$ non-Abelian charges. Since charges are sourced by particles, those **non-Abelian strings are not pure strings but attached with non-Abelian particles.** (For a projection perspective from 3D to 2D, a nonAbelain string of $\mathcal{C}^{3\text{D}}$ is a non-Abelain dyon with both charge and flux of $\mathcal{C}_b^{2\text{D}}$.)

Modular T^{xy} of 3D-

We shall compute $\mathsf{T}^{xy}, \mathsf{S}^{xyz}$ using the formula derived in Sec.III B for Type IV ω_4 theory (for generality, we also include the twists by Type II, III ω_4). Due to the large GSD and the quantum dimension of non-Abelian nature, we focus on a simplest example $G = (Z_2)^4$ theory to have the smallest amount of data. By $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{21}$, we have 2^{21} types of theories, where 2^{20} types with Type IV are endorsed with non-Abelian statistics. (While 2^{20} types are Abelian gauge theories of non-Type IV have their T, S data in Sec.IV B.) For $G = (Z_2)^4$, there are still $\text{GSD}_{\mathbb{T}^3,\text{IV}} = 1576$. Thus both T and S are matrices with the rank 1576. T^{xy} has 1576 components along diagonal.

For $G = (Z_2)^4$, we first define a quantity η_{g_1,g_2,g_3} of convenience from the $\mathsf{C}_{a,b}^{(2)}(c,d)$ in Eq.(15),

$$\eta_{g_1,g_2,g_3} \equiv \begin{cases} 0, & \text{if } \mathsf{C}_{g_1,g_2}^{(2)}(g_3,g_3) = +1 \\ 1, & \text{if } \mathsf{C}_{g_1,g_2}^{(2)}(g_3,g_3) = -1 \end{cases} \tag{50}$$

Below the p_{lm}, p_{lmn} are the shorthand of Type II, III (both 1st, 2nd) topological term labels, the $p_{lm} f_{lm}(a,b,c)$, $p_{lmn} f_{lmn}(a,b,c)$ abbreviate the function forms in the exponents of Type II, III ω^4 in Table I. Namely, we regard their 4-cocycle $\omega_4(a,b,c,d)$ as a trivial 2-cocycle $\mathsf{c}_{a,b}(c,d)$ written as $\mathsf{c}_{a,b}(c,d) = \frac{\eta_{a,b}(c)\eta_{a,b}(d)}{\eta_{a,b}(c+d)}$, where $\eta_{a,b}(c)$ is a 1-cochain: $\eta_{a,b}(c) = \exp(\text{i} p_{lm} f_{lm}(a,b,c)) = \exp(\frac{2\pi \text{i}}{N_{lm} N_m} p_{lm} a_l b_m c_m))$ for Type II case. $\eta_{a,b}(c) = \exp(\text{i} p_{lmn} f_{lmn}(a,b,c)) = \exp(\frac{2\pi \text{i}}{N_{lm} N_n} p_{lmn} a_l b_m c_n))$ for Type III case. We derive $\mathsf{T}^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z}$ of Eq.(31) in Table V.

Excitations (α, a, b)	$\mathsf{T}_\alpha^{a,b}$
$(\alpha, F(j_{Abel}))$	$\exp\big(\sum_{k=1}^4 \pi \text{i}\ \alpha_k a_k\big) \quad \to$ e.g. ± 1
$(((\pm)_a, (\pm)_b), F(j_{nAbel}))$	$e^{\text{i}\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(a,b,a) + p_{lmn} f_{lmn}(a,b,a))} (\pm)_a (\pm)_b (\text{i})^{\eta_{a,b,a}} \quad \to$ e.g. ± 1 or $\pm\text{i}$

TABLE V. $\mathrm{SL}(3,\mathbb{Z})$ modular data $\mathsf{T}^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z}$ for the $(Z_2)^4$ theory with Type IV ω^4. The formula of T^{xy} is separated to two sets: the first set with 736 components (from the sector $\text{GSD}_{\mathbb{T}^3,\text{IV}}^{Abel}$) and another 840 components (from the sector $\text{GSD}_{\mathbb{T}^3,\text{IV}}^{nAbel}$). $F = (a_i, b_i)$ are fluxes with 8 components, $(a_1, a_2, a_3, a_4) \in (Z_2)^4$ and $(b_1, b_2, b_3, b_4) \in (Z_2)^4$. The number of distinct fluxes in $F(j_{Abel})$ is $46 (= \text{Flux}_{\text{IV}}^{Abel})$, the number of distinct fluxes $F(j_{nAbel})$ is $210 (= \text{Flux}_{\text{IV}}^{nAbel})$. This table lists *all* 2^{20} kinds of T^{xy} for the non-Abelian theories in $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{21}$ (half of 2^{21}). $((\pm)_a, (\pm)_b)$ pair makes up the numbers of charge Rep $n^2 = 2^2$ in Eq.(45). Details of the rank-2 matrix Rep are given in Appendix A.

Modular S^{xyz} of 3D-

The S^{xyz} matrix has 1576×1576 components. We organize S^{xyz} into four blocks, denoting $(n)Abel$ for (non)Abelian with 736 (840) components. Defining $\mathsf{S}_{(\alpha,a,b)(\beta,c,d)}^{xyz} \equiv \frac{1}{|G|} \mathsf{S}_{a,b,d}^{\alpha,\beta} \delta_{b,c}$, we obtain:

$$\mathsf{S}^{xyz} = \frac{1}{|G|} \begin{array}{c} \begin{array}{cc} \text{736 components} & \text{840 components} \\ {\scriptstyle (\beta_1,\beta_2,\beta_3,\beta_4,\ c,d)} & {\scriptstyle ((\pm)_c,(\pm)_d,\ c,d)} \end{array} \\ \left(\begin{array}{c:c} \mathsf{S}_{Abel,Abel} & \mathsf{S}_{Abel,nAbel} \\ \hdashline \mathsf{S}_{nAbel,Abel} & \mathsf{S}_{nAbel,nAbel} \end{array}\right) \end{array} \begin{array}{l} {\scriptstyle (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\ a,b)} \\ {\scriptstyle ((\pm)_a,(\pm)_b,\ a,b)} \end{array} \tag{51}$$

$$\begin{cases} \mathsf{S}_{Abel,Abel} = & 1 \cdot \exp(\sum_k \frac{2\pi \text{i}}{N_k}(-\alpha_k d_k + \beta_k a_k)) \cdot \delta_{b,c} = (-1)^{(-\alpha_k d_k + \beta_k a_k)} \cdot \delta_{b,c}, \\ \mathsf{S}_{Abel,nAbel} = & 2 \cdot (-1)^{(-\alpha_k d_k)} \cdot e^{\text{i}\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(b,d,a) + p_{lmn} f_{lmn}(b,d,a))} (\pm 1)_b (\pm 1)_d \cdot (\text{i})^{\eta_{b,d,a}} \delta_{a\in\{\mathbf{1},b,d,bd\}} \cdot \delta_{b,c}, \\ \mathsf{S}_{nAbel,Abel} = & 2 \cdot (-1)^{(\beta_k a_k)} \cdot e^{-\text{i}\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(a,b,d) + p_{lmn} f_{lmn}(a,b,d))} (\pm 1)_a (\pm 1)_b \cdot (\text{i})^{\eta_{a,b,d}} \delta_{d\in\{\mathbf{1},a,b,ab\}} \cdot \delta_{b,c}, \\ \mathsf{S}_{nAbel,nAbel} = & 4 \cdot e^{-\text{i}\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(a,b,d) + p_{lmn} f_{lmn}(a,b,d))} \cdot e^{\text{i}\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(b,d,a) + p_{lmn} f_{lmn}(b,d,a))} \\ & \cdot (\pm 1)_a (\pm 1)_b (\pm 1)_c (\pm 1)_d \cdot (-\text{i})^{\eta_{a,b,d}} \cdot (\text{i})^{\eta_{b,d,a}} \cdot \delta_{a\in\{b,d,bd\}} \cdot \delta_{d\in\{a,b,ab\}} \cdot \delta_{b,c}. \end{cases} \tag{52}$$

The $\exp(\sum_k \frac{2\pi \text{i}}{N_k}(-\alpha_k d_k + \beta_k a_k))$ factor in the top-left block shows the pure-particle pure-string braiding of untwisted Z_N gauge theory (no ω_4 dependence). We define $\delta_{a\in\{b,d,bd\}}=1$ if $a\in\{b,d,bd\}$, otherwise $\delta_{a\in\{b,d,bd\}}=0$. Some other technical details follow: for $G=(Z_2)^4$, the constraint $\delta_{a\in\{b,d,bd\}}\cdot\delta_{d\in\{a,b,ab\}}$ reduces to $\delta_{d\in\{a,ab\}}$. The survival nonzero $\mathsf{S}_{nAbel,nAbel}$ are only in two kinds of forms, either $d=a$ or $d=ab$.

$$\mathsf{S}_{nAbel,nAbel}=\begin{cases}\mathsf{S}^{\alpha,\beta}_{a,b,a}\delta_{b,c}\delta_{d,a}\\ \mathsf{S}^{\alpha,\beta}_{a,b,ab}\delta_{b,c}\delta_{d,ab}\end{cases} \tag{53}$$

Some remarks follow:

(1) **Dimensional reduction from 3D to 2D sectors with** b **flux**: From the above $\mathsf{S}^{xyz},\mathsf{T}^{xy}$, there is no difficulty deriving S^{xy} from Eq.(32). From all these modular data $\mathsf{S}^{xy}_b,\mathsf{T}^{xy}_b$ data, we find consistency with the dimensional reduction of 3D topological order by comparison with induced 3-cocycle ω_3 from ω_4. Let us consider a single specific example, given the Type IV $p_{1234}=1$ and other zero Type II,III indices $p_{..}=p_{...}=0$,

$$\begin{aligned}&\mathcal{C}^{3\text{D}}_{(Z_2)^4,\omega_{4,\text{IV}}}=\oplus_b\mathcal{C}^{2\text{D}}_b \\ &=\mathcal{C}^{2\text{D}}_{(Z_2)^4}\oplus 10\,\mathcal{C}^{2\text{D}}_{(Z_2)\times(Z_2)^3_{(ijl)},\omega^{(ijl)}_{3,\text{III}}}\oplus 5\mathcal{C}^{2\text{D}}_{(Z_2)^4,\omega_{3,\text{III}}\times\omega_{3,\text{III}}\times\dots} \\ &=\mathcal{C}^{2\text{D}}_{(Z_2)^4}\oplus 10\,\mathcal{C}^{2\text{D}}_{(Z_2)\times(D_4)}\oplus 5\mathcal{C}^{2\text{D}}_{(Z_2)^4,\omega_{3,\text{III}}\times\omega_{3,\text{III}}\times\dots}\end{aligned} \tag{54}$$

The $\mathcal{C}^{2\text{D}}_{(Z_2)^4}$ again is the normal $(Z_2)^4$ gauge theory at $b=0$. The 10 copies of $\mathcal{C}^{2\text{D}}_{(Z_2)\times(D_4)}$ have an untwisted dihedral D_4 gauge theory ($|D_4|=8$) product with the normal (Z_2) gauge theory. The duality to D_4 theory in 2D can be expected,[42] see Table VI. (As a byproduct of our work, we go beyond Ref.42 to give the complete classification of all twisted 2D ω_3 of $G=(Z_2)^3$ and their corresponding topological orders and twisted quantum double $D^\omega(G)$ in Appendix.A.) The remaining 5 copies $\mathcal{C}^{2\text{D}}_{(Z_2)^4,\omega_{3,\text{III}}\times\omega_{3,\text{III}}\times\dots}$ must contain the twist on the full group $(Z_2)^4$, not just its subgroup. This peculiar feature suggests the following remark.

(2) Sometimes there may exist a duality between a twisted Abelian gauge theory and a untwisted non-Abelian gauge theory,[42] one may wonder whether one can find a dual non-Abelian gauge theory for $\mathcal{C}^{3\text{D}}_{(Z_2)^4,\omega_{4,\text{IV}}}$? We find that, however, $\mathcal{C}^{\mathbf{3D}}_{(Z_2)^4,\omega_{4,\mathbf{IV}}}$ **cannot be dual to a normal gauge theory (neither Abelian nor non-Abelian), but must be a twisted (Abelian or non-Abelian) gauge theory**. The reason is more involved. Let us first recall the more familiar 2D case. One can consider $G=(Z_2)^3$ example with $\mathcal{C}^{2\text{D}}_{(Z_2)^3,\omega_3}$, with $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^7$. There are 2^6 for non-Abelian types with Type III ω_3 (the other 2^6 Abelian without with Type III ω_3). We find the 64 non-Abelian **types** of 3-cocycles ω_3 go to 5 **classes** labeled $\omega_3[1]$, $\omega_3[3d]$, $\omega_3[3i]$, $\omega_3[5]$ and $\omega_3[7]$, and their twisted quantum double model $D^\omega(G)$ are shown in Table VI. The number in the bracket [..] is related to the number of pairs of $\pm$i in the T matrix and the d/i stand for the linear dependence(d)/independence(i) of fluxes generating cocycles.

Class	Twisted quantum double $D^\omega(G)$	Number of Types
$\omega_3[1]$	$D^{\omega_3[1]}({Z_2}^3)$, $D(D_4)$	7
$\omega_3[3d]$	$D^{\omega_3[3d]}({Z_2}^3)$, $D^{\gamma^4}(Q_8)$	7
$\omega_3[3i]$	$D^{\omega_3[3i]}({Z_2}^3)$, $D(Q_8)$, $D^{\alpha_1}(D_4)$, $D^{\alpha_2}(D_4)$	28
$\omega_3[5]$	$D^{\omega_3[5]}({Z_2}^3)$, $D^{\alpha_1\alpha_2}(D_4)$	21
$\omega_3[7]$	$D^{\omega_3[7]}({Z_2}^3)$	1

TABLE VI. $D^\omega(G)$, the twisted quantum double model of G in 2+1D, and their 3-cocycles ω_3(involving Type III) types in $\mathcal{C}^{2\text{D}}_{(Z_2)^3,\omega_3}$. We classify the 64 types of 2D non-Abelian twisted gauge theories to 5 classes, which agree with Ref.62. Each class has distinct non-Abelian statistics. Both dihedral group D_4 and quaternion group Q_8 are non-Abelian groups of order 8, as $|D_4|=|Q_8|=|(Z_2)^3|=8$. $D^\omega(G)$ data can be found in Ref.62. Details are reserved to Appendix A.

From Table VI, we show that two classes of 3-cocycles for $D^{\omega_3}(Z_2)^3$ of 2D can have dual descriptions by gauge theory of non-Abelian dihedral group D_4, quaternion group Q_8. However, the other three classes of 3-cocycles for $D^{\omega_3}(Z_2)^3$ do not have a dual (untwisted) non-Abelian gauge theory.

Now let us go back to consider 3D $\mathcal{C}^{3\text{D}}_{G,\omega_{4,\text{IV}}}$, with $|Z_2|^4=16$. From Ref.39, we know 3+1D D_4 gauge theory has decomposition by its 5 centralizers. Apply the rule of decomposition to other groups, it implies that for untwisted group G in 3D $\mathcal{C}^{3\text{D}}_G$, we can decompose it into sectors of $\mathcal{C}^{2\text{D}}_{G_b,b}$, here G_b becomes the **centralizer** of the **conjugacy class**(flux) b: $\mathcal{C}^{3\text{D}}_G=\oplus_b\mathcal{C}^{2\text{D}}_{G_b,b}$. Some useful information is:

$$\mathcal{C}^{3\text{D}}_{(Z_2)^4}=16\mathcal{C}^{2\text{D}}_{(Z_2)^4} \tag{55}$$
$$\mathcal{C}^{3\text{D}}_{D_4}=2\mathcal{C}^{2\text{D}}_{D_4}\oplus 2\mathcal{C}^{2\text{D}}_{(Z_2)^2}\oplus\mathcal{C}^{2\text{D}}_{Z_4}, \tag{56}$$
$$\mathcal{C}^{3\text{D}}_{Z_2\times D_4}=4\mathcal{C}^{2\text{D}}_{Z_2\times D_4}\oplus 4\mathcal{C}^{2\text{D}}_{(Z_2)^3}\oplus 2\mathcal{C}^{2\text{D}}_{Z_2\times Z_4}, \tag{57}$$
$$\mathcal{C}^{3\text{D}}_{Q_8}=2\mathcal{C}^{2\text{D}}_{Q_8}\oplus 3\mathcal{C}^{2\text{D}}_{Z_4}, \tag{58}$$
$$\mathcal{C}^{3\text{D}}_{Z_2\times Q_8}=4\mathcal{C}^{2\text{D}}_{Z_2\times Q_8}\oplus 6\mathcal{C}^{2\text{D}}_{Z_2\times Z_4}. \tag{59}$$

and we find that no such decomposition is possible from $|G| = 16$ group to match Eq.(54)'s. Furthermore, if there exists a non-Abelian G_{nAbel} to have Eq.(54), those $(Z_2)^4$, $(Z_2) \times (D_4)$ or the twisted $(Z_2)^4$ must be the centralizers of G_{nAbel}. But one of the centralizers (the centralizer of the identity element as a conjugacy class $b = 0$) of G_{nAbel} must be G_{nAbel} itself, which has already ruled out from Eq.(55),(57). Thus, we prove that $\mathcal{C}^{\mathbf{3D}}_{(Z_2)^4,\omega_{4,\mathbf{IV}}}$ **is not a normal 3+1D gauge theory (not $Z_2 \times D_4$, neither Abelian nor non-Abelian) but must only be a twisted gauge theory.**

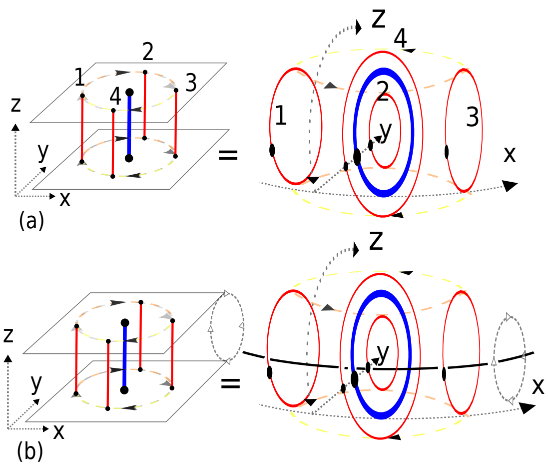


FIG. 9. For 3+1D Type IV $\omega_{4,\text{IV}}$ twisted gauge theory $\mathcal{C}^{\text{3D}}_{G,\omega_{4,\text{IV}}}$: (a) **Two-string statistics in unlink 0_1^2 configuration is Abelian**. (The $b = 0$ sector as $\mathcal{C}^{\text{2D}}_G$.) (b) **Three-string statistics in two Hopf links $2_1^2\#2_1^2$ configuration is non-Abelian**. (The $b \neq 0$ sector in $\mathcal{C}^{\text{2D}}_b = \mathcal{C}^{\text{2D}}_{G,\omega_{3,\text{III}}}$.) The $b \neq 0$ flux sector creates a monodromy effectively acting as the third (black) string threading the two (red,blue) strings.

(3) We discover that, see Fig.9, for any twisted gauge theory $\mathcal{C}^{\text{3D}}_{G,(\omega_{4,\text{IV}}\cdot\omega_{4,..})}$ with Type IV 4-cocycle $\omega_{4,\text{IV}}$ (whose non-Abelian nature is not affected by adding other Type II,III $\omega_{4,..}$), by **threading a third string through two-string unlink 0_1^2 into three-string Hopf links $2_1^2\#2_1^2$ configuration, Abelian two-string statistics is promoted to non-Abelian three-string statistics.** We can see the physics from Eq.(54), the $\mathcal{C}^{\text{2D}}_b$ is Abelian in $b = 0$ sector; but non-Abelian in $b \neq 0$ sector. The physics of Fig.9 is then obvious, by applying our discussion in Sec.III C about the equivalence between string-threading and the $b \neq 0$ monodromy causes a branch cut.

(4) **Cyclic relation for non-Abelian S^{xyz} in 3D**: Interestingly, for the $(Z_2)^4$ twisted gauge theory with non-Abelian statistics, we find that a similar cyclic relation Eq.(41) still holds as long as two conditions are satisfied: (i) the charge labels are equivalent $\alpha = \beta$ and (ii) $\delta_{a\in\{b,d,bd\}} \cdot \delta_{d\in\{a,b,ab\}} \cdot \delta_{b\in\{d,a,da\}} = 1$. However, Eq.(41) is modified with a factor depending on the dimensionality of Rep α:

$$\mathsf{S}^{\alpha,\alpha}_{a,b,d} \cdot \mathsf{S}^{\alpha,\alpha}_{b,d,a} \cdot \mathsf{S}^{\alpha,\alpha}_{d,a,b} \cdot |\dim(\alpha)|^{-3} = 1. \tag{60}$$

This identity should hold for any Type IV non-Abelian strings. This is a cyclic relation of 3D nature, instead of a dimensional-reducing 2D nature of $\mathsf{S}^{\text{2D}\,\alpha,\beta}_{a,c\;(b)}$ in Fig.2.

V. CONCLUSION

A. Knot and Link configuration

Throughout our work, we have been indicating that the mathematics of knots and links may be helpful in organizing our string-braiding patterns in 3D. Here we illustrate them more systematically. We will use Alexander-Briggs notation for the knots and links, see Fig.10.

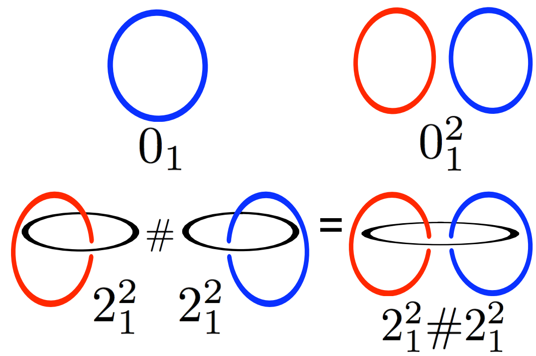


FIG. 10. Under Alexander-Briggs notation, an unknot is 0_1, and two unknots can form an unlink 0_1^2. A Hopf link is 2_1^2, and the connected sum of two Hopf links is $2_1^2\#2_1^2$.

The knots and links for our string-braiding patterns are organized into Table VII. We recall that, in Sec.III C, the **topological spin** for a closed string in the $b = 0$ flux sector of $\mathcal{C}^{\text{2D}}_b$ does a self-2π flipping under the 0_1 unknot configuration. Due to our **spin-statistics relation** of a closed string, we can view the topological spin of the $b = 0$ sector as the **exchange statistics** of two identical strings in 0_1^2 unlink configuration. On the other hand, for the **topological spin** in the $b \neq 0$ flux sector, we effectively thread a (black) string through the (blue) unknot, which forms a Hopf link 2_1^2. Meanwhile, we can view the topological spin of $b \neq 0$ sector as the **exchange statistics** of two identical strings threaded by a third (black) string in a connected sum of two Hopf links $2_1^2\#2_1^2$ configuration. Furthermore, we can promote two-string Abelian statistics under the 0_1^2 unlink of $b = 0$ sector to three-string Abelian (in Sec.IV B) or non-Abelian (in Sec.IV C) statistics under Hopf links $2_1^2\#2_1^2$ of $b \neq 0$ sector.

Nothing prevents us from considering more generic knot and link patterns for three-string or multi-string braiding. Our reason is here - From the full modular $\text{SL}(3,\mathbb{Z})$ group viewpoint, the S^{xyz} is a necessary

$\mathcal{C}_b^{2\text{D}}$	Physics of Strings	Knots and Links
$b=0$	topological spin (T)	0_1
	exchange statistics	0_1^2
	2-string braiding	0_1^2
$b \neq 0$	topological spin (T)	2_1^2
	exchange statistics	$2_1^2\#2_1^2$
	3-string braiding	$2_1^2\#2_1^2, \ldots$

TABLE VII. Various string-braiding patterns in terms of knots and links in Alexander-Briggs notation: the topological spin of a loop, the exchange/braiding statistics of two loops without any background string inserted ($b=0$ sector) or with another background string inserted ($b \neq 0$ sector). Here we effectively view the string braiding statistics of 3D topological order in terms of 2D sectors: $\mathcal{C}^{3\text{D}} = \oplus_b \mathcal{C}_b^{2\text{D}}$.

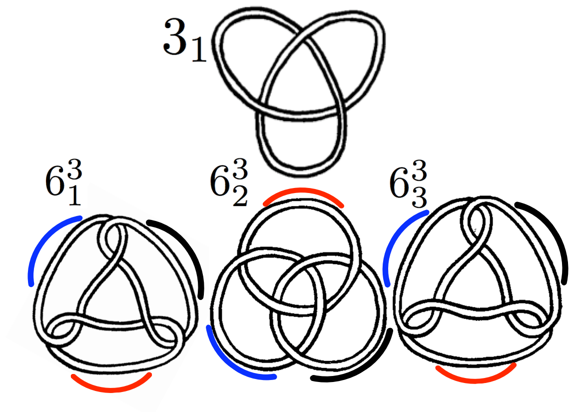


FIG. 11. The trefoil knot is 3_1. Some other simplest 3-string links (beyond Hopf links $2_1^2\#2_1^2$) are shown: 6_1^3, 6_2^3 (Borromean rings), 6_3^3. From the spin-statistics relation of a closed string discussed in Sec.III C, where the topological spin of certain knot/link configurations (0_1 for the monodromy flux $b=0$ and 2_1^2 for $b \neq 0$) is equivalent to the exchange statistics of certain knot/link configurations (0_1^2 for $b=0$ and $2_1^2\#2_1^2$ for $b \neq 0$) under Eq.(35). Therefore, we may further conjecture that the topological spin of a trefoil knot 3_1 may relate to the braiding statistics of 6_1^3, 6_2^3, 6_3^3.

generator to access the full data of the $\mathrm{SL}(3,\mathbb{Z})$ group. However, we have learned that our 3D to 2D reduction by Eq.(4) using $\mathrm{SL}(2,\mathbb{Z})$ subgroup's data S^{xy} and T^{xy} already encode all the physics of braidings under the simplest knots and links in Fig.10 - These include self-flipping topological spin and exchange/braiding statistics (Sec.III C,IV). It suggests that S^{xyz} contains more than these string-braiding configurations. In addition, there are more generic Mapping Class Groups $\mathrm{MCG}(\mathcal{M}_{space})$ beyond $\mathrm{MCG}(\mathbb{T}^3) = \mathrm{SL}(3,\mathbb{Z})$, which potentially encode more exotic multi-string braidings.

Indeed, as noted in Sec.IV, the 3D S matrix essentially contains the information on three fluxes $(d,a,b) = (d_x, a_y, b_z)$ in Eq.(38), $\mathsf{S}^{xyz} = \mathsf{S}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{bc}$. Since strings carry fluxes in 3D, this further suggests that we should look for the braiding involving with three strings, where the 3-loop braiding has also been recently emphasized in Ref.40 and 41.

The configuration we have studied so far with three strings is the Hopf link $2_1^2\#2_1^2$. We propose that using more general three strings pattern, such as the link

$$\mathcal{N}_m^3$$

or its connected sum to study topological states. ($\mathcal{N}_m^3$ is in Alexander-Briggs notation, here 3 means that there are three closed loops, $\mathcal{N}$ means the crossing number, and m is the label for different kinds for $\mathcal{N}^3$ linking.) For example, three-string braiding can include links of 6_1^3, 6_2^3, 6_3^3 in Fig.11. Configurations in Fig.11 are potentially promising for studying the braiding statistics of strings to classify or characterize topological orders.

To examine whether the *multi-string braiding* is topologically well-defined, we propose a way to check that (such as the braiding processes in Fig.9,11):

"*The path that one (red) loop A winds around another (blue) loop B along the time evolution is nontrivial in the complement space of the B and the base (black) loop C. Namely, the path of A needs to be a nontrivial element of the fundamental group for the complement space of B and C. Thus the path needs to be homotopically nontrivial.*"

Before concluding this subsection, another final remark is that in Sec.III C 3, we mention generalizing the framed worldline picture of particles in Fig.5 to the framed worldsheet picture of closed-strings. (ps. The framed worldline is like a worldsheet, the framed worldsheet is like a worldvolume.) Thus, it may be interesting to study how incorporating the framing of particles and strings (with worldline/worldsheet/worldvolume) can provide richer physics and textures into the knot-link pattern.

B. Cyclic identity for Abelian and non-Abelian strings

In Secs.IV B and IV C, we discuss *cyclic identity* for Abelian and non-Abelian strings particularly for 3+1D twisted gauge theories. We find Eq.(60), "*Cyclic identity of 3D's* S^{xyz} *matrix* of Eq.(38) $\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{b,c}$":

$$\boxed{\mathsf{S}^{\alpha,\alpha}_{a,b,d} \cdot \mathsf{S}^{\alpha,\alpha}_{b,d,a} \cdot \mathsf{S}^{\alpha,\alpha}_{d,a,b} \cdot |\dim(\alpha)|^{-3} = 1}. \tag{61}$$

For the Abelian case, the dimension of Rep is simply $\dim(\alpha)=1$, which reduces to Eq.(41).

On the other hand, we find that there is also another cyclic identity, based on 2D's $\mathsf{S}^{xy}_b = \mathsf{S}^{xy}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)}\delta_{b,d}$ matrix of Eq.(34), written in terms of $\mathsf{S}^{\text{2D }\alpha,\beta}_{a,c\ (b)}$, at least for Abelian strings of Type II, III 4-cocycle twists, namely

"*Cyclic identity of 2D's* S^{xy} *matrix*":

$$\boxed{\mathsf{S}^{\text{2D }0,0}_{a_i,c_k\ (b_j)} \cdot \mathsf{S}^{\text{2D }0,0}_{c_k,b_j\ (a_i)} \cdot \mathsf{S}^{\text{2D }0,0}_{b_j,a_i\ (c_k)} = 1}. \tag{62}$$

This Eq.(62) cyclic identity has two additional criteria: (1) Here $\alpha = \beta = 0$ means that all strings must have zero charges. (2) In addition, the $\prod_i Z_{N_i}$ flux labels a_i, b_j, c_k must satisfy that $a_i = |a|\hat{e}_i$, $b_j = |b|\hat{e}_j$, $c_k = |c|\hat{e}_k$, as a multiple of a single unit flux, each only carrying one of $\prod_i Z_{N_i}$ fluxes. Note that $\hat{e}_j \equiv (0,\dots,0,1,0,\dots,0)$ is defined to be a unit vector with a nonzero component in the j-th component for the Z_{N_j} flux. Eq.(62) is true even in the non-canonical basis, such as the case for the b-flux sector in Table IV. Thus, the fact whether in the canonical basis[60] or not does not affect the identity Eq.(62), at least for the example of Abelian Type II, III 4-cocycles.

This 2D's S^{xy}_b cyclic identity in Eq.(62) is indeed the cyclic relation of Ref.40. From the fact that we associate 2D's S^{xy}_b matrix to the dimensional reduction of string braiding in Fig.2, it shows that the Abelian statistical angle $\theta_{a_i,c_k,(b_j)}$ can be defined, up to a basis,[60] as

$$\mathsf{S}^{\text{2D }0,0}_{a_i,c_k\ (b_j)} = \exp(\mathrm{i}\ \theta_{a_i,c_k,(b_j)}). \tag{63}$$

Thus Eq.(62) implies a cyclic relation for Abelian statistical angles:

$$\theta_{a_i,c_k,(b_j)} + \theta_{c_k,b_j,(a_i)} + \theta_{b_j,a_i,(c_k)} = 0 \pmod{2\pi}. \tag{64}$$

In contrast, the 3D cyclic relation works for both Abelian and non-Abelian strings, and it is not restricted to zero charge but only for equal charges $\alpha = \beta$. More importantly, Eq.(61) allows any flux for each a, b, c, instead of being limited to a single unit flux or a multiple of a single unit flux in Eq.(62).

C. Main results

We have studied string and particle excitations in 3+1D twisted discrete gauge theories, which belong to a class of topological orders. These 3D theories are gapped topological systems with topology-dependent ground state degeneracy. The twisted gauge theory contains its data of gauge group G and 4-cocycle twist $\omega_4 \in \mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ of G's fourth cohomology group. Such a data provides many **types** of theories, however several types of theories belong to the same **class** of a topological order. To better characterize and classify topological orders, we use the mapping class group on the $\mathbb{T}^3$ torus, by $\mathrm{MCG}(\mathbb{T}^d) = \mathrm{SL}(d,\mathbb{Z})$, to extract the $\mathrm{SL}(3,\mathbb{Z})$ modular data S^{xyz} and T^{xy} in the ground state sectors, which however reveal information of gapped excitations of particles and strings. We have posed five main questions **Q1-Q5** and other sub-questions throughout our work, and have addressed each of them in some depth. We summarize our results and approaches below, and make comparisons with some recent works:

(1) **Dimensional Reduction**: By inserting a gauge flux b into a compactified circle z of 3D topological order $\mathcal{C}^{\text{3D}}$, we can realize $\mathcal{C}^{\text{3D}} = \oplus_b \mathcal{C}^{\text{2D}}_b$, where $\mathcal{C}^{\text{3D}}$ becomes a direct sum of degenerate states of 2D topological orders $\mathcal{C}^{\text{2D}}_b$ in different flux b sectors. We should emphasize that this dimensional reduction is **not** *real space decomposition along the z direction*, **but** *the decomposition in the Hilbert space of ground states* (excitations basis such as the canonical basis of Eq.(11)). We propose that this decomposition in Eq.(5) will work for a generic topological order without a gauge group description. In the most general case, b becomes the certain basis label of the Hilbert space. The recent study of Ref.39 implements the dimensional reduction idea on the *normal gauge theories* described by the 3D Kitaev Z_N toric code and 3D quantum double models *without cocycle twists* using the *spatial Hamiltonian* approach. In our work, we consider more generic *twisted gauge theories* with a lattice realization in the twisted 3D quantum double models under the framework of Dijkgraaf-Witten theory.[36] We apply both the *spatial Hamiltonian* approach and the *spacetime path integral* approach.

(2) **Modular Data**: We find explicit formula representations of the $\mathrm{SL}(3,\mathbb{Z})$ modular data S and T using (i) path integral and cocycle approach, and (ii) Representation(Rep) theory approach. The Rep theory approach is convenient, and perhaps contains more general and simplified expressions. While recent work either focuses on Abelian statistics[40,41] or focuses on normal gauge theories,[39] our formula embodies generic non-Abelian twisted gauge theories and thus is the most powerful.

(3) **Classification and Characterization**: We use the modular data S and T to characterize the braiding statistics of some 2D and 3D topological orders. We can further use the modular data S and T taking into account excitation-relabeling to classify (or partially classify) topological orders. Explicit 2D examples are $G = (Z_2)^3$ twisted gauge theories, and 3D examples are $G = (Z_2)^4$ twisted gauge theories. Some of our results are compared with the mathematics literature in the Appendix A. Some of 2D results are compared to twisted quantum double models $D^\omega(G)$.

Our result can also facilitate the study of symmetric protected topological states (SPTs) protected by a global symmetry G_s.[52] By gauging the G_s symmetry of SPTs, one can use the induced dynamical gauged theory to study the braiding of excitations and to characterize

Braiding statistics - (G,ω_4) of $\mathcal{C}^{\rm 3D}_{G,\omega_4}=\oplus_b \mathcal{C}^{\rm 2D}_b$	$b=0$ braiding 2-strings 0_1^2	$b\neq 0$ braiding 3-strings $2_1^2\#2_1^2$
$(G_{Abel},1)$	Abelian st	Abelian st
$(G_{Abel},\text{without } \omega_{4,\rm IV})$	Abelian st	Abelian st
$(G_{Abel},\text{with } \omega_{4,\rm IV})$	Abelian st	non-Abelian st
$(G_{nAbel},1)$	non-Abelian st	(non)Abelian st
(G_{nAbel},ω)	non-Abelian st	non-Abelian st

TABLE VIII. Braiding statistics, Abelian or non-Abelian, in terms of (G,ω_4), a gauge group G and a cocycle twist ω_4 of a 3D topological order $\mathcal{C}^{\rm 3D}_{G,\omega_4}$. Here G_{Abel} means Abelian G, G_{nAbel} means non-Abelian G, and st means statistics. A normal gauge theory has $\omega_4=1$ with no cocycle twist. The (non)Abelian st means it can be either non-Abelian or pure Abelian statistics. For example, any $b\neq 0$ sector of an untwisted S_3 gauge theory has pure Abelian statistics, because the S_3's centralizers of non-indentity elements are Abelian. However, some $b\neq 0$ sector of untwisted D_4 and Q_8 gauge theories have non-Abelian statistics, because the D_4 and Q_8's centralizers of non-indentity elements can be non-Abelian. The $b=0$ 2-strings 0_1^2 braiding is the process of Fig.9 (a). The $b\neq 0$ 3-strings $2_1^2\#2_1^2$ braiding is the process of Fig.9 (b)

.

SPTs.[40,63–65]

(4) **Physics of string and particle braiding**: We provide the physics meaning of the **topological spin** and **spin-statistics** relation for a closed string. We also interpret the 3-string braiding statistics first studied in Ref.40 from a new perspective - a dimensional reduction with b flux monodromy. We find that **with Type IV 4-cocycle twist for the twisted gauge theory**, by threading a third string through the two-string unlink 0_1^2 into the three-string Hopf links $2_1^2\#2_1^2$ configuration, **Abelian two-string statistics is promoted to non-Abelian three-string statistics**. In Ref.39, an effect somewhat the opposite of ours is found: where the normal (untwisted) non-Abelian 3D topological order has found with non-Abelian statistics in the $b=0$ sector, but there may be Abelian statistics in the $b\neq 0$ sector. Incorporate this understanding, we have the more unified picture organized in Table VIII, for the string-braiding statistics of twisted/untwisted Abelian/non-Abelian gauge theories as topological orders. Since the string deformation on the lattice can blur the Abelian U(1) phase, our non-Abelian string-braiding statistics provides a better alternative for a robust physical observable than Abelian string-braiding statistics[40,41] to be tested numerically or experimentally in the future. Last but not least, we propose the use of more general patterns, such as $\mathcal{N}_m^3$ (or $\mathcal{N}_m^l\#\dots$) knots/links of Alexander-Briggs to study the three-string (or multi-string) braiding statistics.

VI. ACKNOWLEDGEMENTS

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At the *Symmetry in Topological Phases* workshop at Princeton University, we became aware that the authors of Ref.40 were working on the braiding statistics of 3+1D gapped phases; their studies intersect some of ours, but also further inspire our work. During the long process of preparing our manuscript, two recent works appeared in Ref.40 and 41 dealing with the Abelian braiding statistics of twisted gauge theories, and a recent preprint[66] considering the surface topological order of SPTs with loop braiding statistics.

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Appendix

Appendix A: Group Cohomology and Cocycles

1. Cohomology group

Here we review the cohomology group $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})=\mathcal{H}^{d+1}(G,\mathrm{U}(1))$ by $\mathbb{R}/\mathbb{Z}=\mathrm{U}(1)$, as the $(d+1)$th-cohomology group of G over G module U(1). Each class in $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ corresponds to a distinct $(d+1)$-cocycles. The n-cocycle is a n-cochain additionally satisfying the n-cocycle-conditions $\delta\omega=1$. The n-cochain is a mapping $\omega(A_1,A_2,\dots,A_n)$:

$G^n \to \mathrm{U}(1)$ (which inputs $A_i \in G$, $i = 1, \dots, n$, and outputs a U(1) phase). The n-cochains satisfy the group multiplication rule:

$$(\omega_1 \cdot \omega_2)(A_1, \dots, A_n) = \omega_1(A_1, \dots, A_n) \cdot \omega_2(A_1, \dots, A_n), \tag{A1}$$

thus form a group. The coboundary operator δ

$$\delta \mathsf{c}(g_1, g_2, \dots, g_{n+1}) \equiv \mathsf{c}(g_2, \dots, g_{n+1}) \mathsf{c}(g_1, \dots, g_n)^{(-1)^{n+1}} \cdot \prod_{j=1}^{n} \mathsf{c}(g_1, \dots,\ g_j g_{j+1},\ \dots, g_{n+1})^{(-1)^j}, \tag{A2}$$

defines the n-cocycle-condition $\delta\omega = 1$ (a pentagon relation in 2D). We check the distinct n-cocycles are not equivalent by n-coboundaries. The n-cochain forms a group C^n, the n-cocycle forms a subgroup Z^n of C^n, and the n-coboundary further forms a subgroup B^n of Z^n (since $\delta^2 = 1$). Overall, it shows $\mathrm{B}^n \subset \mathrm{Z}^n \subset \mathrm{C}^n$. The n-cohomology group is exactly a relation of a kernel Z^n (the group of n-cocycles) modding out an image B^n (the group of n-coboundary):

$$\mathcal{H}^n(G, \mathrm{U}(1)) = \mathrm{Z}^n / \mathrm{B}^n. \tag{A3}$$

To derive the expression of $\mathcal{H}^d(G, \mathrm{U}(1))$ in terms of groups explicitly, we apply some key formulas:

(1). Künneth formula:
We denote R as a ring, $\mathbb{M}, \mathbb{M}'$ as the R-modules, X, X' are some chain complex. The Künneth formula shows the cohomology of a chain complex $X \times X'$ in terms of the cohomology of a chain complex X and another chain complex X'. For topological cohomology H^d, we have

$$\begin{aligned} &H^d(X \times X', \mathbb{M} \otimes_R \mathbb{M}') \\ &\simeq \Big[\oplus_{k=0}^{d} H^k(X, \mathbb{M}) \otimes_R H^{d-k}(X', \mathbb{M}') \Big] \oplus \\ &\quad \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^R(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{M}')) \Big]. \end{aligned} \tag{A4}$$

$$\begin{aligned} &H^d(X \times X', \mathbb{M}) \\ &\simeq \Big[\oplus_{k=0}^{d} H^k(X, \mathbb{M}) \otimes_{\mathbb{Z}} H^{d-k}(X', \mathbb{Z}) \Big] \oplus \\ &\quad \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{Z})) \Big]. \end{aligned} \tag{A5}$$

The above is valid for both topological cohomology H^d and group cohomology $\mathcal{H}^d$ (for G' is a finite group):

$$\begin{aligned} &\mathcal{H}^d(G \times G', \mathbb{M}) \\ &\simeq \Big[\oplus_{k=0}^{d} \mathcal{H}^k(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G', \mathbb{Z}) \Big] \oplus \\ &\quad \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{H}^k(G, \mathbb{M}), \mathcal{H}^{d-k+1}(G', \mathbb{Z})) \Big]. \end{aligned} \tag{A6}$$

(2). Universal coefficient theorem(UCT) can be derived from Künneth formula, Eq.(A5), by taking $X = 0$ or Z_1 or a point thus only $H^0(X', \mathbb{M}) = \mathbb{M}$ survives,

$$H^d(X', \mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} H^d(X', \mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{M}, H^{d+1}(X', \mathbb{Z})). \tag{A7}$$

Using UCT, we can rewrite Eq.(A5) as a decomposition below.

(3). Decomposition:

$$H^d(X \times X', \mathbb{M}) \simeq \oplus_{k=0}^{d} H^k[X, H^{d-k}(X', \mathbb{M})]. \tag{A8}$$

The above is valid for both topological cohomology and group cohomology:

$$\mathcal{H}^d(G \times G', \mathbb{M}) \simeq \oplus_{k=0}^{d} \mathcal{H}^k[G, \mathcal{H}^{d-k}(G', \mathbb{M})], \tag{A9}$$

provided that both G and G' are finite groups.

The expression of Künneth formula is in terms of the tensor-product operation $\otimes_R$ and the torsion-product Tor_1^R operation of a base ring R, which we write $\boxtimes_R \equiv \mathrm{Tor}_1^R$ as shorthand, their properties are:

$$\begin{aligned} &\mathbb{M} \otimes_{\mathbb{Z}} \mathbb{M}' \simeq \mathbb{M}' \otimes_{\mathbb{Z}} \mathbb{M}, \\ &\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M}, \\ &\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M}, \\ &\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathrm{U}(1) \simeq \mathrm{U}(1) \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0, \\ &\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\gcd(m,n)}, \\ &(\mathbb{M}' \oplus \mathbb{M}'') \otimes_R \mathbb{M} = (\mathbb{M}' \otimes_R \mathbb{M}) \oplus (\mathbb{M}'' \otimes_R \mathbb{M}), \\ &\mathbb{M} \otimes_R (\mathbb{M}' \oplus \mathbb{M}'') = (\mathbb{M} \otimes_R \mathbb{M}') \oplus (\mathbb{M} \otimes_R \mathbb{M}''); \end{aligned} \tag{A10}$$

and

$$\begin{aligned} &\mathrm{Tor}_1^R(\mathbb{M}, \mathbb{M}') \equiv \mathbb{M} \boxtimes_R \mathbb{M}', \\ &\mathbb{M} \boxtimes_R \mathbb{M}' \simeq \mathbb{M}' \boxtimes_R \mathbb{M}, \\ &\mathbb{Z} \boxtimes_{\mathbb{Z}} \mathbb{M} = \mathbb{M} \boxtimes_{\mathbb{Z}} \mathbb{Z} = 0, \\ &\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathbb{M} = \{m \in \mathbb{M} | nm = 0\}, \\ &\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathrm{U}(1) = \mathbb{Z}_n, \\ &\mathbb{Z}_m \boxtimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\langle m,n \rangle}, \\ &\mathrm{Tor}_1^{\mathbb{Z}}(U(1), U(1)) = 0, \\ &\mathbb{M}' \oplus \mathbb{M}'' \boxtimes_R \mathbb{M} = \mathbb{M}' \boxtimes_R \mathbb{M} \oplus \mathbb{M}'' \boxtimes_R \mathbb{M}, \\ &\mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M}'' = \mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M} \boxtimes_R B. \end{aligned} \tag{A11}$$

For other details, we refer the reader to Ref.52 and references therein.

We summarize some useful facts in Table IX, and some derived results in Table X.

0	$\mathcal{H}^0(G,M)=M$	$\mathcal{H}^0(G,Z)=\mathbb{Z}$	$\mathcal{H}^0(G,\mathrm{U}(1))=\mathrm{U}(1)$
1	$\mathcal{H}^1(G,M)$	$\mathcal{H}^1(G,Z)=\mathbb{Z}_1$	$\mathcal{H}^1(G,\mathrm{U}(1))=G$ (1D Rep of group)
2	$\mathcal{H}^2(G,M)$	$\mathcal{H}^2(G,Z)=\mathcal{H}^1(G,\mathrm{U}(1))$	$\mathcal{H}^2(G,\mathrm{U}(1))$ (Projective Rep of group)
3	$\mathcal{H}^3(G,M)$	$\mathcal{H}^3(G,Z)=\mathcal{H}^2(G,\mathrm{U}(1))$	
$d\geq 2$	$\mathcal{H}^d(G,M)$	$\mathcal{H}^d(G,Z)=\mathcal{H}^{d-1}(G,\mathrm{U}(1))$	

TABLE IX. Some facts about the cohomology group. For a finite Abelian group G, we have $\mathcal{H}^2(G,Z)=\mathcal{H}^1(G,\mathrm{U}(1))=G$.

	Type I	Type II	Type III	Type IV	Type V	Type VI	...	...	
	$\mathbb{Z}_{N_i}$	$\mathbb{Z}_{N_{ij}}$	$\mathbb{Z}_{N_{ijl}}$	$\mathbb{Z}_{N_{ijlm}}$	$\mathbb{Z}_{\mathrm{gcd}\otimes_i^5(N^{(i)})}$	$\mathbb{Z}_{\mathrm{gcd}\otimes_i^6(N_i)}$	$\mathbb{Z}_{\mathrm{gcd}\otimes_i^m(N_i)}$	$\mathbb{Z}_{\mathrm{gcd}\otimes_i^{d-1}N_i}$	$\mathbb{Z}_{\mathrm{gcd}\otimes_i^d N^{(i)}}$
$\mathcal{H}^1(G,\mathrm{U}(1))$	1								
$\mathcal{H}^2(G,\mathrm{U}(1))$	0	1							
$\mathcal{H}^3(G,\mathrm{U}(1))$	1	1	1						
$\mathcal{H}^4(G,\mathrm{U}(1))$	0	2	2	1					
$\mathcal{H}^5(G,\mathrm{U}(1))$	1	2	4	3	1				
$\mathcal{H}^6(G,\mathrm{U}(1))$	0	3	6	7	4	1			
$\mathcal{H}^d(G,\mathrm{U}(1))$	$\frac{(1-(-1)^d)}{2}$	$\frac{d}{2}-\frac{(1-(-1)^d)}{4}$	...	...	...	...	...	$d-2$	1

TABLE X. The table shows the exponent of the $\mathbb{Z}_{\mathrm{gcd}\otimes_i^m(N_i)}$ class in $\mathcal{H}^d(G,\mathrm{U}(1))$ for $G=\prod_{i=1}^n \mathbb{Z}_{N_i}$. We define a shorthand of $\mathbb{Z}_{\mathrm{gcd}(N_i,N_j)}\equiv\mathbb{Z}_{N_{ij}}\equiv\mathbb{Z}_{\mathrm{gcd}\otimes_i^2(N_i)}$, etc, also for other higher gcd. Our definition of the Type m is from its number m of cyclic gauge groups in the gcd class $\mathbb{Z}_{\mathrm{gcd}\otimes_i^m(N_i)}$. The number of exponents can be systematically obtained by adding all the numbers in the previous column from the top row to the row before the number that one wishes to determine. For example, our table shows that we derive that $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})=\prod_{1\leq i<j<l\leq n}\mathbb{Z}_{N_i}\times\mathbb{Z}_{N_{ij}}\times\mathbb{Z}_{N_{ijl}}$ and $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})=\prod_{1\leq i<j<l<m\leq n}(\mathbb{Z}_{N_{ij}})^2\times(\mathbb{Z}_{N_{ijl}})^2\times\mathbb{Z}_{N_{ijlm}}$, etc.

.

2. Derivation of cocycles

To derive Table X, we find that by carrying out the Künneth formula decomposition carefully for a generic finite Abelian group $G=\prod_i Z_{N_i}$, some corresponding structure becomes transparent. See Table XI.

(d+1)dim	$\mathcal{H}^{d+1}(G,\mathrm{U}(1))$	Künneth formula in $\mathcal{H}^{d+1}(G,\mathrm{U}(1))$	path integral forms in "fields"
0+1D	$\mathbb{Z}_{n_1}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1)]$
1+1D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2)]$
2+1D	$\mathbb{Z}_{n_1}$	$\mathcal{H}^3(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1dA_1)]$
2+1D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\otimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1dA_2)]$
2+1D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))]\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2A_3)]$
3+1D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^3(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2dA_2)]$
3+1D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^3(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_2A_1dA_1)]$
3+1D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\otimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))]\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int(A_1dA_2)A_3)]$
3+1D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))]\otimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int(A_1A_2)dA_3)]$
3+1D	$\mathbb{Z}_{n_{1234}}$	$[[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))]\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))]\boxtimes_{\mathbb{Z}}\mathcal{H}^1(\mathbb{Z}_{n_4},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2A_3A_4)]$

TABLE XI. Some derived facts about the cohomology group and its cocycles.

From the known field-theory facts, we know that for 2+1D twisted gauge theories from $\mathcal{H}^3(G,\mathrm{U}(1))=\prod_{1\leq i<j<l\leq m}\mathbb{Z}_{N_i}\times\mathbb{Z}_{N_{ij}}\times\mathbb{Z}_{N_{ijl}}$, their $\mathbb{Z}_{n_i}$ classes are captured by a path integral $\simeq\exp(\mathrm{i}k_{..}\int A_idA_i)$ up to some normalization factor.

(Here we omit the wedge product, denoting $A_i dA_i \equiv A_i \wedge dA_i$. We also schematically denote the quantization factor $k_{..}$, the details of $k_{..}$ level quantizations can be found in Ref.67.) The $\mathbb{Z}_{n_{jl}}$ classes are captured by a path integral $\simeq \exp(\mathrm{i}k_{..}\int A_j dA_l)$, where A is a 1-form gauge field. We deduce that the Künneth formula decomposition in $\mathcal{H}^{d+1}(G,\mathrm{U}(1))$ with the torsion product $\mathrm{Tor}_1^R \equiv \boxtimes_R$ suggests a wedge product $\wedge$ structure in the corresponding field theory, while the tensor product $\otimes_{\mathbb{Z}}$ suggests appending an extra exterior derivative $\wedge d$ structure in the corresponding field theory. For example, $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},U(1)) \to [\exp(\mathrm{i}\int A_1 \wedge A_2)]$, and $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \to [\exp(\mathrm{i}\int A_1)]$, then $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \otimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},U(1)) \to [\exp(\mathrm{i}\int A_1 \wedge dA_2)]$. Such an organization also shows the corresponding form of cocycles for 3+1D in Table I and 2+1D in Table XII. For example: The relation $A_1 \to a_1$, maps a 1-form field to a gauge flux a_1 (or a group element). The relation $dA_2 \to (b_2 + c_2 - [b_2 + c_2])$, maps an exterior derivative to the operation taking on different edges/vertices on the spacetime complex. We use this fact to determine whether two cocycles are the same forms or whether they are up to coboundaries. We comment that such a path integral so far is only suggestive, but not yet being strongly evident enough to formulate a consistent field theoretic path integral. Thus we label them with speculative quotation marks in path integral forms in "fields." The more systematic formulation in terms of field theoretic *partition functions* will be reported elsewhere in the following work in Ref.67 from the perspective of symmetric protected topological states (SPTs).

3. Dimensional reduction from a slant product

In general, for dimensional reduction of cochains, we can use the slant product mapping n-cochain c to $(n-1)$-cochain $i_g\mathsf{c}$:

$$i_g\mathsf{c}(g_1,g_2,\dots,g_{n-1}) \equiv \mathsf{c}(g,g_1,g_2,\dots,g_{n-1})^{(-1)^{n-1}} \cdot \prod_{j=1}^{n-1} \mathsf{c}(g_1,\dots,g_j,(g_1\dots g_j)^{-1}\cdot g\cdot(g_1\dots g_j),\dots,g_{n-1})^{(-1)^{n-1+j}}. \tag{A12}$$

Here we focus on the Abelian group G. For example in 2+1D, we have 3-cocycle to 2-cocycle:

$$\mathsf{C}_A(B,C) \equiv i_A\omega(B,C) = \frac{\omega(A,B,C)\omega(B,C,A)}{\omega(B,A,C)} \tag{A13}$$

In 3+1D, we have 4-cocycle to 3-cocycle:

$$\mathsf{C}_A(B,C,D) \equiv i_A\omega(B,C,D) = \frac{\omega(B,A,C,D)\omega(B,C,D,A)}{\omega(A,B,C,D)\omega(B,C,A,D)} \tag{A14}$$

In order to study the projective representation (the second cohomology group $\mathcal{H}^2$) from 4-cocycles, we do the slant product again:

$$\mathsf{C}^{(2)}_{AB}(C,D) \equiv i_B\mathsf{C}_A(C,D) = \frac{\mathsf{C}_A(B,C,D)\mathsf{C}_A(C,D,B)}{\mathsf{C}_A(C,B,D)} \tag{A15}$$

$$= \frac{\omega(B,A,C,D)\omega(B,C,D,A)}{\omega(A,B,C,D)\omega(B,C,A,D)} \cdot \frac{\omega(A,C,B,D)\omega(C,B,A,D)}{\omega(C,A,B,D)\omega(C,B,D,A)} \cdot \frac{\omega(C,A,D,B)\omega(C,D,B,A)}{\omega(A,C,D,B)\omega(C,D,A,B)} \tag{A16}$$

4. 2+1D topological orders of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$

a. 3-cocycles

Here we organize the known fact about the third cohomology group $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ with $G=\prod_{i=1}^k Z_{N_i}$:

$$\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z}) = \prod_{1\le i<j<l\le m} \mathbb{Z}_{N_i}\times\mathbb{Z}_{N_{ij}}\times\mathbb{Z}_{N_{ijl}}.$$

We study the the 2D's $\mathrm{MCG}(\mathbb{T}^2)=\mathrm{SL}(2,\mathbb{Z})$ modular data: S and T using the Rep theory approach.

$\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$	3-cocycle name	3-cocycle form	Induced $\mathsf{C}_a(b,c)$
$\mathbb{Z}_{N_i}$	Type I p_i	$\omega_{3,\mathrm{I}}^{(i)}(a,b,c) = \exp\Big(\frac{2\pi \mathrm{i} p_i}{N_i^2}\ a_i(b_i+c_i-[b_i+c_i])\Big)$	$\exp\Big(\frac{2\pi \mathrm{i} p_i}{N_i^2}\ a_i(b_i+c_i-[b_i+c_i])\Big)$
$\mathbb{Z}_{N_{ij}}$	Type II p_{ij}	$\omega_{3,\mathrm{II}}^{(ij)}(a,b,c) = \exp\Big(\frac{2\pi \mathrm{i} p_{ij}}{N_i N_j}\ a_i(b_j+c_j-[b_j+c_j])\Big)$	$\exp\Big(\frac{2\pi \mathrm{i} p_{ij}}{N_i N_j}\ a_i(b_j+c_j-[b_j+c_j])\Big)$
$\mathbb{Z}_{N_{ijl}}$	Type III p_{ijl}	$\omega_{3,\mathrm{III}}^{(ijl)}(a,b,c) = \exp\Big(\frac{2\pi \mathrm{i} p_{ijl}}{N_{ijl}}\ a_i b_j c_l\Big)$	$\exp\Big(\frac{2\pi \mathrm{i} p_{ijl}}{N_{ijl}}\ (a_i b_j c_l - b_i a_j c_l + b_i c_j a_l)\Big)$

TABLE XII. The cohomology group $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ and 3-cocycles ω_3 for a generic finite Abelian group $G=\prod_{i=1}^n Z_{N_i}$. The first column shows the classes in $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$. The second column shows the topological term indices for 2+1D *twisted gauge theory*. (When all indices $p_{...}=0$, it becomes the *normal untwisted gauge theory*.) The third column shows explicit 3-cocycle function $\omega_3(a,b,c)$: $(G)^3\to \mathrm{U}(1)$. Here $a=(a_1,a_2,\dots,a_k)$, with $a\in G$ and $a_i \in Z_{N_i}$. The same notation is used for b,c,d. The last column lists induced 2-cocycles from the slant product $\mathsf{C}_a(b,c)$ using Eq.(A13).

b. *Projective Rep and* S, T *for Abelian topological orders*

This subsection simply reviews some known facts for later convenience in discussing new results. Much of the discussions can be absorbed from Ref.42, 50, 51, and 61. Firstly we study the Abelian topological orders from Type I, II 3-cocycles ω_3 of Table XII for 2+1D topological orders. We can determine the C_a projective representation (Rep) and $\widetilde{\rho}^a_\alpha(b)$:

$$\widetilde{\rho}^a_\alpha(b)\widetilde{\rho}^a_\alpha(c)=\mathsf{C}_a(b,c)\widetilde{\rho}^a_\alpha(bc). \tag{A17}$$

Given Z_a is the centralizer of $a\in G$, C_a determines the projective Rep of Z_a. Each C_a classifies a class of projective Rep named C_a-representations $\widetilde{\rho}: Z_a \to \mathrm{GL}(Z_a)$. In Type I, II ω_3, the irreducible C_A-representations $\widetilde{\rho}^g_\alpha$ of Z_g are in the one-to-one correspondence to the irreducible linear representations. The linear Rep originates from the normal untwisted $\prod_i Z_{N_i}$ gauge theory/toric code is: $\exp(2\pi \mathrm{i}(\sum_i \frac{1}{N_i}\alpha_i h_i))$. It has pure-charge (α_i)-pure-flux (h_i) coupling formulated by a BF theory in any dimension (a mutual Chern-Simons theory in 2+1D). The full C_a-representations is:

$$\widetilde{\rho}^g_\alpha(h)=\exp\big(2\pi \mathrm{i}(\sum_i \frac{1}{N_i}\alpha_i h_i)\big)\exp\big(2\pi \mathrm{i}\sum_i \frac{1}{N_i^2}p_i g_i h_i)\big)\exp\big(2\pi \mathrm{i}\sum_{i,j}\frac{1}{N_iN_j}p_{ij}g_i h_j)\big). \tag{A18}$$

We will interpret $(\alpha_1,g_1,\alpha_2,g_2,\alpha_3,g_3)$ and $(\beta_1,h_1,\beta_2,h_2,\beta_3,h_3)$ as the charges α,β and fluxes a,b of particles in a doubled basis $|\alpha,g\rangle$, $|\beta,h\rangle$. The generic T matrix formula of modular $\mathrm{SL}(2,\mathbb{Z})$ data is[42,50]

$$\mathsf{T}_{(\alpha,A)(\beta,B)}=\mathsf{T}_{(\alpha,A)}\delta_{\alpha,\beta}\delta_{A,B}=\frac{\mathrm{Tr}\widetilde{\rho}^{g^A}_\alpha(g^A)}{\dim(\alpha)}. \tag{A19}$$

We obtain:

$$\mathsf{T}_{(\alpha,A)}=\exp\big(2\pi i([\sum_i\frac{1}{N_i}\alpha_i a_i]+\sum_{j=1,2,3}\frac{1}{N_j^2}p_j\ (a_j^2)+\sum_{ij=12,23,13}\frac{1}{N_iN_j}p_{ij}\ (a_i a_j))\big), \tag{A20}$$

which $\mathsf{T}_{(\alpha,A)}=e^{\mathrm{i}\Theta^A_\alpha}$ describe the exchange statistics of two identical particles or the topological spin of the same particle. On the other hand, the generic S matrix formula in 2+1D reads from[42,50]

$$\mathsf{S}_{(\alpha,a)(\beta,b)}=\frac{1}{|G|}\sum_{\substack{g\in C^a,h\in C^b\\ gh=hg}}\mathrm{Tr}\widetilde{\rho}^g_\alpha(h)^*\mathrm{Tr}\widetilde{\rho}^h_\beta(g)^* \tag{A21}$$

yielding

$$\mathsf{S}_{(\alpha,a)(\beta,b)}(p_j,p_{ij})=\frac{1}{|G|}\exp\big(-2\pi \mathrm{i}(\frac{1}{N_i}[\sum_i^2\alpha_i b_i+\beta_i a_i]+2\sum_{j=1,2,3}\frac{1}{N_i^2}p_j\ (a_j b_j)+\sum_{ij=12,23,13}\frac{1}{N_iN_j}p_{ij}\ (a_i b_j+b_i a_j))\big). \tag{A22}$$

One can use a K-matrix Chern-Simons theory of an action $\mathbf{S}=\frac{1}{4\pi}\int K_{IJ}a_I\wedge da_J$ to encode the information of $|\alpha,g\rangle$, $|\beta,h\rangle$ into quasiparticles vectors l,l' respectively, and formulate a K with $\mathsf{S}_{l,l'}(p_j,p_{ij})=\frac{1}{|G|}\exp(-2\pi \mathrm{i} l^T K^{-1} l')$. We can use S,T to study **the classifications of classes of topological orders**. For example, for $G=(Z_2)^2$ twisted theories, simply using T under basis(particles)-relabeling, we find the diagonal eigenvalues of T can be labeled by $(\mathrm{N}_1, \mathrm{N}_{-1}, \mathrm{N}_{\mathrm{i}}, \mathrm{N}_{-\mathrm{i}})$, as numbers of eigenvalues for $\mathsf{T} = 1, -1, \mathrm{i}, -\mathrm{i}$. We show that using the data show in Table XIII is enough to match the classes found in Ref.64. We denote $(\mathrm{n}_{\pm\mathrm{i}}, \mathrm{n}_{\pm 1}, \mathrm{n}_1)$ as the numbers for (the pair of $\pm$i, the pair of ±1, individual 1). Note that $\mathrm{N}_1 + \mathrm{N}_{-1} + \mathrm{N}_{\mathrm{i}} + \mathrm{N}_{-\mathrm{i}} = 2\mathrm{n}_{\pm\mathrm{i}} + 2\mathrm{n}_{\pm1} + \mathrm{n}_1 = \mathrm{GSD}_{\mathbb{T}^2} = |G|^2$. There are 8 **types** of 3-cocycles but there are only 4 **classes** in Table XIII. The number in the bracket [.] of ω_3[.] indicates the number of $+$i (or equivalently the number of a pair of $\pm$i, paired due to the *twisted quantum doubled model* nature).

Class	$(\mathrm{N}_1, \mathrm{N}_{-1}, \mathrm{N}_{\mathrm{i}}, \mathrm{N}_{-\mathrm{i}})$	$(\mathrm{n}_{\pm\mathrm{i}}, \mathrm{n}_{\pm1}, \mathrm{n}_1)$	Number of Types
$\omega_3[0]$	$(10, 6, 0, 0)$	$(0, 6, 4)$	1
$\omega_3[2]$	$(8, 4, 2, 2)$	$(2, 4, 4)$	3
$\omega_3[4]$	$(6, 2, 4, 4)$	$(4, 2, 4)$	3
$\omega_3[6]$	$(4, 0, 6, 6)$	$(6, 0, 4)$	1

TABLE XIII. Phases of $\mathcal{H}^3((Z_2)^2, \mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^3$. 8 **types** of 3-cocycles but there are only 4 **classes**.

For another example, $G = (Z_2)^3$ twisted theories, we find that, in Table XIV, by classifying and identifying the modular S, T data, the **64 Abelian types** 3-cocycles (all with Abelian statistics) in $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ are truncated to only **4 classes**.

Class	$(\mathrm{N}_1, \mathrm{N}_{-1}, \mathrm{N}_{\mathrm{i}}, \mathrm{N}_{-\mathrm{i}})$	$(\mathrm{n}_{\pm\mathrm{i}}, \mathrm{n}_{\pm1}, \mathrm{n}_1)$	Number of Types
$\omega_3[0]$	$(36, 28, 0, 0)$	$(0, 28, 8)$	1
$\omega_3[8]$	$(28, 20, 8, 8)$	$(8, 20, 8)$	21
$\omega_3[16]$	$(20, 12, 16, 16)$	$(16, 12, 8)$	35
$\omega_3[24]$	$(12, 4, 24, 24)$	$(24, 4, 8)$	7

TABLE XIV. Phases of $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^7$. Among 128 types of 3-cocycles, 64 **types** of 3-cocycles with Abelian statistics but there are only 4 **classes**.

c. Projective Rep and S, T for non-Abelian topological orders

For 2+1D $G = (Z_2)^3$ twisted gauge theories of $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z}) = (Z_2)^7$, with 128 types of theories, we have shown that the 64 types of theories with Abelian statistics (from 64 types of 3-cocycles without Type III twist) are truncated to 4 classes in Table XIV. Here we will consider the remaining 64 types 3-cocycles with Type III twist in $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z})$. Although the gauge group G is Abelian, the Type III cocycle twist promotes the theory to have non-Abelian statistics. Our basic knowledge and formalism are rooted in Ref.42, where the dual D_4 and Q_8 gauge theories are found for certain Type III twist. Here we generalize Ref.42's result to all kinds of 3-cocycles twists.

Our expression is the generalized case where 3-cocycles are based on Type III's but can include (or not include) Type I, II 3-cocycles. There are 8 Abelian charged particles with zero flux, and 14 non-Abelian charged particles (which projective Rep $\widetilde{\rho}^{a}_{\alpha}(b)$ is 2 dimensional, described by a rank-2 matrix) with nonzero fluxes as dyons. For $a, b, c \in G = (Z_2)^3$, we will label 8 elements in $G = (Z_2)^3$ by $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. We denote the above 8 elements as the abbreviation: $F(0), F(1), F(2), F(3), F(4), F(5), F(6), F(7)$ accordingly. Let us recall: $\widetilde{\rho}^{g_a}_{\alpha}(g_b)$ contains α meaning the representation as charges, also g_b meaning the flux, and g_a indicating in general the conjugacy class (i.e. flux) as basis. In short, our notation leads to $\widetilde{\rho}^{g_a}_{\alpha}(g_b) = \widetilde{\rho}^{\text{conjugacy class(flux) as basis}}_{\text{representation(charge)}}(\text{flux})$.

- $1 \cdot 8 = 8$ **particles:** $F(0), (\alpha_1, \alpha_2, \alpha_3)$

When the flux is zero flux, $a = F(0)$ is the conjugacy class $C^{F(0)}$. There are 8 linear irreducible representations as charges. These charges can be labeled by $(\alpha_1, \alpha_2, \alpha_3)$ with $(\alpha_1, \alpha_2, \alpha_3) \in (Z_2)^3$, $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$. So we have

$$\widetilde{\rho}^{F(0)}_{F(0),(\alpha_1,\alpha_2,\alpha_3)}(b) = \widetilde{\rho}^{F(0)}_{F(0),(\alpha_1,\alpha_2,\alpha_3)}(b_1, b_2, b_3) = \exp(\frac{2\pi i}{m^2} m(\sum_{j=1,2,3} \alpha_j b_j)). \tag{A23}$$

- $7 \cdot 2 = 14$ **particles:** $F(j), \pm$

The remaining 7 kinds of fluxes are $a = F(j)$ for $j = 1, \ldots, 7$. There are two kinds of representations for each. We can denote these two representations as $+$ or $-$. So together these give 14 more type of particles. In total, there are $1 \cdot 8 + 7 \cdot 2 = 22$ quasi-particle excitations as the GSD on $\mathbb{T}^2$ torus. Generally, the representation is $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(l))$

for some inserting flux $F(l)$. This is a 2-dimensional representation. The identity always assigns to $F(0)$, namely $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will list down three more elements $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1))$, $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2))$, $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3))$. The other remaining $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(l))$ for $l = 4, \dots, 7$ can be determined by Eq.(A17). The representations are adjusted by a 1-dimensional projective Rep by Type I ω_I, Type II ω_{II} 3-cocycles: with topological level quantized coefficients as p_1, p_2, p_3 of Type I and p_{12}, p_{13}, p_{23} of Type II. Under the Type I, Type II twists, the Type III Rep adjusts to:

$$\widetilde{\rho}^{F(j)=a}_{F(j)=a,\pm}(b) \to \widetilde{\rho}^{F(j)}_{F(j),\pm}(b) e^{i\frac{\pi}{2}(\sum_{\substack{j,l\in\{1,2,3\}\\ j<l}} p_l a_l b_l + p_{ln} a_l b_n)}. \tag{A24}$$

- **2 particles:** $F(1), \pm$.
For $j = 1$, here $(a_1, a_2, a_3) = F(1) = (1, 0, 0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(2), \pm$.
For $j = 2$, here $(a_1, a_2, a_3) = F(2) = (0, 1, 0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(3), \pm$.
For $j = 3$, here $(a_1, a_2, a_3) = F(3) = (0, 0, 1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(4), \pm$.
For $j = 4$, here $(a_1, a_2, a_3) = F(4) = (1, 1, 0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(5), \pm$.
For $j = 5$, here $(a_1, a_2, a_3) = F(5) = (1, 0, 1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(6), \pm$.
For $j = 6$, here $(a_1, a_2, a_3) = F(6) = (0, 1, 1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

- **2 particles:** $F(7), \pm$.
For $j = 7$, here $(a_1, a_2, a_3) = F(7) = (1, 1, 1)$, (note in particular this Rep, our choice $\mp$ differs from Ref.42.)

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \mp\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)}, \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \mp\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)}, \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \mp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}$$

With the above projective Rep $\widetilde{\rho}^a_\alpha(b)$, we can derive the analytic form of modular data S, T in 2D. Here for $G = (Z_2)^3$,

$$\mathsf{T}^A_\alpha = e^{\mathrm{i}\frac{\pi}{2}(\sum_{\substack{l,m\in\{1,2,3\}\\ l<m}} p_l {a_l}^2 + p_{lm} a_l a_l)} (\pm)_a (\mathrm{i})^{\eta_{a,a}} \quad \rightarrow \quad \mathsf{T}^A_\alpha = \pm 1 \text{ or } \pm \mathrm{i} \tag{A25}$$

$$\eta_{g_1,g_2} \equiv \begin{cases} 0, & \text{if } \mathsf{C}_{g_1}(g_2, g_2) = +1. \\ 1, & \text{if } \mathsf{C}_{g_1}(g_2, g_2) = -1. \end{cases} \tag{A26}$$

More explicitly, we compute T^A_α in Table A 4 c:

Particle	T^a_α
$((\alpha_1,\alpha_2,\alpha_3),F(0))$	1
$(\pm,F(1))$, $(\pm,F(2))$, $(\pm,F(3))$	$\pm \mathrm{i}^{p_1}$, $\pm \mathrm{i}^{p_2}$, $\pm \mathrm{i}^{p_3}$
$(\pm,F(4))$, $(\pm,F(5))$, $(\pm,F(6))$	$\pm \mathrm{i}^{p_1+p_2+p_{12}}$, $\pm \mathrm{i}^{p_1+p_3+p_{13}}$, $\pm \mathrm{i}^{p_2+p_3+p_{23}}$
$(\pm,F(7))$	$\pm \mathrm{i}\cdot \mathrm{i}^{p_1+p_2+p_3+p_{12}+p_{13}+p_{23}}$

TABLE XV. The modular T^a_α matrix for 2D twisted $(Z_2)^3$ theories with non-Abelian statistics. The table contains all 64 non-Abelian theories in $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})$.

With the modular $\mathsf{S}^{xy}=\mathsf{S}^{xy}_{(\alpha,a)(\beta,b)}$ matrix (of 64 types of 2D twisted $(Z_2)^3$ theories with non-Abelian statistics):

$$\mathsf{S}=\frac{1}{|G|}\begin{array}{cccc} (\beta_j,0) & (+,b_j) & (-,b_j) & \\ \left(\begin{array}{c|c|c} 1 & 2(-1)^{b_1\alpha_1+b_2\alpha_2+b_3\alpha_3} & 2(-1)^{b_1\alpha_1+b_2\alpha_2+b_3\alpha_3} \\ \hline 2(-1)^{a_1\beta_1+a_2\beta_2+a_3\beta_3} & \delta_{a,b}4\cdot(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j+p_{jl}a_j a_l} & -\delta_{a,b}4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j+p_{jl}a_j a_l} \\ \hline 2(-1)^{a_1\beta_1+a_2\beta_2+a_3\beta_3} & -\delta_{a,b}4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j+p_{jl}a_j a_l} & \delta_{a,b}4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j+p_{jl}a_j a_l} \end{array}\right) & \begin{array}{c}(\alpha_j,0)\\ (+,a_j)\\ (-,a_j)\end{array} \end{array} \tag{A27}$$

In Eq.(A27), the factor $(-1)^{\eta_{a,a}}$ is derived from a computation of $(\mathrm{i})^{\eta_{a,b}}\cdot(\mathrm{i})^{\eta_{b,a}}\delta_{a,b}=(-1)^{\eta_{a,a}}\delta_{a,b}$. From Eq.(A26), we notice that $\eta_{a,a}=1$ is nonzero only when $a=(1,1,1)=F(7)$ for the $(Z_2)^3$ flux.

5. Classification of 2+1D twisted $(Z_2)^3$ gauge theories, $D^\omega((Z_2)^3)$ and $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})$.

The twisted $(Z_2)^3$ gauge theories dual to D_4, Q_8 non-Abelian gauge theories were first discovered in Ref.42. Here we present the three other classes which cannot be dual to any non-Abelian gauge theory, but only to be a twisted (Abelian or non-Abelian) gauge theories themselves. We again label the diagonal eigenvalues of T by $(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_{\mathrm{i}},\mathrm{N}_{-\mathrm{i}})$, their number of eigenvalues for $\mathsf{T}=1,-1,\mathrm{i},-\mathrm{i}$. We also use shorthand $(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm1},\mathrm{n}_1)$ instead, which stands for the numbers for (the pair of $\pm\mathrm{i}$, the pair of ±1, individual 1) in the diagonal of T. Note that $\mathrm{N}_1+\mathrm{N}_{-1}+\mathrm{N}_{\mathrm{i}}+\mathrm{N}_{-\mathrm{i}}=2\mathrm{n}_{\pm\mathrm{i}}+2\mathrm{n}_{\pm1}+\mathrm{n}_1=\mathrm{GSD}_{\mathbb{T}^2}=22$. There are 64 **types** of 3-cocycles corresponding to theories with non-Abelian statistics but there are only 5 inequivalent **classes** in Table XIII. The number in the bracket [.] of ω_3[.] 3 (the first column) indicates the number of $+\mathrm{i}$ (or equivalently the number of pairs of $\pm\mathrm{i}$, paired due to the nature of the *quantum double model* nature.).

Class	$(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm1},\mathrm{n}_1)$	$(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_{\mathrm{i}},\mathrm{N}_{-\mathrm{i}})$	Twisted quantum double $D^\omega(G)$	Number of Types
$\omega_3[1]$	(1,6,8)	(14,6,1,1)	$D^{\omega_3[1]}({Z_2}^3)$, $D(D_4)$	7
$\omega_3[3d]$	(3,4,8)	(12,4,3,3)	$D^{\omega_3[3d]}({Z_2}^3)$, $D^{\gamma^4}(Q_8)$	7
$\omega_3[3i]$	(3,4,8)	(12,4,3,3)	$D^{\omega_3[3i]}({Z_2}^3)$, $D(Q_8)$, $D^{\alpha_1}(D_4)$, $D^{\alpha_2}(D_4)$	28
$\omega_3[5]$	(5,2,8)	(10,2,5,5)	$D^{\omega_3[5]}({Z_2}^3)$, $D^{\alpha_1\alpha_2}(D_4)$	21
$\omega_3[7]$	(7,0,8)	(8,0,7,7)	$D^{\omega_3[7]}({Z_2}^3)$	1

TABLE XVI. $D^\omega(G)$ is the twisted quantum double of G with a cocycle twist ω of G's cohomology group. Here we consider a 3-cocycle twist ω_3 in $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^7$, which ω_3 contains a factor of Type III 3-cocycle. We compute the values in the second and the fourth columns, and then compare them with the mathematics literature Ref.62 to match for the third column. We find that the 64 types of non-Abelien theories are truncated to 5 classes.

Although $\omega_3[3d]$ and $\omega_3[3i]$ share the same T matrix data, but they can still be distinguished by the linear dependency of the fluxes which carry three pairs of eigenvalues i. (And, of course, they can be distinguished by the more-involved S matrix.) There are 7 types in the $\omega_3[3d]$ class, whose $\pm\mathrm{i}$ are generated by linear-dependent fluxes. There are another 28 types in the $\omega_3[3i]$ class, whose $\pm\mathrm{i}$ are generated by linear-independent fluxes. In this notation of linear (in)dependency, we have $\omega_3[1]=\omega_3[1i]$, $\omega_3[5]=\omega_3[5d]$, $\omega_3[7]=\omega_3[7d]$. Such a concept is also used in the mathematic literature in Ref.62, where they study the Frobenius-Schur indicators, Frobenius-Schur exponents and the support of cocycle twist, supp ω; and use these data to classify twisted quantum double model $D^\omega(G)$. Remarkably, we find that using our data is enough to match the classes found in the math literature[62] in the quantum double and module category framework.

These findings altogether with Appendix. A 4 b form a complete data set of $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^7$, where **128 types of 3-cocycles fall into 4 distinct classes of Abelian topological orders in Table XIII and 5 distinct classes of non-Abelian topological orders in Table XVI** . In total, there are 9 distinct classes of topological orders within twisted $(Z_2)^3$ gauge theories. We note that $\omega_3[3i]$, $\omega_3[5]$, $\omega_3[7]$ can only be twisted gauge theories, not dual to any untwisted non-Abelian gauge theory.

6. 3+1D topological orders of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$

This subsection continues the discussion and notations from $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ of 2+1D to $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ of 3+1D topological orders. Now we fill in some more information about the data of the projective Rep.

a. Projective Rep and S, T *for Abelian topological orders*

The data of $\widetilde{\rho}_\alpha^{ab}(c)$ is organized below in Table XVII for $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ of the cohomology group $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$. Its modular S, T matrices for this Rep have been presented in Table II, III, IV. In the main text, we provide an example of classifying 3D topological orders from 3+1D $(Z_2)^2$ twisted gauge theories of 4 types (from $\mathcal{H}^4((Z_2)^2, \mathbb{R}/\mathbb{Z}) = (Z_2)^2$), and find out that 4 types are truncated to only 2 distinct classes of topological orders.

$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	4-cocycle	$\widetilde{\rho}_\alpha^{a,b}(c)$
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\widetilde{\rho}_{\mathrm{II},\alpha}^{(1st)a,b}(c) = \exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\ \alpha_k c_k\big) \cdot \exp\big(\frac{2\pi \mathrm{i} p_{\mathrm{II}(12)}^{(1st)}}{(N_{12} \cdot N_2)}\ (a_2 b_1 - a_1 b_2) c_2\big)$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\widetilde{\rho}_{\mathrm{II},\alpha}^{(2nd)a,b}(c) = \exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\ \alpha_k c_k\big) \cdot \exp\big(\frac{2\pi \mathrm{i} p_{\mathrm{II}(12)}^{(2nd)}}{(N_{12} \cdot N_1)}\ (a_1 b_2 - a_2 b_1) c_1\big)$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\widetilde{\rho}_{\mathrm{III},\alpha}^{(1st)a,b}(c) = \exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\ \alpha_k c_k\big) \cdot \exp\big(\frac{2\pi \mathrm{i} p_{\mathrm{III}(123)}^{(1st)}}{(N_{12} \cdot N_3)}\ (a_2 b_1 - a_1 b_2) c_3\big)$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\widetilde{\rho}_{\mathrm{III},\alpha}^{(2nd)a,b}(c) = \exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\ \alpha_k c_k\big) \cdot \exp\big(\frac{2\pi \mathrm{i} p_{\mathrm{III}(123)}^{(2nd)}}{(N_{31} \cdot N_2)}\ (a_1 b_3 - a_3 b_1) c_2\big)$

TABLE XVII. $\widetilde{\rho}_\alpha^{a,b}(c)$ for a 3+1D twisted gauge theory with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$. We derive $\widetilde{\rho}_\alpha^{a,b}(c)$ from the equation introduced in the main text, $\widetilde{\rho}_\alpha^{a,b}(c)\widetilde{\rho}_\alpha^{a,b}(d) = \mathsf{C}_{a,b}^{(2)}(c,d)\widetilde{\rho}_\alpha^{a,b}(cd)$, presenting the projective representation, because the induced 2-cocycle belongs to the second cohomology group $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$. The $\widetilde{\rho}_\alpha^{a,b}(c)$: $(Z_a, Z_b) \to \mathrm{GL}\,(Z_a, Z_b)$ can be written as a general linear matrix.

b. Projective Rep and S, T *for non-Abelian topological orders*

Below we will present the data of twisted gauge theories for those with non-Abelian statistics in $\mathcal{H}^4(G = (Z_2)^4, \mathbb{R}/\mathbb{Z})$ labeled by 4-cocycles ω_4. Among $\mathcal{H}^4((Z_2)^4, \mathbb{R}/\mathbb{Z}) = (Z_2)^{21}$ types of theories, there are 2^{20} types of them endorsed with non-Abelian statistics. In some case, we will write the formula in terms of a slightly generic $G = (Z_n)^4$, for a prime n.

Analogously to Appendix A 4 c, we recall that the 3D triple basis renders: $\widetilde{\rho}_\alpha^{g^a, g^b}(g^c) = \widetilde{\rho}_{\text{representation(charge)}}^{\text{conjugacy class(flux,flux) as basis}}(\text{flux})$. So we understand that the representation $\widetilde{\rho}(c)$ is constrained by the flux a, b. We consider Type IV $\omega_{4,\mathrm{IV}}$ twisted theories, but we include $\omega_{4,\mathrm{IV}}$ further multiplied by Type II $\omega_{4,\mathrm{II}}$ and Type III $\omega_{4,III}$ 4-cocycles. Thus, the representation also relates to their topological terms p_{lm} of Type II $\omega_{4,\mathrm{II}}$ labeling $(Z_2)^{2\binom{4}{2}} = (Z_2)^{12}$ types of theories, p_{lmn} of Type III $\omega_{4,\mathrm{III}}$ labeling $(Z_2)^{2\binom{4}{3}} = (Z_2)^8$ types of theories. In total all these 4-cocycles multiplied by $\omega_{4,\mathrm{IV}}$ yield the 2^{20} types of theories showing non-Abelian statistics. Under the Type II or Type III twists, the Type IV Rep is adjusted to:

$$\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(c) = \widetilde{\rho}_{F(j_1)=a,F(j_2)=b,(\pm,\pm)}^{F(j_1)=a,F(j_2)=b}(c) \cdot e^{i\frac{\pi}{2}\big(\sum_{\substack{l,m,n \in \{1,2,3,4\} \\ l<m<n}} p_{lm} f_{lm}(a,b,c) + p_{lmn} f_{lmn}(a,b,c)\big)}. \tag{A28}$$

Note that the trace $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(c)]$ is nonzero only when (1) $c = a$, $c = b$ or $c = ab$ with $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a}(c)] \neq 0$, or (2) $c = F(0)$ zero flux, i.e. $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(F(0))] \neq 0$. Other cases have zero traces. Among the degeneracy sectors on the

$\mathbb{T}^3$ torus, we have $\mathrm{GSD}_{\mathbb{T}^3} = \big(n^8+n^9-n^5\big)+\big(n^{10}-n^7-n^6+n^3\big)$ (ground state bases in terms of particles and string quasi-excitations), which is 1576 for $n=2$. We can use $|G|^2=(n^4)^2=256$ (doubled) fluxes to do the first labeling. Note the fluxes form a doubled basis (a,b) in $|\alpha,a,b\rangle$. Among 256 fluxes, there are $n^4+n^5-n=46$ fluxes carrying Abelian excitations, while the remaining $(n^8-(n^4+n^5-n))=210$ are non-Abelian excitations. (Note that the bases carry two fluxes and one charge, these bases should **not** be confused with string and particle types.) We may organize the ground state bases in terms of two kinds, which correspond to Abelian and non-Abelian excitations:

- $(n^4+n^5-n)\cdot n^4 = 46\times 16 = 736$ Abelian excitations: $F(j_{ab}),(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$

Here $a=F(j_{ab})$ can be zero fluxes, or nonzero fluxes by satisfying the following conditions:

$$a_1b_2=a_2b_1, a_1b_3=a_3b_1, a_1b_4=a_4b_1, a_2b_3=a_3b_2, a_2b_4=a_4b_2, a_3b_4=a_4b_3 \pmod N \tag{A29}$$

There are (n^4+n^5-n) independent solutions for these sets of a, b. The conjugacy class $C^{F(j_{ab})}$ stands for fluxes. There are n^4 representation as charges; these can be labeled by $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ with $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in(Z_2)^4$, and $Z_2=\{0,1\}$. We will write $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha$. Eq.(A28) becomes

$$\widetilde{\rho}^{F(j_{ab})}_{F(j_{ab}),(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(c)=\widetilde{\rho}^{F(0)}_{F(0),(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(c_1,c_2,c_3,c_4)=\exp\big(\sum_{k=1}^{4}\frac{2\pi i}{N_k}\alpha_k c_k\big). \tag{A30}$$

For $n=2$, there are $(2^4+2^5-2)=46$ (doubled) fluxes contributing Abelian excitations.

- $(n^8-(n^4+n^5-n))\cdot n^2 = 210\times 4 = 840$ non-Abelian excitations: $F(j_{non.ab}),(\pm,\pm)$

For $n=2$, there are $(n^8-(n^4+n^5-n))=210$ (doubled) fluxes contributing non-Abelian excitations. Each of them carries 2-dimensional Rep with two pairs of $(\pm,\pm)$ charge Rep. Thus the number of doubled fluxes multiplied by 4 yields 840 excitations. It is equivalent to count the $\mathsf{C}^{(2)}_{a,b}(c,d)$ class that they belong to. There are six $c_l d_m$ terms in the Type IV 4-cocycles:

$$\begin{aligned}\mathsf{C}^{(2)}_{a,b}(c,d)=\exp\big(&\frac{2\pi i p_{IV(1234)}}{N_{ijlm}}\,(a_4b_3-a_3b_4)c_1d_2+(a_2b_4-a_4b_2)c_1d_3+(a_4b_1-a_1b_4)c_2d_3\\&+(a_3b_2-a_2b_3)c_1d_4+(a_1b_3-a_3b_1)c_2d_4+(a_2b_1-a_1b_2)c_3d_4\big).\end{aligned} \tag{A31}$$

Below each solution will be multiplied by 6, due to $\binom{3}{2}\times 2$, that 3 terms a,b,ab can choose 2 as the generator basis for a, b. Those terms have $\mathrm{Tr}[\widetilde{\rho}^{a}_{a,b,(\pm,\pm)}(c)]\neq 0$ for $c=0,a,b,ab$. And the permutation of a,b results in an extra multiple of 2. We organize the solutions to the following six **styles**. Each style may contain dimensionally reduced 3-cocycles, as "Type III 3-cocycle like" or "mixed-Type III 3-cocycles." Here "*Type III 3-cocycle like*" means that the dimensional reduced 2D theory has an induced 3-cocycle which is a Type III 3-cocycle *within* a subgroup $(Z_2)^3$. "*Mixed-Type III 3-cocycle*" means that the dimensional reduced 2D theory has an induced 3-cocycle which contains several Type III 3-cocycles spanning the full group $(Z_2)^4$. The **six styles** of solutions are:

- **Style 1** (Type III 3-cocycle like) — $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains 1 cd term: $\binom{6}{1}\times 6=36$ non-Abelian fluxes :

- **Style 2** (Type III 3-cocycle like) — $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains 2 cd term: $(\binom{6}{2}-3)\times 6=72$ non-Abelian fluxes :

We have $\binom{6}{2}$ subtract 3, due to it is impossible to have nonzero coefficients cd terms of $\mathsf{C}^{(2)}_{a,b}(c,d)$ for both of the following terms together:

(1) c_3d_4 and c_1d_2 terms, (2) c_2d_4 and c_1d_3 terms, (3) c_2d_3 and c_1d_4 terms.

Style 3 (Type III 3-cocycle like), Style 4 (mixed-Type III 3-cocycles) — $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains 3 cd term: $\binom{4}{3}\times 6+\binom{4}{3}\times 6=48$ non-Abelian fluxes :

- **Style 3** (Type III 3-cocycle like) $\binom{4}{3}\times 6$:

$\binom{4}{3}$ out of 6 have nonzero coefficients for: (1) c_2d_3, c_2d_4, c_3d_4. (2) c_1d_3, c_1d_4, c_3d_4. (3) c_1d_2, c_1d_4, c_2d_4. (3) c_1d_2, c_1d_3, c_2d_3. Each type has 6 possible choices for a,b.

- **Style 4** (mixed-Type III 3-cocycles) $\binom{4}{3}\times 6$:

$\binom{4}{3}$ out of 6 have nonzero coefficients for: (1) c_1d_2, c_1d_3, c_1d_4. (2) c_1d_2, c_2d_3, c_2d_4. (3) c_1d_3, c_2d_3, c_3d_4. (4) c_1d_4,

c_2d_4, c_4d_4. Each type has 6 possible choices for a,b.

• **Style 5** (mixed-Type III 3-cocycles) — $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains 4 cd term: $(\binom{6}{4} - \binom{4}{3}\cdot 3) \times 6 = 3 \times 6 = 18$ non-Abelian fluxes

Among 15 terms (with 4 cd) in $\binom{6}{4} = 15$, there are only 3 terms allowed.
(1) c_1d_2, c_2d_3, c_1d_4, c_3d_4, (2) c_1d_3, c_2d_3, c_1d_4, c_2d_4, (3) c_1d_2, c_1d_3, c_2d_4, c_3d_4.
There are terms from $\binom{4}{3}\cdot 3 = 12$ is not allowed, like c_1d_2, c_1d_3, c_2d_3, c_1d_4. (i.e. choose 3 elements as $\binom{4}{3}$ and choose one of the three, thus times 3, to pair with the remaining unchosen elements.)

• **Style 6** (mixed-Type III 3-cocycles) — $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains 5 cd term: $\binom{6}{5} \times 6 = 36$ non-Abelian fluxes
(1) c_1d_2, c_1d_3, c_1d_4, c_2d_3, c_2d_4, (2) c_1d_2, c_1d_3, c_1d_4, c_2d_3, c_3d_4, (3) c_1d_2, c_1d_3, c_1d_4, c_2d_4, c_3d_4.
(4) c_1d_2, c_1d_3, c_2d_3, c_2d_4, c_3d_4, (5) c_1d_2, c_1d_4, c_2d_3, c_2d_4, c_3d_4, (6) c_1d_3, c_1d_4, c_2d_3, c_2d_4, c_3d_4

Those Style 1, 2, 3 are pure Type III 3-cocycle ω_3 like, which $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$ can be deduced from Appendix.A 4 c's $G = (Z_2)^3$ result. Style 4, 5, 6 are mixed Type III 3-cocycle in the full $G = (Z_2)^4$ group, so one needs to assign the Rep $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$ in slightly different manners. But it turns out that rank-2 matrices are always sufficient to encode the irreducible projective representation of $\mathsf{C}^{(2)}_{ab}(c,d)$. After finding the $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$, we analytically derive their non-Abelian S^{xyz}, T^{xy} of 3D presented in the main text, in Table V, Eq.(51), Eq.(52).

Appendix B: S^{xyz} and T^{xy} calculation in terms of the gauge group G and 4-cocycle ω_4

1. Unimodular Group and SL$(N,\mathbb{Z})$

In the case of the unimodular group, there are the unimodular matrices of rank N forms GL$(N,\mathbb{Z})$. $\mathsf{S_U}$ and $\mathsf{T_U}$ have determinant $\det(\mathsf{S_U}) = -1$ and determinant $\det(\mathsf{T_U}) = 1$ for any general N:

$$\mathsf{S_U} = \begin{pmatrix} 0 & 0 & 0 & \dots & (-1)^N \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{B1}$$

$$\mathsf{T_U} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{B2}$$

Note that $\det(\mathsf{S_U}) = -1$ in order to generate both determinant 1 and -1 matrices.

For the SL$(N,\mathbb{Z})$ modular transformation, we denote their generators as S and T for a general N with $\det(\mathsf{S}) = \det(\mathsf{T}) = 1$:

$$\mathsf{S} = \begin{pmatrix} 0 & 0 & 0 & \dots & (-1)^{N-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{B3}$$

$$\mathsf{T} = \mathsf{T_U}. \tag{B4}$$

Here for simplicity, let us denote S^{xyz} as $\mathsf{S}_{\rm 3D}$, S^{xy} as $\mathsf{S}_{\rm 2D}$, $\mathsf{T}^{xy} = \mathsf{T}_{\rm 3D} = \mathsf{T}_{\rm 2D}$. Recall that SL$(3,\mathbb{Z})$ is fully generated by generators $\mathsf{S}_{\rm 3D}$ and $\mathsf{T}_{\rm 3D}$.

$$\mathsf{S}_{\rm 3D} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{T}_{\rm 3D} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{S}_{\rm 2D} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\mathsf{S}_{\rm 2D} = (\mathsf{T}_{\rm 3D}^{-1}\mathsf{S}_{\rm 3D})^3(\mathsf{S}_{\rm 3D}\mathsf{T}_{\rm 3D})^2\mathsf{S}_{\rm 3D}\mathsf{T}_{\rm 3D}^{-1}. \tag{B5}$$

By dimensional reduction (note $\mathsf{T}_{\rm 2D} = \mathsf{T}_{\rm 3D}$), we expect that,

$$\mathsf{S}_{\rm 2D}^4 = (\mathsf{S}_{\rm 2D}\mathsf{T}_{\rm 3D})^6 = 1, \tag{B6}$$

$$(\mathsf{S}_{\rm 2D}\mathsf{T}_{\rm 3D})^3 = e^{\frac{2\pi \mathrm{i}}{8}c_-}\mathsf{S}_{\rm 2D}^2 = e^{\frac{2\pi \mathrm{i}}{8}c_-}C. \tag{B7}$$

c_- carries the information of central charges. We can express

$$\mathsf{R} \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\mathsf{T}_{\rm 3D}\mathsf{S}_{\rm 3D})^2\mathsf{T}_{\rm 3D}^{-1}\mathsf{S}_{\rm 3D}^2\mathsf{T}_{\rm 3D}^{-1}\mathsf{S}_{\rm 3D}\mathsf{T}_{\rm 3D}\mathsf{S}_{\rm 3D}. \tag{B8}$$

One can check that

$$\mathsf{S}_{\rm 3D}\mathsf{S}_{\rm 3D}^\dagger = \mathsf{S}_{\rm 3D}^3 = \mathsf{R}^6 = (\mathsf{S}_{\rm 3D}\mathsf{R})^4 = (\mathsf{R}\mathsf{S}_{\rm 3D})^4 = 1, \tag{B9}$$

$$(\mathsf{S}_{\rm 3D}\mathsf{R}^2)^4 = (\mathsf{R}^2\mathsf{S}_{\rm 3D})^4 = (\mathsf{S}_{\rm 3D}\mathsf{R}^3)^3 = (\mathsf{R}^3\mathsf{S}_{\rm 3D})^3 = 1, \tag{B10}$$

$$(\mathsf{S}_{\rm 3D}\mathsf{R}^2\mathsf{S}_{\rm 3D})^2\mathsf{R}^2 = \mathsf{R}^2(\mathsf{S}_{\rm 3D}\mathsf{R}^2\mathsf{S}_{\rm 3D})^2 \pmod 3. \tag{B11}$$

Such expressions are known in the mathematic literature, part of them are listed in Ref.37.

2. Rules for the path integral for the spacetime complex of cocycles

For the branching of a spacetime-complex or a simplex, we define that, for any arrow that goes from a small number to a large number, the number ordering

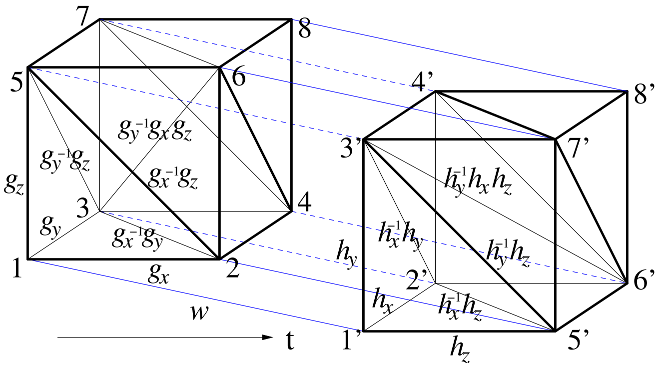


FIG. 12. Spacetime complex $\mathbb{T}^3 \times I$, where $I = [0, 1]$ is the time direction. The figure shows $\mathbb{T}^3 \times \{0\}$ and $\mathbb{T}^3 \times \{1\}$. The blue lines illustrate how the two $\mathbb{T}^3$ are connected for $t \in (0, 1)$. Note that the two $\mathbb{T}^3$'s differ by a rotation S^{xyz}. In other words, when time forms a loop, the two $\mathbb{T}^3$ are glued together by $1 \to 1'$, $2 \to 2'$, $3 \to 3'$, $4 \to 4'$, $5 \to 5'$, $6 \to 6'$, $7 \to 7'$, and $8 \to 8'$.

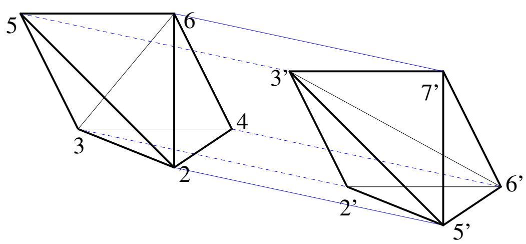


FIG. 13. The complex M_1.

is $1 < 2 < 3 < 4 < \cdots < 0' < 1' < 2' < 2^{*\prime} < 3' < 4' < 5' < 6' < 6^{*\prime} < \cdots$. The time evolves along the fourth direction from the left to the right, or from a smaller number to a larger number. Also we may write: $[01].[12] = [02]$, or, equivalently, $g_{01}.g_{12} = g_{02}$. If $[01] = g$ and $[12] = h$, then $[02] = gh$.

3. Explicit expression of S^{xyz} in terms of (G, ω_4)

The S^{xyz}-matrix can be computed from the amplitude $A^{xyz}(g_x, g_x, g_z, h_x, h_y, h_z; \mathrm{w})$ of the path integral on spacetime complex $\mathbb{T}^3 \times I$ (see Fig.12). Each $\mathbb{T}^3$ is divided into six tetrahegrons. The amplitude $A^{xyz}(g_x, g_x, g_z, h_x, h_y, h_z; \mathrm{w})$ is the product of the four amplitutes A_i for the four shapes M_i, $i = 1, \cdots, 4$, which are given in Fig.13–16).

Each shape M_i can be divided into several 4-simplices. So the amplitude A_i for each M_i is the product of several cocycles on the simplices. We find that, for M_3:

$$A_3 = \frac{\omega_4(g_{12}, g_{23}, g_{35}, g_{51'})\omega_4^{-1}(g_{35}, g_{51'}, g_{1'2'}, g_{2'5'})}{\omega_4(g_{23}, g_{35}, g_{51'}, g_{1'5'})\omega_4(g_{51'}, g_{1'2'}, g_{2'3'}, g_{3'5'})} \tag{B12}$$

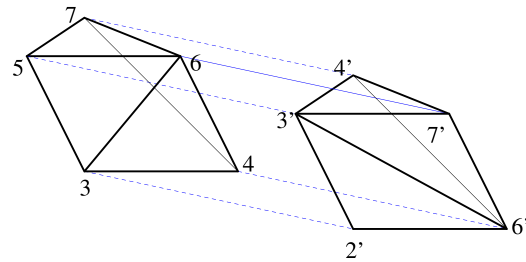


FIG. 14. The complex M_2.

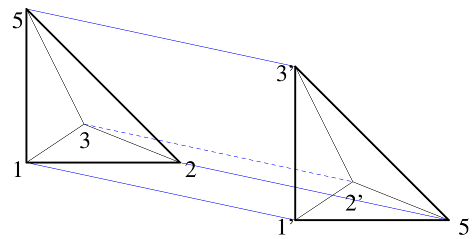


FIG. 15. The complex M_3.

for M_4:

$$A_4 = \frac{\omega_4(g_{67}, g_{78}, g_{86'}, g_{6'7'})\omega_4(g_{84'}, g_{4'6'}, g_{6'7'}, g_{7'8'})}{\omega_4(g_{46}, g_{67}, g_{78}, g_{86'})\omega_4(g_{78}, g_{84'}, g_{4'6'}, g_{6'7'})}. \tag{B13}$$

To compute the amplitude for M_1, we may view M_1 and a composition of M_1' and M_1'' (see Fig. 17 and 18). The amplitude for M_1' is

$$A_1' = \frac{\omega_4(g_{23}, g_{35}, g_{56}, g_{65'})\omega_4(g_{56}, g_{62'}, g_{2'3'}, g_{3'5'})}{\omega_4^{-1}(g_{35}, g_{56}, g_{62'}, g_{2'5'})\omega_4(g_{62'}, g_{2'3'}, g_{3'5'}, g_{5'7'})} \times \frac{\omega_4^{-1}(g_{34}, g_{46}, g_{62'}, g_{2'5'})\omega_4^{-1}(g_{62'}, g_{2'5'}, g_{5'6'}, g_{6'7'})}{\omega_4(g_{23}, g_{34}, g_{46}, g_{65'})\omega_4^{-1}(g_{46}, g_{62'}, g_{2'5'}, g_{5'6'})}. \tag{B14}$$

The above eight cocycles come from eight 4-simplices as illustrated in Fig. 19. The amplitude for M_1'' is

$$A_1'' = \omega_4^{-1}(g_{2'3'}, g_{3'5'}, g_{5'6'}, g_{6'7'}). \tag{B15}$$

and the total amplitude for M_1 is

$$A_1 = A_1' A_1''. \tag{B16}$$

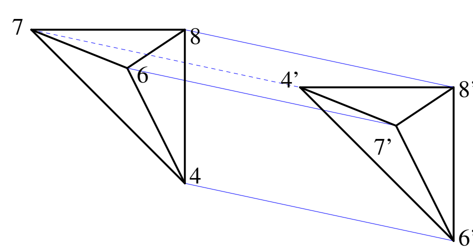


FIG. 16. The complex M_4.

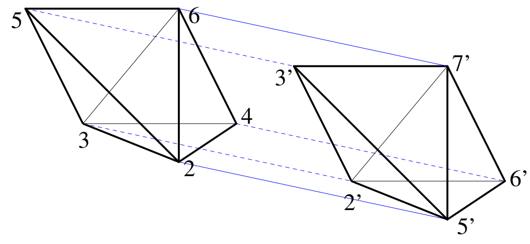


FIG. 17. The complex M_1'.

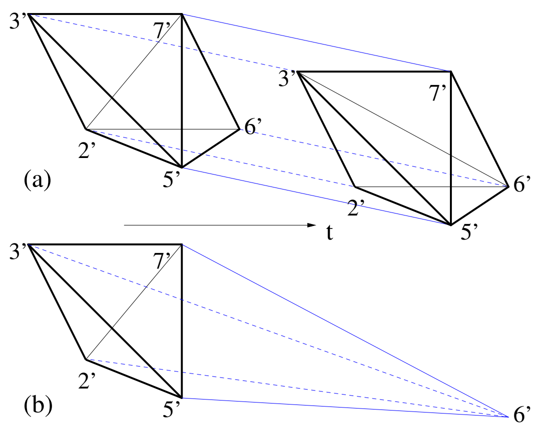


FIG. 18. The complex M_1'', which is formed by one 4-simplex. Note that all the vertices in (a) are on the same time slice but the (curved) edge $(2'7')$ is on an earlier time slice and the (curved) edge $(3'6')$ is on a later time slice. To realize this using straight edges, we put the vertex $6'$ on a later time slice, and this gives us a 4-simplex in (b).

Similarly, for M_2, we find that

$$A_2 = A_2' A_2'', \tag{B17}$$

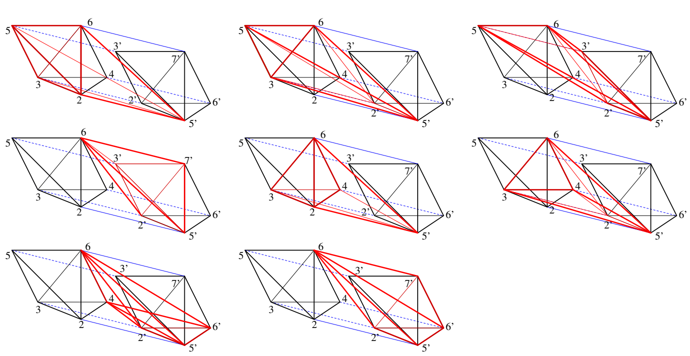


FIG. 19. The complex M_1' is formed by eight 4-simplices.

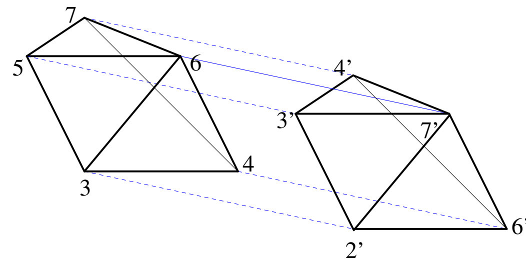


FIG. 20. The complex M_2', which is formed by eight 4-simplices.

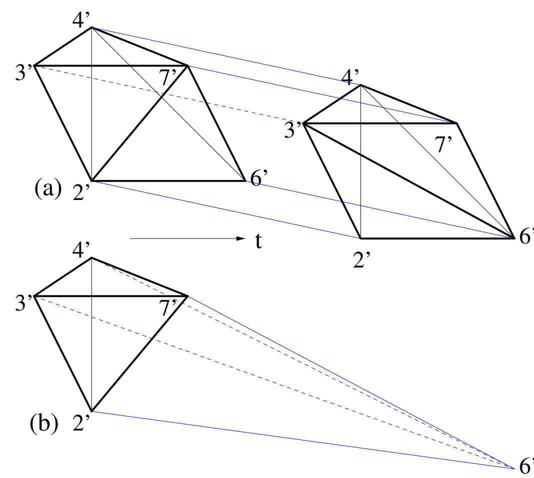


FIG. 21. The complex M_2'', which is formed by one 4-simplex. Note that all the vertices in (a) are on the same time slice but the (curved) edge $(2'7')$ is on an earlier time slice and the (curved) edge $(3'6')$ is on a later time slice. To realize this using straight edges, we put the vertex $6'$ on a later time slice, and this gives us a 4-simplex in (b).

where A_2' is the amplitued for M_2' (see Fig. 20)

$$A_2' = \frac{\omega_4(g_{35}, g_{56}, g_{67}, g_{72'})\omega_4(g_{67}, g_{72'}, g_{2'3'}, g_{3'7'})}{\omega_4(g_{56}, g_{67}, g_{72'}, g_{2'3'})\omega_4^{-1}(g_{72'}, g_{2'3'}, g_{3'4'}, g_{4'7'})} \frac{\omega_4(g_{46}, g_{67}, g_{72'}, g_{2'6'})\omega_4(g_{72'}, g_{2'4'}, g_{4'6'}, g_{6'7'})}{\omega_4(g_{34}, g_{46}, g_{67}, g_{72'})\omega_4(g_{67}, g_{72'}, g_{2'6'}, g_{6'7'})} \tag{B18}$$

and A_2' is the amplitued for M_2'' (see Fig. 21)

$$A_2'' = \omega_4(g_{2'3'}, g_{3'4'}, g_{4'6'}, g_{6'7'}). \tag{B19}$$

Here g_{ij} is the group element on the edge (ij). We have

$$\begin{aligned}
&g_{12} = g_{34} = g_{56} = g_{78} = g_x, \\
&g_{13} = g_{24} = g_{57} = g_{68} = g_y, \\
&g_{15} = g_{26} = g_{37} = g_{48} = g_z, \\
&g_{23} = g_{67} = g_x^{-1} g_y, \quad g_{35} = g_{46} = g_y^{-1} g_z, \\
&g_{25} = g_{47} = g_x^{-1} g_z, \quad g_{36} = g_y^{-1} g_x g_z,
\end{aligned} \tag{B20}$$

$$h_{12} = h_{34} = h_{56} = h_{78} = h_x,$$
$$h_{13} = h_{24} = h_{57} = h_{68} = h_y,$$
$$h_{15} = h_{26} = h_{37} = h_{48} = h_z,$$
$$h_{23} = h_{67} = h_x^{-1} h_y, \quad h_{35} = h_{46} = h_y^{-1} h_z,$$
$$h_{25} = h_{47} = h_x^{-1} h_z, \quad h_{36} = h_y^{-1} h_x h_z. \tag{B21}$$

$$g_{51'} = g_z^{-1} \mathrm{w}, \quad g_{62'} = g_z^{-1} g_x^{-1} g_y \mathrm{w}, \quad g_{84'} = \mathrm{w} h_z^{-1},$$
$$g_{65'} = g_{72'} = g_{86'} = \mathrm{w} h_y^{-1}. \tag{B22}$$

Also if the following conditions are not statisfied, the amplitude $A^{xyz}(g_x, g_x, g_z, h_x, h_y, h_z; \mathrm{w})$ will be zero:

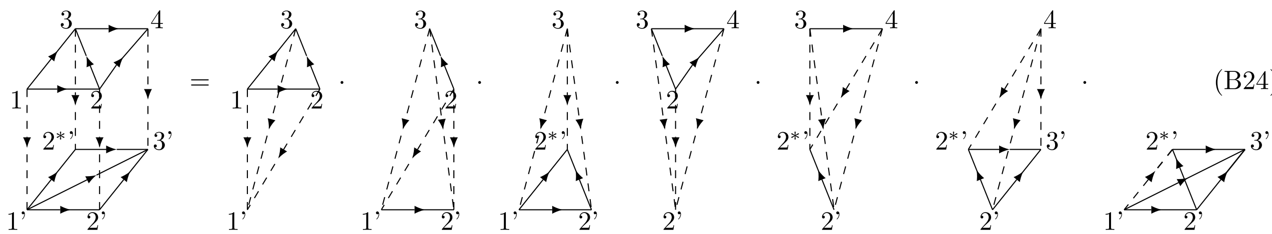
$$g_x \mathrm{w} = \mathrm{w} h_z, \quad g_y \mathrm{w} = \mathrm{w} h_x, \quad g_z \mathrm{w} = \mathrm{w} h_y,$$
$$g_x g_y = g_y g_x, \quad g_y g_z = g_z g_y, \quad g_z g_x = g_x g_z,$$
$$h_x h_y = h_y h_x, \quad h_y h_z = h_z h_y, \quad h_z h_x = h_x h_z, \tag{B23}$$

Note the above has g_x, g_y, g_z commute due to the identification on a $\mathbb{T}^3$ torus.

4. Explicit expression of T^{xy} in terms of (G, ω_4)

Similar to S^{xyz}, we can triangulate T^{xy} on $\mathbb{T}^3 \times I$. It is easier to start with a T^{xy} on $\mathbb{T}^2 \times I$ for 2D, which we denote $\mathsf{T}_{\mathrm{2D}}(\mathrm{w})$ and triangulate in the following $3!+1 = 7$ tetrahedra (3-simplex). Here we have the vertex ordering for the arrows: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 1' < 2' < 2^{*\prime} < 3' < 5' < 6' < 6^{*\prime} < 7'$.

$\mathsf{T}_{\mathrm{2D}}(\mathrm{w}) =$

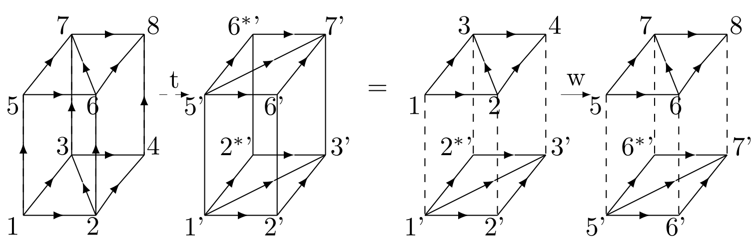


(B24)

The last extra piece is required to change the branching structure of the 3-simplex due to T^{xy} transformation.

For $\mathsf{T}_{\mathrm{3D}}(\mathrm{w})$, we simply have 7 pieces of slant products. Each slant product contains four 4-simplices. So totally there are 28 pieces of 4-cocycles in $\mathsf{T}_{\mathrm{3D}}(\mathrm{w})$.

$\mathsf{T}_{\mathrm{3D}}(\mathrm{w}) =$



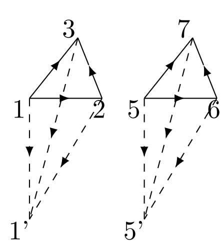
$= (\mathsf{T}_1)(\mathsf{T}_2)(\mathsf{T}_3)(\mathsf{T}_4)(\mathsf{T}_5)(\mathsf{T}_6)(\mathsf{T}_7).$ (B25)

The constraints given by $\mathsf{T}(\mathrm{w})$ are

$$\mathrm{w}^{-1} g_x \mathrm{w} = h_x, \tag{B26}$$
$$\mathrm{w}^{-1} g_x g_y \mathrm{w} = h_y, \tag{B27}$$
$$\mathrm{w}^{-1} g_z \mathrm{w} = h_z. \tag{B28}$$

Below we explicitly write down seven T_i, where we omit the w arrow without drawing it, which shall connect from the left 3-simplex to the right 3-simplex.

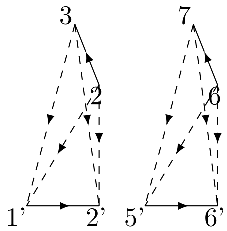
$(\mathsf{T}_1) =$



(B29)

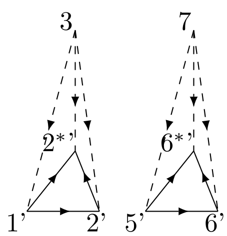
$$= \omega_4([12],[23],[35],[51']) \cdot \omega_4([23],[35],[56],[61'])$$
$$\cdot \omega_4([35],[56],[67],[71']) \cdot \omega_4^{-1}([56],[67],[71'],[1'5']).$$

$$(\mathsf{T}_2) = \tag{B30}$$



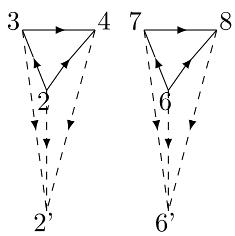
$$= \omega_4^{-1}([23],[36],[61'],[1'2']) \cdot \omega_4([36],[67],[71'],[1'2'])$$
$$\cdot \omega_4^{-1}([67],[71'],[1'2'],[2'5']) \cdot \omega_4([67],[72'],[2'5'],[5'6']).$$

$$(\mathsf{T}_3) = \tag{B31}$$



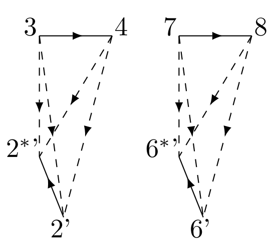
$$= \omega_4([37],[71'],[1'2'],[2'2^{*\prime}]) \cdot \omega_4^{-1}([71'],[1'2'],[2'2^{*\prime}],[2^{*\prime}5'])$$
$$\cdot \omega_4^{-1}([72'],[2'2^{*\prime}],[2^{*\prime}5'],[5'6'])$$
$$\cdot \omega_4^{-1}([72^{*\prime}],[2^{*\prime}5'],[5'6'],[6'6^{*\prime}]).$$

$$(\mathsf{T}_4) = \tag{B32}$$



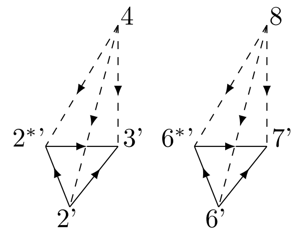
$$= \omega_4^{-1}([23],[34],[46],[62']) \cdot \omega_4^{-1}([34],[46],[67],[72'])$$
$$\cdot \omega_4^{-1}([46],[67],[78],[82']) \cdot \omega_4([67],[78],[82'],[2'6']).$$

$$(\mathsf{T}_5) = \tag{B33}$$



$$= \omega_4([34],[47],[72'],[2'2^{*\prime}]) \cdot \omega_4^{-1}([47],[78],[82'],[2'2^{*\prime}])$$
$$\cdot \omega_4([78],[82'],[2'2^{*\prime}],[2^{*\prime}6']) \cdot \omega_4^{-1}([78],[82^{*\prime}],[2^{*\prime}6'],[6'6^{*\prime}]).$$

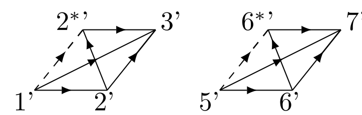
$$(\mathsf{T}_6) = \tag{B34}$$



$$= \omega_4^{-1}([48],[82'],[2'2^{*\prime}],[2^{*\prime}3']) \cdot \omega_4([82'],[2'2^{*\prime}],[2^{*\prime}3'],[3'6'])$$
$$\cdot \omega_4([82^{*\prime}],[2^{*\prime}3'],[3'6'],[6'6^{*\prime}]) \cdot \omega_4([83'],[3'6'],[6'6^{*\prime}],[6^{*\prime}7']).$$

For the tricky T_7, we shift $1'$ to a new later time slice $1''$, and shift $5'$ to a new later time slice $5''$:

$$(\mathsf{T}_7) = \tag{B35}$$



$$= \omega_4^{-1}([1'2'],[2'2^{*\prime}],[2^{*\prime}3'],[3'5'])$$
$$\cdot \omega_4([2'2^{*\prime}],[2^{*\prime}3'],[3'5'],[5'6'])$$
$$\cdot \omega_4^{-1}([2^{*\prime}3'],[3'5'],[5'6'],[6'6^{*\prime}])$$
$$\cdot \omega_4([3'5'],[5'6'],[6'6^{*\prime}],[6^{*\prime}7']).$$

One can also define the projection operator on $\mathbb{T}^3$ as

$$\mathsf{P}_{3\mathrm{D}}(\mathrm{w}) = (\mathsf{T}_1)(\mathsf{T}_2)(\mathsf{T}_3)(\mathsf{T}_4)(\mathsf{T}_5)(\mathsf{T}_6). \tag{B36}$$

Once we have obtained the path integral of 4-cocycles, we can change the flux basis to the canonical basis, and follow the procedure outlined in the Appendix of Ref.50 to derive the Rep theory formula given in our main text Sec.III B. An additional remark - an easier way to check the consistency of formulas for S and T is to use the rules in AppendixB 1 and to apply the discrete Fourier transformation of a finite group such as:

$$\frac{1}{|G|}\sum_{b,d,\beta} \mathrm{tr}\widetilde{\rho}_{\beta}^{b,d}(a)\ \mathrm{tr}\widetilde{\rho}_{\beta}^{b,d}(e)^* = \delta_{a,e}, \tag{B37}$$

$$\frac{1}{|G|}\sum_{a,b,d} \mathrm{tr}\widetilde{\rho}_{\alpha}^{a,b}(d)^*\ \mathrm{tr}\widetilde{\rho}_{\gamma}^{a,b}(d) = \delta_{\alpha,\gamma}. \tag{B38}$$

Use the properties of $\mathsf{C}^{(2)}_{a,b}(c,d)$ and the canonical basis $|\alpha,a,b\rangle$, we can justify that our formulas satisfy the rules (up to some projective representation's complex phases). See also Ref.68 for the derivation.

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