Quantum Statistics and Spacetime Surgery

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We apply the geometric-topology surgery theory on the spacetime manifold to study the constraints of quantum statistics data in 2+1 and 3+1 spacetime dimensions. First we introduce the fusion data for worldline and worldsheet operators capable creating anyon excitations of particles and strings, well-defined in gapped states of matter with intrinsic topological orders. Second we introduce the braiding statistics data of particles and strings, such as the geometric Berry matrices for particle-string Aharonov-Bohm and multi-loop adiabatic braiding process, encoded by submanifold linkings, in the closed spacetime 3-manifolds and 4-manifolds. Third we derive "quantum surgery" constraints analogous to Verlinde formula associating fusion and braiding statistics data via spacetime surgery, essential for defining the theory of topological orders, and potentially correlated to bootstrap boundary physics such as gapless modes, conformal field theories or quantum anomalies.

Decades ago, the exotic phenomenon named fractional quantum Hall effect was discovered [1]. The intrinsic relation between the topological quantum field theories (TQFT) and the topology of manifolds was found [2, 3] years after. The two breakthroughs partially motivated the study of topological order [4] as a new state of matter in quantum many-body systems and in condensed matter systems [5]. Topological orders are defined as the gapped states of matter with physical properties depending on global topology (such as the ground state degeneracy), robust against any local perturbation and any symmetry-breaking perturbation. Accordingly, topological orders cannot be characterized by the old paradigm of symmetry-breaking phases of matter via the Ginzburg-Landau theory [6, 7]. The systematic studies of 2+1 dimensional (2+1D) topological orders enhance our understanding of the real-world plethora phases including quantum Hall states and spin liquids [8]. In this work, we explore the relations between the 2+1D and 3+1D topological orders and the geometric-topology properties of 3- and 4-manifolds.

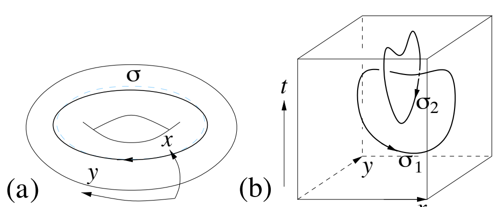


FIG. 1. (a) A topologically-ordered ground state on 2-torus is labeled by a quasiparticle σ. (b) The quantum amplitude of two linked loops is proportional to a complex number $\mathcal{S}_{\sigma_1\sigma_2}$.

Our main tools are the quantum mechanics in physics and the surgery theory in mathematics [9, 10]. Our main results are: (1) We provide the *fusion data* for worldline and worldsheet operators creating excitations of particles (i.e. anyons [11]) and strings (i.e. anyonic strings) in topological orders. (2) We provide the *braiding statistics data* of particles and strings encoded by the submanifold linking, in the 3- and 4-dimensional closed spacetime manifolds. (3) By "cutting and gluing" quantum amplitudes, we derive constraints between the fusion and braiding statistics data analogous to Verlinde formula [12, 13] for 2+1 and 3+1D topological orders.

Quantum Statistics: Fusion and Braiding Data – Imagine a renormalization-group-fixed-point topologically ordered quantum system on a spacetime manifold $\mathcal{M}$. The manifold can be viewed as a long-wavelength continuous limit of certain lattice regularization of the system. We aim to compute the quantum amplitude from "gluing" one ket-state $|R\rangle$ with another bra-state $\langle L|$, such as $\langle L|R\rangle$. A quantum amplitude also defines a path integral or a partition function with the linking of worldlines/worldsheets on a d-manifold $\mathcal{M}^d$, read as

$$\langle L|R\rangle = Z(\mathcal{M}^d; \text{Link}[\text{worldline,worldsheet}, \dots]). \quad (1)$$

For example, the $|R\rangle$ state can represent a ground state of 2-torus T^2_{xy} if we put the system on a solid torus $D^2_{xt} \times S^1_y$ [14] (see Fig.1(a) as the product space of 2-dimensional disk D^2 and 1-dimensional circle S^1). Note that its boundary is $\partial(D^2 \times S^1) = T^2$, and we can view the time t is along the radial direction. We label the trivial vacuum sector without any line insertions along S^1_y as $|0_{D^2_{xt}\times S^1_y}\rangle$, which is trivial respect to the measurement of any contractible line operator along S^1_x. A worldline operator creates a pair of anyon and anti-anyon at its end points, if it forms a closed loop then it can be viewed as creating then annihilating a pair of anyons in a closed trajectory [15]. Inserting a line operator $W_\sigma^{S^1_y}$ in the interior of $D^2_{xt} \times S^1_y$ gives a new state $W_\sigma^{S^1_y}|0_{D^2_{xt}\times S^1_y}\rangle \equiv |\sigma_{D^2_{xt}\times S^1_y}\rangle$. Here σ denotes the anyon type [16] along the oriented line, see Fig.1. Insert all possible line operators of all σ can completely span the ground state sectors for 2+1D topological order. The gluing of $\langle 0_{D^2\times S^1}|0_{D^2\times S^1}\rangle$ computes the path integral $Z(S^2\times S^1)$. If we view the S^1 as a compact time, this counts the ground state degeneracy (GSD) on a 2D spatial sphere S^2 without quasiparticle insertions, thus it is a 1-dimensional Hilbert space with $Z(S^2\times S^1)=1$. Similar relations hold for other dimensions, e.g. 3+1D topological orders on a S^3 without quasi-excitation yields $\langle 0_{D^3\times S^1}|0_{D^3\times S^1}\rangle = Z(S^3\times S^1)=1$.

2+1D Data – In 2+1D, we only need to consider the worldline operators which create particles. We define the *fusion data* via fusing worldline operators:

$$W_{\sigma_1}^{S^1_y}W_{\sigma_2}^{S^1_y}=\mathcal{F}^{\sigma}_{\sigma_1\sigma_2}W_{\sigma}^{S^1_y},\ \text{ and } G^{\alpha}_{\sigma}\equiv\langle\alpha|\sigma_{D^2_{xt}\times S^1_y}\rangle. \quad (2)$$

Here G^{α}_{σ} is read from the projection to a complete basis $\langle\alpha|$. Indeed the $W_{\sigma}^{S^1_y}$ generates all the canonical bases from $|0_{D^2_{xt}\times S^1_y}\rangle$. Thus the canonical projection can be $\langle\alpha| = \langle 0_{D^2_{xt}\times S^1_y}|(W_{\alpha}^{S^1_y})^{\dagger} = \langle 0_{D^2_{xt}\times S^1_y}|(W_{\bar\alpha}^{S^1_y}) = \langle \alpha_{D^2_{xt}\times S^1_y}|$, then we have $G^{\alpha}_{\sigma} = \langle 0_{D^2_{xt}\times S^1_y}|(W_{\bar\alpha}^{S^1_y})W_{\sigma}^{S^1_y}|0_{D^2_{xt}\times S^1_y}\rangle = Z(S^2\times S^1;\bar\alpha,\sigma) = \delta_{\alpha\sigma}$, where a pair of particle-antiparticle σ and $\bar\sigma$ can fuse to the vacuum. We derive

$$\begin{aligned}\mathcal{F}^{\alpha}_{\sigma_1\sigma_2} &= \langle 0_{D^2_{xt}\times S^1_y}|(W_{\bar\alpha}^{S^1_y})W_{\sigma_1}^{S^1_y}W_{\sigma_2}^{S^1_y}|0_{D^2_{xt}\times S^1_y}\rangle\\ &= Z(S^2\times S^1;\bar\alpha,\sigma_1,\sigma_2)\equiv\mathcal{N}^{\alpha}_{\sigma_1\sigma_2}. \quad (3)\end{aligned}$$

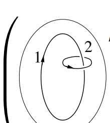
Since this path integral counts the dimension of the Hilbert space on the spatial S^2 (namely the number of channels σ_1 and σ_2 can fuse to α), this shows the fusion data $\mathcal{F}^{\alpha}_{\sigma_1\sigma_2}$ is equivalent to the fusion rule $\mathcal{N}^{\alpha}_{\sigma_1\sigma_2}$, symmetric under exchanging σ_1 and σ_2.

More generally we can glue the T^2_{xy}-boundary of $D^2_{xt}\times S^1_y$ via its mapping class group (MCG), namely $\text{MCG}(T^2)=\text{SL}(2,\mathbb{Z})$ generated by

$$\hat{\mathcal{S}}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \hat{\mathcal{T}}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \quad (4)$$

The $\hat{\mathcal{S}}$ identifies $(x,y)\to(-y,x)$ while $\hat{\mathcal{T}}$ identifies $(x,y)\to(x+y,y)$ of T^2_{xy}. Based on Eq.(1), we write down the quantum amplitudes of the two $\text{SL}(2,\mathbb{Z})$ generators $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ projecting to degenerate ground states. We denote gluing two open-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ along their boundaries $\mathcal{B}$ under the MCG-transformation $\hat{\mathcal{U}}$ to a new manifold as $\mathcal{M}_1\cup_{\mathcal{B};\hat{\mathcal{U}}}\mathcal{M}_2$ [17]. Then it is amusing to visualize the gluing $D^2\times S^1\cup_{T^2;\hat{\mathcal{S}}}D^2\times S^1=S^3$ shows that the $\mathcal{S}_{\bar\sigma_1\sigma_2}$ represents the Hopf linking of two worldlines σ_1 and σ_2 (e.g. Fig.1(b)) in S^3 with the given orientation (in the canonical basis $\mathcal{S}_{\bar\sigma_1\sigma_2}=\langle\sigma_1|\hat{\mathcal{S}}|\sigma_2\rangle$):

$$\mathcal{S}_{\bar\sigma_1\sigma_2}\equiv\langle\sigma_{1\,D^2_{xt}\times S^1_y}|\hat{\mathcal{S}}|\sigma_{2\,D^2_{xt}\times S^1_y}\rangle = Z\left(\ \right). \quad (5)$$

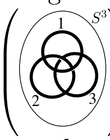


Use the gluing $D^2\times S^1\cup_{T^2;\hat{\mathcal{T}}}D^2\times S^1=S^2\times S^1$, we can derive a well known result written in the canonical bases,

$$\mathcal{T}_{\sigma_1\sigma_2}\equiv\langle\sigma_{1\,D^2_{xt}\times S^1_y}|\hat{\mathcal{T}}|\sigma_{2\,D^2_{xt}\times S^1_y}\rangle=\delta_{\sigma_1\sigma_2}\mathrm{e}^{\mathrm{i}\theta_{\sigma_2}}. \quad (6)$$

Its spacetime configuration is that two unlinked closed worldlines σ_1 and σ_2, with the worldline σ_2 twisted by 2π. The amplitude of a twisted worldline is given by the amplitude of untwisted worldline multiplied by $\mathrm{e}^{\mathrm{i}\theta_{\sigma_2}}$, where $\theta_\sigma/2\pi$ is the spin of the σ excitation. It means that $\mathcal{S}_{\bar\sigma_1\sigma_2}$ measures the *mutual braiding statistics* of σ_1-and-σ_2, while $\mathcal{T}_{\sigma\sigma}$ measures the *spin* and *self-statistics* of σ.

We can introduce additional data, the Borromean rings (BR) linking between three S^1 circles in S^3, written as $Z\left(\ \right) = Z[S^3;\text{BR}[\sigma_1,\sigma_2,\sigma_3]]$.



Although we do not know a bra-ket expression for this amplitude, we can reduce this configuration to an easier one $Z[T^3_{xyt};\sigma'_{1x},\sigma'_{2y},\sigma'_{3t}]$, a path integral of 3-torus with three orthogonal line operators each inserting along a non-contractible S^1 direction. The later is a simpler expression because we can uniquely define the three line insertions exactly along the homology group generators of T^3, namely $H_1(T^3,\mathbb{Z})=\mathbb{Z}^3$. The two path integrals are related by three consecutive modular $\mathcal{S}$ surgeries done along the T^2-boundary of $D^2\times S^1$ tubular neighborhood around three S^1 rings [18]. Namely, $Z[T^3_{xyt};\sigma'_{1x},\sigma'_{2y},\sigma'_{3t}] = \sum_{\sigma_1,\sigma_2,\sigma_3}\mathcal{S}_{\sigma'_{1x}\sigma_1}\mathcal{S}_{\sigma'_{2y}\sigma_2}\mathcal{S}_{\sigma'_{3z}\sigma_3}Z[S^3;\text{BR}[\sigma_1,\sigma_2,\sigma_3]]$.

3+1D Data – In 3+1D, there are intrinsic meanings of braidings of string-like excitations. We need to consider both the worldline and the worldsheet operators which create particles and strings. In addition to the S^1-worldline operator $W_{\sigma}^{S^1}$, we introduce S^2- and T^2-worldsheet operators as $V_{\mu}^{S^2}$ and $V_{\mu'}^{T^2}$ which create closed-strings (or loops) at their spatial cross sections. We consider the vacuum sector ground state on open 4-manifolds: $|0_{D^3\times S^1}\rangle$, $|0_{D^2\times S^2}\rangle$, $|0_{D^2\times T^2}\rangle$ and $|0_{S^4\smallsetminus D^2\times T^2}\rangle$, while their boundaries are $\partial(D^3\times S^1)=\partial(D^2\times S^2)=S^2\times S^1$ and $\partial(D^2\times T^2)=\partial(S^4\smallsetminus D^2\times T^2)=T^3$. Here $\mathcal{M}_1\smallsetminus\mathcal{M}_2$ means the complement space of $\mathcal{M}_2$ out of $\mathcal{M}_1$. Similar to 2+1D, we define the *fusion data* $\mathcal{F}^M$ by fusing operators:

$$W_{\sigma_1}^{S^1}W_{\sigma_2}^{S^1}=(\mathcal{F}^{S^1})^{\sigma}_{\sigma_1\sigma_2}W_{\sigma}^{S^1}, \quad (7)$$

$$V_{\mu_1}^{S^2}V_{\mu_2}^{S^2}=(\mathcal{F}^{S^2})^{\mu_3}_{\mu_1\mu_2}V_{\mu_3}^{S^2}, \quad (8)$$

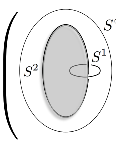
$$V_{\mu_1}^{T^2}V_{\mu_2}^{T^2}=(\mathcal{F}^{T^2})^{\mu_3}_{\mu_1\mu_2}V_{\mu_3}^{T^2}. \quad (9)$$

Notice that we introduce additional upper indices in the fusion algebra $\mathcal{F}^M$ to specify the topology of M for the fused operators [19]. We require normalizing worldline/sheet operators for a proper basis, so that the $\mathcal{F}^M$ is also properly normalized in order for $Z(Y^{d-1}\times S^1;\dots)$

as the GSD on a spatial closed manifold Y^{d-1} always be a positive integer. In principle, we can derive the fusion rule of excitations in any closed spacetime 4-manifold. For instance, the fusion rule for fusing three particles on a spatial S^3 is $Z(S^3 \times S^1; \bar{\alpha}, \sigma_1, \sigma_2) = \langle 0_{D^3\times S^1} | W^{S^1}_{\bar{\alpha}} W^{S^1}_{\sigma_1} W^{S^1}_{\sigma_2} | 0_{D^3\times S^1} \rangle = (\mathcal{F}^{S^1})^{\alpha}_{\sigma_1\sigma_2}$. Many more examples of fusion rules can be derived from computing $Z(\mathcal{M}^4; \sigma, \mu, \dots)$ [20] by using $\mathcal{F}^M$ and Eq.(1), here the worldline and worldsheet are submanifolds *parallel not linked* with each other.

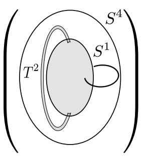
If the worldline and worldsheet are linked as Eq.(1), then the path integral encodes the *braiding data*. Below we discuss the important braiding processes in 3+1D. First, the Aharonov-Bohm particle-loop braiding can be represented as a S^1-worldline of particle and a S^2-worldsheet of loop linked in S^4 spacetime,

$$\mathrm{L}^{(S^2,S^1)}_{\mu\sigma} \equiv \langle 0_{D^2\times S^2} | V^{S^2 \dagger}_{\mu} W^{S^1}_{\sigma} | 0_{D^3\times S^1} \rangle = Z \left(\quad \right), \tag{10}$$



if we design the worldline and worldsheet along the generators of the first and the second homology group $H_1(D^3 \times S^1, \mathbb{Z}) = H_2(D^2 \times S^2, \mathbb{Z}) = \mathbb{Z}$ respectively via Alexander duality. We also use the fact $S^2 \times D^2 \cup_{S^2\times S^1} D^3 \times S^1 = S^4$, thus $\langle 0_{D^2\times S^2} | 0_{D^3\times S^1} \rangle = Z(S^4)$. Second, we can also consider particle-loop braiding as a S^1-worldline of particle and a T^2-worldsheet (below T^2 drawn as S^2 with a handle) of loop linked in S^4,

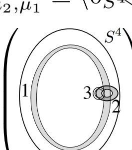
$$\langle 0_{D^2\times T^2} | V^{T^2 \dagger}_{\mu} W^{S^1}_{\sigma} | 0_{S^4 \smallsetminus D^2\times T^2} \rangle = Z \left(\quad \right), \tag{11}$$

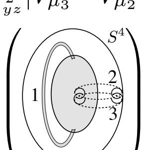


if we design the worldline and worldsheet along the generators of $H_1(S^4 \smallsetminus D^2 \times T^2, \mathbb{Z}) = H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$ respectively. Compare Eqs.(10) and (11), the loop excitation of S^2-worldsheet is shrinkable [21], while the loop of T^2-worldsheet needs not to be shrinkable.

Third, we can represent a three-loop braiding process [22–27] as three T^2-worldsheets *triple-linking* [28] in the spacetime S^4 (as the first figure in Eq.(12)). We find that

$$\mathrm{L}^{\mathrm{Tri}}_{\mu_3,\mu_2,\mu_1} \equiv \langle 0_{S^4 \smallsetminus D^2_{wx}\times T^2_{yz}} | V^{T^2_{zx}\dagger}_{\mu_3} V^{T^2_{xy}\dagger}_{\mu_2} V^{T^2_{yz}}_{\mu_1} | 0_{D^2_{wx}\times T^2_{yz}} \rangle$$
$$= Z \left(\quad \right) = Z \left(\quad \right), \tag{12}$$

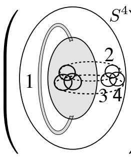




where we design the worldsheets $V^{T^2_{yz}}_{\alpha}$ along the generator of homology group $H_2(D^2_{wx} \times T^2_{yz}, \mathbb{Z}) = \mathbb{Z}$ while we design $V^{T^2_{xy}\dagger}_{\beta}$ and $V^{T^2_{zx}\dagger}_{\gamma}$ along the two generators of $H_2(S^4 \smallsetminus D^2_{wx} \times T^2_{yz}, \mathbb{Z}) = \mathbb{Z}^2$ respectively. We find that Eq.(12) is also equivalent to the spun surgery construction [29] of a Hopf link linked by a third T^2-torus, shown as the second figure in Eq.(12).

Fourth, the four-loop braiding process, where three loops dancing in the Borromean ring trajectory while linked by a fourth loop [30], can characterize certain 3+1D non-Abelian topological orders [25]. We find it is also the spun surgery construction of Borromean rings of three loops linked by a fourth torus in the spacetime picture, and its path integral $Z[S^4; \mathrm{Link}[\mathrm{Spun}[\mathrm{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$ can be transformed:

$$Z \left(\quad \right) \xrightarrow{\text{surgery}} Z[T^4 \# S^2 \times S^2; \mu_4', \mu_3', \mu_2', \mu_1']$$



$$= \langle 0_{T^4\# S^2\times S^2 \smallsetminus D^2_{wx}\times T^2_{yz}} | V^{T^2\dagger}_{\mu_4'} V^{T^2\dagger}_{\mu_3'} V^{T^2\dagger}_{\mu_2'} V^{T^2_{yz}}_{\mu_1'} | 0_{D^2_{wx}\times T^2_{yz}} \rangle, \tag{13}$$

where the surgery contains four consecutive modular $\mathcal{S}$-transformations done along the T^3-boundary of $D^2 \times T^2$ tubular neighborhood around four T^2-worldsheets [31]. The final spacetime manifold is $T^4 \# S^2 \times S^2$, where $\#$ stands for the connected sum.

We can glue the T^3-boundary of 4-submanifolds (e.g. $D^2 \times T^2$ and $S^4 \smallsetminus D^2 \times T^2$) via $\mathrm{MCG}(T^3) = \mathrm{SL}(3, \mathbb{Z})$ generated by [32]

$$\hat{\mathcal{S}}^{xyz} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{\mathcal{T}}^{xy} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

In this work, we define their representations as [32]

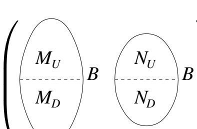
$$\mathcal{S}^{xyz}_{\mu_2,\mu_1} \equiv \langle 0_{D^2_{xw}\times T^2_{yz}} | V^{T^2_{yz}\dagger}_{\mu_2} \hat{\mathcal{S}}^{xyz} V^{T^2_{yz}}_{\mu_1} | 0_{D^2_{xw}\times T^2_{yz}} \rangle, \tag{15}$$

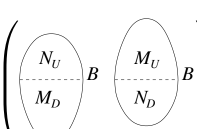
$$\mathcal{T}^{xy}_{\mu_2,\mu_1} \equiv \langle 0_{D^2_{xw}\times T^2_{yz}} | V^{T^2_{yz}\dagger}_{\mu_2} \hat{\mathcal{T}}^{xy} V^{T^2_{yz}}_{\mu_1} | 0_{D^2_{xw}\times T^2_{yz}} \rangle, \tag{16}$$

while $\mathcal{S}^{xyz}$ is a spun-Hopf link in $S^3 \times S^1$, and $\mathcal{T}^{xy}$ is related to the *topological spin* and *self-statistics* of closed strings [25].

Quantum surgery and general Verlinde formulas – Now we like to derive a powerful identity for fixed-point path integral of topological orders. If the path integral formed by disconnected manifolds M and N, denoted as $M \sqcup N$, we have $Z(M \sqcup N) = Z(M)Z(N)$. Assume that (1) we divide both M and N into two pieces such that $M = M_U \cup_{\partial M_D} M_D$, $N = N_U \cup_{\partial N_D} N_D$, and their cut configurations equal $\partial M_D = \partial M_U = \partial N_D = \partial N_U = B$, and (2) the Hilbert space on the spatial slice is 1-dimensional (namely the GSD=1)[33], then we obtain

$$Z \left(\quad \right) = Z \left(\quad \right) \tag{17}$$

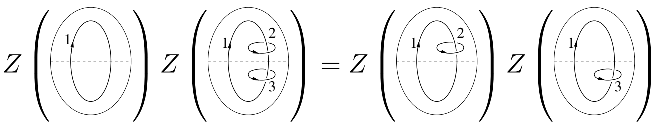




$$\Rightarrow Z(M_U \cup_B M_D) Z(N_U \cup_B N_D) = Z(N_U \cup_B M_D) Z(M_U \cup_B N_D).$$

In 2+1D, we can derive the renowned Verlinde formula [3, 12, 13] by one version of Eq.(17):

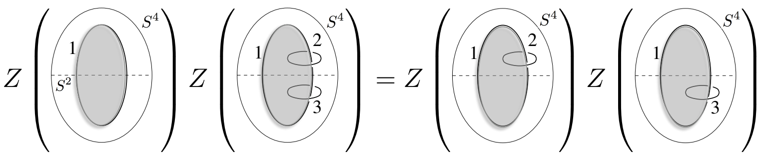
$Z(\cdot)\,Z(\cdot) = Z(\cdot)\,Z(\cdot)$



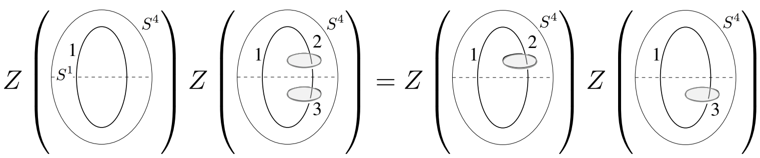
$$\Rightarrow \mathcal{S}_{\bar{\sigma}_1 0}\sum_{\sigma_4}\mathcal{S}_{\bar{\sigma}_1\sigma_4}\mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3} = \mathcal{S}_{\bar{\sigma}_1\sigma_2}\mathcal{S}_{\bar{\sigma}_1\sigma_3}, \quad (18)$$

where each spacetime manifold is S^3, with the line operator insertions such as an unlink and Hopf links. Each S^3 is cut into two D^3 pieces, so $D^3\cup_{S^2} D^3 = S^3$, while the boundary (dashed) cut is $B = S^2$. The GSD for this spatial section S^2 with a pair of particle-antiparticle must be 1, so our surgery satisfies the assumptions for Eq.(17). The second line is derived from rewriting path integrals in terms of our data introduced before – the fusion rule $\mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3}$ comes from fusing $\sigma_2\sigma_3$ into σ_4 which Hopf-linked with σ_1, while Hopf links render $\mathcal{S}$ matrices [34].

In 3+1D, the particle-string braiding in terms of S^4-spacetime path integral Eq.(10) has constraint formulas:



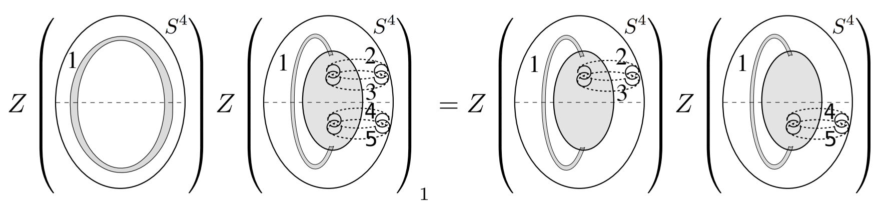
$$\Rightarrow \mathrm{L}^{(S^2,S^1)}_{\mu_1 0}\sum_{\sigma_4}\mathrm{L}^{(S^2,S^1)}_{\mu_1\sigma_4}(\mathcal{F}^{S^1})^{\sigma_4}_{\sigma_2\sigma_3} = \mathrm{L}^{(S^2,S^1)}_{\mu_1\sigma_2}\mathrm{L}^{(S^2,S^1)}_{\mu_1\sigma_3}. \quad (19)$$



$$\Rightarrow \mathrm{L}^{(S^2,S^1)}_{0\sigma_1}\sum_{\mu_4}\mathrm{L}^{(S^2,S^1)}_{\mu_4\sigma_1}(\mathcal{F}^{S^2})^{\mu_4}_{\mu_2\mu_3} = \mathrm{L}^{(S^2,S^1)}_{\mu_2\sigma_1}\mathrm{L}^{(S^2,S^1)}_{\mu_3\sigma_1}. \quad (20)$$

The shaded areas mean S^2-spheres. Notice that Eqs.(19) and (20) are symmetric by exchanging worldsheet/worldline indices: $\mu\leftrightarrow\sigma$, except that the fusion data is different: $\mathcal{F}^{S^1}$ fuses worldlines, $\mathcal{F}^{S^2}$ fuses worldsheets.

We also derive a quantum surgery constraint formula [35] for the three-loop braiding in terms of S^4-spacetime path integral Eq.(12) via the $\mathcal{S}^{xyz}$-surgery and its matrix representation:



$$\Rightarrow \mathrm{L}^{\mathrm{Tri}}_{0,0,\mu_1}\cdot\sum_{\Gamma,\Gamma',\Gamma_1,\Gamma_1'}(\mathcal{F}^{T^2})^{\Gamma}_{\zeta_2,\zeta_4}(\mathcal{S}^{xyz})^{-1}_{\Gamma',\Gamma}(\mathcal{F}^{T^2})^{\Gamma_1}_{\mu_1\Gamma'}\mathcal{S}^{xyz}_{\Gamma_1',\Gamma_1}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\Gamma_1'}$$
$$= \sum_{\zeta_2',\eta_2,\eta_2'}(\mathcal{S}^{xyz})^{-1}_{\zeta_2',\zeta_2}(\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\zeta_2'}\mathcal{S}^{xyz}_{\eta_2',\eta_2}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\eta_2'}\cdot\sum_{\zeta_4',\eta_4,\eta_4'}(\mathcal{S}^{xyz})^{-1}_{\zeta_4',\zeta_4}(\mathcal{F}^{T^2})^{\eta_4}_{\mu_1\zeta_4'}\mathcal{S}^{xyz}_{\eta_4',\eta_4}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\eta_4'}, \quad (21)$$

here ζ_2 relates to fusing μ_2-μ_3-worldsheets, and ζ_4 relates to fusing μ_4-μ_5-worldsheets. Only μ_1,ζ_2,ζ_4 are the fixed indices, other indices are summed over. For all path integrals of S^4 in Eqs.(19), (20) and (21), each S^4 is cut into two D^4 pieces, so $D^4\cup_{S^3} D^4 = S^4$, while the dashed cut is $B = S^3$. Here we require a stronger criterion that all loop excitations are gapped without zero modes, then the GSD is 1 for all above spatial section S^3 (with a loop in Eq.(19), a pair of particle-antiparticle in Eq.(20), and a pair of loop-antiloop in Eq.(21)). Thus all our surgeries satisfy the assumptions for Eq.(17).

The above Verlinde-like formulas constrain the fusion data (e.g. $\mathcal{N}$, $\mathcal{F}^{S^1}$, $\mathcal{F}^{S^2}$, $\mathcal{F}^{T^2}$, etc.) and braiding data (e.g. $\mathcal{S}$, $\mathcal{T}$, $\mathrm{L}^{(S^2,S^1)}$, $\mathrm{L}^{\mathrm{Tri}}$, $\mathcal{S}^{xyz}$, etc.). Moreover, we can derive constraints between the fusion data itself. Since a T^2-worldsheet contains two non-contractible S^1-worldlines along its two homology group generators in $H_1(T^2,\mathbb{Z})=\mathbb{Z}^2$, the T^2-worldsheet operator $V^{T^2}_{\mu}$ contains the data of S^1-worldline operator $W^{S^1}_{\sigma}$. More explicitly, we can compute the state $W^{S^1_y}_{\sigma_1}W^{S^1_y}_{\sigma_2}V^{T^2_{yz}}_{\mu_2}|0_{D^2_{wx}\times T^2_{yz}}\rangle$ by fusing two $W^{S^1}_{\sigma}$ operators and one $V^{T^2}_{\mu}$ operator in different orders, then we obtain a consistency formula [35]:

$$\sum_{\sigma_3}(\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2}(\mathcal{F}^{T^2})^{\mu_3}_{\sigma_3\mu_2} = \sum_{\mu_1}(\mathcal{F}^{T^2})^{\mu_1}_{\sigma_2\mu_2}(\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_1}. \quad (22)$$

We organize our quantum statistics data of fusion and braiding, and some explicit examples of topological orders and their topological invariances in terms of our data in the Supplemental Material.

CONCLUSION

It is known that the quantum statistics of particles in 2+1D begets anyons, beyond the familiar statistics of bosons and fermions, while Verlinde formula [12] plays a key role to dictate the consistent anyon statistics. In this work, we derive a set of quantum surgery formulas analogous to Verlinde's constraining the fusion and braiding quantum statistics of anyon excitations of particle and string in 3+1D.

A further advancement of our work, comparing to the pioneer work Ref.[3] on 2+1D Chern-Simons gauge theory, is that we apply the surgery idea to generic 2+1D and 3+1D topological orders without assuming quantum field theory (QFT) or gauge theory description. Although many lattice-regularized topological orders happen to have TQFT descriptions at low energy, we may not know which topological order derives which TQFT easily. Instead we simply use quantum amplitudes written in the bra and ket (over-)complete bases, obtained from inserting worldline/sheet operators along the cycles of nontrivial homology group generators of a spacetime manifold, to cut and glue to the desired path integrals. Consequently our approach, without the necessity of any QFT description, can be powerful to describe more generic quantum systems. While our result is originally based on studying specific examples of TQFT in Dijkgraaf-Witten gauge theory [35, 36], we formulate the data without using QFT. We have incorporated the necessary generic quantum statistic data and new constraints to characterize some 3+1D topological orders (including Dijkgraaf-Witten's), we will leave the issue of their sufficiency and completeness for future work. Formally, our approach can be applied to any spacetime dimensions.

It will be interesting to study the analogous Verlinde formula constraints for 2+1D boundary states, such as highly-entangled gapless modes, conformal field theories (CFT) and anomalies, for example through the bulk-boundary correspondence [3, 37–39]. The set of consistent quantum surgery formulas we derive may lead to an alternative effective way to *bootstrap* [40, 41] 3+1D topological states of matter and 2+1D CFT.

Note added: The formalism and some results discussed in this work have been partially reported in the first author's Ph.D. thesis [42]. Readers may refer to Ref.[42] for other discussions.

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Supplemental Material

A. Summary of quantum statistics data of fusion and braiding

Quantum statistics data of fusion and braiding
Data for **2+1D** topological orders:
• Fusion data: $\mathcal{N}_{\sigma_1\sigma_2}^{\sigma_3} = \mathcal{F}_{\sigma_1\sigma_2}^{\sigma_3}$ (fusion tensor), • Braiding data: $\mathcal{S}^{xy}$, $\mathcal{T}^{xy}$ (modular SL$(2,Z)$ matrices from MCG(T^2)), $Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$ (or $Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3])$, etc.
Data for **3+1D** topological orders:
• Fusion data: $(\mathcal{F}^{S^1})_{\sigma_1\sigma_2}^{\sigma_3}$, $(\mathcal{F}^{S^2})_{\mu_1\mu_2}^{\mu_3}$, $(\mathcal{F}^{T^2})_{\mu_1\mu_2}^{\mu_3}$. (fusion tensor) • Braiding data: $\mathcal{S}^{xyz}$, $\mathcal{T}^{xy}$ (modular SL$(3,\mathbb{Z})$ matrices from MCG(T^3), including $\mathcal{S}^{xy}$) $\text{L}^{\text{Tri}}_{0,0,\mu}$ (from $\text{L}^{\text{Tri}}_{\mu_3,\mu_2,\mu_1}$), $\text{L}^{\text{Lk}(S^2,S^1)}_{\mu\sigma}$, $Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$ (from $Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]])$, etc.

TABLE I. Some data for 2+1D and 3+1D topological orders encodes their quantum statistics properties, such as fusion and braiding statistics of their quasi-excitations (anyonic particles and anyonic strings). However, the data is not complete because we do not account the degrees of freedom of their boundary modes, such as the chiral central charge $c_- = c_L - c_R$ for 2+1D topological orders.

We organize the quantum statistics data of fusion and braiding introduced in the main text in Table I. We propose using the set of data in Table I to label topological orders. We also remark that Table I may not contain all sufficient data to characterize and classify all topological orders. What can be the missing data in Table I? Clearly, there is the chiral central charge $c_- = c_L - c_R$, the difference between the left and right central charges, missing for 2+1D topological orders. The c_- is essential for describing 2+1D topological orders with 1+1D boundary gapless chiral edge modes. The gapless chiral edge modes cannot be fully gapped out by adding scattering terms between different modes, because they are protected by the net chirality. So our 2+1D data only describes *2+1D non-chiral topological orders*. Similarly, our 2+1D/3+1D data may not be able to fully classify 2+1D/3+1D topological orders whose boundary modes are *protected to be gapless*. We may need additional data to encode boundary degrees of freedom for their boundary modes.

In some case, some of our data may overlap with the information given by other data. For example, the 2+1D topological order data ($\mathcal{S}_{xy}, \mathcal{T}_{xy}$, $\mathcal{N}_{\sigma_1\sigma_2}^{\sigma_3}$) may contain the information of $Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$ (or $Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3])$ already, since we know that we the former set of data may fully classify 2+1D bosonic topological orders.

Although it is possible that there are extra required data beyond what we list in Table I, we find that Table I is sufficient enough for a large class of topological orders, at least for those described by Dijkgraaf-Witten twisted gauge theory [36] and those gauge theories with finite Abelian gauge groups. In the next Appendix, we will give some explicit examples of 2+1D and 3+1D topological orders described by Dijkgraaf-Witten theory, which can be completely characterized and classified by the data given in Table I.

B. Examples of topological orders and their topological invariances in terms of our data

In Table II, we give some explicit examples of 2+1D and 3+1D topological orders from Dijkgraaf-Witten twisted gauge theory. We like to emphasize that our quantum-surgery Verlinde-like formulas apply to generic 2+1D and 3+1D topological orders beyond the gauge theory or field theory description. So our formulas apply to quantum phases of matter or theories beyond the Dijkgraaf-Witten twisted gauge theory description. We list down these examples only because these are famous examples with a more familiar gauge theory understanding. In terms of topological order language, Dijkgraaf-Witten theory describes the low energy physics of certain bosonic topological orders which can be regularized on a lattice Hamiltonian [23, 25, 44] with local bosonic degrees of freedom (without fermions).

We also clarify that what we mean by the correspondence between the items in the same row in Table II: (i). Quantum statistic braiding data, (ii). cocycles and (iii). topological quantum field theory (TQFT). What we mean is that we can distinguish the topological orders of given cocycles of (ii) with the low energy TQFT of (iii) by

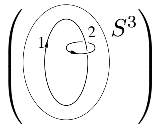
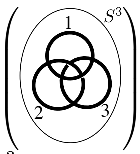
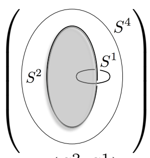
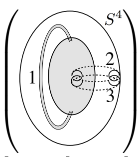
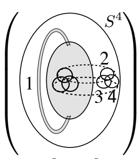
(i). Path-integral linking topological invariances; Quantum statistic braiding data	(ii). Group-cohomology cocycles distinguished by the braiding in (i)	(iii). TQFT actions characterized by the spacetime-braiding in (i)
	2+1D	
Z  $= Z[S^3; \mathrm{Hopf}[\sigma_1, \sigma_2]]$ $= \mathcal{S}_{\bar{\sigma}_1 \sigma_2}$	$\exp\left(\frac{2\pi\, \mathrm{i}\, p_{IJ}}{N_I N_J}\, a_I (b_J + c_J - [b_J + c_J])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \frac{\mathrm{i} p_{IJ}}{2\pi} A^I \wedge \mathrm{d}A^J$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I.$
Z  $= Z[S^3; \mathrm{BR}[\sigma_1, \sigma_2, \sigma_3];$ Also $Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$	$\exp\left(\frac{2\pi\, \mathrm{i}\, p_{123}}{N_{123}}\, a_1 b_2 c_3\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \mathrm{i} c_{123} A^1 \wedge A^2 \wedge A^3$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I + 2\pi \tilde{c}_{IJK} A^J g^K$ $-\pi \tilde{c}_{IJK} A^J \,\mathrm{d}g^K.$
	3+1D	
Z  $= \mathrm{L}^{(S^2, S^1)}_{\mu\sigma}$	1	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge \mathrm{d}A^I$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I.$
Z  $= Z[S^4; \mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3, \mu_2]], \mu_1]]$ $= \mathrm{L}^{\mathrm{Tri}}_{\mu_3, \mu_2, \mu_1}$	$\exp\left(\frac{2\pi\, \mathrm{i}\, p_{IJK}}{(N_{IJ} \cdot N_K)} (a_I b_J)(c_K + d_K - [c_K + d_K])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \frac{\mathrm{i} N_1 N_2\, p_{IJK}}{(2\pi)^2 N_{12}} A^I \wedge A^J \wedge \mathrm{d}A^K$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I + \epsilon_{IJ} \frac{N_1 N_2\, p_{IJK}}{2\pi N_{12}} dg^J \wedge A^K,$ here K is fixed.
Z  $= Z[S^4; \mathrm{Link}[\mathrm{Spun}[\mathrm{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]];$ Also $Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$	$\exp\left(\frac{2\pi\, \mathrm{i}\, p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \mathrm{i} c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + d\eta^I - \pi \tilde{c}_{IJKL} A^J A^K g^L$ $+\pi \tilde{c}_{IJKL} A^J g^K dg^L - \frac{\pi}{3} \tilde{c}_{IJKL} g^J dg^K dg^L.$

TABLE II. Examples of topological orders and their topological invariances in terms of our data in the spacetime dimension $d+1$D. Here some explicit examples are given as Dijkgraaf-Witten twisted gauge theory [36] with finite gauge group, such as $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \mathbb{Z}_{N_4} \times \dots$, although our quantum statistics data can be applied to more generic quantum systems without gauge or field theory description. The first column shows the path integral form which encodes the braiding process of particles and strings in the spacetime. In terms of spacetime picture, the path integral has nontrivial linkings of worldlines and worldsheets. The geometric Berry phases produced from this adiabatic braiding process of particles and strings yield the measurable quantum statistics data. This data also serves as topological invariances for topological orders. The second column shows the group-cohomology cocycle data ω as a certain partition-function solution of Dijkgraaf-Witten theory, where ω belongs to the group-cohomology group, $\omega \in \mathcal{H}^{d+1}[G, \mathbb{R}/\mathbb{Z}] = \mathcal{H}^{d+1}[G, \mathrm{U}(1)]$. The third column shows the proposed continuous low-energy field theory action form for these theories and their gauge transformations. In 2+1D, A and B are 1-forms, while g and η are 0-forms. In 3+1D, B is a 2-form, A and η are 1-forms, while g is a 0-form. Here $I, J, K \in \{1, 2, 3, \dots\}$ belongs to the gauge subgroup indices, $N_{12\dots u} \equiv \gcd(N_1, N_2, \dots, N_u)$ is defined as the greatest common divisor (gcd) of $N_1, N_2, \dots, N_u$. Here $p_{IJ} \in \mathbb{Z}_{N_{IJ}}, p_{123} \in \mathbb{Z}_{N_{123}}, p_{IJK} \in \mathbb{Z}_{N_{IJK}}, p_{1234} \in \mathbb{Z}_{N_{1234}}$ are integer coefficients. The $c_{IJ}, c_{123}, c_{IJK}, c_{1234}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2\, p_{12}}{N_{12}}$, $c_{123} = \frac{1}{(2\pi)^2} \frac{N_1 N_2 N_3\, p_{123}}{N_{123}}$, $c_{1234} = \frac{1}{(2\pi)^3} \frac{N_1 N_2 N_3 N_4\, p_{1234}}{N_{1234}}$. Be aware that we define both $p_{IJ\dots}$ and $c_{IJ\dots}$ as constants with *fixed-indices* $I, J, \dots$ without summing over those indices; while we additionally define $\tilde{c}_{IJ\dots} \equiv \epsilon_{IJ\dots} c_{12\dots}$ with the $\epsilon_{IJ\dots} = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where $I, J, \dots$ are *free indices* needed to be Einstein-summed over, but $c_{12\dots}$ is fixed. The quantization labelings are described and derived in [25, 43].

measuring their quantum statistic Berry phase under the prescribed braiding process in the path integral of (i). For example, the mutual braiding (Hopf linking) measures the $\mathcal{S}$ matrix distinguishing different types of $\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \frac{\mathrm{i} p_{IJ}}{2\pi} A^I \wedge dA^J$ with different p_{IJ} couplings; while the Borromean ring braiding can distinguish different types of $\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{123} A^1 \wedge A^2 \wedge A^3$ with different c_{123} couplings. However, the table does not mean that we cannot use braiding data in one row to measures the TQFT in another row. For example, $\mathcal{S}$ matrix can also distinguish the $\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{123} A^1 \wedge A^2 \wedge A^3$-type theory. However, $Z[S^3; \mathrm{BR}[\sigma_1, \sigma_2, \sigma_3] = Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}] = 1$ is trivial for $\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \frac{\mathrm{i} p_{IJ}}{2\pi} A^I \wedge dA^J$ with any p_{IJ}. Thus Borromean ring braiding cannot measure nor distinguish the nontrivial-ness of p_{IJ}-type theories.

The relevant field theories are also discussed in Ref. [43, 45–48], here we systematically summarize and claim the field theories in Table II third column describe the low energy TQFTs of Dijkgraaf-Witten theory.

C. Derivations of some quantum surgery formulas

In this Appendix, we derive some Verlinde-like quantum surgery formulas, which are constraints of fusion and braiding data of topological orders. We will work out the derivations of Eqs.(18),(19),(20) and then later we will derive Eq.(21) step by step. We will also derive the fusion constraint Eq.(22) more explicitly.

First we derive a generic formula for our use of surgery. We consider a closed manifold M glued by two pieces M_U and M_D so that $M = M_U \cup_B M_D$ where $B = \partial M_U = \partial M_D$. We consider there are insertions of operators in M_U and M_D. We denote the generic insertions in M_U as α_{M_U} and the generic insertions in M_D as β_{M_D}. Here both α_{M_U} and β_{M_D} may contain both worldline and worldsheet operators. We write the path integral as $Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U} | \beta_{M_D} \rangle$, while the worldline/worldsheet may be linked or may not be linked in M. Here we introduce an extra subscript M in $Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U} | \beta_{M_D} \rangle_M$ to specify the glued manifold is $M_U \cup_B M_D = M$. Now we like to do surgery by cutting out the submanifold M_D out of M and reglue it back to M_U via its mapping class group generator $\hat{K} \in \mathrm{MCG}(B) = \mathrm{MCG}(\partial M_U) = \mathrm{MCG}(\partial M_D)$. We now give some additional assumptions.

Assumption 1: The operator insertions in M are well-separated into M_U and M_D, so that no operator insertions cross the boundary B. Namely, at the boundary cut B there are no defects of point or string excitations from the cross-section of α_{M_U}, β_{M_D} or any other operators.

Assumption 2: We can generate the complete bases of degenerate ground states fully spanning the dimension of Hilbert space for the spatial section of B, by inserting distinct operators (worldline/worldsheet, etc.) into M_D. Namely, we insert a set of operators Φ in the interior of $|0_{M_D}\rangle$ to obtain a new state $\Phi|0_{M_D}\rangle \equiv |\Phi_{M_D}\rangle$, such that these states $\{\Phi|0_{M_D}\rangle\}$ are orthonormal canonical bases, and the dimension of the vector space $\dim(\{\Phi|0_{M_D}\rangle\})$ equals to the ground state degeneracy (GSD) of the topological order on the spatial section B.

If both assumptions hold, then we find a relation:

$$
\begin{aligned}
&Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U} | \beta_{M_D} \rangle_M = \sum_\Phi \langle \alpha_{M_U} | \hat{K}\Phi | 0_{M_D} \rangle \langle 0_{M_D} | (\hat{K}\Phi)^\dagger | \beta_{M_D} \rangle \\
&= \sum_\Phi \langle \alpha_{M_U} | \hat{K}\Phi | 0_{M_D} \rangle \langle 0_{M_D} | \Phi^\dagger \hat{K}^{-1} | \beta_{M_D} \rangle = \sum_\Phi \langle \alpha_{M_U} | \hat{K} | \Phi_{M_D} \rangle_{M_U \cup_{B;\hat{K}} M_D} \langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B;\hat{K}^{-1}} M_D} \\
&= \sum_\Phi Z(M_U \cup_{B;\hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D}) \langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B;\hat{K}^{-1}} M_D} = \sum_\Phi K^{-1}_{\Phi,\beta}\, Z(M_U \cup_{B;\hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D}). \quad (23)
\end{aligned}
$$

We note that in the second equality we write the identity matrix as $\mathbb{I} = \sum_\Phi (\hat{K}\Phi)|0_{M_D}\rangle\langle 0_{M_D}|(\hat{K}\Phi)^\dagger$. In the third and fourth equalities that we have $\hat{K}^{-1}$ in the inner product $\langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle$, because $\hat{K}$ as a MCG generator acts on the spatial manifold B directly. The evolution process from the first $\hat{K}^{-1}$ on the right and the second $\hat{K}$ on the left can be viewed as the *adiabatic evolution* of quantum states in the case of *fixed-point* topological orders. In the fifth equality we rewrite $\langle \alpha_{M_U} | \hat{K} | \Phi_{M_D} \rangle_{M_U \cup_{B;\hat{K}} M_D} = Z(M_U \cup_{B;\hat{K}} M_D; \alpha_{M_U}, \Phi_{M_D})$ where α_{M_U} and Φ_{M_D} may or may not be linked in the new manifold $M_U \cup_{B;\hat{K}} M_D$. In the sixth equality, we assume that both $|\beta_{M_D}\rangle$ and $|\Phi_{M_D}\rangle$ are vectors in a canonical basis, then we can define

$$
\langle \Phi_{M_D} | \hat{K}^{-1} | \beta_{M_D} \rangle_{M_D \cup_{B;\hat{K}^{-1}} M_D} \equiv K^{-1}_{\Phi,\beta} \quad (24)
$$

as a matrix element of K^{-1}, which now becomes a representation of MCG in the quasi-excitation bases of $\{|\beta_{M_D}\rangle, |\Phi_{M_D}\rangle, \dots\}$. It is important to remember that

$K^{-1}_{\Phi,\beta}$ is a quantum amplitude computed in the specific spacetime manifold $M_D \cup_{B;\hat{K}^{-1}} M_D$.

To summarize, so far we derive,

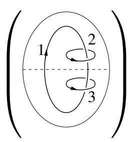
$$\boxed{Z(M;\alpha_{M_U},\beta_{M_D}) = \sum_{\Phi} K^{-1}_{\Phi,\beta}\; Z(M_U \cup_{B;\hat{K}} M_D;\alpha_{M_U},\Phi_{M_D})}. \tag{25}$$

We can also derive another formula by applying the inverse transformation,

$$\boxed{Z(M_U \cup_{B;\hat{K}} M_D;\alpha_{M_U},\Phi'_{M_D}) = \sum_{\Phi'} K_{\beta,\Phi'}\; Z(M;\alpha_{M_U},\beta_{M_D})}. \tag{26}$$

if it satisfies $KK^{-1} = \mathbb{I}$. Again we stress that $K_{\beta,\Phi'}$ is a quantum amplitude computed in the specific spacetime manifold $M_D \cup_{B;\hat{K}^{-1}} M_D$.

We now go back to derive Eqs.(18),(19) and (20). For Eq.(18), the only path integral we need to compute more explicitly is this:

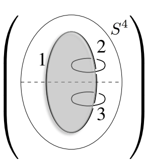
Z


$$= \langle 0_{D^2_{xt}\times S^1_y}|(W^{S^1_y}_{\sigma_1})^\dagger \hat{\mathcal{S}} W^{S^1_y}_{\sigma_2} W^{S^1_y}_{\sigma_3}|0_{D^2_{xt}\times S^1_y}\rangle$$
$$= \langle 0_{D^2_{xt}\times S^1_y}|(W^{S^1_y}_{\sigma_1})^\dagger \hat{\mathcal{S}} W^{S^1_y}_{\sigma_4} \mathcal{F}^{\sigma_4}_{\sigma_2\sigma_3}|0_{D^2_{xt}\times S^1_y}\rangle$$
$$= \sum_{\alpha\sigma_4} (G^{\alpha}_{\sigma_1})^* \mathcal{S}_{\alpha\sigma_4} \mathcal{F}^{\sigma_4}_{\sigma_2\sigma_3} = \sum_{\sigma_4} \mathcal{S}_{\bar{\sigma}_1\sigma_4} \mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3}, \tag{27}$$

where the last equality we use the canonical basis. Together with the previous data, we can easily derive Eq.(18).

Since it is convenient to express in terms of canonical bases, below for all the derivations, we will implicitly project every quantum amplitude into *canonical bases* when we write down its matrix element.

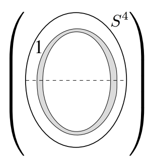
For Eq.(19), the only path integral we need to compute more explicitly is this:

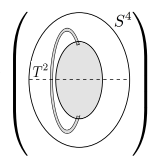
Z


$$= \langle 0_{D^2_{\varphi w}\times S^2_{\theta\phi}}|(V^{S^2_{\theta\phi}}_{\mu_1})^\dagger W^{S^1_\varphi}_{\sigma_2} W^{S^1_\varphi}_{\sigma_3}|0_{D^3_{\theta\phi w}\times S^1_\varphi}\rangle$$
$$= \langle 0_{D^2_{\varphi w}\times S^2_{\theta\phi}}|(V^{S^2_{\theta\phi}}_{\mu_1})^\dagger W^{S^1_\varphi}_{\sigma_4} (\mathcal{F}^{S^1})^{\sigma_4}_{\sigma_2\sigma_3}|0_{D^3_{\theta\phi w}\times S^1_\varphi}\rangle$$
$$= \sum_{\sigma_4} \mathrm{L}^{(S^2,S^1)}_{\mu_1\sigma_4} (\mathcal{F}^{S^1})^{\sigma_4}_{\sigma_2\sigma_3}, \tag{28}$$

again we use the canonical basis. Together with the previous data, we can easily derive Eq.(19). Similarly, we can derive Eq.(20) using the almost equivalent computation.

Now let us derive Eq.(21). In the first path integral, we create a pair of loop μ_1 and anti-loop $\bar{\mu}_1$ excitations and then annihilate them, in terms of the spacetime picture,

Z


$= Z$


$$= \mathrm{L}^{\mathrm{Tri}}_{0,0,\mu_1}, \tag{29}$$

based on the data defined earlier.

In the third path integral $\mathrm{L}^{\mathrm{Tri}}_{\mu_3,\mu_2,\mu_1}$ of Eq.(21), there are two descriptions to interpret it in terms of the braiding process in spacetime. Here is the first description. we create a pair of loop μ_1 and anti-loop $\bar{\mu}_1$ excitations and then there a pair of μ_2-$\bar{\mu}_2$ and another pair of μ_3-$\bar{\mu}_3$ are created while both pairs are thread by μ_1. Then the μ_1-μ_2-μ_3 will do the three-loop braiding process, which gives the most important Berry phase or Berry matrix information into the path integral. After then the pair of μ_2-$\bar{\mu}_2$ is annihilated and also the pair of μ_3-$\bar{\mu}_3$ is annihilated, while all the four loops are threaded by μ_1 during the process. Finally we annihilate the pair of μ_1 and $\bar{\mu}_1$ in the end [23]. The second description is that we take a Hopf link of μ_2-μ_3 linking spinning around the loop of μ_1 [26, 27]. We denote the Hopf link of μ_2-μ_3 as $\mathrm{Hopf}[\mu_3,\mu_2]$, denote its spinning as $\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]]$, and denote its linking with the third T^2-worldsheet of μ_1 as $\mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]],\mu_1]$. Thus we can define $\mathrm{L}^{\mathrm{Tri}}_{\mu_3,\mu_2,\mu_1} \equiv Z[S^4;\mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]],\mu_1]]$. From the second description, we immediate see that $\mathrm{L}^{\mathrm{Tri}}_{\mu_3,\mu_2,\mu_1}$ as $Z[S^4;\mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]],\mu_1]]$ are symmetric under exchanging $\mu_2 \leftrightarrow \mu_3$, up to an overall conjugation due to the orientation of quasi-excitations.

We can view the spacetime S^4 as a $S^4 = \mathbb{R}^4 + \{\infty\}$, the Cartesian coordinate $\mathbb{R}^4$ plus a point at the infinity $\{\infty\}$. Similar to the embedding of Ref.[26], we embed the T^2-worldsheets μ_1,μ_2,μ_3 into the $(X_1,X_2,X_3,X_4) \in \mathbb{R}^4$

as follows:

$$\begin{cases} X_1(u,\vec{x}) = [r_1(u) + (r_2(u) + r_3(u)\cos x)\cos y]\cos z, \\ X_2(u,\vec{x}) = [r_1(u) + (r_2(u) + r_3(u)\cos x)\cos y]\sin z, \\ X_3(u,\vec{x}) = (r_2(u) + r_3(u)\cos x)\sin y, \\ X_4(u,\vec{x}) = r_3(u)\sin x, \end{cases} \tag{30}$$

here $\vec{x} \equiv (x,y,z)$. We choose the T^2-worldsheets as follows.
The T^2-worldsheet μ_1 is parametrized by some fixed u_1 and free coordinates of (z,x) while $y=0$ is fixed.
The T^2-worldsheet μ_2 is parametrized by some fixed u_2 and free coordinates of (x,y) while $z=0$ is fixed.
The T^2-worldsheet μ_3 is parametrized by some fixed u_3 and free coordinates of (y,z) while $x=0$ is fixed.
We can set the parameters $u_1 > u_2 > u_3$. Meanwhile, a T^3-surface can be defined as $\mathcal{M}^3(u,\vec{x}) \equiv (X_1(u,\vec{x}), X_2(u,\vec{x}), X_3(u,\vec{x}), X_4(u,\vec{x}))$ with a fixed u and free parameters $\vec{x}$. The T^3-surface $\mathcal{M}^3(u,\vec{x}) \equiv (X_1(u,\vec{x}), X_2(u,\vec{x}), X_3(u,\vec{x}), X_4(u,\vec{x}))$ encloses a 4-dimensional volume. We define the enclosed 4-dimensional volume as the $\mathcal{M}^3(u,\vec{x}) \times I^1(s)$ where $I^1(s)$ is the 1-dimensional radius interval along r_3, such that $I^1(s) = \{s | s = [0, r_3(u)]\}$, namely $0 \leq s \leq r_3(u)$. Here we can define $r_3(0) = 0$. The topology of the enclosed 4-dimensional volume of $\mathcal{M}^3(u,\vec{x}) \times I^1(s)$ is of course the $T^3 \times I^1 = T^2 \times (S^1 \times I^1) = T^2 \times D^2$. For a $\mathcal{M}^3(u_{\text{large}},\vec{x}) \times I^1(s)$ prescribed by a fixed larger u_{large} and free parameters $\vec{x}$, the $\mathcal{M}^3(u_{\text{large}},\vec{x}) \times I^1(s)$ must enclose the 4-volume spanned by the past history of $\mathcal{M}^3(u_{\text{small}},\vec{x}) \times I^1(s)$, for any $u_{\text{large}} > u_{\text{small}}$. Here we set $u_1 > u_2 > u_3$. And we also set $r_1(u) > r_2(u) > r_3(u)$ for any given u.

One can check that the three T^2-worldsheet μ_1, μ_2 and μ_3 indeed have the nontrivial *triple-linking number* [28]. We can design the triple-linking number to be: $\text{Tlk}(\mu_2,\mu_1,\mu_3) = \text{Tlk}(\mu_3,\mu_1,\mu_2) = 0$, $\text{Tlk}(\mu_1,\mu_2,\mu_3) = +1$, $\text{Tlk}(\mu_3,\mu_2,\mu_1) = -1$, $\text{Tlk}(\mu_2,\mu_3,\mu_1) = +1$, $\text{Tlk}(\mu_1,\mu_3,\mu_2) = -1$.

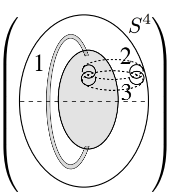
Below we will frequently use the surgery trick by cutting out a tubular neighborhood $D^2 \times T^2$ of the T^2-worldsheet and re-gluing this $D^2 \times T^2$ back to its complement $S^4 \smallsetminus D^2 \times T^2$ via the modular $\mathcal{S}^{xyz}$-transformation. The $\mathcal{S}^{xyz}$-transformation sends

$$\begin{pmatrix} x_{\text{out}} \\ y_{\text{out}} \\ z_{\text{out}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{\text{in}} \\ y_{\text{in}} \\ z_{\text{in}} \end{pmatrix}. \tag{31}$$

Thus, the $\mathcal{S}^{xyz}$-identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (z_{\text{in}}, x_{\text{in}}, y_{\text{in}})$. The $(\mathcal{S}^{xyz})^{-1}$-identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (y_{\text{in}}, z_{\text{in}}, x_{\text{in}})$. The surgery on the initial S^4 outcomes a new manifold,

$$(D^2 \times T^2) \cup_{T^3;\mathcal{S}^{xyz}} (S^4 \smallsetminus D^2 \times T^2) = S^3 \times S^1 \# S^2 \times S^2. \tag{32}$$

In terms of the spacetime path integral picture, use Eqs.(25) and (26), we derive:

$Z\Bigg($ 

$\Bigg) \equiv \mathrm{L}^{\text{Tri}}_{\mu_3,\mu_2,\mu_1} = Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3,\mu_2]],\mu_1]]$

$$= \sum_{\mu_3'} \mathcal{S}^{xyz}_{\mu_3',\mu_3}\, Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \mu_2 \parallel \mu_3') \tag{33}$$

$$= \sum_{\mu_3',\Gamma_2} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'}\, Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \Gamma_2) \tag{34}$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2'} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2}\, Z(S^4; \mu_1, \Gamma_2') \tag{35}$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2',\Gamma_2''} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2'',\Gamma_2'}\, Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \Gamma_2'') \tag{36}$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2',\Gamma_2'',\eta_2} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2'',\Gamma_2'} (\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\Gamma_2''}\, Z(S^3 \times S^1 \# S^2 \times S^2; \eta_2) \tag{37}$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2',\Gamma_2'',\eta_2,\eta_2'} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2'',\Gamma_2'} (\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\Gamma_2''} \mathcal{S}^{xyz}_{\eta_2',\eta_2}\, \mathrm{L}^{\text{Tri}}_{0,0,\eta_2'}. \tag{38}$$

As usual, the repeated indices are summed over. With the trick of $\mathcal{S}^{xyz}$-transformation in mind, here is the step-by-step sequence of surgeries we perform.

Step 1: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2-worldsheet of μ_3 and re-glue this $D^2 \times T^2$

back to its complement $S^4 \smallsetminus D^2 \times T^2$ via the modular $(\mathcal{S}^{xyz})^{-1}$-transformation. The $D^2 \times T^2$ neighborhood of μ_3-worldsheet can be viewed as the 4-volume $\mathcal{M}^3(u_3, \vec{x}) \times I^1(s)$, which encloses neither μ_1-worldsheet nor μ_2-worldsheet. The $(\mathcal{S}^{xyz})^{-1}$-transformation sends $(y_{\text{in}}, z_{\text{in}})$ of μ_3 to $(x_{\text{out}}, y_{\text{out}})$ of μ_2. The gluing however introduces the summing-over new coordinate μ_3', based on Eq.(25). Thus Step 1 obtains Eq.(33).

In Step 1, as Eq.(33) and thereafter, we write down $\mathcal{S}^{xyz}_{\mu_3',\mu_3}$ matrix. Based on Eq.(24), we stress that the $\mathcal{S}^{xyz}_{\mu_3',\mu_3}$ is projected to the $|0_{D^2\times T^2}\rangle$-states with operator-insertions for both bra and ket states.

$$\begin{aligned}\mathcal{S}^{xyz}_{\mu_3',\mu_3} &\equiv \langle \mu'_{3\,D^2\times T^2} | \hat{\mathcal{S}}^{xyz} | \mu_{3\,D^2\times T^2} \rangle_{D^2\times T^2 \cup_{T^3;\hat{\mathcal{S}}^{xyz}} D^2\times T^2} \\ &= \langle 0_{D^2_{xw}\times T^2_{yz}} | V^{T^2_{yz}\dagger}_{\mu_3'} \hat{\mathcal{S}}^{xyz} V^{T^2_{yz}}_{\mu_3} | 0_{D^2_{xw}\times T^2_{yz}} \rangle_{S^3\times S^1} \qquad (39)\end{aligned}$$

Here we use the surgery fact

$$D^2 \times T^2 \cup_{T^3;\hat{\mathcal{S}}^{xyz}} D^2 \times T^2 = S^3 \times S^1. \qquad (40)$$

So our $\mathcal{S}^{xyz}_{\mu_3',\mu_3}$ is defined as a quantum amplitude in $S^3 \times S^1$. Two T^2-worldsheets μ_3' and μ_3 now become a pair of Hopf link resides in S^3 part of $S^3 \times S^1$, while share the same S^1 circle in the S^1 part of $S^3 \times S^1$. We can view the shared S^1 circle as the spinning circle of the spun surgery construction on the Hopf link in D^3, the spun-topology would be $D^3 \times S^1$, then we glue this $D^3 \times S^1$ contains Spun[Hopf[μ_3', μ_3]] to another $D^3 \times S^1$, so we have $D^3 \times S^1 \cup_{S^2 \times S^1} D^3 \times S^1 = S^3 \times S^1$ as an overall new spacetime topology. Hence we also denote

$$\mathcal{S}^{xyz}_{\mu_3',\mu_3} = Z[S^3 \times S^1; \text{Spun}[\text{Hopf}[\mu_3', \mu_3]]]. \qquad (41)$$

Step 2: The earlier surgery now makes the inner μ_3'-worldsheet *parallels* to the outer μ_2-worldsheet, since they share the same coordinates $(x_{\text{out}}, y_{\text{out}}) = (y_{\text{in}}, z_{\text{in}})$. We denote their parallel topology as $\mu_2 \parallel \mu_3'$. So we can fuse the μ_2-worldsheet and μ_3'-worldsheet via the fusion algebra, namely $V^{T^2_{x_{\text{out}},y_{\text{out}}}}_{\mu_2} V^{T^2_{x_{\text{out}},y_{\text{out}}}}_{\mu_3'} = (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} V^{T^2_{x_{\text{out}},y_{\text{out}}}}_{\Gamma_2}$. Thus Step 2 obtains Eq.(34).

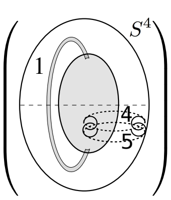
Step 3: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2-worldsheet of Γ_2 and re-glue this $D^2 \times T^2$ back to its complement $S^4 \smallsetminus D^2 \times T^2$ via the modular $\mathcal{S}^{xyz}$-transformation. The $D^2 \times T^2$ neighborhood of Γ_2-worldsheet can be viewed as the 4-volume $\mathcal{M}^3(u_2, \vec{x}) \times I^1(s)$ in the new manifold $S^3 \times S^1 \# S^2 \times S^2$, which encloses no worldsheet inside. After the surgery, the $\mathcal{S}^{xyz}$-transformation sends the redefined $(x_{\text{in}}, y_{\text{in}})$ of Γ_2 back to $(y_{\text{out}}, z_{\text{out}})$ of Γ_2'. The gluing however introduces the summing-over new coordinate Γ_2', based on Eq.(25). We also transform $S^3 \times S^1 \# S^2 \times S^2$ back to S^4 again. Thus Step 3 obtains Eq.(35).

Step 4: We cut out the tubular neighborhood $D^2 \times T^2$ of the T^2-worldsheet of Γ_2' and re-glue this $D^2 \times T^2$ back to its complement $S^4 \smallsetminus D^2 \times T^2$ via the modular $\mathcal{S}^{xyz}$-transformation. The $D^2 \times T^2$ neighborhood of ${\Gamma_2}'$-worldsheet viewed as the 4-volume in the manifold S^4 encloses no worldsheet inside. After the surgery, the $\mathcal{S}^{xyz}$-transformation sends the $(x_{\text{in}}, y_{\text{in}})$ of Γ_2' to $(z_{\text{out}}, x_{\text{out}})$ of μ_1. The gluing however introduces the summing-over new coordinate Γ_2'', based on Eq.(25). We also transform S^4 to $S^3 \times S^1 \# S^2 \times S^2$ again. Thus Step 4 obtains Eq.(36).

Step 5: The earlier surgery now makes the inner Γ_2''-worldsheet *parallels* to the outer μ_1-worldsheet, since they share the same coordinates $(z_{\text{out}}, x_{\text{out}}) = (x_{\text{in}}, y_{\text{in}})$. We denote their parallel topology as $\mu_1 \parallel \Gamma_2''$. We now fuse the μ_1-worldsheet and Γ_2''-worldsheet via the fusion algebra, namely $V^{T^2_{z_{\text{out}},x_{\text{out}}}}_{\mu_1} V^{T^2_{z_{\text{out}},x_{\text{out}}}}_{\Gamma_2''} = (\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\Gamma_2''} V^{T^2_{z_{\text{out}},x_{\text{out}}}}_{\eta_2}$. Thus Step 5 obtains Eq.(37).

Step 6: We should do the inverse transformation to get back to the S^4 manifold. Thus we cut out the tubular neighborhood $D^2 \times T^2$ of the T^2-worldsheet of η_2 and re-glue this $D^2 \times T^2$ back to its complement via the modular $(\mathcal{S}^{xyz})^{-1}$-transformation. We relate the original path integral to the final one $Z(S^4; \eta_2') = \text{L}^{\text{Tri}}_{0,0,\eta_2'}$. Thus Step 6 obtains Eq.(38).

Similarly, in the fourth path integral of Eq.(21), we derive

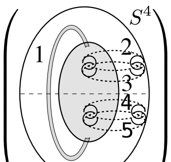
$Z\Big($ 

$\Big)$

$$\begin{aligned} &\equiv \text{L}^{\text{Tri}}_{\mu_5,\mu_4,\mu_1} = Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_5, \mu_4]], \mu_1]] \\ &= \sum_{\mu_5', \Gamma_4, \Gamma_4', \Gamma_4'', \eta_4, \eta_4'} \mathcal{S}^{xyz}_{\mu_5',\mu_5} (\mathcal{F}^{T^2})^{\Gamma_4}_{\mu_4\mu_5'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4',\Gamma_4} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4'',\Gamma_4'} (\mathcal{F}^{T^2})^{\eta_4}_{\mu_1\Gamma_4''} \mathcal{S}^{xyz}_{\eta_4',\eta_4} \; \text{L}^{\text{Tri}}_{0,0,\eta_4'}. \qquad (42)\end{aligned}$$

In the second path integral of Eq.(21), we have the Hopf link of Hopf$[\mu_3,\mu_2]$ and the Hopf link of Hopf$[\mu_5,\mu_4]$. In the spacetime picture, all μ_2,μ_3,μ_4,μ_5 are T^2-worldsheets under the spun surgery construction. We can locate the the spun object named Spun[Hopf$[\mu_3,\mu_2]$, Hopf$[\mu_5,\mu_4]$] inside a $D^3\times S^1$, while this $D^3\times S^1$ is glued with a $S^2\times D^2$ to a S^4. Here the $S^2\times D^2$ contains a T^2-worldsheet μ_1. We can view the T^2-worldsheet μ_1 contains a S^2-sphere of the $S^2\times D^2$ but attached an extra handle. We derive:

$$Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) \equiv Z[S^4;\text{Link}[\text{Spun}[\text{Hopf}[\mu_3,\mu_2],\text{Hopf}[\mu_5,\mu_4]]],\mu_1]]$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2'} \sum_{\mu_5',\Gamma_4,\Gamma_4'} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} \mathcal{S}^{xyz}_{\mu_5',\mu_5} (\mathcal{F}^{T^2})^{\Gamma_4}_{\mu_4\mu_5'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4',\Gamma_4} Z[S^4;\text{Spun}[\Gamma_2',\Gamma_4'],\mu_1] \tag{43}$$

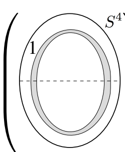
$$= \sum_{\mu_3',\Gamma_2,\Gamma_2'} \sum_{\mu_5',\Gamma_4,\Gamma_4'} \sum_{\Gamma} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} \mathcal{S}^{xyz}_{\mu_5',\mu_5} (\mathcal{F}^{T^2})^{\Gamma_4}_{\mu_4\mu_5'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4',\Gamma_4} (\mathcal{F}^{T^2})^{\Gamma}_{\Gamma_2',\Gamma_4'} Z[S^4;\Gamma,\mu_1] \tag{44}$$

$$= \sum_{\substack{\mu_3',\Gamma_2,\Gamma_2'\\ \mu_5',\Gamma_4,\Gamma_4'}} \sum_{\Gamma,\Gamma',\Gamma_1,\Gamma_1'} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2} \mathcal{S}^{xyz}_{\mu_5',\mu_5} (\mathcal{F}^{T^2})^{\Gamma_4}_{\mu_4\mu_5'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4',\Gamma_4} (\mathcal{F}^{T^2})^{\Gamma}_{\Gamma_2',\Gamma_4'} (\mathcal{S}^{xyz})^{-1}_{\Gamma',\Gamma} (\mathcal{F}^{T^2})^{\Gamma_1}_{\mu_1\Gamma'} \mathcal{S}^{xyz}_{\Gamma_1',\Gamma_1}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\Gamma_1'}. \tag{45}$$

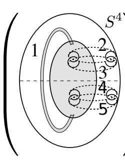
Here we do the Step 1, Step 2 and Step 3 surgeries on Spun[Hopf$[\mu_3,\mu_2]$] first, then do the same 3-step surgeries on Spun[Hopf$[\mu_5,\mu_4]$] later, then we obtain Eq.(43). While in Eq.(43), the new T^2-worldsheets Γ_2' and Γ_4' have no triple-linking with the worldsheet μ_1. Here Γ_2' and Γ_4' are arranged in the $D^3\times S^1$ part of the S^4 manifold, while μ_1 is in the $S^2\times D^2$ part of the S^4 manifold. Indeed, Γ_2' and Γ_4' can be fused together in parallel to a new T^2-worldsheet Γ via the fusion algebra $(\mathcal{F}^{T^2})^{\Gamma}_{\Gamma_2',\Gamma_4'}$, so we obtain Eq.(44). Then we apply the Step 4, Step 5 and Step 6 surgeries on the T^2-worldsheets Γ and μ_1 of $Z[S^4;\text{Spun}[\Gamma],\mu_1] = Z[S^4;\Gamma,\mu_1]$ in Eq.(44), we obtain the final form Eq.(45).

Use Eqs.(29),(38),(42) and (45), and plug them into the path integral surgery relations, we derive a new quantum surgery formula (namely Eq.(19) in the main text):

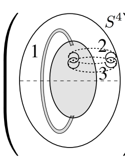
$$Z\left(\vphantom{\int}\right.$$



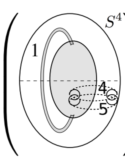
$$\left.\vphantom{\int}\right) Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) Z\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right)$$

$$\Rightarrow \mathrm{L}^{\mathrm{Tri}}_{0,0,\mu_1} \cdot \sum_{\Gamma,\Gamma',\Gamma_1,\Gamma_1'} (\mathcal{F}^{T^2})^{\Gamma}_{\Gamma_2',\Gamma_4'} (\mathcal{S}^{xyz})^{-1}_{\Gamma',\Gamma} (\mathcal{F}^{T^2})^{\Gamma_1}_{\mu_1\Gamma'} \mathcal{S}^{xyz}_{\Gamma_1',\Gamma_1}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\Gamma_1'}$$
$$= \sum_{\Gamma_2'',\eta_2,\eta_2'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2'',\Gamma_2'} (\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\Gamma_2''} \mathcal{S}^{xyz}_{\eta_2',\eta_2}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\eta_2'} \cdot \sum_{\Gamma_4'',\eta_4,\eta_4'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_4'',\Gamma_4'} (\mathcal{F}^{T^2})^{\eta_4}_{\mu_1\Gamma_4''} \mathcal{S}^{xyz}_{\eta_4',\eta_4}\ \mathrm{L}^{\mathrm{Tri}}_{0,0,\eta_4'}, \tag{46}$$

here only $\mu_1,\Gamma_2',\Gamma_4'$ are the fixed indices, other indices are summed over.

Lastly we provide more explicit calculations of Eq.(22), the constraint between the fusion data itself. First, we recall that

$$W^{S^1_y}_{\sigma_1} W^{S^1_y}_{\sigma_2} = (\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2} W^{S^1_y}_{\sigma_3}, \tag{47}$$

$$V^{T^2_{yz}}_{\mu_1} V^{T^2_{yz}}_{\mu_2} = (\mathcal{F}^{T^2})^{\mu_3}_{\mu_1\mu_2} V^{T^2_{yz}}_{\mu_3}, \tag{48}$$

$$W^{S^1_y}_{\sigma_1} V^{T^2_{yz}}_{\mu_2} = (\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_2} V^{T^2_{yz}}_{\mu_3}. \tag{49}$$

Of course, the fusion algebra is symmetric respect to exchanging the lower indices, $(\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_2} = (\mathcal{F}^{T^2})^{\mu_3}_{\mu_2\sigma_1}$. We can regard the fusion algebra $(\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2}$ and $(\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_2}$ with worldlines as a part of a larger algebra of the fusion algebra of worldsheets $(\mathcal{F}^{T^2})^{\mu_3}_{\mu_1\mu_2}$. We compute the state $W^{S^1_y}_{\sigma_1} W^{S^1_y}_{\sigma_2} V^{T^2_{yz}}_{\mu_2} |0_{D^2_{wx}\times T^2_{yz}}\rangle$ by fusing two $W^{S^1}_{\sigma}$ operators and one $V^{T^2}_{\mu}$ operator in different orders.

On one hand, we can fuse two worldlines first, then fuse with the worldsheet,

$$W^{S^1_y}_{\sigma_1} W^{S^1_y}_{\sigma_2} V^{T^2_{yz}}_{\mu_2} |0_{D^2_{wx}\times T^2_{yz}}\rangle$$
$$= \sum_{\sigma_3} (\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2} W^{S^1_y}_{\sigma_3} V^{T^2_{yz}}_{\mu_2} |0_{D^2_{wx}\times T^2_{yz}}\rangle$$
$$= \sum_{\sigma_3,\mu_3} (\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2} (\mathcal{F}^{T^2})^{\mu_3}_{\sigma_3\mu_2} V^{T^2_{yz}}_{\mu_3} |0_{D^2_{wx}\times T^2_{yz}}\rangle. \tag{50}$$

On the other hand, we can fuse a worldline with the worldsheet first, then fuse with another worldline,

$$W^{S^1_y}_{\sigma_1} W^{S^1_y}_{\sigma_2} V^{T^2_{yz}}_{\mu_2} |0_{D^2_{wx}\times T^2_{yz}}\rangle$$
$$= \sum_{\mu_1} W^{S^1_y}_{\sigma_1} (\mathcal{F}^{T^2})^{\mu_1}_{\sigma_2\mu_2} V^{T^2_{yz}}_{\mu_1} |0_{D^2_{wx}\times T^2_{yz}}\rangle$$
$$= \sum_{\mu_1,\mu_3} (\mathcal{F}^{T^2})^{\mu_1}_{\sigma_2\mu_2} (\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_1} V^{T^2_{yz}}_{\mu_3} |0_{D^2_{wx}\times T^2_{yz}}\rangle \quad (51)$$

Therefore, by comparing Eqs.(50) and (51), we derive a consistency condition for fusion algebra:

$$\sum_{\sigma_3} (\mathcal{F}^{S^1})^{\sigma_3}_{\sigma_1\sigma_2} (\mathcal{F}^{T^2})^{\mu_3}_{\sigma_3\mu_2} = \sum_{\mu_1} (\mathcal{F}^{T^2})^{\mu_1}_{\sigma_2\mu_2} (\mathcal{F}^{T^2})^{\mu_3}_{\sigma_1\mu_1}. \quad (52)$$

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[14] This is consistent with the fact that TQFT assigns a complex number to a closed manifold without boundary, while assigns a state-vector in the Hilbert space to an open manifold with a boundary.
[15] Our worldline or worldsheet operator corresponds to the Wilson or 't Hooft operator of the gauge theory, although throughout our work we consider a more generic quantum description without limiting to gauge or field theory.
[16] If there is a gauge theory description, then the quasi-excitation type of particle σ and string μ would be labeled by the representation for gauge charge and by the conjugacy class of the gauge group for gauge flux.
[17] We may simplify the gluing notation $\mathcal{M}_1 \cup_{\mathcal{B};\mathcal{U}} \mathcal{M}_2$ to $\mathcal{M}_1 \cup_{\mathcal{B}} \mathcal{M}_2$ if the mapping class group's generator $\mathcal{U}$ is trivial or does not affect the glued manifold. We may simplify the gluing notation further to $\mathcal{M}_1 \cup \mathcal{M}_2$ if the boundary $\mathcal{B} = \partial\mathcal{M}_1 = \partial\mathcal{M}_2$ is obvious or stated in the text earlier.
[18] The three-step surgeries we describe earlier sequently send the initial 3-sphere configuration with Borromean rings insertion $S^3 \xrightarrow{\text{1st surgery}} S^2 \times S^1 \xrightarrow{\text{2nd surgery}} S^2 \times S^1 \# S^2 \times S^1 \xrightarrow{\text{3rd surgery}} T^3$ to a 3-torus configuration. Here we use the notation $\mathcal{M}_1 \# \mathcal{M}_2$ means the connected sum of manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$.
[19] To be even more specific, we can also specify the whole open-manifold topology $M \times V$ corresponding to the ground state sector $|0_{M\times V}\rangle$, namely we can rewrite the data in new notations to introduce the refined data: $(F^{S^1}) \to (F^{D^3\times S^1})$, $(F^{T^2}) \to (F^{D^2\times T^2})$ and $(F^{S^2}) \to (F^{D^2\times S^2})$. However, because (F^M) are the fusion data in the local neighborhood around the worldline/worldsheet operators with topology M, physically it does not encode the information from the remained product space $M \times V$. Namely, at least for the most common and known theory that we can regularize on the lattice, we understand that $(F^M) = (F^{M\times V_1}) = (F^{M\times V_2}) = \dots$ for any topology $V_1, V_2, \dots$. So here we make a physical assumption that $(F^{S^1}) = (F^{D^3\times S^1})$, $(F^{T^2}) = (F^{D^2\times T^2})$ and $(F^{S^2}) = (F^{D^2\times S^2})$.
[20] Throughout our work, we consistently use σ to represent the quasi-particle label (such as charge or flux) for worldlines $W^{S^1}_\sigma$, and we use μ to represent the quasi-string label for worldsheets, e.g. $V^{S^2}_\mu$ and $V^{T^2}_\mu$.
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[29] By the spun surgery, hereafter we mean that taking an object (such as a Hopf link in this case) inside a $(d-1)$-disk D^{d-1} and spinning it around a fixed axis, so to make it a $D^{d-1} \times S^1$. Then we will glue this $D^{d-1} \times S^1$ with another manifold $\mathcal{M}'^{d-1}$ who shares a common boundary as $\partial(\mathcal{M}'^{d-1}) = \partial(D^{d-1} \times S^1)$. The final manifold is constructed as $D^{d-1} \times S^1 \cup \mathcal{M}'^{d-1}$. In this case, we have

$D^3 \times S^1 \cup S^2 \times D^2 = S^4$, while we insert a T^2 inside the $S^2 \times D^2$ part, so that the T^2 is linked with the spun Hopf link.

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[31] The four-step surgeries here sequently send the initial configuration 4-sphere into: $S^4 = (D^3 \times S^1) \cup (S^2 \times D^2) = ((S^3 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \xrightarrow{\text{1st surgery}} ((S^2 \times S^1 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \xrightarrow{\text{2nd surgery}} ((S^2 \times S^1 \# S^2 \times S^1 \setminus D^3) \times S^1) \cup (S^2 \times D^2) \xrightarrow{\text{3rd surgery}} ((T^3 \setminus D^3) \times S^1) \cup (S^2 \times D^2) = (T^4 \setminus D^3 \times S^1) \cup (S^2 \times D^2) \xrightarrow{\text{4th surgery}} T^4 \# S^2 \times S^2$ to a connected sum of T^4-torus and $S^2 \times S^2$ configuration. Here we use the notation $\mathcal{M}_1 \# \mathcal{M}_2$ means the connected sum of manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$. The outcome $Z[T^4 \# S^2 \times S^2; \mu_4', \mu_3', \mu_2', \mu_1']$ has the wonderful desired property that we can design the worldsheets along the generator of homology groups so the operator insertions are well-defined. Here $V_{\mu_1'}^{T^2_{yz}}$ is the worldsheet operator acting along the only generator of homology group $H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$. And $V_{\mu_2'}^{T^2}, V_{\mu_3'}^{T^2}, V_{\mu_4'}^{T^2}$ are the worldsheet operators acting along three among the seven generators of $H_2(T^4 \# S^2 \times S^2 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^7$ with the hollowed $D^2 \times T^2$ specified in the earlier text. This shows that $Z[T^4 \# S^2 \times S^2; \mu_4', \mu_3', \mu_2', \mu_1']$ is a better quantum number easier to be computed than $Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$.

[32] In this work, we project $\hat{\mathcal{S}}^{xyz}$ and $\hat{\mathcal{T}}^{xy}$ into the braket bases of $|\mu_{D^2 \times T^2}\rangle \equiv V_\mu^{T^2_{yz}} |0_{D^2 \times T^2}\rangle$. Our representation of $\mathcal{S}^{xyz}_{\mu_2, \mu_1} \equiv \langle \mu_{2\,D^2 \times T^2} | \hat{\mathcal{S}}^{xyz} | \mu_{1\,D^2 \times T^2} \rangle = \langle 0_{D^2_{xw} \times T^2_{yz}} | V_{\mu_2}^{T^2_{yz}\dagger} \hat{\mathcal{S}}^{xyz} V_{\mu_1}^{T^2_{yz}} | 0_{D^2_{xw} \times T^2_{yz}} \rangle = Z[S^3 \times S^1; \text{Spun}[\text{Hopf}[\mu_2, \mu_1]]]$ is effectively a path integral of a spun Hopf link in the spacetime manifold $D^2 \times T^2 \cup_{T^3; \hat{\mathcal{S}}^{xyz}} D^2 \times T^2 = S^3 \times S^1$. Our representation of $\mathcal{T}^{xy}_{\mu_2, \mu_1} \equiv \langle \mu_{2\,D^2 \times T^2} | \hat{\mathcal{T}}^{xy} | \mu_{1\,D^2 \times T^2} \rangle = \langle 0_{D^2_{xw} \times T^2_{yz}} | V_{\mu_2}^{T^2_{yz}\dagger} \hat{\mathcal{T}}^{xy} V_{\mu_1}^{T^2_{yz}} | 0_{D^2_{xw} \times T^2_{yz}} \rangle$ is effectively a $Z[S^2 \times S^1 \times S^1]$-path integral in the spacetime manifold $D^2 \times T^2 \cup_{T^3; \hat{\mathcal{T}}^{xy}} D^2 \times T^2 = S^2 \times S^1 \times S^1$. In addition, the worldsheet operator $V_\mu^{T^2_{yz}}$ effectively contains also worldline operators, e.g. $W^{S^1_y}$ and $W^{S^1_z}$ along y and z directions. Namely we mean that $V_\mu^{T^2_{yz}} = W^{S^1_y} W^{S^1_z} V^{T^2_{yz}}$, so $W^{S^1_y}$ and $W^{S^1_z}$ are along the two generators of homology group $H_1(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2$, while $V^{T^2_{yz}}$ is along the unique one generator of homology group $H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$. If there is a gauge theory description, then we project our $\hat{\mathcal{S}}^{xyz}$ and $\hat{\mathcal{T}}^{xy}$ into a one-flux (conjugacy class) and two-charge (representation) basis $|\mu_1, \sigma_2, \sigma_3\rangle \equiv V_{\mu_1}^{T^2_{yz}} W_{\sigma_2}^{S^1_y} W_{\sigma_3}^{S^1_z} |0_{D^2_{xw} \times T^2_{yz}}\rangle$. So our projection here is different from the one-charge (representation) and two-flux (conjugacy class) basis $|\sigma_1, \mu_2, \mu_3\rangle$ used in Ref.[23–25]. The $|\sigma_1, \mu_2, \mu_3\rangle$-bases can be obtained through $W_{\sigma_1}^{S^1_x} V_{\mu_2}^{T^2_{xy}} V_{\mu_3}^{T^2_{xz}} |0_{S_4 \setminus D^2_{xw} \times T^2_{yz}}\rangle$, where $W^{S^1_x}$ is along the generator of homology group $H_1(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$, while $V_{\mu_2}^{T^2_{xy}}$ and $V_{\mu_3}^{T^2_{xz}}$ are along the two generators of homology group $H_2(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}^2$ via the Alexander duality. See also Supplemental Material.

[33] Presumably there may be defect-like excitation of particles and strings on the spatial slice cross-section B. If the dimensional of Hilbert space on the spatial slice B is 1, namely the ground state degeneracy (GSD) is 1, then we can derive the gluing identity $\langle M_U | M_D \rangle = \langle N_U | N_D \rangle \Rightarrow \langle M_U | N_D \rangle = \langle N_U | M_D \rangle$ because the vector space is 1-dimensional and all vectors are parallel in the inner product.

[34] In the canonical basis when $\mathcal{S}$ is invertible, we can massage our formula to $\mathcal{N}^a_{\sigma_2 \sigma_3} = \sum_{\bar{\sigma}_1} \frac{\mathcal{S}_{\bar{\sigma}_1 \sigma_2} \mathcal{S}_{\bar{\sigma}_1 \sigma_3} (\mathcal{S}^{-1})_{\bar{\sigma}_1 a}}{\mathcal{S}_{\bar{\sigma}_1 0}}$.

[35] See further discussions in Supplemental Material.

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